Boğaziçi University
Department of Mathematics
Math 331 Metric Spaces
Fall 2025 – First Midterm Exam

1	2	3	4	5	$\sum$
PROPOSED		SOLUTIONS			
20 pts	20 pts	20 pts	24 pts	18 pts	100 pts

Date: October 27, 2025 Time: 17:00-19:00	Full Name:

In this exam, $X = (X, d)$ denotes an arbitrary metric space; $(V, ||\cdot||_V)$ denotes an arbitrary normed space. A metric g on a set Y that satisfies the stronger axiom $g(a,c) \le \max(g(a,b), g(b,c))$ for all $a,b,c \in Y$ is called an ultrametric on Y.

Recall the expression for the *p-norm* over $\mathbb{R}^n$: $||\mathbf{x}||_p \stackrel{\text{def}}{=} \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$.

1. (a) (p. 5) For any $x, y \in X$, let $d'(x,y) = \min(d(x,y), 1)$. Show that this *bounded* d' is also a metric on X.
   (b) (p. 74) Show that (X, d') is homeomorphic to (X, d) when (X, d) is a bounded space.

(a) • $d'(x,y)=0 \iff d(x,y)=0 \iff x=y$. • $d'(x,y)=d'(y,x)$.
• $d'(x,y) \le \min\left(d(x,z)+d(y,z), 1\right) \le \min\left(d(x,z),1\right) + \min\left(d(y,z),1\right) = d'(x,z)+d'(y,z)$
needs justification

(b) d and d' are equivalent:
$\forall x,y \in X: d'(x,y) \leqslant d(x,y) \leqslant M\cdot d'(x,y)$ where M is a bound for d over X.
By thm, the result follows.

Note. Claim true even if (X,d) is not bounded: $\forall B^d$ is open wrt d' & $\forall B^{d'}$ is open wrt d. } Can you show this?

2. Prove directly from definitions:
   (p. 65) All linear mappings $T : (\mathbb{R}^n, ||\cdot||_1) \to V$ are continuous.

Fix a basis $\{u_1, \dots, u_n\}$ for $\mathbb{R}^n$. Given T, set $M = \max_j Tu_j$.
For $u \in \mathbb{R}^n$, express $u = \sum a_j u_j$. Given $\varepsilon > 0$, we want
$\varepsilon > \|Tu\|_V = \sum |a_j| \cdot T(u_j) \leqslant M \sum_j |a_j| = M \cdot \|u\|_1$.
So just choose $\delta = \varepsilon/M$. Then $\|u\|_1 < \delta \Rightarrow \|Tu\|_V \leqslant M \cdot \|u\|_1 < \varepsilon$.

3. Consider the space $\mathcal{C}$ of all continuous functions from $[0,1]$ to $[-1,+1]$ with the ∞-norm (the supremum norm). Show that the set $P = \{f \in \mathcal{C} : |f(x)| > 0 \text{ for all } x \in [0,1]\}$ is open in $\mathcal{C}$.

Let $g \in P$. Set $s = \inf_{x\in[0,1]} |g(x)| = \min|g(x)|$. Note $s > 0$.

claim: $B_{s/2}(g) \subset P$. pf. Let $h \in B$. Then $\forall x \in [0,1]$, $|g(x)| \leqslant |g(x)-h(x)| + |h(x)| < \frac{s}{2} + |h(x)|$

$\Rightarrow \forall x$, $|h(x)| > |g(x)| - s/2 \geqslant s - \frac{s}{2} = s/2$

So $h \in C$.

For this page do not use any other paper for solutions. Use the spaces provided below.

4. TRUE or FALSE. 8 pts each... Either prove or refute. Refuting is a proof; you can do this by giving an explicit counterexample and proving that that example works.

(a) Every function from X to a discrete metric space is continuous.

FALSE: Let $X = \mathbb{R}$, Y a discrete space with $a, b \in Y$.

Consider $f : \mathbb{R} \to Y$, $f(x) = \begin{cases} a, & x \in \mathbb{Q} \\ b, & x \notin \mathbb{Q} \end{cases}$. OR easier: $id : (\mathbb{R}, \text{eucl}) \to (\mathbb{R}, \text{discrete})$

Observe $\forall \delta > 0$, $f(B_\delta(0)) = \{a, b\}$.

So given $0 < \varepsilon < 1$, there is no $\delta > 0$ s.t. $f(B_\delta(0)) \subset B_\varepsilon(a) = a$

(b) $||\mathbf{x}||_{1/2}$ is a norm on $\mathbb{R}^n$, $n > 0$.

FALSE: For $n=2$ and $(1,0), (0,1) \in \mathbb{R}^2$

$\|(1,0) + (0,1)\|_{1/2} = \|(1,1)\|_{1/2} = (\sqrt{1} + \sqrt{1})^2 = 4$,

while

$\|(1,0)\|_{1/2} + \|(0,1)\|_{1/2} = 1 + 1 = 2$

triangle inequality fails

(c) Let B be an arbitrary open ball in a space Y with an ultrametric g. Then any point of B is a center of B.

TRUE: Let $B = B_r(y) = \{x \in Y \mid d(x,y) < r\}$

Take $z \in B$. Then $\forall x \in B$, $d(x,z) \leq \max(g(x,y), g(y,z))$
$< \max(r, r) = r$.

Hence $B_r(y) \subseteq B_r(x)$. Similarly $B_r(x) \subseteq B_r(y)$.

5. TRUE or FALSE? 3 pts each... No justification required. An incorrect answer cancels a correct one.

1. [F] In any metric space, any finite subset has empty interior.
In a discrete space, the interior of a singleton is nonempty.

2. [T] For any $a \neq b \in X$, there are **open** sets A and B in X such that $a \in A$, $b \in B$, $A \cap B = \emptyset$.
Set $r = d(a,b)$. Then the open balls $B_{r/2}(a)$ and $B_{r/2}(b)$ are disjoint.

3. [F] For the norm-∞ unit sphere in $\mathbb{R}^2$, its diameter in any p-norm is 2.


$d_1(A,B) = 4$ in the 1-norm.

4. [T] With respect to any p-norm on $\mathbb{R}^2$, the sequence $\left((\frac{1}{j}, \frac{1}{j^2})\right)_{j=1}^{\infty}$ converges to $(0,0)$.
The sequence converges to 0 in Euclidean norm. Hence in any p-norm.

5. [F] Let d and d' be **equivalent** metrics on X.
A sequence is Cauchy with respect to d **if and only if** it is Cauchy with respect to d'.
$((0,1), \text{eucl})$ homeom to $(\mathbb{R}, \text{eucl})$. However $(1 - \frac{1}{n})_{n=1}^{\infty}$ is Cauchy in the 1st, not in 2nd.

6. [T] A **linear** mapping from one normed space to another is continuous **if and only if** it is bounded on bounded sets.
a thm