FAILURE OF SINGULAR COMPACTNESS FOR Hom

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ABSTRACT. Assuming Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$, we construct an almost-free abelian group G of singular cardinality, such that for any nontrivial subgroup $G' \subseteq G$ of smaller size, we have $\mathrm{Hom}(G',\mathbb{Z}) \neq 0$, while $\mathrm{Hom}(G,\mathbb{Z}) = 0$. This provides a consistent counterexample to the singular compactness of Hom.

§ 0. INTRODUCTION

Hill [5] proved that if an abelian group G has a singular cardinality with cofinality at most ω_1 and every subgroup of smaller cardinality is free, then G is free. This result serves as a cornerstone for the *Singular Compactness* Theorem by Shelah [8], where he introduced an abstract notion of freeness and get ride of the cofinality restriction. Shelah extended this result by proving that if an abelian group has a singular cardinality with cofinality κ , and every subgroup of smaller cardinality is free, then the group itself must also be free. For more details on singular compactness, see [2, 3], and for its applications, we refer to the book [4].

Compactness (and its counterpart, incompactness) is a central theme in contemporary research. This concept broadly asserts that if every smaller subobject of a given object possesses a particular property denoted by Pr, then the object itself must also exhibit Pr. In this paper, we are interested in the compactness property for the nontrivial duality with respect to the hom-functor $\text{Hom}(-,\mathbb{Z})$ at singular cardinals. Namely, we study the following property:

 $\operatorname{Pr}_{\lambda}$: If G is a group of size λ , and if for any nontrivial subgroup $G' \subseteq G$ of size less than λ , $\operatorname{Hom}(G', \mathbb{Z}) \neq 0$, then $\operatorname{Hom}(G, \mathbb{Z}) \neq 0$.

For a given $\mu \leq \lambda$, recall that $S^{\lambda}_{\mu} = \{\alpha < \lambda \mid \operatorname{cf}(\alpha) = \mu\}$ is a stationary subset of λ . For any stationary set $S \subseteq \lambda$, let \diamondsuit_S denote Jensen's diamond (see Definition 1.3). Now, assuming $\lambda > \aleph_0$ is a regular cardinal and \diamondsuit_S holds for some stationary, non-reflecting set $S \subseteq S^{\lambda}_{\aleph_0}$, one can construct a λ -free abelian group G of size λ such

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that $\operatorname{Hom}(G,\mathbb{Z})=0$ (see [2]). Note that any subgroup G' of G with size less than λ is free, implying that $\operatorname{Hom}(G',\mathbb{Z})\neq 0$. Thus, $\operatorname{Pr}_{\lambda}$ fails for such λ . However, this argument does not extend to singular cardinals.

In this paper, we investigate the consistency of the failure of Pr_{λ} for some singular cardinal λ and show that this can occur in Gödel's constructible universe **L**. The main result of this paper is as follows:

Theorem 0.1. Suppose that:

- (a) $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals with limit λ ,
- (b) $\aleph_0 < \kappa = \operatorname{cf}(\kappa) < \lambda_0$,
- (c) $\lambda_i = \operatorname{cf}(\lambda_i), S_i \subseteq S_{\operatorname{cf}(\mu)}^{\lambda_i}$ is stationary and non-reflecting,
- $(d) \diamondsuit_{S_i} holds,$
- (e) there is no measurable cardinal $\leq \lambda$.

Then there is a λ_0 -free abelian group G of cardinality λ which is counterexample to singular compactness in λ for $\text{Hom}(-,\mathbb{Z}) \neq 0$.

Our work is closely related to the Whitehead property $\operatorname{Ext}_{\mathbb{Z}}(G,\mathbb{Z})=0$, which is arguably more significant but also inherently more complex. In our forthcoming work [1], we investigate singular compactness in the context of Ext. Note that if Gödel's axiom of constructibility $\mathbf{V}=\mathbf{L}$ assumed, this has an easy solution. By Shelah's work [7], for $\lambda > \aleph_0$ and an abelian group G of size λ , the group G is free if and only if $\operatorname{Ext}_{\mathbb{Z}}(G,\mathbb{Z})=0$. Hence, by Shelah's singular compactness theorem for free groups [8], singular compactness holds for the property $\operatorname{Ext}_{\mathbb{Z}}(-,\mathbb{Z})=0$.

In this paper all groups are abelian. For all unexplained definitions from set theoretic algebra see the books by Eklof-Mekler [2] and Göbel-Trlifaj [4]. Also, for unexplained definitions from the group theory see Fuchs' book [3].

§ 1. Preliminaries

In this section, we set out our notation and discuss some facts that will be used throughout the paper and refer to the book of Eklof and Mekler [2] for more information. For abelian groups \mathbb{G} and H, we set $\text{Hom}(G, H) := \text{Hom}_{\mathbb{Z}}(G, H)$.

Notation 1.1. For an index set u, let $\mathbb{Z}_{[u]} := \bigoplus_{\alpha \in u} \mathbb{Z} x_{\alpha}$, so that $\langle x_{\alpha} : \alpha \in u \rangle$ is a basis for $\mathbb{Z}_{[u]}$. For $\eta \in {}^{u}\mathbb{Z}$, let $f_{[\eta]} \in \operatorname{Hom}(\mathbb{Z}_{[u]}, \mathbb{Z})$ be defined as $f_{[\eta]}(\sum_{\alpha \in v} a_{\alpha} x_{\alpha}) = \sum_{\alpha \in v} a_{\alpha} \eta(\alpha)$, for finite $v \subseteq u$.

Definition 1.2. An abelian group G is called \aleph_1 -free if every subgroup of G of cardinality $< \aleph_1$, i.e., every countable subgroup, is free. More generally, an abelian group G is called λ -free if every subgroup of G of cardinality $< \lambda$ is free.

Definition 1.3. Suppose $\lambda > \mu \geq \aleph_0$ are regular and $S \subseteq \lambda$ is stationary.

(1) The Jensen's diamond $\diamondsuit_{\lambda}(S)$ asserts the existence of a sequence $(S_{\alpha} \mid \alpha \in S)$ such that for every $X \subseteq \lambda$ the set $\{\alpha \in S \mid X \cap \alpha = S_{\alpha}\}$ is stationary.

(2) We use the following consequence of $\diamondsuit_{\lambda}(S)$: let $A = \bigcup_{\alpha < \lambda} A_{\alpha}$ and $B = \bigcup_{\alpha < \lambda} B_{\alpha}$ be two λ -filtrations with $|A_{\alpha}|, |B_{\alpha}| < \lambda$. Then there exists a sequence $(g_{\alpha} : A_{\alpha} \to B_{\alpha} \mid \alpha < \lambda)$ such that, for any function $g : A \to B$, the set

$$\{\alpha \in S \mid g \upharpoonright_{A_{\alpha}} = g_{\alpha}\}$$

is stationary in λ .

- (3) S is non-reflecting if for any limit ordinal $\delta < \lambda$ of uncountably cofinality, the set $S \cap \delta$ is non-stationary in δ .
- (4) We set $S^{\lambda}_{\mu} = \{ \alpha < \lambda \mid \operatorname{cf}(\alpha) = \mu \}.$

Definition 1.4. Let \mathcal{K} be the class of objects $\mathbf{k} := (\mu_{\mathbf{k}}, \theta_{\mathbf{k}}, K_{\mathbf{k}})$ consisting of:

- (a) $\mu_{\mathbf{k}}$ is a limit ordinal, and $\theta_{\mathbf{k}} < \mu_{\mathbf{k}}$,
- (b) $K_{\mathbf{k}}$ is an abelian group with the set of elements $\theta_{\mathbf{k}}$, and $0_{K_{\mathbf{k}}} = 0$,
- (c) if $0 \neq K_1 \subseteq K_k$ is a subgroup, then we can find $(H_{\mathbf{k},K_1},\phi_{\mathbf{k},K_1})$ such that:
 - (α) $H_{\mathbf{k},K_1}$ is an abelian group of size $\mu_{\mathbf{k}}$ extending $(K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]}$,
 - (β) $H_{\mathbf{k},K_1}/(K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]}$ is $\mu_{\mathbf{k}}$ -free,
 - $(\gamma) \phi_{\mathbf{k},K_1} \in {}^{\mu_{\mathbf{k}}}(K_1),$
 - (δ) there is no homomorphism $f: H_{\mathbf{k},K_1} \to K_1$ such that $f(x_{\alpha}) = \phi_{\mathbf{k},K_1}(\alpha)$ for $\alpha < \mu_{\mathbf{k}}$:

$$(K_{\mathbf{k}})_{[\{\alpha\}]} \xrightarrow{\subseteq} (K_{\mathbf{k}})_{[\mu_{\mathbf{k}}]} \xrightarrow{\subseteq} H_{\mathbf{k},K_{1}}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad$$

where $\tilde{\phi}(x_{\alpha}) := \phi_{\mathbf{k},K_1}(\alpha)$.

Let us address the existence problem of \mathcal{K} .

Fact 1.5. Let μ be a limit ordinal and $\phi: \mu \to \mathbb{Z}$ be such that $\phi(\xi) \neq 0$, for all $\xi < \mu$. Then there is a free abelian group H equipped with the following three properties:

- (i) $H \supseteq \mathbb{Z}_{[\mu]}$ is of size μ ,
- (ii) $H/\mathbb{Z}_{[\mu]}$ is μ -free,
- (iii) there is no homomorphism $f: H \to \mathbb{Z}$ such that $f(x_{\alpha}) = \phi(\alpha)$ for $\alpha < \mu$.

In particular, identifying the universe of \mathbb{Z} with ω , we have $\mathbf{k} = (\mu, \omega, \mathbb{Z}) \in \mathcal{K}$.

Proof. Let $G_0 = \mathbb{Z}_{[\mu]} \oplus \mathbb{Z}z$, and let G_1 be the \mathbb{Z} -adic completion of G_0 . We define $f: G_0 \to \mathbb{Z}$ by $f(x_\alpha) = \phi(\alpha)$ for $\alpha < \mu$ and f(z) = 1. For any $\vec{a} := \langle a_n : n < \omega \rangle \in \mathbb{Z}$, $\xi < \mu$ and $\ell < \omega$, we set

$$y_{\vec{a},\xi,\ell} = \sum_{n>\ell} \frac{n!}{\ell!} (x_{\xi} - a_n z).$$

It is easily seen that for all ℓ as above,

$$(\dagger): \qquad (\ell+1)y_{\vec{a},\xi,\ell+1} = y_{\vec{a},\xi,\ell} - (x_{\xi} - a_{\ell}z).$$

Let $G_{\vec{a},\xi}$ be the subgroup of G_1 generated by $G_0 \cup \{y_{\vec{a},\xi,n} : n < \omega\}$. Let $\xi < \mu$. We claim that for some \vec{a} , f does not extend to a homomorphism from $G_{\vec{a},\xi}$ into \mathbb{Z} . To this end, we look at

$$\mathcal{A}_{\xi} = \{ \vec{a} \in {}^{\omega}2 : f \text{ has an extension in } \operatorname{Hom}(G_{\vec{a},\xi},\mathbb{Z}), a_0 = a_1 = 0 \}.$$

For $\vec{a} \in \mathcal{A}_{\xi}$ let $h_{\vec{a},\xi} \in \text{Hom}(G_{\vec{a},\xi},\mathbb{Z})$ extends f. For $t \in \mathbb{Z}$ set

$$\mathcal{A}_{t,\xi} = \{ \vec{a} \in \mathcal{A}_{\xi} : h_{\vec{a},\xi}(y_{\vec{a},\xi,0}) = t \}.$$

Clearly, $\mathcal{A}_{\xi} = \bigcup_{t \in \mathbb{Z}} \mathcal{A}_{t,\xi}$. Now, we bring the following claim.

Claim 1.6. For each $t \in \mathbb{Z}$, $|\mathcal{A}_{t,\xi}| \leq 1$.

Proof. Suppose by the way of contradiction that for some $t \in \mathbb{Z}$ we have $|\mathcal{A}_{t,\xi}| > 1$. Let $\vec{a} \neq \vec{b}$ be in $\mathcal{A}_{t,\xi}$ and let n be such that $\vec{a} \upharpoonright n = \vec{b} \upharpoonright n$ and $a_n \neq b_n$. Note that $n \geq 2$. By induction on $\ell \leq n$ we have

$$h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell}).$$

Indeed, the equality holds for $\ell = 0$ by the choice of $\vec{a}, \vec{b} \in \mathcal{A}_{t,\xi}$. For $\ell + 1 \leq n$, by (†) and $a_{\ell} = b_{\ell}$, we have

$$(\ell+1)h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell+1}) = h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell}) - (f(x_{\xi}) - a_{\ell})$$

$$= h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell}) - (f(x_{\xi}) - b_{\ell})$$

$$= (\ell+1)h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell+1}).$$

Hence, on the other hand, we have $h_{\vec{a},\xi}(y_{\vec{a},\xi,\ell+1}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,\ell+1})$. On the other hand, by revisiting (†), and eventuating it with the maps $\{h_{\vec{a},\xi},h_{\vec{b},\xi}\}$, we lead to the following equations:

$$(e_1): (n+1)h_{\vec{a},\xi}(y_{\vec{a},\xi,n+1}) = h_{\vec{a},\xi}(y_{\vec{a},\xi,n}) - (f(x_{\xi}) - a_n).$$

$$(e_2): (n+1)h_{\vec{b},\xi}(y_{\vec{b},\xi,n+1}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,n}) - (f(x_{\xi}) - b_n).$$

Subtracting (e_2) from (e_1) , and noting that $h_{\vec{a},\xi}(y_{\vec{a},\xi,n}) = h_{\vec{b},\xi}(y_{\vec{b},\xi,n})$ we get

$$(n+1)(h_{\vec{a},\xi}(y_{\vec{a},\xi,n+1}) - h_{\vec{b},\xi}(y_{\vec{b},\xi,n+1})) = a_n - b_n.$$

In particular, $n+1 \mid (a_n-b_n)$, contradicting the fact that $|a_n-b_n|=1$. $\square_{1.6}$

Let us proceed the argument of Fact 1.5. In view of Claim 1.6, we deduce that \mathcal{A}_{ξ} is countable. Take any $\vec{a} \in {}^{\omega}2 \setminus \mathcal{A}_{\xi}$. Then \vec{a} is as required. Finally, note that the group $G_{\vec{a},\xi}$ is generated by $\mathcal{B} := \{y_{\vec{a},\xi,\ell} : \ell \in \mathbb{N}\}_{\xi} \cup \{z\}$, because $x_{\xi} = y_{\vec{a},\xi,\ell} - (\ell+1)y_{\vec{a},\xi,\ell+1} + a_{\ell}z$. Since no relations involved in $\{y_{\vec{a},\xi,\ell}\} \cup \{z\}$, we see \mathcal{B} is a base.

We also need the following well-known result of Kurepa.

Fact 1.7. Assume $cf(\lambda) > \aleph_0$ and \mathcal{T} is a tree of height λ , all of whose levels are finite. Then \mathcal{T} has a cofinal branch.

§ 2. Controlling $\text{Hom}(G,\mathbb{Z})$

In this section, we prove our main result (see Theorem 2.6).

Discussion 2.1. Recall that a cardinal κ is called *measurable* if it is uncountable and there exists a non-principal κ -complete ultrafilter \mathcal{D} on κ , meaning that for every subset S of \mathcal{D} with cardinality $< \kappa$, the intersection $\bigcap S$ belongs to \mathcal{D} . It is known that the existence of measurable cardinals cannot be proven from ZFC.

Definition 2.2. Let G be an abelian group. The dual of G is the abelian group $\operatorname{Hom}(G,\mathbb{Z})$, which we denote by G^* . Let $g \in G$, and define $\psi_g : G^* \to \mathbb{Z}$ by the evaluation $\psi_g(G \xrightarrow{f} \mathbb{Z}) := f(g)$. The assignment $g \mapsto \psi_g$ defines a canonical map $\psi : G \to G^{**}$. We say that G is reflexive, if ψ is an isomorphism.

Fact 2.3. (Lös-Eda, Shelah; see [2, 9]). Let $\mu = \mu_{first}$ be the first measurable cardinal. The following hold:

- (a) For any $\theta < \mu$, $\mathbb{Z}^{(\theta)}$ is reflexive. In fact, its dual is \mathbb{Z}^{θ} .
- (b) For any $\lambda \geq \mu$, $\mathbb{Z}^{(\lambda)}$ is not reflexive.
- (c) There exists a reflexive group $G \subset \mathbb{Z}^{\mu}$ of cardinality μ .

Let Pr be any property of abelian groups and λ be a cardinal. Recall that compactness for (λ, \Pr) means that for any group G of cardinality λ and any " $G' \subseteq G \cap |G'| < \lambda \Rightarrow G'$ has Pr" then G has Pr. In this paper we are interested in the following fixed property of abelian groups:

Notation 2.4. By \Pr_{λ} we mean the following property: If G is a group of size λ , and if for any nontrivial subgroup $G' \subseteq G$ of size less than λ , $\operatorname{Hom}(G',\mathbb{Z}) \neq 0$, then $\operatorname{Hom}(G,\mathbb{Z}) \neq 0$.

Let's now turn to the primary framework.

Definition 2.5. (1) Let $\mathbf{M}_{1,\theta}$ be the class of objects

$$\mathbf{m} = (\lambda_{\mathbf{m}}, \langle G_{\alpha}^{\mathbf{m}} : \alpha \leq \alpha_{\mathbf{m}} \rangle, \langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}} \rangle)$$

consisting of:

- (a) (α) $\lambda_{\mathbf{m}} = \operatorname{cf}(\lambda_{\mathbf{m}}) > \aleph_0$,
 - (β) $\lambda_{\mathbf{m}} \geq \alpha_{\mathbf{m}} := \ell g(\mathbf{m}), \text{ the length of } \mathbf{m},$
- (b) (α) $\langle G_{\alpha}^{\mathbf{m}} : \alpha \leq \alpha_{\mathbf{m}} \rangle$ is an increasing and continuous sequence of abelian groups.
 - (β) $|G_{\alpha}^{\mathbf{m}}| < \lambda_{\mathbf{m}} \text{ for } \alpha < \alpha_{\mathbf{m}},$
- (c) $G_{\alpha}^{\mathbf{m}}/G_{0}^{\mathbf{m}}$ is free,
- (d) $\{\beta < \alpha_{\mathbf{m}} : G_{\beta+1}^{\mathbf{m}}/G_{\beta}^{\mathbf{m}} \text{ is not free}\}\$ is a non-reflecting stationary set,
- (e) (a) $S_{\mathbf{m}}$ is a set of cardinality $\leq \theta$,
 - $(\beta) \quad f_{\mathbf{m},s} \in \mathrm{Hom}(G^{\mathbf{m}}_{\alpha_{\mathbf{m}}}, \mathbb{Z}) \text{ for } s \in S_{\mathbf{m}},$
- (f) $\langle f_{\mathbf{m},s} : s \in S_{\mathbf{m}} \rangle$ is a free basis of a subgroup of $\mathrm{Hom}(G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}, \mathbb{Z})$.

- (2) $\mathbf{M}_{2,\theta}$ is defined as above, where item (a)(β) is replaced by $\alpha_{\mathbf{m}} = \lambda_{\mathbf{m}}$, and we further require that:
 - (g) if $f \in \text{Hom}(G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}, \mathbb{Z})$ then for some $h \in \text{Hom}(S_{\mathbf{m}}\mathbb{Z}, \mathbb{Z})$ we have

$$x \in G_{\alpha_{\mathbf{m}}}^{\mathbf{m}} \Rightarrow f(x) = h(\langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}} \rangle),$$

- (h) the mapping $x \mapsto \langle f_{\mathbf{m},s}(x) : s \in S_{\mathbf{m}} \rangle$ defines a homomorphism from $G_{\alpha_{\mathbf{m}}}^{\mathbf{m}}$ onto $S_{\mathbf{m}}\mathbb{Z}$,
- (i) for any $0 \neq G' \subseteq G_{\alpha}^{\mathbf{m}}$ with $\alpha < \alpha_{\mathbf{m}}$, we have $\operatorname{Hom}(G', \mathbb{Z}) \neq 0$.

We are now in a position to state and prove our main result:

Theorem 2.6. Assume that:

- (a) $\langle \lambda_i : i < \kappa \rangle$ is an increasing sequence of regular cardinals with limit λ ,
- (b) $\kappa = \operatorname{cf}(\kappa) < \lambda_0$, and $\aleph_0 < \lambda_0$,
- (c) $\lambda_i = \operatorname{cf}(\lambda_i), S_i \subseteq S_{\operatorname{cf}(\mu)}^{\lambda_i}$ is stationary and non-reflecting,
- $(d) \diamondsuit_{S_i} holds,$
- (e) there is no measurable cardinal $\leq \lambda$.

Then there is a λ_0 -free abelian group G of cardinality λ which is counterexample to singular compactness in λ for \Pr_{λ} .

Proof. We are going to present a λ_0 -free abelian group G of cardinality λ so that for any nontrivial subgroup $G' \subseteq G$ of smaller size, we have $\text{Hom}(G', \mathbb{Z}) \neq 0$, while $\text{Hom}(G, \mathbb{Z}) = 0$. We present the proof is several stages.

Stage A: We define a tree \mathcal{T} of height κ , whose *i*-th level \mathcal{T}_i is defined as follows:

- $(*)_A^i: \mathcal{T}_i$ is the set of η such that:
 - (a) η is a sequence of length i+1,
 - (b) for $j \le i$ we have $\eta(j) = (\eta(j, 1), \eta(j, 2)),$
 - (c) for $j \leq i$, $\eta(j, 1) < \lambda_j$ and $\eta(j, 2) < \kappa$,
 - (d) if $j_1 < j_2 \le i$ then $\eta(j_1, 1) \le \eta(j_2, 1)$, and $\eta(j_1, 2) \le \eta(j_2, 2)$,
 - (e) $Im(\eta)$ is finite,
 - (f) if $j_1 < j_2 \le i$ and $\langle \eta(j, 1) : j \in [j_1, j_2] \rangle$ is constant then $j_2 < \eta(j_1, 2)$.

Let $\mathcal{T} = \bigcup_{i < \kappa} \mathcal{T}_i$, where \mathcal{T} is ordered by end-extension relation \triangleleft . Then it is easily seen that $(\mathcal{T}, \triangleleft)$ is a tree with κ levels whose *i*-th level is \mathcal{T}_i and that if $\eta \in \mathcal{T}_i$, $i < j < \kappa$, then there is $\nu \in \mathcal{T}_i$ such that $\eta \triangleleft \nu$ (so $\eta = \nu \upharpoonright (i+1)$).

Also, we need to introduce the corresponding truncated trees, as follows:

$$\mathcal{T}_{i,\alpha} := \{ \eta \in \mathcal{T}_i : \eta(i,1) \leq \alpha \},$$

where $\alpha \leq \lambda_i$. In particular, $\mathcal{T}_i = \mathcal{T}_{i,\lambda_i}$.

Claim 2.7. $(\mathcal{T}, \triangleleft)$ has no κ -branches.

Proof. Assume by the way of contradiction that $b = \langle \eta_i : i < \kappa \rangle$, where $\eta_i \in \mathcal{T}_i$, is a branch of \mathcal{T} , hence the sequence $\langle \eta_i : i < \kappa \rangle$ is \triangleleft -increasing. It follows that $\langle \eta_i(i,1) : i < \kappa \rangle$ is a non-decreasing sequence of ordinals. As, by clause (e), every initial segment

has finitely many values and $\kappa = \operatorname{cf}(\kappa) > \aleph_0$, necessarily $\langle \eta_i(i,1) : i < \kappa \rangle$ is eventually constant, so for some $i_* < \kappa$, the sequence $\langle \eta_i(i,1) : i \in [i_*,\kappa) \rangle$ is constant. By $(*)_A^i(f)$, $\eta(i_*,2) > i$ for all $i < \kappa$, but on the other hand, $\eta(i_*,2) < \kappa$, a contradiction.

Stage B: We shall choose \mathbf{m}_i by induction on $i < \kappa$ such that:

$$(*)_B^i: (a) \mathbf{m}_i = (\lambda_{\mathbf{m}_i}, \langle G_{\alpha}^{\mathbf{m}_i} : \alpha \leq \alpha_{\mathbf{m}_i} \rangle, \langle f_{\mathbf{m}_i,s} : s \in S_{\mathbf{m}_i} \rangle) \in \mathbf{M}_{1,\lambda_i},$$

(b)
$$\lambda_{\mathbf{m}_i} = \alpha_{\mathbf{m}_i} = \lambda_i$$
, and the set of elements of $G_{\lambda_i}^{\mathbf{m}_i}$ is λ_i ,

(c)
$$G_{< i} := \bigcup \{G_{\lambda_i}^{\mathbf{m}_j} : j < i\} \cup \{0\},\$$

- $(\mathbf{d}) \ G_0^{\mathbf{m}_i} := G_{< i},$
- (e) $S_{\mathbf{m}_i} := \mathcal{T}_i = \mathcal{T}_{i,\lambda_i}$
- (f) if j < i, then $\mathbf{m}_i \leq \mathbf{m}_i$ which means that

$$\eta \in \mathcal{T}_j \land \nu \in \mathcal{T}_i \land \eta \triangleleft \nu \Rightarrow f_{\mathbf{m}_j,\eta} \subseteq f_{\mathbf{m}_i,\nu}.$$

This can be expressed by the following diagram:

$$0 \longrightarrow G_{\lambda_{j}}^{\mathbf{m}_{j}} \xrightarrow{\subseteq} G_{\lambda_{i}}^{\mathbf{m}_{i}}$$

$$f_{\mathbf{m}_{j},\eta} \downarrow \qquad f_{\mathbf{m}_{i},\nu}$$

- (g) $\langle f_{\mathbf{m}_i,\eta} : \eta \in \mathcal{T}_i \rangle$ is an independent subset of $\mathrm{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$,
- (h) $\bigcap \{ \operatorname{Ker}(f_{\mathbf{m}_i,\eta}) : \eta \in \mathcal{T}_i \} = \{ 0 \},$
- (i) if $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$, then for some h, α we have:
 - $(\alpha) \ \alpha < \lambda_i,$
 - (β) $h \in \text{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z}),$
 - (γ) if $x \in G_{\lambda_i}^{\mathbf{m}_i}$, then $f(x) = h(\langle f_{\mathbf{m}_i,\eta}(x) : \eta \in \mathcal{T}_{i,\alpha} \rangle)$.

Remark 2.8. The cardinality of $\mathcal{T}_{i,\alpha}$ is less than λ_i for any $\alpha < \lambda_i$. This will be helpful to show $|G_{\alpha}^{\mathbf{m}_j}| < \lambda_{\mathbf{m}_j}$ for $\alpha < \lambda_{\mathbf{m}_j}$, see subsequent paragraph of (*) below.

For i = 0, we set

- $\mathbf{m}_0 = (\lambda_0, \langle G_{\alpha}^{\mathbf{m}_0} : \alpha \leq \lambda_0 \rangle, \langle f_{\mathbf{m}_0, s} : s \in S_{\mathbf{m}_0} \rangle),$
- $G_{\alpha}^{\mathbf{m}_0} = \bigoplus_{\eta \in \mathcal{T}_{0,\alpha}} \mathbb{Z} x_{\eta},$
- $\bullet \ S_{\mathbf{m}_0} = \mathcal{T}_{0,\alpha},$
- for $\eta \in \mathcal{T}_{0,\alpha}$, $f_{\mathbf{m}_0,\eta}: G_{\alpha}^{\mathbf{m}_0} \to \mathbb{Z}x_{\eta}$ is the projection map.

Note that by Lös theorem [2, Corollary III. 1.5],

$$\operatorname{Hom}({}^{(\mathcal{T}_0)}\mathbb{Z},\mathbb{Z}) \cong \bigoplus_{\eta \in \mathcal{T}_0} \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \cong \bigoplus_{\eta \in \mathcal{T}_0} \mathbb{Z} x_{\eta},$$

from which we can easily conclude $(*)^i_B(i)$. The reason we take $G^{\mathbf{m}_0}_{\lambda_0}$ free is to make sure at the end of the construction, all our groups are at least λ_0 -free, as for the next steps $i < \kappa$ of the construction, we only get a bit more than $\sum_{j < i} \lambda_j$ -freeness, which for i = 0 is not well-defined.

Now assume that $0 < i < \kappa$ and $\langle \mathbf{m}_j : j < i \rangle$ has been defined. Fix a diamond sequence $\langle F_{i,\delta} : \delta \in S_i \rangle$ with $F_{i,\delta} : \delta \to \mathbb{Z}$.

Notation 2.9. Let $\langle \beta_i(\gamma) : \gamma < \lambda_i \rangle$ be an increasing and continuous sequence of ordinals, cofinal in λ_i with $\beta_i(0) = 0$.

We proceed by setting:

- $G_{< i} = \bigcup_{j < i} G_{\lambda_j}^{\mathbf{m}_j} \cup \{0\}$ (so $G_{< i} = \{0\}$, if i = 0), for $\eta \in \mathcal{T}_i$ set $f_{< i, \eta} = \bigcup_{j < i} f_{\mathbf{m}_j, \eta \upharpoonright j + 1}$, hence $f_{< i, \eta} : G_{< i} \to \mathbb{Z}$.

We shall choose $\mathbf{m}_{i,\gamma}$ by induction on $\gamma < \lambda_i$ such that:

- $(*)_C^{\gamma}$: (a) $\mathbf{m}_{i,0}$ is defined as
 - $(\alpha) \ \ell g(\mathbf{m}_{i,0}) = 0,$
 - $\begin{aligned} (\beta) \ \lambda_{\mathbf{m}_{i,0}} &= \sup_{j < i} \lambda_{\mathbf{m}_{j}}, \\ (\gamma) \ G_{0}^{\mathbf{m}_{i,0}} &:= G_{< i}, \end{aligned}$

 - (δ) $S_{\mathbf{m}_{i,0}} = \mathcal{T}_{i,\beta_i(0)},$
 - (ϵ) for $\eta \in \mathcal{T}_{i,\beta_i(0)}, f_{\mathbf{m}_{i,0},\eta} := f_{\langle i,\eta\rangle}$
 - (b) $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$ is an increasing and continuous sequence from \mathbf{M}_{1,λ_i} with $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)}$ and $\ell g(\mathbf{m}_{i,\gamma}) = \alpha_{i,\gamma} < \lambda_i$, which means:
 - (α) if $\rho < \gamma$, then $\mathbf{m}_{i,\rho} \leq \mathbf{m}_{i,\gamma}$,
 - (β) if γ is a limit ordinal, then $\mathbf{m}_{i,\gamma} = \bigcup_{\rho < \gamma} \mathbf{m}_{i,\rho}$, i.e.,

 - $(\beta_1) \ \alpha_{i,\gamma} = \sup_{\rho < \gamma} \alpha_{i,\rho},$ $(\beta_2) \ G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}} = \bigcup_{\rho < \gamma} G_{\alpha_{i,\rho}}^{\mathbf{m}_{i,\rho}},$
 - (β_3) $S_{\mathbf{m}_{i,\gamma}} = \mathcal{T}_{i,\beta_i(\gamma)},$
 - (β_4) if $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$, then $f_{\mathbf{m}_{i,\gamma},\eta} = f_{i,<\eta} \cup \bigcup_{\rho < \gamma} f_{\mathbf{m}_{i,\rho},\eta \upharpoonright \rho + 1}$,
 - (c) if $\rho < \gamma$, and $\rho \notin S_i$, then $G_{\alpha_i,\gamma}^{\mathbf{m}_i,\gamma}/G_{\alpha_i,\rho}^{\mathbf{m}_i,\rho}$ is free,
 - (d) $\bigcap \{ \operatorname{Ker}(f_{\mathbf{m}_{i,\gamma},\eta}) : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \} = \{0\},$
 - (e) $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ has set of elements an ordinal ordinal $\delta_{i}(\gamma) < \lambda_{i}$,
 - (f) Recall that $\langle F_{i,\delta} : \delta \in S_i \rangle$ is the diamond sequence. Suppose we have the following list of notations and assumptions:
 - $(\alpha) \ \gamma = \alpha_{i,\gamma} \in S_i,$
 - (β) the set of elements of $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ is γ ,
 - (γ) Im $(F_{i,\gamma}) \subseteq \mathbb{Z}$ is non-zero. In particular, Im $(F_{i,\gamma}) = n\mathbb{Z} \cong \mathbb{Z}$ for some nonzero $n \in \mathbb{Z}$,
 - (δ) $F_{i,\gamma}$ is a homomorphism from $G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ onto $\mathrm{Im}(\mathrm{F}_{i,\gamma})$,
 - (ϵ) $F_{i,\gamma} \notin \langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$, where $\langle f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$ is the subgroup of $\mathrm{Hom}(G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}, \mathbb{Z})$ generated by $\{f_{\mathbf{m}_{i,\gamma},\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \}$.

<u>Then</u>, we shall choose $\mathbf{m}_{i,\gamma+1}$ such that $F_{i,\gamma}$ has no extension to a homomorphism from $G_{\alpha_{i,\gamma+1}}^{\mathbf{m}_{i,\gamma+1}}$ into K_1 . Namely, we have

$$G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}} \xrightarrow{\subseteq} G_{\alpha_{i,\gamma+1}}^{\mathbf{m}_{i,\gamma+1}}$$

$$F_{i,\gamma} \downarrow \qquad \qquad \downarrow \sharp$$

$$\operatorname{Im}(F_{i,\gamma}) \xrightarrow{=} \operatorname{Im}(F_{i,\gamma})$$

For notational simplicity, we set

- $G_{i,\gamma,\rho} := G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$ for any $\rho \le \alpha_{i,\gamma}$,
- $f_{i,\gamma,\eta} := f_{\mathbf{m}_{i,\gamma},\eta}$ for any $\eta \in \mathcal{T}_{i,\beta_i(\gamma)}$,
- $\bullet \ G_{i,\gamma} := G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}.$

The case $\gamma = 0$ is trivial, and can be defined as in $(*)_C^{\gamma}(a)$. Note that by the induction hypothesis and the way we defined $f_{i,0,\eta}$:

- the sequence $\langle f_{i,0,\eta} : \eta \in \mathcal{T}_{i,\beta_i(0)} \rangle$ is an independent subset of $\text{Hom}(G_{i,0},\mathbb{Z})$,
- each $f_{i,0,\eta}$ extends $f_{\langle i,\eta}$ and

If γ is a limit ordinal, set $\alpha_{i,\gamma} = \sup_{\rho < \gamma} \alpha_{i,\rho}$ and define $\mathbf{m}_{i,\gamma}$ as in clause $(*)_C^{\gamma}(\mathbf{a})$. Suppose we have defined $\mathbf{m}_{i,\gamma}$. Assume one of the following hypotheses hold:

- $(h_1): \gamma \notin S_i$ or at least one of the hypotheses $(*)_C^{\gamma}(i)(\alpha)$ - (δ) are not satisfied, or
- $(h_2): \gamma \in S_i$, the hypotheses in $(*)_C^{\gamma}(i)(\alpha)$ - (δ) are all satisfied, but either
 - $-G_{i,\gamma}$ doesn't have domain γ , or
 - $-F_{i,\gamma} \notin \text{Hom}(G_{i,\gamma},\mathbb{Z})$ or
 - $-F_{i,\gamma} \in \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle.$

<u>Then</u>, we define $\mathbf{m}_{i,\gamma+1}$ as the following table:

- (1) $\alpha_{i,\gamma+1} = \alpha_{i,\gamma} + 1$,
- (2) $\mathbf{m}_{i,\gamma} \leq \mathbf{m}_{i,\gamma+1}$,
- (3) $S_{\mathbf{m}_{i,\gamma+1}} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- (4) $G_{i,\gamma+1} = G_{i,\gamma+1,\alpha_{i,\gamma}+1} := G_{i,\gamma} \oplus \mathbb{Z}_{[u_{i,\gamma}]}$, where $u_{i,\gamma} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- (5) for $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, $f_{i,\gamma+1,\eta} := f_{i,\gamma,\eta} \oplus \pi_{\eta}$, where $\pi_{\eta} : \mathbb{Z}_{[u_{i,\gamma}]} \to \mathbb{Z}x_{\eta}$ is the projection map, and for $\eta \notin \mathcal{T}_{i,\beta_i(\gamma)}$, we demand $f_{i,\gamma,\eta}$ is the zero-map.

Finally, suppose that $\mathbf{m}_{i,\gamma}$ is defined, $\gamma \in S_i$ and the hypotheses in $(*)_C^{\gamma}(i)(\alpha)$ - (δ) are all satisfied. Also, suppose that $G_{i,\gamma}$ has domain γ , and $F_{i,\gamma} \in \text{Hom}(G_{i,\gamma}, \mathbb{Z})$ is such that $F_{i,\gamma} \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma)} \rangle$. In this case, we define $\mathbf{m}_{i,\gamma+1}$ so that the conclusion of $(*)_R^{\gamma}(i)$ is satisfied.

Let $\alpha_{i,\gamma+1} := \alpha_{i,\gamma} + 1$, and naturally set

$$G_{i,\gamma,\rho} = G_{\rho}^{\mathbf{m}_{i,\gamma+1}} := G_{\rho}^{\mathbf{m}_{i,\gamma}} (= G_{i,\gamma,\rho}) \quad \forall \rho \le \alpha_{i,\gamma}.$$

We have to define

- $G_{i,\gamma+1} = G_{\alpha_{i,\gamma}+1}^{\mathbf{m}_{i,\gamma+1}}$ and
- $f_{i,\gamma+1,\eta} = f_{\mathbf{m}_{i,\gamma+1},\eta} : G_{i,\gamma+1} \to \mathbb{Z} \text{ for } \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}.$

For every $\beta < \lambda_i$, set

$$G_{i,\gamma}^{[\beta]} = \{ x \in G_{i,\gamma} : \eta \in \mathcal{T}_i \land \eta(i,1) < \beta \Rightarrow f_{i,\gamma,\eta}(x) = 0 \}.$$

Then the sequence $\langle G_{i,\gamma}^{[\beta]} : \beta < \lambda_i \rangle$ is increasing, hence as $|G_{i,\gamma}| < \lambda_i$, there is $\beta_{i,\gamma} < \lambda$ such that

$$G_{i,\gamma}^{[\beta]} = G_{i,\gamma}^{[\beta_{i,\gamma}]}, \quad \forall \beta \in (\beta_{i,\gamma}, \lambda_i).$$

Let $K_{i,\gamma} = \operatorname{Im}(F_{i,\gamma} \upharpoonright G_{i,\gamma}^{[\beta_{i,\gamma}]})$, and define

$$\mu_{i,\gamma} = \sum_{j < i} \lambda_j + |\mathcal{T}_{i,\beta_i(\gamma+1)}| + \aleph_0 < \lambda_i.$$

Set $\mathbf{k}_{i,\gamma} := (\mu_{i,\gamma}, \omega, \mathbb{Z})$. According to Fact 1.5, we know $\mathbf{k}_{i,\gamma} \in \mathcal{K}$. This gives us

$$(H_*, \phi_*) := (H_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}, \phi_{\mathbf{k}_{i,\gamma}, K_{i,\gamma}}),$$

so that:

- $H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ is a free abelian group of size $\mu_{i,\gamma}$, which extends $(K_{i,\gamma})_{[u_{i,\gamma}]}$, by recalling that $u_{i,\gamma} = \mathcal{T}_{i,\beta_i(\gamma+1)}$,
- $\phi_* = \phi_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} : u_{i,\gamma} \to K_{i,\gamma},$
- $H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}/(K_{i,\gamma})_{[\mu_{i,\gamma}]}$ is $\mu_{i,\gamma}$ -free,
- there is no homomorphism $f: H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \to K_{i,\gamma}$ such that $f(x_{\eta}) = \phi_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}(\eta)$ for $\eta \in \mathcal{T}_{i,\beta_{i}(\gamma+1)}$.

Also, for every β , with $\beta_{i,\gamma} \leq \beta < \lambda_i$, and $b \in K_{i,\gamma}$ there is $y_{b,\beta} \in G_{i,\gamma}$ such that

$$(*)_{\beta,b} \quad \text{(a)} \quad \eta \in \mathcal{T}_i \wedge \eta(i,1) < \beta \Rightarrow f_{i,\gamma,\eta}(y_{b,\beta}) = 0,$$

$$\text{(b)} \quad F_{i,\gamma}(y_{b,\beta}) = b.$$

Since $|G_{i,\gamma}| < \lambda_i$, for each $b \in K_{i,\gamma}$ as above, there exists some fixed $y_b \in G_{i,\gamma}$ such that the set

$$X_b = \{ \beta < \lambda_i : \beta_{i,\gamma} \le \beta \text{ and } y_{b,\beta} = y_b \}$$

is stationary in λ_i .

The assignment $x_{\eta} \mapsto y_{\phi_*(\eta)}$ induces a morphism $g_{i,\gamma} : (K_{i,\gamma})_{[u_{i,\gamma}]} \to G_{i,\gamma}$. Recall that id: $(K_{i,\gamma})_{[u_{i,\gamma}]} \to H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ is the natural inclusion. Let us summarize these data with the following notation

$$H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$$

$$\mathrm{id} \uparrow$$

$$(K_{i,\gamma})_{[u_{i,\gamma}]} \xrightarrow{g_{i,\gamma}} G_{i,\gamma},$$

The group that we were searching for it, is the pushout of the above data. Namely,

$$G_{i,\gamma+1} := G_{i,\gamma} \oplus_{(K_{i,\gamma})_{[u_{i,\gamma}]}} H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}.$$

In other words, $G_{i,\gamma+1}$ has the following presentation:

$$G_{i,\gamma+1} = \frac{G_{i,\gamma} \times H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}}{\left\langle (\mathrm{id}(k), -g_{i,\gamma}(k)) : k \in (K_{i,\gamma})_{[u_{i,\gamma}]} \right\rangle} \quad (*)$$

Recall that $H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ is of size $\mu_{i,\gamma} = \sum_{j<i} \lambda_j + |\mathcal{T}_{i,\beta_i(\gamma+1)}| + \aleph_0 < \lambda_i$. We combine this with an inductive argument along with (*), to concluded that group $G_{i,\gamma+1}$ is of size less than λ_i .

Notation 2.10. For $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ and $g \in G_{i,\gamma}$, let $[(h,g)] \in G_{i,\gamma+1}$ denote the equivalence class of [(h,g)].

This push-out construction, gives us two embedding maps $h_{i,\gamma}: G_{i,\gamma} \to G_{i,\gamma+1}$ and $k_{i,\gamma}: H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \to G_{i,\gamma+1}$ so that $h_{i,\gamma} \circ g_{i,\gamma} = k_{i,\gamma}$. Let us depict all things together:

$$H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \xrightarrow{k_{i,\gamma}} G_{i,\gamma+1}$$

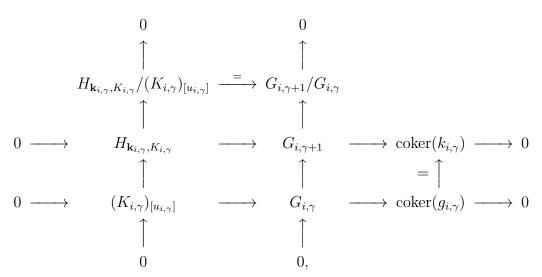
$$\downarrow d \qquad \qquad \downarrow h_{i,\gamma}$$

$$(K_{i,\gamma})_{[u_{i,\gamma}]} \xrightarrow{g_{i,\gamma}} G_{i,\gamma}$$

We now show that $h_{i,\gamma}: G_{i,\gamma} \to G_{i,\gamma+1}$, is an embedding. Indeed, the assignment $x \in G_{i,\gamma} \mapsto [(0,x)]$ defines $h_{i,\gamma}$. Suppose $0 = h_{i,\gamma}(x) = [(0,x)]$. By the above equivalence relation, there is a $k \in (K_{i,\gamma})_{[\mu_{i,\gamma}]}$ so that $(k,g_{i,\gamma}(k)) = (0,x)$. Hence k=0 and $x = g_{i,\gamma}(0) = 0$. This shows that $h_{i,\gamma}$ is an embedding, as desired. Thus, by simplicity, we may assume that $G_{i,\gamma} \subseteq G_{i,\gamma+1}$ and $h_{i,\gamma}$ is the inclusion map. We now show that $F_{i,\gamma}$ does not extend to a homomorphism from $G_{i,\gamma+1}$ into $K_{i,\gamma}$. Indeed if $F: G_{i,\gamma+1} \to K_{i,\gamma}$ extends $F_{i,\gamma}$, then $f: H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}} \to \mathbb{Z}$ defined by $f = F \circ k_{i,\gamma}$ satisfies

$$f(x_{\eta}) = F \circ k_{i,\gamma} = F \circ (h_{i,\gamma} \circ g_{i,\gamma})(x_{\eta}) = F_{i,\gamma} \circ g_{i,\gamma}(x_{\eta}) = \phi_*(\eta),$$

for all $\eta \in u_{i,\gamma}$. This contradicts the choice of $(H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}},\phi_{\mathbf{k}_{i,\gamma},K_{i,\gamma}})$ and Definition 1.4(δ). Now, by the following well-known diagram



we are able to deduce that

$$G_{i,\gamma+1}/G_{i,\gamma} \cong H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}/(K_{i,\gamma})_{[u_{i,\gamma}]},$$

which is $\mu_{i,\gamma}$ -free.

We next define the map $f_{i,\gamma+1,\eta}$. Take any $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$. For any $h \in H_{\mathbf{k}_{i,\gamma},K_{i,\gamma}}$ and $g \in G_{i,\gamma}$, the assignment $[(h,g)] \mapsto f_{i,\gamma,\eta}(g)$, defines a morphism

$$f_{i,\gamma+1,\eta} := f_{\mathbf{m}_{i,\gamma+1},\eta} : G_{i,\gamma+1} \longrightarrow \mathbb{Z}.$$

Let us show that $f_{i,\gamma+1,\eta}$ is well-defined, by arguing that $f_{i,\gamma,\eta} \circ g_{i,\gamma} = 0$. Given any $\eta \in \mathcal{T}_{i,\beta_i(\gamma+1)}$, choose $\beta \in X_{\phi_*(\eta)}$ such that $\eta(i,1) < \beta$. In view of $(*)_{\beta,\phi_*(\eta)}(a)$, we have

$$f_{i,\gamma,\eta} \circ g_{i,\gamma}(x_{\eta}) = f_{i,\gamma,\eta}(y_{\phi_*(\eta)}) = f_{i,\gamma,\eta}(y_{\phi_*(\eta),\beta}) = 0.$$

Clearly, $\langle f_{i,\gamma+1,\eta} : \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)} \rangle$ are independent and also

$$\bigcap \{ \operatorname{Ker}(f_{i,\gamma+1,\eta}) : \eta \in \mathcal{T}_{i,\beta_i(\gamma+1)} \} = \{0\}.$$

Having finished the construction, for $\eta \in \mathcal{T}_i$ we set $f_{i,\eta} := \bigcup_{\gamma < \lambda_i} f_{i,\gamma,\eta}$. Now, we are ready to bring the following claim.

Claim 2.11. $\langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle$ generates $\operatorname{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$.

Proof. Suppose $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z}) \setminus \langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle$. Take $\gamma \in S$ such that $G_{i,\gamma}$ has domain γ , $f \upharpoonright \gamma = F_{i,\gamma}$, and $f \upharpoonright \gamma \notin \langle f_{i,\gamma,\eta} : \eta \in \mathcal{T}_i \rangle$. Then by our construction, $f \upharpoonright \gamma$ does not extend to a homomorphism from $G_{i,\gamma+1}$ into \mathbb{Z} , a contradiction. $\square_{2.11}$

Also note that clause $(*)_C^{\gamma}(i)$ holds by Claim 2.11, indeed, given any $f \in \text{Hom}(G_{\lambda_i}^{\mathbf{m}_i}, \mathbb{Z})$, we can find some $\eta_0, \dots, \eta_{n-1} \in \mathcal{T}_i$ and some $\alpha_0, \dots, \alpha_{n-1}$ such that

$$f = \sum_{k < n} \alpha_k f_{i, \eta_k}.$$

Let us to pick $\alpha < \lambda_i$ large enough such that for each $k < n, \eta_k(i, 1) \le \alpha$. By Lös' theorem [2, Corollary III. 1.5],

$$\operatorname{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_{i,\alpha})} \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \cong \bigoplus_{\eta \in (\mathcal{T}_{i,\alpha})} \mathbb{Z} x_{\eta}.$$

In particular, there is $h \in \text{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z})$ such that

$$h(\langle f_{i,\eta} : \eta \in \mathcal{T}_i \rangle) = \sum_{k < n} \alpha_k f_{i,\eta_k}.$$

Clause $(*)_C^{\gamma}(b)$ follows from the fact that S_i is non-reflecting, hence for some club $C \subseteq \gamma$ with $\min(C) = \rho$, such that $C \cap S_i = \emptyset$, hence $G_{i,\gamma}/G_{i,\rho}$ is the union of the increasing and continuous sequence $\langle G_{i,\tau}/G_{i,\rho} : \tau \in C \rangle$, and by the induction hypothesis, each $G_{i,\tau}/G_{i,\mu}$ is free for all $\mu < \tau$ from C, so easily $G_{i,\gamma}/G_{i,\rho}$ is free.

Having defined $\langle \mathbf{m}_{i,\gamma} : \gamma < \lambda_i \rangle$, set $\mathbf{m}_i = \bigcup_{\gamma < \lambda_i} \mathbf{m}_{i,\gamma}$. This completes the inductive construction of $\langle \mathbf{m}_i : i < \kappa \rangle$.

Stage C: In this step, we show that for each i, $\mathbf{m}_i \in \mathbf{M}_{2,\lambda_i}$ (see Definition 2.5(2)). Items (a)-(e) of Definition 2.5(1) and $\alpha_{\mathbf{m}_i} = \lambda_i = \lambda_{\mathbf{m}_i}$ are obvious.

For clause (i), suppose $\alpha < \lambda_i$ and $0 \neq G' \subseteq G_{\alpha}^{\mathbf{m}_i}$. Then for some $\gamma < \lambda_i, G' \subseteq G_{\alpha_{i,\gamma}}^{\mathbf{m}_{i,\gamma}}$. Let $0 \neq x \in G'$. In view of $(*)_C^{\gamma}(\mathbf{c})$, we have

$$\bigcap \{ \operatorname{Ker}(f_{i,s}) : s \in \mathcal{T}_i \} = \{ 0 \}.$$

This in turns imply that $f_{i,s}(x) \neq 0$ for some $s \in \mathcal{T}_i$. In particular, $f_{i,s} \upharpoonright G' \in \text{Hom}(G',\mathbb{Z})$ is non-zero.

Stage D: In this stage we conclude the proof of Theorem 2.6. for each $i < \kappa$ set $G_i = G_{\mathbf{m}_i,\lambda_i}$ and $G_{< i} = \bigcup \{G_j : j < i\}$. Then the sequence $\langle G_i : i < \kappa \rangle$ is increasing continuous and for each $i, G_i/G_{< i}$ is $\sum_{j < i} \lambda_j$ -free (as for each $\gamma < \lambda_i$, $\mu_{i,\gamma} \geq \sum_{j < i} \lambda_j$ and each $G_{i,\gamma+1}/G_{i,\gamma}$ is $\mu_{i,\gamma}$ -free). Define the group

$$G := \bigcup \{G_i : i < \kappa\}.$$

From this, G is an abelian group of size λ .

We first show that G is λ_0 -free. Thus suppose that H is a subgroup of G of size less than λ_0 . Then the sequence $\langle H \cap G_i : 0 < i < \kappa \rangle$ is increasing, continuous and for each $0 < i < \kappa$,

$$(H \cap G_i)/(H \cap G_{< i}) \cong ((H \cap G_i) + G_{< i})/G_{< i}$$

is free as $G_i/G_{< i}$ is $\sum_{j < i} \lambda_j$ -free and hence λ_0 -free. It then easily follows that $H = \bigcup_{i < \kappa} (H \cap G_i)$ is free.

Next, suppose H is a non-zero subgroup of G of size less than λ . We show that $\operatorname{Hom}(H,\mathbb{Z}) \neq 0$. Let $i < \kappa$ be such that $H \cap G_{< i} \neq \{0\}$ and $|H| < \lambda_i$. According to Definition 2.5(2)(i), we must have $\operatorname{Hom}(H \cap G_i,\mathbb{Z}) \neq 0$. Furthermore, by an argument as above, $(H \cap G_i)/(H \cap G_{< i}) \cong ((H \cap G_i) + G_{< i})/G_{< i}$ is free. It then clearly follows that $\operatorname{Hom}(H,\mathbb{Z}) \neq 0$.

Finally, let us show that $\operatorname{Hom}(G,\mathbb{Z}) = 0$. Suppose, by the way of contradiction that $f \in \operatorname{Hom}(G,\mathbb{Z})$ and f is non-zero. By $(*)_B^i(h)$, for each $i < \kappa$, we can find some $\alpha_i < \lambda_i$ and $h_i \in \operatorname{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z})$ such that

$$x \in G_i \Rightarrow f(x) = h_i(\langle f_{\mathbf{m}_i,\eta}(x) : \eta \in \mathcal{T}_{i,\alpha} \rangle).$$

Thanks to Lös' theorem [2, Corollary III. 1.5], for each $i < \kappa$,

$$\operatorname{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z})\cong\bigoplus_{\eta\in(\mathcal{T}_{i,\alpha})}\operatorname{Hom}(\mathbb{Z},\mathbb{Z})\cong\bigoplus_{\eta\in(\mathcal{T}_{i,\alpha})}\mathbb{Z}x_{\eta}.$$

In particular, $\operatorname{Hom}({}^{(\mathcal{T}_{i,\alpha})}\mathbb{Z},\mathbb{Z})$ is free, and it has a natural basis $\langle f_{\eta} : \eta \in (\mathcal{T}_{i,\alpha}) \rangle$. It then follows from [2, Corollary III. 3.3], that for some finite set $u_i \subseteq \mathcal{T}_{i,\alpha}$, the following holds:

$$x \in G_i$$
 and $(\forall \eta \in u_i)(f_{\mathbf{m}_i,\eta}(x) = 0) \Rightarrow f(x) = 0.$

As $\kappa = \operatorname{cf}(\kappa) > \aleph_0$ for some n_* the set $\mathcal{V}_1 = \{i < \kappa : |u_i| = n_*\}$ is unbounded in κ .

For any $i < j < \kappa$, we define the projection map $\operatorname{prj}_{i,j} : \mathcal{T}_j \to \mathcal{T}_i$ in the natural way by $\operatorname{prj}_{i,j}(\eta) = \eta \upharpoonright (i+1)$. Clearly, $\operatorname{pr}_{i,j}$ maps u_j onto u_i . By Kurepa's theorem, see Fact 1.7, \mathcal{T} has a cofinal branch, which contradicts Claim 2.7.

References

- [1] M. Asgharzadeh, M. Golshani and S. Shelah, Failure of singular compactness for Ext, work in progress.
- [2] P. C. Eklof and A. Mekler, *Almost free modules: Set theoretic methods*, Revised Edition, North–Holland Publishing Co., North–Holland Mathematical Library, **65**, 2002.
- [3] L. Fuchs, Abelian groups, Springer Monographs in Mathematics. Springer, Cham, 2015.
- [4] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, vol. i, ii, de Gruyter Expositions in Mathematics, Walter de Gruyter, 2012.
- [5] P. Hill, On the freeness of abelian groups: a generalization of Pontryagin's theorem, Bull. Amer. Math. Soc. **76** (1970), 1118–1120.
- [6] T. Jech, Set theory, The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [7] S. Shelah, Infinite abelian groups, Whitehead problem and some constructions, Israel J. Math. 18 (1974), 243–256.
- [8] S. Shelah, A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals, Israel Journal of Mathematics 21 (1975), 319–349.
- [9] S. Shelah, Reflexive abelian groups and measurable cardinals and full MAD families, Algebra Universalis 63 (2010), 351–366.
- [10] S. Shelah, Quite free complicated abelian groups, pcf and black boxes, Israel J. Math. 240 (2020), 1–64.

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