Last time.

- linear combination ; span
- linear independence of vectors $v_1, \dots, v_k$:
  whenever $c_1 v_1 + \dots + c_k v_k = 0$ $(c_i \in \mathbb{R})$
  these c_i's are forced to be zero.
- basis of a vector space.
  $u_1, \dots, u_n \in V$ is a basis if
  - $V = \operatorname{span}(u_1, \dots, u_n)$
  - $u_1, \dots, u_n$ are linearly independent.
- dimension of V is the # of vectors in a basis.

ex. $\mathbb{R}^n$ with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

because if for $c_i \in \mathbb{R}$

$c_1 e_1 + \dots + c_n e_n = 0$ then:

$$\begin{pmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ \vdots \\ c_n = 0 \end{matrix}$$

Example for today: $A_{4\times 6}$

$$A \xrightarrow{\text{row reduction}} \begin{array}{cccccc} \begin{bmatrix} 1 & -1 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \end{array}$$

Recall

$$(*) \quad \underset{4\times 4}{F} \cdot A = U \quad \& \quad A = F^{-1} \cdot U$$

① $\boxed{\text{null}(A)} = \text{null}(U)$ because

fact 1. If $P = Q \cdot R$ & Q invertible then $\text{null}\,P = \text{null}\,R$

Now $\text{null}(U) = \{ x \in \mathbb{R}^6 \mid Ux = 0 \}$

$$= \{ x_5 = -x_6,\ x_3 = -x_4 - x_6,\ x_1 = x_2 - 2x_4 + 3x_5 - x_6 \}$$

$$= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{=u_2} + x_4 \underbrace{\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=u_4} + x_6 \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{=u_6} \right\}$$

$= \{ x \in \mathbb{R}^6 \mid x = x_2 u_2 + x_4 u_4 + x_6 u_6,$
$x_2, x_4, x_6 \in \mathbb{R} \}$

$= \operatorname{span}(u_2, u_4, u_6)$

Moreover u_2, u_4, u_6 are linearly independent: Assume for some $c_1, c_2, c_3 \in \mathbb{R}$ $c_1 u_2 + c_2 u_4 + c_3 u_6 = 0$

Then
$$\begin{cases} c_1 - 2c_2 + 2c_3 = 0 \\ c_1 = 0 \\ -c_2 - c_3 = 0 \\ c_2 = 0 \\ -c_3 = 0 \\ c_3 = 0 \end{cases} \Rightarrow \begin{matrix} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0. \end{matrix}$$

<u>thm 1.</u>
- A basis for $\operatorname{null}(A_{p\times q})$ is the collection of vectors in $\mathbb{R}^q$ found by the above algorithm.
- $\dim(\operatorname{null}(A_{p\times q})) = \#$ of free variable $= q - \#$pivots

The argument above proves also:

<u>fact 2:</u> The following collection is linearly independent:

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}*\\\vdots\\*\\1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}*\\\vdots\\*\\1\\0\end{pmatrix}$$

(2) $\boxed{\mathrm{col}(A)} = ?$

ex: $\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}$

$\mathrm{col} = \begin{pmatrix}1\\1\end{pmatrix} \neq \mathrm{col} = \mathrm{span}\begin{pmatrix}1\\0\end{pmatrix}$

Although $\mathrm{col}(A) \neq \mathrm{col}(U)$ in general, they are related as below:

$\mathrm{col}(U)$ has basis $\begin{pmatrix}1\\0\end{pmatrix}$. $\mathrm{col}(A)$ has basis $\begin{pmatrix}1\\1\end{pmatrix}$.

$$\operatorname{col}(U) = \operatorname{span}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \right)$$

$$= \operatorname{span}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\1\\0\end{pmatrix} \right)$$

Moreover by fact 2, these constitute a basis. Hence we proved:

<u>thm2a.</u> $\dim(\operatorname{col}(U)) = \#$ pivots.

<u>defn:</u> <u>rank</u> of A is the $\#$ pivots

Then $\dim(\operatorname{col}(U)) = r = \operatorname{rank}(A)$.

What about $\operatorname{col}(A)$?

Recall: $F \cdot A = U$, $A = F^{-1} \cdot U$.

By fact 1, $\mathrm{null}(A) = \mathrm{null}(U)$

i.e.
all possible linear combinations of the columns of A that give $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the same as all possible linear combs of U that give $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

i.e.
If a nonzero linear comb of column of U gives 0 the the same nonzero comb of columns of A gives 0 too (and vice versa)

i.e. the linearly indep set of columns of A is exactly the correspondin lin. indep columns of U.

thm 26. • The col(A) has a basis which is constituted of the columns corresp to the columns of U that are basis for col(U).

• dim(col(A)) = dim(col(U)) = rank(A)

In the example:
col(U) has basis the 1^{st}, 3^{rd} & 5^{th} columns (the ones with pivots)
So col(A) has basis the 1^{st}, 3^{rd}, 5^{th} columns of A.

③ $\boxed{\text{row}(A)} = ?$

$=$ span of rows of A

$=$ span of rows of $U = \text{row}(U)$

because rows of U are lin. combos of rows of A & vice versa.

An elegant way of saying this:

$F \cdot A = U$ is nothing but taking linear combinations of rows of A to produce rows of U.

In our example:

$$\underset{\mathbb{R}^6}{\text{row}(U)} = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

By fact 2, the first 3 rows are linearly independent. This proves:

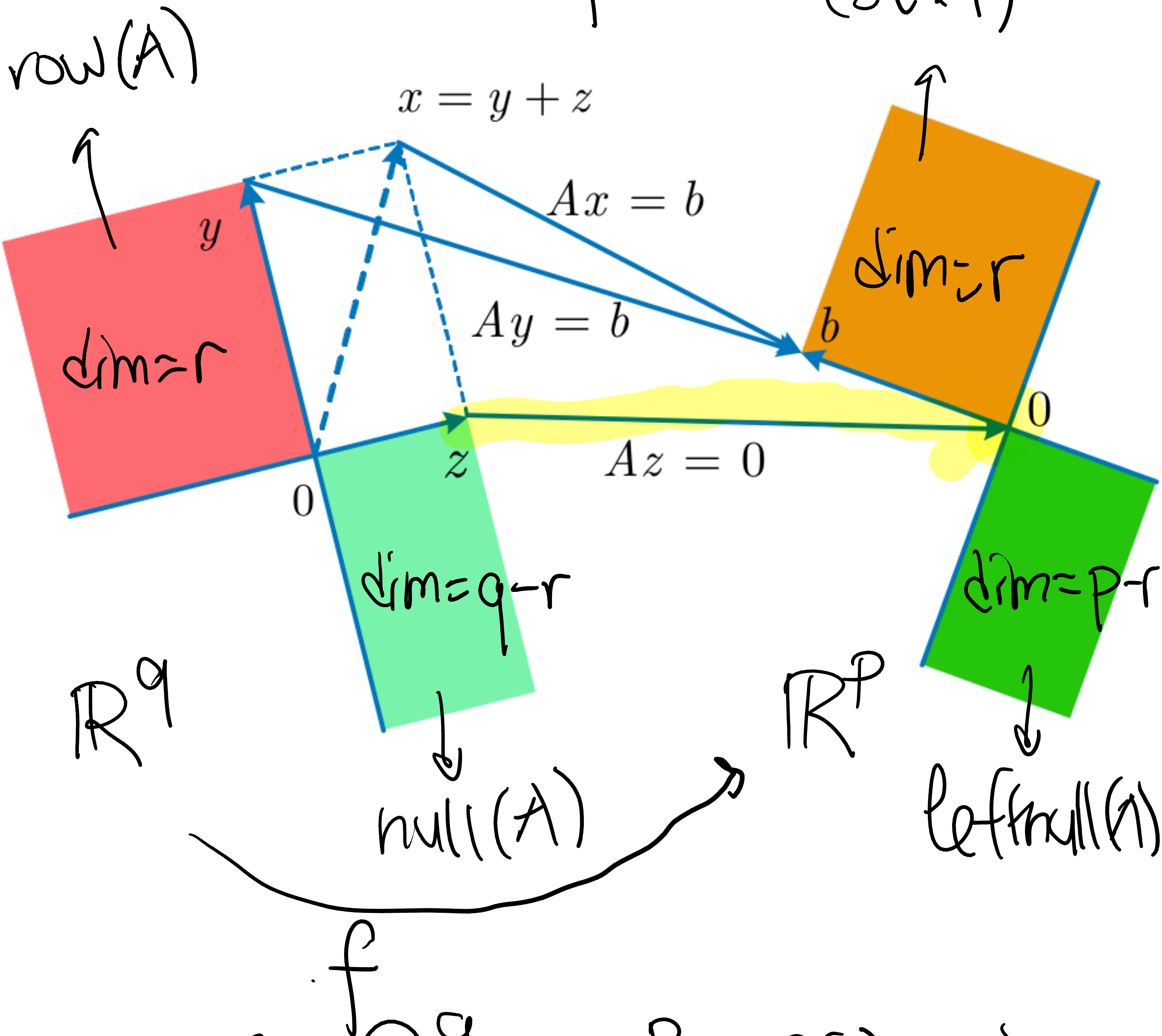
<u>thm 3</u>
- row(A) has basis the rows corresponding to the pivots.
- $\dim(\text{row}(A)) = r = \text{rank}(A)$.

<u>Rank-Nullity Theorem</u> (or the Fundamental Thm of lin. Alg.)

(i) null(A), row(A) are subspaces of $\mathbb{R}^q$ $p \times q$ with dim $q-r$ & r respectively.

(ii) $\text{null}(A^T)$, $\text{col}(A)$ $\text{row}(A^T)$ $\subset \mathbb{R}^p$ with dimensions $p-r$ & r respectively

For $A_{p \times q}$



Define $f : \mathbb{R}^q \to \mathbb{R}^p$, $f(x) = Ax$.

Observe: image(f) = col(A)

We'll see later:

- f sends row(A) to col(A) in a 1-1, onto fashion.
- row(A) is orthogonal to null(A)

Dimension re(ally)-visited.

If there were two bases with different # of vectors then dimension would not be defined. We claim that is impossible.

thm 4. Let $V \overset{=\mathbb{R}^2?}{}$ be a vector space & $(u_1, \dots, u_n)$, $(v_1, \dots, v_k)$ be two bases. Then $n = k$ (so that dim. is well-defined.

proof. Suppose $n < k$ &

$$v_j = a_{1j} u_1 + \dots + a_{nj} u_n \quad \text{for each } j.$$

i.e. $\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \underbrace{\begin{pmatrix} a_{11} & & a_{k1} \\ \vdots & \ddots & \vdots \\ a_{1n} & & a_{kn} \end{pmatrix}}_{A}$

Since $n<k$ & $n \geq \#$pivots, there are free variables for $Ax=0$

i.e. $\mathrm{null}(A) \neq \{0\}$; in particular, there is some $x \in \mathbb{R}^k$ $x \neq 0$ s.t. RHS$\cdot x=0$

So LHS$\cdot x=0 \Leftrightarrow$ the columns $v_1, \ldots, v_k$ are linearly dependent. This is a contrafiction. To get rid of that we give up the assumption $n<k$. With similar argument for $n>k$, we get $n=k$.

$\blacksquare$