

CE203

ساختمان داده ها و الگوریتم ها

سجاد شیرعلی شمرضا
پاییز 1400

جلسه چهارم: مرتب سازی درجی و ادغامی

شنبه، 10 مهر 1400

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 2.1، 2.3
- امتحانک اول
 - دوشنبه هفته آینده، 18 اسفند
 - به صورت برخط (آنلاین)
 - از طریق سامانه دروس دانشگاه
 - در طی ساعت کلاس
 - به صورت انفرادی!

مرتب سازی درجی

الگوریتم، اثبات درستی، تحلیل زمان اجرا

THE SORTING TASK

INPUT: a list of n elements (for today, we'll assume all elements are distinct)

3	2	6	8	1	5	4	7
---	---	---	---	---	---	---	---



1	2	3	4	5	6	7	8
---	---	---	---	---	---	---	---

OUTPUT: a list with those n elements in sorted order!

INSERTION SORT: PSEUDOCODE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

```
InsertionSort(A):  
  for i in range(1, len(A)):  
    cur_value = A[i]  
    j = i - 1  
    while j >= 0 and A[j] > cur_value:  
      A[j+1] = A[j]  
      j -= 1  
    A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

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            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



At the start, our growing sorted list only has one element (the first element):
3 is in its “correct” place within the growing list (shaded)

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```


INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Now we look at $A[1] = 2$. We'll move 2 into its “correct” place in the growing sorted list.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

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Now we look at $A[1] = 2$. We'll move 2 into its “correct” place in the growing sorted list.

In other words, move 2 towards the start of the list until it hits something smaller (or if it can't go any further).

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Now we look at $A[2] = 5$. We'll move 5 into its “correct” place in the growing sorted list.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Now we look at $A[2] = 5$. We'll move 5 into its “correct” place in the growing sorted list.

It's already where it should be in the growing sorted list, so we don't need to move it anywhere. Moving on!

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Now we look at $A[3] = 1$. We'll move 1 into its “correct” place in the growing sorted list.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Now we look at $A[3] = 1$. We'll move 1 into its “correct” place in the growing sorted list.
We move it all the way to the front, since that's its “correct” position in this growing sorted list.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Finally, we look at $A[4] = 4$. We'll move 4 into its “correct” place in the growing sorted list.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.



Finally, we look at $A[4] = 4$. We'll move 4 into its “correct” place in the growing sorted list.
It just needs to squeeze in right before 5.

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```


INSERTION SORT: EXAMPLE

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And, that's it! We've finished performing Insertion Sort on this example array of five elements.
Now we ask... does it work?

```
InsertionSort(A):  
    for i in range(1, len(A)):  
        cur_value = A[i]  
        j = i - 1  
        while j >= 0 and A[j] > cur_value:  
            A[j+1] = A[j]  
            j -= 1  
        A[j+1] = cur_value
```



سوال؟

اثبات درستی مرتب سازی درجی

آیا واقعا ورودی را مرتب میکند؟

INSERTION SORT: DOES IT WORK?

Since the algorithm isn't too complex, it might feel pretty obvious... but it won't be so obvious later, so let's take some time now to see how to prove the correctness of this algorithm rigorously..

3	2	5	1	4
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We verified Insertion Sort worked for this particular input list.
However, we need to prove that the algorithm works for *all* possible input lists.

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3	2	5	1	4
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We verified Insertion Sort worked for this particular input list.
However, we need to prove that the algorithm works for *all* possible input lists.

HERE'S WHAT WE FOCUS ON:

Insertion Sort is an *iterative* algorithm - what does each iteration promise?

INSERTION SORT: DOES IT WORK?

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Each iteration of the algorithm promises to add one more element to the sorted region.

In other words: by the end of iteration i , we're guaranteed that the first $i+1$ elements in the array are sorted.

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THIS IS A JOB FOR: PROOF BY INDUCTION!

4 INGREDIENTS OF INDUCTION

INDUCTIVE HYPOTHESIS (IH)

This is a statement that's basically what you're trying to prove, except it's written in terms of some variable (e.g. i). We need to set up the inductive hypothesis clearly, and our goal in the next three steps is to prove that the IH holds for a whole *range* of values for i .

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BASE CASE

First establish that the inductive hypothesis holds for some base case value(s) of i .

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First establish that the inductive hypothesis holds for some base case value(s) of i .

INDUCTIVE STEP (*weak induction version*)

Next, assume that the inductive hypothesis holds when i takes on some value k .
Now prove that the IH holds as well when i takes on the value $k+1$.

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CONCLUSION

By induction, conclude that the IH holds across the range of i you're dealing with.

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BASE CASE

First establish that the inductive hypothesis holds for some base case value(s) of i .

INDUCTIVE STEP (*strong/complete induction version*)

Next, assume that the IH holds when i takes on any value *between* [base case value(s)] and *some number* k . Now prove that the IH holds as well when i takes on the value $k+1$.

CONCLUSION

By induction, conclude that the IH holds across the range of i you're dealing with.

INSERTION SORT: INDUCTION PROOF

INDUCTIVE HYPOTHESIS (IH)

After iteration i of the outer for-loop, $A[:i+1]$ is sorted.

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INDUCTIVE STEP (*weak induction*)

Let k be an integer, where $0 < k < n$. Assume that the IH holds for $i = k-1$, so $A[:k]$ is sorted after the $(k-1)^{\text{th}}$ iteration. We want to show that the IH holds for $i = k$, i.e. that $A[:k+1]$ is sorted after the k^{th} iteration.

Let j^* be the largest position in $\{0, \dots, k-1\}$ such that $A[j^*] < A[k]$. Then, the effect of the inner while-loop is to turn:

$[A[0], A[1], \dots, A[j^*], \dots, A[k-1], \mathbf{A[k]}]$ into $[A[0], A[1], \dots, A[j^*], \mathbf{A[k]}, A[j^*+1] \dots, A[k-1]]$

We claim that the second list on the right is sorted. This is because $A[k] > A[j^*]$, and by the inductive hypothesis, we have $A[j^*] \geq A[j]$ for all $j \leq j^*$, so $A[k]$ is larger than everything positioned before it. Similarly, we also know that $A[k] \leq A[j^*+1] \leq A[j]$ for all $j \geq j^*+1$, so $A[k]$ is also smaller than everything that comes after it. Thus, $A[k]$ is in the right place, and all the other elements in $A[:k+1]$ were already in the right place.

Thus, after the k^{th} iteration completes, $A[:k+1]$ is sorted, and this establishes the IH for k .

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CONCLUSION

By induction, we conclude that the IH holds for all $0 \leq i \leq n-1$. In particular, after the algorithm ends, $A[:n]$ is sorted.

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After iteration 0 of the outer loop (i.e. start of algorithm), the list $A[:1]$ is sorted (only 1 element). Thus, IH holds for $i = 0$.

INDUCTIVE STEP (*weak induction*)

Let k be an integer such that $0 \leq k < n-1$. Assume that the IH holds for all $i \leq k-1$. We want to show that the IH holds for $i = k$.

Let j^* be the index of the element $A[k]$ in the sorted subarray $A[:k]$. Then, $j^* \leq k$.

By the IH, the subarray $A[:k]$ is sorted. Thus, $A[j^*] \leq A[k]$.

We claim that $A[j^*+1] \leq A[k]$. If $j^* = k$, then $A[j^*+1]$ is the first element in the unsorted subarray $A[k+1:n]$. Thus, $A[j^*+1] \leq A[k]$.

If $j^* < k$, then $A[j^*+1]$ is the first element in the sorted subarray $A[j^*+1:k]$. Thus, $A[j^*+1] \leq A[k]$.

Thus, $A[j^*+1] \leq A[k]$. Therefore, $A[k]$ is in the right place, and the growing list on the left of A is still going to be properly sorted.

Thus, after the k^{th} iteration completes, $A[:k+1]$ is sorted, and this establishes the IH for k .

CONCLUSION

By induction, we conclude that the IH holds for all $0 \leq i \leq n-1$. In particular, after the algorithm ends, $A[:n]$ is sorted.

INSERTION SORT: DOES IT WORK?

We just used induction to prove that the Insertion Sort algorithm correctly produces a sorted array given *any input array of length n* .

(This is also what we mean by worst case analysis - even if a “bad guy” comes up with a worst-case input for our algorithm, we’ve proven that our algorithm will work).



سوال؟

زمان اجرای مرتب سازی درجی

چقدر سریع است؟

INSERTION SORT: IS IT FAST?

THE POINT OF ASYMPTOTIC NOTATION

suppress constant factors and lower-order terms

too system dependent

irrelevant for large inputs

FROM SATURDAY!

- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.
 - We want to reason about high-level algorithmic approaches rather than lower-level details

INSERTION SORT: IS IT FAST?

Instead of counting every little operation, we can think about:

How many iterations take place

How much work happens within each iteration

InsertionSort(A):

```
for i in range(1, len(A)):
    cur_value = A[i]
    j = i - 1
    while j >= 0 and A[j] > cur_value:
        A[j+1] = A[j]
        j -= 1
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        j -= 1
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```

At most n
inner while-loop
iterations

At most n
outer for-loop
iterations

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        A[j+1] = A[j]
        j -= 1
    A[j+1] = cur_value
```

At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

We have n for-loop iterations. Each iteration does $O(n)$ work.

(Each for-loop iteration performs an inner-while loop which iterates up to n times and does $O(1)$ work in each iteration).

INSERTION SORT: IS IT FAST?

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```

At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

OVERALL RUNTIME OF INSERTION SORT: $O(n^2)$

INSERTION SORT: IS IT FAST?

Instead of counting every little operation, we can think about:

How many times we place

How much work per iteration

Insertion

for

for

for

while

while

while

while

while

while

while

while

THE QUESTION IS...
**CAN WE DO
BETTER?**

At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

_value:

while

while

while

while

while

while

OVERALL RUNTIME OF INSERTION SORT: **$O(n^2)$**



سوال؟

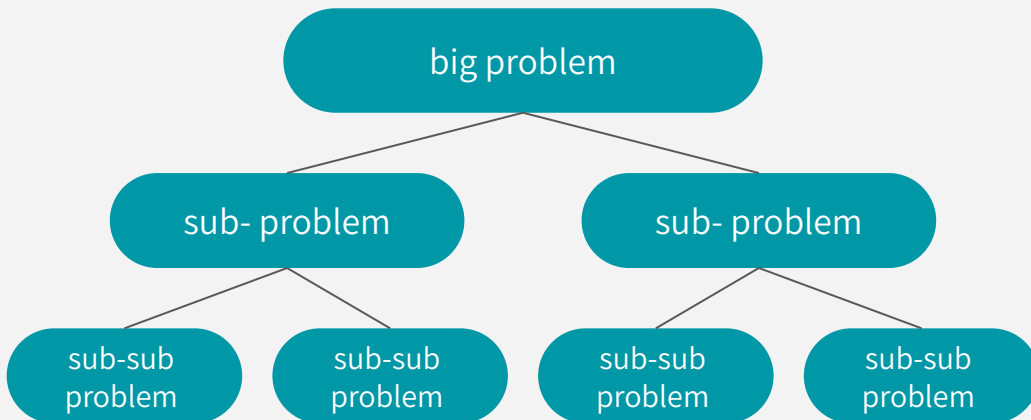
مرتب سازی ادغامی

الگوریتم، اثبات درستی، تحلیل زمان اجرا

MERGESORT

FROM PREVIOUS WEEKS!

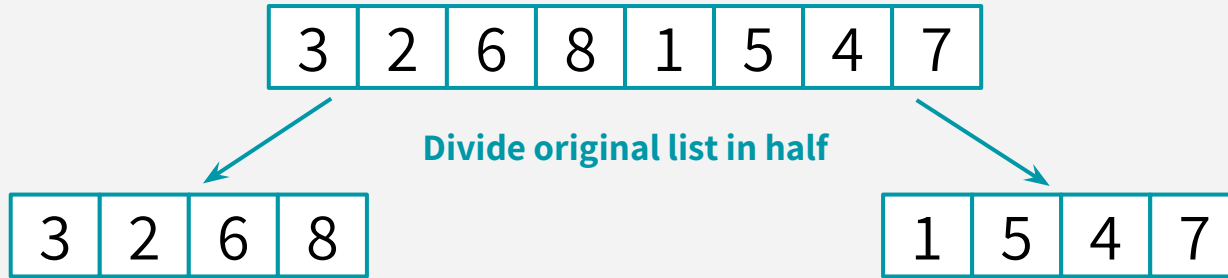
- **DIVIDE-AND-CONQUER: an algorithm design paradigm**
 1. break up a problem into smaller subproblems
 2. solve those subproblems *recursively*
 3. combine the results of those subproblems to get the overall answer



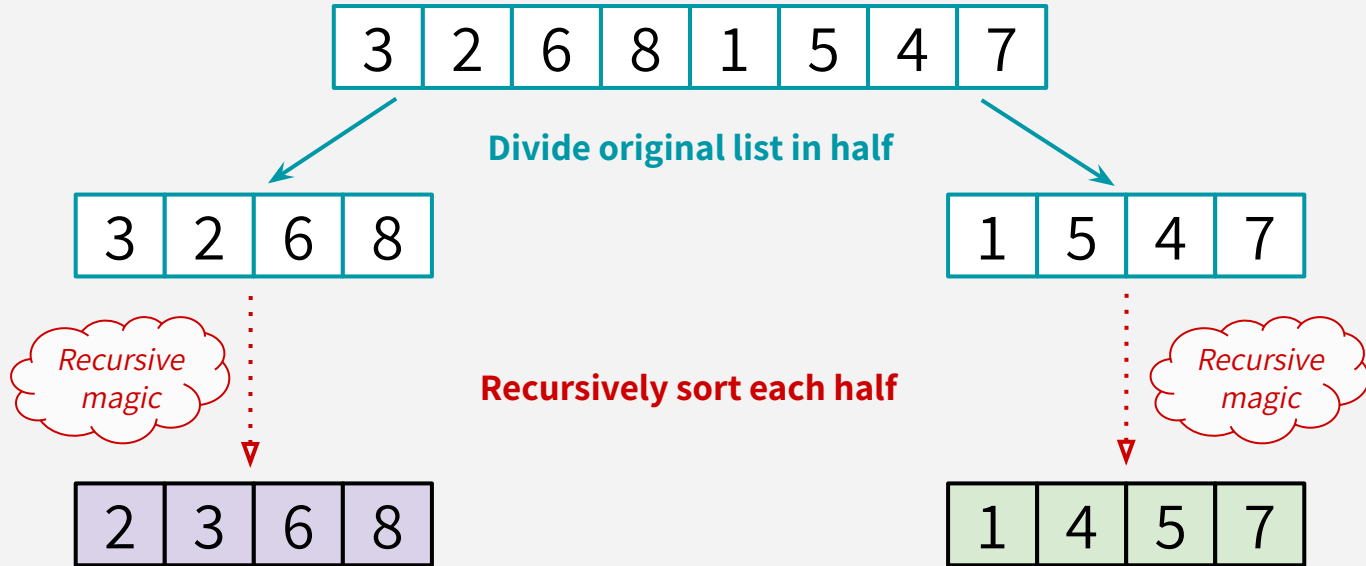
MERGESORT

3	2	6	8	1	5	4	7
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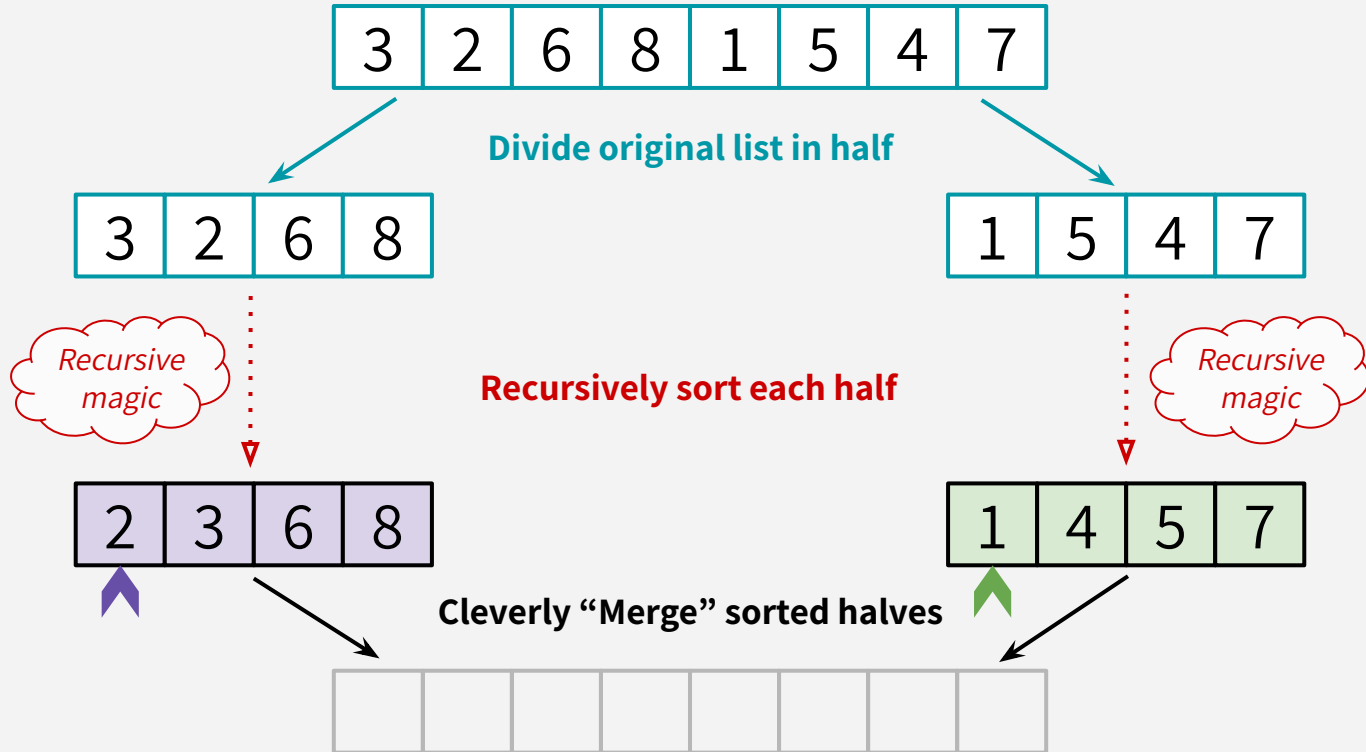
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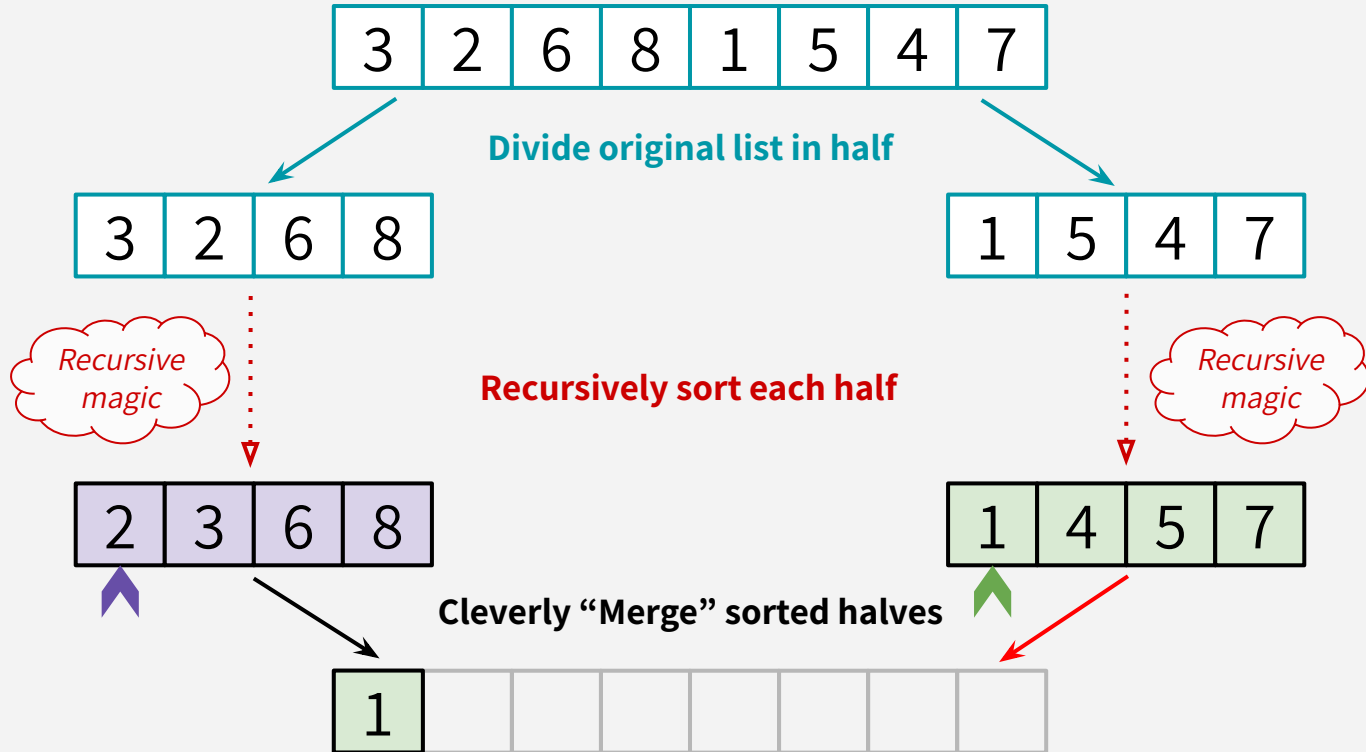
MERGESORT



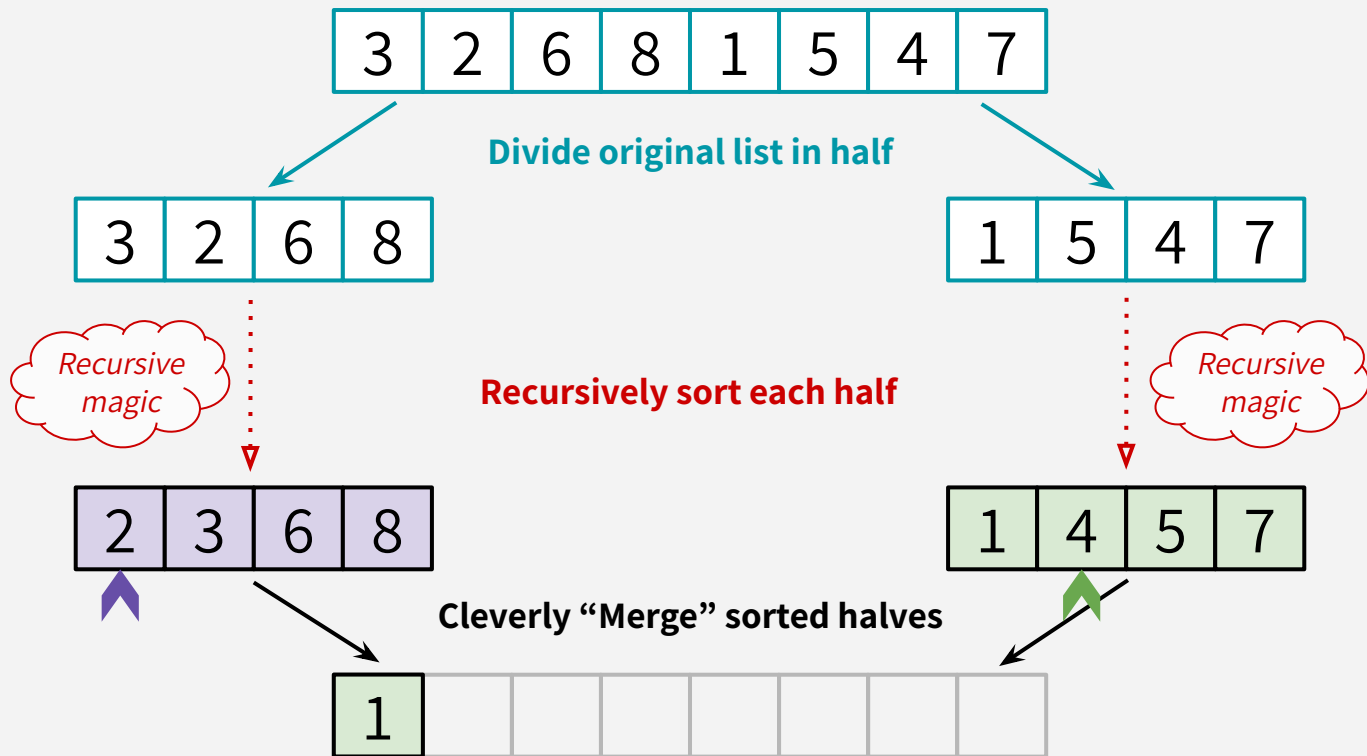
MERGESORT



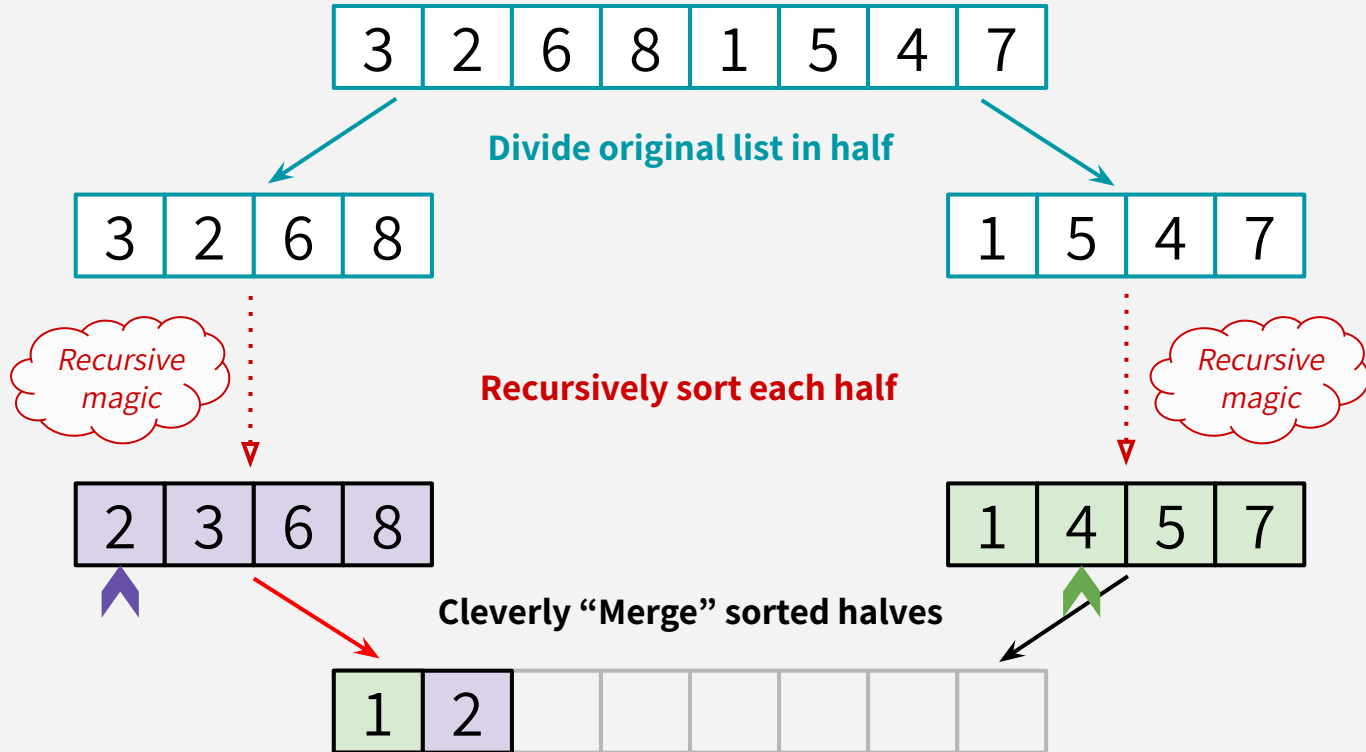
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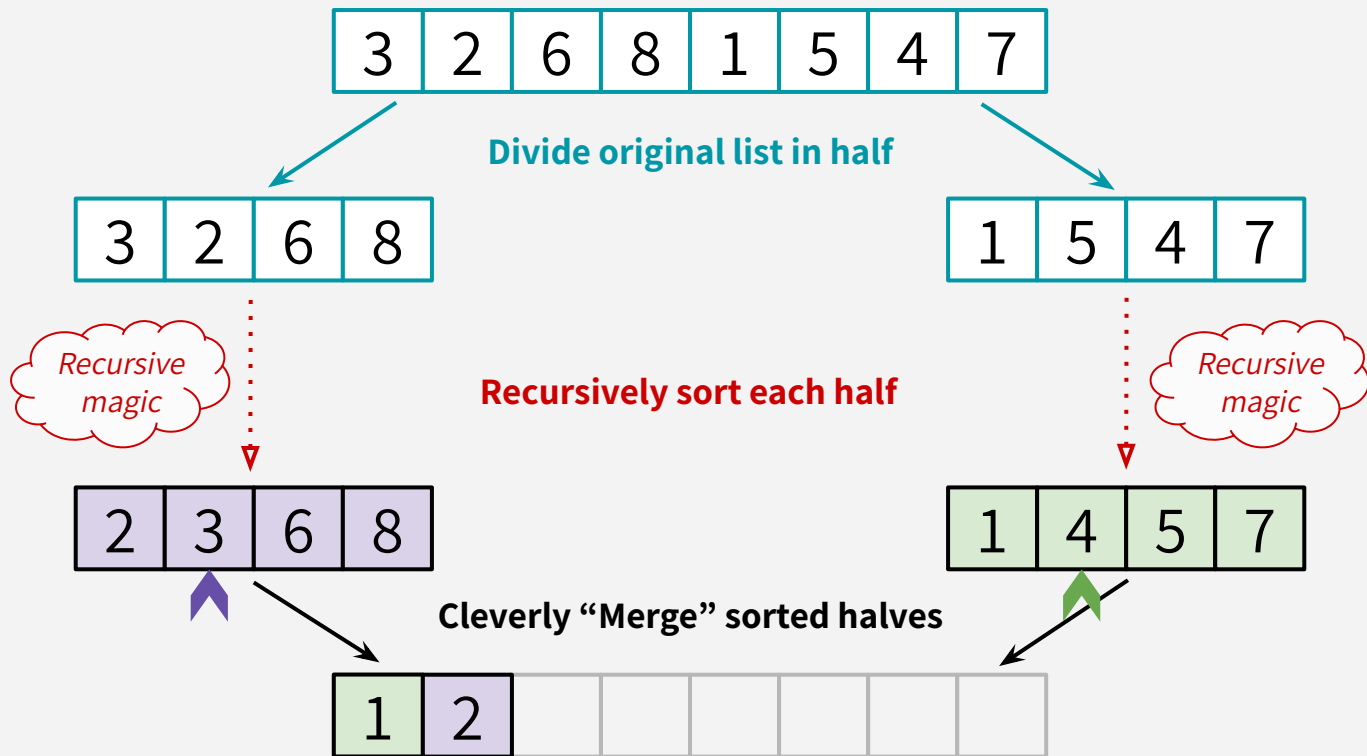
MERGESORT



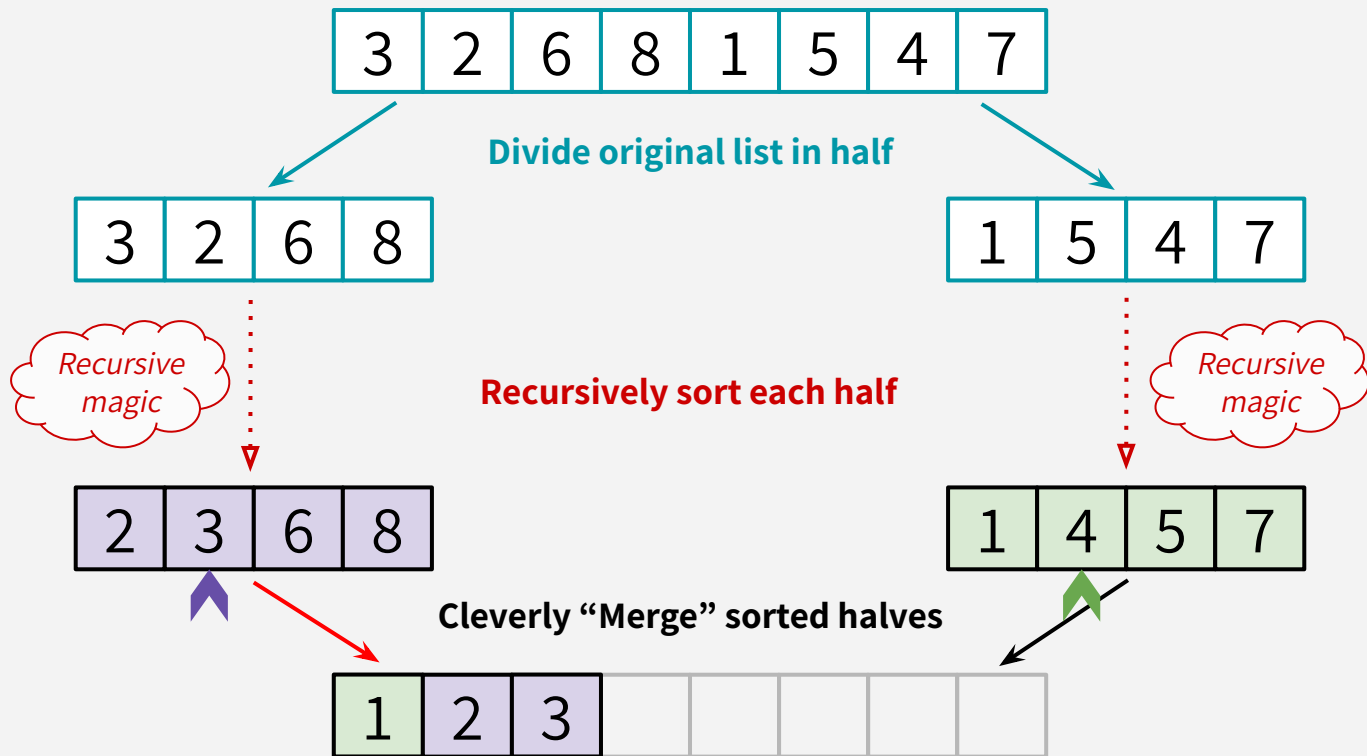
MERGESORT



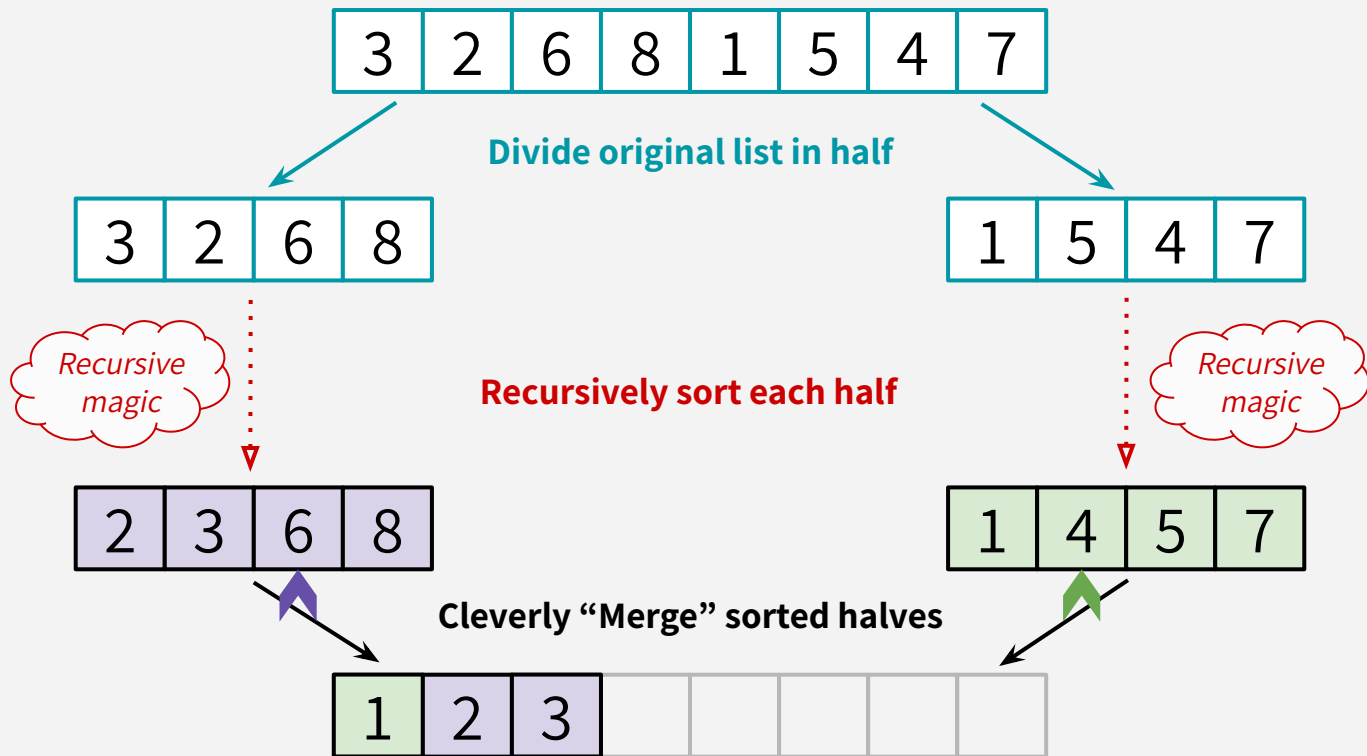
MERGESORT



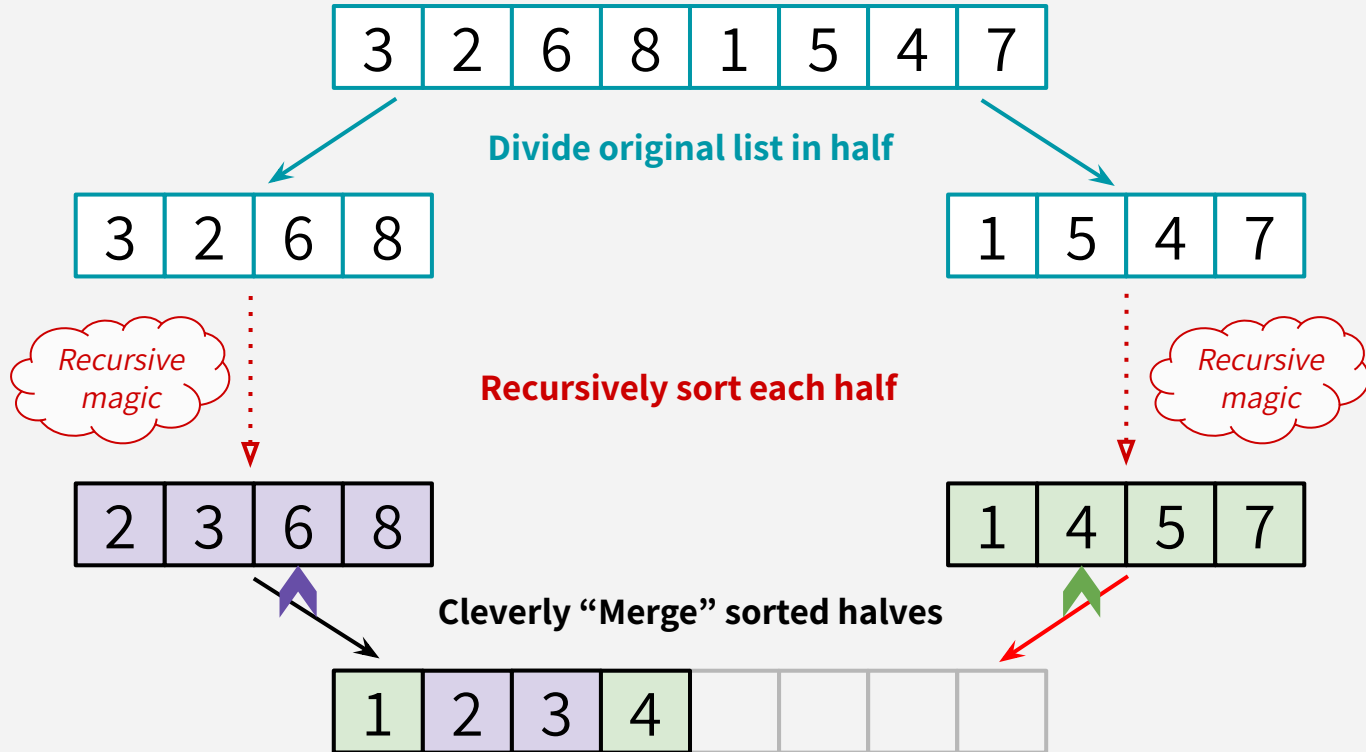
MERGESORT



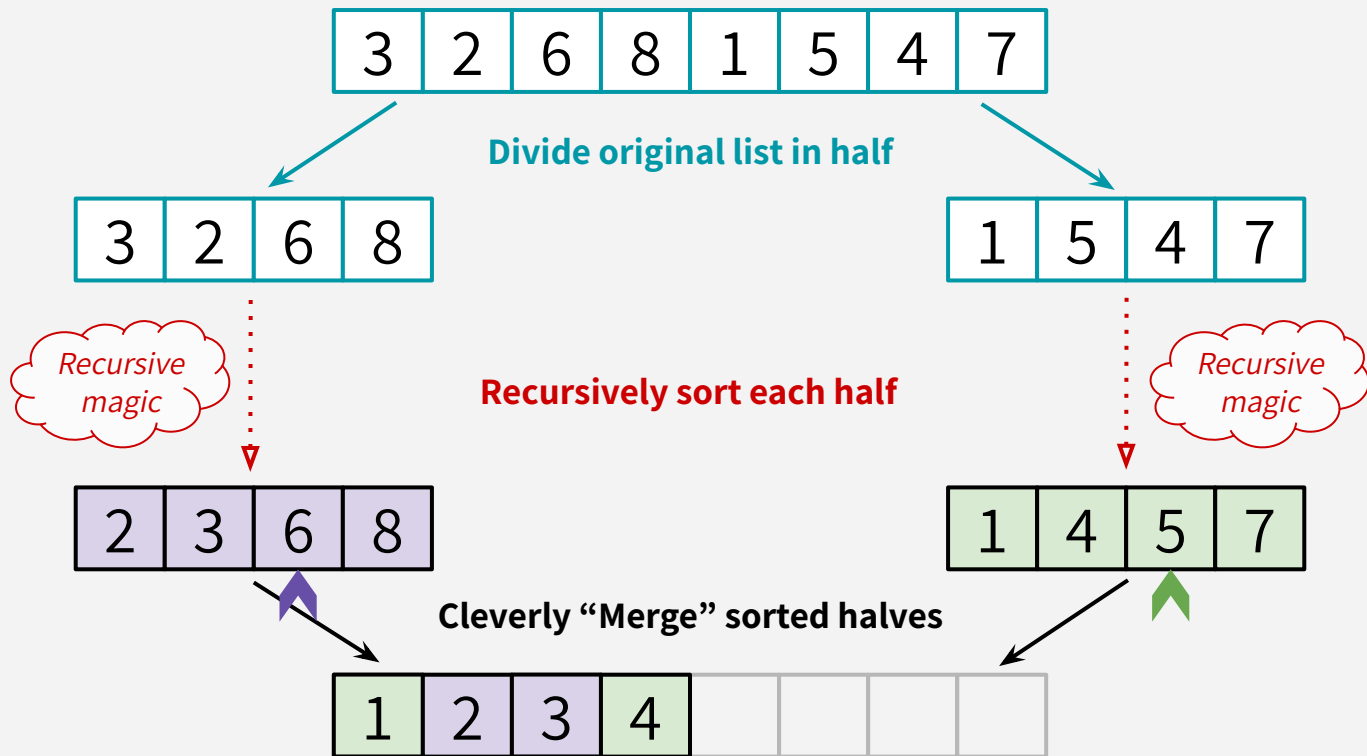
MERGESORT



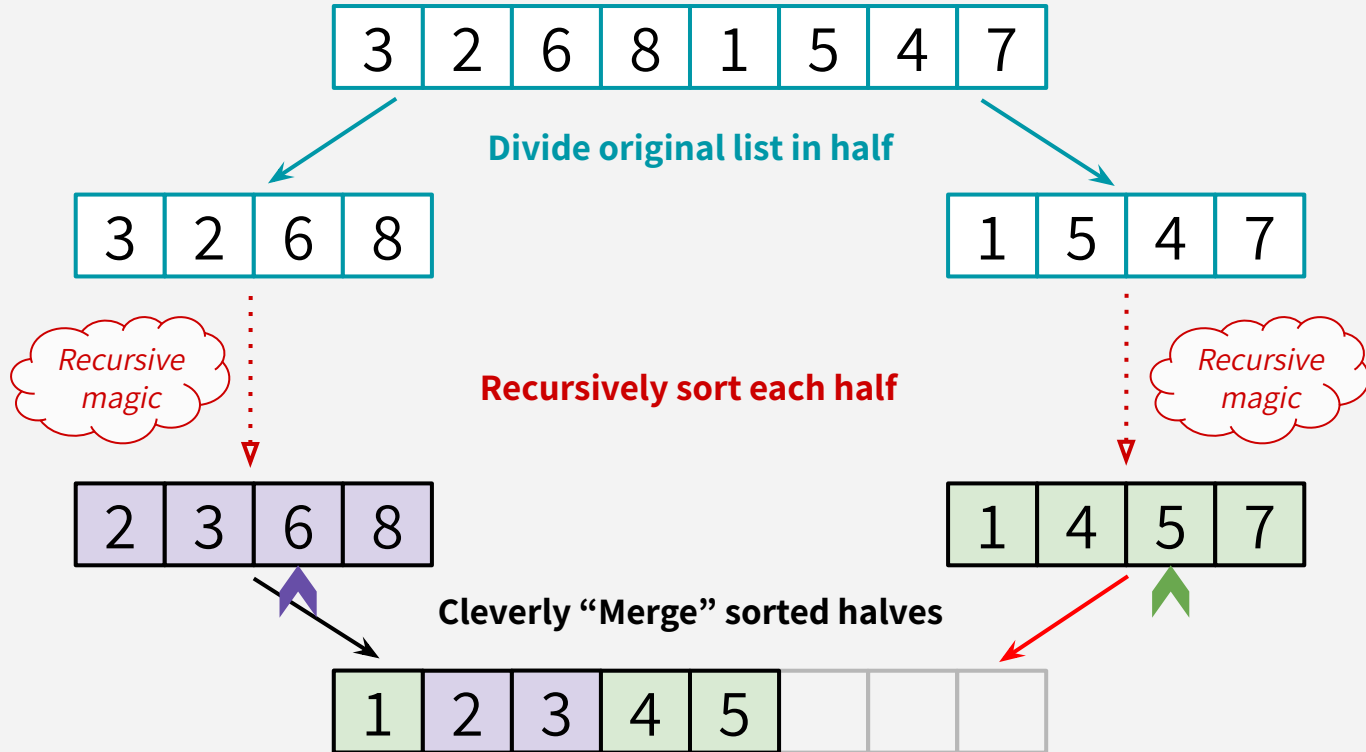
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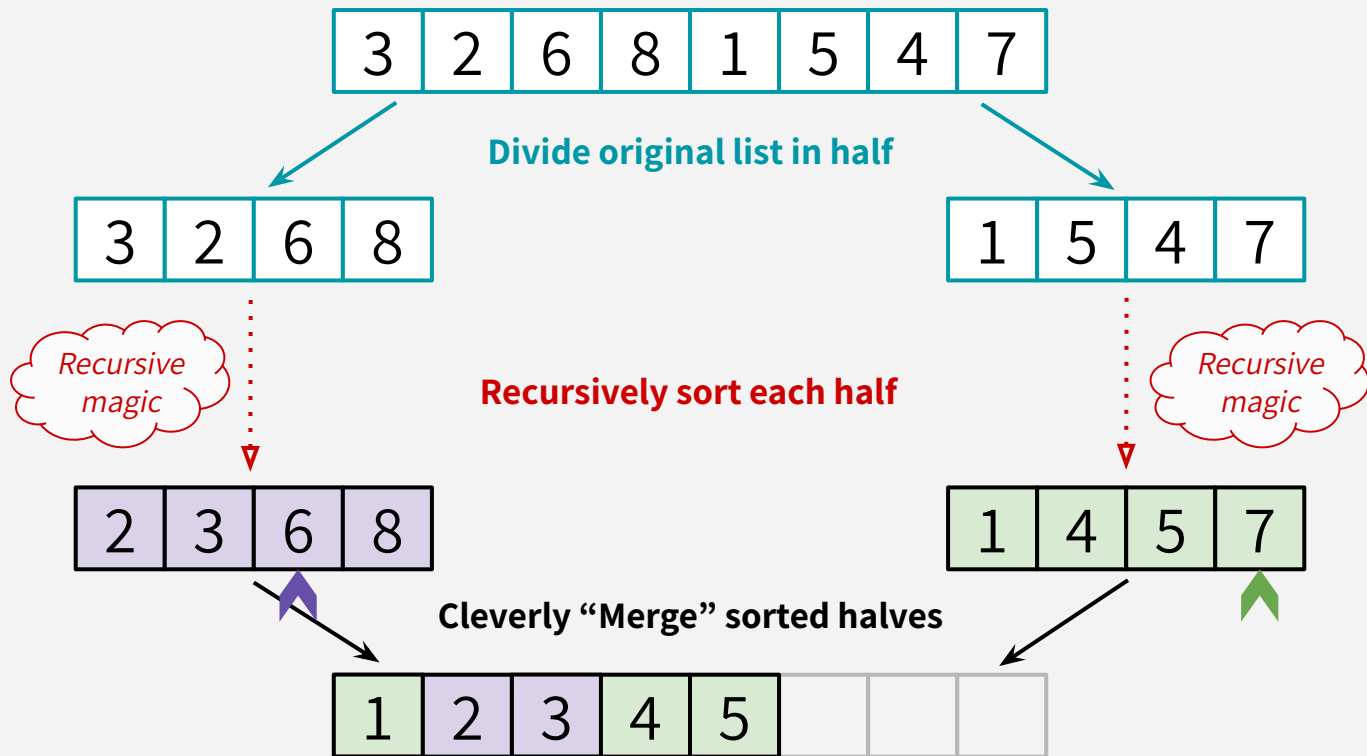
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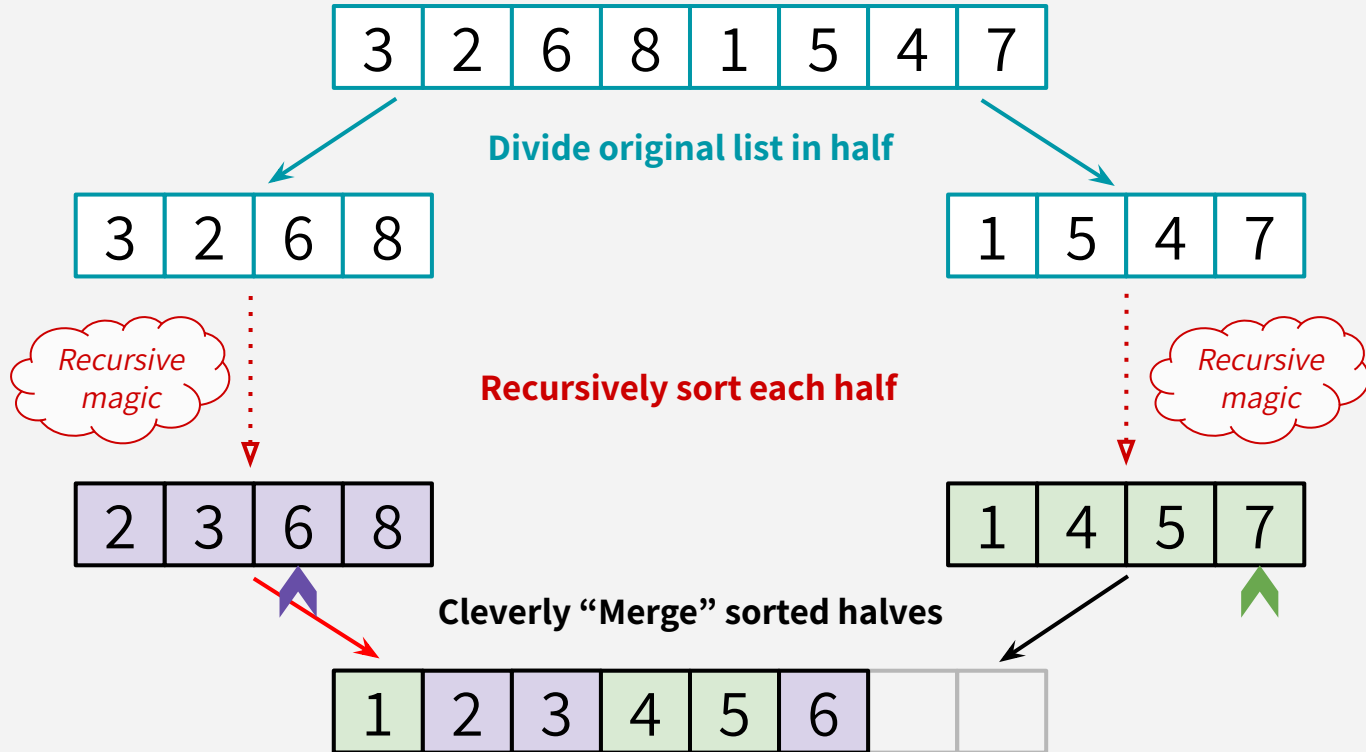
MERGESORT



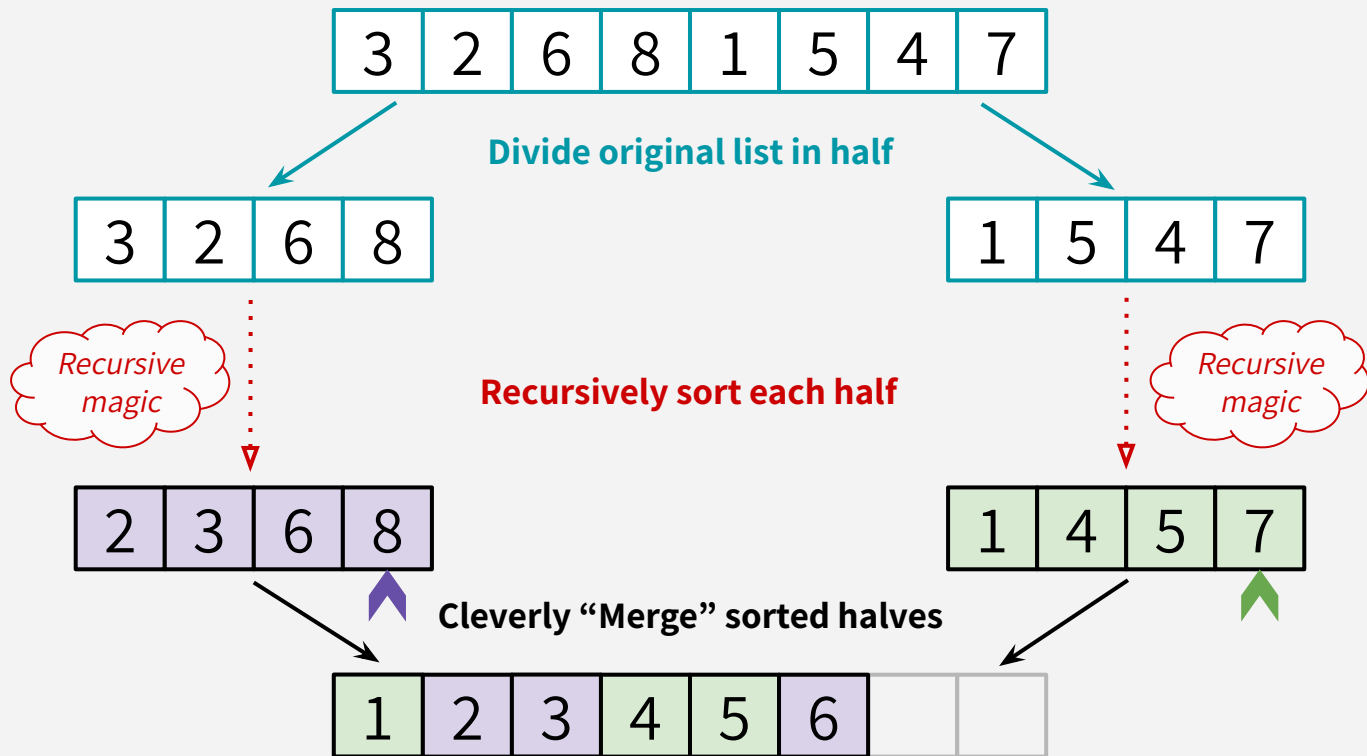
MERGESORT



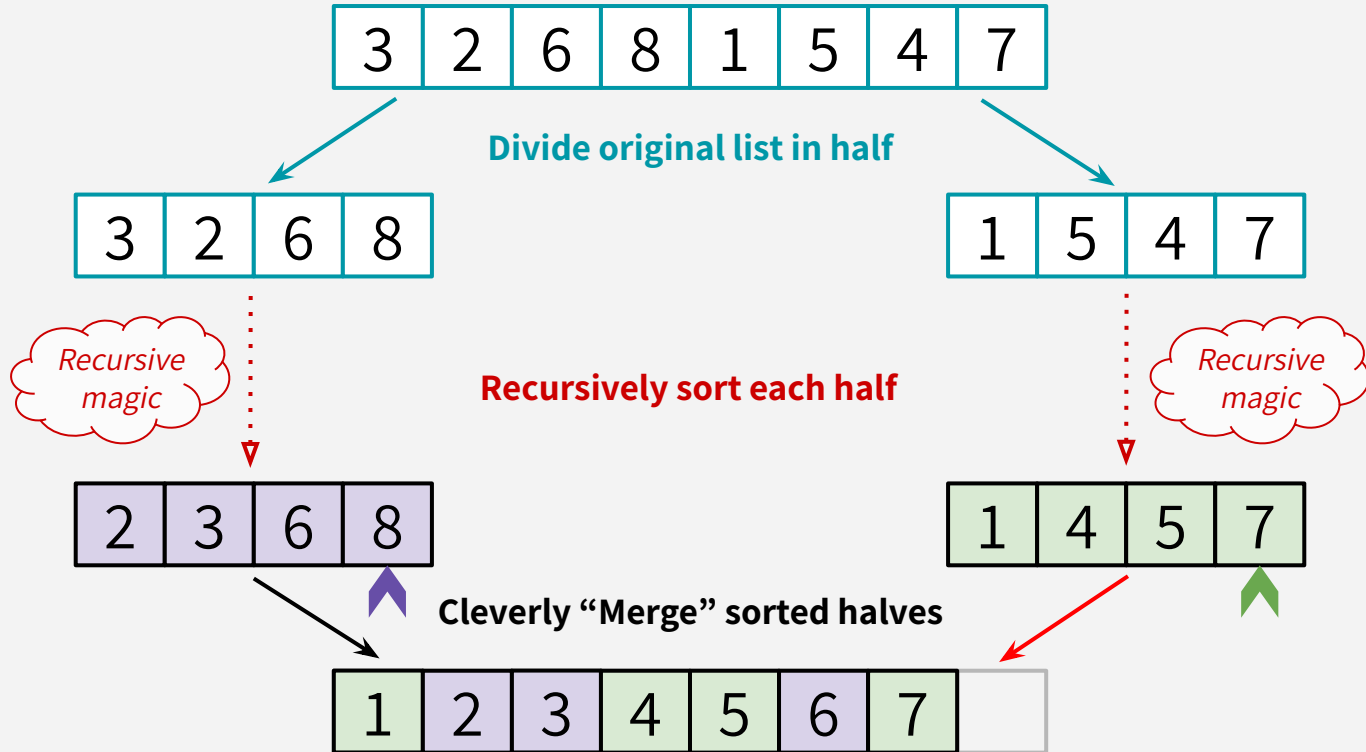
MERGESORT



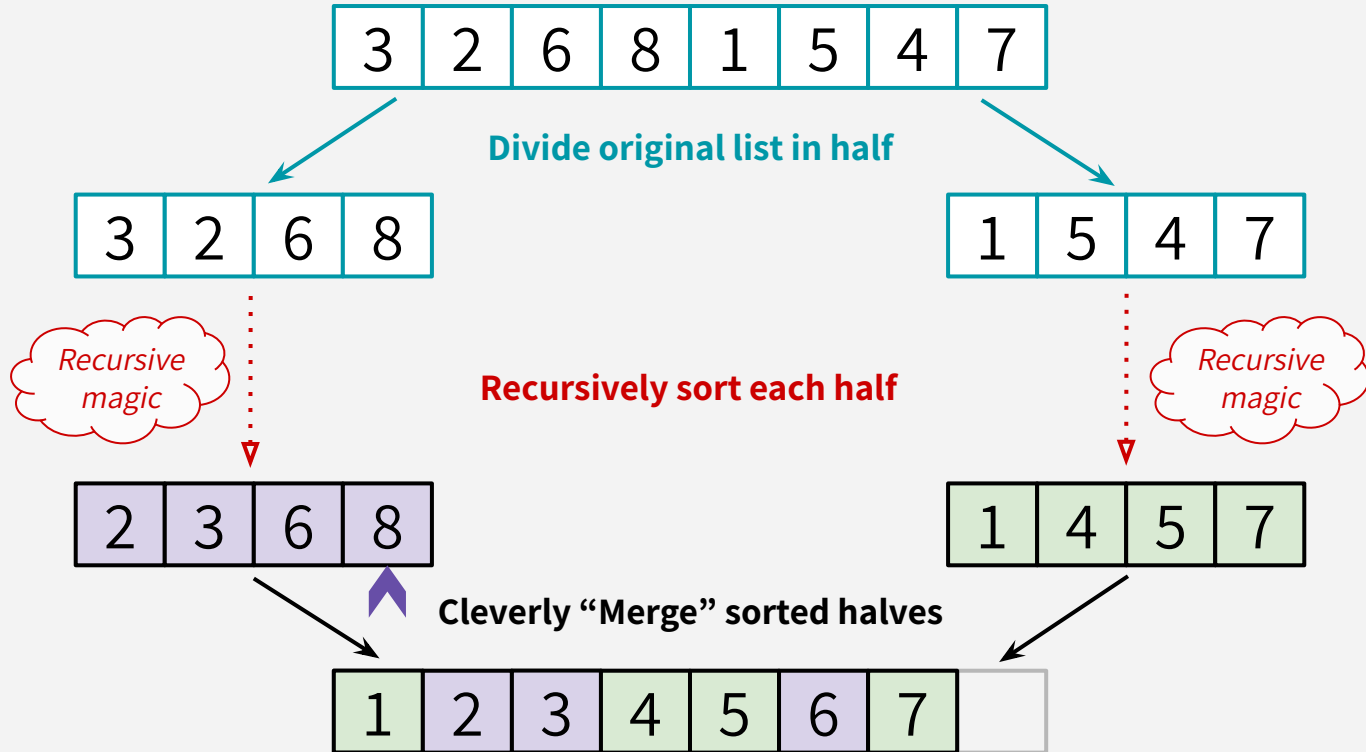
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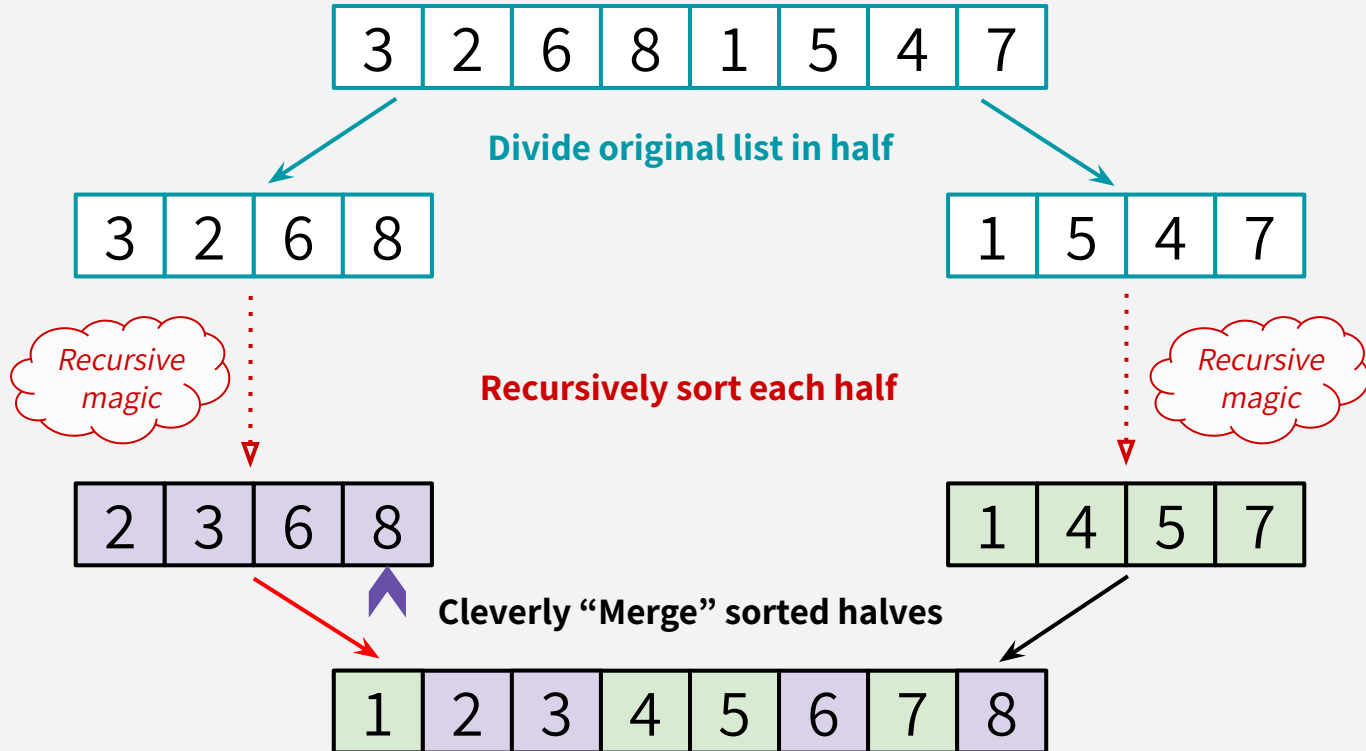
MERGESORT



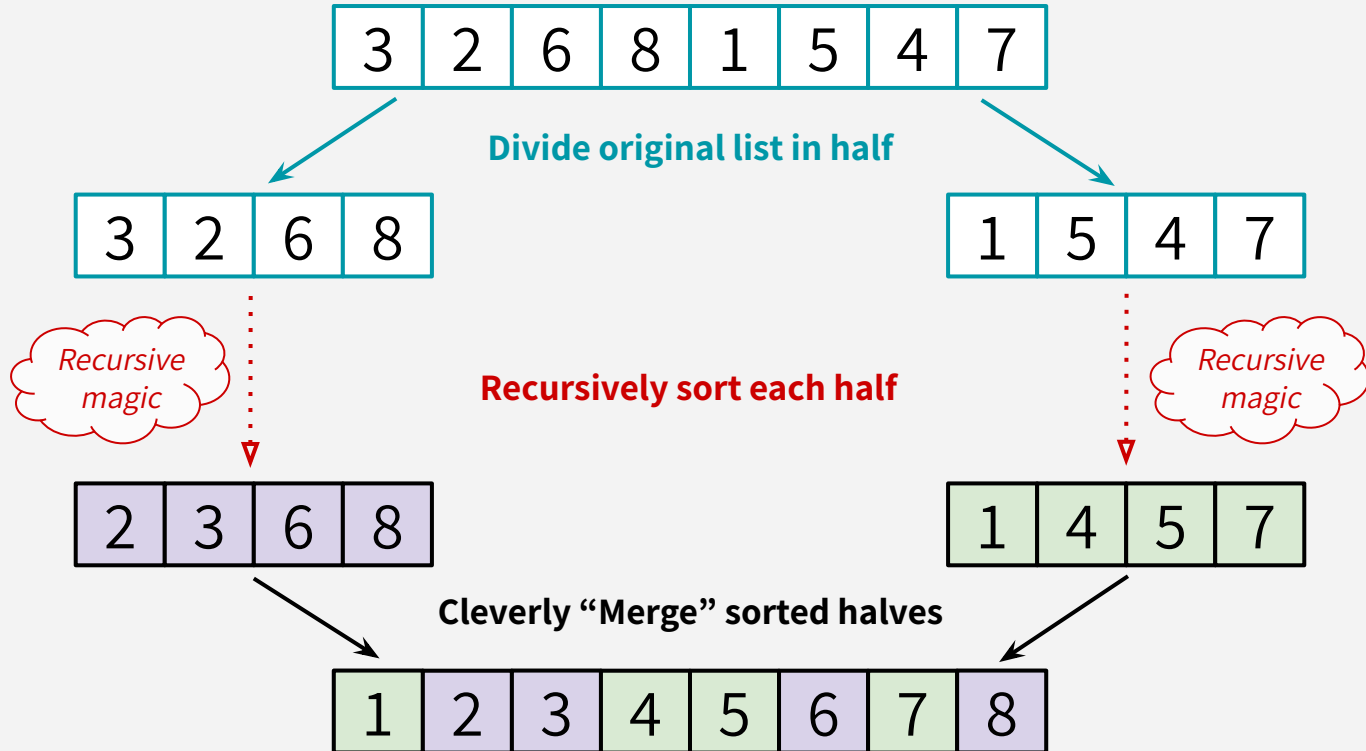
MERGESORT



MERGESORT



MERGESORT



MERGESORT: PSEUDOCODE

Intuition: Divide and Conquer. If you sort your left and right halves, it's easier to “Merge” them into a sorted list.

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For today, let's
assume that n
is a power of 2.

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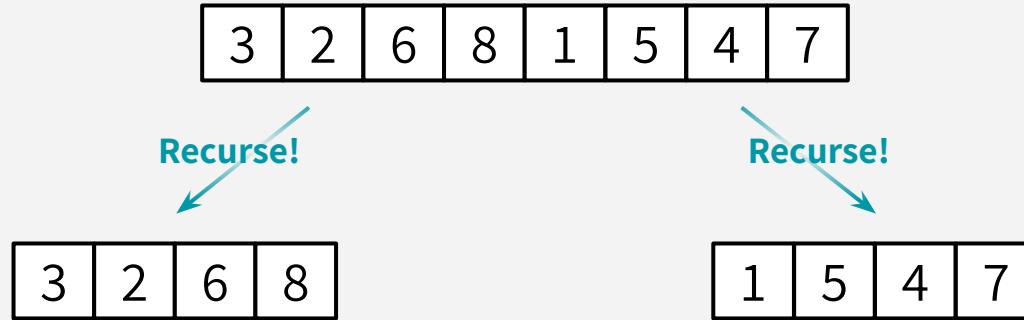
 return result

^{*} Not complete! Some corner cases are missing.

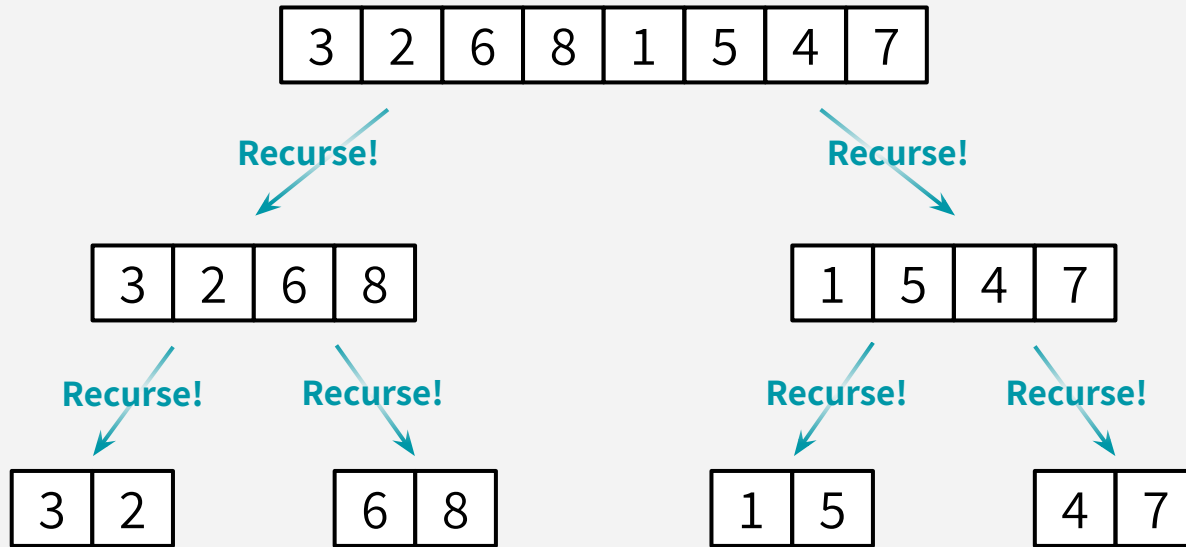
MERGESORT: RECURSIVE CALLS

3	2	6	8	1	5	4	7
---	---	---	---	---	---	---	---

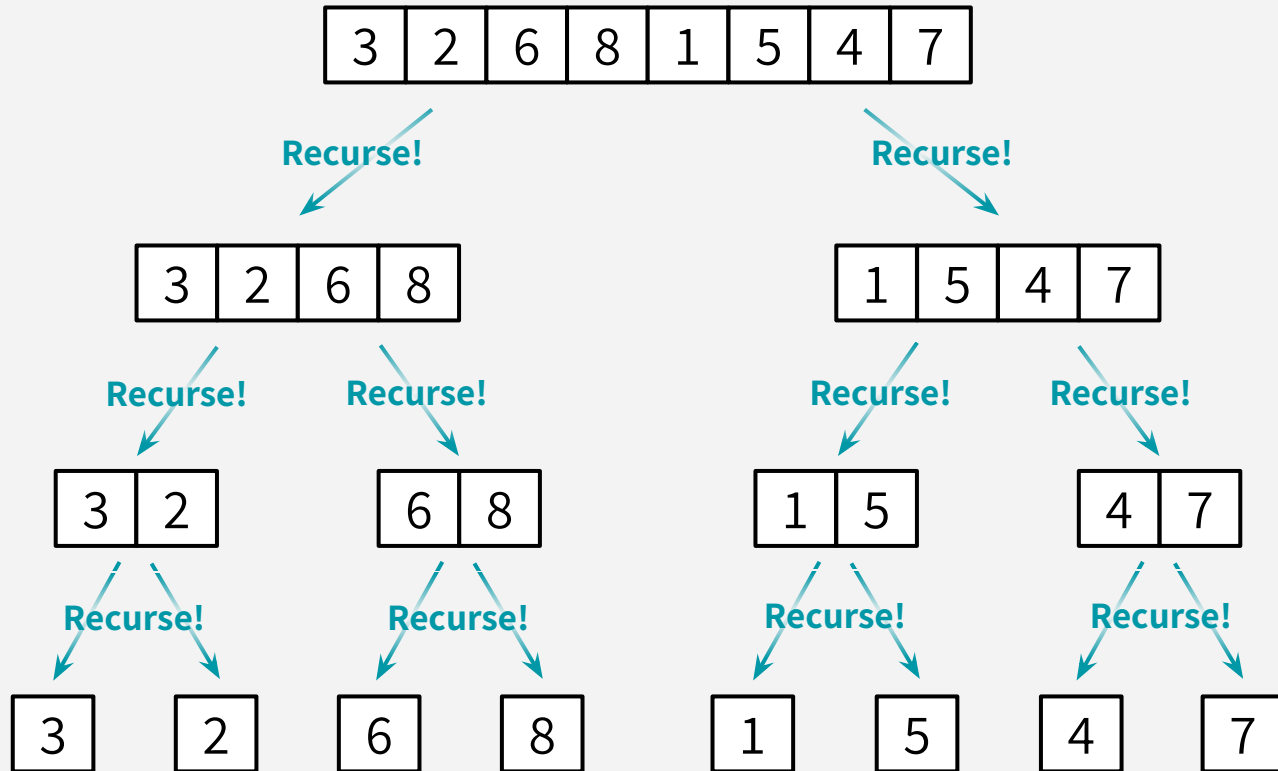
MERGESORT: RECURSIVE CALLS



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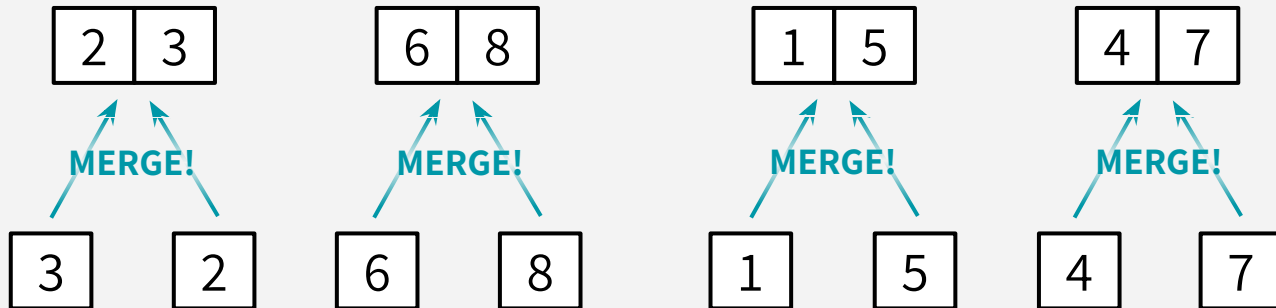


This is where
we hit our
base case!

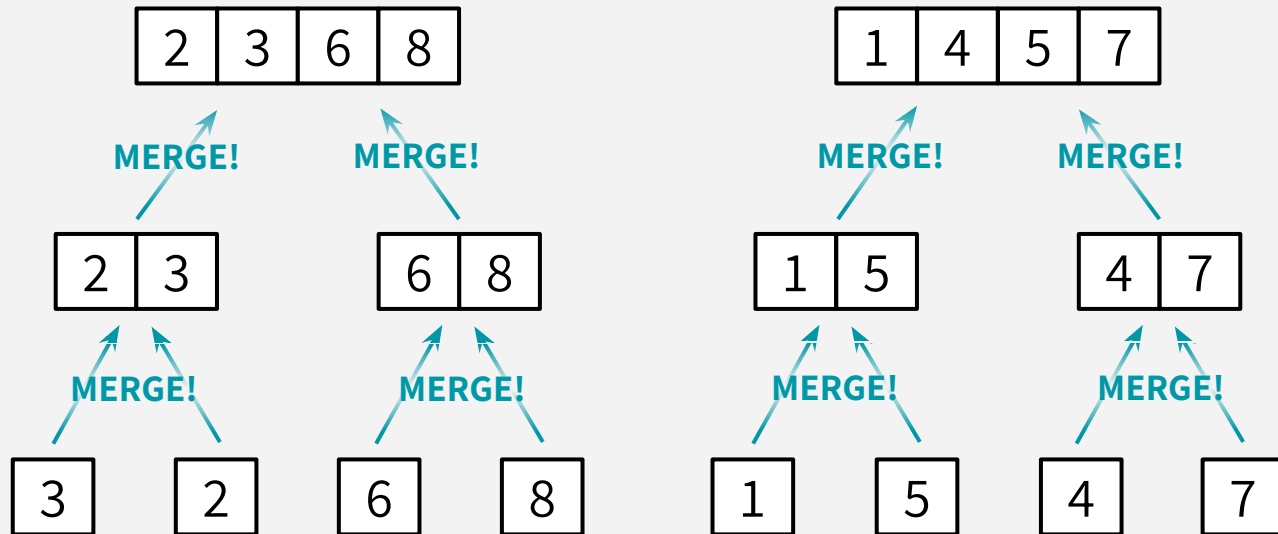
MERGESORT: MERGE STEPS

3 2 6 8 1 5 4 7

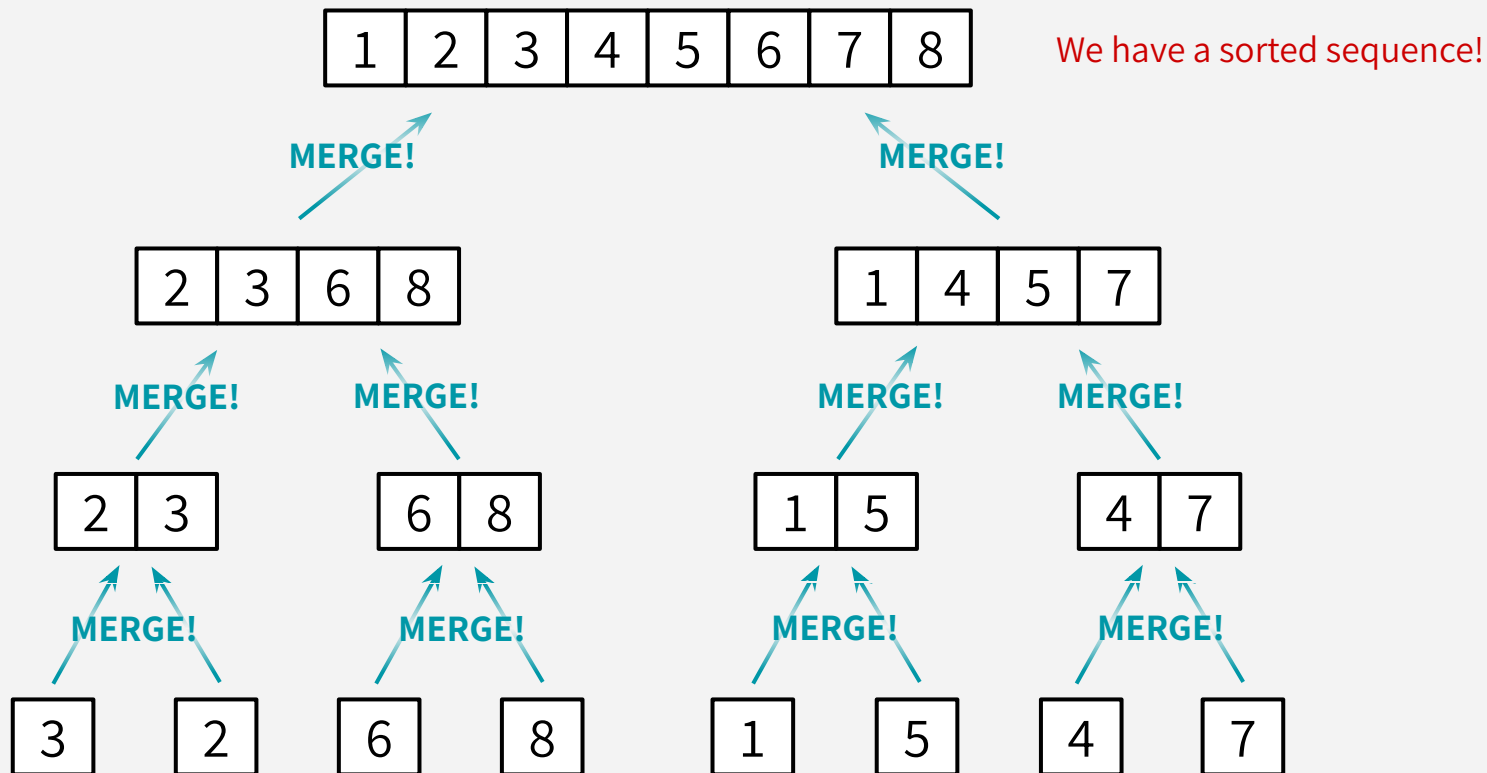
MERGESORT: MERGE STEPS



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MERGESORT: MERGE STEPS





سوال؟

اثبات درستی مرتب سازی ادغامی

آیا واقعا ورودی را مرتب میکند؟

MERGESORT: DOES IT WORK?

HERE'S WHAT WE FOCUS ON:

Whenever we make two “child” recursive calls, as long as those calls successfully sort our left and right halves, we'll safely merge them to create a fully sorted array.

In other words: as long as our recursive calls work on arrays of smaller lengths, then our algorithm will correctly return a sorted array.

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THIS IS A JOB FOR: PROOF BY INDUCTION!

(This time, we perform induction on the *length of input list*, rather than # of iterations)

MERGESORT: INDUCTION PROOF

INDUCTIVE HYPOTHESIS (IH)

In every recursive call on an array of length *at most* i , MERGESORT returns a sorted array.

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INDUCTIVE STEP (*strong/complete induction*)

Let k be an integer, where $1 < k \leq n$. Assume that the IH holds for $i < k$, so MERGESORT correctly returns a sorted array when called on arrays of length less than k . We want to show that the IH holds for $i = k$, i.e. that MERGESORT returns a sorted array when called on an array of length k .

[INSERT INDUCTION PROOF TO PROVE THE MERGE SUBROUTINE IS CORRECT WHEN GIVEN TWO SORTED ARRAYS]

Since the two “child” recursive calls are executed on arrays of length $k/2$ (which is strictly less than k), then our inductive hypothesis tells us that MERGESORT will correctly sort the left and right halves of our length- k array. Then, since the MERGE subroutine is correct when given two sorted arrays, we know that MERGESORT will ultimately return a fully sorted array of length k .

Try out
this inner
proof on
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MERGESORT: INDUCTION PROOF

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CONCLUSION

By induction, we conclude that the IH holds for all $1 \leq i \leq n$. In particular, it holds for $i = n$, so in the top recursive call, MERGESORT returns a sorted array.

PROVE CORRECTNESS w/ INDUCTION

ITERATIVE ALGORITHMS

RECURSIVE ALGORITHMS

PROVE CORRECTNESS w/ INDUCTION

ITERATIVE ALGORITHMS

1. **Inductive hypothesis:** some state/condition will always hold throughout your algorithm by any iteration i
2. **Base case:** show IH holds for iteration 0 (i.e. start of algorithm)
3. **Inductive step:** Assume IH holds for $k \Rightarrow$ prove $k+1$
4. **Conclusion:** IH holds for $i = \#$ total iterations \Rightarrow yay!

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RECURSIVE ALGORITHMS

1. **Inductive hypothesis:** your algorithm is correct for sizes *up to* i
2. **Base case:** IH holds for $i < \text{small constant}$
3. **Inductive step:**
 - assume IH holds for $k \Rightarrow$ prove $k+1$, OR
 - assume IH holds for $\{1, 2, \dots, k-1\} \Rightarrow$ prove k .
4. **Conclusion:** IH holds for $i = n \Rightarrow$ yay!



سوال؟

زمان اجرای مرتب سازی ادغامی

چقدر سریع است؟

MERGESORT: IS IT FAST?

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MERGESORT(A):  
    n = len(A)  
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    return MERGE(L,R)
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CLAIM: MergeSort runs in time **$O(n \log n)$**

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

$\log(n)$ grows very slowly! (Much more slowly than n)

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***ALL LOGARITHMS
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ARE BASE 2***

$$\log(2) = 1$$

$$\log(4) = 2$$

...

$$\log(64) = 6$$

$$\log(128) = 7$$

...

$$\log(4096) = 12$$

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$$\log(\text{\# particles in the universe}) < 280$$

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

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$n \log n$ grows much more slowly than n^2

Punchline: A running time of $O(n \log n)$ is a LOT better than $O(n^2)$

MERGESORT: $O(n \log n)$ PROOF

Instead of counting every little operation and tracing all recursive calls, we can think about:

THE RECURSION TREE!

(and we'll add up all the work done across levels to compute the Big-O runtime)

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n iterations,
O(1) work
per iteration

We can see that MERGE is **$O(n)$**

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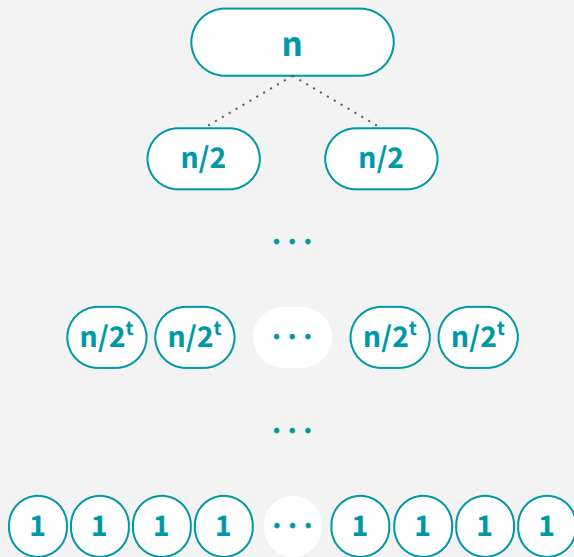
 result = length n array

This means that within one recursive call that processes an array/subarray of length n , the work done in that subproblem (creating subproblems & “merging” those results) is $O(n)$.

 return result

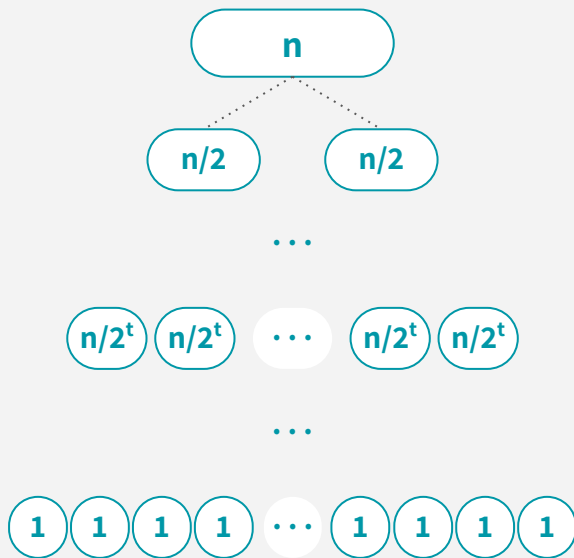
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MERGESORT RECURSION TREE



Level	# of Problems	Size of each Problem	Work done per Problem	Total work on this level
0				
1				
...				
t				
...				
$\log_2 n$				

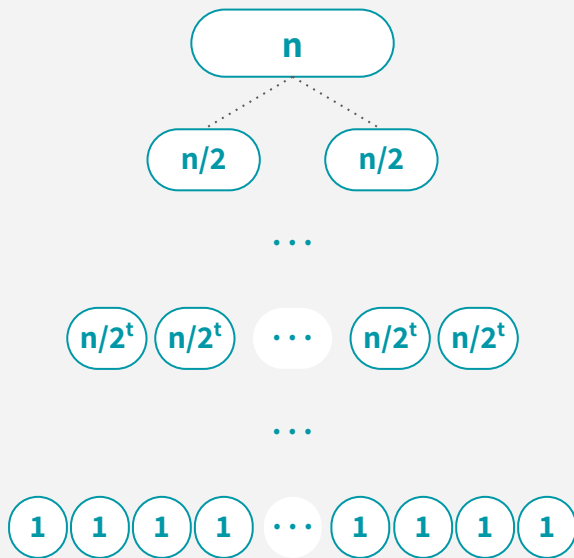
MERGESORT RECURSION TREE



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0	1	n		
1	2^1	$n/2$		
...				
t	2^t	$n/2^t$		
...				
$\log_2 n$	$2^{\log_2 n} = n$	1		

MERGESORT RECURSION TREE

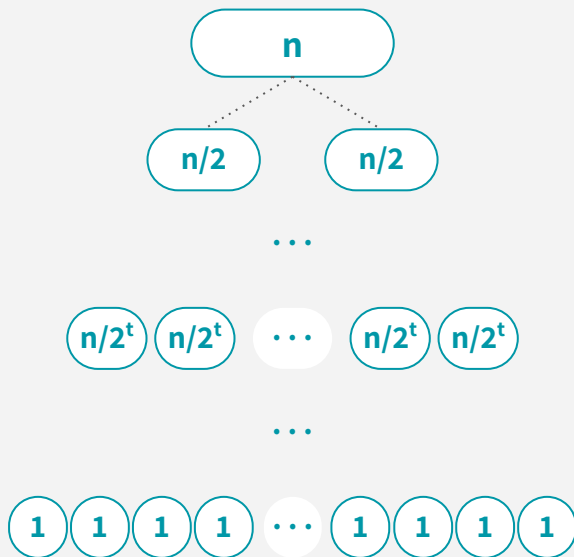
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 $\Rightarrow \text{Work} \leq c \cdot n$ (c is a constant)



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...				
t	2^t	$n/2^t$	$c \cdot (n/2^t)$	
...				
$\log_2 n$	$2^{\log_2 n} = n$	1	$c \cdot (1)$	

MERGESORT RECURSION TREE

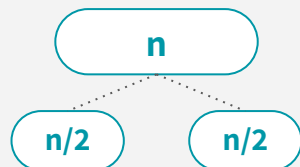
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...				
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...

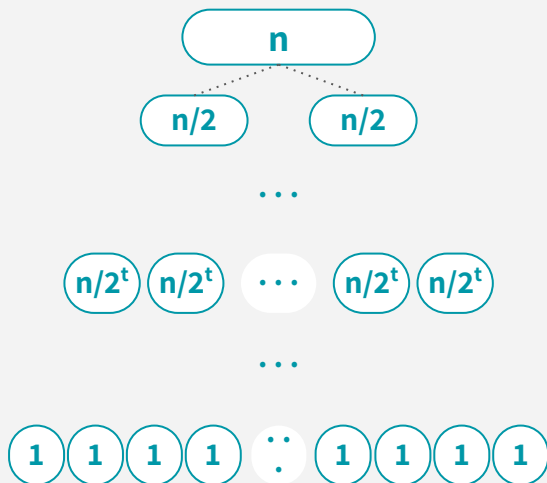


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...				
$\log_2 n$	$2^{\log_2 n} = n$	1	$c \cdot (1)$	$n \cdot c \cdot (1) = O(n)$

We have $(\log_2 n + 1)$ levels, each level has $O(n)$ work total $\Rightarrow O(n \log n)$ work overall!

MERGESORT: $O(n \log n)$ RUNTIME

Using the “Recursion Tree Method” (i.e. drawing the tree & filling out the table),
we showed that the runtime of MergeSort is **$O(n \log n)$**



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سوال؟