Boğaziçi University
Department of Mathematics
Math 234 Advanced Calculus II
Spring 2025 – Final·

1	2	3	4	5	$\sum$
12 pts	10 pts	25 pts	28 pts	25 pts	100 pts

Date: May 29th, 2025 Time: 16:00-18:20	Full Name: PROPOSED SOLUTIONS

1. TRUE or FALSE. No justification needed. An incorrect answer cancels a correct one.

1. [T] If $S \subset \mathbb{R}^k$ is bounded and $f : S \to \mathbb{R}$ is integrable then $|f|$ is integrable over S too. A theorem; proven.
2. [F] Every bounded open set in $\mathbb{R}^k$ is J. measurable. See midterm 1 bis.
3. [F] A continuous function on a compact subset of $\mathbb{R}^k$ is integrable. There are cpct, nonmeasurable sets.
4. [T] On a smooth (orientable) surface in $\mathbb{R}^3$, there are exactly two possible orientations.
5. [F] Given any divergent infinite series (a_n) and any real number t, there is a rearrangement of (a_n) with series sum equal to t. counterexample $\sum(-1)^n$
6. [T] If $\sum f_n(x)$ and $\sum g_n(x)$ are uniformly convergent on $E \subset \mathbb{R}^k$ then $\sum f_n(x) + g_n(x)$ is uniformly convergent on E too. needs a small proof.

2. Tell 5 instances that you learned in Advanced Calculus where you can swap two operations/processes (i.e. taking limits, derivatives, integrals, infinite sequences and series). Write your claims and express carefully and very shortly when each is valid. You can refer any of these as lemmas in the following questions.

	Assumption	Claim
A.	f_n cont at a, $f_n \xrightarrow{unif} f$	$\lim\limits_{n\to\infty} \lim\limits_{x\to a} f_n(x) = \lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x)$ $\quad\; \text{''}\; f(a) \quad \lim\limits_{x\to a} f(x)\;\text{''}$
B.	$f_n \to f$, $\frac{df_n}{dx}$ converges unif.	$\lim\limits_{n\to\infty} \frac{df_n}{dx} = \frac{d}{dx} \lim\limits_{n\to\infty} f_n = \frac{df}{dx}$
C.	$f_n \xrightarrow{unif} f$, f_n integrable over S	$\lim\limits_{n\to\infty} \int_S f_n = \int_S \lim f_n = \int_S f$
D.	Series analogue of B	$\sum \frac{d}{dx} f_n = \frac{d}{dx} \sum f_n = \frac{ds}{dx}$
E.	series analogue of C	$\sum \int f_n = \int \sum f_n = \int s$

F. Continuity via sequences G. And why not Fund. Thm. Calculus ...

3. (a) [10pts] Check convergence: $\sum_{n=1}^{\infty}\sqrt{\frac{\sqrt{n+1}-\sqrt{n}}{n+1}}$.

$b_n = \left((\sqrt{n+1}+\sqrt{n})(n+1)\right)^{-1/2}$. Convergent:

Limit compare with $n^{-3/4}$:

$$\frac{b_n}{n^{-3/4}} = \left(\frac{n^{3/2}}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+1\right)\left(1+\frac{1}{n}\right)}\right)^{1/2} \xrightarrow[n\to\infty]{} 1.$$

(b) [15pts] Determine the values of x at which the series converges absolutely or conditionally:

$$\sum_{0}^{\infty}\underbrace{\frac{1\cdot 3\cdots(2n+1)}{2\cdot 5\cdots(3n+2)}x^n}_{a_n x^n}.$$

$$\left|\frac{a_{n+1}\cdot x^{n+1}}{a_n\cdot x^n}\right| = \left|\frac{2n+3}{3n+5}\cdot x\right| \xrightarrow{n\to\infty} \frac{2}{3}|x|$$

- Ratio test: converges for $|x|<\frac{3}{2}$. Radius of convergence of this power series is $3/2$.

By thm, the convergence is absolute for each $|x|<3/2$.

- For $x=+3/2$, Raabe: $n\cdot\left(1-\frac{a_{n+1}x^{n+1}}{a_n x^n}\right) = n\cdot\frac{6n+10-6n-9}{6n+10} \xrightarrow{n\to\infty} 1/6<1$

↳ At $x=3/2$, divergent.

- For $x=-3/2$, absolutely diverges

Meanwhile in $\sum \underbrace{a_n\cdot\left(\frac{3}{2}\right)^n}_{d_n}\cdot(-1)^n$

d_n is decreasing: $\frac{d_{n+1}}{d_n} = \frac{6n+9}{6n+10} < 1$.

So by Dirichlet's (or alternating series test), we have convergence at $x=-3/2$.

4. *Lambert series.* Suppose $c_n \in \mathbb{R}$ and $\sum_1^\infty c_n$ converges. Consider for $x \in \mathbb{R} - \{\pm 1\}$ the series

$$(L) \qquad \sum_1^\infty c_n \frac{x^n}{1-x^n}.$$

In your answers below state carefully what facts you use, step by step.
(a) [6] Show that for any $0 < a < 1$, (L) converges absolutely and uniformly on $[-a, +a]$.
(b) [10] Show that for any $b > 1$, (L) converges uniformly on $(-\infty, -b]$ and $[+b, +\infty)$. (Hint: Considering part (c), apparently no direct application of Weierstrass M-test here. Instead observe that $\frac{x^n}{1-x^n} = \frac{1}{1-x^n} - 1$. Now express the series as the sum of two infinite series and investigate each.)
(c) [6] Show that in part (b) the convergence is absolute if and only if $\sum_1^\infty c_n$ converges absolutely. (Hint: You need to prove a small lemma to conclude.)
(d) [6] Let $s(x)$ denote the series sum of (L), whenever the sum is finite. What is the domain of s? At what points is s continuous?

(a) For $0<a<1$ and $x \in [-a,+a]$: $\forall n$, $\left| c_n \cdot \frac{\overset{\leq a^n}{x^n}}{\underbrace{1-x^n}_{\geq 1-x \geq 1-a}} \right| \leqslant |c_n| \frac{a^n}{1-a} \underset{\uparrow}{<} \frac{a^n}{1-a} =: M_n$

ultimately because $\sum c_n$ converges

By Weierstrass M-test, (L) converges absolutely and uniformly on $[-a,+a]$, $\forall a \in (0,1)$.

(b) Let $b>1$ and $x \in [+b, +\infty)$.

$c_n \cdot \frac{x^n}{1-x^n} = \frac{c_n}{1-x^n} - c_n$. Now, • $\sum c_n$ is convergent, independent from x.

• $\sum \frac{c_n}{1-x^n}$: $\left| \frac{c_n}{1-x^n} \right| \leqslant \frac{|c_n|}{b^n - 1} \leqslant \frac{b^{-n}}{1-b^{-n}} \leqslant 2 \cdot b^{-n}$. By Weierstrass,

ultimately

this converges absolutely and uniformly on $[+b, +\infty)$

Then $\sum \frac{c_n}{1-x^n} - c_n$ converges uniformly since convergence of $\sum c_n$ does not depend on x.

(c) lemma. $\sum u_n$ & $\sum v_n$ converges abs. $\Rightarrow \sum u_n + v_n$ converges absolutely

(d) $s(x)$ is defined whenever (L) converges: $\mathbb{R} - \{\pm 1\}$

Since the convergence is uniform, $s(x)$ is continuous on its domain, by 2A.

5. Consider the series $\sum_1^\infty \frac{1}{x^2 - n^2}$. Do either (a) or (a'), not both! State carefully what facts you use, step by step.

(a) [8pts] Show the series converges uniformly on $(-1, +1)$.

(a') [12pts] Show the series converges uniformly on any compact interval that does not contain a nonzero integer.

(b) [13pts] For $x \in (-1, +1)$ let $f(x) = x^2 \sum_1^\infty \frac{1}{x^2 - n^2}$, whenever defined. What is the domain A of f? Show that f is C^1 on A. Compute $f'(x)$ on A.

(a) $\left|\frac{1}{x^2-n^2}\right| \leqslant n^2/4$, for $x \in (-1,+1)$. Then by Weierstrass...

(b) $A = (-1, +1)$.

$$(f_n(x))' = \frac{2x}{x^2-n^2} - \frac{x^2 \cdot 2x}{(x^2-n^2)^2} = \frac{-2xn^2}{(x^2-n^2)^2}$$

On $(-1,+1)$: $|(f_n(x))'| \leqslant \frac{2n^2}{(x^2-n^2)^2} \leqslant \frac{2n^2}{n^4/2} \leqslant 4n^{-2} =: M$

So $\sum (f_n(x))'$ converges uniformly on $(-1,+1)$.

Hence by 2D, $f'(x) = \sum (f_n(x))'$ on $(-1,+1)$.

Since each $(f_n(x))'$ is continuous on $(-1,+1)$, and $\sum f_n'(x)$ converges uniformly, $f'(x)$ is cont by 2A so that $f(x)$ is C^1.