ISOMORPHISM OF ÉTALE FUNDAMENTAL GROUPS LIFTS TO ISOMORPHISM OF STRATIFIED FUNDAMENTAL GROUP

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Abstract. It is shown that if a finite generically smooth morphism $f : Y \longrightarrow X$ of smooth projective varieties induces an isomorphism of the étale fundamental groups, then the induced map of the stratified fundamental groups $\pi_1^{str}(f) : \pi_1^{str}(Y, y) \longrightarrow \pi_1^{str}(X, f(y))$ is also an isomorphism.

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1. Introduction

Let X be an irreducible smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. Gieseker conjectured that if the étale fundamental $\pi_1^{et}(X, x)$ is trivial then the stratified fundamental group $\pi_1^{str}(X, x)$ must be trivial (see [Gi]). This conjecture was proved by Esnault and Mehta in [EM]. Let $f : Y \longrightarrow X$ be a morphism of smooth projective varieties. In [ES], it was shown that if the induced homomorphism $\pi_1^{et}(f) : \pi_1^{et}(Y, y) \longrightarrow \pi_1^{et}(X, f(y))$ between the étale fundamental groups is trivial, then the induced homomorphism $\pi_1^{str}(f) : \pi_1^{str}(Y, y) \longrightarrow \pi_1^{str}(X, f(y))$ between the stratified fundamental groups is also trivial. In [Sun] the proof of these two results were simplified.

We prove another relative version of Gieseker conjecture which can be viewed as a generalization of the Gieseker conjecture. The main result of this article is the following:

Theorem 1.1. *Let $f : Y \longrightarrow X$ be a finite generically smooth morphism of smooth projective varieties over an algebraically closed field k of characteristic $p > 0$ such that the induced homomorphism $\pi_1^{et}(f) : \pi_1^{et}(Y, y) \longrightarrow \pi_1^{et}(X, f(y))$ is an isomorphism. Then the induced homomorphism $\pi_1^{str}(f) : \pi_1^{str}(Y) \longrightarrow \pi_1^{str}(X)$ of stratified fundamental groups is also an isomorphism.*

Note that Theorem 1.1 implies the conjecture of Gieseker. Indeed, if $\pi_1^{et}(X)$ is trivial, then consider a finite generically smooth morphism $f : X \longrightarrow \mathbb{P}^d$, where $d = \dim X$, which can be constructed using Noether normalization; since $\pi_1^{et}(\mathbb{P}^d)$ and $\pi_1^{str}(\mathbb{P}^d)$ are trivial, Theorem 1.1 implies that $\pi_1^{str}(X)$ is trivial.

Note that in the earlier results ([EM], [ES], [Sun]), one only needs to show that certain stratified bundles are trivial bundles. In our work, we show that the pullback functor between the category of stratified bundles on X to that of Y is an equivalence of categories.

The fact that the functor is fully faithful is proved using Proposition 2.3 (a generalization of [BP, Lemma 4.3]). Note that this only requires the map f to be genuinely ramified. Hence we also get $\pi_1^{et}(f)$ is surjective implies $\pi_1^{str}(f)$ is also surjective (see Remark 5.1).

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Other key ingredients in the proof of Theorem 1.1 are Theorem 4.2 (which may be of independent interest) Lemma 2.2 (a result of Hrushovski) and the theory of representation spaces developed by Simpson ([Sim]) and its generalization to positive characteristics ([Sun]).

2. Stratified vector bundles

Let k be an algebraically closed field of characteristic p, with $p > 0$. Let X be an irreducible smooth projective variety over k. Denote by $\mathcal{D}_X$ the sheaf of differential operators, in the sense of Grothendieck, on X [Gr], [BO]. A stratified vector bundle on X is an $\mathcal{O}_X$–coherent $\mathcal{D}_X$–module [Gi].

Let $F_X : X \longrightarrow X$ be the absolute Frobenius morphism for X. So for any vector bundle V on X, the pullback $F_X^* V \longrightarrow X$ is the subbundle of $V^{\otimes p}$ defined by $\{v^{\otimes p} \in V^{\otimes p} \bigm| v \in V\}$. A F–divisible vector bundle on X is a sequence of vector bundles $\{E_i\}_{i\geq 0}$ on X indexed by the nonnegative integers together with an isomorphism $E_i \longrightarrow F_X^* E_{i+1}$ for every $i \geq 0$ [Gi], [Sa]. There is a natural equivalence of categories between the stratified vector bundles and the F–divisible vector bundles. The underlying vector bundle for the stratified bundle corresponding to a F–divisible vector bundle $\{E_i\}_{i\geq 0}$ is E_0 [Gi], [Sa]. Similarly, the rank of $\{E_i\}_{i\geq 0}$ is the common rank of E_i.

Let $\mathrm{Vect^{str}(X)}$ be the category of F–divisible vector bundles on X, which, as mentioned above, is equivalent to the category of stratified vector bundles on X. Henceforth, by a stratified bundle we will mean a F–divisible vector bundle. The category of finite dimensional k–vector spaces will be denoted by Vect(k). Fix a closed point x of X. We have the fiber functor

$$\omega_x : \mathrm{Vect}^{str}(X) \longrightarrow \mathrm{Vect(k)}, \quad \mathrm{\{E_i\}_{i\geq 0}} \longmapsto \mathrm{(E_0)_x}. \tag{2.1}$$

Then the pair $(\mathrm{Vect^{str}(X)}, \omega_\mathrm{x})$ forms a neutral Tannakian category. Its Tannaka dual is the stratified fundamental group $\pi_1^{str}(X, x)$ [Sa] [Gi] (see [SR], [DM], [No1], [No2] for Tannaka dual).

Let $\mathrm{Vect^{et}(X)}$ denote the category of étale trivializable vector bundles on X. Note that any étale trivial vector bundle on X gives rise to a stratified bundle and this induces a fully faithful functor from $\mathrm{Vect^{et}(X)}$ to $\mathrm{Vect^{str}(X)}$. This functor Induces an epimorphism $\pi_1^{str}(X, x) \longrightarrow \pi_1^{et}(X, x)$.

Let $f : Y \longrightarrow X$ be a generically smooth morphism of irreducible smooth projective varieties over k. Fix a closed point y of Y, and set $x = f(y)$. Then the pullback functor

$$f^* : (\mathrm{Vect}^{str}(X), \omega_x) \longrightarrow (\mathrm{Vect}^{str}(Y), \omega_y)$$

is a functor of Tannakian categories, and it induces a homomorphism of group schemes

$$\pi_1^{et}(f) : \pi_1^{str}(Y, y) \longrightarrow \pi_1^{str}(X, x)$$

between the corresponding Tannaka duals.

Let $\mathcal{O}_X(1)$ be a fixed very ample line bundle X. For a torsionfree sheaf V of X,

$$P(V, m) := \chi(V(m))$$

is a polynomial in m which is called the Hilbert polynomial of V. We say that V is semistable if for any nonzero subsheaf $W \subset V$, the inequality

$$\frac{P(W,m)}{\mathrm{rank}(W)} \leq \frac{P(V,m)}{\mathrm{rank}(V)}$$

holds for all m sufficiently large.

Let $\mathcal{E} = (E_n)_{n\geq 0}$ be a stratified vector bundle on X of rank r. For all $n \geq 0$, the Hilbert polynomial of E_n is same as that of the trivial vector bundle $\mathcal{O}_X^{\oplus r}$, and there exists an integer $n_0 \geq 1$ (which depends on $\mathcal{E}$) such that for all $j \geq n_0$, the vector bundle E_j is semistable. This is because if W is a subsheaf of E_n, then $(F_X^n)^*W$ is subsheaf of $(F_X^n)^*E_n = E_0$; on the other hand, we have $c_i((F_X^n)^*W) = p^{ni}c_i(W)$. Now from the boundedness of the destabilizing subsheaves of E_0 it follows that E_n is semistable for sufficiently large n. This also shows that $c_i(E_j) = 0$ for all $i \geq 1$ and all $j \geq 0$.

Let X be a smooth irreducible projective variety over k. Fix a closed point

$$\xi : \mathrm{Spec}(k) \longrightarrow X$$

of X. Recall from [Sun] that a representation space $\mathcal{R}(X, \xi, P)$ parametrizes all pairs (V, β) where V is a semistable vector bundle with Hilbert polynomial P and $\beta : \xi^*V \longrightarrow \mathcal{O}_{\mathrm{Spec}(k)}^{\oplus r}$ is an isomorphism. This was constructed by Simpson in characteristic zero and it was extended to positive characteristics by Sun. In particular, in [Sun, Theorem 2.3] it was shown that $\mathcal{R}(X, \xi, P)$ is in fact a fine moduli space.

Proposition 2.1. *Let $f : Y \longrightarrow X$ be a finite generically smooth morphism. Let ζ be a closed point in Y, and $\xi = f(\zeta)$. Then f induces a morphism $\Phi : \mathcal{R}(X, \xi, P_X) \longrightarrow \mathcal{R}(Y, \zeta, P_Y)$ defined by $(V, \beta) \longmapsto (f^*V, f^*\beta\big|_\zeta)$, where P_X and P_Y are the Hilbert polynomials of $\mathcal{O}_X^{\oplus r}$ and $\mathcal{O}_Y^{\oplus r}$ respectively.*

Proof. Since f is finite generically smooth morphism, for a semistable vector bundle V on X with Hilbert polynomial P_X, the pullback f^*V is a semistable vector bundle of Hilbert polynomial P_Y on Y. Denote $\mathcal{R} = \mathcal{R}(X, \xi, P)$, and let $f_{\mathcal{R}}$ be the base change of f to $\mathrm{Spec}(\mathcal{R})$, so

$$f_{\mathcal{R}} = f \times \mathrm{Id}_{\mathrm{Spec}(\mathcal{R})} : Y \times \mathrm{Spec}(\mathcal{R}) \longrightarrow X \times \mathrm{Spec}(\mathcal{R}).$$

Let $(\mathcal{V}, \beta_{\mathcal{R}}) \longrightarrow X \times \mathcal{R}$ be the universal vector bundle. Then $f_{\mathcal{R}}^*\mathcal{V}$ is a semistable vector bundle on $Y_{\mathcal{R}} \times \mathrm{Spec}(\mathcal{R})$. The pair $(f_{\mathcal{R}}^*\mathcal{V}, f_{\mathcal{R}}^*\beta_{\mathcal{R}}\big|_{\zeta\times\mathcal{R}})$ is a family of semistable vector bundles on Y together with an isomorphism of the fiber over ζ with $k^{\oplus r}$. Hence by universal property of moduli spaces there is a morphism $\Phi : \mathcal{R} \longrightarrow \mathcal{R}(Y, \zeta, P_Y)$ as in the statement of the proposition. □

The following is a consequence of [Hr, Corollary 1.2] (also see [Var, Corollary 0.4]).

Lemma 2.2. *Take an irreducible variety defined Z over $\overline{\mathbb{F}}_p$, and let $\Psi : Z \longrightarrow Z$ be a rational dominant map. Then the subset*

$$S := \{z \in Z \mid \Psi^n(z) = z \ \textit{for some}\ n\} \subset Z$$

is dense in Z.

A finite generically smooth map $f : Y \longrightarrow X$ of two irreducible projective varieties of the same dimension is called *genuinely ramified* if the induced homomorphism of étale fundamental groups

$$\pi^{et}(f) : \pi_1^{et}(Y, y) \longrightarrow \pi_1^{et}(X, f(y))$$

is surjective.

The following proposition is proved in Section 6.

Proposition 2.3. *Let $f : Y \longrightarrow X$ be a genuinely ramified map between irreducible smooth projective varieties. Let V and W be semistable vector bundles on X of same slope. Then the natural map*

$$\mathrm{Hom}_X(V, W) \longrightarrow \mathrm{Hom}_Y(f^*V, f^*W)$$

is an isomorphism.

We will assume Proposition 2.3 and defer its proof to Section 6.

3. The case of $k = \overline{\mathbb{F}}_p$

In this section we prove the main theorem when $k = \overline{\mathbb{F}}_p$.

Theorem 3.1. *Let $f : Y \longrightarrow X$ be a genuinely ramified map of smooth projective varieties defined over $\overline{\mathbb{F}}_p$. If the induced homomorphism $\pi^{et}(f) : \mathrm{Vect}^{\mathrm{et}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{et}}(\mathrm{Y})$ is essentially surjective, then so is the induced homomorphism $\pi_1^{str}(f) : \mathrm{Vect}^{\mathrm{str}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{str}}(\mathrm{Y})$.*

Proof. Let $\mathcal{E} = (E_n)_{n\geq 0}$ be a stratified vector bundle on Y of rank r. As noted before, there exists an integer m such that for all $n \geq m$, the vector bundle E_n is semistable of rank r, and also $c_i(E_j) = 0$ for all $i \geq 1$ and $j \geq 0$. Let $\eta \in Y$ be a closed point. For every $n > m$, by [Sun, Lemma 3.3] there is an isomorphism

$$\beta_n : \eta^* E_n \longrightarrow \mathcal{O}_{\mathrm{Spec}(k)}^{\oplus r}$$

such that $F_Y^* \beta_{n+1} = \beta_n$. Let N be the closure of

$$T := \{(E_n, \beta_n) \bigm| n > m\} \tag{3.1}$$

in the representation space $\mathcal{R}(Y, \eta, P_Y)$ parametrizing the isomorphism classes of pairs (V, β), where V is a vector bundle on Y with Hilbert polynomial P_Y while $\beta : \eta^* V \longrightarrow \mathcal{O}_{\mathrm{Spec}(k)}^{\oplus r}$ is an isomorphism. From [Sun, Theorem 2.3 (1)] it follows that (E_n, β_n) is a point of $\mathcal{R}(Y, \eta, P_Y)$ for every $n > m$.

In [Sun, Proposition 2.5] it is shown that the analog of the rational map Verschiebung exists on $\mathcal{R}(Y, \eta, P_Y)$. Let

$$\mathcal{V} : N \longrightarrow N \tag{3.2}$$

be the restriction of this Verschiebung. Note that the image of $\mathcal{V}$ contains T (see (3.1)), and hence $\mathcal{V} : N \longrightarrow N$ is a dominant rational map.

Let $N_1, \cdots, N_\ell$ be the positive dimensional irreducible components of N. The generic point of N_j, $1 \leq j \leq \ell$, will be denoted by η_j. Since the map $\mathcal{V}$ in (3.2) is dominant, it permutes $\eta_1, \cdots, \eta_\ell$. Hence there exists $a \geq 1$ such that $\mathcal{V}^a$ fixes η_i for all $1 \leq i \leq \ell$.

Recall that each N_i is irreducible. Hence by Lemma 2.2, the set

$$S_i = \{(E, \beta) \in N_i \mid \mathcal{V}^b((E, \beta)) = (E, \beta) \text{ for some } b \geq 1\}$$

is dense in N_i. Consequently,

$$S = \{(E, \beta) \in N \mid \mathcal{V}^b((E, \beta)) = (E, \beta) \text{ for some } b \geq 1\}$$

is dense in N.

For any $(E, \beta) \in S$, we have E to be a semistable vector bundle of rank r on Y such that $(F_Y^j)^*E = E$ for some $j \geq 1$. Hence a theorem of Lange and Stulher says that E is an étale trivial vector bundle on Y [LS]. Since $f^* : \mathrm{Vect}^{\mathrm{et}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{et}}(\mathrm{Y})$ is essentially surjective,

$$E = f^*E'$$

for some étale trivial vector bundle E' on X. Moreover E' is a semistable vector bundle of rank r whose Hilbert polynomial is the Hilbert polynomial of $\mathcal{O}_X^{\oplus r}$.

Consider the morphism

$$\Phi : \mathcal{R}(X, f(\eta), P_X) \longrightarrow \mathcal{R}(Y, \eta, P_Y)$$

defined by the pullback using $f : Y \longrightarrow X$ as in Proposition 2.1. Let $N' = \Phi^{-1}(N)$. Note that the restriction $\Phi\big|_{N'} : N' \longrightarrow N$ is dominant, because S is in $\mathrm{Im}(\Phi)$ and S is dense in N. Consequently, there exists an open dense subset U of N contained in $\mathrm{Im}(\Phi)$. As T is dense in N, there exists a subsequence $\{n_j \mid n_j > m\}$ such that $\{(E_{n_j}, \alpha_j) \mid j \geq 1\} \subset U$. Consequently, there are semistable vector bundles E'_{n_j} on X such that $f^*E'_{n_j} = E_{n_j}$. We have $f \circ F_Y = F_X \circ f$, and therefore,

$$E_{n_j-1} = F_Y^* E_{n_j} = F_Y^* f^* E'_{n_j} = f^* F_X^* E'_{n_j}.$$

Set $n_0 = 0$, and define $E'_i := (F_X^{n_j - i})^* E'_{n_j}$, where $j \geq 0$ and $n_{j-1} < i < n_j$. Then for all n, we have $E_n = f^*E'_n$ for some vector bundles E'_n on X. Also for $j \geq 0$,

$$f^* F_X^* E'_{n_j+1} \cong f^* E'_{n_j}.$$

But f is genuinely ramified and for $j \geq 1$, the vector bundle E'_{n_j} is semistable. Consequently, we have $F_X^* E'_{n_j+1} \cong E'_{n_j}$ by Proposition 2.3. Therefore, $\mathcal{E}' = (E'_{n_j})_{j \geq 0}$ is a stratified bundle on X and it satisfies the condition $f^*\mathcal{E}' = \mathcal{E}$. ☐

Corollary 3.2. *Let $f : Y \longrightarrow X$ be a finite generically smooth morphism of smooth projective varieties over $\overline{\mathbb{F}}_p$. Let $y \in Y$ be a closed point, and $x = f(y)$. It is given that the induced homomorphism $\pi^{et}(f) : \pi_1^{et}(Y, y) \longrightarrow \pi_1^{et}(X, x)$ is an isomorphism. Then the induced homomorphism $\pi^{str}(f) : \pi_1^{str}(Y, y) \longrightarrow \pi_1^{str}(X, x)$ is also an isomorphism.*

Proof. The hypothesis implies that f is genuinely ramified, and the pullback functor $f^* : \mathrm{Vect}^{\mathrm{et}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{et}}(\mathrm{Y})$ is an equivalence of categories. By Theorem 3.1, the pullback functor on stratified bundles $f^* : \mathrm{Vect}^{\mathrm{str}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{str}}(\mathrm{Y})$ is essentially surjective.

Let $g : (V_n)_{n \geq 0} \longrightarrow (W_n)_{n \geq 0}$ be a morphism in the category $\mathrm{Vect}^{\mathrm{str}}(\mathrm{X})$. Then by definition, g consist of morphisms of vector bundles

$$g_n : V_n \longrightarrow W_n$$

such that $F_X^*(g_{n+1}) = g_n$ for all $n \geq 0$. Therefore, g is uniquely determined by $\{g_n \mid n \text{ sufficiently large}\}$. Note that there exists an integer m such that for $n \geq m$, the vector bundles V_n and W_n are semistable. Since f is genuinely ramified, by Proposition 2.3,

$$\mathrm{Hom}_X(V_n,\, W_n) \,\cong\, \mathrm{Hom}_Y(f^*V_n,\, f^*W_n)$$

for all $n \geq m$. Hence $f^* \,:\, \mathrm{Vect}^{\mathrm{str}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{str}}(\mathrm{Y})$ is also fully faithful. Since f^* is a tensor functor which is an equivalence of Tannakian categories, the induced homomorphism between the Tannakian duals is an isomorphism. □

4. The general case

In this section, we are given a family of morphisms $f_T \,:\, Y_T \longrightarrow X_T$ of irreducible smooth projective varieties parametrized by an integral k–scheme T such that the restriction of f_T over the generic geometric point of T is genuinely ramified (respectively, the restriction of f_T induces an isomorphism of étale fundamental groups). We show that the fiber f_t for any t in an open dense subset of T is genuinely ramified (respectively, f_t induces an isomorphism of étale fundamental groups). See Theorem 4.2 for the precise statement.

For a connected scheme X, let

$$\mathrm{Cov}(X) \tag{4.1}$$

denote the category of finite étale covers of X, so the objects of $\mathrm{Cov}(X)$ are finite étale morphisms $Z \longrightarrow X$, and the morphisms from an object $Z \longrightarrow X$ to another object $Z' \longrightarrow X$ are the X-morphisms $Z \longrightarrow Z'$. Note that a morphism of varieties $f \,:\, Y \longrightarrow X$ induces a pullback functor $\mathrm{Cov}(X) \longrightarrow \mathrm{Cov}(Y)$ given by the base change from X to Y.

Proposition 4.1. *Let R be a complete discrete valuation ring with algebraically closed residue field k, and denote by K^s the separable closure of the fraction field of R. Set $T = \mathrm{Spec}(R)$, and let $\eta \,:\, \mathrm{Spec}(K^s) \longrightarrow T$ be the generic geometric point of T. Let*

$$f_T \,:\, Y_T \longrightarrow X_T$$

be a finite flat generically smooth morphism of smooth proper integral T–schemes X_T and Y_T such that the closed fiber

$$f_0 \,:\, Y_0 \longrightarrow X_0$$

is also a finite flat generically smooth morphism of smooth connected proper schemes over k. Let

$$f_\eta \,:\, Y_\eta \longrightarrow X_\eta$$

be the fiber of f_T over the geometric generic point. If f_η is genuinely ramified, then so is f_0.

Let y_0 (respectively, y) be a geometric point of Y_0 (respectively, Y_η), with $x_0 := f_0(y_0)$ and $x := f_\eta(y)$. Also, assume that X_0 is not contained in the branch locus of f_T. If the homomorphism

$$\pi_1^{et}(f_\eta) \,:\, \pi_1^{et}(Y_\eta,\, y) \longrightarrow \pi_1^{et}(X_\eta,\, x)$$

induced by f_η is an isomorphism, then the induced homomorphism

$$\pi_1^{et}(f_0) \,:\, \pi_1^{et}(Y_0,\, y_0) \longrightarrow \pi_1^{et}(X_0,\, x_0)$$

is an isomorphism.

Proof. We first note that the specialization map $\pi_1^{et}(X_\eta,\, x) \longrightarrow \pi_1^{et}(X_0,\, x_0)$ is surjective ([SGA1], [Mur, 9.2]). Also, by the functoriality of π_1^{et}, the following diagram is commutative:

$$\begin{tikzcd}
\pi_1^{et}(Y_\eta,\, y) \arrow[r] \arrow[d] & \pi_1^{et}(X_\eta,\, x) \arrow[d] \\
\pi_1^{et}(Y_0,\, y_0) \arrow[r] & \pi_1^{et}(X_0,\, x_0).
\end{tikzcd}$$

Therefore, if f_η is genuinely ramified, then the homomorphism

$$\pi_1^{et}(Y_\eta,\, y) \longrightarrow \pi_1^{et}(X_0,\, x_0)$$

induced by f_η is also surjective. Consequently, the map $\pi_1^{et}(f_0) : \pi_1(Y_0) \longrightarrow \pi_1(X_0)$ is also surjective, i.e., the map f_0 is genuinely ramified. This is also equivalent to the statement that the pullback functor $\mathrm{Cov}(X_0) \longrightarrow \mathrm{Cov}(Y_0)$ (see (4.1)) is fully faithful ([SGA1], [SP, Lemma 58.4.1]).

Now to prove that $\pi_1^{et}(f_0) : \pi_1^{et}(Y_0) \longrightarrow \pi_1^{et}(X_0)$ is an isomorphism under the additional hypothesis that $\pi_1^{et}(f_\eta)$ is an isomorphism, it is enough to show that the pullback functor $\mathrm{Cov}(X_0) \longrightarrow \mathrm{Cov}(Y_0)$ is essentially surjective.

There is an equivalence of categories $\mathrm{Cov}(Y_T) \xrightarrow{\sim} \mathrm{Cov}(Y_0)$ (see [SP, Lemma 58.9.1] or [SGA1]). Let $Z_0 \longrightarrow Y_0$ be a finite étale connected cover. This induces a finite étale connected covering $Z_T \longrightarrow Y_T$. Since $\pi_1(Y_\eta) \longrightarrow \pi_1(X_\eta)$ is an isomorphism, the functor $\mathrm{Cov}(X_\eta) \longrightarrow \mathrm{Cov}(Y_\eta)$ is an equivalence of categories. So we conclude that $Z_\eta \longrightarrow Y_\eta$ is the pull-back of an étale connected covering $W_\eta \longrightarrow X_\eta$.

There exists a finite separable extension $K/k(T)$ such that

$$Z_T \times_T \mathrm{Spec}(K) \longrightarrow Y_T \times_T \mathrm{Spec}(K)$$

is the pullback of an étale connected cover $W \longrightarrow X_T \times_T \mathrm{Spec}(K)$ with

$$W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s) = W_\eta.$$

Since X_T is integral, it follows that W is also integral. The normalization of T in K will be denoted by $\widehat{T}$. Denote by $X_{\widehat{T}}$, $Y_{\widehat{T}}$ and $Z_{\widehat{T}}$ the base change — to $\widehat{T}$ — of X_T, Y_T and Z_T respectively. Let $W_{\widehat{T}} \longrightarrow X_{\widehat{T}}$ be the normalization of $X_{\widehat{T}}$ in $k(W)$. So we get the following commutative diagram:

$$\begin{tikzcd}
Z_{\widehat{T}} \arrow[r] \arrow[d] & W_{\widehat{T}} \arrow[d] \\
Y_{\widehat{T}} \arrow[r] & X_{\widehat{T}}.
\end{tikzcd}$$

Note that $Z_T \longrightarrow Y_T$ is étale, and the branch locus of $Y_T \longrightarrow X_T$ does not contain X_0. Consequently, the branch locus of $Z_T \longrightarrow X_T$ does not contain X_0. Therefore, the branch locus of $W_{\widehat{T}} \longrightarrow X_{\widehat{T}}$ does not contain X_0; moreover, this map is étale over the generic point of $\widehat{T}$. Thus by the purity of branch locus ([SGA1, Exp. X, Thm. 3.1], [SP, Lemma 53.20.4]), the map $W_{\widehat{T}} \longrightarrow X_{\widehat{T}}$ is étale, and therefore this map restricted to the closed fiber $W_0 \longrightarrow X_0$ is also étale. Note that the pullback of $W_{\widehat{T}} \longrightarrow X_{\widehat{T}}$ to $Y_{\widehat{T}}$ is $Z_{\widehat{T}} \longrightarrow Y_{\widehat{T}}$. Thus the pullback of $W_0 \longrightarrow X_0$ to Y_0 is $Z_0 \longrightarrow Y_0$. This completes the proof of the statement that the pullback functor $\mathrm{Cov}(X_0) \longrightarrow \mathrm{Cov}(Y_0)$ is essentially surjective. As mentioned before, this implies that $\pi_1^{et}(f_0) : \pi_1^{et}(Y_0) \longrightarrow \pi_1^{et}(X_0)$ is an isomorphism. This completes the proof of the proposition. □

Theorem 4.2. *Let k be an algebraically closed field, and let T be a finite type connected integral scheme defined over k. Denote by K^s the separable closure of the function field of T, and let $\eta : \mathrm{Spec}(K^s) \longrightarrow T$ be the generic geometric point of T. Let $f_T : Y_T \longrightarrow X_T$ be a finite generically smooth morphism of proper integral smooth schemes over T such that the morphism $f_\eta : Y_\eta \longrightarrow X_\eta$ is genuinely ramified. Then there is an open dense subset U of T such that for all closed points $t \in U$, the morphism of the fibers over t,*

$$f_t : Y_t \longrightarrow X_t,$$

is a genuinely ramified map of smooth proper varieties.

Moreover, if f_η induces an isomorphism of the étale fundamental groups, then there is open dense subset U' of T such that for every closed points $t \in U'$, the morphism of the fibers over t,

$$f_t : Y_t \longrightarrow X_t,$$

induces an isomorphism $\pi_1^{et}(Y_t) \xrightarrow{\sim} \pi_1^{et}(X_t)$ of the étale fundamental groups.

Proof. By replacing T by an open dense subscheme, we may assume that f_T is flat, and for all closed points $t \in T$,

$$f_t : Y_t \longrightarrow X_t$$

is a finite flat generically smooth morphism of smooth proper varieties. Also, since f_T is generically étale, the branch locus of f_T is a proper closed subscheme of X_T. Consequently, by shrinking T further, if necessary, we may assume that for all $t \in T$, the fiber X_t is not contained in the branch locus of f_T.

Fix a closed point $t \in T$. Let R be the completion of a discrete valuation ring with fraction field $k(T)$ dominating $\mathcal{O}_{T,t}$. Denote $\widehat{T} = \mathrm{Spec}(R)$, and let $f_{\widehat{T}} : Y_{\widehat{T}} \longrightarrow X_{\widehat{T}}$ be the pullback of f_T along the natural morphism $\widehat{T} \longrightarrow T$. Denote by $\widehat{K}$ the separable closure of the fraction field of R. Then K^s is a subfield of $\widehat{K}$. Note that we have

$$\pi_1^{et}(X_\eta) \cong \pi_1^{et}(X_\eta \otimes_{K^s} \widehat{K})$$

([SGA1], [Mur, Proposition 7.3.2]). Now the given condition that $Y_\eta \longrightarrow X_\eta$ is genuinely ramified implies that $Y_{\widehat{K}} \longrightarrow X_{\widehat{K}}$ is also genuinely ramified. Hence by Proposition 4.1, $Y_t \longrightarrow X_t$ is genuinely ramified.

Finally, if $Y \longrightarrow X$ induces an isomorphism of étale fundamental groups, then

$$Y \otimes \widehat{K} \longrightarrow X \otimes \widehat{K}$$

also induces an isomorphism $\pi_1^{et}(Y_{\widehat{K}}) \longrightarrow \pi_1^{et}(X_{\widehat{K}})$. Now apply Proposition 4.1 to conclude that $\pi_1^{et}(Y_t) \longrightarrow \pi_1^{et}(X_t)$ is an isomorphism. □

5. Proof of Theorem 1.1

Let $f : Y \longrightarrow X$ be a finite separable morphism of smooth projective varieties defined over an algebraically closed field k of characteristic $p > 0$. Let ξ be a closed point of Y. Let $\mathcal{R}(X, f(\xi), P_X)$ and $\mathcal{R}(Y, \xi, P_Y)$ be the representation spaces over X and Y respectively, as in the proof of Theorem 3.1. Let $\mathcal{E} = \{E_n\}_{n\geq 0}$ be a stratified vector bundle on Y.

There exists a finite type smooth integral $\overline{\mathbb{F}}_p$–scheme T with function field $\mathbb{F}_p(T) \subset K$ satisfying the following three statements:

(1) There is a finite generically smooth T–morphism $f_T : Y_T \longrightarrow X_T$ such that the base change of f_T to k is f.
(2) The representation spaces $\mathcal{R}(X, f(\xi), P_X)$ and $\mathcal{R}(Y, \xi, P_Y)$ are both defined over $k(T)$.
(3) There exists a stratified vector bundle $\mathcal{E}_T = \{E_{T,n}\}_{n\geq 1}$ on Y_T such that the pullback of $\mathcal{E}_T$ along the morphism $Y \longrightarrow Y_T$ is $\mathcal{E}$.

This is because of the following: Since X, Y, f, $\mathcal{E}$ and the representation spaces are all of finite type, there exist finitely many elements $z_1, \cdots, z_m \in k$ such that Y, X, f, $\mathcal{R}(X, f(\xi), P_X)$, $\mathcal{R}(Y, \xi, P_Y)$ and $\mathcal{E}$ are all defined over $\mathrm{Spec}(\overline{\mathbb{F}}_p[z_1, \cdots, z_m])$. Take T to be a smooth affine open subset of $\mathrm{Spec}(\overline{\mathbb{F}}_p[z_1, \cdots, z_m])$.

Here we view $\mathcal{E}$ as a locally free $\mathcal{O}_Y$–coherent $\mathcal{D}_Y$–module. For any integer $n \geq 1$, let $\mathcal{D}_Y^{<n}$ denote the subsheaf of $\mathcal{D}_Y$ of differential operators of order strictly less than n. Note that for any open $U \subset Y$ and $n \geq 0$,

$$E_n(U) = \{s \in \mathcal{E}(U) \mid D(s) = 0 \ \forall\ D \in \mathcal{D}_Y^{<p^n} \text{ with } D(1) = 0\}$$

[Gi, Theorem 1.3]. We obtain a sequence of subsheaves

$$\mathcal{E} = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_j \supset E_{j+1} \supset \cdots.$$

In the proof of [Gi, Theorem 1.3], it is shown that E_n has an $\mathcal{O}_Y$–module structure with respect to which E_n is a free $\mathcal{O}_Y$–module and F^*E_{n+1} is canonically isomorphic to E_n. For an open set U of Y_T, we define a similar sequence of subsheaves of the locally free $\mathcal{E}_T$:

$$E_{T,n}(U) = \{s \in \mathcal{E}_T(U) \mid D(s) = 0 \ \forall\ D \in \mathcal{D}_{Y_T/T}^{<p^n} \text{ with } D(1) = 0\},$$

where $\mathcal{D}_{Y_T/T}$ is the sheaf of algebra of relative differential operators for the projection $Y_T \longrightarrow T$. Since the differential operators in $\mathcal{D}_{Y_T/T}$ are $\mathcal{O}_T$–linear, the sheaves $E_{T,n}$ have an $\mathcal{O}_T$–linear structure as well. Moreover, the inverse image of $E_{T,n}$ along $Y \longrightarrow Y_T$ tensored with k over $\mathcal{O}_T$ is the sheaf E_n. Also, the $\mathcal{O}_Y$–module structure on E_n induces an $\mathcal{O}_{Y_T}$–module structure on $E_{T,n}$. Since E_n is a locally free $\mathcal{O}_Y$–module, it follows that $E_{T,n}$ is also a locally free $\mathcal{O}_{Y_T}$–module. Finally, $F^*E_{T,n+1}$ is isomorphic to $E_{T,n}$ where F is the absolute Frobenius morphism.

Using Theorem 4.2, by shrinking T if necessary, we may also assume that for all closed points $t \in T$, the homomorphism

$$\pi_1^{et}(f_t) : \pi_1(Y_t) \longrightarrow \pi_1(X_t)$$

induced by $f_t : Y_t \longrightarrow X_t$ is an isomorphism.

Now for a given n, by [Sh, Lemma 7] (also see [Ni, Theorem 5]) there exists an open dense subset U of T such that the Harder–Narasimhan filtration of f_*E_n is compatible with the Harder–Narasimhan filtration of the fibers $(f_t)_*E_{t,n}$ for all $t \in U$, where $E_{t,n}$ is the restriction of $E_{T,n}$ to Y_t. Consequently, there is a Harder–Narasimhan filtration of the family of vector bundles $(f_T)_*E_{U,n}$. Let $F_{U,n}$ be the maximal degree 0 subsheaf of $(f_U)_*E_{U,n}$. Then the restriction $F_{t,n}$ of $F_{U,n}$ is the maximal degree 0 subsheaf of $(f_t)_*E_{t,n}$.

Now we apply Theorem 3.1 to $f_t : Y_t \longrightarrow X_t$ and the stratified vector bundle $\{E_{t,m}\}$ to conclude that $E_{t,n} = f_t^*G_{t,n}$ for some $\{G_{t,n}\}$ in $\mathrm{Vect^{str}(X_t)}$, for all $t \in U$.

By the projection formula, we obtain

$$(f_t)_* E_{t,n} = (f_t)_* f_t^* G_{t,n} = G_{t,n} \otimes_{\mathcal{O}_{X_t}} (f_t)_* \mathcal{O}_{Y_t} \tag{5.1}$$

Also, the genuine ramification of f_t implies that the maximal degree zero subsheaf of $(f_t)_*\mathcal{O}_{Y_t}$ is $\mathcal{O}_{X_t}$. This and (5.1) together imply that $G_{t,n}$ is the maximal degree zero subsheaf of $(f_t)_*E_{t,n}$. Consequently, we obtain that

$$G_{t,n} = F_{t,n}, \quad E_{t,n} = f_t^* F_{t,n}$$

for all $t \in U$, and hence it follows that $E_{U,n} = (f_U)^* F_{U,n}$. Restricting to the generic point of U yields that $E_n = f^* F_n$.

So we obtain a stratified vector bundle $\mathcal{F} = \{F_n\}$ on X such that $f^*\mathcal{F} = \mathcal{E}$. Therefore, the functor

$$f^* : \mathrm{Vect}^{\mathrm{str}}(\mathrm{X}) \longrightarrow \mathrm{Vect}^{\mathrm{str}}(\mathrm{Y})$$

is essentially surjective. That it is fully faithful is already proved in Corollary 3.2 without any assumption on the base field. This completes the proof of Theorem 1.1.

Remark 5.1. Note that the above argument, together with the proof of Theorem 3.1, also shows that if $f : Y \longrightarrow X$ is a genuinely ramified map, then the induced homomorphism $\pi_1^{str} : \pi_1^{str}(Y) \longrightarrow \pi_1^{str}(X)$ is surjective (see [DM, p. 139, Proposition 2.21]).

6. Proof of Proposition 2.3

Let $f : Y \longrightarrow X$ be a finite generically smooth morphism between two irreducible projective varieties of the same dimension. Denote by $\mathrm{Aut}(Y/X)$ the group of automorphisms of Y over the identity map of X. The morphism f will be called *Galois* if there is a reduced finite subgroup $\Gamma \subset \mathrm{Aut}(Y/X)$ such that $X = Y/\Gamma$.

Note that we have $\mathcal{O}_X \subset f_*\mathcal{O}_Y$, because $f^*\mathcal{O}_X = \mathcal{O}_Y$ (use adjunction).

Lemma 6.1. *Let $f : Y \longrightarrow X$ be a generically smooth morphism between irreducible smooth projective varieties of the same dimension. Assume that f is Galois of degree d. Then*

$$f^*((f_*\mathcal{O}_Y)/\mathcal{O}_X) \subseteq \mathcal{O}_X^{\oplus(d-1)}$$

Proof. The proof of the lemma is exactly identical to the proof of [BP, p. 12831, Proposition 3.3]. In [BP, Proposition 3.3] it is assumed that $\dim X = 1 = \dim Y$, because [BP] is entirely dedicated to curves but this assumption that $\dim X = 1 = \dim Y$ is not used in the proof of [BP, Proposition 3.3]. $\square$

Let $f : Y \longrightarrow X$ be a genuinely ramified map between irreducible smooth projective varieties. As in Lemma 6.1, assume that f is a Galois map of degree d. Let

$$\Gamma := \mathrm{Gal}(f) \tag{6.1}$$

be the Galois group of f, so $X = Y/\Gamma$. For any element $\sigma \in \Gamma$, let

$$Y_\sigma := \{(y, \sigma(y)) \in Y \times_X Y \bigm| y \in Y\} \subset Y \times_X Y \tag{6.2}$$

be the irreducible component of $Y \times_X Y$.

Lemma 6.2. *There is an ordering of the elements of the group* Γ *in* (6.1)

$$\Gamma = \{\gamma_1, \cdots, \gamma_d\},$$

and a self–map $\eta : \{1, \cdots, d\} \longrightarrow \{1, \cdots, d\}$, *such that the following four statements hold:*

(1) $\gamma_1 = e$ *(the identity element of the group* Γ*),*
(2) $\eta(1) = 1$,
(3) $\eta(j) < j$ *for all* $j \in \{2, \cdots, d\}$, *and*
(4) $Y_{\gamma_j} \bigcap Y_{\gamma_{\eta(j)}} \neq \emptyset$ *(see* (6.2) *for notation).*

Proof. The proof of the lemma is exactly identical to the proof of [BP, p. 12835, Lemma 3.4]. The proof of [BP, Lemma 3.4] is combinatorial and only the connectedness of $Y \times_X Y$ is used. Note that this is true even when $\dim X > 1$ (see [BDP, p. 6, Theorem 2.4(3)]). The assumption that $\dim X = 1 = \dim Y$ is not used in the rest of the proof of [BP, Lemma 3.4]. □

Remark 6.3. The intersection $Y_{\gamma_j} \bigcap Y_{\gamma_{\eta(j)}}$ in the fourth statement of Lemma 6.2 coincides with the fixed–point locus for the element $(\gamma_j)^{-1}\gamma_{\eta(j)} \in \Gamma = \mathrm{Gal}(f)$. Since X and Y are both smooth, from purity of branch locus we know that $Y_{\gamma_j} \bigcap Y_{\gamma_{\eta(j)}}$ is a divisor on Y.

The following lemma constitutes a key input in the proof of Proposition 2.3.

Lemma 6.4. *Let* $f : Y \longrightarrow X$ *be a genuinely ramified map between irreducible smooth projective varieties. Assume that* f *is Galois of degree* d. *Then there are line bundles*

$$\mathcal{L}_j \subsetneq \mathcal{O}_Y, \quad 1 \leq j \leq d-1,$$

such that

$$f^*((f_*\mathcal{O}_Y)/\mathcal{O}_X) \subseteq \bigoplus_{j=1}^{d-1} \mathcal{L}_j.$$

Proof. The proof is rather identical to the proof of [BP, p. 12837, Lemma 3.5]. The details are given for the benefit of the reader. As in (6.1), the Galois group $\mathrm{Gal}(f)$ is denoted by Γ. Let $\nu : \widetilde{Y \times_X Y} \longrightarrow Y \times_X Y$ be the normalization of $Y \times_X Y$. For $i = 1, 2$, let

$$\widetilde{\pi}_i := \pi_i \circ \nu : \widetilde{Y \times_X Y} \longrightarrow Y \tag{6.3}$$

be the composition of maps, where $\pi_i : Y \times_X Y \longrightarrow Y$ is the projection to the i–th factor. We have an isomorphism

$$Y \times \Gamma \longrightarrow \widetilde{Y \times_X Y}, \quad (y, \gamma) \longmapsto (y, \gamma(y)). \tag{6.4}$$

So, for the map $\widetilde{\pi}_1$ in (6.3),

$$(\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y \times_X Y}} = \mathcal{O}_Y \otimes_k k[\Gamma]. \tag{6.5}$$

Using adjunction, the identity map $\pi_1^*\mathcal{O}_Y = \mathcal{O}_{Y \times_X Y} \xrightarrow{\mathrm{Id}_{\mathcal{O}_{Y \times_X Y}}} \mathcal{O}_{Y \times_X Y}$ produces a homomorphism

$$\zeta : \mathcal{O}_Y \longrightarrow (\pi_1)_*\mathcal{O}_{Y \times_X Y}$$

On the other hand, since $\pi_1 \circ \nu = \widetilde{\pi}_1$ (see (6.3)), and ν is surjective, we have an injective homomorphism

$$\varphi \,:\, (\pi_1)_*\mathcal{O}_{Y\times_X Y} \,\longrightarrow\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}}. \tag{6.6}$$

Let

$$\xi \,:=\, \varphi\circ\zeta \,:\, \mathcal{O}_Y \,\longrightarrow\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}} \,=\, \mathcal{O}_Y\otimes_k k[\Gamma] \tag{6.7}$$

be the composition of homomorphisms (see (6.5)).

The ordering, in Lemma 6.2, of the elements of Γ produces an isomorphism of $k[\Gamma]$ with $k^{\oplus d}$. Consequently, from (6.5) we have

$$(\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}} \,=\, \mathcal{O}_Y\otimes_k k[\Gamma] \,=\, \mathcal{O}_Y^{\oplus d}\,. \tag{6.8}$$

Let

$$\Phi \,:\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}} \,=\, \mathcal{O}_Y^{\oplus d} \,\longrightarrow\, \mathcal{O}_Y^{\oplus d} \,=\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}}$$

be the homomorphism defined by

$$(f_1,\, f_2,\, \cdots,\, f_d) \,\longmapsto\, (f_1 - f_{\eta(1)},\, f_2 - f_{\eta(2)},\, \cdots,\, f_d - f_{\eta(d)})\,, \tag{6.9}$$

where η is the map in Lemma 6.2; in other words, the j-th component of $\Phi(f_1,\, f_2,\, \cdots,\, f_d)$ is $f_j - f_{\eta(j)}$. The image

$$\mathcal{F} \,:=\, \Phi(\mathcal{O}_Y^{\oplus d}) \,\subset\, \mathcal{O}_Y^{\oplus d} \,=\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}}$$

is a trivial subbundle of rank $d-1$; the first component of $\Phi(f_1,\, f_2,\, \cdots,\, f_d)$ vanishes identically, because $\eta(1) \,=\, 1$. More precisely,

$$\mathcal{F} \,=\, \mathcal{O}_Y^{\oplus(d-1)} \,\subset\, \mathcal{O}_Y^{\oplus d} \,=\, (\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}}, \tag{6.10}$$

where $\mathcal{O}_Y^{\oplus(d-1)}$ is the subbundle of $\mathcal{O}_Y^{\oplus d}$ generated by all $(f_1,\, f_2,\, \cdots,\, f_d)$ such that $f_1 \,=\, 0$.

From (6.10) it follows immediately that

$$(\widetilde{\pi}_1)_*\mathcal{O}_{\widetilde{Y\times_X Y}} \,=\, \mathcal{O}_Y^{\oplus d} \,=\, \mathcal{F}\oplus\xi(\mathcal{O}_Y)\,, \tag{6.11}$$

where $\xi(\mathcal{O}_C)$ is the homomorphism in (6.7).

We have the commutative diagram

$$\begin{tikzcd}
\widetilde{Y\times_X Y} \arrow[dr, "\nu"] \arrow[drr, bend left, "\widetilde{\pi}_2"] \arrow[ddr, bend right, "\widetilde{\pi}_1"'] & & \\
& Y\times_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd} \tag{6.12}$$

By flat base change [Ha, p. 255, Proposition 9.3],

$$f^*(f_*\mathcal{O}_Y) \,\cong\, (\pi_1)_*(\pi_2^*\mathcal{O}_Y) \,=\, (\pi_1)_*\mathcal{O}_{Y\times_X Y}. \tag{6.13}$$

From (6.11) and (6.13) we get an injective homomorphism of coherent sheaves

$$\Psi \,:\, f^*((f_*\mathcal{O}_Y)/\mathcal{O}_X) \,\longrightarrow\, \mathcal{F} \,=\, \mathcal{O}_Y^{\oplus(d-1)}. \tag{6.14}$$

Note that Ψ is an isomorphism over the open subset of C where the map f is a submersion.

Consider the map η in Lemma 6.2. For every $1 \leq i \leq d-1$, define

$$D_i := Y_{\gamma_{i+1}} \bigcap Y_{\gamma_{\eta(i+1)}} \tag{6.15}$$

(see (6.2)); from the fourth property in Lemma 6.2 and Remark 6.3 it follows that D_i is a nonzero effective divisor on Y. So

$$D_i^0 := \{y \in Y \mid (y, \gamma_{i+1}(y)) \in D_i\} \subset Y \tag{6.16}$$

is a nonzero effective divisor on Y. Let

$$\mathcal{L}_i := \mathcal{O}_Y(-D_i^0) \subset \mathcal{O}_Y$$

be the lie bundle on Y given by the divisor $-D_i^0$.

For every $1 \leq i \leq d-1$, let

$$P_i : \mathcal{O}_Y^{\oplus(d-1)} \longrightarrow \mathcal{O}_Y \tag{6.17}$$

be the natural projection to the i-th factor. Consider the composition of homomorphisms $P_i \circ \Psi$, where P_i and Ψ are constructed in (6.17) and (6.14) respectively. We will show that $P_i \circ \Psi$ vanishes when restricted to D_i^0 in (6.16). To see this, for any $1 \leq j \leq d$, let

$$\widehat{P}_j : \mathcal{O}_Y^{\oplus d} \longrightarrow \mathcal{O}_Y$$

be the natural projection to the j-th factor. Recall the homomorphism Φ constructed in (6.9). If $(f_1, f_2, \cdots, f_d)$ in (6.9) actually lies in the image of $(\pi_1)_*\mathcal{O}_{Y\times_X Y}$ by the inclusion map φ in (6.6), then from (6.15) we have

$$(\widehat{P}_{i+1} \circ \Phi)(f_1, f_2, \cdots, f_d)(y, \gamma_{i+1}) = f_{i+1}(y, \gamma_{i+1}) - f_{\eta(i+1)}(y, \gamma_{\eta(i+1)}) = 0, \tag{6.18}$$

where $y \in D_i^0$ (see (6.16)), and also

$$(\widehat{P}_{i+1} \circ \Phi)(f_1, f_2, \cdots, f_d)(y, \gamma_{\eta(i+1)}) = f_{i+1}(y, \gamma_{i+1}) - f_{\eta(i+1)}(y, \gamma_{\eta\circ\eta(i+1)}) = 0$$

for $y \in D_i^0$. From (6.18) it follows that $P_i \circ \Psi$ vanishes when restricted to D_i^0, where Ψ and P_i are constructed in (6.14) and (6.17) respectively.

Since $P_i \circ \Psi$ vanishes when restricted to the divisor D_i^0, we have

$$P_i \circ \Psi(f^*((f_*\mathcal{O}_Y)/\mathcal{O}_X)) \subset \mathcal{L}_i = \mathcal{O}_Y(-D_i^0) \subset \mathcal{O}_Y. \tag{6.19}$$

From (6.14) and (6.19) it follows immediately that

$$f^*((f_*\mathcal{O}_Y)/\mathcal{O}_X) \hookrightarrow \bigoplus_{i=1}^{d-1} \mathcal{L}_i.$$

This completes the proof of the proposition. □

Lemma 6.5. *Let $f : Y \longrightarrow X$ be a genuinely ramified map between irreducible smooth projective varieties. For any semistable vector bundle V on X,*

$$\mu_{\max}(V \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X)) < \mu(V).$$

Proof. In view of Lemma 6.4, the proof is exactly identical to the proof of [BP, p. 12840, Lemma 4.1]. Note that in Lemma 6.4 we have

$$\text{degree}(\mathcal{L}_j) < 0$$

for all $1 \leq j \leq d-1$, because $\mathcal{L}_j \subsetneq \mathcal{O}_Y$ and $\mathcal{L}_j$ is locally free. More, precisely, $-\text{degree}(\mathcal{L}_j)$ coincides with the degree of the divisor whose ideal sheaf is $\mathcal{L}_j$. □

Proof of Proposition 2.3. The proof is very similar to the proof of [BP, p. 12844, Lemma 4.3]. The details are given for the benefit of the reader.

Using the projection formula, and the fact that f is a finite map, we have

$$H^0(Y, \mathrm{Hom}(f^*V, f^*W)) \cong H^0(X, f_*\mathrm{Hom}(f^*V, f^*W)) \cong H^0(X, f_*f^*\mathrm{Hom}(V, W))$$

$$\cong H^0(X, \mathrm{Hom}(V, W) \otimes f_*\mathcal{O}_Y) \cong H^0(X, \mathrm{Hom}(V, W \otimes f_*\mathcal{O}_Y)). \tag{6.20}$$

Let

$$0 = B_0 \subset B_1 \subset \cdots \subset B_{m-1} \subset B_m = W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X)$$

be the Harder–Narasimhan filtration of $W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X)$. Since W is semistable, and f is genuinely ramified, from Lemma 6.5 we know that

$$\mu(B_i/B_{i-1}) \le \mu(B_1) = \mu_{\max}(W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X)) < \mu(W) = \mu(V)$$

for all $1 \le i \le m$. Since both V and B_i/B_{i-1} are semistable, and $\mu(B_i/B_{i-1}) \le \mu(V)$, we have

$$H^0(X, \mathrm{Hom}(V, B_i/B_{i-1})) = 0$$

for all $1 \le i \le m$. This implies that

$$H^0(X, \mathrm{Hom}(V, W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X))) = 0. \tag{6.21}$$

Now consider the short exact sequence of sheaves

$$0 \longrightarrow \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, W \otimes f_*\mathcal{O}_Y) \longrightarrow \mathrm{Hom}(V, W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X)) \longrightarrow 0,$$

and the corresponding exact sequence of cohomologies

$$0 \longrightarrow H^0(X, \mathrm{Hom}(V, W)) \longrightarrow H^0(X, \mathrm{Hom}(V, W \otimes f_*\mathcal{O}_Y)) \tag{6.22}$$

$$\longrightarrow H^0(\mathrm{Hom}(V, W \otimes ((f_*\mathcal{O}_Y)/\mathcal{O}_X))).$$

Combining (6.21) and (6.22) it follows that

$$H^0(X, \mathrm{Hom}(V, W)) = H^0(X, \mathrm{Hom}(V, W \otimes f_*\mathcal{O}_Y)).$$

From this and (6.20) it follows that

$$H^0(X, \mathrm{Hom}(V, W)) = H^0(Y, \mathrm{Hom}(f^*V, f^*W)).$$

This completes the proof. □

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