Last time.

Let A, M be digble.

☐ $u_{n+1} = A_{p\times p}\, u_n$: $u_{n+1} = A^n u_1$

$$= S\Lambda^n S^{-1} u_1$$

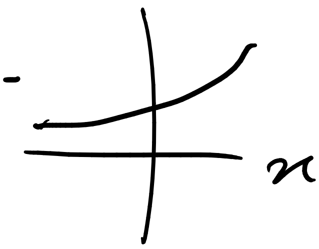
☐ $f'(x) = cf(x)$: $f(x) = \alpha e^{cx}$.

☐ $\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}' = M_{2\times 2} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \rightsquigarrow u(x)$:

$$u(x) = \underbrace{S \begin{pmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{pmatrix} S^{-1}}_{= e^{Mx}} \cdot u(0)$$

Recall:

$$e^x \overset{\text{defn}}{=} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$



$$e^{Mx} \overset{\text{defn}}{=} 1 + \frac{1}{1!} \cdot Mx + \frac{1}{2!} M^2 x^2 + \dots + \frac{1}{n!} M^n x^n + \dots$$

$M = S\Lambda S^{-1}$: $= 1 + S\Lambda S^{-1} x + \frac{1}{2} (S\Lambda S^{-1})^2 x^2 + \dots$

in general false but here true! $\overset{?}{=} S\left(1 + \Lambda x + \Lambda^2/2 \cdot x^2 + \dots + \frac{x^n}{n!}\Lambda^n + \dots\right) S^{-1}$

$$= S \cdot e^{\Lambda x} \cdot S^{-1}$$

COMPLEX NUMBERS

<u>dfn</u>: $(F, +, \cdot)$ such that

- $(F, +)$ satisfies G1, G2, G3, G4.
  assoc. there's an identity element 0, inverse, comm
- $(F - 0, \cdot)$ satisfies G1, G2, G3, G4.
- $\cdot$ distributes over $+$: $a(b+c) = ab + ac$.

Such $(F, +, \cdot)$ is called a <u>field</u>.

<u>ex</u>: * $(\mathbb{R}, +, \cdot)$

* $(\mathbb{Q}, +, \cdot)$ * $\mathbb{I}$rrationals is not.
* $(\mathbb{Z}, +, \cdot)$ is not.
* Arithmetics in mod 2 (OR 3 OR p) prime.
* Arithmetics mod 4 is not.

ex: $F = \mathbb{R}\times\mathbb{R}$ with vector addition;

& $(a,b)\cdot(c,d) \overset{\text{defn}}{=} (ac-bd,\ ad+bc)$

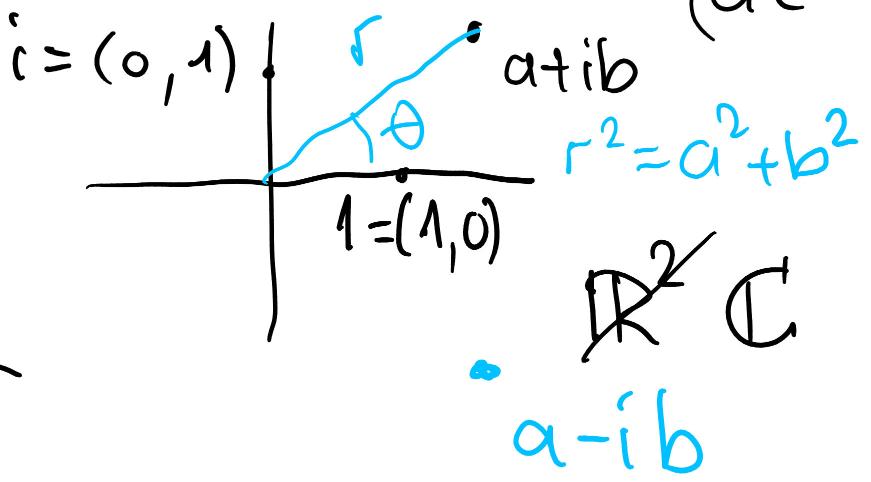
$(1,0)$ is identity for $\cdot$.

One can show that this F is a field. It's called the field of <u>complex numbers</u>, denoted by $\mathbb{C}$.

A more friendly notation: write $a+ib$ for (a,b) where $i^2=-1$.

$$(a+ib)+(c+id) = (a+c)+i(b+d)$$

$$(a+ib)\cdot(c+id) = ac+i^2bd+iad+icb$$
$$= (ac-bd)+i(ad+cb)$$



One can write:

false in general

$$a+ib = r(\cos\theta + i\sin\theta)$$

$$= r\cdot\left[1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots\right]$$

$$= r\cdot e^{i\theta}$$

exercise!

e.g. $b=0$, $a=-1$: $\boxed{0 = 1 + e^{i\pi}}$: Euler's identity.

Extending our theory over $\mathbb{C}$.

1) In **vector space** definition, we let scalars come from $\mathbb{C}$.

ex: $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

2) In **row reduction**, scalars are $\mathbb{C}$.

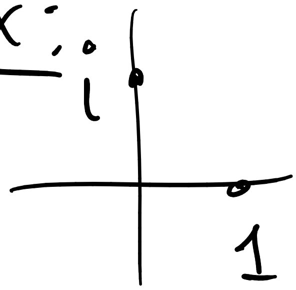
3) row space, col space etc are in $\mathbb{C}^n$.

4) We say $u_1, \dots, u_k \in \mathbb{C}^n$ are **lin. indep** over $\mathbb{C}$ if the following is satisfied:

whenever $c_1u_1 + \dots + c_ku_k = 0$ for $c_1, \dots, c_k \in \mathbb{C}$

we must have $c_1 = \dots = c_k = 0$.

ex:



$1, i \in \mathbb{C} \rightsquigarrow$ is a vector space over $\mathbb{C}$.

They're lin dependent:

$$(i)\cdot 1 + (-1)\, i = 0$$

$\{1\}$ is a basis for $\mathbb{C}$ and complex $\dim \mathbb{C} = 1$.

ex: complex $\dim \mathbb{C}^n = n$

(5) • Length of a cx number $z = a+ib$:

$$\|z\|^2 = \|a+ib\|^2 = \underbrace{(a-ib)}_{\text{conjugate of } z\text{, denoted } \bar{z}} (a+ib) = a^2+b^2$$

i.e. $\|z\|^2 = \bar{z} z$.

• $\boxed{\text{Length}}$ of a vector:

$(\underset{a+ib}{w}, \underset{c+id}{z}) \in \mathbb{C}^2$: $\|(w,z)\|^2 \overset{\text{defn}}{=} \|w\|^2 + \|z\|^2$

$$= \bar{w} w + \bar{z} z \in \mathbb{R}_{\geq 0}$$

$$= a^2+b^2+c^2+d^2$$

• $\boxed{\text{Inner product}}$ of $u, v \in \mathbb{C}^n$

is $\bar{u}^T \cdot v \in \mathbb{C}$

This gives our old inner product when restricted to $\mathbb{R}$ because conjugate of a real # is itself.

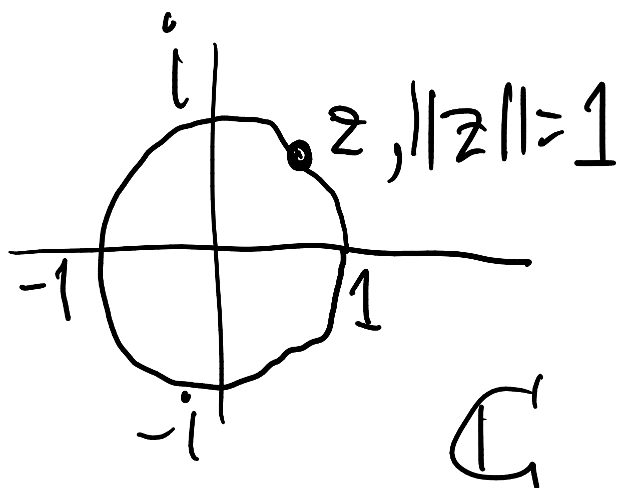
<u>defn</u>: The conjugate transpose of a matrix A is called the <u>Hermitian</u> of A, denoted by A^H.

after Hermite, a French math.

<u>Note</u>: $u^H v \neq v^H u$

⑥ Cx #s with unit length:

$$\|a+ib\|^2 = 1 = a^2+b^2$$



(7) $u, v \in \mathbb{C}^n$ are said to be $\mathbb{C}$ $\boxed{\text{orthogonal}}$ if $u^H \cdot v = 0$. We write $u \perp_{\mathbb{C}} v$.

<u>ex:</u> $1, i \in \mathbb{C}$ are not orthogonal!

(8) $\boxed{\text{Rank-Nullity Theorem}}$ works similarly.

(9) $\boxed{\text{Gram-Schmidt}}$ works.

(10) If for $A_{n \times n}$, $A^H = A$, A is called a <u>Hermition matrix</u>.

(Confined to reals, A is $\boxed{\text{symmetric}}$.)

<u>ex:</u> $A = \overline{\begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}^T} = \overline{\begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}} = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$

is Hermitian.

(11) Evalues are cx; evectors can have cx entries:

<u>ex. above:</u>

evalues of A:

char polyn $= \lambda^2 - \operatorname{tr}A \cdot \lambda + \det A$

$= \lambda^2 - 1 \cdot \lambda + i^2$

$\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{5}) \in \mathbb{R}$.

<u>exercise</u>. find espaces of A.

(12) Let $A_{n\times n}$ be Hermitian & $u \in \mathbb{C}^n$ an evector for evalue $\lambda \in \mathbb{C}$, i.e. $Au = \lambda u$.

$\underset{1\times n}{u^H} \underset{n\times n}{A} \underset{n\times 1}{u} \in \mathbb{C}$ is real:

$(u^H A u)^H = \overline{u^T}\, A^H \overline{(u^H)^T} = u^H A \cdot u$

thm: The evalues of a Hermitian matrix are all real.

corollary: The eigenvalues of a $\boxed{\text{real symmetric}}$ matrix are real.

proof of thm. A Hermitian, u evector for evalue λ: $Au = \lambda u$

$$\Rightarrow \underbrace{u^H A u}_{\in \mathbb{R}\ (11)} = u^H(\lambda u) = \lambda(\underbrace{u^H u}_{=\|u\|^2 \in \mathbb{R}\ (5)})$$

For the equality to hold, λ must be real too.