ORTHOGONALITY

One can supply a vector space with a "metric" to introduce the notions "length" and "angle". In 201, we put an inner product on $\mathbb{R}^n$ to build the Euclidean geometry.

<u>defn</u>: • The (Euclidean) <u>inner product</u> on $\mathbb{R}^n$ takes a pair of vectors and gives out a real # as follows:

$$\underset{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}{\underset{\parallel}{u}}, \underset{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}{\underset{\parallel}{v}} \in \mathbb{R}^n \rightsquigarrow u \cdot v = u^T v \in \mathbb{R}.$$

$$u^T v \overset{\text{defn}}{=} \boxed{u_1 v_1 + \cdots + u_n v_n.}$$

• The (Euclidean) <u>norm</u> (length, magnitude, modulus) of u is

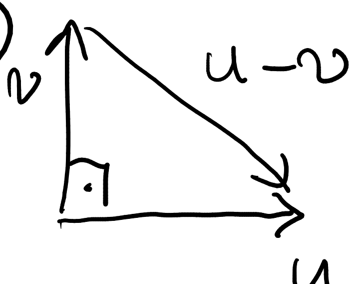
$$\|u\| \underset{\text{defn}}{=} \sqrt{u^T u} = \left(u_1^2 + \cdots + u_n^2\right)^{1/2}$$

Observations

① $\|u\| \geq 0$ & $\|u\| = 0 \iff u = 0$.

② $u^T v = v^T u$.

③



$$\begin{aligned}
&\|u-v\|^2 \\
&= (u-v)^T \cdot (u-v) \quad \text{(by defn)} \\
&= (u^T - v^T) \cdot (u-v) \\
&= u^T u + v^T v - u^T v - v^T u \\
&= \|u\|^2 + \|v\|^2 - 2u^T v \\
&\overset{?}{=} \|u\|^2 + \|v\|^2
\end{aligned}$$

we fancy

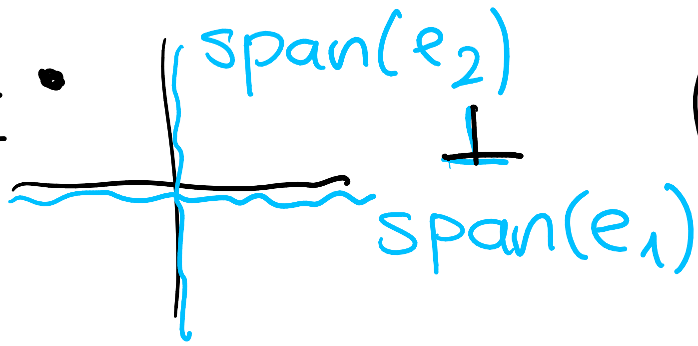
Equality holds if & only if $u^T v = 0$.

defn: If $u, v \in V$ satisfy $u^T v = 0$ then u, v are said to be orthogonal to each other. We write $u \perp v$.

Pythagorean thm: If $u \perp v$ then
$$\|u-v\|^2 = \|u\|^2 + \|v\|^2$$

- Let $u \in V$ & W be a subspace of V. We say u is <u>orthogonal to W</u> if $u \perp$ every $w \in W$. We write $u \perp W$
- For U, W subspaces of V, <u>U is orthogonal to W</u> if every $u \in U \perp$ every $w \in W$: $U \perp W$.

ex: •



$\left(e_1 \perp e_2 : \begin{pmatrix}1\\0\end{pmatrix}^T \begin{pmatrix}0\\1\end{pmatrix} = 0\right)$

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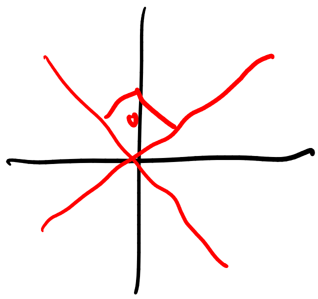

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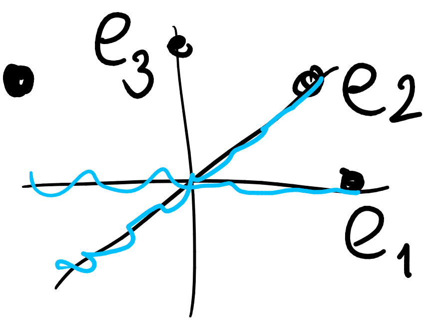

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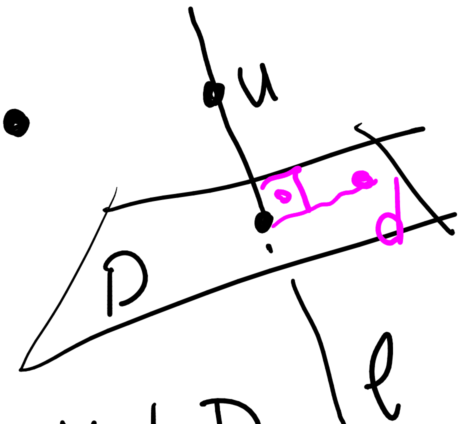


$u \perp D$

$\ell = \text{span}(u) \perp D$

④ $\overset{\in V}{u} \perp \text{span}(v_1, \ldots, v_k) \subset V$

$\iff u \perp v_i$, for every i.

Proof: $\Longrightarrow$: by defn

$\Longleftarrow$: Let $v \in$ span, i.e.

$v = c_1 v_1 + \dots + c_k v_k$ for some $c_j \in \mathbb{R}$.

Then $u^T v = u^T (c_1 v_1 + \dots + c_k v_k)$

$$= c_1 u^T v_1 + \dots + c_k u^T v_k$$

$$\overset{u \perp v_i}{=} c_1 \cdot 0 + \dots + c_k \cdot 0 = 0.$$

Hence $u \perp$ span.

(5) Suppose $u \perp W$. If $u \neq 0$ then $u \notin W$.

proof. Assume $u \perp W$ & $u \in W$.

Then $u \perp u$. In that case

$$\|u\|^2 = u^T u = 0 \overset{(1)}{\Longrightarrow} \|u\| = 0.$$

(6) If $v_1, \dots, v_k \in V$ is a pairwise orthogonal collection (i.e. $v_i \perp v_j$ $\forall i, j, i \neq j$) of nonzero vectors

then that collection is linearly independent.

ex: Converse is not true : $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}$

proof of 6: Assume $v_i \perp v_j$ for $\forall i,j$

and $c_1 v_1 + \dots + c_k v_k = \vec{0}$.

$v_1^T (c_1 v_1 + \dots + c_k v_k) = v_1^T \vec{0} = 0$

$\Rightarrow c_1 v_1^T v_1 + c_2 v_1^T v_2 + \dots + c_k v_1^T v_k = 0$

By assumption $v_1^T v_j = 0 \quad j \neq 1$.

Then $c_1 \|v_1\|^2 = 0 \Rightarrow c_1 = 0$. ($\|v_1\|^2 \neq 0$)

Similary: $c_j = 0 \ \forall j$.

7 claim: $row(A) \perp null(A)$

proof: Take any $x \in null(\underset{p\times q}{A})$; i.e.

$Ax = 0$; i.e. inner product of x

and any row of A is zero.
Then by ④ $x \perp$ span(rows) $=$ row(A)
Hence null$(A) \perp$ row(A).

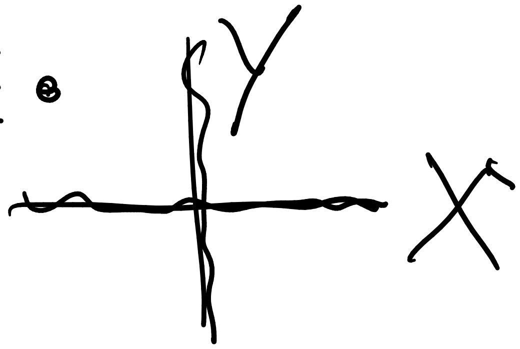
⑧ By ⑤, null$(A) \cap$ row$(A) = \{0\}$.

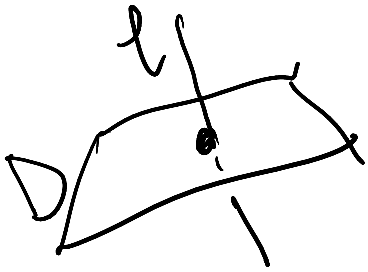
<u>defn</u>: $U, W \subset V$ are said to be orthogonal complements if $U \perp W$ & $\dim U + \dim W = \dim V$.

<u>Rank-Nullity Thm part II</u>. For $A_{p \times q}$

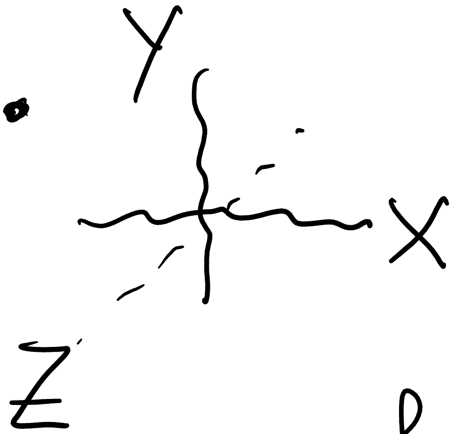
- null(A) & row(A) are orthog. complements in $\mathbb{R}^q$.
- leftnull(A) & col(A) are orthog. complements in $\mathbb{R}^p$.

ex:

- 

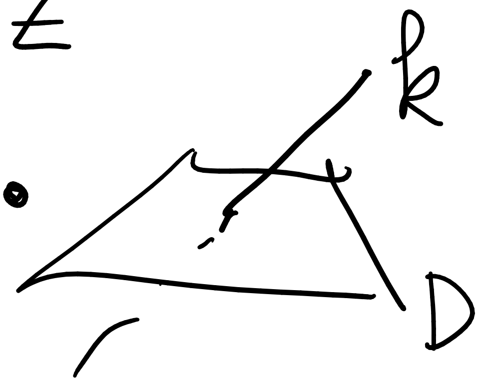
- 

are orthog. complements

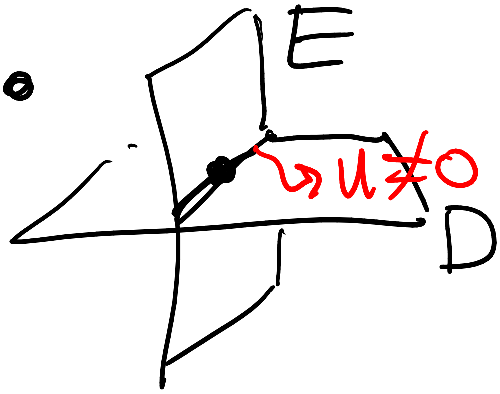
- 

X & Y are not.

$X \perp Y$ but $1+1=2<3$

- 

$1+2=3$ but $k \not\perp D$.

- 

not.

$D \not\perp E$