Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2024 – First Midterm Exam

1	2	3	4	$\sum$
25 pts	25 pts	25 pts	25 pts	100 pts

Date: October 22, 2024
Time: 17:00-18:30

Full Name: PROPOSED SOLUTIONS

1. Mark the statements below as TRUE or FALSE. No justification is needed in this part. EACH INCORRECT ANSWER CANCELS A CORRECT ONE.

1. $\boxed{T}$ A line in $\mathbb{R}^2$ is closed in $\mathbb{R}^2$.
2. $\boxed{T}$ Any finite set in $\mathbb{R}^m$ is compact. Because a finite set is bounded & is a finite union of (closed) singletons.
3. $\boxed{T}$ Every Cauchy sequence is bounded.
4. $\boxed{F}$ If a bounded sequence (a_n) in $\mathbb{R}^m$ has a convergent subsequence then (a_n) is convergent too.
   Counterexample: $(-1)^n$

A sequence $(c_n)_{n=1}^{\infty}$ is said to be **Cauchy** if the following condition is satisfied (write in the box below):

$\forall\, \varepsilon > 0$, there is some N s.t. $\forall k, \ell \geq N$, $|c_k - c_\ell| < \varepsilon$.

2. (a) Show: If A, B are bounded sets of $\mathbb{R}$ then $A \times B$ is bounded in $\mathbb{R}^2$.

A lies in a large interval I_A; B lies in I_B. Then $A \times B \subset I_A \times I_B$.

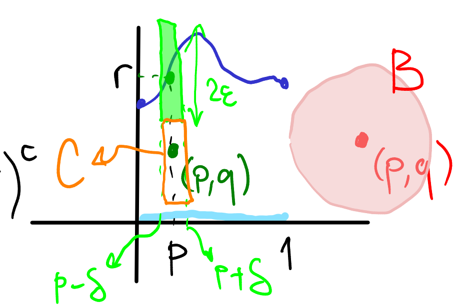
(b) Consider the compact interval $I = [0,1] \in \mathbb{R}$ and a function $f : I \to \mathbb{R}$. The **graph** $\Gamma_f \subset \mathbb{R}^2$ of f is defined as

$$\Gamma_f = \{(x,y) \in \mathbb{R}^2 \,|\, y = f(x)\} \subset \mathbb{R}^2.$$

Show: If f is continuous on I then Γ_f is compact.

- I is compact. Since f is cont, $f(I)$ is compact too.
  By part (a), $I \times f(I)$ is bounded. So $\Gamma_f \subset I \times f(I)$ is bounded too.

- Γ_f is closed because $(\Gamma_f)^c$ is open in $\mathbb{R}^2$:
  Let $(p,q) \notin \Gamma_f$; i.e.
  either $p \notin I$, say $p > 1$. Then $B(p-1, (p,q)) \subset (\Gamma_f)^c$
  OR $p \in I$ but $r = f(p) \neq q$. Then
  we'll use the continuity of f as follows:
  Given $\varepsilon = \frac{|f(p) - r|}{2} > 0$, take $\delta > 0$ so that $B(\delta, p) \subset B(\varepsilon, r)$.
  Then the open box $C = B(\delta, p) \times B(\varepsilon, f(p))$ does not intersect Γ_f.



3. Prove that if (a_n) and (b_n) are Cauchy sequences in $\mathbb{R}^m$, then the sequence of distances $|a_n - b_n|$ converges.

See the defn above for being Cauchy, to fix N_a & N_b, given $\frac{\varepsilon}{2}>0$.

We show the sequence $c_n = |a_n - b_n|$ is Cauchy ($\underset{\text{thm}}{\Longrightarrow}$ convergent)

That is, given $\varepsilon > 0$, $\overset{\text{SHOW}}{\exists}$ N s.t. $k,n > N \Rightarrow |c_k - c_n| < \varepsilon$:

Given $\varepsilon>0$, choose $N = \max(N_a, N_b)$. Then

$$\begin{aligned} k,n > N \Rightarrow |c_k - c_n| &= |a_k - a_n + b_k - b_n| \\ &\leq |a_k - a_n| + |b_k - b_n| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

4. For an arbitrary pair of real numbers $b_0 > a_0 \geq 0$, we consider the recurrence:

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2};$$

i.e. the next a_{n+1} is the geometric mean of the previous a_n and b_n, and the next b_{n+1} is the arithmetic mean of the previous a_n and b_n.
(a) Show: For every $n \in \mathbb{Z}^{\geq 0}$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. (A hint: Start with proving $a_n \leq b_n$. For this you might want to consider $b_n^2 - a_n^2$.)
(b) Show that the sequences (a_n) and (b_n) converge, and they converge to the same limit. (You can commit this part assuming that part (a) is true.)

(a) Observe $b_{n+1}^2 - a_{n+1}^2 = \frac{1}{4}(a_n+b_n)^2 - a_n b_n = \frac{1}{4}(b_n - a_n)^2 > 0$

So $b_{n+1} - a_{n+1} > 0$, $\forall n$.

Also, $a_{n+1}^2 = a_n b_n \geq a_n \cdot a_n \Rightarrow a_{n+1} \geq a_n$;

and $b_{n+1} = \frac{1}{2}(a_n + b_n) \leq \frac{1}{2}(b_n + b_n) = b_n$.

(since all a_n, b_n are nonnegative)

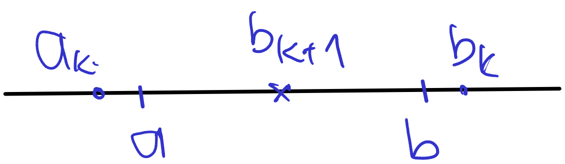
(b) By Mon.Seq.Property, $a_n \longrightarrow \sup a_n =: a$ & $b_n \longrightarrow \inf b_n =: b$.

Observe $b < a$ would contradict with (a). (Work this out)

Now, given $0 < \varepsilon < b - a$, $\exists$ some index k s.t.

$a - a_k < \varepsilon$ & $b - b_k < \varepsilon$. Then $b - b_{k+1} = \frac{1}{2}(b - a_k + b - b_k)$

$\geq (b - a - \varepsilon)/2 > 0$.



This contradicts with $b \leq b_{k+1}$.