Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2024 – Final Exam

1	2	3	4	5	$\sum$
20 pts	20 pts	20 pts	20 pts	21 pts	100 pts

Date: January 4, 2025 Time: 13:00-15:45	Full Name: PROPOSED SOLUTIONS

- For $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^n$, the degree-$k$ Taylor polynomial of f at $\mathbf{b}$ is $P^f_{\mathbf{b},k}(\mathbf{h})$. The Lagrange remainder $R^f_{\mathbf{b},k}(\mathbf{h}) = f(\mathbf{b}+\mathbf{h}) - P_{\mathbf{b},k}(\mathbf{h})$ is given by $\sum_{|\alpha|=k+1} \partial^\alpha f(\mathbf{b}+c\mathbf{h})\frac{\mathbf{h}^\alpha}{\alpha!}$, for some $c \in (0,1)$. Recall $|R^f_{\mathbf{b},k}(\mathbf{h})| \le M||\mathbf{h}||^{k+1}/(k+1)!$ where M is an upper bound for all partials of f of order $k+1$.
- A function $g : \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* if for every $\mathbf{x} \in \mathbb{R}^n$ its Laplacian is zero, that is, $\Delta g(\mathbf{x}) = (\partial_{11}g + \partial_{22}g + \ldots + \partial_{nn}g)(\mathbf{x}) = 0$.
- I wish you keep on having fun with maths in 2025.

1. Suppose for a C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$. Prove that there is a point $\mathbf{b} \in \mathbb{R}^3$ such that $\partial_{11}f(\mathbf{b}) + \partial_{22}f(\mathbf{b}) + \partial_{33}f(\mathbf{b}) \ge 0$.

By thm, there is an absolute min $a \in \mathbb{R}^3$ of f. Since a is a local min, $\nabla f(a) = 0$ and $Jf(a) \geq 0$ and trace $Hf(a) \geq 0$.
But this last inequality is exactly what's asked with $b = a$.

2. Consider the function $f(x,y,z,w) = (\underbrace{z^3 + xw - y}_{f_1}, \underbrace{w^3 + yz - x}_{f_2})$ and its zero level set $S_0 = \{f = 0\} \in \mathbb{R}^4$.
(a) Determine the set of all points $(a,b,c,d) \in S_0$ near which, on S_0, (y,w) can be written as a function of (x,z). Write down all relations which a,b,c,d must satisfy.
(b) Suppose on S_0, $(y,w) = \varphi(x,z)$ around the point $(1,1,0,1) \in S_0$. Compute $\partial_z\varphi(1,0)$.

(a) An application of ImpFT... Note f is C^1 (polynomial).
Consider $\begin{pmatrix} \partial_y f_1 & \partial_w f_1 \\ \partial_y f_2 & \partial_w f_2 \end{pmatrix} = \begin{pmatrix} -1 & w \\ z & 3w^2 \end{pmatrix}$. Its det at (a,b,c,d) is $-3d^2 - cd$.

By ImpFT, (y,w) is a fnc of (x,z) on S_0 whenever det $\neq 0$ on S_0: $\boxed{d(3d+c) \neq 0 \ \& \ c^3 + ad = b,\ d^3 + bc = a}$

(b) Let $P = (1,1,0,1)$. Suppose $(y,w) = \varphi(x,z) = (\varphi_1(x,z), \varphi_2(x,z))$ around P on S_0. We're asked $\partial_z\varphi = (\partial_z\varphi_1, \partial_z\varphi_2) = (\partial_z y, \partial_z w)$ at P.

On S_0, $0 = \partial_z f_1 = 3z^2 + x \cdot w_z - y_z$ $\;= w_z - y_z \longrightarrow y_z(1,0) = -1/3$
$0 = \partial_z f_2 = 3w^2 \cdot w_z + y_z \cdot z + y$ $\;= 3w_z + 1 = w_z(1,0) = -1/3$
at P. Hence $\boxed{\partial_z\varphi(P) = (-1/3, -1/3)}$

3. (a) Find $P^f_{\mathbf{0},5}(x,y)$ for the function $f(x,y) = x\sin(x+y)$.
   (b) Give an upper bound for $|R^f_{\mathbf{0},5}(1,1)|$.

(a) Observe $P^f_{\mathbf{0},k}(x,y) = x\cdot P^{\sin u}_{\mathbf{0},k-1}(x+y)$. So $P^f_{\mathbf{0},0}(x,y) = 0 = P^f_{\mathbf{0},1}(x,y)$

Why? $P^f_{\mathbf{0},2} = x\cdot(x+y) = P^f_{\mathbf{0},3}$; $P^f_{\mathbf{0},4} = x\left[x+y-\frac{(x+y)^3}{3!}\right] = P_{\mathbf{0},5}$

(b) By thm, $|R^f_{\mathbf{0},5}(1,1)| \leq M\cdot\frac{(1+1)^6}{6!}$. For all index pair $\alpha = (\alpha_1,\alpha_2)$ with $|\alpha| = \alpha_1+\alpha_2 = 6$, the partial $\partial^\alpha f(x,y)$ is a sum of sines, cosines, x sines, x cosines. There are at most 7 such terms in $\partial^\alpha f$. So $|\partial^\alpha f(1,1)| \leq 7$. Hence $|R^f_{\mathbf{0},5}(1,1)| \leq 7\cdot\frac{2^6}{6!} = \boxed{28/45}$. Of course, much better can be done. Do it in your leisure.

4. (a) Find the degree-3 Taylor polynomial of $\tan u$ at 0.
   (b) Find the degree-3 Taylor polynomial of $\ln(1+u)$ at 0.
   (c) Using (a) and (b), evaluate the limit $\lim_{x\to+\infty}\frac{\tan^2\frac{1}{x}}{\ln^2(1+\frac{4}{x})}$.

(a) $\tan u = u + u^3/3$ (differentiating three times)

(b) $P_{0,k}(u)$ for $1/(1+u)$ is $\sum_{n=0}^{k}(-1)^n u^n$

So $P_{0,k+1}(u)$ for $\ln(1+u)$ is $\sum_{n=0}^{k}(-1)^n\cdot\frac{u^{n+1}}{n+1} = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots + (-1)^k\frac{u^{k+1}}{k+1}$

(c) $\lim_{x\to+\infty}\frac{\tan^2\frac{1}{x}}{\ln^2(1+\frac{4}{x})} = \lim_{u\to0^+}\left(\frac{\tan u}{\ln(1+4u)}\right)^2 = \lim_{u\to0^+}\left[\frac{u-\frac{u^2}{2}+R^{\tan}_{0,2}(u)}{4u+R^{\ln}_{0,2}(u)}\right] = \lim_{u\to0^+}\left(\frac{1-\frac{u}{2}+\cancel{\frac{R^{\tan}}{u}}}{4+\cancel{\frac{R^{\ln}}{u}}}\right)^2 = \frac{1}{16}$

5. Say True or False. Then prove or disprove... Solve on the spaces provided in this page.
   (a) Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be of class C^1 and $\det Df(\mathbf{0})\neq 0$. Then there is a neighborhood of $\mathbf{0}\in\mathbb{R}^n$ in which f is one-to-one.

TRUE

By InvFT, f has a (C^1) inverse around 0. Hence around 0, f is 1-1.

   (b) Suppose that $g: I\to\mathbb{R}^q$ is differentiable on the open interval $I\subset\mathbb{R}$ and that $|g(t)| = 1$ for all $t\in I$. Then the vector $g(t)\in\mathbb{R}^q$ is perpendicular to the vector $Dg(t)\in\mathbb{R}^q$ for all $t\in I$.

TRUE

$1 = |g(t)|^2 = g(t)\cdot g(t)$. Differentiate wrt t:

$0 = Dg(t)\cdot g(t) + g(t)\cdot Dg(t) \Rightarrow g(t)\cdot Dg(t) = 0$.

   (c) For a harmonic function $h:\mathbb{R}^2\to\mathbb{R}$, if h has a local minimum at the point $\mathbf{b}\in\mathbb{R}^2$ then all second order partial derivatives of h vanish at $\mathbf{b}$.

TRUE

At b, $\partial_{11}h(b)$ & $\partial_{22}h(b) \geq 0$. Then both are 0 since h is harmonic. Also $Jh(b) = (\partial_{11}\partial_{22} - \partial_{12}\partial_{21})h(b) \geq 0$. Because $\partial_{12}h(b) = \partial_{21}h(b)$, we must have both 0.