Positive Transvections

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Abstract

In this paper, we study some simple maps as generators of symplectic group. They will be called positive transvections. We can see from Grove's book [1] that transvections are generators of symplectic group. Positive transvections are a very simple form of transvections. By positive transvections, we can prove two known theorems in easier way.

1 Introduction

Let V be a vector space and W be a hyperplane in V. This means that W is $m-1$ dimensional where V is m dimensional. Let f be a linear map on V to itself. Then f is called transvection with a fixed hyperplane W if

- $f(x) = x$ where $x \in W$
- $f(x) - x \in W$ for all $x \in V$.

By these properties, the reader may think that transvections are reflections (since they have fixed hyperplanes). However, the second property never holds for reflections.

Throughout this paper, we focus only on the transvections which are in the symplectic group. For the reader who are not familiar with that word, let us give little background.

1.1 Symplectic Vector Space

Let V be a vector space and ω be a nondegenerate skew symmetric bilinear form on V. Then the vector space V is symplectic vector space, and denoted by (V, ω).

The symplectic complement of a subspace W of V is defined as

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \quad \forall w \in W\}$$

Let u and v be elements of V. Then u and v are symplectically orthogonal if $\omega(u, v) = 0$. For any two subspaces U and W, we say U and W are symplectically orthogonal if $\omega(u, v) = 0$ for all $u \in U$ and $v \in W$.

Theorem 1.1. *Symplectic vector spaces are even dimensional.*

Proof. Suppose that V is an m dimensional symplectic vector space. For the case $m = 1$, all vectors have to be a multiple of any nonzero vector, i.e. for any two nonzero vectors v and w, there exists a scalar number k such that $v = kw$. This implies that $\omega(v, w) = k\omega(w, w) = 0$, nondegeneracy of the symplectic form is violated. For the case $m = 2$ is trivial. Assume that $m > 2$, then there must be a symplectic vector space with dimension $m-2$ by the following way. Nondegeneracy of the symplectic form shows that there exist u and v satisfying $\omega(u, v)$ is nonzero. Let W be the space spanned by u and v. It is easy to see that W^ω is $m - 2$ dimensional symplectic vector space. After applying this method many times, we will get a 2 dimensional symplectic vector space if m is even, and a one dimensional symplectic vector space if m is odd, which is impossible. $\square$

By Theorem 1.1, we call the dimension of V $2n$ instead of m.

Theorem 1.2. [2, Theorem 2.1.3] *Let (V, ω) be a symplectic vector space with dimension 2n. Then there exists a basis $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ such that $\omega(u_i, u_j) = 0$, $\omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$. This basis is called a symplectic basis for V.*

Definition 1.3. *$\mathbb{R}^{2n}$ has a symplectic structure and $(\mathbb{R}^{2n}, \omega_0)$ denotes the symplectic vector space with the symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ such that $\omega_0(e_i, f_j) = \delta_{ij}$ and $\omega_0(e_i, e_j) = \omega_0(f_i, f_j) = 0$ for all $i, j \in \{1, ..., n\}$.*

By Theorem 1.2, we can say that all symplectic vector spaces with same dimensions are isomorphic. This lead us to consider $(\mathbb{R}^{2n}, \omega_0)$ as a representation of all symplectic vector spaces with dimension $2n$.

Theorem 1.4. [2, Lemma 2.1.1] *dim W + dim W^{ω} = dim V for all subspaces W of symplectic vector space V.*

2 Symplectic Group

Let (V, ω) be a symplectic vector space and $\Psi : V \to V$ be linear isomorphism. Then Ψ be a symplectomorpism if it preserves the form structure, $\Psi^* \omega = \omega$, in other words

$$\omega(\Psi v, \Psi w) = \omega(v, w) \quad \text{for all } \ v, w \in V$$

Theorem 2.1. *Let (V,ω) be a symplectic vector space. The set of all symplectomorphisms of (V,ω) is a group under composition of maps, called symplectic group and denoted by $Sp(V)$.*

Proof. Since identity map preserves the form, it is included in that group. So $\mathrm{Sp}(V)$ is not empty set. Let Ψ and Φ be in $\mathrm{Sp}(V)$, then

$$\omega(\Psi\Phi v, \Psi\Phi w) = \omega(\Phi v, \Phi w) = \omega(v, w)$$

This implies $\Psi\Phi \in \mathrm{Sp}(V)$. Associativity and identity properties are satisfied obviously. All elements in $\mathrm{Sp}(V)$ are linear isomorphisms, so their inverses are also linear isomorphisms and it can be easily shown that they preserve the form structure. So, inverse property is also satisfied. $\square$

Corollary 2.2. *Because any $2n$ dimensional symplectic vector space is isomorphic to $(\mathbb{R}^n, \omega_0)$, $Sp(V)$ is isomorphic to the symplectic group of $\mathbb{R}^{2n}$, and denoted by $Sp(2n)$.*

Theorem 2.3. *$Sp(2n) = \{\Psi \in GL(2n,\mathbb{R}) \mid \ \Psi^T J \Psi = J\}$ where J is $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.*

Proof. Let $\Psi \in \mathrm{Sp}(2n)$ be chosen arbitrarily. For any $\alpha, \beta \in \mathbb{R}^{2n}$, we know that $\omega_0(\alpha, \beta) = \omega_0(\Psi\alpha, \Psi\beta)$. Due to $\omega_0(\alpha, \beta) = \alpha^T J \beta$, $\Psi \in \mathrm{Sp}(2n)$ is equivalent to $\Psi^T J \Psi = J$. $\square$

Theorem 2.4. *det* $\Psi = 1$

We will prove Theorem 2.4 by using positive transvections.

3 Mapping Class Group

Let Σ be a compact connected orientable surface. This implies that Σ has some holes and some cut parts. These are called *genus* and *boundary parts*, respectively. We denote such a surface by Σ_g^b, g genus b boundary parts. Throughout this paper, we use the term *surface* in referring to the compact connected orientable surfaces, and we are mainly concerned about the closed (no boundary parts) surfaces. Therefore we usually use the notation Σ_g instead of Σ_g^0.

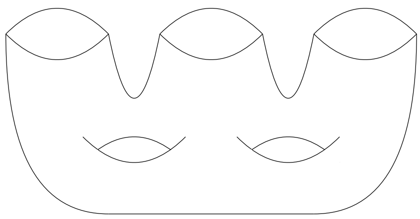

Figure 1: Σ_2^3 surface

Definition 3.1. *Let $f : \Sigma_g^b \to \Sigma_g^b$ be an orientation preserving homeomorphism and induced map on the boundary is identity. Such maps create a group called mapping class group up to isotopy. The mapping class group of Σ_g^b is denoted by Mod(Σ_g^b).*

In other words, $\mathrm{Mod}(\Sigma_g^b)$ is the group of all isotopy classes of orientation preserving self diffeomorphisms of Σ_g^b, which is identity on $\partial\Sigma_g^b$. We use the term diffeomorphism instead of homeomorphism due to the fact that any homeomorphism is homotopic to a diffeomorphism in the surface.

In this paper, we are mainly interested in the generator set of the mapping class groups, which are called *Dehn twists.* Therefore we will talk more about Dehn twists under the next heading.

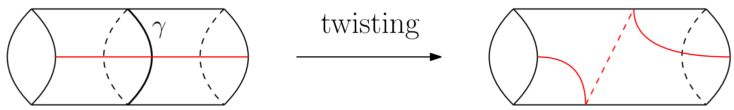

Figure 2: Dehn twist D_γ

3.1 Dehn Twists

Consider a simple closed curve γ on a surface. In the closed tubular neighborhood of γ, Dehn twist about γ is a self-homeomorphism on that surface, made by regluing after $360°$ twisting of that closed tubular neigborhood, as in Figure 2.

Twisting right or left gives two different Dehn twists. The former is called right Dehn twist and the latter is called left Dehn twist. Twisting right and left gives identity map in the mapping class group. So we can say that left Dehn twists are inverse of right Dehn twists.

Let γ be a simple closed curve on a surface. Then γ is called separating if the new surface is disconnected after cutting γ, and nonseparating if the new surface is still connected.

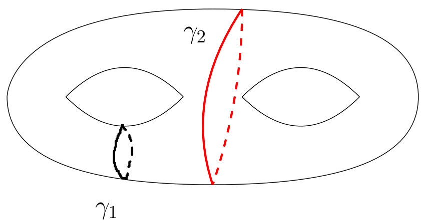

Figure 3: γ_1 is nonseparating curve and γ_2 is separating curve

For any simple closed curve γ in a surface, D_γ represents right Dehn twists and D_γ^{-1} represents left Dehn twists. We are mainly interested in right Dehn twists.

By Dehn-Lickorish theorem [3, Theorem 4.1], Dehn twists generate the mapping class group of a surface. For a closed surface, although there are many studies giving a finite number of Dehn twists, Humphries [4] showed that $2g + 1$ Dehn twists generate $\mathrm{Mod}(\Sigma_g)$ and this number is the most favorable one in the literature.

3.2 Symplectic Representation

Definition 3.2. *For given Σ_g, the algebraic intersection map $\hat{i}(.,.)$ is defined on the first homology group of Σ_g, denoted by $H_1(\Sigma_g, \mathbb{Z})$ and satisfy the following properties*

1. *For all $a, b \in H_1(\Sigma_g, \mathbb{Z})$, $\hat{i}(a, b) = -\hat{i}(b, a)$, $\hat{i}$ is skew symmetric.*
2. *For all $a, b \in H_1(\Sigma_g, \mathbb{Z})$, let α and β represent a and b respectively. Then $\hat{i}(a, b)$ is the sum of signed intersections of α and β curves.*

Theorem 3.3. *Let Σ_g be given and γ be separating curve, then the sum of the signed intersection with any curve in the surface is zero.*

For convention, let $[a] \in \mathrm{H}_1(\Sigma_g, \mathbb{Z})$ be the homology class representing the oriented simple closed curve a. For any separating curve γ, $[\gamma] = 0$ (Look Figure 4. Since the coloured curves, including the points, are homotopic, their homology classes are same which is trivial element). By Theorem 3.3, we can get a result that $\hat{i}$ is nondegenerate.

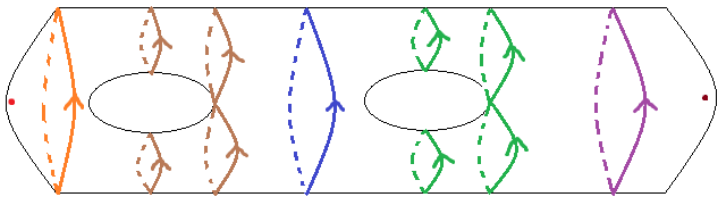

Figure 4: The coloured oriented curves on this Σ_2 are homotopic

Let us consider the ordered basis $\{[a_1], [b_1], ..., [a_g], [b_g]\}$ for $\mathrm{H}_1(\Sigma_g, \mathbb{R})$ in Figure 5. The algebraic intersection map extends to a nondegenerate skew symmetric bilinear form

$$\hat{i} : \mathrm{H}_1(\Sigma_g, \mathbb{R}) \oplus \mathrm{H}_1(\Sigma_g, \mathbb{R}) \to \mathbb{R} \tag{1}$$

With this structure, $(\mathrm{H}_1(\Sigma_g, \mathbb{R}), \hat{i})$ is $2g$ dimensional symplectic vector space and the basis $\{[a_1], [b_1], ..., [a_g], [b_g]\}$ is symplectic basis.

This collection of the curves $a_1, ..., a_g, b_1, ..., b_g$ is called *geometric symplectic basis* for $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$.

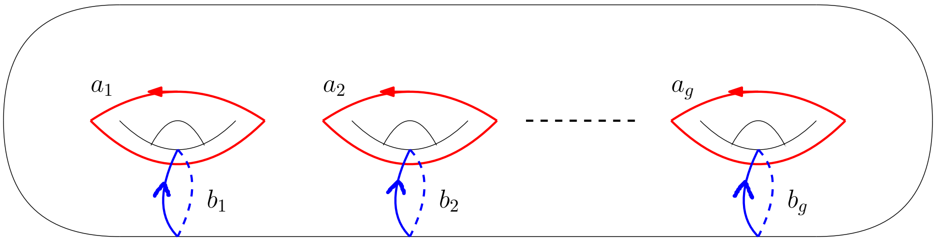

Figure 5: Geometric symplectic basis for $\mathrm{H}_1(\Sigma_g, \mathbb{R})$

The action of $\mathrm{Mod}(\Sigma_g)$ on $\mathrm{H}_1(\Sigma_g, \mathbb{R})$ preserves the structure of $\hat{i}$. This yields a representation

$$Mod(\Sigma_g) \to Sp(2g)$$

and this representation is called the symplectic representation of $\mathrm{Mod}(\Sigma_g)$.

Theorem 3.4. *The representation $Mod(\Sigma_g) \to Sp(2g)$ is surjective.*

We will prove Theorem 3.4 by using positive transvections.

4 Transvections

Definition 4.1. *Let $T : V \to V$ be a linear map. T is transvection with fixed hyperplane W if $T|_W = id_W$ and $T(v) - v \in W$ for all $v \in V$.*

Theorem 4.2. *Let W be a hyperplane, then there exists a nonzero vector u in V such that $W = u^\omega$.*

Proof. By Theorem 1.4, $\dim(W^\omega) = 1$. This means $W^\omega = \mathrm{span}\{u\}$ for some nonzero $u \in V$. Thus $u^\omega = (W^\omega)^\omega = W$. □

Theorem 4.3. [1, p. 22,23] *Any transvection in a symplectic group can be written in the following way*

$$T_{a,u} := v \longrightarrow v + a\omega(v, u)u.$$

Here $W = u^\omega$ and $T(v) - v = a\omega(v, u)u \in u^\omega$. Conversely, for any nonzero a, and any nonzero vector u, $T_{a,u}$ is a transvection in symplectic group.

Proof. Let f be a transvection with fixed hyperplane u^ω on V. So the linear map $h = f - id$ has kernel W. So image of h is one dimensional. Therefore we can write $h(x) = a_x v$ where v is a vector in V and a_x is a linear function mapping x to a real number. So $a_x \in V^*$.

Nondegeneracy of ω implies that all elements in V^* can be written as a scalar of $\omega(.,y)$ for $y \in V$. So $a_x = c\omega(x,y)$ for some $y \in V$. In this case kernel of h becomes y^ω. So, $u^\omega = y^\omega$. Then

$$u^\omega = y^\omega \to (u^\omega)^\omega = (y^\omega)^\omega \to span\{u\} = span\{y\} \to y = ku$$

Then, $h(x) = r\omega(x,u)v$. The rest is showing v is a multiple of u. In this case, we benefit from being symplectomorphism of f.

$$\begin{aligned}
\omega(x,y) &= \omega(f(x), f(y)) \\
&= \omega(x + h(x), y + h(y)) \\
&= \omega(x,y) + \omega(x,h(y)) + \omega(h(x),y) + \omega(h(x),h(y)) \\
0 &= \omega(x,h(y)) + \omega(h(x),y) + 0 \\
\omega(h(y),x) &= \omega(h(x),y) \\
\omega(r\omega(y,u)v,x) &= \omega(r\omega(x,u)v,y) \\
\omega(y,u)\omega(v,x) &= \omega(x,u)\omega(v,y)
\end{aligned}$$

Let x be an element of $u^\omega \setminus v^\omega$. So $\omega(x,u) = 0$ and $\omega(x,v) \neq 0$. Then,

$$\begin{aligned}
\omega(y,u)\omega(v,x) &= \omega(x,u)\omega(v,y) \\
\omega(y,u)\omega(v,x) &= 0 \quad \text{Select } y \text{ be hyperbolic pair of } u \\
\omega(v,x) &= 0 \quad \text{Contradiction!}
\end{aligned}$$

Thus, $u^\omega = v^\omega$. So v is a multiple of u. Hence $f(x) = x + a\omega(x,u)u = T_{a,u}(x)$.

Let $T(v) = v + a\omega(v,u)u$ be given and v_1 and v_2 be arbitrarily chosen elements of V. Then

$$
\begin{aligned}
\omega(T(v_1),T(v_2)) &= \omega(v_1+a\omega(v_1,u)u, v_2+a\omega(v_2,u)u)\\
&= \omega(v_1, v_2+a\omega(v_2,u)u)+\omega(a\omega(v_1,u)u, v_2+a\omega(v_2,u)u)\\
&= \omega(v_1,v_2)+a\omega(v_2,u)\omega(v_1,u)+a\omega(v_1,u)\omega(u,v_2)\\
&= \omega(v_1,v_2)+a\omega(v_2,u)\omega(v_1,u)-a\omega(v_1,u)\omega(v_2,u)\\
&= \omega(v_1,v_2)
\end{aligned}
$$

By induction, one can show that composition of transvections is also a symplectomorphism. □

Theorem 4.4. [1, THEOREM 3.4] *The symplectic group is generated by transvections.*

Let $v \neq w \in V \backslash \{0\}$ be given. It will be shown that there is a composition of transvections which map v to w. Actually with at most 2 transvections, one can do this.

Case 1: $\omega(v,w) \neq 0$

Let a be $\frac{1}{\omega(v,w)}$ and u be $v-w$. Then

$$
\begin{aligned}
T_{a,u}(v) &= v+a\omega(v,u)u\\
&= v+\frac{1}{\omega(v,w)}\omega(v,v-w)(v-w)\\
&= v+\frac{1}{\omega(v,w)}\omega(v,-w)(v-w)\\
&= v+\frac{\omega(v,-w)}{\omega(v,w)}(v-w)\\
&= v-(v-w)\\
&= w.
\end{aligned}
$$

So, in this case, we have a transvection mapping v to w.

Case 2: $\omega(v,w)=0$

In this case, we want to select $z \in V$ such that $\omega(v,z) \neq 0$ and $\omega(w,z) \neq 0$. By nondegeneracy of ω, there exist z_1 and z_2 in V such that $\omega(v,z_1) \neq 0$ and $\omega(w,z_2) \neq 0$. If $\omega(v,z_2) \neq 0$, then z can be z_2. Assume not, and $\omega(w,z_1) \neq 0$, then we can choose $z=z_1$. There is one case left, which is

$\omega(v, z_2) = 0 = \omega(w, z_1)$. Then we can choose $z = z_1 + z_2$. In all cases, such z can be found. By Case 1, there exist T_1 and T_2 such that $T_1(v) = z$ and $T_2(z) = w$. $T_2T_1(v) = w$. Thus, we proved the next lemma.

Lemma 4.5. [1, Proposition 3.2] *For all v and w in V, there exists an element in $\mathcal{T}$ mapping v to w.*

Lemma 4.6. [1, Proposition 3.3] *$\mathcal{T}$ is transitive on hyperbolic pairs.*

Proof. Let (α_1, β_1) and (α_2, β_2) be hyperbolic pairs in V ($\omega(\alpha_1, \beta_1) = \omega(\alpha_2, \beta_2) = 1$). We want to find a composition of transvections such that α_1 will be mapped to α_2 and β_1 will be mapped to β_2. By Lemma 4.5, there is $T_1 \in \mathcal{T}$ such that $T_1(\alpha_1) = \alpha_2$. Now our aim is to find $T_2 \in \mathcal{T}$ such that

$$T_2(\alpha_2) = \alpha_2 \quad \text{and} \quad T_2(T_1(\beta_1)) = \beta_2$$

Case 1: $\omega(T_1(\beta_1), \beta_2) \neq 0$

Let $a = \frac{1}{\omega(T_1(\beta_1), \beta_2)}$ and $u = T_1(\beta_1) - \beta_2$, then

$$\begin{aligned}
T_{u,a}(\alpha_2) &= \alpha_2 + a\omega(\alpha_2, u)u \\
&= \alpha_2 + \frac{\omega(\alpha_2, u)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_2, T_1(\beta_1) - \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_2, T_1(\beta_1)) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(T_1(\alpha_1), T_1(\beta_1)) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{\omega(\alpha_1, \beta_1) - \omega(\alpha_2, \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2 + \frac{0}{\omega(T_1(\beta_1), \beta_2)}u \\
&= \alpha_2
\end{aligned}$$

and

$$\begin{aligned}
T_{u,a}(T_1(\beta_1)) &= T_1(\beta_1) + a\omega(T_1(\beta_1), u)u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), u)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), T_1(\beta_1) - \beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) + \frac{\omega(T_1(\beta_1), -\beta_2)}{\omega(T_1(\beta_1), \beta_2)}u \\
&= T_1(\beta_1) - u \\
&= T_1(\beta_1) - (T_1(\beta_1) - \beta_2) \\
&= \beta_2.
\end{aligned}$$

Let this transvection be denoted by T_2. In this case, $T_2T_1 \in \mathcal{T}$ and $T_2T_1(\alpha_1) = \alpha_2$, $T_2T_1(\beta_1) = \beta_2$.

Case 2: $\omega(T_1(\beta_1), \beta_2) = 0$

By the similar way we have made before, we want to find a z such that $\omega(T_1(\beta_1), z) \neq 0$ and $\omega(\beta_2, z) \neq 0$. For this $\alpha_2 + T_1(\beta_1)$ is very good choice since

$$\begin{aligned}
&\omega(T_1(\beta_1), \alpha_2 + T_1(\beta_1)) = \omega(T_1(\beta_1), T_1(\alpha_1)) = -1 \quad \text{and} \\
&\omega(\beta_2, \alpha_2 + T_1(\beta_1)) = \omega(\beta_2, \alpha_2) = -1.
\end{aligned}$$

We want to find the transvections T_3 and T_4 such that

$$\begin{array}{ll}
T_3(\alpha_2) = \alpha_2 \qquad & T_3(T_1(\beta_1)) = T_1(\beta_1) + \alpha_2 \\
T_4(\alpha_2) = \alpha_2 \qquad & T_4(T_1(\beta_1) + \alpha_2) = \beta_2
\end{array}$$

Finally, we will see that $T_4T_3T_1 \in \mathcal{T}$ and $T_4T_3T_1(\alpha_1) = \alpha_2$, $T_4T_3T_1(\beta_1) = \beta_2$.

For T_3, let a be -1 and u be $-\alpha_2$, then

$$\begin{aligned}
T_3(\alpha_2) &= \alpha_2 + (-1).\omega(\alpha_2, -\alpha_2)(-\alpha_2) \\
&= \alpha_2 \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
T_3(T_1(\beta_1)) &= T_1(\beta_1) + (-1).\omega(T_1(\beta_1), -\alpha_2)(-\alpha_2) \\
&= T_1(\beta_1) + (-1).\omega(T_1(\beta_1), -T_1(\alpha_1))(-\alpha_2) \\
&= T_1(\beta_1) + (-1).(-1).(-1)(-\alpha_2) \\
&= T_1(\beta_1) + \alpha_2.
\end{aligned}$$

For T_4, let a be 1 and u be $\alpha_2 + T_1(\beta_1) - \beta_2$, then

$$\begin{aligned}
T_4(\alpha_2) &= \alpha_2 + \omega(\alpha_2, \alpha_2 + T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + \omega(\alpha_2, T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + [\omega(T_1(\alpha_1), T_1(\beta_1)) - \omega(\alpha_2, \beta_2)](\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + 0.(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
T_4(\alpha_2 + T_1(\beta_1)) &= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2 + T_1(\beta_1), \alpha_2 + T_1(\beta_1) - \beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2 + T_1(\beta_1), -\beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + [\omega(\alpha_2, -\beta_2) + \omega(T_1(\beta_1), -\beta_2)](\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + \omega(\alpha_2, -\beta_2)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \alpha_2 + T_1(\beta_1) + (-1)(\alpha_2 + T_1(\beta_1) - \beta_2) \\
&= \beta_2.
\end{aligned}$$

□

Remark 4.7. *Let T_1 be a transvection, then there is T_2 such that $T_2T_1 = id$. If $T_1 = T_{a,u}$, then T_2 must be $T_{-a,u}$. Assume θ_1 is an element of $\mathcal{T}$ and θ_2 is inverse of θ_1. So if $\theta_1 = T_1T_2...T_n$, then θ_2 must be $T_n^{-1}T_{n-1}^{-1}...T_1^{-1}$. Thus, every element in $\mathcal{T}$ has an inverse in $\mathcal{T}$.*

Proof of Theorem 4.4. If $\Psi \in Sp(V)$ and $\{u_1, v_1, u_2, v_2, ..., u_n, v_n\}$ is a symplectic basis of V, then there is an element in $\mathcal{T}$, let it be denoted by θ_1, such that $\theta_1(\Psi u_1) = u_1$ and $\theta_1(\Psi v_1) = v_1$. Let $W_1 := span\{u_1, v_1\}$. $\theta_1\Psi$ is identity on W_1. W_1^ω is symplectic vector space with basis $\{u_2, v_2, ..., u_n, v_n\}$. So, $\theta_1\Psi \in Sp(W_1^\omega)$.

Assume that there are similar k elements such that $\theta_k...\theta_1\Psi$ is identity on $W_k := span\{u_1, ..., u_k, v_1, ..., v_k\}$. So, $\theta_k...\theta_1\Psi \in \mathrm{Sp}(W_k^\omega)$. There is an element in $\mathcal{T}$, $\theta_{k+1} \in \mathrm{Sp}(W_k^\omega)$ mapping Ψu_{k+1} to u_{k+1}, and Ψv_{k+1} to v_{k+1}. Notice that any vector in W_k^ω is symplectically orthogonal to W_k, this means hyperplanes of the transvections in W_k^ω cover W_k. This implies that these transvections are identity on W_k. So $\theta_{k+1}\theta_k...\theta_1\Psi$ is identity on $W_{k+1} := \{u_1, ..., u_{k+1}, v_1, ..., v_{k+1}\}$. By induction, we showed $\theta_n\theta_{n-1}...\theta_1\Psi$ is identity. By Remark 4.7, $\Psi \in \mathcal{T}$. □

Definition 4.8. *The transvections T_{-1,u_1} and T_{1,u_2} are called positive and negative transvections respectively.*

Although the signs of coefficients and the names of this transvections are opposite, there is a fair reason for it. We will see this reason later.

Theorem 4.9. *Any transvection on the symplectic vector space is a positive transvection or a negative transvection.*

Proof. If $a = 0$, $T_{a,u}$ is identity, so not transvection. Assume that a is nonzero.

Let $a > 0$ and $u \in V$, then

$$T_{a,u}(v) = v + a\omega(v,u)u = v + \omega(v, \sqrt{a}u)\sqrt{a}u = T_{1,\sqrt{a}u}(v)$$

Similarly,

$$T_{-a,u}(v) = v - a\omega(v,u)u = v - \omega(v, \sqrt{-a}u)\sqrt{-a}u = T_{-1,\sqrt{-a}u}(v)$$

Thus, the proof is completed. □

Theorem 4.10. *The symplectic group is generated by positive transvections.*

Proof. Theorem 4.4 and 4.9 implies that $\mathrm{Sp}(V)$ is generated by positive and negative transvections. Remark 4.8 says that negative transvections are inverse of positive transvections. □

Proof of Theorem 2.4. By Theorem 4.10, determinant of any symplectomorphism is product of the determinants of transvections. It is well known that determinant equals to product of eigenvalues. Let us find an eigenvalue of a positive transvection.

$$T_{-1,u}(x) = \lambda x \to x - \omega(x,u)u = \lambda x \to \omega(u,x)u = (\lambda - 1)x$$

From this equality, we conclude that x must be a multiple of u. In that case, $\omega(u,x)$ must be zero, so $\lambda = 1$. Assume that u and x are linearly independent. This implies that the coefficients of that vectors must be zero. Thus, in all cases, there is only one eigenvalue which is zero. Hence determinant of positive transvections is 1, so all symplectomorphisms. □

Proof of Theorem 3.4. Dehn twists generate mapping class group of Σ_g. The symplectic representation is related to the actions of mapping class elements on homology. Therefore we should look at the action of Dehn twists on homology.

By [3, Proposition 6.3],

$$D_\gamma(a) = a - \hat{i}(a, [\gamma]).[\gamma] \text{ where } a \in \mathrm{H}_1(\Sigma_g, \mathbb{Z})$$

This is exactly what positive transvections do in a symplectic vector space. We get the result that a right Dehn twist about a nonseparating curve is a positive transvection and left Dehn twist about nonseparating curve is a negative transvection on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$.

Theorem 4.10 says that the images of all Dehn twists about nonseparating curves generate $\mathrm{Sp}(V)$. □

References

[1] Grove, L. C., *Classical groups and geometric algebra*, Vol. 39 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2002.

[2] McDuff, D. and D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995, Oxford Science Publications.

[3] Farb, B. and D. Margalit, *A primer on mapping class groups*, Vol. 49 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, 2012.

[4] Humphries, S. P., "Generators for the mapping class group", *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, Vol. 722 of *Lecture Notes in Mathematics*, pp. 44–47, Springer, Berlin, 1979.