# THE POLARIZED DEGREE OF IRRATIONALITY OF $K3$ SURFACES

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Abstract. Given a polarized variety $(X, L)$, we construct and study projections of low degree $X \dashrightarrow \mathbb{P}(H^0(L^\vee)) \dashrightarrow \mathbb{P}^n$ using the associated kernel bundles. As an elementary application, we can show that the degree of irrationality of a very general $(1,6)$ abelian surface, as well as that of a very general $K3$ surface of genus 6 is 3. We also give new upper bounds for $K3$ surfaces of any genus. Moreover, in the case of surfaces, this observation can be used to show that maps of the degree at most $d$ move in families. We study the family of projections of minimal degree of a very general $K3$ surface of genus $4, 5, 6$. As a different application of our construction, we exhibit new rational maps of low degree for some hyper-Kähler varieties, abelian varieties and Gushel–Mukai threefolds.

## Introduction

Given a quasi-projective variety $X$ over an algebraically closed field, a very natural invariant to measure how far $X$ is from being rational is the degree of irrationality

$$\operatorname{irr}(X) = \min\{\deg(\varphi) \mid \varphi : X \dashrightarrow \mathbb{P}^{\dim(X)} \text{ is a rational, generically finite map}\}.$$

The case when $X$ is a curve leads to the classical notion of gonality, whereas for $\dim(X) \geq 2$ very little is known in general and the problem has recently received considerable attention. This invariant was originally defined in [MH82]. The main results in this direction are for hypersurfaces [BCDP14, BDPE+17, CS20, Sta17, Yan19], abelian varieties [AP92, Che19, CMNP22, Mar22, TY95, Voi18, Yos96], hyper-Kähler varieties [Voi22], or more specific examples [ABL23, GK19]. Giving a sharp lower bound is, in general, a very difficult problem. In this paper, among other things, we compute the degree of irrationality of a very general $K3$ surface of genus 6 and of a very general $(1,6)$ abelian surface. The latter was one of the open cases for very general abelian surfaces in view of [Che19],[Mar22]. The only other case where the degree of irrationality of a very general $(1,d)$ abelian surface was known to be 3 was for $d = 2$, as proven in [TY95].

More generally, we introduce and study a natural invariant very close to the degree of irrationality one may attach to a polarized variety. Let $(X, L)$ be a polarized projective variety of dimension $n \geq 2$, for any $V \subset H^0(L)$ of dimension $n+1$ we get a rational map $\varphi_V : X \dashrightarrow \mathbb{P}^n$, we may define

$$\operatorname{irr}_L(X) = \inf\{\deg(\varphi_V) \mid V \in \operatorname{Gr}(n+1, H^0(L)), \text{ defined in codimension 2, generically finite}\}.$$

In other words, given $X \dashrightarrow \mathbb{P}H^0(L)^\vee$, we aim to study the most special projections, i.e., those of minimal degree.

For a general curve of genus $g$ and a general line bundle $L \in \operatorname{Pic}^d(C)$ of degree $d \geq g+1$ one has to drop the assumption that $\varphi_V$ is defined in codimension 2 in order to get a non-trivial notion. One can compute $\operatorname{irr}_L(C) = \max\{2g+2-d, \lfloor \frac{g+3}{2} \rfloor\}$.

For a surface $S$ and an indecomposable class $L \in \operatorname{Pic}(S)$ (i.e. $L$ can't be written as the tensor product of two effective line bundles) the rational map $\varphi_V : S \dashrightarrow \mathbb{P}^2$ is defined in codimension 2 and generically finite for any $V \in \operatorname{Gr}(3, H^0(L))$. For $K3$ surfaces we obtain the following.

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**Theorem A.** *Let $(S,L)$ be a $K3$ surface of genus $g = 2 + 2n(n+1) + k$, then*

$$\mathrm{irr}_L(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

*Moreover, if $g = 3, 4, 6$ and $S$ is very general, then $\mathrm{irr}_L(S) = 3$. If $g = 5$ and $S$ is very general then $\mathrm{irr}_L(S) = 4$.*

In the paper we present the bound for $K3$ surfaces of Picard rank 1 and the same result for any $K3$ surface follows from general specialization results (c.f. [CS20, Corollary C]). The bounds we obtain are better than the ones given in [Sta17]. In low genus they are optimal up to genus 8 (in [MR25] it is proven that $\mathrm{irr}_L(S) = 4$ for $g = 7, 8, 9, 11$) and asymptotically the improvement is by a factor between 3 and 6 with respect to [Sta17] (depending on the genus). Let us point out that after the first appearance of the first version of this preprint the bounds have been further improved: in [MR25] the bounds have been improved for $9 \leq g \leq 14$ combining the method presented in this paper with derived category tools (Fourier–Mukai transforms and Bridgeland stability conditions); in [Iba24] the method of the present paper have been optimized to get better asymptotic bounds for $k >> 0$ in Theorem A.

Our technique relies on an elementary yet powerful observation, which we briefly outline below. It can be seen as a natural variation of the Lazarsfeld–Mukai bundles introduced for studying linear series on curves on $K3$ surfaces ([Laz86]). For a variety $X$ of dimension $n$ and a linear system $|V|$, inducing a rational map $\varphi_V : X \dashrightarrow \mathbb{P}^n$, one may construct the kernel reflexive sheaf, sitting in the exact sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L.$$

Then, via the identification $\mathrm{Gr}(n, V) = \mathbb{P}V^\vee \subset \mathbb{P}(H^0(E))$, the fibers of the map may be seen as zero loci of sections of $E$. More precisely, the fiber above $[s] \in \mathbb{P}V^\vee = \mathbb{P}^n$ (the target projective space) is $Z(s)$ outside of the base locus of the map (see Proposition 1.5). This gives a cheap way to compute the degree of the rational map. The main idea of this paper is that this construction can be reversed (with some hypothesis on $E$): if one is given a sufficiently general vector bundle $E$ of rank $n$ and at least $n+1$ linearly independent global sections generating $E$ in codimension 2, then one may produce rational maps of degree $\leq c_n(E)$. Moreover, if the vector bundle has many global sections, one can impose base points on them and make the degree drop. The basic example to keep in mind is that of a general $K3$ surface of genus 6. In this case there is a stable vector bundle of rank 2 with $h^0(E) = 5$ and $c_2(E) = 4$, then for any $P \in S$ the map given by $V = H^0(E \otimes \mathcal{I}_P)^\vee \subset H^0(L)$ is of degree $c_2(E) - 1 = 3$ (see also Theorem 2.4). More generally, Theorem A is just a simple application of Proposition 1.5. We now outline some more applications.

For a surface $S$ with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$, this technique can be used to give a scheme structure to the locus of special maps, see Theorem 6.1. We give the structure of an algebraic variety to the locus

$$W^r_d(S, L) = \{V \in \mathrm{Gr}(r+1, H^0(L)) \quad | \deg(\varphi_V) \leq d\}.$$

Some explicit computations are carried out:

**Theorem B.** *Let $(S,L)$ be a $K3$ surface of genus $g$ with $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$. The following hold:*

*(1) if $g = 4$, then $W^2_3(S, L) = \mathbb{P}^3$;*
*(2) if $g = 5$, then $W^2_3(S, L) = \emptyset$ and $W^2_4(S, L)$ has two components (one birational to $S$ and one birational to a $\mathbb{P}^3$ bundle over the moduli space of stable rank 2 vector bundles with $c_2 = 4$);*
*(3) if $g = 6$, then $W^2_3(S, L) = S$.*

A more geometric interpretation of the above theorem is the following: for $S \subset \mathbb{P}^4$ a very general $K3$ surface of degree 6 the locus of trisecant lines is isomorphic to $\mathbb{P}^3$; if $S \subset \mathbb{P}^5$ is a very general $K3$ surface of degree 8 there are no 4−secant planes tangent to $S$; if $S \subset \mathbb{P}^6$ is a very general $K3$ surface of degree

10 for any $p \in S$ there exists a unique $6-$secant codimension 3 linear space tangent to $S$ at $p$. A similar theorem for $K3$ surfaces up to genus 14 has been developed in [MR25, Theorem A and B], suggesting interesting patterns for the dimension of the loci $W^2_d(S, L)$ (c.f. [MR25, Section 8]). Other kinds of polarized surfaces are interesting to study from this point of view, for example, the case of $(S, 3\Theta)$ for a very general principally polarized abelian surface $(S, \Theta)$ should be accessible with the methods of the present paper.

We apply the technique also for higher-dimensional varieties, let us mention here the most surprising results. Recall that a Gushel–Mukai threefold is a Fano threefold of genus 6 and index 1. Moreover, for a surface $S$ we denote with $S^{[n]}$ the Hilbert scheme of length $n$ subschemes.

**Theorem C.** *The following estimates hold:*

*(1) if $X$ is a general Gushel–Mukai threefold, then $\mathrm{irr}(X) \leq 3$;*
*(2) if $(S, L)$ is a very general $(1, 6)$ abelian surface then $\mathrm{irr}_L(S) = 3$;*
*(3) if $(S, L)$ is a very general $K3$ surface of genus $6$ then $\mathrm{irr}(S^{[2]}) \leq 6$;*
*(4) if $(A, L)$ is a very general abelian threefold of type $(1, 3, 12)$ or $(1, 6, 6)$ then $\mathrm{irr}_L(A) \leq 8$;*
*(5) if $(S, L)$ is a very general $K3$ surface of genus $10$ then $\mathrm{irr}(S^{[3]}) \leq 20$.*

Let us remark that the case of the $(1, 6)$ abelian surface settles one of the three remaining open cases for abelian surfaces. Indeed, $\mathrm{irr}(S) = 4$ if $(S, L)$ is a very general $(1, d)$ abelian surface with $d \neq 1, 2, 3, 6$ (c.f. [Che19, Mar22]) and $\mathrm{irr}(S) = 3$ for $d = 2$ (c.f. [TY95]). For the very general abelian threefold the best lower bound known is 5 (c.f. [CMNP22]).

**Structure of the paper.** In section 1 we explain how to construct and compute the degree of a rational map starting from the kernel sheaf. In section 2 the technique is applied to study projections of the lowest possible degree for a $S \subset \mathbb{P}^g$ a very general $K3$ surface of genus $4, 5, 6$ (the content of Theorem B). In the following section 3, we study the case of the $(1, 6)$ abelian surface. In section 4 the asymptotic bound of Theorem A is obtained. We take care of Theorem C in section 5. Finally, in section 6, we prove that the loci $W^r_d(S, L)$ are algebraic varieties (c.f. Theorem 6.1).

Each section presents an independent application of the method introduced in section 1, and the sections are mutually independent. The contents of section 1 and of section 3 hold over any algebraically closed field. The contents of the other sections hold over $\mathbb{C}$.

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## 1. Constructing rational maps via the kernel bundle.

We begin with some notation and definitions. We will denote with $X$ a smooth projective variety of dimension $n$, $E$ a rank $n$ reflexive sheaf and $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$. Let $\mathrm{Bl}(V^\vee) \subset X$ be the union of the locus where the sections of $V^\vee$ do not generate $E$ and of the locus where $E$ is not locally free. We will sometimes write Bl instead of $\mathrm{Bl}(V^\vee)$, if the dependence on $V^\vee$ is clear from the context. We can now define

**Definition 1.1.** Let $X, E, V^\vee$ as above, we define $c_n(V^\vee)$ as the degree of the zero-dimensional part of the zero locus of a general section $s \in V^\vee$ supported in $X - \mathrm{Bl}(V^\vee)$. Moreover, if the general section $s \in V^\vee$ vanishes along a zero-dimensional subscheme we define $bl(V^\vee) = \deg(Z_{cycle}(s) \cap Bl(V^\vee))$ for a general $s \in V^\vee$.

**Remark 1.2.** In general $Z_{cycle}(s) \cap Bl(V^\vee) \subsetneq \mathrm{Bl}(V^\vee)$.

Moreover, we will be interested in couples $(E, V^\vee)$ that satisfy some special properties. For the purpose of this paper we will refer to them as *good pairs.*

**Definition 1.3.** A couple $(E, V^\vee)$ with $E$ a reflexive sheaf and $V^\vee \in \mathrm{Gr}(n+1, H^0(E))/\mathrm{Aut}(E)$ is a *good pair* if $V^\vee$ generates $E$ in codimension 2 and $H^0(E^\vee) = 0$.

Given a good pair $(E, V^\vee)$, the kernel of the evaluation $\mathrm{ev} : V^\vee \otimes \mathcal{O}_X \to E$ is $L^\vee$, where $L = \det(E)$. Indeed, since the evaluation is surjective in codimension 2, we get $c_1(\mathrm{Im}(ev)) = c_1(E)$ and $\mathrm{Im}(ev)^\vee = E^\vee$ (recall that $E$ is reflexive). Dualizing, we get an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L.$$

In particular, to any good pair $(E, V^\vee)$, we can associate the rational map induced by $(L, V)$ (notice that since $h^0(E^\vee) = 0$ we have $V \subset H^0(L)$, therefore, the map is non-degenerate). We will denote the induced rational map by $\varphi_V : X \dashrightarrow \mathbb{P}V^\vee$. It is easy to see that the set-theoretic support of the base locus of the rational map coincides with Bl, defined before: for instance, one can observe that $V$ coincides with the image of $\bigwedge^2 V^\vee \to H^0(L)$ induced by the determinant map on the global sections of $E$. In fact, the locus where $V$ does not generate $L$ coincides with the union of the locus where $V^\vee$ does not generate $E$ and the locus where $E$ is not locally free.

On the other hand, as mentioned in the introduction, if one starts with a non-degenerate rational map $X \dashrightarrow \mathbb{P}^n$ (hence, a line bundle $L \in \mathrm{Pic}(X)$ together with $V \in \mathrm{Gr}(n+1, H^0(L))$ generating $L$ is codimension 2), one may construct the kernel sheaf $E^\vee = \ker(V \otimes \mathcal{O}_X \to L)$ together with $V^\vee \subset H^0(E)$. We have the following.

**Lemma 1.4.** *Given a rational non-degenerate map $X \dashrightarrow \mathbb{P}^n$ the pair $(E, V^\vee)$ is a good pair.*

*Proof.* On a smooth variety any rational map is well defined in codimension 2, hence the sequence

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L,$$

is exact in codimension 2. Dualizing, we get that $V^\vee$ generates $E$ in codimension 2. Moreover, $E$ is the kernel of a morphism between vector bundles, so it is reflexive. The morphism $V \to H^0(L)$ is injective, this implies $H^0(E^\vee) = 0$. We showed that $(E, V^\vee)$ is a good pair. □

We can now prove the key proposition that will be used throughout the paper: there is a bijective correspondence between rational maps of degree $d$ induced by $L$ and good pairs $(E, V^\vee)$ with $c_n(V^\vee) = d$. The invariant $c_n(V^\vee)$ will make the computation of the degree of certain rational maps much easier.

**Proposition 1.5.** *Let $X$ be a smooth projective variety of dimension $n$. There is a* one-to-one *correspondence*

$$\{\textit{rational non-degenerate maps } X \dashrightarrow \mathbb{P}^n \textit{ of degree } d\} \leftrightarrow \{\textit{good pairs } (E, V^\vee) \textit{ with } c_n(V^\vee) = d\},$$
$$(L, V) \longrightarrow (E, V^\vee).$$

*Moreover, the fibers of the rational map $\varphi_V : X \dashrightarrow \mathbb{P}V^\vee$ are computed by $\varphi_V^{-1}[s] \cap (X - \mathrm{Bl}) = Z(s) \cap (X - \mathrm{Bl})$, where $s$ can be seen both as a point of the target projective space (left hand side of the equality) and as a section of the vector bundle $E$ (right hand side of the equality). In particular, if $E$ is a vector bundle and the general element of $V^\vee$ vanishes along a zero dimensional subscheme*

$$c_n(V^\vee) = \deg(\varphi_V) = c_n(E) - bl(V^\vee).$$

*Proof.* The map $(E, V^\vee) \to (L, V)$ is clearly a bijection between good pairs and non-degenerate rational maps $X \dashrightarrow \mathbb{P}^n$. Indeed, given a rational map one can always reconstruct the kernel sheaf $E^\vee = \ker(V \otimes \mathcal{O}_X \to L)$ with $V^\vee \subset H^0(E)$ (uniquely up to the action of $\mathrm{Aut}(E)$ on $H^0(E)$) and Lemma 1.4 proves that $(E, V^\vee)$ is a good pair.

We need to show the second part of the lemma (the fact that degree $d$ rational maps corresponds to good pairs with $c_n(V^\vee) = d$ follows from it). We abuse notation and denote with $\varphi_V$ the induced morphism $\varphi_V : (X - \mathrm{Bl}) \to \mathbb{P}V^\vee$. The sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}V^\vee} \longrightarrow T\mathbb{P}V^\vee(-1) \longrightarrow 0$$

pulls back to

$$0 \longrightarrow L^\vee_{|X-\mathrm{Bl}} \longrightarrow V^\vee \otimes \mathcal{O}_{X-\mathrm{Bl}} \longrightarrow E_{|X-\mathrm{Bl}} \longrightarrow 0,$$

on $X - \mathrm{Bl}$. In particular $E = \varphi^* T\mathbb{P}V^\vee(-1)$ and pullback induces an identification $\varphi_V^* : H^0(T\mathbb{P}V^\vee(-1)) = V^\vee \subset H^0(E)$. The general section $s$ of $T\mathbb{P}V^\vee(-1)$ vanishes at the point $[s] \in \mathbb{P}V^\vee$ (and at no other point, since $c_n(T\mathbb{P}V^\vee(-1)) = 1$). Hence, the fiber over $[s]$ of the map $\varphi_V : X - \mathrm{Bl} \to \mathbb{P}V^\vee$ is $Z(\varphi_V^* s)$. The facts concerning the degree follows since $c_n(V^\vee)$ is precisely the degree of the zero-dimensional part of the base locus of a general section of $V^\vee$ outside the base locus. □

Moreover, it is sometimes easy to understand when a pair is a good pair.

**Corollary 1.6.** *Let $X$ be a smooth projective variety such that $\mathrm{NS}(X) = \mathbb{Z} \cdot L$, then there is a* one-to-one *correspondence*

$$\{V \in \mathrm{Gr}(n+1, H^0(L)) \textit{ with } \deg(\varphi_V) = d\} \leftrightarrow \{(E, V^\vee) \quad E \textit{ stable reflexive of rank } n, c_1 = L, c_n(V^\vee) = d\}.$$

*In particular: for any $V \in \mathrm{Gr}(n+1, H^0(L))$ the associated kernel sheaf is stable; for any stable rank $n$ reflexive sheaf $E$ and $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$ the couple $(E, V^\vee)$ is a good pair.*

*Proof.* Consider $V \in \mathrm{Gr}(n+1, H^0(L))$. Since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$ the evaluation $V \otimes \mathcal{O}_X \to L$ is surjective in codimension 2 (if the base locus was of dimension 1 then $L$ would have an effective sub-line bundle). Now, if $E$ is not stable we get an exact sequence destabilizing $E$

$$0 \longrightarrow M \longrightarrow E \longrightarrow N,$$

with $c_1(M) \geq L$ numerically. This implies that $c_1(N) \leq \mathcal{O}_S$ numerically. Now, $E$ is generically globally generated, hence all of its quotients are generically globally generated. A generically globally generated torsion free sheaf with $c_1 \leq 0$ must be of the form $N = \mathcal{O}_X^{\oplus k}$. Dualizing we get a map $\mathcal{O}_S^{\oplus k} \to E^\vee$ contradicting the fact that $(E, V^\vee)$ is a good pair (it is by Lemma 1.4).

On the other hand, if we start with a stable reflexive sheaf $E$, we get by stability that $h^0(E^\vee) = 0$. Consider any $V^\vee \in \mathrm{Gr}(3, H^0(E))$ suppose by contradiction that $ev : V^\vee \otimes \mathcal{O}_X \to E$ is not surjective in codimension 2. If $\mathrm{Im}(ev)$ is of generic rank $\leq n-1$, we get $c_1(\mathrm{Im}(ev)) > 0$, hence $c_1(\mathrm{Im}(ev)) \geq L$ numerically (by $\mathrm{NS}(X) = \mathbb{Z} \cdot L$). This contradicts the stability of $E$. If $\mathrm{Im}(ev)$ is of generic rank $n$ then again $c_1(\mathrm{Im}(ev)) \geq L$, this implies[1] $c_1(\mathrm{Im}(ev)) = L$ and that the map is surjective in codimension 2 (if the cokernel is supported on a divisor, then $\mathrm{Im}(ev)$ and $E$ would not have the same $c_1$). Hence, for any stable rank $n$ reflexive sheaf with $c_1(E) = L$ and any $V^\vee \in \mathrm{Gr}(n+1, H^0(E))$ the couple $(E, V^\vee)$ is a good pair.

□

In the case of surfaces, everything is simpler.

**Corollary 1.7.** *Let $S$ be a smooth surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$ and a good pair $(E, V^\vee)$ with $c_1(E) = L$, then*

*(i) the sheaf $E$ is a vector bundle;*
*(ii) the cycle $\cap Z_{cycle}(s) \cap \mathrm{Bl}(V^\vee)$ is rigid for general $s \in V^\vee$, in particular $bl(V^\vee) = \deg \cap_{s \in V^\vee} Z_{cycle}(s)$ (does not change moving $s \in V^\vee$ for a general $s$);*
*(iii) the degree of $\varphi_V$ can be computed as $d = c_2(E) - bl(V^\vee) > 0$*

*Proof.* Reflexive implies locally free on a surface, hence $E$ is a vector bundle. The base locus $\mathrm{Bl}(V^\vee)$ is of dimension 0, in this case. Hence, for a general $s \in V^\vee$ the cycle $Z_{cycle}(s) \cap \mathrm{Bl}(V^\vee)$ is rigid. Therefore, $bl(V^\vee)$ can be calculated as $\deg \cap_{s \in V^\vee} Z_{cycle}(s)$. The computation of the degree, then, follows from Proposition 1.5 together with the fact that any section of $E$ vanishes along a zero-dimensional subscheme (if not, it would induce an effective line sub-bundle of $L$). The fact that $d > 0$ follows from the fact that $L$ is indecomposable. □

We end this section with a couple of remarks.

**Remark 1.8.** Note that for (ii) the zero locus of a section $s \in V^\vee$ may change scheme theoretically. Moreover, the degree of this cycle may jump for some special $s \in V^\vee$.

**Remark 1.9.** In the case of surfaces, the length of the base locus length(Bl) can be computed as $L^2 - c_2(E)$. Hence $c_2(E)$ gives information on *how secant* the linear space $\mathbb{P}(\ker(H^0(L)^\vee \to V^\vee)) \subset \mathbb{P}H^0(L)^\vee$ is with respect to $S$. On the other hand $\mathrm{bl}(V^\vee)$ gives the number of points (counted with multiplicity) this linear space is tangent to $S$.

## 2. Brill–Noether loci for the very general $K3$ surface of genus $4, 5, 6$.

In this section $(S, L)$ will denote a polarized smooth surface. We will use the framework provided by the previous section to study the following loci.

$$W^2_d(S, L) = \{V \in \mathrm{Gr}(3, H^0(L)) \text{ with } \deg(\varphi_V \leq d)\}$$

In the last section of the paper we will see that as soon as $\mathrm{NS}(S) = \mathbb{Z} \cdot L$ they can always be given a scheme structure. We study some examples: the case of a very general $K3$ surface of genus $4, 5, 6$. We start with the case of the $K3$ surface of genus 4. We recall the following preliminary, essentially due to Mukai (see [Muk87, Corollary 2.5]). This can be seen also as a corollary of [Laz86], saying that curves with indecomposable class on gl $K3$ surfaces do not carry linear series with negative Brill–Noether number. The inequality we write in the following remark follows directly from $\rho(g, r-1, d) \geq 0$.

[1]We have an induced non-zero map $\det(\mathrm{Im}(ev)) \to \det(E) = L$, hence $c_1(\mathrm{Im}(ev)) \leq c_1(E)$

**Remark 2.1.** Let $F$ be a stable vector bundle on a $K3$ surface of rank $r$, $c_1(F)^2 = 2g-2$ and $c_2(F) = d$, then $d \geq \frac{(r-1)(g+r)}{r}$. Moreover, when equality holds, there exists a unique vector bundle with those invariants.

**Lemma 2.2.** *Let $(S,L)$ be a $K3$ surface of genus $4$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot L$. Then, $W^2_3(S,L) = \mathbb{P}^3$.*

*Proof.* By Mukai's theory of vector bundles on $K3$ surfaces, there is a unique stable rank 2 vector bundle with $c_2(E) = 3$ and $c_1(E) = L$ (c.f. [Muk87]). I claim that, through the correspondence of Corollary 1.6, we have $W^2_3(S,L) = \mathrm{Gr}(3, H^0(E)) = \mathbb{P}H^0(E)^\vee \simeq \mathbb{P}^3$. On the one hand, we clearly have $\mathrm{Gr}(3,H^0(E)) \subset W^2_3(S,L)$ through $(E,V^\vee) \to (L,V)$. On the other hand, giving a degree 3 projection $S \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ is the same as giving a $3-$secant line $l \subset \mathbb{P}^4$. Therefore, we get an exact sequence

$$0 \longrightarrow E_l^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I}_{l\cap S} \longrightarrow 0.$$

The kernel $E_l$ is a stable rank 2 vector bundle with $c_2(E_l) = L^2 - \deg(l \cap S) = 6-3 = 3$ and $c_1(E_l) = L$ (see again Corollary 1.6). Therefore, we get $E_l = E$ and $V^\vee \in \mathrm{Gr}(3, H^0(E))$ as desired. $\square$

We now move to $K3$ surfaces of genus $5, 6$. The following is probably well known.

**Lemma 2.3.** *Let $(S,L)$ be a $K3$ surface of genus $5,6$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot L$ and $E$ be a rank $2$ stable vector bundle such that $c_1(E) = L, c_2(E) = 4$. Then $E$ is globally generated.*

*Proof.* The fact that $E$ is globally generated is implied by $h^1(E \otimes \mathcal{I}_P) = 0$ for any $P \in S$. Assume by contradiction $h^1(E\otimes \mathcal{I}_P) \geq 1$, then, there exists a non-trivial extension

$$0 \longrightarrow E^\vee \longrightarrow F \longrightarrow \mathcal{I}_P \longrightarrow 0.$$

We get that $F$ is a stable rank 3 vector bundle with $c_2(F) = 5$. This is a contradiction because there does not exist any non-trivial stable rank 3 vector bundle with $c_2 < 6$ (the bundle $F$ is stable because $F^\vee$ is generated in codimension 2 with $c_1(F)$ generating $\mathrm{NS}(S)$ and a similar argument of the proof of Corollary 1.6 applies).

$\square$

**Theorem 2.4.** *Let $(S,L)$ be a very general $K3$ surface of genus $6$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot L$. Then $W^2_3(S,L) = S$. In particular, $\mathrm{irr}(S) = \mathrm{irr}_L(S) = 3$.*

*Proof.* By Mukai's theory of vector bundles on $K3$ surfaces (c.f. [Muk87]) on such a $K3$ surface there is a unique stable rank 2 vector bundle $E$ with invariants $c_1(E) = L, c_2(E) = 4$ and $h^0(E) = 5$. Note that, since $E$ is globally generated, $h^0(E \otimes \mathcal{I}_P) = 3$. In view of the correspondence of Corollary 1.6, for any $P \in S$ the pair $(E, H^0(E\otimes\mathcal{I}_P))$ is a good pair that induces a map of degree $d = c_2(E) - bl(H^0(E\otimes\mathcal{I}_P)) = 4-1 = 3$. This gives an inclusion $S \subset W^2_3(S,L)$. Moreover, it can be proven that any map $(V,L)$ of degree $\leq 3$ induced by the primitive linear system has a kernel bundle with $c_2(E) \leq 4$ (c.f. [MR25, corollary 2.3]). Hence, for any $(L,V) \in W^2_3(S,L)$, we find that the kernel bundle $E$ has $c_2(E) \leq 4$. Since $E$ has to be stable (see Corollary 1.6) we get by Mukai's theory that $c_2(E) \geq 4$. Hence, $E$ must be the unique stable rank 2 vector bundle with $c_1 = L, c_2 = 4$. Moreover, since $d = 3$ we have $bl(V^\vee) = c_2(E) - d = 1$, this forces the sections of $V^\vee$ to have a base point. We deduce $V^\vee = H^0(E \otimes \mathcal{I}_P)$, so $(L,V) \in S$. We proved $W^2_3(S,L) = S$. The fact that $\mathrm{irr}(S) \neq 2$ follows from the fact that a very general $K3$ surface does not possess any non-trivial birational automorphism. $\square$

**Remark 2.5.** The geometric interpretation of this projection is transparent. We have an exact sequence

$$0 \longrightarrow E^\vee \longrightarrow H^0(E\otimes\mathcal{I}_P)^\vee \otimes \mathcal{O}_S \longrightarrow L \otimes \mathcal{I}_{\mathrm{Bl}} \longrightarrow 0,$$

here we are considering the base locus Bl scheme theoretically. It is easy to see that the degree of the base locus is $L^2 - c_2(E) = 6$. Hence, we are projecting from a $6-$secant codimension $3-$linear space. Moreover, the map $H^0(E \otimes \mathcal{I}_P) \otimes k(P) \to E \otimes k(P)$ is 0, hence, $\mathcal{I}_{\mathrm{Bl}} \otimes k(P)$ is $3-$dimensional. This implies that the codimension $3-$linear space is tangent to $S$ at $P$.

We now study projections $S \subset \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ for a very general $K3$ surface of genus 5.

**Theorem 2.6.** *Let $(S, L)$ be a $K3$ surface genus $g = 5$ whose Nèron-Severi group is of rank 1. Then, there are no projections $S \subset \mathbb{P}(H^0(L)^\vee) \dashrightarrow \mathbb{P}^2$ of degree $\leq 3$.*

*Proof.* Consider $S$ a $K3$ of genus 5 and suppose that we have a rational map $\varphi_{|L\otimes\mathcal{I}|} : S \dashrightarrow \mathbb{P}^2$ of degree 3. Consider the associated kernel bundle $E$. We may suppose $h^0(E) \geq 3$ and $h^1(E) = 0$. This means that we only have to analyze the cases $c_2(E) = 4, c_2(E) = 5$. In the first case according to the previous lemma for any $P$ the grassmannian $\mathrm{Gr}(3, H^0(E \otimes \mathcal{I}_P))$ is empty. Hence, for any $V \in \mathrm{Gr}(3, H^0(E))$ the map $\varphi_V$ is of degree 4 by Corollary 6.5. If $c_2(E) = 5$ the problem is even easier because then $\dim(\mathcal{O}_S/\mathcal{I}) = 3$, hence the sections of $E$ may have at most one common zero (in that case $\mathcal{I} = m_P^2$). We get $\deg(\varphi_{|L\otimes\mathcal{I}|}) \geq c_2(E) - 1 = 4$. The proof is complete.

□

**Remark 2.7.** The content of the previous theorem can be rephrased by saying that a $4-$secant plane to $S \subset \mathbb{P}^5$ is never tangent to $S$ (more precisely, the intersection with $S$ is always a local complete intersection subscheme).

We can also describe the Brill-Noether locus $W_4^2(S)$ in this case.

**Corollary 2.8.** *The locus $W_4^2(S) \subset \mathrm{Gr}(3, H^0(L))$ consists of two disjoint components birational to $S$ and to a $\mathrm{Gr}(3, 4)$-bundle over $\mathcal{M}$, the moduli space of stable rank 2 vector bundles with $c_2 = 4$.*

*Proof.* The component birational to $S$ is given by

$$P \to H^0(L \otimes \mathcal{I}_P^2) \in W_4^2(S) \subset \mathrm{Gr}(3, H^0(L)).$$

This corresponds to the case where the base locus of the map is of length 3. For the other component, consider $\mathcal{M}$ the moduli space of stable vector bundles with $c_2 = 4$ and $c_1 = L$. Let $\mathcal{G} \to \mathcal{M}$ be the relative Grassmanian of hyperplanes (i.e. $\mathcal{G}_E = \mathrm{Gr}(3, H^0(E))$). By duality, we get a morphism $\mathcal{G} \to W_4^2(S)$ given by

$$(E, V) \to V^\vee \in W_4^2(S) \subset \mathrm{Gr}(3, H^0(L)).$$

This corresponds to the case where the base locus of the rational map is of degree 4. The fact that the two components are disjoint is a consequence of the previous Theorem (if they would intersect, we get a map of degree 3, i.e. we cannot have a tangent plane to $S \subset \mathbb{P}^5$ meeting $S$ at another point). There cannot be any other component because if the degree of the base locus is $\leq 2$, then the map is of degree $\geq 6$. If the degree of the base locus is 3 then the base locus must be of the form $\mathrm{Spec}\mathcal{O}_S/\mathcal{I}_P^2$, otherwise the map is of degree $\geq 4$ (so we are in the component birational to $S$). If the base locus is of degree 4 then the kernel is a stable rank 2 vector bundle with $c_2 = 4$, hence we are in the other component (the base locus cannot be of higher degree because there are no stable rank 2 vector bundles with $c_2 \leq 3$ by [Muk87, Corollary 2.5]). The proof is complete. □

## 3. The case of the $(1,6)$ abelian surface.

In this section, we consider $(A, L_A)$ an abelian surface of type $(1,6)$ in $\mathbb{P}^5$. Every such abelian surface admits a description as $(B, L_B)/\langle b\rangle$, where $(B, L_B)$ is a $(1,3)$ abelian surface and $b$ is a $2-$torsion point[2]. We will furthermore assume that $L_B$ has no sub-bundles of the form $\mathcal{O}_B(nF)$ with $n \geq 2$ for some elliptic curve $F \subset B$. This assumption is satisfied for the very general abelian surface. We will use the rank 2 vector bundle $E = \pi_* L_B$ to construct rational maps of degree 3. We begin with a preliminary lemma on the properties of $E$.

**Lemma 3.1.** *The push-forward $E = \pi_* L_B$ is a rank 2 vector bundle with invariants $c_1(E) = L_A$ and $c_2(E) = 3$. Moreover, $E$ is globally generated in codimension 2 and the determinant map*

$$\bigwedge^2 H^0(E) \to H^0(L_A)$$

*is injective.*

*Proof.* We observe that $\pi^* E = L_B \oplus t_b^* L_B$. The computations of the Chern classes follow. Moreover, under the identification $i : H^0(L_B) = H^0(E)$ we get for $s \in H^0(L_B)$

$$\pi^* i(s) = (s, t_b^* s);$$

hence, in particular, $Z(\pi^* i(s)) = Z(s) \cap t_b^* Z(s)$.

First, let us prove that the determinant map $\bigwedge^2 H^0(E) \to H^0(L_A)$ is injective. Since $h^0(E) = 3$ we get that all the elements of $\bigwedge^2 H^0(E)$ are simple tensors. If $i(s_1) \wedge i(s_2) = 0$, we deduce that $i(s_1), i(s_2)$ span a subline bundle $M \subset E$. In particular, this implies that $Z(i(s_1))$ is a divisor moving in a linear system. Now $Z(\pi^* i(s_1)) = Z(s_1) \cap t_b^* Z(s_1)$. We get $\pi^* M \subset L_B$ is a line sub-bundle with a $2-$dimensional space of invariant (or anti-invariant) global sections under translation by $b$. This is only possible if $M^2 = 0$. Hence $M = \mathcal{O}_B(nF)$ for some $n \geq 2$ (recall that $h^0(M) \geq 2$). This gives a contradiction. This in particular implies that the general section of $E$ vanishes along a zero-dimensional subscheme of length $c_2(E) = 3$.

Now we prove that $E$ is globally generated in codimension 2. Suppose by contradiction that this is false. Then, the base locus Bl of the space of sections $\bigwedge^2 H^0(E) \subset H^0(L_A)$ has a one-dimensional component $D_0$. Now consider $\langle i(s_1), i(s_2)\rangle \subset H^0(E)$ such that $i(s_1)$ is vanishing along a zero-dimensional subscheme (so this is the case for the general element). The zero cycle of the sections induces a morphism

$$\mathbb{P}\langle i(s_1), i(s_2)\rangle = \mathbb{P}^1 \to \mathrm{Sym}^3 Z(i(s_1) \wedge i(s_2)) \subset \mathrm{Sym}^3(A).$$

Now $Z(i(s_1) \wedge i(s_2)) = D_0 \cup D_1$ and

$$\mathrm{Sym}^3 Z(i(s_1) \wedge i(s_2)) = \mathrm{Sym}^3 D_0 \cup (\mathrm{Sym}^2 D_0 \times D_1) \cup (D_0 \times \mathrm{Sym}^2 D_1) \cup \mathrm{Sym}^3 D_1.$$

Now the zero locus of a general section in $\langle i(s_1), i(s_2)\rangle$ need to be supported in $D_0 \cup D_1$ with at least a point in each component[3] (otherwise the wedge product would be just one of the two components). We get that the previous morphism factors as $\mathbb{P}^1 \to (\mathrm{Sym}^2 D_0 \times D_1) \cup (D_0 \times \mathrm{Sym}^2 D_1) \subset \mathrm{Sym}^3(D_0 \cup D_1)$. This implies that either $D_0$ or $D_1$ is rational, contradiction. □

**Corollary 3.2.** *The pair $(E, H^0(E))$ is a good pair. In particular, the degree of irrationality of a very general $(1,6)$ abelian surface is 3.*

[2] In fact, there are 3 such descriptions corresponding to the 3 non-trivial $2-$torsion points in the kernel of the dual polarization of $\mathrm{Pic}^0(A)$.

[3] More precisely, for any point of $D_0 \cup D_1$ we can find a section vanishing at that point.

*Proof.* Since $(E, H^0(E))$ is a good pair, it induces, via the correspondence of Proposition 1.5, a rational map of degree $\leq 3$. Let us prove that the map is of positive degree. In fact, if the image was a (non-degenerate) curve $C$ in $\mathbb{P}^2$, then, we would get numerically $L_A = M^{\otimes n}$ where $M = \pi^*\mathcal{O}_C(p)$ (since the base locus of the map is zero dimensional). This contradicts the fact that the polarization is primitive. Finally, the degree cannot be 2 by [AP92].

□

**Remark 3.3.** In [GM25] the geometry of the degree 3 rational map $A \dashrightarrow \mathbb{P}^2$ is studied in detail. The invariants of the Galois closure (which is a Lagrangian surface in $A \times A$) are computed. A similar program was carried out for $(1, 2)$ abelian surface and the degree 3 map constructed by Tokunaga and Yoshioka ([TY95]) in the paper [BPS10].

With some extra effort it is possible to prove a theorem similar to Theorem B for the very general $(1, 6)$ abelian surface $A \subset \mathbb{P}^5 = \mathbb{P}H^0(L_A)$ . Using some of the geometry of the stable rank 2 vector bundles with $c_2 = 3$ and $c_1 = L_A$ (which turn out to be always pushforward of a line bundle on a $(1, 3)$ abelian surface by work of Mukai, [Muk78]) and symmetry properties of the degree 3 morphism $A \dashrightarrow \mathbb{P}^2$ just constructed (see also [GM25, Section 2]) one can show that given $A \subset \mathbb{P}^5 = \mathbb{P}H^0(L_A)$ there are exactly four distinct 9−secant planes and describe their intersection with $A$.

## 4. The upper bound

We use the construction of the previous section to prove the existence of rational maps of low degree from $K3$ surfaces of genus $g$. Our construction improves asymptotically by a factor between 3 and 6 the best bound previously known, due to Stapleton (depending on the genus), see [Sta17]. We report a tabular of the upper bound we get in low genus

| $g$ | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 42 | 62 | $2+2n(n+1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{irr}(S)$ | 3 | 4 | 4 | 5 | 4 | 5 | 5 | 6 | 6 | 7 | 5 | 6 | 7 | $2+n$ |

First we deal with the case $g = 2 + 2n(n + 1)$, where our construction is optimal. The second Chern class of the minimal rank 2 vector bundle $E$ is $c_2(E) = 2 + n(n + 1)$. We will prove that a $K3$ surface $S$ with these invariants has degree of irrationality $\mathrm{irr}(S) \leq 2 + \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = 2 + n \sim \frac{1}{\sqrt{2}}\sqrt{g}$ (for this series of genera the bound is asymptotically $\frac{1}{6}$ of Stapleton's bound) .

The theorem is an easy consequence of the construction in the previous section.

**Theorem 4.1.** *Let $(S, L)$ be a $K3$ surface of genus $g = 2 + 2n(n + 1)$, then* $\mathrm{irr}(S) \leq 2 + n$.

*Proof.* Consider the minimal rank 2 vector bundle $E$ with invariants

$$c_1(E) = L, \quad c_2(E) = 2 + n(n + 1), \quad h^0(E) = 3 + n(n + 1).$$

Let $P \in S$ be any point then there exists a vector space $V_P^\vee \subset H^0(E \otimes m_P^n)$ of dimension 3. Any section of $V_P^\vee$ vanishes at $P$ with order $n^2$. We deduce that $\mathrm{bl}(V_P^\vee) = \deg(\cap_{s\in V_P^\vee} Z_{cycle}(s)) \geq n^2$. By Proposition 1.5

$$\deg(\varphi_{V_P}) = c_2(V_P)^\vee \leq c_2(E) - n^2 = 2 + n,$$

where $\varphi_{V_P}$ is the map $S \dashrightarrow \mathbb{P}^2$ induced by $V_P \subset H^0(L)$. □

The general case is an easy consequence.

**Corollary 4.2.** *Let $(S, L)$ be a $K3$ surface of genus $g = 2 + 2n(n + 1) + k < 2 + 2(n + 1)(n + 2)$ (i.e. $k < 4n + 4$), then*

$$\mathrm{irr}(S) \leq 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

*Proof.* A minimal rank 2 vector bundle $E$ has invariants

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil, \quad h^0(E) = 3 + n(n+1) + \left\lceil \frac{k}{2} \right\rceil.$$

Let

$$V^\vee \in \mathrm{Gr}(3, H^0(E \otimes m_P^n \otimes m_{Q_1} \otimes \cdots \otimes m_{Q_{\lfloor \frac{k}{4} \rfloor}}),$$

then $\varphi_V$ is of degree

$$\deg(\varphi_V) = c_2(V^\vee) \leq c_2(E) - n^2 - \left\lfloor \frac{k}{4} \right\rfloor = 2 + n + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor.$$

□

We end this section with a remark.

**Remark 4.3.** The scheme-theoretic zero locus of sections $V^\vee \in H^0(E \otimes m_p^n)$ is given by two equations $f, g \in m_P^n$ locally at $P$. Hence, the scheme structure at $P$ of the zero locus of a general section of $H^0(E \otimes m_P^n)$, may, in principle, change scheme-theoretically (the locus of complete intersection subschemes in $S^{[n^2]}$ containing $m_p^n$ is $\mathrm{Gr}(2, n+1)$).

## 5. An application to higher-dimensional varieties.

In this section, we apply Proposition 1.5 to higher-dimensional varieties. First, we give some sufficient conditions for a good pair $(E, V^\vee)$ to induce a rational map of finite degree.

**Lemma 5.1.** *Let $(E, V^\vee)$ be a good pair on an $n-$dimensional variety $X$. Suppose that for any $k \geq 0$ the locus $\{s \in V^\vee$ such that $Z(s)$ has a component of dimension $\geq k\}$ is of dimension $< n - k$. Then, $c_n(V^\vee) > 0$ (i.e. $\varphi_V$ is of positive degree). In particular, if $E$ is a vector bundle and all sections of $V^\vee$ vanish along a zero-dimensional subscheme, then $c_n(V^\vee) > 0$.*

*Proof.* The locus $\cup_{s \in V^\vee}(Z(s) \cap (X - \mathrm{Bl}))$ covers $(X - \mathrm{Bl})$ (by Proposition 1.5 the fibers of $\varphi_V : X - \mathrm{Bl} \to \mathbb{P}V^\vee$ are precisely the zero locus of sections in $V^\vee$). The lemma follows by a dimension count. □

**Corollary 5.2.** *The following estimates hold:*

- *if $(X, H)$ is a general Gushel–Mukai threefold (i.e. a Fano threefold of genus 6 and index 1) then $\mathrm{irr}(X) \leq 3$;*
- *if $(S, L)$ is a general $K3$ surface of genus 6 then $\mathrm{irr}(S^{[2]}) \leq 6$;*
- *if $(A, L)$ is a general abelian threefold of type $(1, 3, 12)$ or $(1, 6, 6)$ then $\mathrm{irr}_L(A) \leq 8$;*
- *if $(S, L)$ is a general $K3$ surface of genus 10 then $\mathrm{irr}(S^{[3]}) \leq 20$.*

*Proof.*
- For the Gushel–Mukai threefold one may just fix a pencil in $|H|$ giving a map $X \dashrightarrow \mathbb{P}^1$. We can suppose that a general element $S \in |H|$ is a $K3$ surface of Picard rank 1 (and of genus 6). On the Gushel–Mukai threefold there is a tautological rank 2 vector bundle $E$, such that $E_{|S}$ is the stable rank 2 vector bundle with $c_2(E) = 4$. Now fix $P$ is the base locus of the pencil $\mathbb{P}^1 \subset |H|$. The map $\psi : X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}H^0(E \otimes m_P)$ is easily checked to be of degree 3. Indeed $\psi^{-1}(\{S\} \times \mathbb{P}H^0(E \otimes \mathcal{I}_P)) = S$ and $\psi_{|_S}$ is easily checked to be one of the degree 3 maps described in Theorem 2.4.
- For $S^{[2]}$ consider the minimal vector bundle $E$ of rank 2 on $S$ (i.e. $c_1(E) = L, c_2(E) = 4, h^0(E) = 5$). This vector bundle induces a tautological rank 4 vector bundle $\mathcal{E}$ on $S^{[2]}$ (locally defined as $\mathcal{E} \otimes k(\xi) = E \otimes \mathcal{O}_\xi$ for any $\xi \in S^{[2]}$). Moreover, we have $c_4(\mathcal{E}) = 6$ and an identification $i : H^0(E) = H^0(\mathcal{E})$. It is also easy to understand the zero locus of sections of $\mathcal{E}$: if $Z(s) = \zeta \in S^{[4]}$,

then $Z(i(s)) \subset \{\xi \in S^{[2]}$ such that $\xi \subset \zeta\}$ which is finite if $\zeta$ is curvilinear. Note that for general $s$ we have that $Z(s)$ is curvilinear. Hence, for the general $i(s)$, we find that $Z(i(s))$ is zero dimensional. This implies that $H^0(\mathcal{E})$ generically generates $\mathcal{E}$ (otherwise, the general section would be the section of a rank 3 subsheaf and would vanish along a subscheme of dimension $\geq 1$). In particular, the pair $((\mathrm{Im}(ev)^\vee)^\vee, H^0(\mathcal{E}))$ is a good pair.

Moreover, if $\zeta$ is not curvilinear, then $\mathcal{I}_\zeta = (x^2, y^2)$ for some local coordinates at $\mathrm{Supp}(\zeta)$ and $\{\xi \in S^{[2]}$ such that $\xi \subset \zeta\}$ is $1-$dimensional (consists of the length 2 subschemes supported at $P$). The vector space $H^0(E \otimes m_P^2)$ is at most $1-$dimensional (otherwise, $Z(\bigwedge^2 H^0(E \otimes m_P^2))$ gives a rational curve with a point of multiplicity 4 in $|L|$, contradiction by [Che02]). for any $P \in S$. Hence, by Lemma 5.1 we get $c_4(H^0(\mathcal{E})) > 0$. Moreover, since the general section of $(H^0(\mathcal{E})) \subset (\mathrm{Im}(ev)^\vee))^\vee$ vanishes at 6 points outside of the base locus we get $c_4(H^0(\mathcal{E})) \leq 6$. Hence, by Proposition 1.5 the pair $((\mathrm{Im}(ev)^\vee)^\vee, H^0(\mathcal{E}))$ induce a rational map $\varphi$ of degree $2 \leq \deg(\varphi_V) \leq 6$.

- A very general abelian threefold $(A, L_A)$ of type $(1,3,12)$ (or $(1,6,6)$) admits a description ad $(B, L_B)/\langle b \rangle$, where $(B, L_B)$ is a very general $(1,1,4)$ (or $(1,2,2)$) abelian threefold and $b$ is a $3-$torsion point. The vector bundle $E = \pi_* L_B$ has Chern classes $c_1(E) = L_A$ and $c_3(E) = 8$ (since there is an isomorphism $\pi^* E = L_B \oplus t_b^* L_B \oplus t_{2b}^* L_B$). In particular, if $\mathrm{NS}(A) = \mathbb{Z} \cdot L_A$ (as it is the case for the very general abelian threefold), we have that $(E, H^0(E))$ is a good pair in view of Corollary 1.6. Moreover, one can check that for the very general abelian threefold any section $s \in H^0(E)$ vanishes along a zero dimensional subscheme (this can be seen for instance degenerating the abelian threefold $B$ to $E \times S$ for $E$ an elliptic curve and $S$ a very general $(1,4)$ abelian surface. In this case the sections of $E$ vanish either along a zero dimensional scheme or along a disjoint union of smooth elliptic curves hence their zero locus cannot be the flat limit of the zero locus of sections vanishing along a curve on the very general (simple) abelian threefold). By Lemma 5.1 and Proposition 1.5 we get that the pair $(E, H^0(E))$ is a good pair inducing a map of positive degree $\leq 8$.
- The case of $S^{[3]}$ is similar to that of $S^{[2]}$ and it is left to the reader.

□

**Remark 5.3.** The case of the abelian varieties and of the Hilbert schemes of surfaces can easily be generalized to higher dimensions. Note that for abelian threefolds we are using as kernel bundle a semi-homogenous vector bundle, i.e. the pushforward under an isogeny of a line bundle (as in the case of the $(1,6)$ abelian surface; see Remark 3.3). For more details on semi-homogenous vector bundles on abelian varieties, the interested reader may consult the work of Mukai, [Muk78]. This implies, in particular, some equivariance and symmetry properties for the map we construct.

## 6. Towards a Brill-Noether theory for surfaces.

We fix a surface $S$ such that $\mathrm{NS}(S) = \mathbb{Z} \cdot L$ and $h^0(L) \geq 3$. The goal of this section is to show that the locus of special projections

$$W_d^r(S, L) = \{V \in \mathrm{Gr}(r+1, H^0(L)) \quad | \deg(\varphi_V) \leq d\}$$

is a scheme. Notice that the definition of a general member of $W_d^r$ is different from the curve case, in the sense that in the above definition $d$ may drop at special points. We should remark here that Mendes-Lopes, Pardini and Pirola defined and studied Brill-Noether loci for irregular varieties in [MLPP14] as cohomology jump loci in $\mathrm{Pic}^0(S)$. Our definition is completely different.

6.1. **The correspondence.** Let us recall some notation and generalize some notation. We say that $\varphi$ is associated with $(V, L) = (V^\vee, E)$ where

$$0 \longrightarrow E^\vee \longrightarrow V \otimes \mathcal{O}_S \longrightarrow L.$$

Let us start by generalizing some definitions (recall that on a surface reflexive implies locally free).

**Definition 6.1.** A couple $(E, V^\vee)$ with $E$ a rank $r$ vector bundle and $V^\vee \in \mathrm{Gr}(r+1, H^0(E))/\mathrm{Aut}(E)$ is a *good $r$-pair* if $V$ generates $E$ in codimension 2 and $H^0(E^\vee) = 0$.

We have the following analogue of Corollary 1.6.

**Lemma 6.2.** *Let $S$ be a smooth surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$. Then, there is a* one-to-one *correspondence*

$$\{V \in \mathrm{Gr}(r+1, H^0(L))\} \leftrightarrow \{(E, V^\vee), E \text{ stable vector bundle of rank } r, c_1 = L, V^\vee \in \mathrm{Gr}(r+1, H^0(E))\}.$$

*Proof.* Same proof as Corollary 1.6. □

Given a good $r-$pair $(E, V^\vee)$ and $W \in \mathrm{Gr}(r-1, V^\vee)$. One can consider the evaluation morphism $\mathrm{ev}_W : W \otimes \mathcal{O}_S \to E$. The morphism $ev_W$ is injective, otherwise, $\mathrm{Im}(ev_W)$ would have $c_1 \geq L$ (numerically) and would destabilize $E$ (here we are using $\mathrm{NS}(S) = \mathbb{Z} \cdot L$). Therefore, we have an exact sequence

$$0 \longrightarrow W \otimes \mathcal{O}_S \longrightarrow E \longrightarrow L \otimes \mathcal{I}_\xi \longrightarrow 0,$$

for some $\xi \in S^{[c_2(E)]}$. We will now generalize and introduce some more notation.

**Definition 6.3.** Let $(E, V^\vee)$ be a good $r-$pair with $c_1(E) = L$, we define:

- $\mathrm{Bl}(V^\vee) \in \mathrm{Sym}^{L^2 - c_2(E)}$ the locus where $V^\vee$ does not generate $E$;
- for any $W \in \mathrm{Gr}(r-1, V^\vee)$
$$\begin{aligned} Z(W) &:= \mathrm{Spec}\, L/\mathrm{coker}(\mathrm{ev}_W) \in S^{[c_2(E)]}, \\ Z_{cycle}(W) &:= \mathrm{Supp}(Z(W)) \in \mathrm{Sym}^{c_2(E)} S; \end{aligned}$$
- $c_2(V^\vee)$ as the degree of the zero locus of $Z(W)$ supported in $S - \mathrm{Bl}$ for general $W \in \mathrm{Gr}(r-1, V^\vee)$;
- $bl(V^\vee) = \bigcap_{W \in \mathrm{Gr}(r-1, V^\vee)} Z_{cycle}(W)$.

**Remark 6.4.** Note that since $\mathrm{Bl}(V^\vee)$ is zero dimensional we get $bl(V^\vee) = \deg(Z_{cycle}(W) \cap \mathrm{Bl}(V^\vee))$ for general $W$. We also remark that $Z(W)$ is well defined for any $W$ since $ev_W : W \otimes \mathcal{O}_S \to E$ is injective for any $W \in \mathrm{Gr}(r-1, V^\vee)$.

The above definitions globalize to suitable relative Grassmannian over the moduli space of stable bundles. More precisely, let:

- $\mathcal{M}_{r,L,c}$ be the moduli of stable rank $r$ vector bundles of rank $r$, with $c_1 = L$ and $c_2 = c$,
- $\mathcal{G}_{r,l,c} \to \mathcal{M}_{r,c,l}$ be the morphism whose fiber[4] over $E \in \mathcal{M}_{r,L,c}$ is $\mathrm{Gr}(r+1, H^0(E))$,
- $\mathcal{W}_{r,L,c} \to \mathcal{G}_{r,L,c}$ be the morphism whose fiber over the pair $(E, V^\vee)$ is $\mathrm{Gr}(r-1, V^\vee)$,
- $\mathcal{G}_r = \bigcup_c \mathcal{G}_{r,L,c}$ and $\mathcal{W}_r = \bigcup_c \mathcal{W}_{r,L,c}$.

Note that, in view of Lemma 6.2, the space $\mathcal{G}_r$ parametrizes good pairs and comes equipped with a natural morphism (a bijection) $\mathcal{G}_r \to \mathrm{Gr}(r+1, H^0(L))$. The above Definition 6.3 globalizes

$$Z : \mathcal{W}_{r,L,c} \to S^{[c]}, \quad Z_{cycle} : \mathcal{W}_{r,L,c} \to \mathrm{Sym}^c(S), \quad \mathrm{Bl} : \mathcal{G}_{r,L,c} \to \mathrm{Sym}^{L^2 - c}(S).$$

The construction is based on the following observation. This is a generalization of Proposition 1.5 in the case of surfaces.

[4]the morphism can be defined locally analitically on $\mathcal{M}$, since there always exists a universal bundle locally analitically on $\mathcal{M}$. For more details, in the $K3$ case, see [Muk87]. It can be globalized even if there is no global universal bundle.

**Corollary 6.5.** *Let $\varphi_V : S \dashrightarrow \mathbb{P}^r = \mathbb{P}V^\vee$ be a rational map with kernel bundle $E$ and base locus $\mathrm{Bl}(V^\vee)$, then for any $W \in Gr(r-1, V^\vee)$ there is an identification $\varphi_V^{-1}[W] \cap (S - \mathrm{Bl}) = Z(W) \cap (S - \mathrm{Bl})$. In particular*

$$\deg(\varphi_V) = c_2(V^\vee) = c_2(E) - bl(V^\vee).$$

*Proof.* We abuse notation and denote with $\varphi_V$ the induced morphism $\varphi_V : (S - \mathrm{Bl}) \to \mathbb{P}V^\vee$. The sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}V^\vee} \longrightarrow T\mathbb{P}V^\vee(-1) \longrightarrow 0$$

pulls back to

$$0 \longrightarrow L^\vee \longrightarrow V^\vee \otimes \mathcal{O}_X \longrightarrow E \longrightarrow 0,$$

on $S - \mathrm{Bl}$. In particular $E = \varphi^* T\mathbb{P}^\vee(-1)$ and pullback induces an identification $\varphi_V^* : H^0(T\mathbb{P}V^\vee(-1)) = V^\vee \subset H^0(E)$ and more generally inclusions of $\mathrm{Gr}(r-1, H^0(T\mathbb{P}V^\vee(-1))) \subset \mathrm{Gr}(r-1, H^0(E))$. The general element $W \otimes \mathcal{O}_X \to T\mathbb{P}^n(-1)$ vanishes at the linear space $[W] \subset \mathbb{P}V^\vee$ (and at no other point, since $c_n(T\mathbb{P}V^\vee(-1)) = \mathcal{O}(1)^{r-n}$). Hence, for $W \in \mathrm{Gr}(r-1, V^\vee)$ the fiber above $[W]$ of the map $\varphi_V : S - \mathrm{Bl} : X \to \mathbb{P}V^\vee$ is $Z_{cycle}(W)$. The facts concerning the degree follow since $c_2(V^\vee)$ is precisely the degree of $Z_{cycle}(W)$ outside of the base locus Bl (see also Remark 6.4) and the degree of the map is precisely the degree of the pullback of a codimension 2 linear subspace $W \in \mathrm{Gr}(r-1, V^\vee)$.

□

We are now ready to show that the loci $W^r_d(S, L)$ are quasi-projective algebraic varieties.

**Theorem 6.6.** *Let $(S, L)$ be a polarized surface with $\mathrm{NS}(S) = \mathbb{Z} \cdot L$. Then, the locus*

$$W^r_d(S, L) = \{V \in \mathrm{Gr}(r+1, H^0(L))\} \quad | \deg(\varphi_V) \le d\},$$

*is a quasi-projective algebraic variety.*

*Proof.* In view of Lemma 6.2 we have a bijection $\mathcal{G}_r \to \mathrm{Gr}(r+1, H^0(L))$. We will construct these loci in $\mathcal{G}_r = \cup_c \mathcal{G}_{r,L,c}$ (this is a filtration on $\mathrm{Gr}(r+1, H^0(L))$ according to the second Chern class of the kernel bundle). Recall the relative grassmanian $\pi : \mathcal{W}_{r,L,c} \to \mathcal{G}_{r,L,c}$. To any point $(E, V^\vee, W) \in \mathcal{W}_{r,L,c}$ we can naturally associate two cycles: $\mathrm{Bl}(V^\vee)$ and $Z_{cycle}(W)$. In view of Corollary 6.5 the point $(E, V^\vee)$ gives a rational map of degree $\le d$ if and only if for any $W \in \mathrm{Gr}(r-1, V^\vee)$ there are $c - d$ points of $Z_{cycle}(W)$ lying in $\mathrm{Bl}(V^\vee)$. Therefore, if we define[5]

$$\Delta_{c,d} = \{(P_1 + \cdots + P_c, Q_1 + \ldots Q_{L^2-c}) \in \mathrm{Sym}^c(S) \times \mathrm{Sym}^{L^2-c}(S) \text{ such that } P_1 = Q_{i_1}, \ldots, P_{c-d} = Q_{i_{c-d}}\},$$

we get that the locus

$$W^r_d(S, L)_c := \{(E, V^\vee) \in \mathcal{G}_{r,L,c} \text{ such that } \pi^{-1}(E, V^\vee) \subset (Z_{cycle}, \mathrm{Bl} \circ \pi)^{-1}(\Delta_{c,d})\}$$

parametrizes good $r-$pair $(E, V^\vee)$ such that $c_2(E) = c$ and $\deg(\varphi_V) \le c - (c - d) = d$. The loci just defined are algebraic varieties (by general facts on flag varieties). Therefore, the finite union

$$W^r_d(S, L) = \bigcup_c W^r_d(S, L)_c$$

is an algebraic variety. □

**Remark 6.7.** These loci are often non-projective in $\mathcal{G}$ (as $\mathcal{G}$ is itself non-projective). In [MR25] several computations of these loci for $r = 2$ are carried out.

[5] The following definition reads: the locus of pairs of cycles in $\mathrm{Sym}^c(S) \times \mathrm{Sym}^{L^2-c}(S)$ such that there are $c - d$ points of the first cycle belonging to the support of the second cycle.

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