Q. When do A & B have common evectors?

thm. Let A & B be diagble.
All evectors of A & B are the same if & only if A & B commute; i.e. $AB = BA$.

proof. $\Rightarrow$: Assume common evectors. Then the same S diagonalizes both A and B:

$S^{-1}AS = \Lambda_A$, $S^{-1}BS = \Lambda_B$. Then

$$
\begin{aligned}
AB &= S\Lambda_A S^{-1} \cdot S\Lambda_B S^{-1} \\
&= S\Lambda_A\Lambda_B S^{-1} = S\Lambda_B\Lambda_A S^{-1} \\
&= S\Lambda_B S^{-1} \cdot S\Lambda_A S^{-1} \\
&= B \cdot A
\end{aligned}
$$

$\blacksquare$

$\Leftarrow$: Assume $AB=BA$ and an extra condition: A has distinct evalues.

Research exercise: Find a proof without this extra assumption.

Then espaces are all 1-dim for A.

Take an evector u of A for evalue c.

$$A(Bu) \underset{\substack{\downarrow \\ AB=BA}}{=} BAu = c(Bu).$$

Both u & Bu are evectors of A for evalue c. By the extra assumption, Bu is a multiple of u, i.e. u is an evector of B too! $\blacksquare$

DIFFERENCE EQUATIONS

Given $f: \mathbb{R} \to \mathbb{R}$ & $n \in \mathbb{R}$,

$$\frac{df}{dx}(n) = \lim_{a \to 0} \frac{1}{a}\left(f(n+a) - f(n)\right)$$

is called the derivative of f at n, if lim exists.

If you discretize your space, taking $a=1$, deleting lim, we get

$$\frac{\Delta f}{\Delta x}(n) = f(n+1) - f(n) = f_{n+1} - f_n$$

This motivates us talk about the difference eqns: for a sequence $(a_n)_{n\geq 0}^{\infty}$

$$a_{n+1} = g(a_n, \dots, a_0).$$

ex: Fibonacci sequence.

$$a_{n+1} = a_n + a_{n-1}$$: a recursion (or recursive relation)

with $\underbrace{a_0 = 0, \; a_1 = 1}_{\text{an initial condition}}$

Find a_{201}. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = ?$

Observe:

$$u_{n+1} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = u_n$$

Then $$u_n = A u_{n-1} = A^n u_1$$

Is A diagble? YES.

char polyn $= \lambda^2 - \operatorname{tr} A \cdot \lambda + \det A$

$$= \lambda^2 - \lambda - 1$$

evalues: $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{5})$

evectors: $\mathcal{E}_+ = \operatorname{null}\begin{pmatrix} 1-\lambda_+ & 1 \\ 1 & -\lambda_+ \end{pmatrix} = \left\{ \alpha \begin{pmatrix} 1 \\ \lambda_+ - 1 \end{pmatrix} \right\}$

$1 - \lambda_+^2 + \lambda_+ = 0$

$$\lambda_+ - 1 = \frac{1}{2} + \frac{\sqrt{5}}{2} - 1 = -\lambda_-$$

$$\mathcal{E}_- = \operatorname{null}\begin{pmatrix} 1-\lambda_- & 1 \\ 1 & -\lambda_- \end{pmatrix} = \left\{ \beta \begin{pmatrix} 1 \\ \lambda_- - 1 \end{pmatrix} \right\} = \left\{ \beta \begin{pmatrix} 1 \\ -\lambda_+ \end{pmatrix} \right\}$$

Then $u_n = A^n u_1 = S \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}^n S^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -1 \\ \lambda_- & \lambda_+ \end{pmatrix} \begin{pmatrix} \lambda_+^n & \\ & \lambda_-^n \end{pmatrix} \begin{pmatrix} \lambda_+ & 1 \\ -\lambda_- & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\lambda_- - \lambda_+}$$

$$= \frac{-1}{\sqrt{5}} \begin{pmatrix} -1 & -1 \\ \lambda_- & \lambda_+ \end{pmatrix} \begin{pmatrix} \lambda_+^{n+1} \\ -\lambda_-^{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

Hence $a_n = \frac{1}{\sqrt{5}} \left(\lambda_+^{n+1} - \lambda_-^{n+1} \right) \in \mathbb{Z}$

Observe $-1 < \lambda_- \cong -0{,}61 < 0$

& $\lambda_-^{n+1} \sim 0$ as n gets large

So $a_n = \left[\frac{\lambda_+^{n+1}}{\sqrt{5}} \right] \rightsquigarrow$ take the closest integer.

$$\frac{a_{n+1}}{a_n} = \frac{\lambda_+^{n+2} - \lambda_-^{n+2}}{\lambda_+^{n+1} - \lambda_-^{n+1}} \sim \lambda_+$$ as n gets larger.

$$\lambda_+ = \frac{1+\sqrt{5}}{2} \sim 1.618 \leftarrow$$ called the golden ratio

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- Let u_1 be an evector of A for λ.

Then $u_n = A^n u_1 = \lambda^n u_1$.

- Let $\lambda_1, \dots, \lambda_k$ be evalues of $A_{p \times p}$ & $v_1, \dots, v_k$ be corresp (lin indep) evectors. If $u_1 = c_1 v_1 + \dots + c_k v_k$, then $u_n = A^n \cdot u_1$.

$$= c_1 \lambda_1^n v_1 + \dots + c_k \lambda_k^n v_k$$

$$= \underbrace{\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}}_{S} \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

- If u_0 is given explicitly:

$$u_0 = c_1 v_1 + \dots + c_k v_k = \underbrace{\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}}_{S} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

- If $k = n$ then

$$u_n = S \cdot \Lambda^n \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = S \Lambda^n \cdot S^{-1} u_1$$

as expected.

DIFFERENTIAL EQUATIONS

ex: Consider for $f:\mathbb{R}\to\mathbb{R}$,

$$\boxed{\frac{df}{dx}(x)=cf(x)}$$

. This is a differential equation.

$f(x)=be^{cx}$ solves this.

$$\boxed{f(x_0)=f_0\in\mathbb{R}}$$

This is an initial condition.

$$be^{cx_0}=f_0 \Rightarrow b=f_0/e^{cx_0}$$

and $f(x)=\frac{f_0}{e^{cx_0}}\cdot e^{cx}$ is the soln

for the diff eqn with initial condn.

ex: Suppose $f,g:\mathbb{R}\to\mathbb{R}$ unknown.

$$\frac{df}{dx}(x)=a\cdot f(x)+b\cdot g(x) \quad \text{with } a,b,c,d\in\mathbb{R}$$

$$\frac{dg}{dx}(x)=cf(x)+d\cdot g(x)$$

This is a system of linear differential eqns.

$$\underbrace{\begin{pmatrix} df/dx \\ dg/dx \end{pmatrix}}_{u'(x)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}}_{u(x)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = M$$

fix this below

We want to solve $u(x)$ in

(*) $u'(x) = M.u(x)$ with the initial condn $u(0) = u_0$. (**)

$$u_0 = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

Let's try a soln of the form:

$$u(x) = e^{\alpha x} \cdot \begin{pmatrix} A \\ B \end{pmatrix} \cdot \text{ for some } \alpha, A, B \in \mathbb{R}.$$

Insert in (*):

$$\alpha \cancel{e^{\alpha x}} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = M \cdot \underbrace{\cancel{e^{\alpha x}}}_{\neq 0} \begin{pmatrix} A \\ B \end{pmatrix}$$

Then if α is an evalue & $\begin{pmatrix} A \\ B \end{pmatrix}$ is an evector for α, we have a solution!

For M: char polyn $= \lambda^2 - 4\lambda - 5$

evalues: $\lambda_1 = 5$, $\lambda_2 = -1$.

evectors: $\mathcal{E}_5 = \text{null}\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$\mathcal{E}_{-1} = \text{null}\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} = \left\{ \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

So $u(x) = e^{5x}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are two solutions for $u(x)$.

Moreover $c_1 e^{5x}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{S} \begin{pmatrix} e^{5x} & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

is a soln of $(*)$ for any $c_1, c_2 \in \mathbb{R}$.

Satisfy also $(**)$:

$$u_0 = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = S\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S^{-1} u_0$$

Hence

$$u(x) = S \overbrace{\begin{pmatrix} e^{5x} & 0 \\ 0 & e^{-x} \end{pmatrix}}^{e^{\Lambda x}\text{: next time}} S^{-1} u_0 \rightarrow \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

is a soln for $(*)$ & $(**)$.

That is,

$$\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{5x} & \\ & e^{-x} \end{pmatrix} \begin{pmatrix} +1 & +1 \\ +2 & -1 \end{pmatrix} \Big/ {+3} \cdot \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 13e^{5x} \\ 17e^{-x} \end{pmatrix}$$

$$\Rightarrow f(x) = \frac{1}{3}\left(13e^{5x} + 17e^{-x}\right)$$

$$g(x) = \frac{1}{3}\left(26e^{5x} - 17e^{-x}\right)$$

solve together the diff eqn $(*)$ with the initial condn $(**)$.