Constraints On Short, Hard Gamma-Ray Burst Beaming Angles From Gravitational Wave Observations

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(Dated: January 29, 2014)

Apologies in advance for inconsistent conditioning statements in probabilities ...

Contents

I. INTRODUCTION

It is common in the literature to draw inferences on the rate of binary coalescence $\mathcal{R}$, given some estimate for the beaming angle θ and the observed rate of sGRBs $\mathcal{R}_{\rm grb}$. In this work, we investigate what statements can *currently* be made on the beaming angle itself using the upper limits placed on $\mathcal{R}$ from all-sky, all-time gravitational wave searches and explore the potential for direct inference of sGRB beaming angles in the advanced detector era.

II. INFERRING THE SGRB BEAMING ANGLE FROM RATE MEASUREMENTS

Assuming that at least some fraction of sGRBs are due to compact binary coalescence, the observed rate of sGRBs may be written,

$$\mathcal{R}_{\rm grb} = \epsilon\mathcal{R}(1-\cos\theta), \tag{1}$$

where $\mathcal{R}$ is the rate of binary coalescence, θ is the beaming angle of the outflow from the GRB and ϵ is the (generally unknown) probability that a binary coalescence results in an observed sGRB. In this work, we assume $\mathcal{R}_{\rm grb} = 10\,{\rm Gpc}^{-3}\,{\rm yr}^{-1}$ [1, 2].

Inferences of the GRB beaming angle are made from the posterior probability density on the beaming angle $p(\theta|D,I)$. Our goal then, is to transform the measured posterior probability density on the rate $\mathcal{R}$ to a posterior on the beaming angle. First, note that the PDF $p(\theta|D,I)$ may be written as a marginalisation over ϵ in the the joint PDF $p(\theta,\epsilon|D,I)$:

$$p(\theta) = \int_\epsilon p(\theta,\epsilon)\ {\rm d}\epsilon, \tag{2}$$

where we have dropped the conditioning statements temporarily for notational convenience. Next, we note that the joint probability $p(\theta, \mathcal{R})$ can be written in terms of ϵ and $\mathcal{R}$ according to,

$$p(\theta, \epsilon) = p(\mathcal{R}, \epsilon) \left|\left| \frac{\partial(\mathcal{R}, \epsilon)}{\partial(\theta, \epsilon)} \right|\right|, \tag{3}$$

where the Jacobian matrix is given by,

$$\frac{\partial(\mathcal{R}, \epsilon)}{\partial(\theta, \epsilon)} = \begin{bmatrix} \frac{\partial \mathcal{R}}{\partial \theta} & \frac{\partial \mathcal{R}}{\partial \epsilon} \\ \frac{\partial \epsilon}{\partial \theta} & \frac{\partial \epsilon}{\partial \epsilon} \end{bmatrix}. \tag{4}$$

Bringing all of these terms together then, and writing out the Jacobian matrix determinant, we have

$$p(\theta) = \frac{2\mathcal{R}_{\mathrm{grb}} \sin\theta}{(\cos\theta - 1)^2} \int_{\epsilon} \frac{p(\epsilon)p(\mathcal{R})}{\epsilon} \, \mathrm{d}\epsilon, \tag{5}$$

where we have assumed ϵ and $\mathcal{R}$ are logically independent such that,

$$p(\epsilon, \mathcal{R}) = p(\epsilon|\mathcal{R})p(\mathcal{R}) = p(\epsilon)p(\mathcal{R}). \tag{6}$$

To find the target PDF $p(\theta|D, I)$ then, we require the measured coalescence rate posterior $p(\mathcal{R}|D, I)$, and we must specify a prior PDF for ϵ which encapsulates assumptions made about the relative rates of compact binary mergers and sGRBs.

Following [3, 4], the posterior on the binary coalescence rate may be determined from the loudest event in the gravitational wave analysis. Specifically, for a foreground event rate due to binary coalescence $\mathcal{R}$, the probability of obtaning no events with ranking statistic ρ greater than the observed loudested event ρ^* is,

$$P_F(\rho^*|\mathcal{R}, C_L, T) = e^{-\mathcal{R}C_L(\rho^*)T}, \tag{7}$$

where $C_L(\rho^*)$ is the total luminosity to which the search is sensitive and T is the duration of the search. The overall probability of obtaining no events with ranking statistic $\rho > \rho^*$ is the product of obtaining no such events from foreground *and* the probability of obtaining no such events from the background in the detector, denoted $P_B(\rho^*)$,

$$P(\rho^*|\mathcal{R}, I) = P_B(\rho^*|I)e^{-\mathcal{R}C_L(\rho^*)T} \tag{8}$$

Using a uniform prior on $\mathcal{R}$ and inverting the overall probability with Bayes' theorem, we arrive at,

$$p(\mathcal{R}|C_L(\rho^*, T, \Lambda) \propto p(\mathcal{R}) \left[\frac{1 + \Lambda C_L(\rho^*)T}{1 + \Lambda} \right] e^{-\mathcal{R}C_L(\rho^*)T} \tag{9}$$

where $p(\mathcal{R})$ is the prior probability distribution on the rate. The quantity $\hat{\Lambda}$ measures the relative probability of detecting an event with ranking statistic x due to gravitational waves versus the probability of an equally loud event arising in the background distribution. The reader is directed to section 3 of [4] for a full discussion of this quantity.

III. BEAMING ANGLE LIMITS FROM PAST GW SEARCHES

In this section, we focus attention on the interpretation of rate upper limits from past gravitational wave analyses. We first reconstruct the measured posterior on the binary coalescence rate and go on to compute the posterior and upper limit on the sGRB beaming angle under a range of assumptions regarding the sGRB efficiency ϵ.

The upper limit on the binary coalescence rate at confidence α is found by integrating the rate posterior from zero to α. Assuming a uniform prior on the rate $\mathcal{R}$ and using the rate posterior given by equation 9, the upper limit on the rate $\mathcal{R}_\alpha$ is given by equation 21 in [4]:

$$1 - \alpha = e^{-\mathcal{R}_\alpha C_L(\rho^*)T)} \left[1 + \left(\frac{\Lambda}{1 + \Lambda} \right) \mathcal{R}_\alpha T C_L(\rho^*) \right]. \tag{10}$$

In the event that no gravitational wave signal has been observed and the loudest event is umabiguously due to background noise fluctuations, we are in the limit in which $\Lambda \to 0$. In this case, we simply have,

$$C_L(\rho^*)T = -\frac{\log(1 - \alpha)}{\mathcal{R}_\alpha}, \tag{11}$$

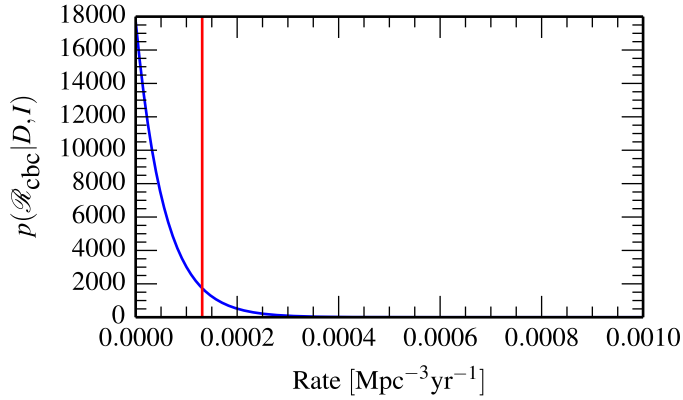

FIG. 1: Rate posterior for S6/VSR2,3 Low-mass Search For Compact Binary Coalescence.

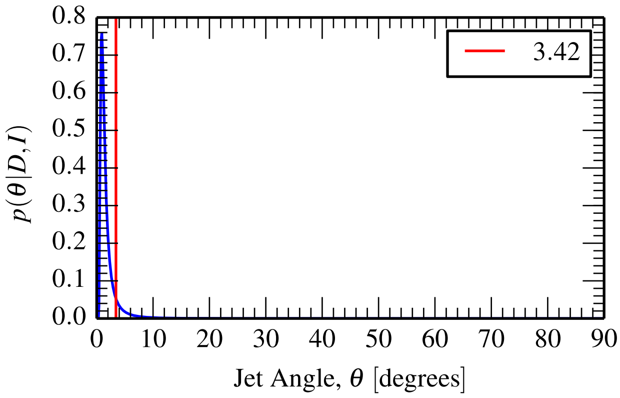

FIG. 2: Jet angle posterior derived from S6/VSR2,3 Low-mass Search For Compact Binary Coalescence posterior / upper limit on compact binary coalescence rate, assuming *all* BNS result in sGRBs.

and the value of $\mathcal{R}$ can be taken straight from the literature[1]

Using the most stringent 90% confidence upper limit from gravitational wave observations on the rate of binary neutron star coalescences to date, $\mathcal{R}^{90\%}_{\rm bns} = 1.3 \times 10^{-4}\,\mathrm{Mpc}^{-3}\mathrm{yr}^{-1}$ [5], gives $C_L(\rho^*)T = 17712$. Similarly, for NS-BH systems, $\mathcal{R}^{90\%}_{\rm nsbh} = 3.1 \times 10^{-5}\,\mathrm{Mpc}^{-3}\mathrm{yr}^{-1}$ gives $C_L(\rho^*)T = 74277$. The posteriors on the rates, assuming these values and $\Lambda = 0$, are shown in figure 1.

A. Known GRB Efficiency, ϵ

Figure 2 shows the resulting posterior on the GRB beaming angle, using the transformation in equation **??** and our results so far. Now, the quantity of interest is the upper limit on the jet angle $\theta^{90\%}$:

$$0.9 = \int_{\theta^{90\%}}^{\infty} p(\theta|\mathcal{R}^{90\%})\ \mathrm{d}\theta \tag{12}$$

This is indicated with the vertical red line in figure 2. We find $\theta^{90\%} = 3.3°$.

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[1] Note that this procedure necessarily confines our jet angle inferences based on progenitor systems for which the rate upper limits are available. Given that the binary coalesence rate limits are quoted for canonical binary neutron star and neutron star-black hole systems, both plausible sGRB progenitors, this simply means our inferences on the jet angle are specific to each system and treated separately.

B. Unknown GRB Efficiency, ϵ

C. Astrophysical Interpretation & Comparison With Other Limits

The comparison with other limits is straightforward. I think it looks pretty consistent with Dietz, Holtz etc (but should double check).

More importantly, however, we've demonstrated how to get a limit on the beaming angle. That's all well and good but we need some astrophysical interpretation, really. The most obvious thing (to James) is that the beaming angle is really a proxy for the Lorentz factor of the outflow (I think):

$$\theta \sim \frac{1}{\Gamma}. \tag{13}$$

So, what *physics* of the progenitor can we constrain with this?

IV. BEAMING ANGLE INFERENCES IN THE DETECTION ERA

In this section, we discuss an approach to measuring the GRB beaming angle based on the binary coalescence rate as measured from a pair of detection scenarios. In the first instance, we consider an imagined 'early' science run with the the advanced LIGO detectors where $\mathcal{O}(1)$ gravitational wave events from binary neutron star coalescence are detected. In the second secnario, we consider the measurements possible with multiple gravitational wave detections. This section will draw heavily on the observing scenarios section (4) in [6].

A. For discussion: Post-detection Coalescence Rate Posterior

Here, we determine the formalism for the construction of a rate posterior in the event of single or multiple gravitational wave detections. There needs to be no specific discussion of GRBs here, it's more general than that. In fact, it's worth remembering *nothing* in this paper / analysis (so far) has assumed contemporaneous GRB-GW detections![2]. All we actually do is assume we have independent measurements of BNS and sGRB rates.

Rather, the point of this discussion is to determine/describe a simple, natural way to construct the posterior on the rate $\mathcal{R}$, given multiple GW detections, as expected in the advanced detector era.

Recall the rate posterior from Fairhurst & Brady [4]:

$$p(\mathcal{R}|\rho_m, T, B) \propto p(R) \left[\frac{1 + \Lambda R C_L(\rho_m) T}{1+\Lambda} e^{-R C_L(\rho_m) T} \right] \tag{14}$$

Equation 14 is the posterior on the rate, as measured by the single loudest event in the search conducted. In the equation above, Λ is the likelihood of the *loudest event* being from GW vs background and C_L is the cumulative luminosity the search which produced loudest event ρ_m is sensitive to. It's a neat, sensible approach to estimating event rates given very rare measurements. That is, it seems like a Good Thing / natural for constructing a rate posterior where we only expect a single signal trigger from a population of potentially trigger-inducing events. It also seems sensible:

- in the absence of convincing detections (loudest event is unlikely to be a signal; $\Lambda \to 0$),
- for a single, convincing detection (loudest event is very likely to be a signal; $\Lambda \to \infty$).

It's worth highlighting here that there is absolutely nothing *preventing* one from using the loudest event statistic when we expect more than one signal trigger; regardless of how many events we have, there's always going be a loudest one and we can always determine the probability of obtaining a trigger that loud, given the sensitivity of the search and a population model. Indeed, consider the key components in the ocnstruction of the L.E. posterior:

1. The loudest event, characterised by, say, single IFO signal-to-noise ratio ρ. Label this loudest value ρ_m.

[2] although that does seem like a potential avenue for measuring the CBC-sGRB efficiency ...

2. The probability of obtaining zero events with $\rho > \rho_m$ from a population of signals; this is formed from knowledge of the sensitivity of the analysis and a population model. It does not require that there be only a single signal observed in the experiment.

3. The probability of obtaining zero events with $\rho > \rho_m$, from a population of background fluctuations (Gaussian or otherwise).

The (potential) issue at hand is the following: suppose the most optimistic rate estimates are correct and aLIGO instantaneously reaches full design sensitivity. In one year, we will have acquired $\mathcal{O}(100)$ BNS detections. That's an entire year of measurements and a sizeable population of SNR measurements. Here's the big question:

> **If we form the rate posterior from the loudest event, are we in any sense 'throwing away' information by considering only the loudest trigger?**

Equally importantly (and, at the risk of sounding glib, not necessarily the same point. . .):

> **Does the L.E. formalism reflect what we'll actually do with, say, 100 BNS detections in aLIGO?**

Without thinking about it too hard, it 'feels' like we're literally throwing away 99% of the information gathered in the experiment. But is that true? That is, if we use the observation time of the experiment together with an estimate of the distance / volume sensitivity of the search, is there actually any more information to be gleaned from a sample of GW triggers than is already present in the value of the loudest trigger? Is it perhaps the case that the loudest event probes the *tail* of the SNR distribution, while a sample of triggers allows one to better estimate the full distribution of SNRs?

B. Alternative Rate Posterior Construction From §14 in Gregory

Gregory's book, 'Logical Data Analysis For The Physical Sciences' has an entire chapter (§14) devoted to 'Bayesian Inference With Poisson Sampling'. This seems to match our problem rather well. In particular, he derives expressions for a Poisson rate posterior in §14.3, 'Signal + known background' and, even better, §14.4, 'Analysis of ON/OFF measurements' ("we want to infer the source rate, s, when the background rate, b, is imprecisely measured").

I will not reproduce the derivations here - see the book. Instead, I'll write down the posterior for the source rate s when events lie in a background with event rate b which is imprecisely measured (e.g., estimated from time-slides). Note that the discussion and examples in Gregory are based on things like 'on-source' and 'off-source' photon counts in high-energy astrophysics or 'foreground' and 'background' in particle physics. In our case, 'on-source' is the zero-lag data for a science run and 'off-source' is the time-slides.

The posterior on the rate of GW events *measured in the science run* is,

$$p(s|N_{\rm on}, I) = \sum_{i=0}^{N_{\rm on}} C_i \frac{T_{\rm on}(sT_{\rm on})^i e^{-sT_{\rm on}}}{i!}, \tag{15}$$

where,

$$C_i \approx \frac{\left(1+\frac{T_{\rm off}}{T_{\rm on}}\right)^i \frac{(N_{\rm on}+N_{\rm off}-i)!}{(N_{\rm on}-i)!}}{\sum_{j=0}^{N_{\rm on}} \left(1+\frac{T_{\rm off}}{T_{\rm on}}\right)^j \frac{(N_{\rm on}+N_{\rm off}-j)!}{(N_{\rm on}-j)!}}, \tag{16}$$

and the quantities in both terms are as follows,

- $N_{\rm on}$ is the number of zero-lag events observed in a science run of duration $T_{\rm on}$.
- $N_{\rm off}$ is the number of time-slide events observed in total background time $T_{\rm off}$.

Now, I placed emphasis on s being the event rate measured in the data with good reason. We are interested in the binary coalescence rate $\mathcal{R}$. This is not the same thing as the signal rate s. However, the conversion between signal rate and coalescence rate is precisely what the 'rates' paper [7] was all about!

In particular, the detection rate for a given type of binary coalescence in LIGO-Virgo is given by equation 1 in [7],

$$\dot{N} = \mathcal{R} \times N_G, \tag{17}$$

where $\mathcal{R}$ is the coalescence rate of that type of binary per galaxy and N_G is the number of galaxies accessible with a search for the relevant binary type. N_G is well approximated at large distances by,

$$N_G = \frac{4}{3}\pi \left(\frac{D_{\rm horizon}}{\rm Mpc}\right)^3 (2.26)^{-3}(0.0116). \tag{18}$$

The reader is directed to [7] for a discussion of the numerical factors in the equation above.

Finally, we recognise that $\dot{N}$ is the signal rate s in equation 15 so that we arrive at the desired posterior on the binary coalescence rate,

$$\begin{aligned} p(\mathcal{R}|N_{\rm on}, I) &= p(s|N_{\rm on}, I)\left|\frac{{\rm d}s}{{\rm d}\mathcal{R}}\right| & (19)\\ &= N_G.p(s|N_{\rm on}, I) & (20)\end{aligned}$$

If the above reasoning is sound, then 'all' we require to move forward are sensible numbers of zero-lag and background events and observation times. These numbers *should* be quite straightforward to piece together if we assume central χ^2 distributed background triggers and non-central χ^2 distributed foreground triggers. In fact, I've already got some codes kicking around from my Cardiff days which I think will compute exactly these numbers (this would be a great way to tidy up some loose ends with Steve, too!)

Final remark: if the Gregory approach is sane, it'd be tempting to try this for the null-detection scenario, too. Note, however, that would require using closed data . . .

C. Early aLIGO Detection Scenario

D. Late aLIGO Detection Sceneario

V. CONCLUSION

Appendix A: Jacobian Calculation

This doesn't need to be in the publication, these are just notes for James' benefit and possibly verification.

$$\mathcal{R} = \frac{\mathcal{R}_{\rm grb}}{\epsilon(1-\cos\theta)}, \tag{A1}$$

$$p(\theta) = \int_\epsilon p(\theta,\epsilon)\ {\rm d}\epsilon, \tag{A2}$$

$$p(\theta,\epsilon) = p(\mathcal{R},\epsilon)\left|\left|\frac{\partial(\mathcal{R},\epsilon)}{\partial(\theta,\epsilon)}\right|\right|, \tag{A3}$$

$$\frac{\partial(\mathcal{R},\epsilon)}{\partial(\theta,\epsilon)} = \begin{bmatrix} \frac{\partial\mathcal{R}}{\partial\theta} & \frac{\partial\mathcal{R}}{\partial\epsilon} \\ \frac{\partial\epsilon}{\partial\theta} & \frac{\partial\epsilon}{\partial\epsilon}\end{bmatrix}. \tag{A4}$$

$$\begin{aligned} \frac{\partial\mathcal{R}}{\partial\theta} &= -\frac{\mathcal{R}_{\rm grb}\sin\theta}{\epsilon(\cos\theta-1)^2} & (A5)\\ \frac{\partial\mathcal{R}}{\partial\epsilon} &= \frac{\mathcal{R}_{\rm grb}}{\epsilon^2(\cos\theta-1)} & (A6)\\ \frac{\partial\epsilon}{\partial\theta} &= -\frac{\mathcal{R}_{\rm grb}\sin\theta}{\mathcal{R}(\cos\theta-1)^2} & (A7)\\ \frac{\partial\epsilon}{\partial\epsilon} &= 1 & (A8)\\ & & (A9)\end{aligned}$$

$$\left|\frac{\partial(\mathcal{R},\epsilon)}{\partial(\theta,\epsilon)}\right| = \frac{\partial\mathcal{R}}{\partial\theta}\frac{\partial\epsilon}{\partial\epsilon} - \frac{\partial\mathcal{R}}{\partial\epsilon}\frac{\partial\epsilon}{\partial\theta} \tag{A10}$$

$$= -\frac{2\sin\theta}{\epsilon(\cos\theta-1)^2} \tag{A11}$$

Finally:

$$\left\|\frac{\partial(\mathcal{R},\epsilon)}{\partial(\theta,\epsilon)}\right\| = \frac{2\mathcal{R}_{\rm grb}\sin\theta}{\epsilon(\cos\theta-1)^2} \tag{A12}$$

Appendix B: Null-detection Jet Angle Posteriors With Different Priors

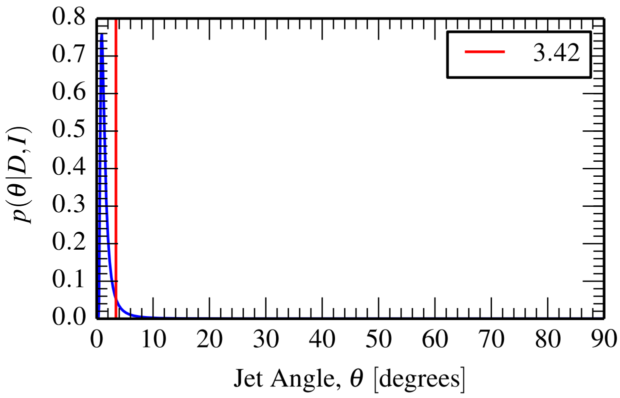

FIG. 3: $p(\epsilon|I) = \delta(\epsilon - 1)$: efficiency assumed known: $\epsilon = 1$

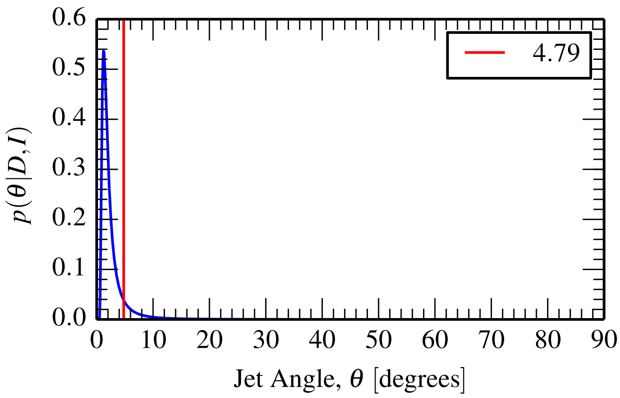

FIG. 4: $p(\epsilon|I) = \delta(\epsilon - 0.5)$: efficiency assumed known: $\epsilon = 0.5$

[1] E. Nakar, Physics Reports **442**, 166 (2007).
[2] A. Dietz, Astron. Astrophys. **529**, A97 (2011), 0802.0874.
[3] R. Biswas, P. R. Brady, J. D. E. Creighton, and S. Fairhurst, Class. Quantum Grav. **26**, 175009 (2009), gr-qc/0308069).
[4] P. R. Brady and S. Fairhurst, Classical and Quantum Gravity **25**, 105002 (2008), 0707.2410.
[5] (????).

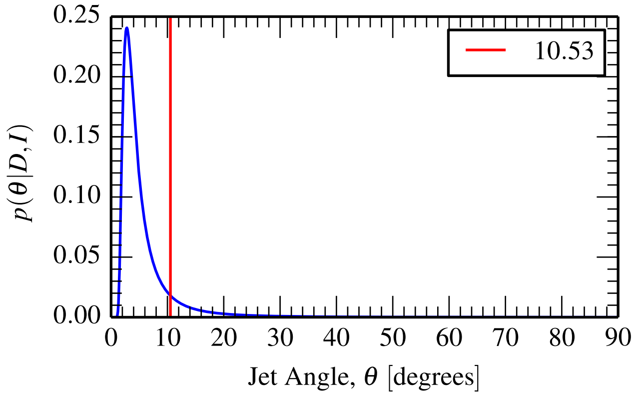

FIG. 5: $p(\epsilon|I) = \delta(\epsilon - 0.1)$: efficiency assumed known: $\epsilon = 0.1$

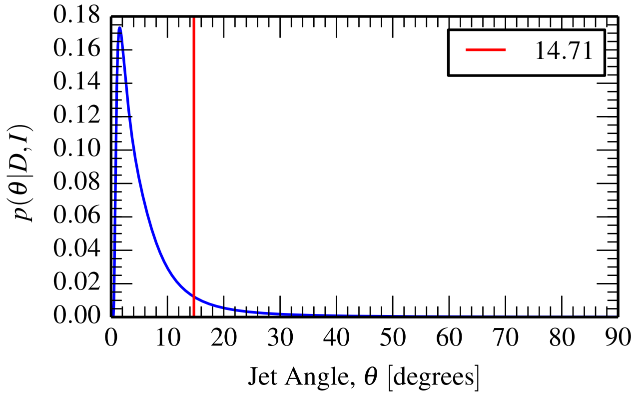

FIG. 6: $p(\epsilon|I) \propto 1$, $\epsilon \in (0, 1]$. Here, we assume the GRB efficiency is $\epsilon \in (0, 1]$, with no preference where: the linearly uniform prior. The reason for choosing a non-zero lower bound is that the Jacobian determinant goes to inf as $\epsilon \to 0$. James: I'm very open to suggestions here as it's not clear to me that this is expected. In practice, $\epsilon \in [0.01, 1]$.

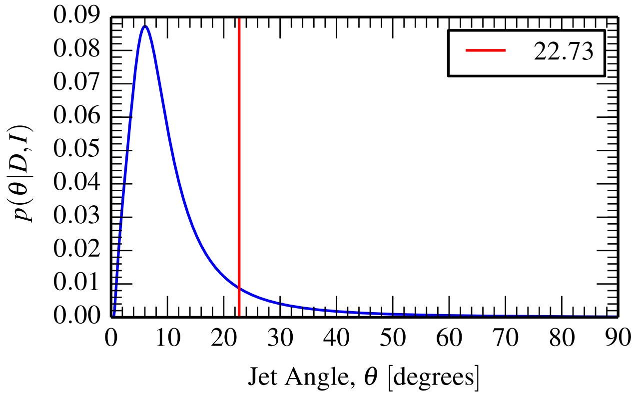

FIG. 7: $p(\epsilon|I) \propto 1/\epsilon$, $\epsilon \in (0, 1]$ Now assume the GRB efficiency is $\epsilon \in (0, 1]$, with a log-uniform prior (commonly but incorrectly referred to as the Jeffrey's prior). Here, the requirement of a non-zero lower bound is imposed by the normalisation for this prior.

[6] LIGO Scientific Collaboration and Virgo Collaboration, ArXiv e-prints (2013), 1304.0670.

[7] LIGO Scientific Collaboration and Virgo Collaboration, Classical and Quantum Gravity **27**, 173001 (2010), URL `http://stacks.iop.org/0264-9381/27/i=17/a=173001`.

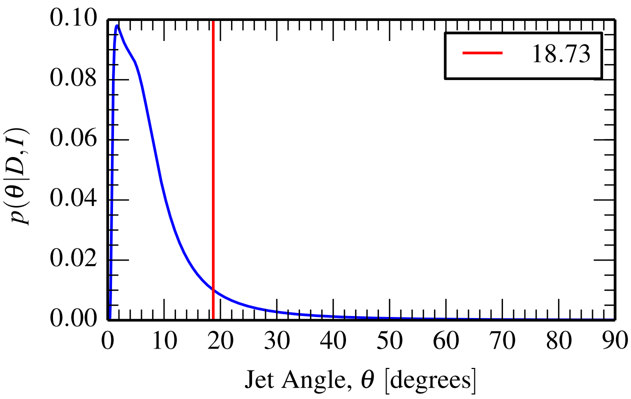

FIG. 8: $p(\epsilon|I) = \beta(1/2, 1/2),\ \epsilon \in (0, 1)$. Here, we use a Beta distribution with shape and scale parameters of 1/2. That prior has a slightly surprising shape with a minimum at $\epsilon = 0.5$ and shoots off to ∞ as $\epsilon \to 0, 1$. It seems *this* is, in fact, the Jeffrey's prior for a Bernoulli trial (if we say a toin coss = BNS →GRB). See http://en.wikipedia.org/wiki/Jeffreys_prior

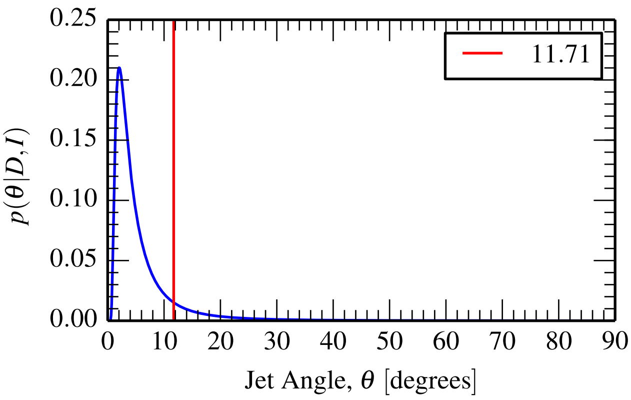

FIG. 9: $p(\epsilon|I) = \beta(2, 5),\ \epsilon \in (0, 1)$. The GRB efficiency prior is, this time, a β distribution with shape and scale parameters 2 and 5, respectively. This is now completely ad hoc but yields a maximum (in the prior) around $\epsilon = 0.2$ and goes to 0 as $\epsilon \to 0, 1$. So it's a nice example of an asymmetric distribution with quite sensible (and presumably tuneable) behaviour.

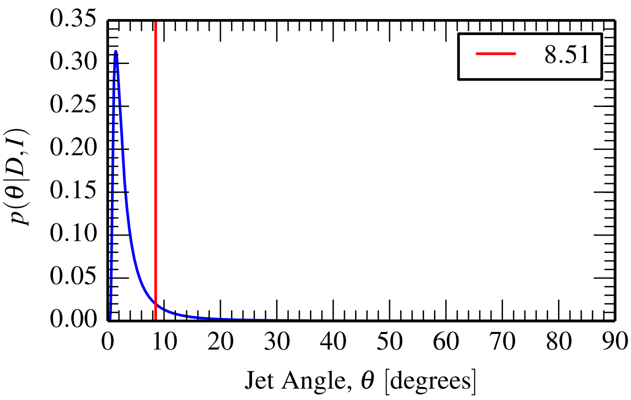

FIG. 10: $p(\epsilon|I) = \beta(2, 2),\ \epsilon \in (0, 1)$. Finally, we have for the GRB efficiency prior a β distribution with shape and scale parameters 2 and 2, respectively. Again, this is ad hoc but symmetric about $\epsilon = 0.5$ with a cosine-like shape.