Rank-Nullity Theorem. For $A_{p\times q}$

$\dim \operatorname{row} A + \dim \operatorname{null} A = q$

$\|$

$\dim \operatorname{col} A + \dim \operatorname{null} A^T = p$

$= \operatorname{rank} A = \#\text{pivots}$

thm: The following sentences are equivalent for a square matrix $A_{p\times p}$:

(1) After row exchanges one can get an upper triangular U with all pivots nonzero, i.e. no pivot is missing.

(2) A has an inverse.

(3) For every $b \in \mathbb{R}^p$, $Ax=b$ has unique solution.

(4) $\operatorname{null}(A) = \{0\}$ (4.5) $\dim \operatorname{null}(A) = 0$

(5) $\#\text{pivots} = \operatorname{rank}(A) = p$

(6) $\dim \operatorname{leftnull}(A) = 0$

(7) $\dim \operatorname{row}(A) = p$

(8) the rows of A is a basis for $\mathbb{R}^p$.

(9) $\dim \operatorname{col}(A) = p$

(10) the columns of A is a basis for $\mathbb{R}^p$.

LINEAR TRANSFORMATIONS

<u>defn</u>: A function $f: V \to W$ is a linear transformation if

vector spaces

(a) For any $u, v \in V$,

$$f(u +_V v) = f(u) +_W f(v)$$

(b) For any $u \in V$, $t \in \mathbb{R}$,

$$f(t \cdot_V u) = t \cdot_W f(u)$$

<u>examples.</u>

1. Let $V = \mathbb{R} = W$. Fix $a, b \in \mathbb{R}$.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$.

(a) $f(x+y) = a(x+y) + b$

$$f(x) + f(y) = (ax+b) + (ay+b)$$

Are equal iff $b = 0$.

(b) $\left.\begin{aligned} f(tx) &= atx + b \\ tf(x) &= atx + tb \end{aligned}\right\} =$ iff $b=0$

When $b=0$, f is a linear transf.

② $V=\mathbb{R}^q$, $W=\mathbb{R}^p$; fix $A_{p\times q}$.

$f(v)=A\cdot v$ is a linear transf.

Multiplying by a matrix is a linear transformation.

③ Any lin transf $f: V \to W$ takes 0_V to 0_W: because

$$f(0_V)=f(\underset{\underset{\mathbb{R}}{\uparrow}}{0} \cdot_V v) \underset{(b)}{=} \overset{\in\mathbb{R}}{0} \cdot_W f(v) = 0_W$$

Alternatively one could prove using (a):

$$f(0_V)=f(\underset{\underset{V}{\uparrow}}{u} +_V (-u)) \underset{(a)}{=} f(u)+f(-u)$$

$$\underset{=}{\overset{(b)}{}} f(u)+(-1)\cdot f(u) = 0_W$$

4) $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $g((x,y)) = (x+y, y, 2x-y)$

is a lin. transf. because

(b) $g(t(x,y)) = g((tx, ty))$

$= (tx+ty, ty, 2tx-ty)$

$= t(x+y, y, 2x-y) = tg((x,y))$

(a) $(x,y) \in \mathbb{R}^2$, $(z,w) \in \mathbb{R}^2$

$g((x,y)+(z,w)) = g((x+z, y+w))$

$= (x+z+y+w, y+w, 2x+2z-y-w)$

$= (x+y, y, 2x-y) + (z+w, w, 2z-w)$

$= g((x,y)) + g((z,w))$.

⑤ P_n = set of all polynomials with real coeffs with degree $\leq n$

$$= \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$$

with the polynomial addition

& the scalar mult: $t\left(\sum_{i=0}^{n} a_i x^i\right) = \sum t a_i x^i$

P_n is a vector space because

G2: 0 polynomial is the zero of $+$.

G3: $\left(\sum_{i=1}^{n} a_i x^i\right) + \left(\sum_{i=1}^{n} (-a_i) x^i\right) = 0$.

G1, G4, S1, S2, S3 are satisfied too.

Define:

$$\Delta : P_3 \longrightarrow P_4$$

Δ produces a degree ≤ 2 polynomial

It goes to any P_n with $n \geq 2$.

$$\Delta(a_0 + a_1 x + \cdots + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2$$

(i.e. derivative of the polynomial)

claim: Δ is linear:

(a) $\Delta\left(\sum a_i x^i + \sum b_i x^i\right)$

$= \Delta\left(\sum (a_i + b_i) x^i\right)$

$= (a_1+b_1) + 2(a_2+b_2)x + 3(a_3+b_3)x^2$

$= \Delta\left(\sum a_i x^i\right) + \Delta\left(\sum b_i x^i\right)$

(b) $\Delta(t\underbrace{p}_{\in P_3}) = t\Delta(p)$

Now fact: Any linear transformation is multiplication with a matrix. We'll build this below, slowly...

thm: If you know what $f: V \to W$ does on a basis of V then you know all f.

proof. Assume $v_1, \dots, v_n$ is a basis $\mathcal{V}$ for V. Suppose we know $f(v_1), \dots, f(v_n)$. Then for any $v \in V$, we know its image: first $v = c_1 v_1 + \dots + c_n v_n$. Then

$$
\begin{aligned}
f(v) &= f(c_1 v_1 + \dots + c_n v_n) \\
&\overset{(a)}{=} f(c_1 v_1) + \dots + f(c_n v_n) \\
&\overset{(b)}{=} c_1 f(v_1) + \dots + c_n f(v_n).
\end{aligned}
$$

Notation: When we write

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_{\mathcal{V}}$$

it denotes the element $c_1 v_1 + \dots + c_n v_n$ in V.

back to ex 4.

$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \ g((x,y)) = (x+y, y, 2x-y)$

Consider the standard bases S & S' in $\mathbb{R}^2$ & in $\mathbb{R}^3$; i.e.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes e_1: unit vector along $+x$-axis etc.

Now:

$g(e_1) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ means $1e_1 + 2e_3$ in $\mathbb{R}^3$.

$\| \quad g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

& $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. What about $ae_1 + be_2$?

$$g(ae_1 + be_2) = a\,g(e_1) + b\,g(e_2)$$

$$= a\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

That is:

$$g: \begin{pmatrix} a \\ b \end{pmatrix}_{\mathcal{S}} \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

↳ takes vectors in $\mathcal{S}$ and gives out vectors in $\mathcal{S}'$

With these conventions, g is represented by the 3×2 matrix.

<u>back to ex5.</u>

First we fix a basis for P_3 & P_4

$\mathcal{R}$: $p_1 = 1$

$p_2 = x$

$p_3 = x^2$

$p_4 = x^3$

$$\boxed{\dim P_3 = 4}$$

- $P_3 = \text{span}(p_1, \dots, p_4)$
- Whenever $c_1p_1 + \dots + c_4p_4 = 0$, c_i's must be 0.

$\mathcal{P}': p_1 = 1, \ldots, p_4 = x^3, p_5 = x^4$

is a basis for $\mathbb{P}_4$. $(\dim \mathbb{P}_4 = 5)$

Now:

$$\Delta(p_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{P}'} \qquad \Delta(p_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{P}'}$$

$$\Delta(p_3) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{P}'}, \quad \Delta(p_4) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{P}'}$$

So

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

takes vectors in $\mathbb{P}_3$ expressed in basis $\mathcal{P}$ & gives out vectors in $\mathbb{P}_4$ expressed in basis $\mathcal{P}'$.

Let's check:

$$p = 3 + x + 2x^2 + 4x^3$$

$$\Delta(p) = D \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 4 \\ 12 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}'} = 1 + 4x + 12x^2$$