

A system of linear equations

↳ In every term there is at most one unknown and with Power 1.

↪ real coefficients

$$\begin{cases} x + 2y + z = 0 \\ 3x - y + 2z = 0 \\ 5x + y - z = 0 \\ 7x - z = 0 \end{cases}$$

"Solve for  $x, y, z$ " means:

- Are there values for  $x, y, z$  which, when inserted in the system, satisfy the equalities?
- How many solutions are there? Find all possible solutions.

ex: Solve  $x+2y+z=0$

soln. set =  $\{ (x,y,z) \in \mathbb{R}^3 \mid \underbrace{\quad}_{\text{are free}} \}$   
 $x = -2y - z, y \in \mathbb{R}, z \in \mathbb{R} \}$   
 $=$  a plane thru 0 in  $\mathbb{R}^3$

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ex:  
 $x+2y+z=0$

$$3x - y + 2z = 0$$

$$5x + y - z = 0$$

$$\downarrow -3 \cdot R_1 + R_2 \rightarrow R_2$$

$$x+2y+z=0 \quad -5R_1 + R_3 \rightarrow R_3$$

$$0x - 7y - z = 0$$

$$5x + y - z = 0$$

This algorithm  
is called  
row reduction  
OR Gaussian  
elimination.

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & -9 & -6 \end{pmatrix}$$

$$\xrightarrow{-\frac{9}{7}R_2 + R_3 \rightarrow R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & -\frac{33}{7} \end{pmatrix} \rightsquigarrow$$

$$\text{soln set} = \{ (0,0,0) \}$$

$$\begin{aligned} x+2y+z &= 0 \\ -7y-z &= 0 \\ -\frac{33}{7}z &= 0 \end{aligned}$$

## Gaussian elimination:

- multiply rows with real #s
- add rows to other (below)
- swap the rows

ex:  $x + 2y + z = 0$

$$3x - y + 2z = 0$$

$$5x + y + az = 0 \quad (a = 5 - \frac{9}{7})$$

↓ Gaussian elimination

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & -9 & a-5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x + 2y + z = 0 \\ -7y - z = 0 \\ 0 = 0 \end{array}$$

Suppose  $y$  is free:

$$R2: z = -7y$$

$$R1: x = 5y$$

$$\text{soln set} = \{ (x, y, z) \mid y \in \mathbb{R}, \\ z = -7y, x = 5y \}$$

Observe. Given a homogeneous system  
of linear equation  $\rightarrow$  the right hand

↳ the right hand sides are all zero

You apply the Gaussian elimination to obtain

EITHER

A diagram of a matrix with rows and columns. Some elements are circled in pink and crossed with an 'X'. A blue line traces a path through the matrix, starting from the top-left, moving right, then down, then right again, and finally down to the bottom-right. A green circle is at the bottom-right corner.

→ a ladder: row reduced echelon form

as a pivot

$\otimes$ : nonzero real #

Here # of pivots = # of unknowns,  
hence no free variables

There is a unique solution: the zero solution.

A hand-drawn diagram of a matrix enclosed in large parentheses. The matrix has a staircase pattern of asterisks (\*) along the main diagonal. The bottom-right element is highlighted with a blue box and labeled "bottom right" with a blue arrow. The matrix is partitioned into four quadrants by a horizontal and vertical line. The top-left and bottom-right quadrants contain asterisks, while the top-right and bottom-left quadrants contain circles.

Here # of pivots  $<$  # of unknowns,  
hence there are free variables

There are only many solutions,

ex:  $x + 2y + z = 1$   
 $3x - y + 2z = 0$   
 $5x + y - z = 0$   $\rightarrow$  a nonhomog system

row reductions

$$\left( \begin{array}{ccc|c} 1 & 2 & 1 & +1 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & -\frac{33}{7} & -5 + \left( -\frac{9}{7} \cdot (-3) \right) \end{array} \right)$$

$= -8/7$

$$x + 2y + z = 1$$

$$-7y - z = -3$$

$$-\frac{33}{7}z = -\frac{8}{7}$$

$$R3: z = 8/33$$

$$R2: y = (-3 + z)/(-7)$$

$$R1: x = 1 - 2y - z$$

$$\text{Soln set} = \left\{ (x, y, z) \mid z = 8/33, y = \frac{3-z}{7}, x = \dots \right\}$$

ex:  $x + 2y + z = 1$

$$3x - y + 2z = 0$$

$$5x + y + b \cdot z = 0$$

( $b$  to be decided  
to make this zero)

...

$$\begin{pmatrix} 1 & 2 & 1 & | & +1 \\ 0 & -7 & -1 & | & -3 \\ 0 & 0 & \textcircled{0} & | & -8/7 \end{pmatrix}$$



$$x + 2y + z = 1$$

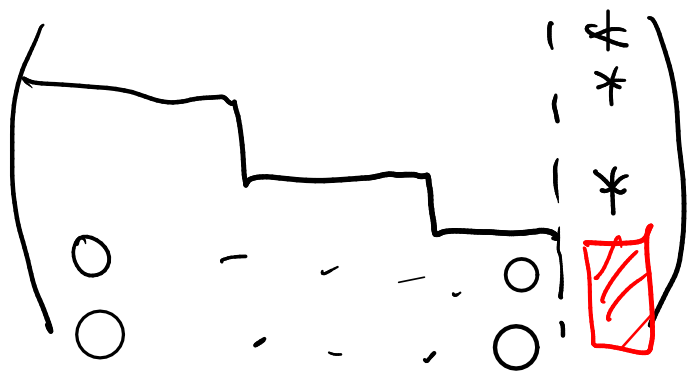
$$-7y - z = -3$$

$$\boxed{0 = -8/7}$$

this is an inconsistency

the system is not  
consistent.

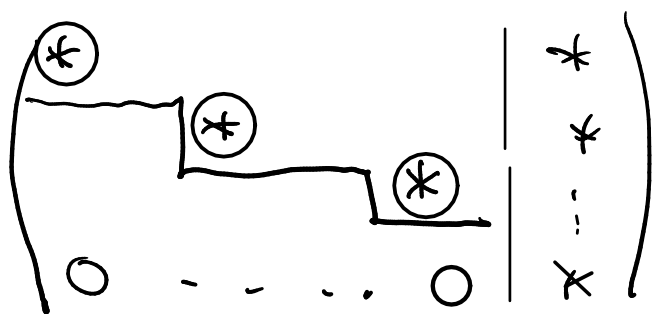
For a nonhomog system:  
EITHER inconsistent:



$$\boxed{\neq 0}$$

no solutions  
at all

OR



# of pivots = # of unknowns  
then a unique soln.

# of pivots < # of unknowns  
then only many solutions