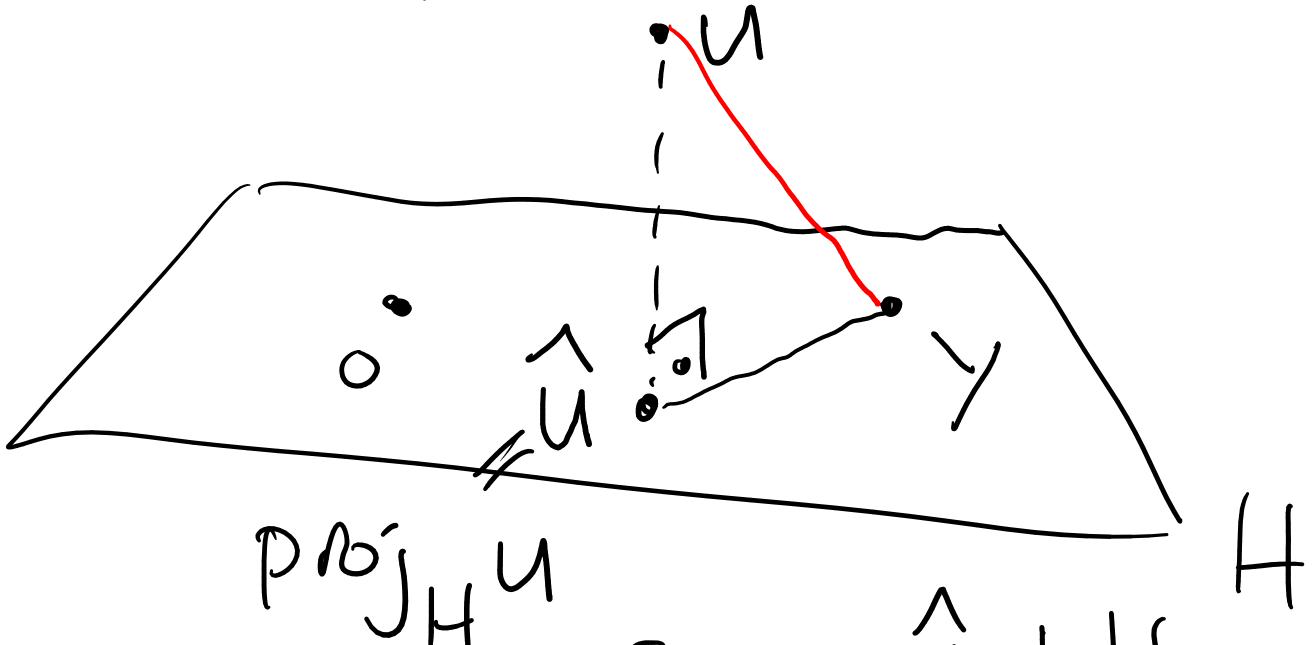
- Given $H \subset \mathbb{R}^n$ a subspace; $(k \leq n)$
$A_{n\times k}$ with $\mathrm{col}(A) = H$ & colums linearly independent.
$\underset{n\times n}{P} = A(A^TA)^{-1}A^T$ projects orthogonally vectors in $\mathbb{R}^n$ to $H$.
- $P^2 = P$ & $P^T = P$; i.e. $P$ is a projection matrix.
- So $\mathrm{col}(P) = \mathrm{col}(A)$.

  & $\mathrm{rank}(P) = \mathrm{rank}(A) = k$.
- $\dim \mathrm{null}(P) = n-k$.
- If $Pu = \hat{u}$ then for any $n \in \mathrm{null} P$, $u+n$ projects orthog to $\hat{u}$ too.
- 


so $u - \hat{u} \perp H$.

Therefore for any $y \in H$, the distance from $u$ to $y \geq$ the distance from $u$ to $\hat{u}$.

<u>thm1.</u> $\text{proj}_H u$ is the closest point in $H$ to $u$ (in the Euclidean distance)

<u>thm2.</u> Any projection matrix $Q_{n\times n}$ orthogonally projects to a subspace of $\mathbb{R}^n$.

<u>proof.</u> For any $x$, $Qx \in H = \text{col}(Q)$

We claim that $Qx = \text{proj}_H x$.

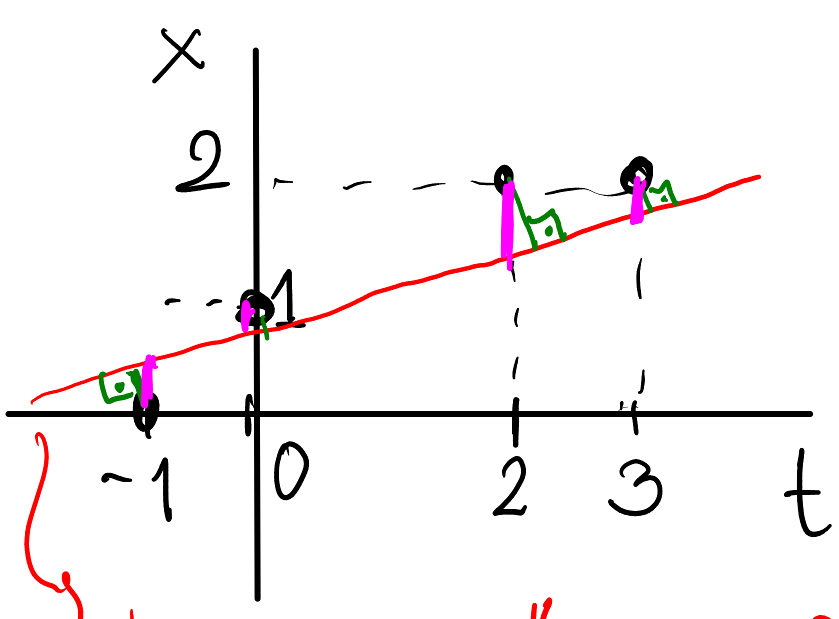
$Qx \in H$ already. We must prove:

$$x - Qx \perp H \quad \text{i.e. } x - Qx \perp \text{every column of } Q$$

$$\Leftrightarrow Q^T(x - Qx) = 0$$

$$Q^T x - Q^T Q x \overset{Q^T = Q}{=} Qx - Q^2 x \overset{Q^2 = Q}{=} Qx - Qx = 0$$

# LEAST SQUARES APPROXIMATION




| t | x |
|---|---|
| -1 | 0 |
| 0 | 1 |
| 2 | 2 |
| 3 | 2 |

L is the "best fitting" line.

L: $x = a + bt$ ; $a, b$ unknown.

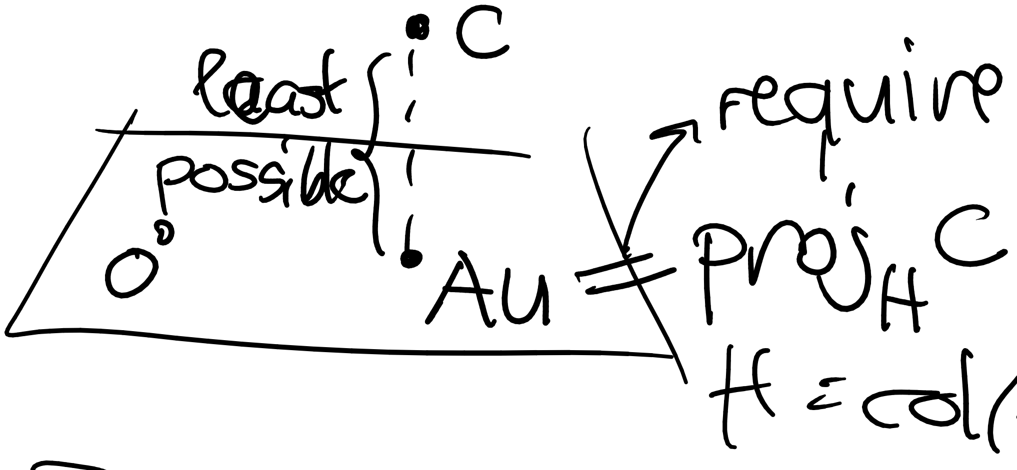
Find them, so that the sum of the square of the vertical distances of the data points to L is made minimum.

(minimizing the orthog. distance to the line is another problem.)

We fancy a line L:

$$\begin{aligned} a + (-1)b &= 0 \\ a + 0.b &= 1 \\ a + 2b &= 2 \\ a + 3b &= 2 \end{aligned} \iff \overbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} a \\ b \end{pmatrix}}^{u} = \overbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}}^{c}$$

There is no soln for u because apparently the data points are not collinear. Instead we are content to solve u such that Au is closest to c.




This solves the minimizing the sum of the square of vertical distances problem.

Let's solve.

$$P = A(A^TA)^{-1}A^T$$

$$= A \cdot \begin{pmatrix} 4 & 4 \\ 4 & 14 \end{pmatrix}^{-1} \cdot A^T = A \cdot \frac{1}{40}\begin{pmatrix} 14 & -4 \\ -4 & 4 \end{pmatrix} A^T$$

$$= A \cdot \frac{1}{40} \cdot \begin{pmatrix} 18 & 14 & 6 & 2 \\ -8 & -4 & 4 & 8 \end{pmatrix}$$

Thus solve $\underset{4\times 2}{A}\, u = Pc \in \mathrm{col}(A)$

The soln for $u$ is unique because $A_{4\times 2}$ has rank 2.

$$Au = Pc = A\underbrace{(A^TA)^{-1}A^Tc}$$

So the unique soln $u$ for $u$ is

$$\boxed{u = (A^TA)^{-1}A^Tc}$$

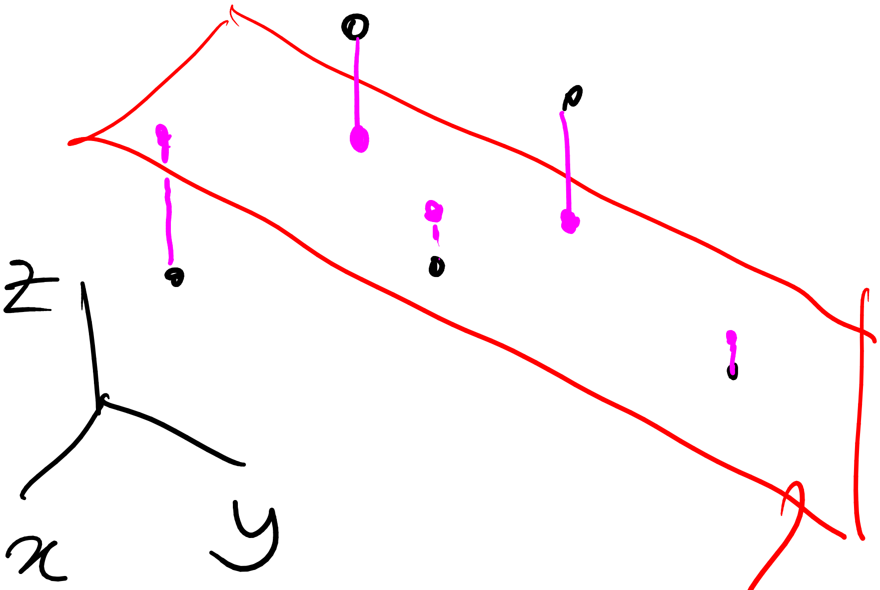
$$= \frac{1}{20}\begin{pmatrix} 9 & 7 & 3 & 1 \\ -4 & -2 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 15 \\ 10 \end{pmatrix}$$

so that $a = 3/4$, $b = 1/2$.

ex: Measurement:

| $x$ | $y$ | $z$ |
|---|---|---|
| $x_1$ | $y_1 \to$ | $z_1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $x_k$ | $y_k \to$ | $z_k$ |

Here $z = z(x,y)$




best fitting plane

$z = a + bx + cy$

We solve

$$\underbrace{\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_k & y_k \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{u} = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} = z$$

Instead solve $Au = \mathrm{proj}_{\mathrm{col}(A)}\, z$:

$$\boxed{u = (A^T A)^{-1} A^T z}$$

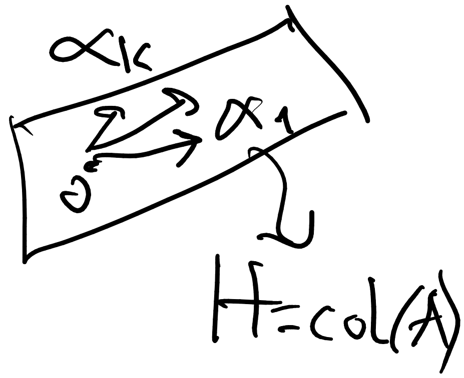
# ORTHONORMALITY & ORTHOGONAL MATRICES

Back to projection matrices.

$$P = A(A^TA)^{-1}A^T$$

with $A = (\alpha_1 \vdots \cdots \vdots \alpha_k)$

& $\{\alpha_1, \dots, \alpha_k\}$ a basis for $H$. Observe:




$$A^TA = \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \vdots \\ \text{---} \alpha_k^T \text{---} \end{pmatrix} (\alpha_1 \vdots \cdots \vdots \alpha_k)$$

$$= \begin{pmatrix} \alpha_1^T\alpha_1 & \alpha_1^T\alpha_2 & \cdots \\ \alpha_2^T\alpha_1 & \ddots & \\ \vdots & & \end{pmatrix} = I$$

we'd have

Assume $\alpha_1, \dots, \alpha_k$ is an orthogonal collection with lengths 1.

i.e. $\alpha_i^T\alpha_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Kronecker delta

In that case $P = AA^T$.

<u>defn</u>: An orthogonal collection with all lengths 1 is called an <u>orthonormal</u> collection.

<u>thm 3</u>. Let $\alpha_1, \dots, \alpha_k$ be orthonormal basis for $\mathbb{R}^k$. Then for $u \in \mathbb{R}^k$,

$$u = \mathrm{proj}_{\alpha_1} u + \cdots + \mathrm{proj}_{\alpha_k} u.$$

<u>proof</u>: Let $u = c_1\alpha_1 + \cdots + c_k\alpha_k$.

Then $\alpha_1^T u = c_1 \underbrace{\alpha_1^T\alpha_1}_{1} + \cdots + c_k \underbrace{\alpha_1^T\alpha_k}_{0}$

$= c_1$

Similarly $c_j = \alpha_j^T u$ so that

$$u = (\underbrace{\alpha_1^T u}_{\alpha_1^T\alpha_1})\alpha_1 + \cdots + (\alpha_k^T u)\alpha_k$$

$$= \mathrm{proj}_{\alpha_1} u + \cdots + \mathrm{proj}_{\alpha_k} u.$$