What does similarity mean geometrically?

$$S^{-1}AS = B$$

Let's consider a basis $\mathcal{V} = \{v_1, \dots, v_n\}$ & the standart basis $\mathcal{S} = \{e_1, \dots, e_n\}$. Let S turn vectors expressed in $\mathcal{V}$ into vectors in $\mathcal{S}$.

$$\underset{\mathcal{S}\to\mathcal{V}}{S^{-1}} \cdot \underset{\mathcal{S}\to\mathcal{S}}{A} \cdot \underset{\mathcal{V}\to\mathcal{S}}{S} = \underset{\mathcal{V}\to\mathcal{V}}{B}$$

Setting up like that, A and B are two matrices for the same linear transformation.

In particular if A is diagble, the lin. transf. represented by A is nothing but a stretch/compress in suitably chosen n lin. indep directions

JORDAN CANONICAL FORM

If A is not diagble, what is the "next best" situation?

ex:

$$J = \begin{bmatrix} 3 & 1 & 0 & & & & & \\ & 3 & 1 & & & & & \\ & & 3 & & & & & \\ & & & 3 & 1 & & & \\ & & & & 3 & & & \\ & & & & & 2 & 1 & \\ & & & & & & 2 & \\ & & & & & & & 2 \end{bmatrix}$$

a Jordan canonical form

THM. Any square matrix is similar to a matrix in the form:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$$

where each J_ℓ is

a Jordan block

$$\begin{bmatrix} c & 1 & & 0 \\ & c & \ddots & \\ & & \ddots & 1 \\ 0 & & & c \end{bmatrix}$$

Here c is an evalue.

Rmk. If all Jordan blocks are 1x1 then J is diagonal.

ex. cont'd. Which matrices are similar to this J?

$$A \cdot [s_1 | \cdots | s_8] = AS = SJ$$

$$= [s_1 | \cdots | s_8] \cdot J$$

$$= [3s_1 \,|\, s_1 + 3s_2 \,|\, s_2 + 3s_3 \,|\, 3s_4 \,|\, s_4 + 3s_5 \,|\, 2s_6 \,|\, s_6 + 2s_7 \,|\, 2s_8]$$

$\Rightarrow$ s_1, s_4 evectors of A for 3

s_6, s_8 evectors of A for 2

$As_2 = 3s_2 + s_1$, $As_5 = 3s_5 + s_4$

$As_3 = 3s_3 + s_2$, $As_7 = 2s_7 + s_6$

Let's play:

$$As_2 = 3s_2 + s_1 \iff \boxed{(A-3I)s_2 = s_1}$$

was evector for 3
$s_1 \in null(A-3I)$

$$\Rightarrow (A-3I)^2 s_2 = 0$$
$$\iff s_2 \in null(A-3I)^2$$

Similarly $s_3 \in null(A-3I)^3$.

defn: If $(A-\lambda I)u = v_1$ (generalized) an evector then u is called a generalized evector. The space $null(A-\lambda I)^m$ (m = mult) is called the generalized espace for λ.

thmA: Given A, find its evalues & lin. indep evectors. For each evector, one can choose a chain of generalized evectors that're lin. indep. The matrix with columns the gen. evectors takes A to a Jordan form.

thm B. Jordan form for a matrix is unique up to row/column permutations.

ex: 1 Let $A_{5\times5}$ have evalues $0, 1, 1, 2, 2$. (so A is singular).

A has at least 3 lin. indep. evectors: u_0, u_1, u_2 for $0, 1, 2$.

The list of possible Jordan forms:

$\dim \mathcal{E}_1$	1	2	1	2
$\dim \mathcal{E}_2$	1	1	2	2

$$\begin{bmatrix} 0 & & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 2 & 1 \\ & & & & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & 1 \\ & & & & 2 \end{bmatrix} \quad \vdots \quad \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 2 \end{bmatrix}$$

We can say nothing more.

② $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. evalues $= 1, 1$

$$E_1 = \text{null}(B - 1 \cdot I) = \text{null}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \text{span}\overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{v_1}$$

Second vector w satisfies:

$$w \in \text{null}(B - I)^2 = \text{null}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2 .$$

More precisely w satisfies:

$$Bw = 1 \cdot w + v_1 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then $w = \begin{pmatrix} 1 \\ * \end{pmatrix}$ & $S = (v_1 \,|\, w)$

$*$ → choose whatever you want

$$S^{-1} = \frac{1}{\det S}\begin{pmatrix} * & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -* & 1 \\ 1 & 0 \end{pmatrix}$$

Check: $S^{-1} B S = S^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1+* \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \checkmark$$

Note: If you choose $* = 0$, S is a permutation

③ How to produce a non-diagble 2×2 matrix which is not already in the Jordan form:

We want C to be similar to $J=\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$; i.e. $S^{-1}CS=J$. Since $C=SJS^{-1}$, any wild choice for S, S^{-1} gives some C.

COMPLEXITY OF OUR ALGORITHM

↳ # of real multiplications

<u>Echelon form.</u> For $A_{n\times n}$,

$$ech = n(n-1) + (n-1)(n-2) + \cdots$$

$$= (n-1)\cdot 2\cdot(n-1) + (n-3)\cdot 2(n-3) + \cdots$$

$$= 2\left[(n-1)^2 + (n-3)^2 + \cdots\right]$$

n odd

$$= 8\cdot\left[\left(\frac{n-1}{2}\right)^2 + \left(\frac{n-3}{2}\right)^2 + \cdots\right]$$

$$= 8\cdot\frac{1}{6}\cdot\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\cdot\left(n-1+1\right)$$

$$= \frac{1}{3}\cdot(n-1)(n+1)\cdot n \sim n^3/3$$

<u>LU decomposition</u>

$$lu = 2ech \sim \frac{2}{3}n^3.$$

Matrix multiplication.

$M = n^3$

determinant.

using initial formula: $\underbrace{n \cdot n!}_{\text{very large}}$!

use echelon form: $\delta = ech + n \sim n^3/3$.

matrix inversion.

Use Gauss-Jordan: $gj = 4ech \sim 4/3\, n^3$.

$\boxed{\sim 1974:}$ If there's an algorithm for matrix mult. with complexity M, then there is an algorithm for determinant with complexity M.

$\boxed{\sim 2016:}$ $M \sim n^{2.373}$.

THE STORY IS YET TO BEGIN.
KEEP IN TOUCH (WITH MATHS)
& HAVE FUN!