thm: The following sentences are equivalent for a square matrix $A_{p\times p}$:

1. After row exchanges one can get an upper triangular U with all pivots nonzero, i.e. no pivot is missing.
2. A has an inverse.
3. For every $b \in \mathbb{R}^p$, $Ax=b$ has unique solution.
4. $\text{null}(A) = \{0\}$ (4.5) $\dim \text{null}(A) = 0$
5. $\#\text{pivots} = \text{rank}(A) = p$
6. $\dim \text{leftnull}(A) = 0$
7. $\dim \text{row}(A) = p$
8. the rows of A is a basis for $\mathbb{R}^p$.
9. $\dim \text{col}(A) = p$
10. the columns of A is a basis for $\mathbb{R}^p$.
11. $\det A \neq 0$

today:

12. All eigenvalues of A are nonzero.

Recall: $A_{n\times n}$; $0\neq u \in \mathbb{R}^n$, λ a complex number

☐ $$Au = \lambda u$$

λ ↳ an evalue of A

u ↳ an evector for λ.

☐ $\mathcal{E}_\lambda$ = set of evectors for λ $\cup \{0\}$

is a subspace of $\mathbb{R}^n$, called the <u>eigenspace</u> for λ.

$$Au = \lambda \overset{\neq 0}{u} \iff Au - \lambda u = 0$$

$$\iff (A-\lambda I)\overset{\neq 0}{u} = 0$$

$$\iff u \in \text{null}(A-\lambda I),\ u \neq 0$$

$$\iff \text{null}(A-\lambda I) \neq 0$$

$$\iff \det(A-\lambda I) = 0 \quad \text{for some } \lambda \in \mathbb{C}$$

We can compute now!

ex. 1 $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. An evalue λ of A satisfies $0=|A-\lambda I|$

$$=\det\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right]$$

$$=\begin{vmatrix}1-\lambda&0\\0&-\lambda\end{vmatrix}=\underbrace{\lambda(\lambda-1)}_{\text{char. polyn of } A}\quad \deg=2$$

$\lambda=0$ OR 1 : two eigenvalues

$$E_0=\operatorname{null}(A-0\cdot I)=\operatorname{null}\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$=\left\{b\begin{pmatrix}0\\1\end{pmatrix}\;\middle|\; b\in\mathbb{R}\right\}=y\text{-axis}$$

$$E_1=\operatorname{null}(A-1\cdot I)=\operatorname{null}\begin{pmatrix}0&0\\0&-1\end{pmatrix}$$

$$=\left\{a\begin{pmatrix}1\\0\end{pmatrix}\;\middle|\; a\in\mathbb{R}\right\}=x\text{-axis}$$

Geometrically these were obvious: A projects to the x-axis.

② Given $A_{n\times n} = (a_{ij})_{1\leq i,j\leq n}$.

$$0 = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & \\ a_{21} & a_{22}-\lambda & \cdots & \\ & \vdots & \ddots & \\ & \vdots & & a_{nn}-\lambda \end{vmatrix}$$

wrt the 1st row.

$$= (a_{11}-\lambda)\cdot \begin{vmatrix} a_{22}-\lambda & a_{23} & \cdots \\ a_{32} & \ddots & \\ \vdots & & a_{nn}-\lambda \end{vmatrix}$$

$$- a_{12}\cdot \begin{vmatrix} a_{21} & a_{23} & \cdots \\ a_{31} & a_{33}-\lambda & \\ \vdots & & \ddots \end{vmatrix} + \text{other}$$

$= (a_{11}-\lambda)\cdot\ldots\cdot(a_{nn}-\lambda) +$ other terms which do not contain λ^n.

$= (-\lambda)^n +$ other terms with λ^k, $0\leq k<n$.

This is a polynomial in λ with degree $= n$.
It is called the characteristic polynomial of A.

So an evalue of A is a root of the char. polyn. of A.

<u>THM</u> (Fundamental Theorem of Algebra)

Every polyn with deg n (& with real coefficients) has exactly n roots, counted with <u>multiplicity</u>.

how many times the value is the root.

Among these n cx roots some may be real; how many we don't know.

Our evalues $\lambda \in \mathbb{C}$ ~ cx numbers

$$\{a+i\cdot 0 \mid a\in\mathbb{R}\} = \mathbb{R} \subset \left\{a+ib \,\middle|\, \begin{matrix} a,b\in\mathbb{R} \\ i^2=-1 \end{matrix}\right\}$$

③ char polyn of $I_{n\times n}$

$$= (1-\lambda)^n = 0$$

$\lambda = 1$ with multiplicity n.

④ $D = \mathrm{diag}(d_1, \dots, d_n)$

char. polyn. $= \begin{vmatrix} d_1-\lambda & & 0 \\ & d_2-\lambda & \\ 0 & & \ddots \\ & & & d_n-\lambda \end{vmatrix}$

$$= (d_1-\lambda) \cdot \ldots \cdot (d_n-\lambda)$$

evalues are $d_1, \dots, d_n$.

⑤ $U = \begin{pmatrix} u_1 & & * \\ & u_2 & \\ 0 & & \ddots \; u_n \end{pmatrix}$

char polyn $= (u_1-\lambda) \cdot \ldots \cdot (u_n-\lambda)$

evalues of U: $u_1, \dots, u_n$

Also true for lower tri. matrices.

⑥ Consider $P_{n\times n}$, $P^2=P$; let $0 \neq u \in \mathbb{R}^n$ be an evector for an evalue λ. i.e. $Pu=\lambda u$. Then

$$\lambda^2 u = \lambda P u = P(\lambda u) = P^2 u = Pu = \lambda u$$

$$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0 \text{ OR } 1.$$

(Counted with multiplicity the total is n)

A particular example is a proj matrix (it is also symmetric).

$\mathcal{E}_1 = \mathrm{col}(P)$; $\dim \mathcal{E}_1 = \mathrm{rank}(P)$

multiplicity of 1

$\mathcal{E}_0 = (\mathrm{col}(P))^{\perp} = \mathrm{leftnull}(P)$

$\dim \mathcal{E}_0 = n - \mathrm{rank}(P)$.

multiplicity of 0

⑦ $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. char. polyn $= (1-\lambda)^2$.

$\lambda = 1$ is an evalue with mult $= 2$.

$$\mathcal{E}_1 = \text{null} \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}_{\lambda=1} = \text{null} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$= \left\{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$; $\dim \mathcal{E}_1 = 1 < 2$!

Where is the missing dimension?!!

⑧ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\text{char. polyn.} = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$= \lambda^2 - (\underbrace{a+d}_{\substack{\text{sum of} \\ \text{diag entries of } A \,=\, \text{trace of } A}})\lambda + (\underbrace{ad-bc}_{\det A}) = (\lambda_1 - \lambda) \cdot (\lambda_2 - \lambda)$$

Then $\lambda_1 + \lambda_2 = \text{trace}(A)$.

$\lambda_1 \cdot \lambda_2 = \det(A)$

(9) $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\lambda_1 + \lambda_2 = 2\cos\theta$, $\lambda_1 \cdot \lambda_2 = 1$.

$\lambda_{1,2} = \frac{1}{2}\left(+2\cos\theta \pm \sqrt{4\cos^2\theta - 4}\right)$

Observe $4\cos^2\theta - 4 \leq 0$.

If $\cos^2\theta = 1$ i.e. $\theta = 0, \pi$

then $\lambda_1 = \lambda_2 = +\cos\theta$; mult $= 2$.

$= \begin{cases} 1, & \theta = 0 \\ -1, & \theta = \pi \end{cases}$

Otherwise, there are 2 cx evalues.

The observation in (8) works always.

<u>thm</u>: Given $A_{n\times n}$. Let $\lambda_1, \dots, \lambda_n$ be its evalues.

(a) $\lambda_1 \cdot \dots \cdot \lambda_n = \det A$

(b) $\lambda_1 + \dots + \lambda_n = \operatorname{trace}(A)$

Proof. char polyn of A

$$= (\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$$

$$\overset{(*)}{=} \begin{vmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & & \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{vmatrix}$$

(a) Insert $\lambda = 0$ above:

$\lambda_1 \cdot \ldots \cdot \lambda_n = \det A$.

(b) In $(*)$ there are 2 polynomials of deg n which are equal to each other; i.e. their corresp coeffs are equal.

Let's compute the coefficients of $(-\lambda)^{n-1}$ in these polynomials.

$\Rightarrow$ LHS $:= (-\lambda)^n + \lambda_1(-\lambda)^{n-1} + \lambda_2(-\lambda)^{n-1}$
$+ \cdots + \lambda_n(-\lambda)^{n-1}$ + other terms with less degree

The coeff of $(-\lambda)^{n-1}$ is $\lambda_1 + \cdots + \lambda_n$.

RHS $= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$
+ others that contain $(-\lambda)^k$ with $0 \leq k \leq n-2$

$= (-\lambda)^n + (a_{11} + \cdots + a_{nn})(-\lambda)^{n-1}$
+ others that contain $(-\lambda)^k$ with $0 \leq k \leq n-2$

The coeff of $(-\lambda)^{n-1}$ here is

$a_{11} + \cdots + a_{nn} = \text{trace}(A)$

$\blacksquare$

ex: $A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$, trace $= 4$, det $= -90$

char. polyn. $= \begin{vmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{vmatrix}$

$$= \begin{vmatrix} -\lambda & -4 & 2 \\ 0 & 1-\lambda & 2 \\ 9-\lambda & 2 & 5-\lambda \end{vmatrix}$$

$$= -\lambda \cdot \left(\lambda^2 - 6\lambda + \overbrace{5-4}^{1} \right)$$

$$+ (9-\lambda) \cdot \left(\overbrace{-8-2}^{-10} + 2\lambda \right)$$

$$= -\lambda^3 + 4\lambda^2 + 27\lambda - 90 = 0$$

$\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = -5$.

exercise. Find $\mathcal{E}_3$, $\mathcal{E}_6$, $\mathcal{E}_{-5}$.