CE203
ساختمان داده ها و الگوریتم ها

سجاد شیرعلی شهرضا

پاییز 1400

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

جلسه پنجم:
رابطه بازگشتی و قضیه اصلی

دوشنبه، 12 مهر 1400

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 4.3، 4.4، 4.5
- امتحانک اول
  - دوشنبه 19 مهر
  - به صورت برخط (آنلاین)
  - از طریق سامانه دروس دانشگاه
  - در طی ساعت کلاس
  - به صورت **انفرادی**!

رابطه بازگشتی

RUNTIMES FOR RECURSIVE ALGOS

Previously, we used the "Recursion Tree Method" (i.e. drawing the tree & filling out the table) to manually add up all the work in the tree and find that the runtime of MergeSort is **O(n log n)**.

Drawing the tree & doing all that adding kind of takes a lot of work...
Here's another way to reason about the runtime of a recursive algorithm like Mergesort:

INTRODUCING...

RECURRENCE RELATIONS

RECURRENCE RELATIONS

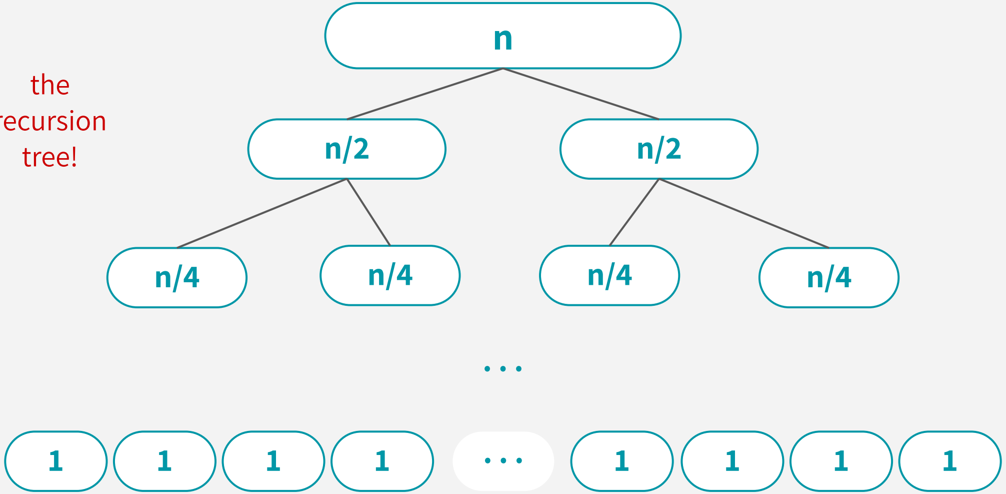
Basically, Recurrence Relations give us a *recursive* way to express runtimes for *recursive* algorithms!

We can then employ some math-ier approaches to analyze these recurrence relations.

RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

the recursion tree!



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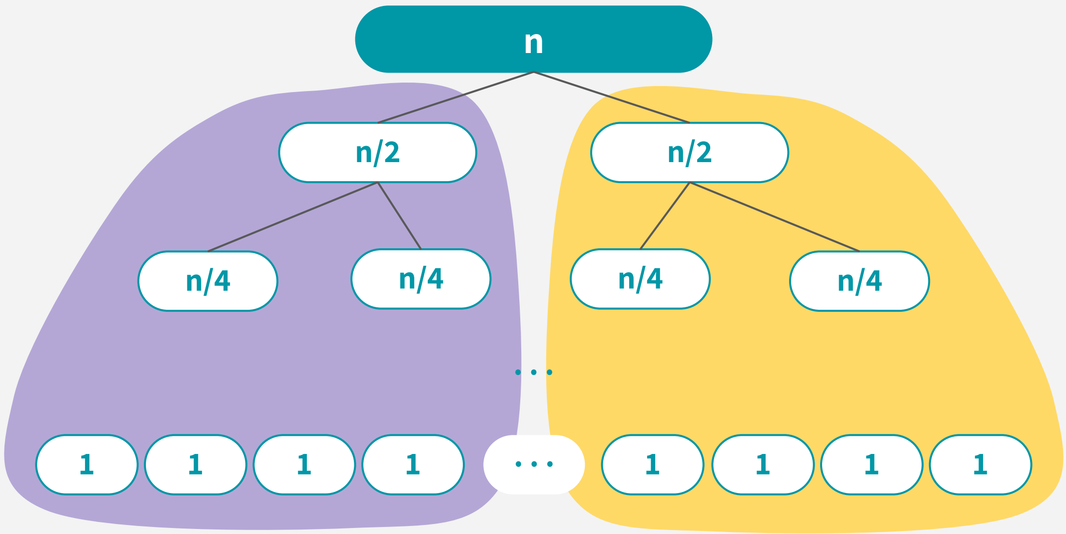

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total work in LEFT recursive call (left subtree)

+

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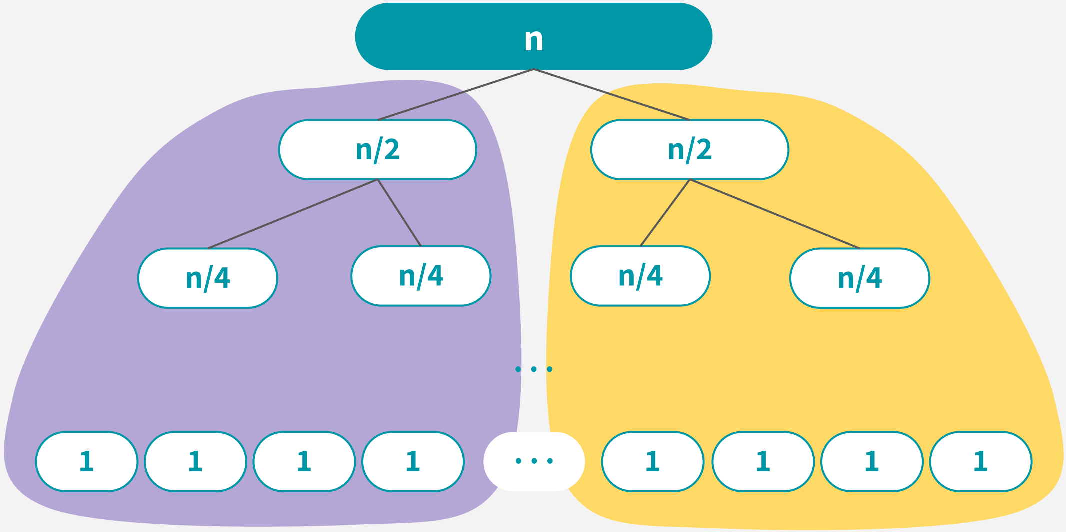

Work in the whole tree =

total work in LEFT recursive call
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total work in RIGHT recursive call
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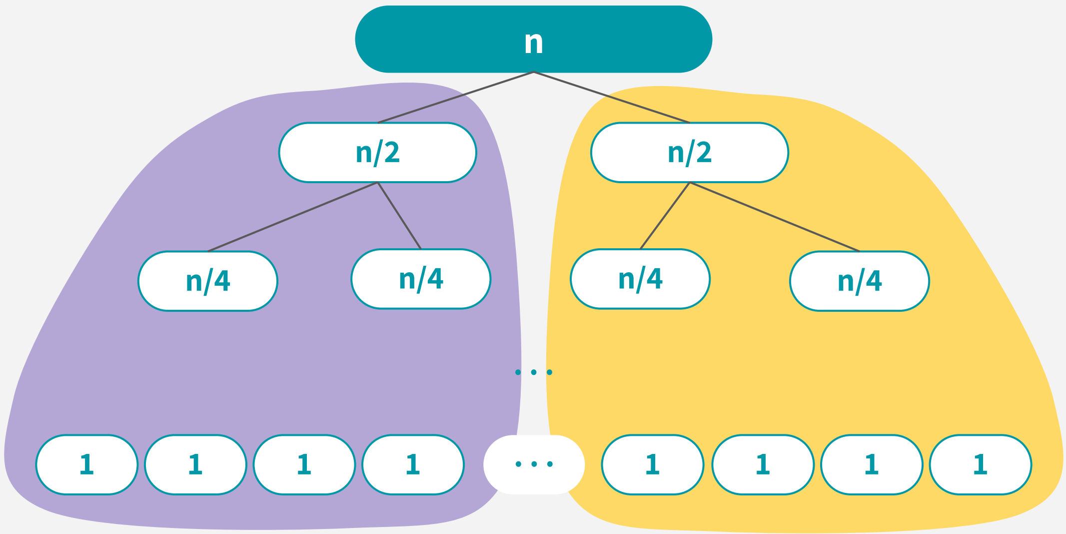

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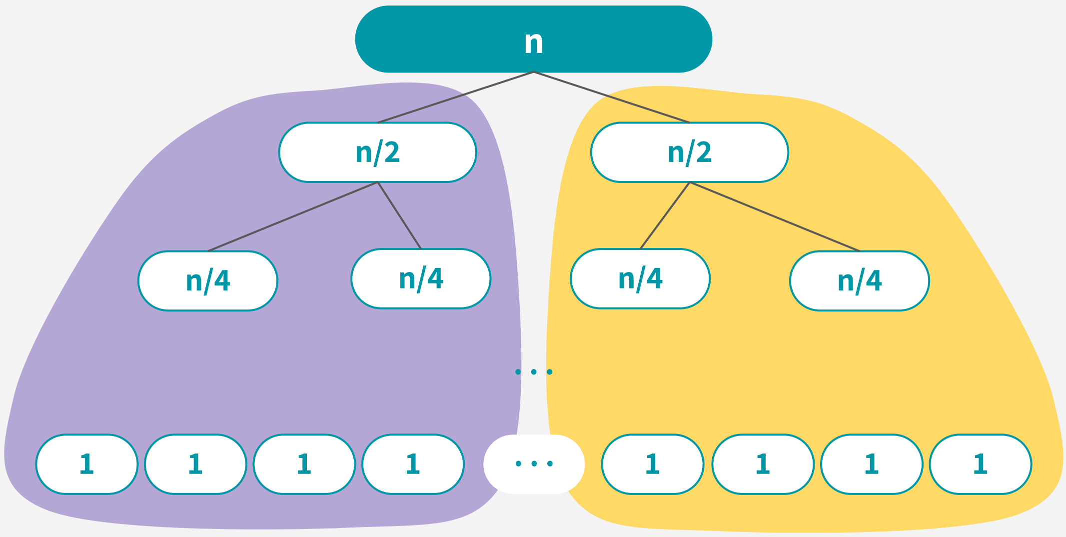

Work in the whole tree =

total work in LEFT recursive call (left subtree)

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work done *within* top problem

work to create suproblems & “merge” their solutions

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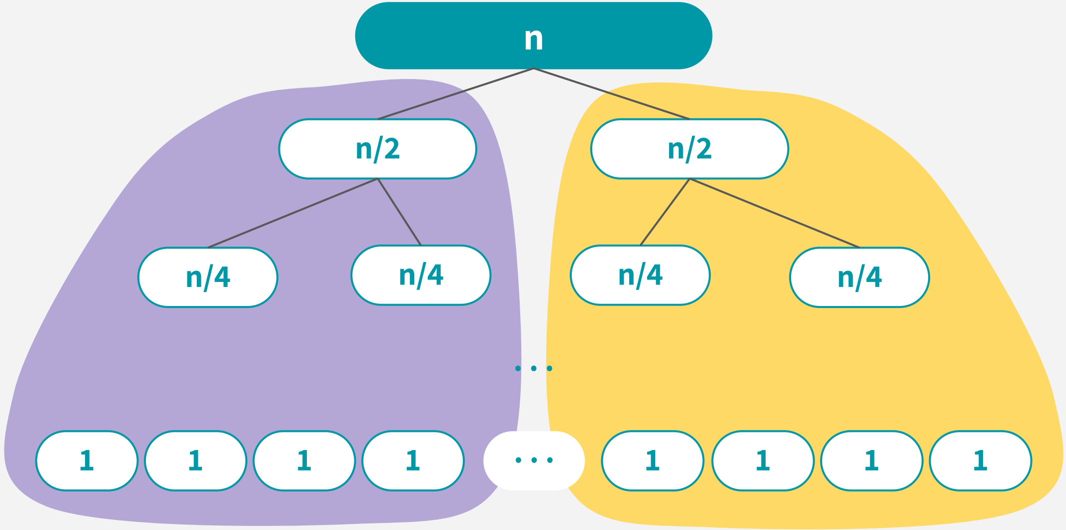

T(n) =

T(n/2)

+

T(n/2)

+

O(n)

RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

A note:

We're making a simplifying assumption here that **n** is a perfect power of two (otherwise, we should use floors and ceilings).

Turns out that if we do incorporate floors and ceilings, we still get constant size subproblems at level $\lfloor \log_b n \rfloor$, and generally, the stuff we'll do in this class with Recurrence Relations will still work if we forget about floors and ceilings here.

1 1

T(n) =

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RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

$$\mathbf{T(n)} = \underbrace{\boxed{T(n/2)} + \boxed{T(n/2)}}_{} + \boxed{\mathbf{O(n)}}$$

since the subproblems are equal sizes, we can also write this as ***2 · T(n/2)***

This is a *recursive* definition for T(n), so we also need a BASE CASE:

$$\mathbf{T(1) = O(1)}$$

No matter what T is, T(1) = O(1). If it's greater than O(1), then the problem size wouldn't actually be 1.

Since we already used the Recursion Tree to compute the runtime of MergeSort, we know that **T(n) = O(n log n)**.

EXAMPLE RECURRENCE RELATIONS

Useless Divide-and-Conquer Multiplication

$$T(n) = 4 \cdot T(n/2) + O(n)$$
$$T(n) = O(n^{\log_2 4}) = \mathbf{O(n^2)}$$

Karatsuba Integer Multiplication

$$T(n) = 3 \cdot T(n/2) + O(n)$$
$$T(n) = O(n^{\log_2 3}) \approx \mathbf{O(n^{1.6})}$$

MergeSort

$$T(n) = 2 \cdot T(n/2) + O(n)$$
$$T(n) = \mathbf{O(n \log n)}$$

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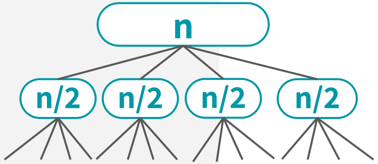

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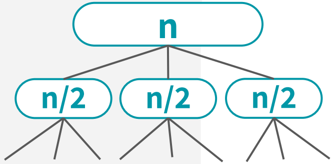

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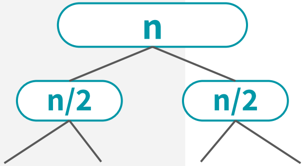

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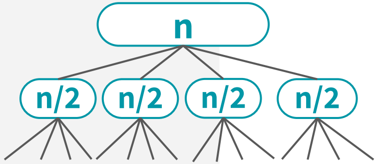

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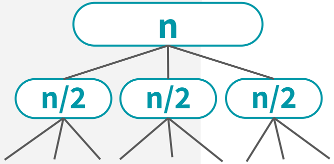

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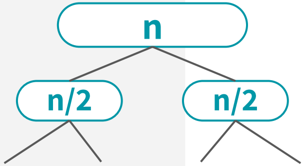

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IS THERE A
PATTERN
???

قضیه اصلی

فرمولی برای حل بسیاری از روابط بازگشتی
(اما نه همه آنها!)

THE MASTER THEOREM

Suppose that $\mathbf{a \geq 1}$, $\mathbf{b > 1}$, and $\mathbf{d}$ are constants (i.e. independent of $\mathbf{n}$).

Suppose $\mathbf{T(n) = a \cdot T(n/b) + O(n^d)}$. The Master Theorem states:

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Suppose $\mathbf{T(n) = a \cdot T(n/b) + O(n^d)}$. The Master Theorem states:

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

d: need to do $O(n^d)$ work to create subproblems + “merge” their solutions

MASTER THEOREM EXAMPLES

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

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USELESS DIVIDE & CONQUER MULTIPLICATION	$T(n) = 4 \cdot T(n/2) + O(n)$	$a = 4$ $b = 2$ $d = 1$	$a > b^d$

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KARATSUBA INTEGER MULTIPLICATION

$\mathbf{T(n) = 3 \cdot T(n/2) + O(n)}$

a = 3
b = 2
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MASTER THEOREM EXAMPLES

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MASTER THEOREM "INTUITION"

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

amount of work at each level ~ same

highest level "dominates": work per level decreases (subproblem work shrinks more!)

leaves "dominate": work per level increases (branch more!)

a: number of subproblems (branching factor)

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To see why this is true:

We need to look at a "generalized" recursion tree

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

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MASTER THEOREM "INTUITION"

Original set up: $T(n) = a \cdot T(n/b) + O(n^d)$

We're going to suppose that $T(n) \le a \cdot T(n/b) + c \cdot n^d$

Wait a minute: Is it okay to just replace $O(n^d)$ with $c \cdot n^d$? What about n_0? Do we just drop that?
For simplicity, we're going to assume that we're using $\mathbf{n_0 = 1}$ in the definition of big-O. (The proof of the Master Theorem" will work for larger n_0 too - verify this yourself!)

GENERALIZED RECURRENCE TREE

1. Draw the tree for **$T(n) = a \cdot T(n/b) + c \cdot n^d$**
2. Fill out the table & sum up last column (from $t = 0$ to $t = \log_b n$)

LEVEL	# OF PROBLEMS	SIZE OF EACH PROBLEM	WORK PER PROBLEM	TOTAL WORK AT THIS LEVEL
0	1	n	$c \cdot n^d$	$1 \cdot c \cdot n^d$
1	a	n/b	$c \cdot (n/b)^d$	$a \cdot c \cdot (n/b)^d$
. . .				
t	a^t	n/b^t	$c \cdot (n/b^t)^d$	$a^t \cdot c \cdot (n/b^t)^d$
. . .				
$\log_b n$	$a^{\log_b n}$	$n/b^{\log_b n} = 1$	$c \cdot (n/b^{\log_b n})^d$	$a^{\log_b n} \cdot \overbrace{c \cdot (n/b^{\log_b n})^d}^{\text{This} = 1}$

Total amount of work:

$$\boxed{c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}$$

(add work across all levels up, then factor out the c & n^d terms & write in summation form)

GENERALIZED RECURRENCE TREE

1. Draw the tree for **$T(n) = a \cdot T(n/b) + c \cdot n^d$**
2. Fill out the table & sum up last column (from $t = 0$ to $t = \log_b n$)

So $T(n) \leq$ $$\boxed{c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}$$

We can verify that for each of the three cases (**$a =, <,$ or $> b^d$**), this equation above gives us the desired results:

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

CASE 1: $a = b^d$

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}} \right)^t$$

$$= c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} 1$$

This is equal to 1!

$$= c \cdot n^d \cdot (\log_b(n) + 1)$$

$$= c \cdot n^d \cdot \left(\frac{\log(n)}{\log(b)} + 1 \right)$$

$$= \Theta(n^d \log(n))$$

CASE 2: $a < b^d$

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ \mathbf{O(n^d)} & \textbf{if } \mathbf{a < b^d} \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}} \right)^t$$

$$= c \cdot n^d \cdot [\text{ some constant }]$$

$$= \Theta(n^d)$$

This is less than 1!

Geometric series with the "multiplier" < 1 & constant!

CASE 2: $a > b^d$

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ \mathbf{O(n^{\log_b(a)})} & \textbf{if } \mathbf{a > b^d} \end{cases}$$

This is greater than 1!

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}} \right)^t$$

$$= \Theta\left(n^d \left(\frac{a}{b^d} \right)^{\log_b(n)} \right)$$

$$= \Theta\left(n^{\log_b(a)} \right)$$

The n^d term cancels with (b^d)^{$\log_b n$}!
And $\mathbf{a^{\log_b n} = n^{\log_b a}}$!

Use the geometric series formula to convince yourself that this is legitimate!

WE CHECKED ALL THREE CASES!

$$T(n) = a \cdot T(n/b) + O(n^d)$$

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$