Last time:

- $$PA = LDU$$
  - $U \to$ upper
  - $D \to$ diagonal
  - $L \to$ lower
  - $A \to$ square
  - $P \to$ permutation: shuffles the rows

$P = P_k \cdot \ldots \cdot P_2 \cdot P_1$ where each P_j exchanges two rows.

Note: $P_j^{-1} = P_j$. Therefore P is invertible with

$$P^{-1} = (P_k \cdots P_1)^{-1} = P_1^{-1} \cdot P_2^{-1} \cdots P_k^{-1}$$

Remark. Instead of applying P in the beginning, you can do that after you got L to obtain:

$$A = LPU' = LPDU$$

□ Gauss-Jordan algorithm computes A^{-1} if you can obtain U with no pivots missing; i.e. if U has zeros instead of pivots then G-J is stuck.

$$\left[\begin{array}{c|c} A & I \end{array}\right] \longrightarrow \left[\begin{array}{c|c} A_1 & \cdots \ 0 \ 0 \end{array}\right]$$ → in that case stuck

$$\longrightarrow \left[\begin{array}{c|c} I & B \end{array}\right]$$ in that case $A^{-1} = B$.

<u>thm</u>: The following cases are equivalent:

1. There is a permutation of rows of $A_{p\times p}$ so that G-J is successful; i.e. there is no missing pivot.
2. A has an inverse
3. For every $b \in \mathbb{R}^p$ the system $Ax = b$ has a unique solution.

Proof: One has to prove $① \Rightarrow ②$, $② \Rightarrow ①$, $① \Rightarrow ③$, $③ \Rightarrow ①$, $② \Rightarrow ③$, $③ \Rightarrow ②$

PLAN: We'll prove

$① \Rightarrow ② \Rightarrow ③ \Rightarrow ①$

$① \Rightarrow ②$: Assume ①; i.e. after some permutation, A has the inverse B:

$$(PA)^{-1} = B$$

$$\Leftrightarrow PA = B^{-1}$$

$$\Leftrightarrow P^{-1}PA = P^{-1}B^{-1}$$

$$\Leftrightarrow A = P^{-1}B^{-1} = (BP)^{-1}$$

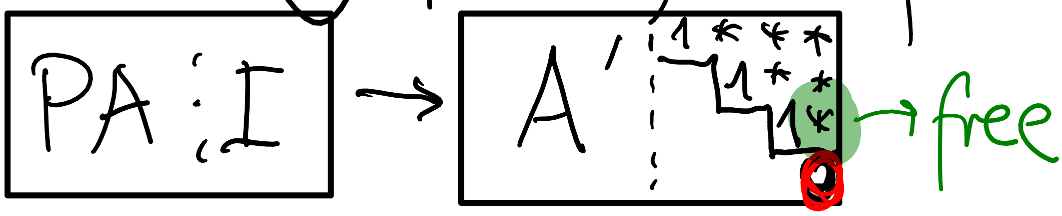
$$\Leftrightarrow A^{-1} = BP$$

$② \Rightarrow ③$: Assume A^{-1} exists & $b \in \mathbb{R}^p$ given. Then $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b$

$\Rightarrow x = A^{-1}b$ is the unique soln.

③ ⇒ ① : Let us prove the equivalent statement NOT ① ⇒ NOT ③

Assume NOT ① : For some permutation there is a missing pivot, i.e. for some P,



Now

Prove NOT ③ : For some $b \in \mathbb{R}^p$ it is not true that there's a unique soln.

In fact:

- If $b = \begin{pmatrix} * \\ \vdots \\ * \\ 1 \end{pmatrix}$, then there's no soln. → inconsistency
- If $b = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \end{pmatrix}$, then there are only → consistent many solution.

SUBSPACES

Start with a vector space $V = \mathbb{R}^n$.

<u>defn:</u> A <u>(linear) subspace</u> W

is a subset of $\mathbb{R}^n$ which is closed under $+$ & scalar multiplication, i.e. for $u, v \in W$, $k \in \mathbb{R}$:

(i) $u + v \in W$ too.

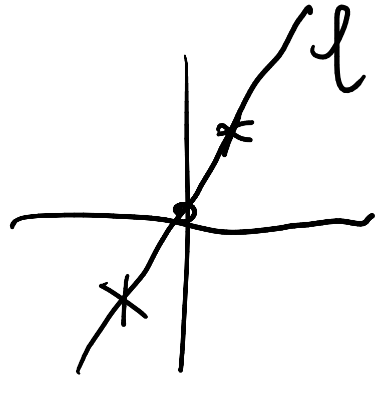
(ii) $ku \in W$ too.

<u>ex:</u> ① A line in $\mathbb{R}^2$ thru the origin is a subspace of $\mathbb{R}^2$. Because:

$\ell : \{(x,y) \mid ax + by = 0\}$ satisfies:

(i) For $(x,y), (x',y') \in \ell$

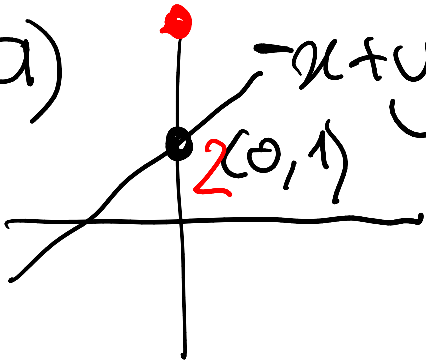
$(x+x', y+y') \in \ell$.



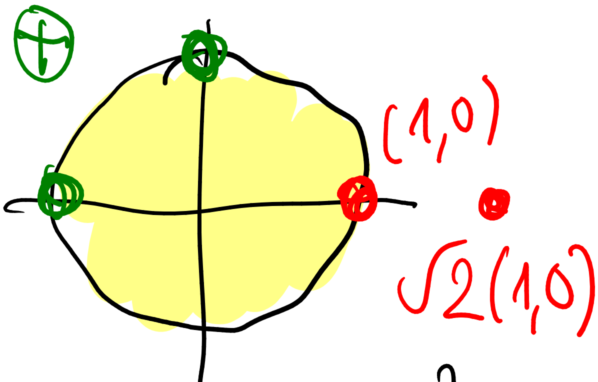
(ii) $k(x,y) = (kx, ky) \in \ell$

② These are not subspaces of $\mathbb{R}^2$:

(a)

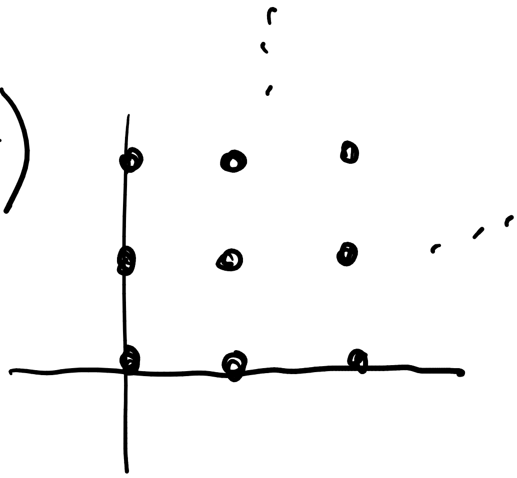


(b)



$S=\{x^2+y^2\leq 1\}$

(c)



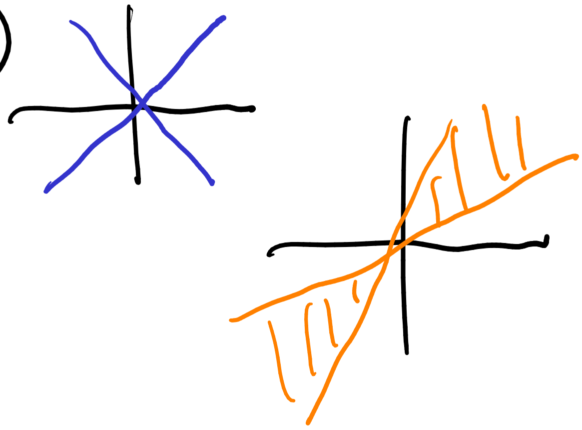
$T=\{$all pairs with nonnegative integer coordinates$\}$

(i) closed under $+$:

$(3,8)+(1,5)\in T$

(ii) not under scalar mult.

(d)



These satisfy (ii) but not (i).