Boğaziçi University
Department of Mathematics
Math 234 Advanced Calculus II
Spring 2025 – Second Midterm Exam.

1	2	3	4	$\sum$
30 pts	24 pts	24 pts	24 pts	100 pts

Date: May 6, 2025 Time: 17:00-18:45	Full Name: PROPOSED SOLUTIONS

Below all curves and surfaces that appear are orientable and nice enough. Figures are from Thomas'.

1. (a) [2pts] What does an orientation mean on a surface $S \subset \mathbb{R}^3$?

(b) [2pts] What is the induced orientation on the boundary of an oriented surface?

(c) [8pts] State Stokes's Theorem in $\mathbb{R}^3$. Explain every object that appears in the expression.

TRUE or FALSE... 6 points each... Either justify shortly or refute. Refuting is a proof; you can do this by giving a counterexample. Here $\Omega \subset \mathbb{R}^2$ open, $f : \Omega \to \mathbb{R}$, $\mathbf{F} : \Omega \to \mathbb{R}^2$; $D \subset \mathbb{R}^3$ open, $g : D \to \mathbb{R}$. The functions all are many times differentiable.

(d) If f is the div of some vector field $\mathbf{F}$ then grad $f = \mathbf{0}$.

FALSE: $\vec{F} = (x^2, y^2)$, $f = \operatorname{div}\vec{F} = 2x + 2y$, $\operatorname{grad} f = (2,2) \neq 0$.

(e) If S is a closed surface (i.e. S is topologically closed with $\partial S = \emptyset$) then the flux of any vector field through S is zero.

FALSE: $\vec{F}(x,y,z) = (x,y,z)/\|(x,y,z)\|$, $S =$ unit sphere, oriented outwards

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \iint_S dA = 4\pi \neq 0.$$

(f) If $\operatorname{curl} \mathbf{G} = 0$ then for some function g, $\mathbf{G} = \operatorname{grad} g$ on $D \subset \mathbb{R}^3$.

FALSE: Existence of g depends on the topology of D.

Let $G = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q}, 0\Big)$. (Here $\operatorname{dom} G = \mathbb{R}^3 - \{z\text{-axis}\}$)

↳ not "simply connected"

$$\operatorname{curl} G = (Q_x - P_y)\vec{k} = \frac{2}{x^2+y^2} + \frac{-2x^2}{(x^2+y^2)^2} + \frac{-2y^2}{(x^2+y^2)^2} = 0$$

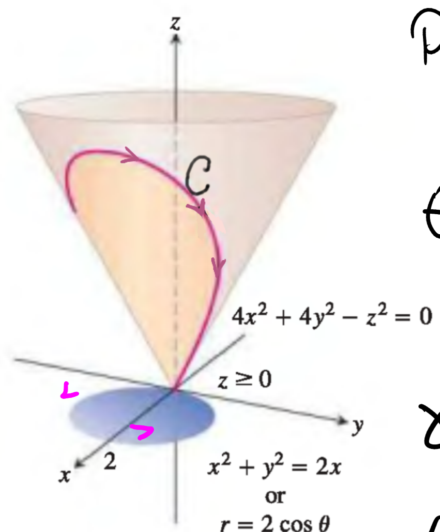
However, for $C = \{x^2+y^2=1, z=0\}$ oriented positively,

$$\int_C \vec{G} \cdot d\vec{x} = \int_0^{2\pi} \left(\frac{-s\theta}{1} (c\theta)' + \frac{c\theta}{1} \cdot (s\theta)' \right) d\theta = 2\pi \neq 0$$

$C : (c\theta, s\theta)$

Hence $\vec{G}$ is not conservative. $\vec{G}$ cannot be a gradient.

2. Find the work done by the vector field $\mathbf{H} = \mathbf{i} + z\mathbf{j} - z/4\,\mathbf{k}$ along the curve $\mathcal{C}$ on the cone in the figure, which projects down to the boundary of a planar 2-disk. The curve $\mathcal{C}$ is oriented as in the figure.



Parametrize C respecting the orientation:

$$\theta \in [-\pi/2, \pi/2],\ \gamma(\theta) = \Big(rc\theta, rs\theta, \underbrace{\sqrt{4x^2+4y^2}}_{2\sqrt{2x} = 2\sqrt{4c^2\theta}}\Big)$$

$$= (2c^2\theta, \underbrace{2c\theta s\theta}_{s2\theta}, 4c\theta)$$

$$\gamma'(\theta) = (4c\theta(-s\theta), 2c2\theta, -4s\theta)$$

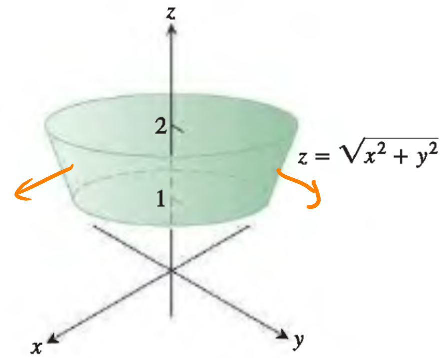
$$\int_C \vec{H} \cdot d\vec{x} = \int_{-\pi/2}^{\pi/2} \left(1, 4c\theta, \frac{-4c\theta}{4}\right) \cdot \gamma'(\theta)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (-4c\theta s\theta + 8c\theta \cdot c2\theta + 4c\theta s\theta)\, d\theta$$

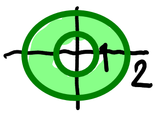
$$= 8 \cdot \int_{-\pi/2}^{+\pi/2} c\theta(1 - 2s^2\theta)\, d\theta = 8\left(2 - \frac{2}{3} s^3\theta \Big|_{-\pi/2}^{\pi/2}\right)$$

$$= 16/3$$

3. (a) Consider the **outwards** oriented surface K in the figure (a portion of a cone, called a *cone frustum*). K is a surface with boundary, which is the disjoint union of two circles.
Determine a one-to-one C^1 parametrization φ [6 pts] for (at least a portion of) K supplying the given orientation [6 pts], and with the condition that the domain of φ is open in $\mathbb{R}^2$. What is the domain of your parametrization?



$$\varphi : \Omega = \{1 < u^2+v^2 < 4\} \to \mathbb{R}^3$$



$$\varphi(u,v) = (-u, v, \sqrt{u^2+v^2})$$

this provides the right orientation.

Note: $\varphi_u = (-1, 0, u/\sqrt{})$, $\varphi_v = (0, 1, v/\sqrt{})$

$$\varphi_u \times \varphi_v = (-u/\sqrt{}, v/\sqrt{}, -1)$$

(b) Compute the area of K using a surface integral. (OK, everybody knows the result from high school. Here I want you to compute the area using math234 technology.)

$$\text{area}(K) = \iint_K dA = \iint_\Omega \|\varphi_u \times \varphi_v\| \, dA_{u,v}$$

$$= \iint \left(1 + \frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2}\right)^{1/2} du\,dv = \sqrt{2}\,(4\pi - \pi) = 3\sqrt{2}\,\pi$$

4. Consider the surface K in the previous question. We build a closed surface Σ by gluing a pair of horizontal 2-disks to K along their boundaries. Then we orient Σ **outwards**. Now compute the flux

$$\iint_{\Sigma} \mathbf{G} \cdot \mathbf{n}\, dA$$

for the vector field $\mathbf{G} = (x^2 + \tan(yz/4))\mathbf{i} + (e^{x^2 z} - 2xy)\mathbf{j} + xz\mathbf{k}$.

Observe G is defined in $W \stackrel{\text{def}}{=}$ solid bounded by Σ.
Moreover Σ is oriented correctly. Then we can use

orientation of Σ is right

Divergence thm: $\displaystyle\iint_{\Sigma} G \cdot n = \iiint_W \operatorname{div} G \, dV$

$\operatorname{div} G = 2x - 2x + x$

$$= \iiint_W x \, dV = \int_1^2 \int_0^{2\pi} \int_0^z x \cdot r \, dr \, d\theta \, dz$$

cylindrical coords

$$= \int_1^2 \int_0^{2\pi} \int_0^z r^2 \cos\theta \, dr \, d\theta = 0$$