Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2024 – Second Midterm Exam·

1	2	3	4	5	$\sum$
18 pts	20 pts	20 pts	22 pts	20 pts	100 pts

Date: November 28, 2024 Time: 17:00-18:45	Full Name: PROPOSED SOLUTIONS

You may use every fact that we have already proven in the class. You cannot use the solutions of PS and quiz questions. If needed you must reproduce them.

1. Mark the statements below as TRUE or FALSE. No justification is needed in this part. EACH INCORRECT ANSWER CANCELS A CORRECT ONE.

   In this question $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}^k$.

1. $\boxed{T}$ If f is differentiable at a, then its directional derivatives at a along every direction exist.
2. $\boxed{F}$ If all partial derivatives of f exist at a but are not continuous at a, then f is not differentiable at a.
3. $\boxed{T}$ If g is C^1 and $g'(0) > 0$ then g is increasing in a neighborhood of 0.
4. $\boxed{F}$ If γ is differentiable and $\gamma'(t_1) = \gamma'(t_2)$ for some $t_1 < t_2$ then there is $c \in (t_1, t_2)$ such that $\gamma'(c) = 0$.
5. $\boxed{T}$ An open connected set in $\mathbb{R}^n$ is arc-wise connected.
6. $\boxed{F}$ If the closure of a set in $\mathbb{R}^n$ is connected then the set is connected too.

2. Let $n \geq k$ and $\pi : \mathbb{R}^n \to \mathbb{R}^k, \pi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$ be the projection onto the first k coordinates.
   (a) [10] Show that at each $a \in \mathbb{R}^n$, π is differentiable. What is $D\pi(a)$? Compute it.
   (b) [10] Let $p \geq n$ and $f : \mathbb{R}^p \to \mathbb{R}^n$ be differentiable. Use the Chain Rule to determine $D(\pi \circ f)(x)$ for each $x \in \mathbb{R}^p$. (I want to see explicitly what Chain Rule is and how you use it.) What is each entry of the matrix $D(\pi \circ f)(x)$?

(a) Either use directly the definition of differentiability;
Or: set $\pi = (\pi_1, \ldots, \pi_k)$ with $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\pi_j(x_1, \ldots, x_n) = x_j$.
Observe $\partial_i \pi_j = \begin{cases} 1, & i=j \\ 0, & \text{else} \end{cases}$. Since all these partials exist and are cont, by thm π is diffble.
Moreover $D\pi(x) = \begin{pmatrix} 1 & & 0 & \vdots & 0 \\ & 1 & & \vdots & \\ & & \ddots & \vdots & \ddots \\ 0 & & 1 & \vdots & 0 \end{pmatrix}_{k\times n} = (I_{k\times k} \mid \underline{0}_{k\times(n-k)})$

(b) $D(\pi\circ f)(x) = D\pi(f(x)) \cdot Df(x)$
$= (I \,\vdots\, \underline{0})_{k\times n} \cdot \begin{pmatrix} Df_1(x) \\ \vdots \\ Df_n(x) \end{pmatrix}_{n\times p} = (I \,\vdots\, \underline{0}) \cdot \begin{pmatrix} \partial_1 f_1 & \cdots & \partial_p f_1 \\ & & \\ \partial_1 f_n & \cdots & \partial_p f_n \end{pmatrix}(x) = \begin{pmatrix} \partial_1 f_1 & & 0 & \vdots & \\ & \ddots & & \vdots & \underline{0} \\ 0 & & \partial_k f_k & \vdots & \end{pmatrix}_{k\times p}$

$f_j : \mathbb{R}^p \to \mathbb{R}$

3. (a) [16] Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable and $\phi'(x) > 1$ for every $x \in \mathbb{R}$. Prove that there can be at most one point $p \in \mathbb{R}$ such that $\phi(p) = p$.
(b) [4] Show that the graphs in $\mathbb{R}^2$ of the functions $\alpha(x) = \tan x : (-\pi/2, +\pi/2) \to \mathbb{R}$ and $\beta(x) = x$ intersect exactly once.

(a) Suppose $\varphi(p) - p = 0 = \varphi(q) - q$. The fnc $\varphi - id$ is diffble so by Rolle's thm $\exists$ some c btw p & q such that $\varphi' - 1 = 0$. To avoid a contradiction, either $\nexists$ p OR q; OR $p = q$.

(b) Note $(\tan x)' = \sec^2 x > 1$ whenever $x \neq 0$. The graphs intersect at $x = 0$. They cannot intersect elsewhere since otherwise we'd get a contradiction

4. (Fill in the boxes –here! not somewhere else– and solve the questions. Each box is worth 1 point.)
Considering $\arctan : \mathbb{R} \to (-\pi/2, +\pi/2)$ and a differentiable function $\psi : \mathbb{R}^{\boxed{3}} \to \mathbb{R}^{\boxed{1}}$, define $w = \arctan(\psi(y^2, 2x - y, -4))$. Then $w : \mathbb{R}^{\boxed{2}} \to \mathbb{R}^{\boxed{1}}$. Answer the following questions by defining explicitly a new function. In this way w becomes the composition of several functions so that you can employ the Chain Rule.
(a) [8] Why is w differentiable?
(b) [8] Compute the $\boxed{1} \times \boxed{2}$ matrix Dw in terms of the partial derivatives $\partial_1\psi, \partial_2\psi, \partial_3\psi$.

(a) Define $u : \mathbb{R}^2 \to \mathbb{R}^3$, $u(x,y) = (y^2, 2x - y, -4)$. Then $w = \arctan \circ \psi \circ u$. Since each of these fncs is diffble, by Chain Rule w is diffble too.

(b) $Dw(x,y) = (\arctan)'(\psi \circ u(x,y)) \cdot \psi'(u(x,y)) \cdot u'(x,y)$

$$= \frac{1}{1 + [\psi(y^2, 2x - y, -4)]^2} \cdot \underbrace{\begin{bmatrix} \partial_1\psi & \partial_2\psi & \partial_3\psi \end{bmatrix}}_{1\times 3}(y^2, 2x-y, -4) \cdot \begin{bmatrix} 0 & 2y \\ 2 & -1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

$$= \frac{1}{1 + \psi^2} \cdot \begin{bmatrix} 2\partial_2\psi & 2y\partial_1\psi - \partial_2\psi \end{bmatrix}_{1\times 2}(y^2, 2x - y, -4)$$