A system of linear equations

In every term there is at most one unknown and with power 1.

real coefficients

$$\begin{cases} x + \boxed{2}y + z = 0 \\ 3x - y + 2z = 0 \\ 5x + y - z = 0 \\ 7x \quad\quad - z = 0 \end{cases}$$

"Solve for x, y, z" means:

- Are there values for x, y, z which, when inserted in the system, satisfy the equalities
- How many solutions are there? Find all possible solutions.

ex: Solve $x+2y+z=0$

soln. set $= \{ (x,y,z) \in \mathbb{R}^3 \mid x = -2y-z, \underbrace{y \in \mathbb{R}, z \in \mathbb{R}}_{\text{are free}} \}$

$=$ a plane thru 0 in $\mathbb{R}^3$.

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ex:

$$\begin{aligned} x+2y+z&=0 \\ 3x-y+2z&=0 \\ 5x+y-z&=0 \end{aligned}$$

$-3 \cdot R1 + R2 \to R2$

This algorithm is called row reduction or Gaussian elimination.

$$\begin{aligned} x+2y+z&=0 \\ 0x-7y-z&=0 \\ 5x+y-z&=0 \end{aligned} \xrightarrow{-5R1+R3 \to R3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & -9 & -6 \end{pmatrix}$$

$$\xrightarrow{-\frac{9}{7}R2+R3 \to R3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & -\frac{33}{7} \end{pmatrix} \rightsquigarrow \begin{aligned} x+2y+z&=0 \\ -7y-z&=0 \\ -\frac{33}{7}z&=0 \end{aligned}$$

soln set $= \{(0,0,0)\}$

Gaussian elimination:

- multiply rows with real #s
- add rows to other (below)
- swap the rows

ex: $x+2y+z=0$

$3x-y+2z=0$

$5x+y+az=0 \quad (a=5-9/7)$

Gaussian elimination

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & -9 & a-5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{r} x+2y+z=0 \\ -7y-z=0 \\ 0=0 \end{array}$$

Suppose y is free:

R2: $z=-7y$

R1: $x=5y$

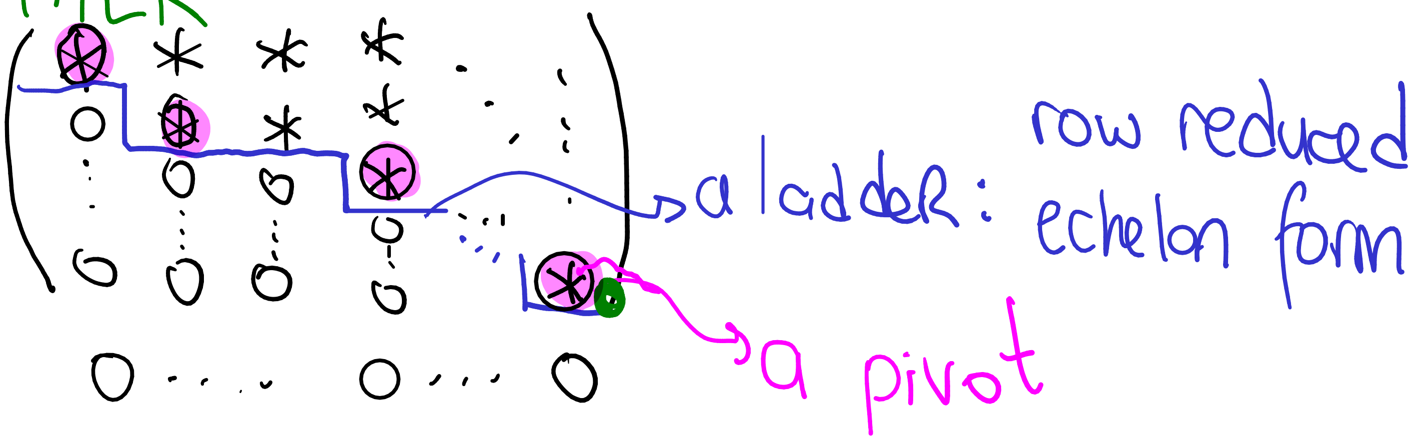
soln set $=\{(x,y,z) \mid y\in\mathbb{R},\ z=-7y,\ x=5y\}$

Observe. Given a homogeneous system of linear equation

↳ the right hand sides are all zero

You apply the Gaussian elimination to obtain

EITHER

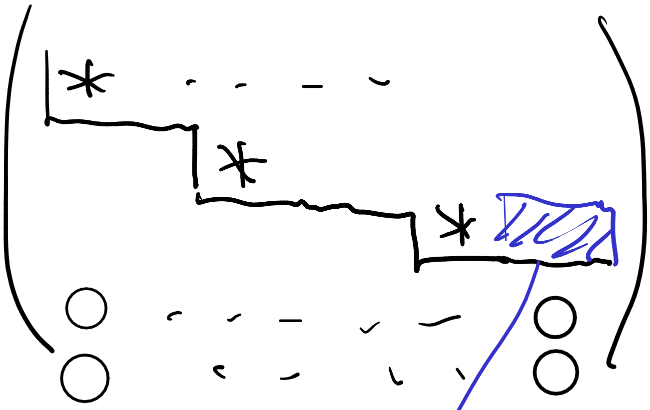


$\circledast$: nonzero real #

Here # of pivots = # of unknowns, hence no free variables

There is a unique solution: the zero solution.

OR



Here # of pivots < # of unknowns, hence there are free variables

There are ∞ly many solutions,

ex:

$$\begin{aligned} x+2y+z&=1\\ 3x-y+2z&=0\\ 5x+y-z&=0 \end{aligned}$$

$\rightsquigarrow$ a nonhomog system

row reductions

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & +1 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & -\frac{33}{7} & \underbrace{-5+\left(-\frac{9}{7}\cdot(-3)\right)}_{=-8/7} \end{array}\right)$$

$$\begin{aligned} x+2y+z&=1\\ -7y-z&=-3\\ -\tfrac{33}{7}z&=-\tfrac{8}{7} \end{aligned}$$

R3: $z=8/33$

R2: $y=(-3+z)/(-7)$

R1: $x=1-2y-z$

Soln set $=\left\{(x,y,z)\;\middle|\; z=8/33,\ y=\frac{3-z}{7},\ x=\dots\right\}$

ex:
$$x+2y+z=1$$
$$3x-y+2z=0$$
$$5x+y+b\,z=0$$

(b to be decided to make this zero)

...

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & +1 \\ 6 & -7 & -1 & -3 \\ 0 & 0 & 0 & -8/7 \end{array}\right)$$

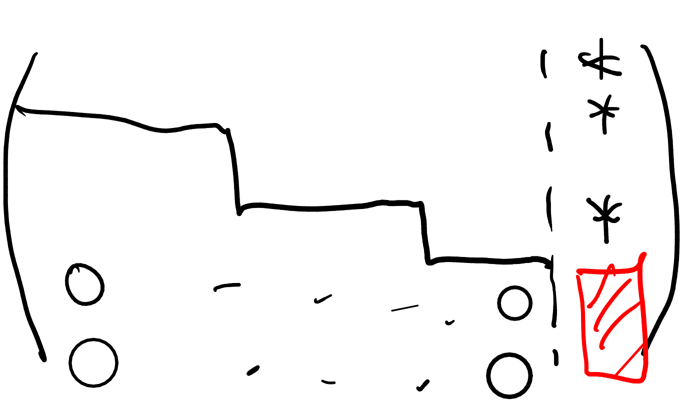
$$x+2y+z=1$$
$$-7y-z=-3$$
$$\boxed{0=-8/7}$$

this is an inconsistency

the system is not consistent.

For a nonhomog system:

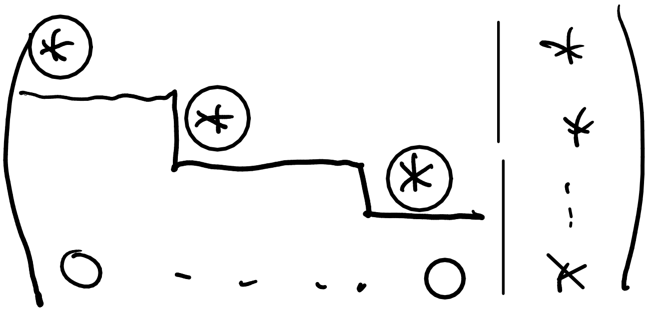
EITHER inconsistent:



$\neq 0$

no solutions at all

OR



\# of pivots = # of unknowns then a unique soln.

\# of pivots < # of unknowns then ∞ly many solutions