Spring 2024 Math 584 - Singularity Theory
Homework 8 - Milnor Fibration
Due: 6/6/2024

MIL: Singular points of complex hypersurfaces

1. The Milnor fibration is in fact a *mapping torus*. For a k-manifold F with boundary (if you are confused take $k = 2$ in this question) and a homeomorphism $\varphi : F \to F$ which is the identity map on ∂F, *the mapping torus of F with monodromy φ* is the quotient manifold-with-boundary $M(F;\varphi) = (F \times I)/\left((x,1) \sim (\varphi(x),0)\right)$.
(a) Recalling HW#7 Q2, show that in case F is oriented and φ is **orientation preserving** then M is orientable. (Recall for an oriented manifold-with-boundary $X \subset R^N$ of dimension m, the corresponding orientation on ∂X is given as follows: Let $p \in \partial X$ and $(u_1, \ldots, u_n)$ be a choice of ordered basis (an orientation) on T_pX such that u_1 is outward directed. Then $(u_2, \ldots, u_n)$ is the induced orientation on $T_p\partial X$.)
(b) Show that if $\varphi, \psi : F \to F$ are isotopic homeomorphisms then $M(F;\varphi)$ is homeomorphic to $M(F;\psi)$.

2. (a) Show: If 0 is a regular point of $f : \mathbb{C}^{n+1} \to \mathbb{C}$ then the fiber of the Milnor fibration is diffeomorphic to $\mathbb{R}^{2n}$. More precisely work out the details of the proof of MIL, p. 23, Lemma 2.13, assuming Lemma 2.12, which we have already proven.
(b) Then using Q1(b) show that the Milnor fibration in part (a) is trivial, i.e. $S_\varepsilon - K$ is homeomorphic to $\mathbb{R}^{2n} \times S^1$.

3. For a vector space V over $\mathbb{C}$, a Hermitian inner product $H : V \times V \to \mathbb{C}$ is complex linear in its first parameter, conjugate symmetric (i.e. $H(u,v) = \overline{H(v,u)}$) and positive definite (i.e. $H(u,u) > 0, \forall u \neq 0$). We have observed in the class that the real part of H is a real inner product on the underlying real space $V_\mathbb{R}$.
Now show that the imaginary part of H is a *symplectic form* $w : V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$, i.e. a **skew-symmetric** ($w(u,v) = -w(v,u)$), **non-degenerate** ($w(u,v) = 0, \forall v \in V_\mathbb{R} \Rightarrow u = 0$) **bilinear** form.

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