ORTHONORMALITY & ORTHOGONAL MATRICES

- $v_1, \ldots, v_k \in \mathbb{R}^n$ is called an <u>orthogonal collection</u> if $v_i \perp v_j$, $\forall i \neq j$, i.e. $v_i^T v_j = 0$.
- They're called an <u>orthonormal collection</u> if $v_i^T v_j = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$
- If a square matrix has its columns orthonormal then it's called an <u>orthogonal matrix</u>.
- For $Q_{n \times n} = (\alpha_1 | \cdots | \alpha_n)$ orthogonal,

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 | \cdots | \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \\ \alpha_2^T \alpha_1 & \ddots \\ \vdots & \\ \alpha_n^T \alpha_1 & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \vdots \\ \vdots & 0 & \ddots \\ & \vdots & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Alternative defn: Q is an <u>orthogonal matrix</u> if $\boxed{Q^{-1} = Q^T}$

- So $QQ^T = I$ too. This shows that the rows of Q are an orthonormal collection too!
- If $A_{n\times k}$ has orthonormal columns (A not necess. square) then $\underset{k\times n}{A^T}\underset{n\times k}{A} = \underset{k\times k}{I}$
- If $A_{n\times k}$ has orthonormal columns then $P = A(A^TA)^{-1}A^T = AA^T$.

GRAM-SCHMIDT ORTHOGONALIZATION

Q: Is it true that any vector space V with an inner product has an orthonormal basis?

ex: • On $\mathbb{R}^n$: $\{e_1, e_2, \ldots, e_n\}$
• On $\mathbb{R}^2$: $\left\{\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}\right\}$

thm: Given V with a basis $\{v_1, \dots, v_k\}$ one can turn this basis into an orthonormal basis $\{p_1, \dots, p_k\}$

ex: $V \subset \mathbb{R}^4$, $V = \text{span}(v_1, v_2, v_3)$ where $v_1 = (1\ 1\ 0\ 1)^T$, $v_2 = (1\ 0\ 0\ 1)$, $v_3 = (1\ {-2}\ 1\ 1)^T$.

Set: $q_1 = v_1$

$$q_2 = v_2 - \text{proj}_{q_1} v_2 = v_2 - \frac{q_1^T v_2}{q_1^T q_1} q_1$$

$$q_3 = v_3 - \text{proj}_{\text{span}(q_1, q_2)} v_3$$

$$= v_3 - \text{proj}_{q_1} v_3 - \text{proj}_{q_2} v_3$$

$$= v_3 - \frac{q_1^T v_3}{q_1^T q_1} q_1 - \frac{q_2^T v_3}{q_2^T q_2} q_2$$

Then $\{q_1, q_2, q_3\}$ is an orthogonal basis.

So $\{p_1 = \frac{q_1}{\|q_1\|}, \ldots, p_3 = \frac{q_3}{\|q_3\|}\}$ is an orthonormal basis for V.

In our example:

$$q_1 = (1\ 1\ 0\ 1)^T, \ \|q_1\| = \sqrt{3}$$

$$\frac{1}{3} q_2 = (1\ 0\ 0\ 1)^T - \frac{2}{3}(1\ 1\ 0\ 1)^T$$

$$= \left(\frac{1}{3}, -\frac{2}{3}, 0, \frac{1}{3}\right)^T = \frac{1}{3}\underbrace{(1, -2, 0, 1)^T}_{q_2}$$

$$\|q_2\| = \sqrt{6}$$

$$q_3 = (1\ -2\ 1\ 1)^T - \frac{0}{3} q_1 - \frac{6}{6}(1\ -2\ 0\ 1)^T$$

$$= (0\ 0\ 1\ 0)^T$$

Check: $q_i^T q_j = 0, \ i \neq j$.

So $p_1 = \frac{1}{\sqrt{3}}(1\ 1\ 0\ 1)^T, \ p_2 = \frac{1}{\sqrt{6}}(1\ -2\ 0\ 1)^T$

$p_3 = (0\ 0\ 1\ 0)^T$ is an orthonormal basis for V.

This is the Gram-Schmidt Orthogonalization Process.

Remark. 1) Even if $\{v_1, \dots, v_k\}$ is not linearly independent, the process runs. If at some j,

$$v_j \in \operatorname{span}(v_1, \dots, v_{j-1}) = \operatorname{span}(q_1, \dots, q_{j-1})$$ then

$$q_j = v_j - \operatorname{proj}_{\operatorname{span}(v_1, \dots, v_{j-1})} v_j = v_j - v_j = 0.$$

Hence the number of nonzero q_j's at the end of the process is equal to the dim of $\operatorname{span}(v_1, \dots, v_k)$

2) Given $\{v_1, \dots, v_k\}$, assume we extract $p_1, \dots, p_m$ from $v_1, \dots, v_m$ & the

rest v_j, $j>m$, depend on the previous ones. Let's express v_j's in terms of $p_1, \dots, p_m$.

$$v_1 = \mathrm{proj}_{p_1} v_1 = \frac{p_1^T v_1}{\cancel{p_1^T p_1} \to 1} p_1 = (p_1^T v_1) p_1$$

$$v_2 = \mathrm{proj}_{p_1} v_2 + \mathrm{proj}_{p_2} v_2$$

$$= (p_1^T v_2) p_1 + (p_2^T v_2) p_2$$

$$v_m = (p_1^T v_m) p_1 + \dots + (p_m^T v_m) p_m$$

$$v_{m+1} = (p_1^T v_{m+1}) p_1 + \dots + (p_m^T v_{m+1}) p_m$$

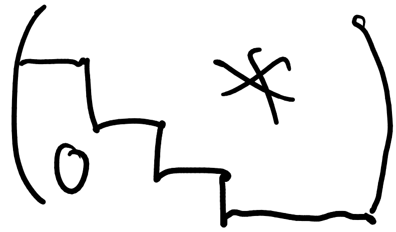
Thus we obtain:

$$\underbrace{\left(v_1 \,\vdots\, \cdots \,\vdots\, v_k \right)}_{A} = \underbrace{\left(p_1 \,\vdots\, \cdots \,\vdots\, p_m \right)}_{= Q} \cdot \underbrace{\begin{pmatrix} p_1^T v_1 & p_1^T v_2 & \cdots & p_1^T v_m & p_1^T v_{m+1} & \cdots \\ 0 & p_2^T v_2 & \cdots & \vdots & \vdots & \cdots \\ \vdots & 0 & & \vdots & & \\ 0 & 0 & \cdots & p_m^T v_m & p_m^T v_{m+1} & \cdots \end{pmatrix}}_{R =} \qquad (m \leq k)$$

thm: • Any matrix A can be written as $A = Q \cdot R$

Q: has orthonormal columns.

R:



• If the given A is square with lin. indep. columns then

$A = Q \cdot R$

Q: orthogonal

R: upper triangular

This is called a QR-decomposition of A.

Facts on orthogonal matrices.

① $Q^T = Q^{-1}$.

② Q has orthonormal columns & orthonormal rows.

(3) $Q_{n\times n} : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Qx$.

This transformation preserves lengths:

$$\|Qx\|^2 = (Qx)^T(Qx) = x^T(Q^TQ)x$$
$$= x^Tx = \|x\|^2$$

(4) Q preserves the inner products:

$$(Qx)^T(Qy) = x^TQ^TQy = x^Ty.$$

(5) Hence Q preserves the angles.

(6) $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal

It rotates by $+\pi/2$.

(7) $\begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix}$ is orthogonal; rotates by $+\theta$.

(8) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is orthogonal; reflects wrt the subspace $\{x=y\}$

(9) Any 2×2 orthog. matrix is either a rotation or a reflection wrt a line thru O.

(10) Let $Q_{3\times3}$ be orthogonal.

<u>THM</u>: It's either a rotation of $\mathbb{R}^3$ by some θ about some line thru O; OR it's a reflection wrt a plane thru O.