Discrete Mathematics

Session III

The Principle of Inclusion and Exclusion

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March, 2020

Introduction

Consider the equation $x_1 + x_2 + x_3 + x_4 = 20$ where x_i's are nonnegative integers not greater than 8 ($0 \leq x_i \leq 8$.) How many solutions does this equation have? Example solutions are $(6, 4, 6, 4)$, $(2, 8, 8, 2)$, and $(0, 8, 7, 5)$.

Note that transforming the variables does not work here (The problem can directly be reduced to the problem of combination with repetitions only if we can convert the given equation to the one where the variables are nonnegative integers.)

We extend the theory (of counting) so that we can solve some problems that otherwise cannot be solved. The extended theory may also help us solve some problems more easily. One such an extension is the introduction of ***the principle of inclusion and exclusion***.

Using the principle of inclusion and exclusion, one can solve some important and useful counting problems; **the number of onto functions** with finite domains and codomains and the values of **Euler's phi (totient) function** are examples of such problems.

The principle involves some basic knowledge of set theory.

Setting up the Scene

Let S be a finite set whose cardinality is N, that is $|S| = N$. Assume also that $c_1, c_2, \ldots, c_t$ are t ***conditions***. Each condition is satisfied by none or some of the elements of S.

For example, $S = \{1, 2, 4, 5, 12, 18, 19, 21, 25, 37, 91, 96\}$, $t = 3$, c_1: The element is divisible by 2, c_2: The element is divisible by 3, and c_3: The element is a prime number.

For $1 \le i \le t$, $N(c_i)$ denotes the number of elements of S that satisfy c_i. Similarly, $N(c_i c_j)$ with $i \ne j$ denotes the number of elements of S that satisfy both c_i and c_j. More generally, $N(c_{i_1} c_{i_2} \ldots c_{i_k})$ is the number of elements of S that satisfy each of the conditions c_{i_1}, c_{i_2}, …, and c_{i_k} ($1 \le i_1, \ldots, i_k \le t$.)

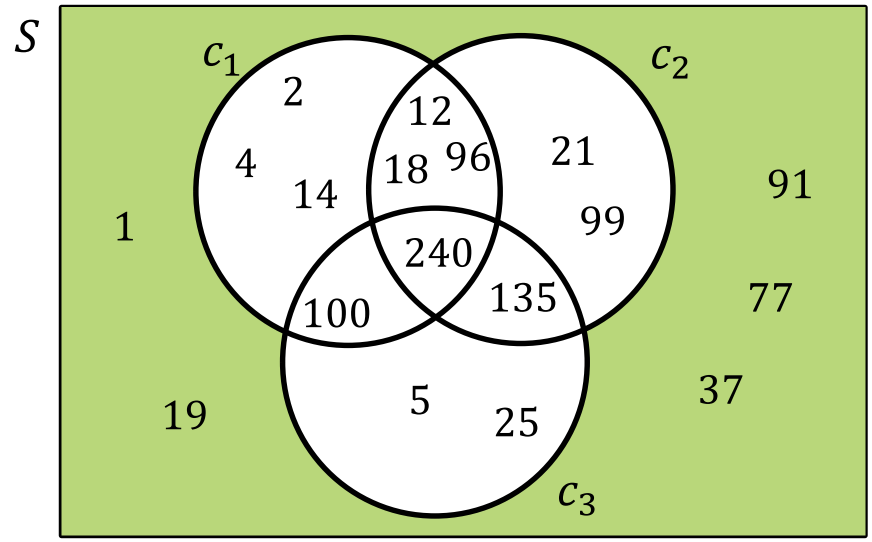
In the above example, $N(c_1) = 5$, $N(c_2) = 4$, $N(c_3) = 4$, $N(c_1 c_2) = 3$, $N(c_1 c_3) = 1$, $N(c_2 c_3) = 0$, and $N(c_1 c_2 c_3) = 0$.

For $1 \le i \le t$, $N(\bar{c}_i)$ denotes the number of elements of S that do not satisfy c_i. Evidently, $N(\bar{c}_i) = N - N(c_i)$. For $1 \le i, j \le t$ with $i \ne j$, $N(\bar{c}_i \bar{c}_j)$ denotes the number of elements of S that do not satisfy either of the conditions c_i or c_j (satisfy neither c_i nor c_j.) Similarly, $N(\bar{c}_{i_1} \bar{c}_{i_2} \ldots \bar{c}_{i_k})$ denotes the number of elements of S that satisfy none of the conditions c_{i_1}, c_{i_2}, …, and c_{i_k}.

In the example above, $N(\bar{c}_1) = 7$, $N(\bar{c}_2) = 8$, $N(\bar{c}_3) = 8$, $N(\bar{c}_1 \bar{c}_2) = 6$, $N(\bar{c}_1 \bar{c}_3) = 4$, $N(\bar{c}_2 \bar{c}_3) = 4$, and $N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 3$.

An Example

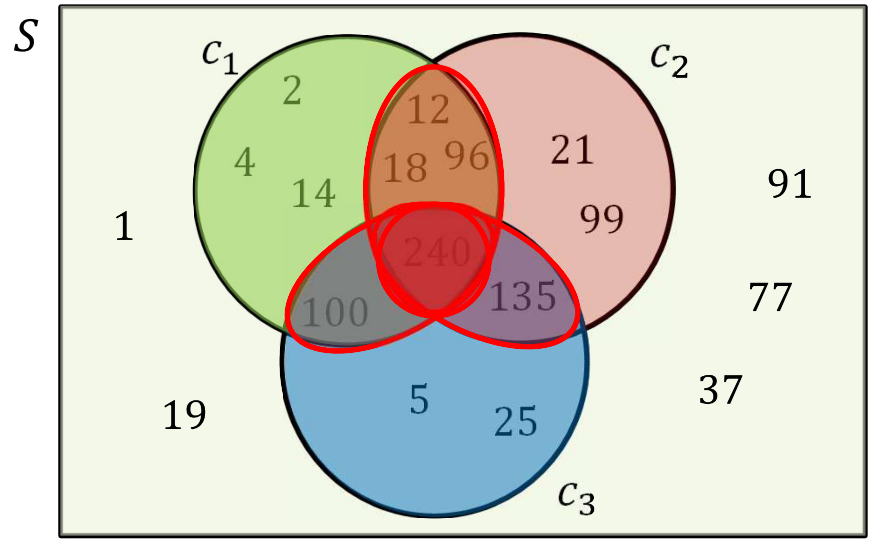
Consider the set $S = \{1, 2, 4, 5, 12, 14, 18, 19, 21, 25, 37, 77, 91, 96, 99, 100, 135, 240\}$ and the conditions c_1: The element is divisible by 2, c_2: The element is divisible by 3, and c_3: The element is divisible by 5. How many elements of S satisfy none of the three conditions? That is, how to calculate $N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$?



As seen, $N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 5$. It is the number of elements of S in the green area.

How can one calculate $N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$ in a systematic manner?

An Example (Ctd.)

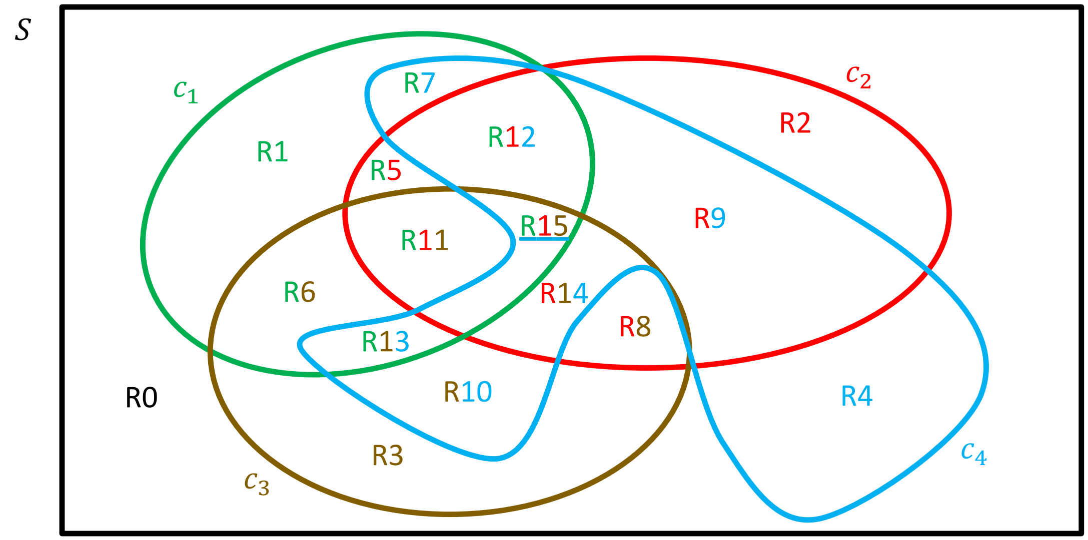


$$N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3)$$

$$18 - 8 - 7 - 5 + 4 + 2 + 2 - 1 = 5$$

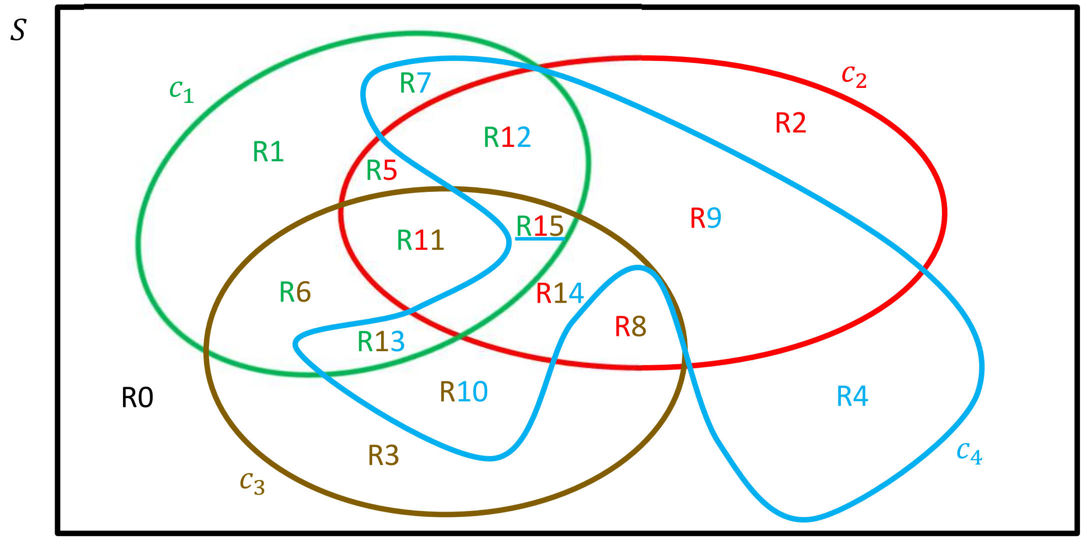
The Case $t = 4$

Consider the case $t = 4$ and calculate $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$.



$N(c_1)$:	R1+R5+R6+R7+R11+R12+R13+R15	$N(c_2c_4)$:	R9+R12+R14+R15
$N(c_2)$:	R2+R5+R8+R9+R11+R12+R14+R15	$N(c_3c_4)$:	R10+R13+R14+R15
$N(c_3)$:	R3+R6+R8+R10+R11+R13+R14+R15	$N(c_1c_2c_3)$:	R11+R15
$N(c_4)$:	R4+R7+R9+R10+R12+R13+R14+R15	$N(c_1c_2c_4)$:	R12+R15
$N(c_1c_2)$:	R5+R11+R12+R15	$N(c_1c_3c_4)$:	R13+R15
$N(c_1c_3)$:	R6+R11+R13+R15	$N(c_2c_3c_4)$:	R14+R15
$N(c_1c_4)$:	R7+R12+R13+R15	$N(c_1c_2c_3c_4)$:	R15
$N(c_2c_3)$:	R8+R11+R14+R15		

The Case $t = 4$ (Ctd.)



$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N$$
$$- N(c_1) - N(c_2) - N(c_3) - N(c_4)$$
$$+ N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)$$
$$- N(c_1c_2c_3) - N(c_1c_2c_4) - N(c_1c_3c_4) - N(c_2c_3c_4)$$
$$+ N(c_1c_2c_3c_4)$$

The Principle of Inclusion and Exclusion

Theorem (The Principle of Inclusion and Exclusion). Assume that S is a finite set with $|S| = N$ and that $c_1, c_2, \ldots, c_t$ are t conditions. Then, the number of elements of S that satisfy none of the t conditions, denoted $\overline{N}$ or $N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_t)$, is given by

$$N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_i c_j) - \sum_{1\le i<j<k\le t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \ldots c_t).$$

Proof. The proof will be given later.

To simplify the formula, we use the notation S_0 for N and S_k for

$$\sum_{1\le i_1<i_2<\cdots<i_k\le t} N(c_{i_1}\, c_{i_2} \ldots c_{i_k})$$

for every $1 \le k \le t$.

Thus,

$$\overline{N} = N(\bar{c}_1\bar{c}_2 \ldots \bar{c}_t) = S_0 - S_1 + S_2 - \cdots + (-1)^t S_t = \sum_{k=0}^{t} (-1)^k S_k.$$

- The number of elements of S that satisfy at least one of the t conditions is $N - \overline{N}$.
- The number of summands in S_k is $\binom{t}{k}$.

Applications of the Principle

$$N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \dots c_t).$$

Example 1. Determine the number of positive integers not greater than 1000 that are not divisible by 2, 3, or 5.

Solution. Take $S = \{1, 2, 3, \dots, 1000\}$ and consider the three conditions c_1: the element is divisible by 2, c_2: the element is divisible by 3, and c_3: the element is divisible by 5. The answer to the problem is $N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$.

According to the principle of inclusion and exclusion, we have

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3)$$

$$= N - \big(N(c_1) + N(c_2) + N(c_3)\big) + \big(N(c_1c_2) + N(c_1c_3) + N(c_2c_3)\big) - N(c_1c_2c_3)$$

$$= 1000 - \left(\left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor\right) + \left(\left\lfloor\frac{1000}{6}\right\rfloor + \left\lfloor\frac{1000}{10}\right\rfloor + \left\lfloor\frac{1000}{15}\right\rfloor\right) - \left\lfloor\frac{1000}{30}\right\rfloor$$

$$= 1000 - (500 + 333 + 200) + (166 + 100 + 66) - 33$$

$$= 266$$

Applications of the Principle (Ctd.)

$$N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \ldots c_t).$$

Example 2. Let A and B be finite sets and $|A| = m \ge n = |B|$. Determine the number of onto functions from A to B.

Solution. Let S be the set of all functions from A to B. We have $|S| = n^m$ (why?) Assume also that $B = \{b_1, b_2, \ldots b_n\}$. For $1 \le i \le n$, define c_i to be the condition

b_i **is not** in the range of the function.

The set of onto functions from A to B equals $N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_n)$, that is,

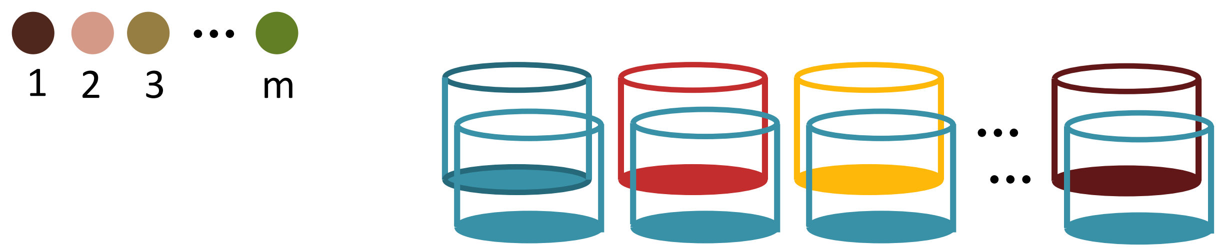
$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \cdots + (-1)^n \binom{n}{n}(n-n)^m.$$

It can be written as

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m .$$

Applications of the Principle (Ctd.)

The number of ways that one can put m distinct objects into n distinct containers such that no container remains empty equals the number of onto functions from A to B where $|A| = m$ and $|B| = n$.



What about **indistinguishable** containers?

$$\frac{1}{n!}\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^{m}.$$

This number, denoted $S(m,n)$, is called the ***Stirling number of the second kind***.

The number of onto functions from A to B, where $|A| = m \geq |B| = n$, is $n!\,S(m,n)$.

The equality $S(m,n) = S(m-1,n-1) + n \cdot S(m-1,n)$ holds.

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - \sum_{1\le i\le 4} N(c_i) + \sum_{1\le i<j\le 4} N(c_ic_j) - \sum_{1\le i<j<k\le 4} N(c_ic_jc_k) + N(c_1c_2c_3c_4).$$

Example 3. Find the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = 20$ where x_i's are nonnegative integers not greater than 8 ($0 \le x_i \le 8$.)

Solution. Let S be the set of all nonnegative integer solutions to the equation. Note that each solution to the equation is indeed a quadruple such as $(1, 9, 0, 10)$ and $(4, 5, 8, 3)$. It is immediate that $N = |S| = \binom{4+20-1}{20} = \binom{23}{20} = 1{,}771$. We define conditions c_i as $x_i \ge 9$ for $1 \le i \le 4$. A solution to the equation where x_i's are nonnegative integers not greater than 8 is then an element of S that satisfies none of the conditions, that is, $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$.

Now, we obtain $N(c_1)$, for example. It is the number of elements of S where the first component is greater than 8. That is, the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$ where $x_1 \ge 9$ and $x_2, x_3, x_4 \ge 0$. Thus, $N(c_1)$ is the number of nonnegative integer solutions to $y_1 + x_2 + x_3 + x_4 = 11$, that is, $\binom{14}{11}$. $N(c_1c_2)$ is The number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$ where $x_1, x_2 \ge 9$ and $x_3, x_4 \ge 0$, which is $\binom{5}{2}$. Therefore, the answer is

$$\binom{23}{20} - \binom{4}{1}\binom{14}{11} + \binom{4}{2}\binom{5}{2} - \binom{4}{3}\cdot 0 + \binom{4}{4}\cdot 0 = 495.$$

Applications of the Principle (Ctd.)

Two integers are said to be **relatively prime**, also called **coprime**, if their only common factor (a positive integer that divides both of them) is 1. In other words, two integers a and b are relatively prime if theirs **greatest common divisor** (a, b) is 1; The pairs $(4, 9)$, (100, 91), and $(7, 1024)$ are example pairs of relatively prime integers. **Euler's totient function**, also called **Euler's phi function**, denoted ϕ, takes every positive integer n to the number of positive integers not greater than n that are relatively prime to n. Consider the following table showing the values of $\phi(n)$ for $1 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$\phi(n)$	1	1	2	2	4	2	6	4	6	4

For a prime number p, we have $\phi\left(p^k\right) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$ for every positive integer k.

The Fundamental Theorem of Arithmetic. Every positive integer $n > 1$ can be uniquely represented as a product of one or more primes, irrespective of the order of the factors $(n = p_1^{k_1} p_1^{k_1} \cdots p_m^{k_m}.)$

For example, $100 = 2^2 \cdot 5^2$, $425 = 5^2 \cdot 17^1$, and $1524 = 2^2 \cdot 3 \cdot 127$.

$$N - \sum_{1 \le i \le m} N(c_i) + \sum_{1 \le i < j \le m} N(c_i c_j) - \sum_{1 \le i < j < k \le m} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \ldots c_m).$$

Example 4. Find a formula for $\phi(n)$ where $n > 1$.

Solution. By the fundamental theorem of arithmetic, a positive integer n greater than 1 can be uniquely expressed as a product of primes, say $n = p_1^{k_1} p_1^{k_1} \cdots p_m^{k_m}$. Take $S = \{1, 2, 3, \ldots, n\}$ and, for $1 \le i \le m$, define conditions c_i as "being divisible by p_i." Then, $\phi(n)$ is equal to the number of elements of S that satisfy none of the m conditions c_i. That is, $N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_m)$. We have,

$$N(c_{i_1} c_{i_2} \ldots c_{i_r}) = \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_r}},$$

for $1 \le i_1 < i_2 < \cdots < i_r \le m$. Thus,

$$\begin{aligned}
\phi(n) &= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_m}\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + \frac{n}{p_{m-1} p_m}\right) \\
&\quad - \left(\frac{n}{p_1 p_2 p_3} + \frac{n}{p_1 p_2 p_4} + \cdots + \frac{n}{p_{m-2} p_{m-1} p_m}\right) + \cdots + (-1)^m \frac{n}{p_1 p_2 \ldots p_m} \\
&= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_m}\right) \\
&= n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)
\end{aligned}$$

A Proof of the Principle

Theorem (The Principle of Inclusion and Exclusion). Assume that S is a finite set with $|S| = N$ and that $c_1, c_2, \dots, c_t$ are t conditions. Then, the number of elements of S that satisfy none of the t conditions, denoted $\overline{N}$ or $N(\bar{c}_1\bar{c}_2 \dots \bar{c}_t)$, is given by

$$N(\bar{c}_1\bar{c}_2 \dots \bar{c}_t) = N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_i c_j) - \sum_{1\le i<j<k\le t} N(c_i c_j c_k) + \dots + (-1)^t N(c_1 c_2 \dots c_t).$$

Proof. Assume that x is an arbitrary element of S. Such an element either is counted once in the left-hand side of the equality or it is not counted in the left side at all (it does not contribute to $N(\bar{c}_1\bar{c}_2 \dots \bar{c}_t)$.) We show that this element is counted the same number of times in the right side of the equality as it is counted in the left side. We have two cases: <u>x satisfies none of the conditions</u> and <u>x satisfies exactly r of the t conditions</u> where $1 \le r \le t$. If x satisfies none of the conditions, it is counted once in the left side. In the right side, it is counted once in N and zero times in other terms, as it does not satisfy any of the conditions. Thus, x is counted the same number of times (once) in both sides of the equality. If x satisfies exactly r of the t conditions (the second case,) it is not counted in left side of the equality. In the right side, it is counted once in N, $\binom{r}{1}$ times in $\sum_{1\le i\le t} N(c_i)$, $\binom{r}{2}$ times in $\sum_{1\le i<j\le t} N(c_i c_j)$, and in general, $\binom{r}{k}$ times in $\sum_{1\le i_1<i_2<\dots<i_k\le t} N(c_{i_1} c_{i_2} \dots c_{i_k})$. Thus, the number of times that such an element is counted in the right-hand side of the equality is

$$1 - \binom{r}{1} + \binom{r}{2} - \dots + (-1)^r \binom{r}{r} = (1-1)^r = 0.$$

Textbook: Ralph P. Grimaldi, Discrete and Combinatorial Mathematics

You may begin doing exercises of Chapter 8.