

What does similarity mean geometrically?

$$S^{-1}AS = B$$

Let's consider a basis  $\mathcal{V} = \{v_1, \dots, v_n\}$   
& the standard basis  $\mathcal{S} = \{e_1, \dots, e_n\}$ .  
Let  $S$  turn vectors expressed in  $\mathcal{V}$   
into vectors in  $\mathcal{S}$ .

$$\underset{\mathcal{S} \rightarrow \mathcal{V}}{S^{-1}} \cdot \underset{\mathcal{S} \rightarrow \mathcal{S}}{A} \cdot \underset{\mathcal{V} \rightarrow \mathcal{S}}{S} = \underset{\mathcal{V} \rightarrow \mathcal{V}}{B}$$

Setting up like that,  $A$  and  $B$   
are two matrices for the same  
linear transformation.

In particular if  $A$  is diagonal,  
the lin. transf. represented by  $A$   
is nothing but a stretch/compress  
in suitably chosen  $n$  lin. indep directions.

# JORDAN CANONICAL FORM

If  $A$  is not diagonalizable, what is the "next best" situation?

ex:

$J =$

3	1	0					
	3	1					
		3					
			3	1			
				3			
					2	1	
						2	
							2

a Jordan canonical form

THM. Any square matrix is similar to a matrix in the form:

$J =$

$J_1$			
	$J_2$		
		$\ddots$	
			$J_k$

where each  $J_\ell$  is

a Jordan block

$c$	1	0	
	$c$	$\ddots$	
		$\ddots$	
0			$c$

Here  $c$  is an eigenvalue.

Rank. If all Jordan blocks are  $1 \times 1$   
[then  $J$  is diagonal.

ex. cont'd. Which matrices are  
similar to this  $J$ ?

$$\begin{aligned} A \cdot [s_1 \mid \dots \mid s_8] &= AS = SJ \\ &= [s_1 \mid \dots \mid s_8] \cdot J \\ &= \begin{bmatrix} 3s_1 & s_1 + 3s_2 & s_2 + 3s_3 & \\ 3s_4 & s_4 + 3s_5 & \\ 2s_6 & s_6 + 2s_7 & \\ 2s_8 \end{bmatrix} \end{aligned}$$

$\Rightarrow$   $s_1, s_4$  e vectors of  $A$  for 3  
 $s_6, s_8$  e vectors of  $A$  for 2

$$\begin{aligned} AS_2 &= 3S_2 + S_1, & AS_5 &= 3S_5 + S_4 \\ AS_3 &= 3S_3 + S_2, & AS_7 &= 2S_7 + S_6 \end{aligned}$$

Let's play:

$$As_2 = 3s_2 + s_1 \Leftrightarrow (A - 3I)s_2 = s_1$$

$$\Rightarrow (A - 3I)^2 s_2 = 0$$

was evector for 3

$$s_1 \in \text{null}(A - 3I)$$

$$\Leftrightarrow s_2 \in \text{null}(A - 3I)^2$$

$$\text{Similarly } s_3 \in \text{null}(A - 3I)^3$$

defn: If  $(A - \lambda I)u = v_1 \rightarrow$  an evector (generalizing)

then  $u$  is called a generalized

evector. The space  $\text{null}(A - \lambda I)^m$  is called the generalized space <sup>mult</sup>

thm A. Given  $A$ , find its evals <sup>for  $\lambda$</sup>

& lin. indep eectors. For each evector, one can choose a chain of generalized eectors that're lin. indep. The matrix with columns the gen. eectors takes  $A$  to a Jordan form.

thm B. Jordan form for a matrix is unique up to row/column permutations.

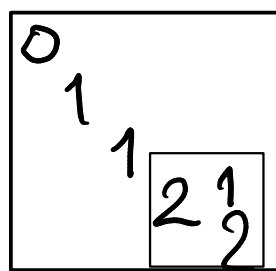
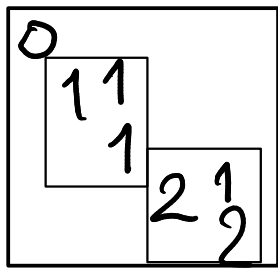
ex: ① Let  $A_{5 \times 5}$  have values  $\underline{0}, 1, 1, 2, 2$ . (so  $A$  is singular).

$A$  has at least 3 lin. indep.

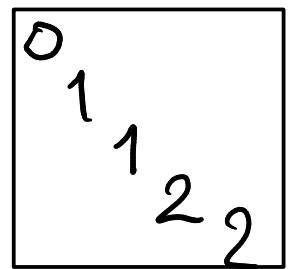
eigenvectors:  $u_0, u_1, u_2$  for  $0, 1, 2$ .

The list of possible Jordan forms:

$\dim \mathcal{E}_1$	1	2	1	2
$\dim \mathcal{E}_2$	1	1	2	2



⋮



We can say nothing more.

$$\textcircled{2} B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \text{eigenvalues} = 1, 1 \quad \textcolor{violet}{v_1}$$

$$E_1 = \text{null}(B - 1 \cdot I) = \text{null} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Second vector  $w$  satisfies:

$$w \in \text{null}(B - I)^2 = \text{null} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2.$$

More precisely  $w$  satisfies:

$$Bw = 1 \cdot w + v_1 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Then } w = \begin{pmatrix} 1 \\ * \end{pmatrix} \& S = (v_1; w)$$

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} * & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -* & 1 \\ 1 & 0 \end{pmatrix} \quad \textcolor{green}{\rightarrow \text{choose whatever you want}}$$

$$\begin{aligned} \text{Check: } S^{-1}BS &= S^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 1+* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \checkmark \end{aligned}$$

Note: If you choose  $*=0$ ,  $S$  is a permutation

③ How to produce a non-diagble  $2 \times 2$  matrix which is not already in the Jordan form:

We want  $C$  to be similar to  $J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ ; i.e.  $S^{-1}CS = J$ . Since

$C = SJS^{-1}$ , any wild choice for  $S, S^{-1}$  gives some  $C$ .

# COMPLEXITY OF OUR ALGORITHM

↳ # of real multiplications

Echelon form. For  $A_{n \times n}$ ,

$$\text{ech} = n(n-1) + (n-1)(n-2) + \dots$$

$$= (n-1) \cdot 2 \cdot (n-1) + (n-3) \cdot 2(n-3) + \dots$$

$$= 2 \left[ (n-1)^2 + (n-3)^2 + \dots \right]$$

$$\stackrel{n \text{ odd}}{\rightarrow} = 8 \cdot \left[ \left( \frac{n-1}{2} \right)^2 + \left( \frac{n-3}{2} \right)^2 + \dots \right]$$

$$= 8 \cdot \frac{1}{6} \cdot \left( \frac{n-1}{2} \right) \left( \frac{n-1}{2} + 1 \right) \cdot (n-1+1)$$

$$= \frac{1}{3} \cdot (n-1)(n+1) \cdot n \sim n^3/3$$

LU decomposition


$$\text{lu} = 2\text{ech} \sim \frac{2}{3}n^3.$$



# Matrix multiplication.

$$\mu = n^3$$

## determinant.

using initial formula:  $n \cdot n!$  <sup>very large</sup> 

use echelon form:  $S = ech + n$   
 $\sim n^3/3.$

## matrix inversion.

Use Gauss-Jordan:  $gj = 4ech$   
 $\sim 4/3 n^3.$

~1974: If there's an algorithm for matrix mult, with complexity  $\mu$ , then there is an algorithm for determinant with complexity  $\mu$ .

~2016:  $\mu \sim n^{2.373}$

THE STORY IS YET TO BEGIN.  
KEEP IN TOUCH (WITH MATHS)  
& HAVE FUN!