Recall: $u = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}_S$ denotes a vector in $\mathbb{R}^4$ expressed in the standard basis.

if we write S or "std" or nothing

i.e. $u = 0 \cdot e_1 + 1 \cdot e_2 + 2e_3 + 3e_4$

$w = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{B}}$ is a vector in $\mathbb{R}^4$ expressed in the basis $\mathcal{B} = \{\beta_1, \ldots, \beta_4\}$

i.e. $w = 1 \cdot \beta_1 + (-1) \cdot \beta_2 + 0 \cdot \beta_3 + 1 \cdot \beta_4$

$v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_{\mathcal{S}}$ in the vector space P_3 is the vector

$\mathcal{S} = \{1, x, x^2, x^3\}$

$v = 1 \cdot 1 + 2 \cdot x + 3x^2 + 4x^3$

ex: 1 (a) $\mathcal{B}: \beta_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}7\\1\\0\\0\end{pmatrix}, \begin{pmatrix}6\\3\\2\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \beta_4$

is a basis of $\mathbb{R}^4$. One can prove that by showing that

$$B = \begin{pmatrix}1 & 7 & 6 & 1\\0 & 1 & 3 & 2\\0 & 0 & 2 & 3\\0 & 0 & 0 & 4\end{pmatrix}$$ has rank $= 4$

$\Updownarrow$

$\mathrm{col}(B) = \mathbb{R}^4$

$\Updownarrow$

$\beta_1, \dots, \beta_4$ lin indep.

Here $\operatorname{rank} B = \#\text{pivots} = 4$.

(b) Let's assume that $A_{4\times 4}$ has row reduced form B. Then columns of A is a basis for $\mathbb{R}^4$.

② $f: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$,

$f(x,y,z,t) = (x+y, z+t)$ is a linear transformation. We want to find a matrix for f that takes vectors in $\mathbb{R}^4$ expressed in standard basis, & gives out vectors in $\mathbb{R}^2$ in std basis,

$f(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f(e_3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$\overset{''}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$ $f(e_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For <u>any</u> $v = c_1 e_1 + \cdots + c_4 e_4 \in \mathbb{R}^4$:

$$f(v) = c_1 f(e_1) + \cdots + c_4 f(e_4)$$

$$= \underbrace{\left(f(e_1) \,\vdots\, f(e_2) \,\vdots\, f(e_3) \,\vdots\, f(e_4) \right)}_{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}_{2\times 4} \; S \to S} \begin{pmatrix} c_1 \\ \vdots \\ c_4 \end{pmatrix}_S$$

(3) Observe in ex. 1:

$$B\cdot\underbrace{\begin{pmatrix}c_1\\ \vdots\\ c_4\end{pmatrix}_{\mathcal{B}}}_{=u} = c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}7\\1\\0\\0\end{pmatrix}+\dots = u_{\mathcal{S}}.$$

i.e. B changes vectors in $\mathcal{B}$ to vectors expressed in $\mathcal{S}$.

(4) If I give you $w=\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}_{\mathcal{S}}$ how to express this in $\mathcal{B}$:

find $c_1,\dots,c_4\in\mathbb{R}$ such that

$$c_1\beta_1+\dots+c_4\beta_4 = w = \begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}$$

$$\left(\beta_1 \,\middle|\, \cdots \,\middle|\, \beta_4\right)\cdot\begin{pmatrix}c_1\\ \vdots\\ c_4\end{pmatrix} = \begin{pmatrix}1\\0\\-1\\-2\end{pmatrix} \iff Bc = w$$

$$\iff c = B^{-1}w$$

i.e. B^{-1} is a matrix that takes vectors in $\mathcal{S}$ & gives out their expression in $\mathcal{B}$.

(5) back to ex 2:

What's the matrix for f that takes vectors in $\mathcal{B}$ & gives out vectors in $\mathcal{S}$?

$$A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_4 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} * \\ * \end{pmatrix}_{\mathcal{S}}$$

$f(\beta_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f(\beta_2) = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$, $f(\beta_3) = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$

$f(\beta_4) = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

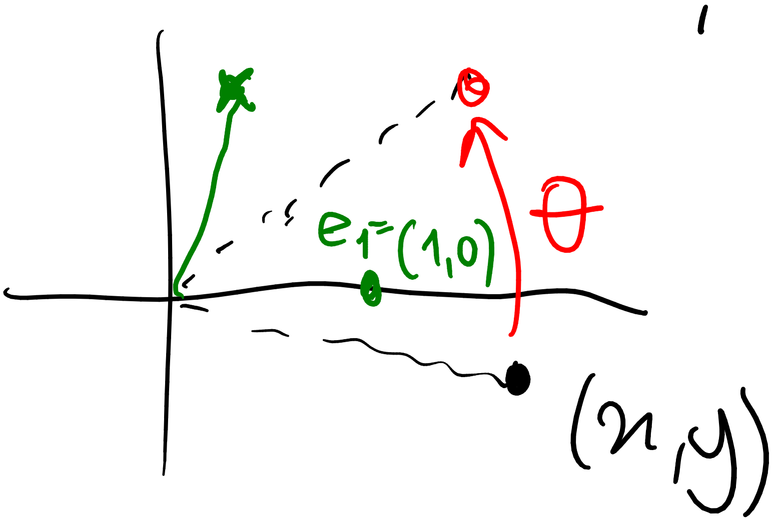
Then $A = \begin{pmatrix} 1 & 8 & 9 & 3 \\ 0 & 0 & 2 & 7 \end{pmatrix}_{\mathcal{B} \to \mathcal{S}}$ is the required matrix.

<u>Conclusion</u>: The columns of A is the images of basis vectors of the given basis.

⑥ Let $\theta \in \mathbb{R}$ & ρ_θ be the rotation of $\mathbb{R}^2$ by θ counterclockwise around 0.

<u>claim</u>: ρ_θ is a linear transformation.
(proof later)

What is the matrix R_θ of ρ_θ in standard basis?



$$\rho_\theta(e_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\rho_\theta(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix}$$

$\rho \to \rho$

e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotates by $\pi/2$.

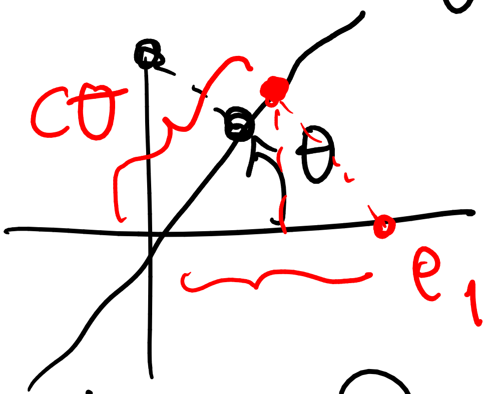
& $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ rotates by π.

7 (a) What does $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0$ do?

It projects onto the x-axis.

$(x,y) \mapsto (x,0)$.

(b) Let L_θ be the θ-line.



What's the matrix of projection P_θ onto L_θ?

claim: P_θ is a linear transf.

proof:

$$\frac{1}{2}\left(I(u) + S_\theta(u)\right) = P_\theta(u) \quad (*)$$

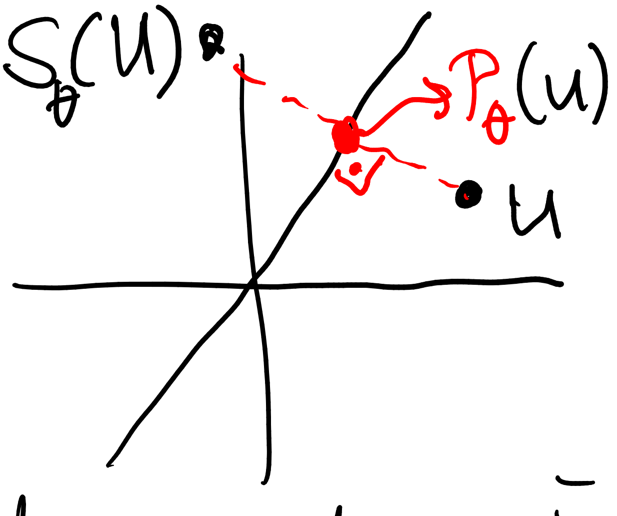
$$P_\theta(e_1) = \begin{pmatrix} c\theta \cdot c\theta \\ c\theta \cdot s\theta \end{pmatrix}, \quad P_\theta(e_2) = \begin{pmatrix} s\theta \cdot c\theta \\ s^2\theta \end{pmatrix}$$

So $P_\theta = \begin{pmatrix} c^2\theta & c\theta s\theta \\ c\theta s\theta & s^2\theta \end{pmatrix}$

(c) $P_\theta^2 = P_\theta$ because projecting twice is the same thing as projecting once.

(8) Fix L_θ. What is the matrix of symmetry S_θ wrt L_θ?

claim: S_θ is a linear transf (proof later)



Then: from (*)

$$S_\theta(u) = (2P_\theta - I)(u)$$

$$= \begin{pmatrix} 2c^2\theta - 1 & 2c\theta s\theta \\ 2c\theta s\theta & 2s^2\theta - 1 \end{pmatrix} u$$

this is the required matrix

⑨ $S_\theta^2 = I$:

$$S_\theta^2 = (2P_\theta - I)^2$$
$$= 4P_\theta^2 - 2P_\theta I - I2P_\theta + I^2$$
$$= \cancel{4P_\theta} - \cancel{2P_\theta} - \cancel{2P_\theta} + I$$
$$= \underline{0} + I = I.$$

Geometrically symmetry inverse is itself.