Last time:

- $\det(A) = \begin{cases} 0 & , A \text{ is not invertib} \\ (-1)^k \cdot d_1 \cdot \ldots \cdot d_n & , \text{otherwise} \end{cases}$

  where $PA = L \cdot \mathrm{diag}(d_1, \ldots, d_n) \cdot U$

  k: how many row swaps P requires.

- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(A^T) = \det(A)$.

How to compute.

① $\det A \overset{(*)}{=} a_{11} \cdot C_{11} + a_{12} C_{12} + \ldots + a_{1n} C_{1n}$

where $C_{1k} = (-1)^{1+k} \cdot \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \vdots & & \\ a_{n1} & & \ldots & a_{nn} \end{vmatrix}$

delete 1st row & k th column

$(n-1) \times (n-1)$

called the $(1,k)$ $\boxed{\text{cofactor}}$

We say $(*)$ computes det using the first row.

$\det A = a_{m1} C_{m1} + a_{m2} C_{m2} + \dots + a_{mn} C_{mn}$

computes det using the m^{th} row

$\det A = a_{1k} C_{1k} + a_{2k} C_{2k} + \dots + a_{nk} C_{nk}$

computes using the k^{th} column

ex: $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 1 \\ 7 & 0 & 5 \end{pmatrix}$

$\det A = 2 \cdot C_{11} + 3 C_{12} + 4 C_{13}$

1st row

$= 2 \cdot (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} + 3 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$

$+ 4 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 1 & 0 \\ 7 & 0 \end{vmatrix}$

$= 2 \cdot 0 - 3 \cdot (5-7) + 4 \cdot 0 = 6$

$\det A = -3 \cdot \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + 0 \cdot * + 0 \cdot *$

2nd column

$= 6$

ex: $B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 4 & 3 & -1 & 1 \end{pmatrix}$

$$\det B = 0 \cdot C_{14} + 0 \cdot C_{24} + 1 \cdot C_{34} + 1 \cdot C_{44}$$

wrt 4th column

$$= (-1)^7 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & 3 & -1 \end{vmatrix} + (-1)^8 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 3 & 0 \end{vmatrix} = 3 - 5 = -2$$

② $A \cdot A^{-1} = I$.

$$1 = \det I = \det(AA^{-1}) = \det A \cdot \det A^{-1}$$

$$\Rightarrow \det(A^{-1}) = 1/\det A.$$

③ Q is orthogonal, i.e. $Q^T Q = I$.

Then $1 = \det Q \cdot \det Q^T = (\det Q)^2$

$$\Rightarrow \det Q = \pm 1.$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

④ Let $A_{n\times n} = Q \cdot R$ upper

orthog.

$A = (a_1 \mid \cdots \mid a_n)$ $\qquad Q = (p_1 \mid \cdots \mid p_n)$

$$\det A = (\pm 1)\cdot \det R = \pm (p_1^T a_1)\cdots(p_n^T a_n)$$

Applications.

$\boxed{1}$ Computing A^{-1}.

$$\det A = a_{11}\cdot C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$$

$$a_{11}\cdot C_{21} + a_{12} C_{22} + \dots + a_{1n} C_{2n}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

wrt 1st row

Therefore

$$
\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A}
\underbrace{\begin{pmatrix} C_{11} & C_{21} & & C_{n1} \\ C_{12} & C_{22} & \ddots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{pmatrix}}_{C^T}
$$

$$
= \begin{pmatrix} \det A & 0 & & 0 \\ 0 & \det A & \ddots & 0 \\ \vdots & 0 & & 0 \\ & \vdots & & \vdots \\ 0 & 0 & & \det A \end{pmatrix}
$$

$$
= \det A \cdot I
$$

Hence if $\det A \neq 0$

$$
A \cdot \left(\frac{1}{\det A} \cdot C^T \right) = I
$$

so that

$$
A^{-1} = C^T / \det A
$$

Here C is the cofactors matrix, with ij entry $= C_{ij}$.

ex:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{pmatrix} \quad C_{11} = -2, \ C_{12} = +3, \ C_{13} = +1, \ C_{21} = +2$$

$C_{22} = -3$, $C_{23} = -2$, $C_{31} = -1$, $C_{32} = +1$, $C_{33} = +1$.

$$\text{cofactors matrix } C = \begin{pmatrix} -2 & 3 & 1 \\ 2 & -3 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -1 \neq 0$$

Hence

$$A^{-1} = (-1) \cdot \begin{pmatrix} -2 & 2 & -1 \\ 3 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix} = C^T / \det A$$

$\boxed{2}$ CRAMER'S RULE.

We want to solve $Ax=b$ with A invertible:

$$x = A^{-1}b = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ & & \ddots & \\ & & & C_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Thus

$$x_1 = \frac{1}{\det A}\left(b_1 C_{11} + b_2 C_{21} + \dots + b_n C_{n1}\right)$$

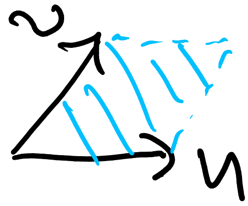
$$= \frac{1}{\det A} \cdot \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Similarly

CRAMER'S RULE

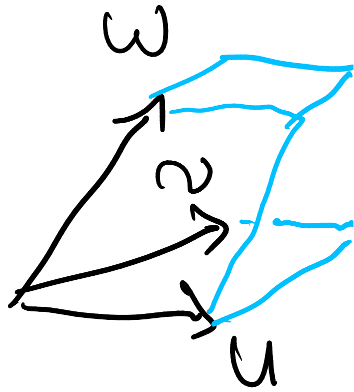
$$\boxed{x_j = \frac{1}{\det A} \cdot \det\left(\begin{array}{l}\text{matrix obtained} \\ \text{from } A \text{ by deleting} \\ \text{its } j\text{th column \& instead} \\ \text{writing the vector } b\end{array}\right)}$$

$\boxed{3}$ area of a parallelogram



$$= \|u \times v\| = \left|\det(u \,\vdots\, v)\right|$$

volume of a parallelpiped



$$= u \cdot (v \times w) = \left|\det\left[\, u \,\vdots\, v \,\vdots\, w \,\right]\right|$$

"volume of a solid in $\mathbb{R}^n$ spanned by" $u_1, \dots, u_n$ is defined to be

$$\left| u_1 \,\vdots\, \cdots \,\vdots\, u_n \right|$$

Observe that if $u_1, \dots, u_n$ are lin. dependent then the volume $= 0$.

$\boxed{4}$

$$\begin{vmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & & \\ 0 & * & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix}$$

(Assume no permutation needed to reach the echelon form.)

The determinants of the 2 top left 2×2 matrices are equal too.

This shows that A is invertible $\Leftrightarrow$ every $k \times k$ top left submatrix is invertible

I.e.

$\det A \neq 0 \Leftrightarrow$ every $k \times k$ top left submatrix has $\det \neq 0$.

EIGENVALUES & EIGENVECTORS

<u>dfn</u>: For $A_{n\times n}$ square, a nonzero vector $v \in \mathbb{R}^n$ & a $\lambda \in \mathbb{R}$ satisfying

$$Av = \lambda v$$

to be modified later.

are called an <u>eigenvector</u> v corresponding to <u>eigenvalue</u> λ.

<u>ex</u>: • $A = I_{n\times n}$. Every $v \in \mathbb{R}^n$ is an evector & there is just one evalue: 1.

$$I \cdot v = 1 \cdot v$$

• For any rotation of $\mathbb{R}^3$ with a rotation axis span(u).

Any cu $\neq 0$ is an evector with the corresponding evalue 1.

- If $Av=\lambda v$, $v\neq 0$ then for any $c\in\mathbb{R}-\{0\}$, $A(cv)=\lambda(cv)$

i.e. if v is evector corresp to λ then $\forall\, cv$ is " " λ.

- If u & v are evector corresp to λ then

$$A(cu+v)=cAu+Av$$
$$=c\lambda u+\lambda v$$
$$=\lambda(cu+v)$$

i.e. $cu+v$ is also an evector corresp to λ.

<u>thm</u>: Let λ be an evalue for A

Then the set of all evectors corr. to λ is a subspace of $\mathbb{R}^n$.

We call that the <u>eigenspace</u> corresponding to λ, denoted by $\mathcal{E}_\lambda$.