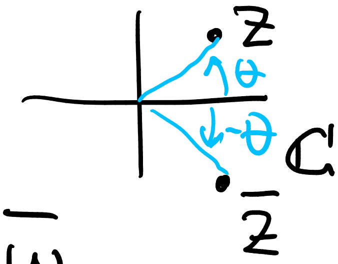
Last time:



- $\overline{z} = \overline{(a+ib)} = \overline{re^{i\theta}} = a-ib = re^{-i\theta}$
- $\overline{z\cdot w} = \overline{(re^{i\theta})(qe^{i\alpha})} = rq\, e^{-i(\theta+\alpha)} = \overline{z}\cdot\overline{w}$
- $\overline{z+w} = \overline{z}+\overline{w}$
- $z \in \mathbb{R} \iff z = \overline{z}$.

thm 1. Evalues of a Hermitian matrix are real.

Real versus Complex

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	$|x_1|^2 = \overline{x}_1 x_1$ length: $\|x\|^2 = |x_1|^2 + \cdots + |x_n|^2$
transpose: $A^{\mathrm{T}}_{ij} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A^{\mathrm{H}}_{ij} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	conjugate transpose $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}}y = \overline{x}_1y_1 + \cdots + \overline{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $|\lambda| = 1$

ex: When is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Hermitian?

if and only if $\begin{cases} \left.\begin{array}{l} a = \bar{a} \\ d = \bar{d} \end{array}\right\} \begin{array}{l} a \in \mathbb{R} \\ d \in \mathbb{R} \end{array} \\ c = \bar{b} \end{cases}$

ex: $A = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$ is Hermitian.

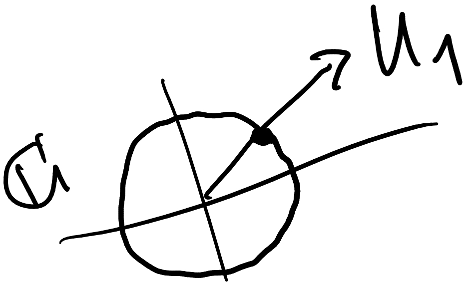
evalues: $\lambda_{\pm} = \frac{1}{2}(1 \pm \sqrt{5})$

espaces have cx $\dim_{\mathbb{C}} = 1$.

evectors: $\text{null} \begin{pmatrix} 1-\lambda_+ & +i \\ -i & -\lambda_+ \end{pmatrix} = \Big\{ c \overbrace{\begin{pmatrix} -i \\ \lambda_- \end{pmatrix}}^{u_1} ; \; c \in \mathbb{C} \Big\}$

Is there a unit evector?

Yes: $u_1 / \|u_1\| = u_1 / \sqrt{1+\lambda_-^2}$.



espace_+ has ∞ly many unit evectors.

exercise. There is no totally real evector in this espace.

We'll prove:

<u>thm2.</u> For a Hermitian matrix A let $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ be two evalues, and u_1 & u_2 be two corresponding evectors. Then $u_1 \perp_{\mathbb{C}} u_2$ i.e. $u_1^H u_2 = 0$.

which immediately implies:

<u>thm3.</u> A Hermitian matrix A with distinct evalues is diagonalizable via a cx matrix whose columns can be chosen to be orthonormal.

<u>defn:</u> A square matrix U with unit column vectors that are cx. orthogonal to each other is called a <u>unitary matrix</u>.

$$U^H \cdot U = I = UU^H.$$

thm 3 (again). Every Hermitian matrix A with distinct evalues is diagble via a unitary matrix:

$$U^{-1}AU = U^{H}AU = \Lambda.$$

thm 4. Every real symmetric matrix A with distinct evalues is diagble via an orthogonal matrix:

$$Q^{T}AQ = \Lambda.$$

proof. thm 3 gives a unitary matrix U. By thm 1, evalues of A are real So it has real evectors for each evalue. Such unit real evectors builds U, a real unitary matrix i.e an orthogonal matrix.

proof of thm 2. Let $Au_1 = \lambda_1 u_1$, $Au_2 = \lambda_2 u_2$. Then

$$\lambda_1 u_1^H u_2 = (\bar{\lambda}_1 u_1)^H u_2 \overset{\lambda_1 \in \mathbb{R}}{=} (\lambda_1 u_1)^H u_2 = (Au_1)^H u_2 = u_1^H A^H u_2 = u_1^H A u_2 = \lambda_2 u_1^H u_2.$$

Since $\lambda_1 \neq \lambda_2$, $u_1^H u_2 = 0$

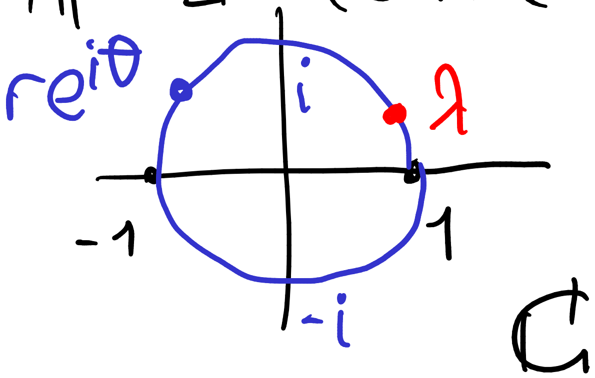
Properties of unitary matrices.

1) $U^H U = I = U U^H$
2) U is invertible with inverse U^H.
3) Considering $U: \mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Ux$ preserves inner product:
$$x^H y = x^H (U^H U) y = (Ux)^H (Uy)$$
4) U preserves lengths: $\|Ux\| = \|x\|$.
5) Every evalue of U is a unit cx #:

for $Ux = \lambda x$, $\|x\|^2 = \|Ux\|^2 = \|\lambda x\|^2$

$$= (\lambda x)^H (\lambda x) = \bar{\lambda} \cdot \lambda \cdot x^H x = \|\lambda\|^2 \cdot \|x\|^2$$

$\Rightarrow \|\lambda\|^2 = 1$ (since $\|x\| \neq 0$).



⑥ U^{-1} is unitary too:

$$(U^{-1})^H \cdot U^{-1} = (U^H)^H \cdot U^H = UU^H = I$$

⑦ U, V unitary $\Rightarrow U \cdot V$ unitary:

$$(UV)^H \cdot (UV) = V^H U^H U V = I.$$

<u>Rmk for ⑤</u>: Every evalue of an orthogonal matrix is a unit cx #.

defn: $A_{n\times n}$ & $B_{n\times n}$ are similar matrices if $S^{-1}AS = B$ for some S. The LHS is called a similarity transformation

Schur's lemma. Any square matrix A is similar to an upper triangular matrix T via a unitary matrix U:

$$U^{-1}AU = U^{H}AU = T$$

Moreover the diagonal entries of T are evalues of A.

Spectral Theorem. Every Hermitian is diagonalizable via a unitary matrix:

$$U^{H}AU = \Lambda$$

Corol. Every real symmetric matrix is diagble via an orthogonal matrix.

Proof of Spectral Theorem. Let A be Hermitian. By Schur's Lemma $T = U^H A U$. Then
$$T^H = U^H A^H U = U^H A U = T.$$
So T is diagonal (with real diagonal)

Proof of Schur's lemma. Let $\lambda_1 \in \mathbb{C}$ be an evalue & v_1 a unit evector, $Av_1 = \lambda_1 v_1$. Then there is some (*) unitary matrix $U_1 = [v_1 \mid * \mid \cdots \mid *]$ such that
$$A \cdot U_1 = [Av_1 \mid * \mid \cdots \mid *]$$
$$= [\lambda_1 v_1 \mid * \mid \cdots \mid *]$$

$$= \left[\begin{array}{c|c|c|c} v_1 & * & \cdots & * \end{array}\right] \cdot \left[\begin{array}{c|c|c|c} \lambda_1 & U_1^{-1} * & \cdots & U_1^{-1} * \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

$$= U_1 \cdot \left[\begin{array}{c|c} \lambda_1 & \\ 0 & \\ \vdots & * \\ 0 & \end{array}\right]$$

(*) Such U_1 can be built as follows: insert $v_1, e_1, \ldots, e_n$ into Gram-Schmidt. Set the output vectors $v_1, v_2, \ldots, v_n$ as columns of U_1.

We finish the proof next time.