Last time.

* Equivalent statements to have inverse of a matrix;
* Definition of subspaces:

defn: A (linear) subspace $W \subset \mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ which is closed under $+$ & scalar multiplication. That is for all $u, v \in W$ and $k \in \mathbb{R}$

(i) $u + v \in W$

(ii) $ku \in W$

Observe:

① If $W \subset \mathbb{R}^n$ is a subspace then $0_{\mathbb{R}^n} \in W$: For $v \in W$, $0 \cdot v \in W$

$$0 \cdot v \overset{\boxed{S3}}{=} 0_{\mathbb{R}^n}$$

Be careful: $0 \in W \not\Rightarrow W$ a subspace

(2) $\{0\} \subset \mathbb{R}^n$ is a subspace.

(3) A subspace $W \subset \mathbb{R}^n$ is itself a vector space: to show this one must prove W satisfies G1–G4, S1–S3.

G1, G4, S1, S2, S3 are automatically satisfied. G2: $0_{\mathbb{R}^n} \in W$ by (1).
G3: every $v \in W$ has an ~~inverse~~ opposite in W because $-v = (-1) \cdot v \in W$ by (ii)

Null space.

defn & claim. For $A_{p \times q}$, the null space of A, denoted by $\text{null}(A)$, is the set of all solns to $\underset{p \times q}{A} x = 0$.

Here $x_{q\times 1} \in \mathbb{R}^q$. So $null(A) \subset \mathbb{R}^q$.

$null(A)$ is a subspace of $\mathbb{R}^q$.

claim

Proof of claim. We have to satisfy the conditions (i) & (ii). So

(i) Take $u, v \in null(A)$; i.e. $Au = 0$ and $Av = 0$. Then

$$A\cdot(u+v) = Au + Av = 0 + 0 = 0$$

$$\left(A(u+v)\right)_{i1} = \sum_{k=1}^{q} A_{ik}(u+v)_k$$

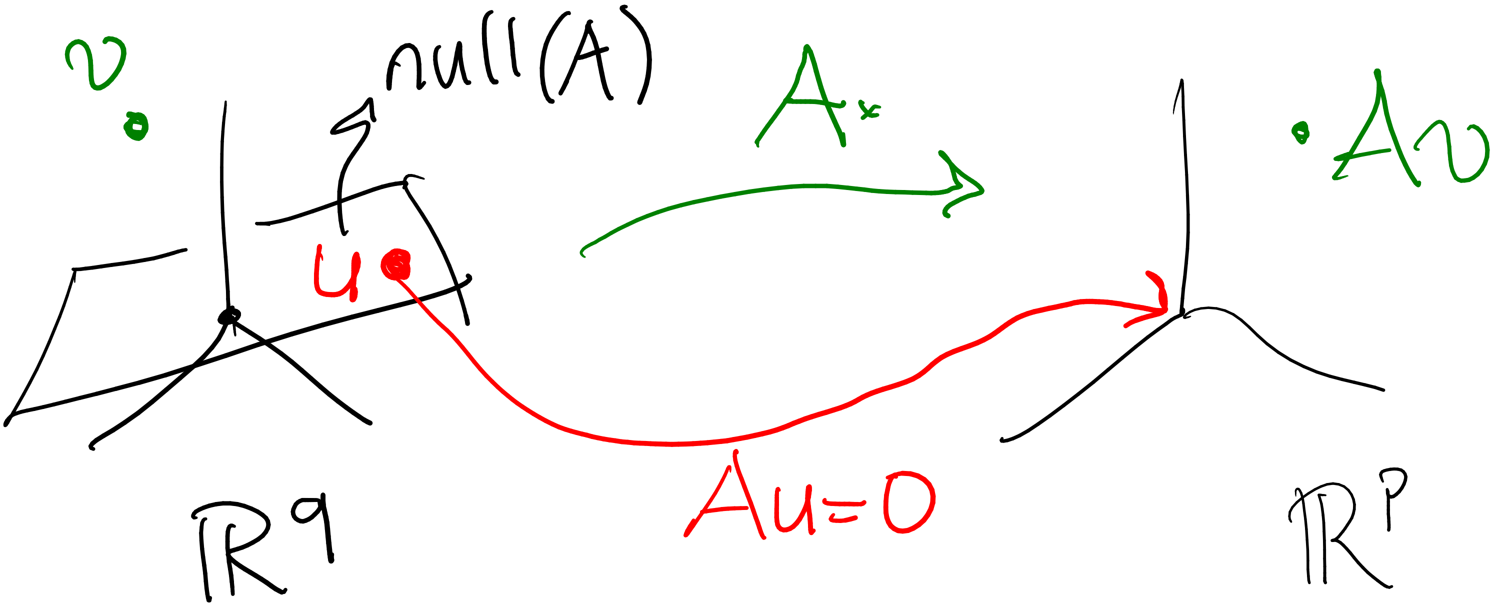
$$= \sum A_{ik}(u_k + v_k)$$

$$= \sum A_{ik}u_k + \sum A_{ik}v_k$$

$$= Au + Av$$

needs proof

(ii) Take $t \in \mathbb{R}$. $tv \in null(A)$ because $A(tv) = t(Av) = t\cdot 0 = 0$



Big observation: Multiplying with A "transforms" vectors in $\mathbb{R}^q$ to vectors in $\mathbb{R}^p$:

$$A_{p\times q}\, \overset{\in \mathbb{R}^q}{u_{q\times 1}} \text{ is } p\times 1, \in \mathbb{R}^p.$$

So $\text{null}(A) \subset \mathbb{R}^q$. Similarly:

<u>defn.</u> The $\text{null}(A^T)$ is a subspace of $\mathbb{R}^p$, called the <u>left null space</u> of A.

$$u \in \text{null}(A^T) \Leftrightarrow A^T u = 0 \Leftrightarrow u^T A = 0$$

Observe:

(1) $\underset{p\times p}{A}$ is invertible $\Leftrightarrow$ null$(A)=\{0\}$.

$\Rightarrow$: If A^{-1} exists then $Ax=0$ has a unique soln: $0_{\mathbb{R}^p}$.

$\Leftarrow$: If A had no inverse, we claim $Ax=0$ would have more than 1 solution. This is true because in that case you would have missing pivots & free variables.

ex: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 12 & 7 & 8 & 9 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\swarrow -6R1+R2 \rightarrow R2$

$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -11 & -16 & -21 \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-11x_3 = +16x_4 + 21x_5$$

$$-x_1 = 2x_2 + 3x_3 + 4x_4 + 5x_5$$

$$\Rightarrow x_1 = -2x_2 - \frac{4}{11}x_4 - \frac{8}{11}x_5$$

$$\text{null}(A) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R}^{\text{free}} \right.$$

$$\& \quad x_1 = -2x_2 - \frac{4}{11}x_4 - \frac{8}{11}x_5$$

$$\left. \& \quad x_3 = -\frac{16}{11}x_4 - \frac{21}{11}x_5 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{u_1} + x_4 \underbrace{\begin{pmatrix} -4/11 \\ 0 \\ -16/11 \\ 1 \\ 0 \end{pmatrix}}_{u_2} + x_5 \underbrace{\begin{pmatrix} -8/11 \\ 0 \\ -21/11 \\ 0 \\ 1 \end{pmatrix}}_{u_3} \right.$$

$$\left. \& \quad x_2, x_4, x_5 \in \mathbb{R} \right\}$$

So $\text{null}(A) < \mathbb{R}^5$ is the set of linear combinations of u_1, u_2, u_3.

② Assume $E_{p\times p}$ is invertible & $D_{p\times q} = E \cdot F_{p\times q}$. Then $\text{null}(D) = \text{null}(F)$.

<u>proof</u>. We must show

$$x \in \text{null}(D) \Leftrightarrow x \in \text{null}(F).$$

Assume: $\Leftarrow$

$Dx = 0$. Then $(EF)x = 0$

$$\Rightarrow E(Fx) = 0 \Rightarrow Fx \in \text{null}(E)$$

By ①, $Fx = 0$. So $x \in \text{null}(F)$.

<u>Conclusion</u>. Assume with Gaussian elimination we have

$$E_k \cdots E_1 PA = U$$

By ②, $\text{null}(A) = \text{null}(U)$. So solving $Ax = 0$ gives the same soln set as $Ux = 0$.

③ Solving $Ax=b$. (*)

If u, v solve (*) then $u-v$ is in null(A).

<u>proof.</u> If $Au=b$ & $Av=b$ then

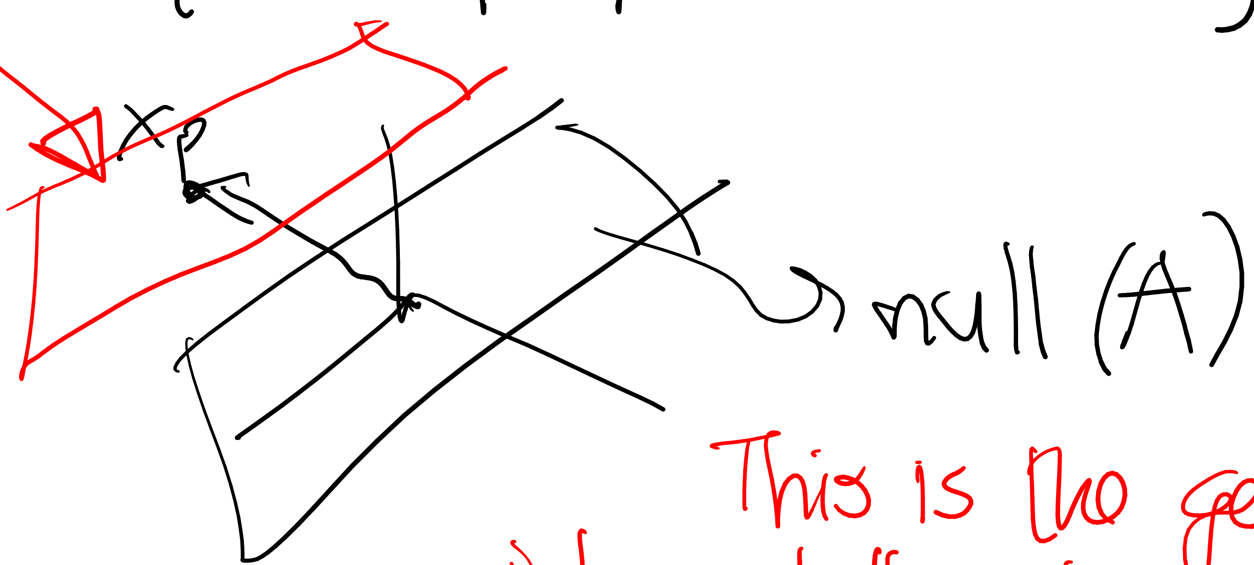
$A(u-v) = Au - Av = b - b = 0$

$\Leftrightarrow u-v \in$ null(A).

④ Take a particular soln x_p for (*).

Then soln set S of $Ax=b$

$$= \{u + x_p \mid u \in \text{null}(A)\} = C$$



This is the geometric interpretation!

<u>proof of claim</u>. We must prove:

$$x \in S \Longleftrightarrow x \in C$$

$\Leftarrow$: If $x \in C$ then $x = u + x_p$.

So $A(u+x_p) = Au + Ax_p$

$$= 0 + b = b.$$

$\Rightarrow$: If $x \in S$ then $Ax = b$.

Also $Ax_p = b$. Then

$$Ax - Ax_p = A(x - x_p)$$

$$= b - b = 0$$

$\Rightarrow x - x_p \in \operatorname{null} A$ i.e.

$x - x_p = u$ for some $u \in \operatorname{null}(A)$.

<u>Rmk</u>. In general, S is not subspace because $0 \notin S$ (unless $x_p \in \operatorname{null}(A)$ $\Leftarrow$ $b = 0$)

back to ex: Solve $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 12 & 7 & 8 & 9 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Observe $x_p = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Then by Ⓐ

$$\text{soln set} = \{ x_p + u \mid u \in \operatorname{null} A \}$$

$$= \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \cdot u_1 + x_4 u_2 + x_5 u_3 \; ; \; x_2, x_4, x_5 \in \mathbb{R} \right\}$$

<u>exercise</u>: Go back to first weeks, solve $Ax = b$ using Gaussian elimination & see that our result here is the same as yours.