Last time.

☑ linear combination ; span

☑ linear independence of vectors $v_1, \ldots, v_k$:

whenever $c_1 v_1 + \ldots + c_k v_k = 0$ $(c_i \in \mathbb{R})$

these c_i's are forced to be zero.

☑ basis of a vector space.

$u_1, \ldots, u_n \in V$ is a basis if

- $V = \text{span}(u_1, \ldots, u_n)$
- $u_1, \ldots, u_n$ are linearly independent.

☑ dimension of V is the # of vectors in a basis.

ex. $\mathbb{R}^n$ with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

because if for $c_i \in \mathbb{R}$

$c_1 e_1 + \ldots + c_n e_n = 0$ then:

$\begin{pmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies$

$c_1 = 0$

$\vdots$

$c_n = 0$

Example for today: $A_{4\times 6}$

$A \xrightarrow{\text{Row reduction}} \begin{bmatrix} 1 & -1 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$

$\quad\quad\quad\quad\quad\quad\quad\quad x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6$

Recall

$(*) \quad F \cdot A = U \quad \& \quad A = F^{-1} U$
$\quad\quad 4\times 4$

① $\boxed{\text{null}(A)} = \text{null}(U)$ because

fact 1. If $P = Q \cdot R$ & Q invertible
then $\text{null } P = \text{null } R$

Now $\text{null}(U) = \{ x \in \mathbb{R}^6 \mid Ux = 0 \}$

$= \{ x_5 = -x_6 , \; x_3 = -x_4 - x_6 , \; x_1 = x_2 - 2x_4 + 3x_5 - x_6 \}$

$= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = U_2 \; + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = U_4 \; + x_6 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = U_6 \right\}$

$$= \left\{ x \in \mathbb{R}^6 \mid \begin{array}{l} x = x_2 u_2 + x_4 u_4 + x_6 u_6, \\ x_2, x_4, x_6 \in \mathbb{R} \end{array} \right\}$$

$$= \text{span}(u_2, u_4, u_6)$$

Moreover u_2, u_4, u_6 are linearly independent: Assume for some $c_1, c_2, c_3 \in \mathbb{R}$ $\quad c_1 u_2 + c_2 u_4 + c_3 u_6 = 0$

Then
$$\begin{cases} c_1 - 2c_2 + 2c_3 = 0 \\ \quad c_1 = 0 \\ -c_2 - c_3 = 0 \\ \quad c_2 = 0 \\ -c_3 = 0 \\ \quad c_3 = 0 \end{cases} \Rightarrow \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0. \end{array}$$

thm 1. • A basis for $\text{null}(A)$ is the collection of vectors in $\mathbb{R}^q$ found by the above algorithm.

• $\dim(\text{null}(A_{p \times q})) = \#$ of free variable $= q - \#$ pivots

The argument above proves also:

fact 2: The following collection is linearly independent:

$$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \end{pmatrix} \right]$$

② $\boxed{\text{col}(A)} = ?$

ex:
$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\text{col} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \text{col} = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Although $\text{col}(A) \neq \text{col}(U)$ in general they are related as below:

$\text{col}(U)$ has basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\text{col}(A)$ has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{col}(U) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$= \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right)$$

Moreover by fact 2, these constitute a basis. Hence we proved:

thm2a. $\dim(\text{col}(U)) = \#$ pivots.

defn: rank of A is the $\#$ pivots

Then $\dim(\text{col}(U)) = r = \text{rank}(A)$.

What about $\text{col}(A)$?

Recall: $F \cdot A = U$, $A = F^{-1} \cdot U$.

By fact 1, $\text{null}(A) = \text{null}(U)$

i.e.
all possible linear combinations of the columns of A that give $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the same as all possible linear combos of U that give $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

i.e.
If a nonzero linear comb of columns of U gives 0 the the same nonzero comb of columns of A gives 0 too.
(and vice versa)

i.e. the linearly indep set of columns of A is exactly the corresponding lin. indep columns of U.

thm 26.

- The col (A) has a basis which is constituted of the columns corresp to the columns of U that are basis for col (U).

- $\dim(\text{col}(A)) = \dim(\text{col}(U)) = \text{rank}(A)$

In the example:

col (U) has basis the $1^{st}, 3^{rd} \& 5^{th}$ columns (the ones with pivots)

So col (A) has basis the $1^{st}, 3^{rd}, 5^{th}$ columns of A.

③ $\boxed{\text{row}(A)} = ?$

= span of rows of A

= span of rows of U = row(U)
↓
because rows of U are lin. combos.
of rows of A & vice versa.

An elegant way of saying this:

$F \cdot A = U$ is nothing but taking
linear combinations of rows of A
to produce rows of U.

In our example:

$$\text{row}(U) = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 0 \\ 6 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$
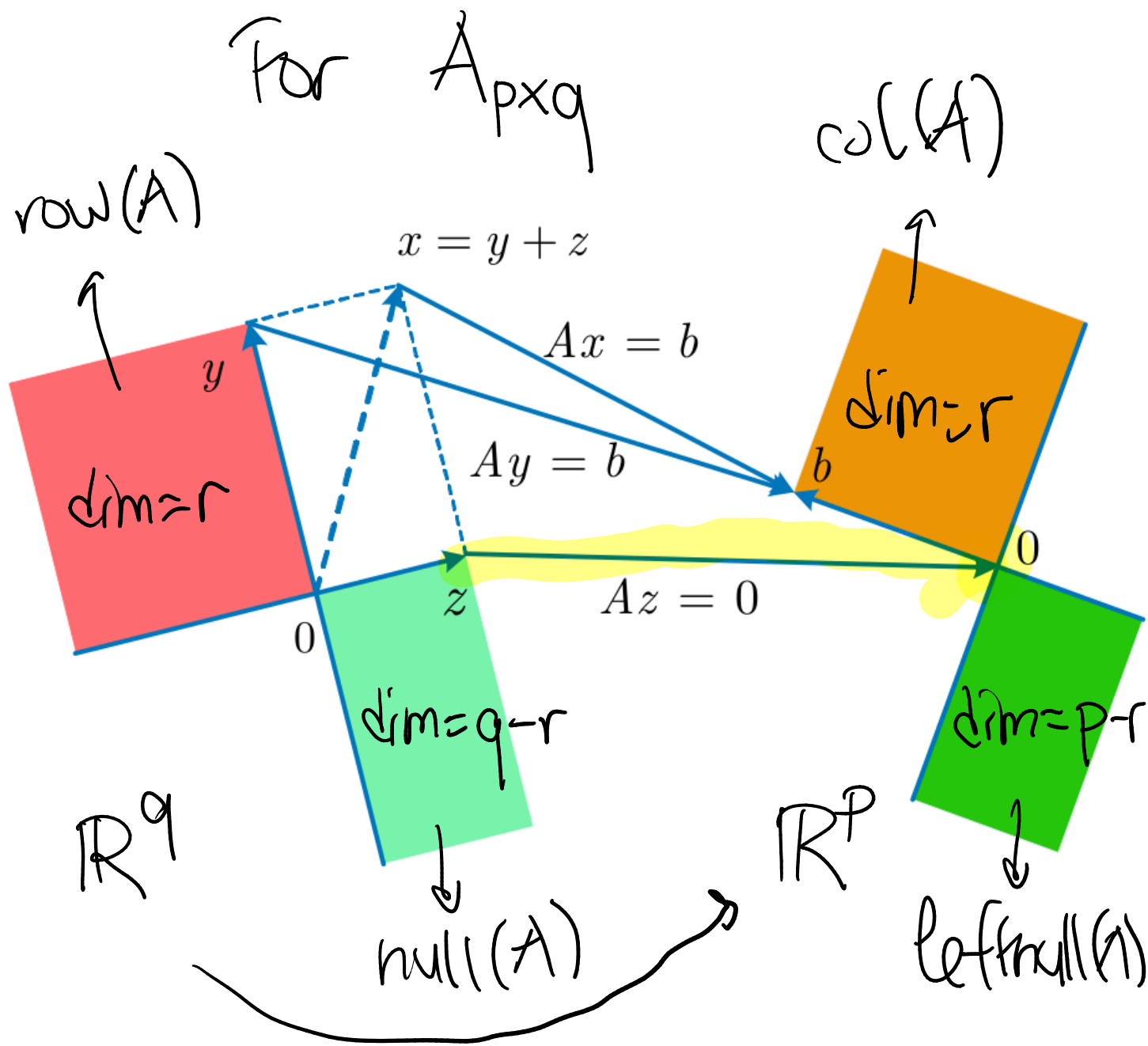
$\nearrow$
$\mathbb{R}^6$

By fact 2, the first 3 rows are linearly independent. This proves:

<u>thm 3</u> • row(A) has basis the rows corresponding to the pivots.

• $\dim(\text{row}(A)) = r = \text{rank}(A)$.

<u>Rank-Nullity Theorem</u> (or the Fundamental Thm of lin. Alg.)

(i) null(A), row(A) are subspaces of $\mathbb{R}^q$ $^{p \times q}$ with $\dim$ $q-r$ & r respectively.

(ii) null(A^T), $\overset{\text{row}(A^T)}{\underset{\uparrow}{\text{col}(A)}} \subset \mathbb{R}^P$ with dimensions $p-r$ & r respectively

for $A_{p \times q}$

row(A)

col(A)

$x = y + z$

$Ax = b$

$Ay = b$

y

dim = r

b

dim = r

$Az = 0$

z

0

0

dim = q-r

dim = p-r

$\mathbb{R}^q$

null(A)

$\mathbb{R}^p$

leftnull(A)

f

Define $f: \mathbb{R}^q \to \mathbb{R}^p$, $f(x) = Ax$.
Observe: image$(f) = $ col(A)
We'll see later:

• f sends row(A) to col(A) in a "1-1" onto fashion.
• row(A) is orthogonal to null(A)

Dimension re(ally)-visited.

If there were two bases with different # of vectors then dimension would not be defined. We claim that is impossible.

thm 4. Let $V \overset{=\mathbb{R}^?}{}$ be a vector space & $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_k)$ be two bases. Then $n = k$ (so that dim. is well-defined.

proof. Suppose $n < k$ &

$$v_j = a_{1j} u_1 + \cdots + a_{nj} u_n \quad \text{for each } j.$$

i.e.

$$\left(v_1 \,\vdots\, \cdots \,\vdots\, v_k\right) = \left(u_1 \,\vdots\, \cdots \,\vdots\, u_n\right)\begin{pmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & \ddots & \vdots \\ a_{1n} & & a_{kn} \end{pmatrix}$$

Since $n < k$, $\dim \operatorname{null}(A) = k - \operatorname{rank}(A)$
$$\geq k - n > 0; \text{ i.e.}$$

$\operatorname{null}(A) \neq \{0\}$; in particular, there is some $0 \neq x \in \mathbb{R}^k$ s.t. RHS·x=0

So LHS·x=0 $\Longleftrightarrow$ the columns $v_1, \cdots, v_k$ are linearly dependent. This is a contradiction. To get rid of that we give up the assumption $n < k$. With similar argument for $n > k$, we get $n = k$.