

HYPERPLANE ARRANGEMENTS AND THE GAUSS MAP OF A PENCIL

T. FASSARELLA AND N. MEDEIROS

ABSTRACT. We show that the coefficients of the characteristic polynomial of a central affine complex hyperplane arrangement \mathcal{A} , coincide with the multidegrees of the Gauss map of a pencil of hypersurfaces naturally associated to \mathcal{A} . As a consequence, we obtain a proof of the Heron-Rota-Welsh conjecture for matroids representable over a field of characteristic zero.

1. INTRODUCTION

This short paper is devoted to the study of the log-concavity and unimodality of the sequence of coefficients of the characteristic polynomial $\chi_{\mathcal{A}}(t)$ associated with a central affine arrangement \mathcal{A} over the field of complex numbers \mathbb{C} . A sequence of real numbers a_0, \dots, a_n is *log-concave*, if for all $i \in \{1, \dots, n-1\}$, the following inequality holds

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

It is called *unimodal* if there exists an index $i \in \{0, \dots, n\}$ such that

$$a_0 \leq \dots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \dots \geq a_n.$$

The Heron-Rota-Welsh conjecture [Rot70, Her72, Wel76], for matroids representable over a field of characteristic zero, can be reduced to showing that the sequence formed by the coefficients of $\chi_{\mathcal{A}}(t)$ is log-concave. This conjecture generalizes those of Read [Rea68] and Hoggar [Hog74], extending their scope to matroids: Read's conjecture posits that the absolute values of the coefficients of a chromatic polynomial of a graph form a unimodal sequence, while Hoggar's conjecture asserts that this sequence is log-concave.

J. Huh [Huh12, Corollary 27] proved the Read-Hoggar conjecture and the Heron-Rota-Welsh conjecture for matroids representable over a field of characteristic zero. The proof is quite sophisticated: it involves works of singularity theory, algebraic geometry, and convex geometry; it relies on a theorem of Dimca and Papadima [DP03] and three technical lemmas from commutative algebra. In his proof, J. Huh connected the coefficients of the characteristic polynomial with the multidegrees of the polar map associated with a hyperplane arrangement that naturally emerges from the matroid. The present work is motivated by this connection: we show, in our Theorem 1.1 below, that the coefficients of the characteristic polynomial are precisely the multidegrees of the Gauss map of a certain pencil of hypersurfaces associated to the arrangement. The log-concavity follows immediately from this. Our approach is based on the results of [FP07], which hinges on logarithmic foliations.

2020 *Mathematics Subject Classification.* Primary 14E05, 05B35; Secondary 14B05, 37F75.

Key words and phrases. Hyperplane arrangements, characteristic polynomial, Gauss map, multidegrees.

The authors are supported by CNPq Universal 10/2023. The first author is supported by CAPES-COFECUB programme project number: Ma 1017/24.

Before going into specifics, we remark that K. Adiprasito, J. Huh, and E. Katz [AHK18] proved the Heron-Rota-Welsh conjecture for matroids in full generality. For an excellent overview of the key developments leading up to their final result, we refer to the introduction of [AHK18].

Let \mathcal{A} be a central hyperplane arrangement in \mathbb{C}^{n+1} , consisting of a collection of $k \geq 1$ subspaces of dimension n . Let $\chi_{\mathcal{A}}(t)$ be its characteristic polynomial, as defined in Section 2. The central arrangement \mathcal{A} defines a projective arrangement, whose union of hyperplanes may be written as $V(f) \subset \mathbb{P}^n$, where f is a product of k homogeneous linear forms in the variables x_0, \dots, x_n .

By introducing an additional variable x_{n+1} , we can associate to \mathcal{A} a pencil \mathcal{F} of hypersurfaces in \mathbb{P}^{n+1} , generated by f and x_{n+1}^k . The members of this pencil are the hypersurfaces $V_{a,b}$ defined by the equation $af + bx_{n+1}^k$, where $(a : b) \in \mathbb{P}^1$. Now, consider the Gauss map $\mathcal{G}(\mathcal{F}) : \mathbb{P}^{n+1} \dashrightarrow \check{\mathbb{P}}^{n+1}$ which sends a smooth point p of a member $V_{a,b}$ to its tangent space $T_p V_{a,b}$ in the dual space $\check{\mathbb{P}}^{n+1}$. This leads us to a sequence of integers, called *multidegrees*, associated to this rational map, see Section 3 or [Dol24, Section 7.1.3] for further details.

Theorem 1.1. *With notations as above, let d_0, \dots, d_{n+1} be the sequence of multidegrees of $\mathcal{G}(\mathcal{F})$. Then*

$$\chi_{\mathcal{A}}(t) = d_0 t^{n+1} - d_1 t^n + \dots + (-1)^{n+1} d_{n+1}.$$

In the main text, this is Theorem 4.3.

It is straightforward to check that the sequence d_0, \dots, d_{n+1} of multidegrees has no internal zeros, see Proposition 4.1. Moreover, it follows from the Hodge-Khovanskii-Teissier type inequalities [Laz04, Example 1.6.4] that this sequence is log-concave. See also [Huh12, Theorem 21] for a different proof and a complete characterization of representable homology classes of a product of two projective spaces. Then the next result follows from Theorem 1.1.

Corollary 1.2. ([Huh12]) *Let M be a matroid representable over \mathbb{C} . Then the coefficients of the characteristic polynomial $\chi_M(t)$ form a log-concave sequence of integers with no internal zeros.*

As mentioned earlier, Corollary 1.2 was previously conjectured by Heron, Rota and Welsh. It is worth noting that a matroid which is representable over a field of characteristic zero is also representable over \mathbb{C} , see the proof of [Huh12, Corollary 27] for further details. In the same context, the chromatic polynomial $\chi_G(t)$ of a finite graph G , coincides with the characteristic polynomial of the graph arrangement \mathcal{A} associated to G . Hence, since any sequence of nonnegative numbers that is log-concave and has no internal zeros is unimodal, Theorem 1.1 yields the following result, which was previously conjectured by Read and Hoggar.

Corollary 1.3. ([Huh12]) *The coefficients of the chromatic polynomial $\chi_G(t)$ of any finite graph G form a log-concave sequence with no internal zeros, hence their absolute values form a unimodal sequence.*

2. CHARACTERISTIC POLYNOMIAL OF AN ARRANGEMENT

We refer to [Sta07] and [OT92] for general background on hyperplane arrangements.

Let \mathcal{A} be a central arrangement given by a finite collection of affine hyperplanes in \mathbb{C}^{n+1} . We denote by $L_{\mathcal{A}}$ the set of all nonempty intersections of hyperplanes of \mathcal{A} , with the ambient space \mathbb{C}^{n+1} included as one of its elements. Let us endow $L_{\mathcal{A}}$ with a structure of *poset* (partially ordered set): define $C_1 \leq C_2$ in $L_{\mathcal{A}}$ if $C_2 \subseteq C_1$. Note that $\mathbb{C}^{n+1} \leq C$ for all $C \in L_{\mathcal{A}}$.

Let $\text{Int}(L_{\mathcal{A}})$ be the set of all closed intervals $[C_1, C_2]$ of $L_{\mathcal{A}}$. In order to define the characteristic polynomial of the arrangement, we need to introduce an important tool, the *Möbius function*. It is a function $\mu: \text{Int}(L_{\mathcal{A}}) \rightarrow \mathbb{Z}$, defined by the following conditions:

- $\mu(C, C) = 1$ for all $C \in L_{\mathcal{A}}$.
- $\mu(C_1, C_2) = -\sum_{C_1 \leq C < C_2} \mu(C_1, C)$ for all $C_1 < C_2$ in $L_{\mathcal{A}}$

We write $\mu(C) = \mu(\mathbb{C}^{n+1}, C)$ for all $C \in L_{\mathcal{A}}$. The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of the arrangement is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{C \in L_{\mathcal{A}}} \mu(C) \cdot t^{\dim C}.$$

Given a hyperplane H in \mathcal{A} , we define a triple of arrangements $(\mathcal{A}, \mathcal{A}^*, \mathcal{A}^{**})$, where $\mathcal{A}^* = \mathcal{A} - H$ is obtained from \mathcal{A} by deleting H and \mathcal{A}^{**} is the arrangement in $H \simeq \mathbb{C}^n$ given by the restriction $\mathcal{A}^*|_H$. The characteristic polynomials of such a triple satisfy the fundamental deletion-restriction principle:

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}^*}(t) - \chi_{\mathcal{A}^{**}}(t). \quad (1)$$

See [Sta07, Lemma 2.2].

Closely related to the characteristic polynomial, the *chromatic polynomial* $\chi_G(t)$ of a graph G counts the number of proper colorings using t colors. A *proper coloring* assigns colors to the vertices such that adjacent ones receive different colors. For a simple graph G with vertex set $\{v_0, \dots, v_n\}$, there exists an associated arrangement \mathcal{A}_G in \mathbb{C}^{n+1} that satisfies

$$\chi_G(t) = \chi_{\mathcal{A}_G}(t). \quad (2)$$

This arrangement is defined by the union of hyperplanes $V(x_i - x_j)$, $i < j$, for all adjacent vertices v_i and v_j . It turns out that $\chi_G(t)$ satisfies, similarly, the deletion-contraction principle for graphs

$$\chi_G(t) = \chi_{G^*}(t) - \chi_{G^{**}}(t).$$

One approach to proving identity (2) is by induction on the number of edges of G , then on the number of hyperplanes of \mathcal{A}_G . See [Sta07, Theorem 2.7]. In Section 4, we will apply the same method to prove that $\chi_{\mathcal{A}}(t)$ coincides with the polynomial $\chi_{\mathcal{F}}(t)$, where the absolute value of each coefficient is a multidegree of the Gauss map of a pencil associated to the arrangement.

3. LOGARITHMIC FOLIATIONS ON PROJECTIVE SPACES

A *logarithmic foliation* \mathcal{F} on \mathbb{P}^n is induced by a rational 1-form

$$\omega = \sum_{i=1}^s \lambda_i \frac{df_i}{f_i} \quad (3)$$

where f_1, \dots, f_s are irreducible homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ and $\lambda_1, \dots, \lambda_s$ are nonzero complex numbers satisfying the relation

$$\sum_{i=1}^s \lambda_i \deg(f_i) = 0.$$

This condition ensures that the contraction $i_R \omega$ with the radial vector field $R = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$ vanishes identically. Furthermore, ω is *integrable*, in the sense that

$$d\omega \wedge \omega = 0.$$

The 1-form ω defines a global section of the sheaf $\Omega_{\mathbb{P}^n}^1(\log D)$, whose sections are 1-forms with at most simple poles along the divisor

$$D = V(f_1 \cdots f_s) \subset \mathbb{P}^n$$

called the *polar locus* of ω . In a sufficiently small analytic neighborhood of a nonsingular point of ω , the integrability condition gives rise to a holomorphic fibration whose relative tangent sheaf coincides with the subsheaf of $T\mathbb{P}^n$ determined by the kernel of ω . The analytic continuation of a fiber of this local fibration defines a *leaf* of \mathcal{F} .

At a nonsingular point $p \in \mathbb{P}^n$ of ω , we denote by $T_p \mathcal{F}$ the *tangent space* of the leaf of \mathcal{F} passing through p . The assignment $p \mapsto T_p \mathcal{F}$ determines a rational map

$$\mathcal{G}(\mathcal{F}): \mathbb{P}^n \dashrightarrow \check{\mathbb{P}}^n$$

called the *Gauss map* of the foliation \mathcal{F} . Here, $\check{\mathbb{P}}^n$ denotes the dual space, formed by hyperplanes in \mathbb{P}^n . Using the natural identification $\check{\mathbb{P}}^n \simeq \mathbb{P}^n$, we can see the Gauss map as a rational map $\mathcal{G}(\mathcal{F}): \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Moreover, if we write ω as

$$\omega = \sum_{i=0}^n a_i dx_i$$

for suitable rational functions a_0, \dots, a_n , then $\mathcal{G}(\mathcal{F})$ is given by $(a_0 : \dots : a_n)$.

We associate to \mathcal{F} a sequence of integers, called *multidegrees*, as follows. If $\mathbb{P}^{n-j} \subset \mathbb{P}^n$ is a general linear subspace of codimension $0 \leq j \leq n$, then set $d_j(\mathcal{G}(\mathcal{F}))$, or simply d_j , to be the degree of the closed subset $\overline{\mathcal{G}(\mathcal{F})^{-1}(\mathbb{P}^{n-j})}$. In particular,

$$d_0 = 1 \quad \text{and} \quad d_n = \deg(\mathcal{G}(\mathcal{F}))$$

where $\deg(\mathcal{G}(\mathcal{F}))$ is the topological degree of the rational map $\mathcal{G}(\mathcal{F})$.

In order to apply some results of [FP07], we need to introduce some notation. Let ω be a rational logarithmic 1-form as in (3). Let $\pi: X \rightarrow \mathbb{P}^n$ be a sequence of blowups, along smooth centers of the polar locus D , such that $\pi^*(D)$ has only normal crossings. We can see $\pi^*(\omega)$ as a global section of $\Omega_X^1(\log(\pi^*(D)))$. Given an irreducible component E of $\pi^*(D)$, one can locally express $\pi^*(\omega)$ at a general point of E as

$$\pi^*(\omega) = \lambda(E) \cdot \frac{dh}{h} + \eta$$

where $\lambda(E) \in \mathbb{C}$, h is a local equation for E and η is a holomorphic 1-form. We say that $\lambda(E)$ is the *residue* of $\pi^*(\omega)$ with respect to E . It is known that there are nonnegative integers

m_1, \dots, m_s such that

$$\lambda(E) = \sum_{i=1}^s m_i \lambda_i.$$

See for example [FP07, Lemma 3]. We say that $(\lambda_1, \dots, \lambda_s)$ is *nonresonant*, with respect to π , if $\lambda(E) \neq 0$ for every irreducible component E of $\pi^*(D)$.

Let $\mathcal{F}|_{\mathbb{P}^i}$ be the restriction of the foliation \mathcal{F} to a general linear subspace $\mathbb{P}^i \subset \mathbb{P}^n$ of dimension i . We note that $\mathcal{F}|_{\mathbb{P}^i}$ is still a logarithmic foliation, defined by the 1-form $\omega|_{\mathbb{P}^i}$. In the next result, we describe the topological degree of the Gauss map $\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})$ of this foliation.

Lemma 3.1. *Let the notation be as above. Assume that $(\lambda_1, \dots, \lambda_s)$ is nonresonant. Let $\mathbb{P}^i \subset \mathbb{P}^n$ denote a general linear space of dimension i and let $\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})$ be the Gauss map of the restriction $\mathcal{F}|_{\mathbb{P}^i}$. Then, for $i = 2, \dots, n$, we have*

$$d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) = (-1)^{i-1} \chi_{\text{top}}(\mathbb{P}^{i-1} \setminus D).$$

Proof. Since $d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i}))$ is the topological degree of $\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})$, the result follows from [FP07, Proposition 1] and the logarithmic Gauss-Bonnet theorem (see [Nor78], [Sil96] or [Kaw78, p. 262]). \square

We will apply the above lemma as follows. Let f be a nonconstant homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$, of degree k . By introducing an additional variable x_{n+1} , we associate to f a logarithmic foliation \mathcal{F} on \mathbb{P}^{n+1} given by

$$\omega = \frac{df}{f} - k \frac{dx_{n+1}}{x_{n+1}}. \quad (4)$$

We can see \mathcal{F} as a pencil of hypersurfaces in \mathbb{P}^{n+1} , generated by f and x_{n+1}^k . Its Gauss map $\mathcal{G}(\mathcal{F})$ sends a general point $p \in \mathbb{P}^{n+1}$ to the tangent space of the member of \mathcal{F} passing through p .

Proposition 3.2. *Let $\mathcal{F} \subset \mathbb{P}^{n+1}$ be the logarithmic foliation associated to a nonconstant polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ of degree k , defined by the rational 1-form (4). Let $\mathcal{G}(\mathcal{F})$ be its Gauss map. Then, for $i = 2, \dots, n+1$, we have*

$$d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) = (-1)^{i-1} \chi_{\text{top}}(\mathbb{P}^{i-1} \setminus (V(f) \cup \mathbb{P}^{i-2}))$$

where $\mathbb{P}^{i-2} \subset \mathbb{P}^{i-1} \subset \mathbb{P}^i$ are general linear spaces.

Proof. Let $V_i = V(f) \cap \mathbb{P}^i$ and define $\mathbb{P}^{i-1} = V(x_{n+1}) \cap \mathbb{P}^i$, for any i . Consider a resolution $\pi: X \rightarrow \mathbb{P}^i$ of V_i , that is, a sequence of blowups such that the total transform $\pi^*(V_i)$ has only normal crossings. This process also yields a resolution of $V_i \cup \mathbb{P}^{i-1}$. Now, write $f = f_1^{n_1} \cdots f_s^{n_s}$ as product of irreducible factors. Then \mathcal{F} is defined by

$$\omega = \sum_{i=1}^s n_i \frac{df_i}{f_i} - k \frac{dx_{n+1}}{x_{n+1}}.$$

We claim that the vector of residues $(n_1, \dots, n_s, -k)$ is nonresonant. Indeed, $\pi^*(\mathbb{P}^{i-1})$ has a unique irreducible component, with residue $-k$, because no center of the blowups lies in \mathbb{P}^{i-1} . Besides this, any other irreducible component of $\pi^*(V_i)$ has residue of the form $\sum_{i=1}^s m_i n_i$, for suitable nonnegative integers m_1, \dots, m_s ; see [FP07, Lemma 3].

The conclusion now follows from Lemma 3.1 by taking $D = V(f \cdot x_{n+1})$ and $\mathbb{P}^{i-2} = V(x_{n+1}) \cap \mathbb{P}^{i-1}$. \square

4. THE PENCIL OF AN ARRANGEMENT

We keep the notations of Sections 2 and 3. We associate to a central arrangement \mathcal{A} in \mathbb{C}^{n+1} the pencil \mathcal{F} of hypersurfaces in \mathbb{P}^{n+1} generated by $f(x_0, \dots, x_n)$ and x_{n+1}^k , where $f = h_1 \cdots h_k$, the product of the homogeneous linear forms defining the members of the arrangement. Then \mathcal{F} has $V(af + bx_{n+1}^k)$, for $(a : b) \in \mathbb{P}^1$, as its members. The pencil \mathcal{F} can be seen as a logarithmic foliation defined by the rational 1-form

$$\frac{df}{f} - k \frac{dx_{n+1}}{x_{n+1}}.$$

Let $\mathcal{G}(\mathcal{F}) : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}$ be the Gauss map of \mathcal{F} , given by the linear system

$$\langle x_{n+1} \frac{\partial f}{\partial x_0}, \dots, x_{n+1} \frac{\partial f}{\partial x_n}, -kf \rangle.$$

As in Section 3, we consider a sequence of integers $d_0(\mathcal{G}(\mathcal{F})), \dots, d_{n+1}(\mathcal{G}(\mathcal{F}))$, or simply, d_0, \dots, d_{n+1} , called multidegrees of the rational map $\mathcal{G}(\mathcal{F})$. They satisfy

$$d_0 = 1, \quad d_1 = k \quad \text{and} \quad d_{n+1} = \deg(\mathcal{G}(\mathcal{F}))$$

where $\deg(\mathcal{G}(\mathcal{F}))$ is the topological degree of $\mathcal{G}(\mathcal{F})$. The following result is an easy and useful fact, which holds for any rational map.

Proposition 4.1. *The above sequence d_0, \dots, d_{n+1} has no internal zeros.*

Proof. Since $d_0 = 1$, we assume that $d_j > 0$ for some $j \geq 1$. This implies that the dimension of the image $\text{Im } \mathcal{G}(\mathcal{F})$ of the rational map $\mathcal{G}(\mathcal{F})$ is at least j . Then the intersection between $\text{Im } \mathcal{G}(\mathcal{F})$ and a general \mathbb{P}^{n+1-i} is nonempty for all $i \leq j$. This gives $d_i > 0$ for all $i \leq j$. \square

We consider the following polynomial defined from the sequence of multidegrees

$$\chi_{\mathcal{F}}(t) = d_0 t^{n+1} - d_1 t^n + \cdots + (-1)^{n+1} d_{n+1}.$$

Note that for an arrangement with a single hyperplane we have

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{F}}(t) = t^{n+1} - t^n. \tag{5}$$

Let us setup a bit more of notation. Given $H \in \mathcal{A}$, denote by $\mathbb{P}_H^{n-1} \subset \mathbb{P}^n$ the corresponding hyperplane in \mathbb{P}^n . As we have done in Section 2 for affine arrangements, we consider a triple (A, A^*, A^{**}) of projective arrangements, where $A, A^* \subset \mathbb{P}^n$ are the union of hyperplanes in $\mathcal{A}, \mathcal{A}^*$ respectively, and $A^{**} = A^* \cap \mathbb{P}_H^{n-1}$. Let \mathcal{F}^* and \mathcal{F}^{**} be the corresponding pencils associated to \mathcal{A}^* and \mathcal{A}^{**} , respectively.

Now we arrive at the central result of this note: The deletion-restriction principle holds for the polynomial $\chi_{\mathcal{F}}(t)$.

Proposition 4.2. *With above notations, we have*

$$\chi_{\mathcal{F}}(t) = \chi_{\mathcal{F}^*}(t) - \chi_{\mathcal{F}^{**}}(t). \tag{6}$$

Proof. This is equivalent to show that the corresponding multidegrees satisfy the identity:

$$d_i(\mathcal{G}(\mathcal{F})) = d_i(\mathcal{G}(\mathcal{F}^*)) + d_{i-1}(\mathcal{G}(\mathcal{F}^{**})) \quad (7)$$

for $i = 1, \dots, n+1$. Since d_1 is the number of hyperplanes of the arrangement and $d_0 = 1$, the identity holds for $i = 1$.

Let us start with the case $i = n+1$. Our strategy is to use the inclusion-exclusion principle for the topological Euler characteristic. We have

$$\begin{aligned} A \cup \mathbb{P}^{n-1} &= A^* \cup \mathbb{P}_H^{n-1} \cup \mathbb{P}^{n-1} \\ &= (A^* \cup \mathbb{P}^{n-1}) \cup [\mathbb{P}_H^{n-1} \setminus (A^* \cup \mathbb{P}^{n-1})] \\ &= (A^* \cup \mathbb{P}^{n-1}) \cup [\mathbb{P}_H^{n-1} \setminus (A^{**} \cup \mathbb{P}^{n-2})] \end{aligned} \quad (8)$$

where \mathbb{P}^{n-1} is a general hyperplane and $\mathbb{P}^{n-2} = \mathbb{P}^{n-1} \cap \mathbb{P}_H^{n-1}$. Since this is a disjoint union, we have

$$(-1)^n \chi_{\text{top}}(\mathbb{P}^n \setminus (A \cup \mathbb{P}^{n-1})) = (-1)^n \chi_{\text{top}}(\mathbb{P}^n \setminus (A^* \cup \mathbb{P}^{n-1})) + (-1)^{n-1} \chi_{\text{top}}(\mathbb{P}_H^{n-1} \setminus (A^{**} \cup \mathbb{P}^{n-2}))$$

hence by Proposition 3.2 we obtain

$$d_{n+1}(\mathcal{G}(\mathcal{F})) = d_{n+1}(\mathcal{G}(\mathcal{F}^*)) + d_n(\mathcal{G}(\mathcal{F}^{**}))$$

therefore concluding that (7) holds for $i = n+1$.

Now let us assume that $i \in \{2, \dots, n\}$. The argument in this case goes in a similar fashion. The key result here is [FP07, Theorem 1], which express the degree $d_i(\mathcal{G}(\mathcal{F}))$ in terms of degrees of restrictions to \mathbb{P}^i and \mathbb{P}^{i+1} , namely

$$d_i(\mathcal{G}(\mathcal{F})) = d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) + d_{i+1}(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^{i+1}})). \quad (9)$$

Our task is to evaluate the terms on the right hand side. By Proposition 3.2 we have

$$d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) = (-1)^{i-1} \chi_{\text{top}}(\mathbb{P}^{i-1} \setminus (A_{i-1} \cup \mathbb{P}^{i-2}))$$

where $A_{i-1} = A \cap \mathbb{P}^{i-1}$. Now, let $\mathbb{P}_H^i = \mathbb{P}_H^{n-1} \cap \mathbb{P}^i$, $A_i^* = A^* \cap \mathbb{P}^i$ and $A_i^{**} = A^{**} \cap \mathbb{P}^i$. By restricting to \mathbb{P}^{i-1} , in the same manner as we did in (8), we can write $A_{i-1} \cup \mathbb{P}^{i-2}$ as a disjoint union

$$A_{i-1} \cup \mathbb{P}^{i-2} = (A_{i-1}^* \cup \mathbb{P}^{i-2}) \cup [\mathbb{P}_H^{i-2} \setminus (A_{i-2}^{**} \cup \mathbb{P}^{i-3})]$$

and from that we get

$$\begin{aligned} (-1)^{i-1} \chi_{\text{top}}(\mathbb{P}^{i-1} \setminus (A_{i-1} \cup \mathbb{P}^{i-2})) &= (-1)^{i-1} \chi_{\text{top}}(\mathbb{P}^{i-1} \setminus (A_{i-1}^* \cup \mathbb{P}^{i-2})) \\ &\quad + (-1)^{i-2} \chi_{\text{top}}(\mathbb{P}_H^{i-2} \setminus (A_{i-2}^{**} \cup \mathbb{P}^{i-3})) \end{aligned}$$

where we agree that A_{i-2}^{**} and \mathbb{P}^{i-3} are empty if $i = 2$. Again by Proposition 3.2,

$$d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) = d_i(\mathcal{G}(\mathcal{F}^*|_{\mathbb{P}^i})) + d_{i-1}(\mathcal{G}(\mathcal{F}^{**}|_{\mathbb{P}^{i-1}})).$$

In the same way we obtain the analogous expression for $d_{i+1}(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^{i+1}}))$. By substituting into (9), we have

$$\begin{aligned} d_i(\mathcal{G}(\mathcal{F})) &= [d_i(\mathcal{G}(\mathcal{F}^*|_{\mathbb{P}^i})) + d_{i-1}(\mathcal{G}(\mathcal{F}^{**}|_{\mathbb{P}^{i-1}}))] + [d_{i+1}(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^{i+1}})) + d_i(\mathcal{G}(\mathcal{F}^{**}|_{\mathbb{P}^i}))] \\ &= [d_i(\mathcal{G}(\mathcal{F}^*|_{\mathbb{P}^i})) + d_{i+1}(\mathcal{G}(\mathcal{F}^*|_{\mathbb{P}^{i+1}}))] + [d_{i-1}(\mathcal{G}(\mathcal{F}^{**}|_{\mathbb{P}^{i-1}})) + d_i(\mathcal{G}(\mathcal{F}^{**}|_{\mathbb{P}^i}))] \\ &= d_i(\mathcal{G}(\mathcal{F}^*)) + d_{i-1}(\mathcal{G}(\mathcal{F}^{**})) \end{aligned}$$

where, in the last equality, we applied (9) for \mathcal{F}^* and \mathcal{F}^{**} . Therefore (7) holds, and that concludes the proof. \square

The following result is Theorem 1.1 of the introduction.

Theorem 4.3. *Let d_0, \dots, d_{n+1} be the sequence of multidegrees of the Gauss map $\mathcal{G}(\mathcal{F})$ associated to the arrangement \mathcal{A} in \mathbb{C}^{n+1} . Then*

$$\chi_{\mathcal{A}}(t) = d_0 t^{n+1} - d_1 t^n + \dots + (-1)^{n+1} d_{n+1}.$$

Proof. The same deletion-restriction principle of Proposition 4.2 holds for $\chi_{\mathcal{A}}(t)$, see (1). Since the initialization process is assured by (5), the proof of the theorem follows by induction on the number of hyperplanes of the arrangement \mathcal{A} . \square

We emphasize the geometric meaning of the coefficients of $\chi_{\mathcal{A}}(t)$ when interpreted as multidegrees of the Gauss map.

Remark 4.4. Let U denote the complement of $V(f \cdot x_{n+1})$ in \mathbb{P}^{n+1} and let φ be the rational function on \mathbb{P}^{n+1} defined by f/x_{n+1}^k . We fix $a_0 = 1$ and for each $i = 1, \dots, n$, let a_i represent the number of *critical points* of the rational function $\varphi|_{\mathbb{P}^i}$ in U . These critical points correspond to the singularities of the logarithmic 1-form $d \log(\varphi|_{\mathbb{P}^i})$, where $\mathbb{P}^i \subset \mathbb{P}^{n+1}$ is a general subspace of dimension i . The topological degree d_{n+1} of the Gauss map $\mathcal{G}(\mathcal{F})$ equals the total number of tangencies between a general $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and the members of \mathcal{F} . This gives

$$d_{n+1} = a_n. \quad (10)$$

Similarly, for the topological degree $d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i}))$ of the Gauss map of the restriction $\mathcal{F}|_{\mathbb{P}^i}$ to a general \mathbb{P}^i , we have $d_i(\mathcal{G}(\mathcal{F}|_{\mathbb{P}^i})) = a_{i-1}$. In particular, the identity (9) implies

$$d_i = a_{i-1} + a_i \quad (11)$$

for all $i = 1, \dots, n$. Consequently,

$$\chi_{\mathcal{A}}(t) = (t-1)(a_0 t^n - a_1 t^{n-1} + \dots + (-1)^n a_n). \quad (12)$$

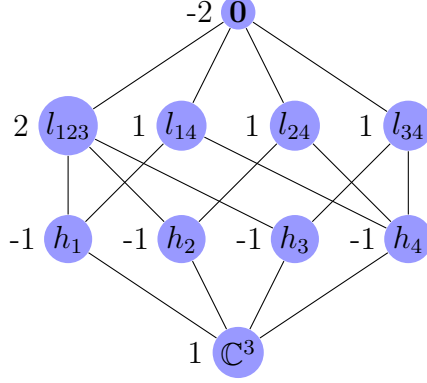
It turns out that a_i is the i -th multidegree of the gradient map $\text{grad} f: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, see [FP07, Corollary 2]. Then (12) recovers one of the two identities of [Huh12, Corollary 25]. \lrcorner

Example 4.5. Here is a basic example, taken from [Sta07, Lecture 1]. Figure 1 shows the values of the Möbius function for the arrangement in \mathbb{C}^3 defined by $V(xy(x+y)z)$, where $h_1 = x$, $h_2 = y$, $h_3 = x+y$, $h_4 = z$ and the l_{ij} represent their pairwise intersections. This arrangement has $\chi_{\mathcal{A}}(t) = t^3 - 4t^2 + 5t - 2$ as characteristic polynomial.

We let w denote the extra variable, and let \mathcal{F} be pencil in \mathbb{P}^3 generated by $f = xy(x+y)z$ and w^4 . The Gauss map $\mathcal{G}(\mathcal{F}): \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ has polynomials of degree 4 as its coordinates, so we have $d_1 = 4$. Since we always have $d_0 = 1$, the only unknowns are d_2 and d_3 . According to Remark 4.4, d_3 corresponds to the number of critical points of the rational function $(f/w^4)|_{\mathbb{P}^2}$ in U . A straightforward computation shows that $d_3 = a_2 = 2$. Therefore $d_2 = a_1 + a_2 = 3 + 2 = 5$. \lrcorner

As mentioned in the Introduction, it is known that the sequence d_0, \dots, d_{n+1} is *log-concave*, meaning that for all $i \in \{1, \dots, n\}$, the following inequality holds

$$d_i^2 \geq d_{i-1} d_{i+1}$$

FIGURE 1. The arrangement $V(xy(x+y)z)$.

See [Laz04, Example 1.6.4]. From this and Proposition 4.1 we get the next result.

Corollary 4.6. *Given a central affine arrangement \mathcal{A} over \mathbb{C} , the coefficients of the characteristic polynomial $\chi_{\mathcal{A}}(t)$ form a log-concave sequence of integers with no internal zeros.*

Given a matroid M which is representable over \mathbb{C} , let \mathcal{A} be the central affine arrangement representing M (see for example [Sta07, Lecture 3]). Since $\chi_M(t) = \chi_{\mathcal{A}}(t)$, then Corollary 1.2 of the introduction follows from Corollary 4.6. In the same spirit, we get Corollary 1.3 via identity (2).

REFERENCES

- [AHK18] K. Adiprasito, J. Huh and E. Katz, *Hodge theory for combinatorial geometries*. Annals of Mathematics 188 (2018), 381–452.
- [DP03] A. Dimca and S. Papadima, *Hypersurfaces complements, Milnor fibres and higher homotopy groups of arrangements*. Annals of Mathematics 158 (2003), 473–507.
- [Dol24] I. Dolgachev, *Classical Algebraic Geometry: a modern view I*. Available at <https://dept.math.lsa.umich.edu/~idolga/CAG-1.pdf> (2024).
- [FP07] T. Fassarella and J. Pereira, *On the degree of polar transformations. An approach through logarithmic foliations*. Sel. Math. New Series 13 (2007), 239–252.
- [Her72] A. P. Heron, *Matroid polynomials, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*. Inst. of Math. and its Appl., Southend-on-Sea, (1972), 164–202.
- [Hog74] S. Hoggar, *Chromatic polynomials and logarithmic concavity*. Journal of Combinatorial Theory, Series B 16 (1974), 248–254.
- [Huh12] J. Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*. J. Amer. Math. Soc. (2012) 25:907–927.
- [Kaw78] Y. Kawamata, *On deformations of compactifiable complex manifolds*, Math. Ann. 235 (1978), 247–265.
- [Laz04] R. Lazarsfeld, *Positivity in algebraic geometry*, vol. I and vol. II, Ergebnisseder Mathematik und ihrer Grenzgebiete. 3. Folge, 49. Springer-Verlag, Berlin, (2004).
- [Nor78] Y. Norimatsu, *Kodaira Vanishing Theorem and Chern Classes for ∂ -Manifolds*. Proc. Japan Acad. 54, Ser. A. (1978), 107–108.
- [OT92] P. Orlik and H. Terao, *Arrangements of Hyperplanes*. Grundlehren der Mathematischen Wissenschaften, 300, Springer-Verlag, Berlin, 1992.
- [Rea68] R. C. Read, *An introduction to chromatic polynomials*. J. Combinatorial Theory 4 (1968), 52–71.

- [Rot70] G.-C. Rota, *Combinatorial theory, old and new*, *Actes du Congrès International des Mathématiciens (Nice, 1970)*. Tome 3, Gauthier-Villars, Paris (1971), 229–233.
- [Sil96] R. Silvotti, *On a conjecture of Varchenko*. *Invent. Math.* **126** (1996), 235–248.
- [Sta07] R. P. Stanley, *An Introduction to Hyperplane Arrangements*. Geometric Combinatorics, IAS/Park City Math. Ser., 13, Amer. Math. Soc., Providence, RI, 2007, 389–496.
- [Wel76] D. Welsh, *Matroid Theory*. London Mathematical Society Monographs, 8, Academic Press, London-New York, (1976).

UNIVERSIDADE FEDERAL FLUMINENSE, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA.
RUA ALEXANDRE MOURA 8, SÃO DOMINGOS, 24210-200 NITERÓI RJ, BRAZIL.

E-mail address: `tfassarella@id.uff.br`

E-mail address: `nivaldomedeiros@id.uff.br`