Last time.

$A_{n\times n}$; $\overset{\neq 0}{u} \in \mathbb{R}^n$;

$\lambda \in \mathbb{C} = \{a+ib \mid a,b \in \mathbb{R},\ i^2=-1\}$

$$Au = \lambda u$$

u → an eigenvector for λ

λ → an eigenvalue of A

$$\Leftrightarrow (A-\lambda I)u = 0$$

$$\Leftrightarrow \dim \operatorname{null}(A-\lambda I) > 0.$$

$$\Leftrightarrow \underbrace{\det(A-\lambda I)}_{\text{The characteristic polynomial of } \lambda \text{ with degree}=n.} = 0$$

The # of evalues (roots) is exactly n, counted with multiplicity.

These roots are complex. They can be real sometimes.

ex: $A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$ evalues $= 3, 6, -5$

$\dim \mathcal{E}_3 = 1 = \dim \mathcal{E}_6 = \dim \mathcal{E}_{-5}$.

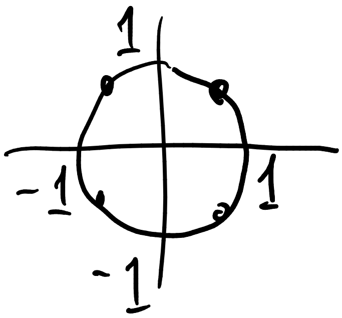
ex: Let $Q_{n\times n}$ be an orthogonal matrix & $Qu = \lambda u$. Recall Q preserves lengths.

$$\|Qu\| = \|u\| = \|\lambda u\|$$

If λ were real then $\lambda = \pm 1$.

In general $\lambda \in \mathbb{C}$. We'll see later that λ is complex "with length" equal to 1.

later: next week



$\mathbb{C} = \mathbb{R}^2$

DIAGONALIZATION

Assume $A_{n\times n}$ has n lin. indep evectors $u_1, \dots, u_n$ for evalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct).

$$A\underbrace{\begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}}_{S} = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}}_{S} \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\Leftrightarrow AS = S\Lambda$$

$$\Leftrightarrow S^{-1}AS = \Lambda \quad \left(\begin{array}{l} S^{-1} \text{ exists because} \\ \operatorname{rank}(S) = n. \end{array}\right)$$

Conversely if there is such an S, the computation above shows that columns of S are evectors of A and the diagonal entries of Λ are the corresponding evalues.

<u>defn</u>: If such an S exists

$$(S^{-1}AS = \Lambda)$$

OR equally if there are n linearly independent evectors for A, A is called <u>diagonalizable</u>.

<u>ex</u>: • $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has evalue $\underline{1}$ with multiplicity 2.

$\mathcal{E}_1 = \{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \}$, $\dim \mathcal{E}_1 = 1$.

Hence A is not diagonalizable.

• $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; evalues $1, 0$.

$\dim \mathcal{E}_1 = 1 = \dim \mathcal{E}_0$. Hence P is diagble. (P is already diagonal)

- $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; evalues are 0 with mult 2.

  $\mathcal{E}_0 = \mathbb{R}^2$; B is diagble. (It's already diagonal).
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagble.

Q: Which nxn matrices are diagble?
I.e. which have n lin. indep. evectors?

We'll give more&more general answers to this question.

thm: Suppose $A_{n\times n}$ has n distinct evalues. Then A is diagonalizable, i.e. it has n lin. indep. evectors.

Beware! The converse is not true in general. See the examples above!

back to 1st ex:

$$A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$$ is diagonalizable; There is S s.t.

$$S^{-1}AS = \begin{pmatrix} 3 & & \\ & 6 & \\ & & -5 \end{pmatrix}.$$

<u>proof of thm.</u> Given n distinct evalues $\lambda_1, \dots, \lambda_n$, choose evectors $u_1, \dots, u_n$.

Assume $c_1 u_1 + \dots + c_n u_n = 0$. (1)

$\boxed{n=3}$

$$A(c_1u_1 + c_2u_2 + c_3u_3) = c_1\lambda_1 u_1 + c_2\lambda_2 u_2 + c_3\lambda_3 u_3$$
$$= 0 \quad (2)$$

(2) $- \lambda_3 \cdot$ (1): $c_1(\lambda_1 - \lambda_3)u_1 + c_2(\lambda_2 - \lambda_3)u_2 \overset{(3)}{=} 0$

$$A \cdot \left(c_1(\lambda_1 - \lambda_3)u_1 + c_2(\lambda_2 - \lambda_3)u_2\right)$$
$$= c_1(\lambda_1 - \lambda_3)\lambda_1 u_1 + c_2(\lambda_2 - \lambda_3)\lambda_2 u_2 \overset{(4)}{=} 0$$

$(4) - \lambda_2 \cdot (3)$:

$$c_1 \underbrace{(\lambda_1 - \lambda_3)}_{\neq 0} \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \overset{\neq 0}{u_1} = 0 \qquad \lambda_i\text{'s distinct}$$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$ by (3)

$\Rightarrow c_3 = 0$ by (1)

This proves linear independence. $\blacksquare$

ex: $A_{3\times 3}$ as above. Compute A^{201}.

Recall $S^{-1}AS = \Lambda \Leftrightarrow A = S\Lambda S^{-1}$

$$A^{201} = (S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})$$

$$= S\Lambda^{201}S^{-1} = S \cdot \begin{pmatrix} 3^{201} & & \\ & 6^{201} & \\ & & -5^{201} \end{pmatrix} S^{-1}$$

lemma (Işın Özgen)

$$S = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 6 & 3 \\ 1 & 16 & 1 \end{pmatrix}$$

ex: Is it true that $A_{n\times n}$ & A^3 have the same evalues?

$$Au = \lambda u \Rightarrow A^2 u = \lambda A u = \lambda^2 u$$
$$\Rightarrow A^3 u = \lambda^3 u.$$

The answer is NO unless evalues of A are 0, 1, −1.

E.g. projection matrices.

orthog. matrix with real evalues

Observe that the computation above shows that their evectors are the same.

ex: Let A have an evalue λ & B an evalue μ. Is it true that $\mu \cdot \lambda$ is an evalue of $B \cdot A$?

$$Au = \lambda u \Rightarrow BAu = \lambda B u$$
$$\neq \lambda \mu u$$

In general not equal, unless u is a common evector of A & B.