Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2025 – First Midterm Exam

1	2	3	4	$\sum$
28 pts	25 pts	25 pts	22 pts	100 pts

Date: October 31, 2025
Time: 17:00-19:00

Full Name: PROPOSED SOLUTIONS

1. Prove or give a counter-example:

(a) Every bounded sequence in $\mathbb{R}^k$ is Cauchy.

FALSE: $a_n = (-1)^n$ bounded. For $\forall i$ odd, j even $|a_i - a_j| = 2$.
So given $\varepsilon < 2$, one cannot find N to satisfy the defn of being Cauchy.

(b) Consider a sequence $(u_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$. If $||u_{n+1} - u_n|| \longrightarrow 0$ as $n \longrightarrow \infty$ then the sequence (u_n) is convergent.

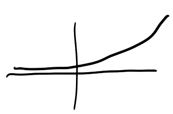
FALSE: Consider $u_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $u_{n+1} - u_n = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
but $\forall M > 0$ $\exists$ some N : $u_N = \sum_{k=1}^{N} \frac{1}{k} > M$.
Hence (u_n) diverges (to $+\infty$).

(c) If a nonempty set $A \subset \mathbb{R}^k$ is bounded then $\text{diam}(A) \in \mathbb{R}$. (Recall: $\text{diam}(A) = \sup\{||x-y|| : x, y \in A\}$.)

TRUE: A bdd $\Leftrightarrow A \subset B(R, 0)$. Then $\forall x, y \in A$, $x, y \in B(R, 0)$ so that
$\|x - y\| \leqslant \|x\| + \|y\| \leqslant 2R$. Hence diam $A \leqslant 2R \in \mathbb{R}$.

(d) Let $n, k \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}^k$ be defined **over all** $\mathbb{R}^n$. If f is **continuous** and $D \subset \mathbb{R}^n$ is **closed**, then $f(D)$ is closed in $\mathbb{R}^k$.

FALSE: $n = k = 1$; $f(x) = e^x$.



$D = \mathbb{R}$ closed in $\mathbb{R}$;
$f(\mathbb{R}) = (0, +\infty)$ not closed in $\mathbb{R}$.

2. For a bounded sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, define the set $S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many } n\text{'s}\}$.

(a) Show that S is nonempty by explicitly giving an element in S.

(a_n) is bounded. Any lower bound of the set $\{a_n\}$ is in S.

(b) Why is $\sup S$ finite? (Denote this sup by a.)

S is bounded from above by any upper bound of $\{a_n\}$.
Then by Completeness, sup exists.

(c) Show that (a_n) has a subsequence which converges to a. Do this by first constructing a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ (tell why the indices can be chosen in an increasing manner); then you must show that the subsequence you constructed converges to a. (Prove all these from scratch! Do not use Bolzano-Weierstrass theorems here. Because your proof will be a new proof of the Bolzano-Weierstrass Theorem I: *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*)

$a = \sup S$. Equivalently a is an upper bound for S and $\forall \varepsilon > 0$ $(a-\varepsilon, a]$ contains an element of S. So given $\frac{1}{k} > 0$, choose an element s_k of S in $(a-\varepsilon, a]$.
$s_k \in S \iff \exists \infty$ly many elts of (a_n) in $(a-\varepsilon, a]$. Choose one with index n_k greater than the previously chosen n_{k-1}.

claim. The constructed $(a_{n_k})_{k=1}^{\infty}$ converges, with limit $= a$. PROVE IT!

3. Consider a sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$, $k \geq 1$. Suppose there is some real $c \in (0,1)$ such that for all n, $|b_{n+1} - b_n| \leq c|b_n - b_{n-1}|$. Prove that the sequence (b_n) is convergent. (Hint: • Bound the differences in terms of $|b_2 - b_1|$. • Recall the sum formula for $1 + r + \ldots + r^k$. • Cauchy.)

Observe $\forall j \geq 2: |b_{j+1} - b_j| \leq c \cdot |b_j - b_{j-1}| \leq \cdots \leq c^{j-1} \cdot |b_2 - b_1|$.

i.e. $N-2 > \frac{\log \varepsilon/\Delta}{\log c}$

Then $\forall i > j \geq 2: \underbrace{|b_i - b_j| \leq |b_i - b_{i-1}| + \cdots + |b_{j+1} - b_j|}_{\text{triangle ineq}}$

$\leq (c^{i-2} + c^{i-3} + \cdots + c^{j-2}) \cdot |b_2 - b_1|$

$= \frac{1 - c^{i-1} - (1 - c^{j-2})}{1-c} \cdot \underbrace{\frac{|b_2 - b_1|}{1-c}}_{= \Delta : \text{fixed}}$

$= c^{j-2} \cdot (1 - c^{i-j+1}) \cdot \Delta \leq c^{j-2} \cdot \Delta$

Thus given $\varepsilon > 0$, choose N s.t $c^{N-2} \cdot \Delta < \varepsilon$

so that $\forall i,j > N: |b_i - b_j| \leq c^{j-2} \cdot \Delta < c^{N-2} \Delta < \varepsilon$

So (b_n) is Cauchy, therefore convergent in $\mathbb{R}$.

4. Let K be closed and P be open in $\mathbb{R}^n$. Then $K - P$ is closed in $\mathbb{R}^n$. Fill in the blank and prove that statement.

Observe $K - P = K \cap P^c$ is closed because K & P are closed.