Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2025 – Second Midterm Exam

1	2	3	4	$\sum$
25 pts	25 pts	25 pts	25 pts	100 pts

`Date: December 3, 2025` `Time: 17:00-19:00`	Full Name:

Below, $\rightsquigarrow$ means "write here; nowhere else!"

1. For a compact set $A \in \mathbb{R}^n$ and a function $f : A \to \mathbb{R}$, the graph Γ_f of f is defined as the set

$$\Gamma_f = \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \,|\, y = f(\mathbf{x})\} \subset \mathbb{R}^{n+1}.$$

(a) [10] Show: if f is continuous on A then Γ_f is compact.
(b) [5] Remind me the compactness in $\mathbb{R}^n$ in terms of sequences:

$\rightsquigarrow$ $A \subset \mathbb{R}^n$ is compact if and only if ... something about sequences every sequence in A has a convergent subsequence with limit in A.

(c) [10] Show: if Γ_f is compact then f is continuous on A. (Hint: I recommend proving the contrapositive. Assume f is not continuous at $\mathbf{x} \in A$. This means something in terms of sequences. Now aim at failing your definition in part (b).)

(a) Observe the function $F: \mathbb{R}^n \to \mathbb{R}^{n+1}$, $F(x) = (x, f(x))$ is a continuous function. This is because its component fncs id & f are cont. Therefore $f(A) = \Gamma_f$ is compact.

(b) Suppose f is not cont at x. Then there is* a sequence (x_n) in A that converges to x but $(f(x_n))$ converges to $\alpha \neq f(x)$. Equivalently the sequence $u_n = (x_n, f(x_n))$ in Γ_f converges to $(x, \alpha) \neq (x, f(x))$. So $\lim u_n \notin \Gamma_f$.

*: is there such a sequence really? Yes; think about it.

2. (p.38) Suppose S is a connected set in $\mathbb{R}^2$ that contains the points $(1, 3)$ and $(4, -1)$. Show that S contains at least one point on the line $x = y$. (Hint: Consider the function $h(x, y) = x - y$.)

h is a continuous function on S & S is connected.
Therefore $h(S)$ is an interval I. Moreover $h(1,3) = -2$, $h(4,-1) = 3$.
Hence $0 \in I$, i.e. $\exists\ (x, y) \in S$ s.t. $h(x,y) = x - y = 0$.

3. (a) [4] Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$. Give the explicit definition of differentiability of φ at $\vec{a}$. (I started the definition below; you go on. In your definition put a vector on top of every letter which denotes something in $\mathbb{R}^2$. In your definition we must also see the little-o notion and its explanation.)

$\rightsquigarrow$ φ `is differentiable at` $\vec{a} = (a_1, a_2)$ `if there exists some` $m \in \mathbb{R}^2$ satisfying

for all $\vec{h} \in \mathbb{R}^2$: $\varphi(\vec{a}+\vec{h}) = \varphi(\vec{a}) + \vec{m} \bullet \vec{h} + E(\vec{h})$
where $E : \mathbb{R}^2 \to \mathbb{R}$ is $o(|\vec{h}|)$, i.e. $E(\vec{h})/|\vec{h}| \xrightarrow[\vec{h}\to 0]{} 0$.

Now consider the function $g(x) = \begin{cases} \dfrac{x^2y^2}{x^4+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

(b) [6] Compute $\partial_1 g(0,0)$ and $\partial_2 g(0,0)$.
(c) [5] If g were differentiable at $(0,0)$ what would be its derivative $\nabla g(0,0)$?
(d) [10] Assume g satisfies your differentiability definition in part (a). In that case show that your claim for $\nabla g(0,0)$ in part (c) causes the contradiction that the error made does not satisfy the little-o condition in part (a). Show your computation about this little-o contradiction explicitly and finish your discussion with a clear, explicit conclusion. (Side note: This part proves that g is not differentiable at $(0,0)$.)

(b) $\partial_1 g(0,0) = \lim\limits_{t\to 0} \frac{1}{t}\left[f(0+t,0) - f(0,0)\right] = 0$
$\partial_2 g(0,0) = \lim \frac{1}{t}\left[f(0,0+t) - f(0,0)\right] = 0$

(c) If g were differentiable at $(0,0)$, its derivative $\nabla g(0,0)$ would be $\nabla g(0,0) = (0,0)$.

(d) Suppose g is diff'ble at $(0,0)$. Then

$$g(h_1,h_2) = g(0,0) + (0,0)\bullet(h_1,h_2) + E(\vec{h}), \quad E \text{ is } o(|\vec{h}|).$$

$$\Leftrightarrow E(\vec{h}) = \frac{h_1^2 h_2^2}{h_1^4 + h_2^3} \text{ is } o(|\vec{h}|), \text{ i.e. } \lim_{\vec{h}\to 0} \frac{h_1^2h_2^2}{h_1^4+h_2^3} \Big/ \sqrt{h_1^2+h_2^2} = 0.$$

However for $h_2 = kh_1$ $(k\in\mathbb{R})$, $\lim\limits_{\substack{\vec{h}\to 0 \\ (h_2 = kh_1 \\ h_1 \to 0)}} \left| \frac{h_1^2 k^2 h_1^2}{(h_1^4 + k^2h_1^3)\cdot|h_1|\sqrt{1+k^2}} \right| = \lim\limits_{h_1\to 0} \left|\frac{k^2}{(h_1+k)\sqrt{1+k^2}}\right|$

$= \dfrac{k^2}{|k|\sqrt{1+k^2}}$. Since this limit depends on k, lim does not exist.
$E(\vec{h})$ cannot be $o(|\vec{h}|)$.

4. (p.94) (a) [15] Show that $|\sin x - x + x^3/6| < 0.09$ for $|x| \leq \pi/2$.
(b) [10] How large do you have to take k so that the k^{th}-order Taylor polynomial $P_{0,k}(x)$ of $\sin x$ centered at 0 approximates $\sin x$ to within 0.01 for $|x| \leq \pi/2$?

(a) $\sin x = P_{0,4}(x) + R_{0,4}(x)$ where $P_{0,4}(x) = x - x^3/6$ is the deg-4 Taylor polynomial of $\sin x$ centered at 0, and

$R_{0,4}(x) = \frac{f^{(5)}(c)}{5!} x^5$, c btw 0 & x, is the Lagrange remainder.

$f^{(4)}(c) \approx \sin c$. So $|R_{0,4}(x)| \leqslant x^5/5! \overset{|x| \leqslant \pi/2}{\leqslant} \pi^5/2^5 \cdot 120 < \frac{320}{2^5 \cdot 120} = 1/12$

(b) We need $\frac{(\pi/2)^n}{n!} < 0.01$; so $100\pi^n < 2^n \cdot n!$. I think $n \geqslant 7$ would do.

So we need the 6th order Taylor polynomial: $x - \frac{x^3}{6} + \frac{x^5}{120}$.