Last time:

Spectral Theorem. Every Hermitian matrix is diagonalizable via a unitary matrix:

$$U^H A U = \Lambda$$

For the proof we used:

Schur's lemma. Every square matrix is similar to an upper triangular matrix via a unitary matrix:

$$U^H \cdot A \cdot U = T$$ where diag. entries of T are the evalues of A.

proof. There is some unitary matrix U_1 such that

$$(*)\quad U_1^H A U_1 = \left(\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \\ \vdots & \boxed{\begin{matrix} * & & \\ & \ddots & \\ \dots & & * \end{matrix}}_{\,M_2} \\ 0 & \end{array}\right)$$

Here λ_1 is an evalue of A!

Let M_2 be the right bottom $(n-1)\times(n-1)$ submatrix of RHS of $(*)$ & λ_2 be an evalue of M_2 with an evector unit $v_2 \in \mathbb{C}^{n-1}$ for $\lambda_2 \in \mathbb{C}$:

$$M_2 v_2 = \lambda_2 v_2$$

- λ_2 is an evalue of the RHS too:

char. polyn of RHS $= \begin{vmatrix} \lambda_1-\lambda & * & \\ 0 & *-\lambda & \ddots \\ \vdots & & \\ 0 & * & \end{vmatrix}$

$\overset{\text{wrt 1st column}}{=} (\lambda_1-\lambda)\cdot\det(\lambda I - M_2)$

$= (\lambda_1-\lambda)\cdot(\text{char. polyn of } M_2)$ ■

- λ_2 is an evalue of A too because similar matrices have the same evalues: let $B = S^{-1}AS$ and β be an evalue of A with an evector v. Then:

$$B(S^{-1}v) = (S^{-1}A\cancel{S)(S^{-1}}v)$$
$$= S^{-1}Av = \beta(S^{-1}v) \qquad \blacksquare$$

- Construct an $n \times n$ unitary matrix

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & \boxed{v_2} & * & \ddots & \\ 0 & & & & * \end{pmatrix}$$

by feeding $v_2, e_1, \dots, e_{n-1} \in \mathbb{C}^{n-1}$ into Gram-Schmidt.

- $$(U_1^H A U_1) U_2 = \begin{pmatrix} \lambda_1 & \overset{M_2 v_2 =}{*} & \cdots & \\ 0 & & & \\ \vdots & \boxed{\lambda_2 v_2} & * \cdots * & \\ 0 & & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \\ 0 & & \\ \vdots & \boxed{v_2} & * \\ 0 & & \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & * & \\ 0 & \lambda_2 & \\ \vdots & 0 & \underbrace{\boxed{*}}_{(n-2)\times(n-2)} \\ 0 & \vdots & \\ & 0 & \end{pmatrix}$$

$$= U_2 \cdot \begin{pmatrix} \lambda_1 & * & \\ 0 & \lambda_2 & \\ \vdots & 0 & * \\ 0 & \ddots\, 0 & \end{pmatrix}$$

- Inductively we get an upper triangular matrix T:

$$\underbrace{U_n^H \cdots U_2^H U_1^H}_{U^H} A \underbrace{U_1 U_2 \cdots U_n}_{=U \text{ : a unitary matrix}} = T$$

NORMAL MATRICES

<u>defn</u>: A square matrix A is called <u>normal</u> if $A^H A = A A^H$.

<u>ex</u>:
- Hermitian matrices:
  $A^H = A$; $A^H A = A^2 = A A^H$.
- Real symmetric matrices:
  $B^T = B$; $B^T B = B^2 = B B^T$.
- Unitary matrices:
  $U^H U = I = U U^H$.

- Orthogonal matrices:
  $$Q^TQ = I = QQ^T$$
- Skew-Hermitian matrices:
  $$C^H = -C\,;\ C^HC = -C^2 = CC^H.$$
- (Real) skew-symmetric matrices:
  $$D^T = -D\,;\ D^TD = -D^2 = DD^T.$$

Rmk: Diagonal of a skew-symmetric matrix is zero.
Diagonal of a skew-Hermitian is either zero or imaginary.

thm: A is normal
$\iff$ A is diagble via unitary matrices

corol. All matrix families in the example are diagble via unitary matrices. Real symmetric ones are diagble via orthogonal matrices since they have real evalues.

proof of thm.

$\Leftarrow$: Suppose $U^H A U = \Lambda$. Then

$$
\begin{aligned}
A^H A &= (U\Lambda U^H)^H \cdot (U \Lambda U^H) \\
&= U \Lambda^H \cancel{U^H \cdot U} \Lambda U^H \\
&= U \Lambda \Lambda^H U^H \\
&= (U \Lambda U^H)(U \Lambda^H U^H) = A A^H \quad \blacksquare
\end{aligned}
$$

$\Rightarrow$: Suppose $A^H A = A A^H$ & $U^H A U = T$.

- Then T is normal too in Schur's lemma:

$$T^H T = (UAU^H)^H \cdot (UAU^H)$$
$$= UA^H \cancel{U^H U} AU^H$$
$$= UAA^H U^H = T \cdot T^H. \quad \blacksquare$$

- An upper triangular normal matrix must be diagonal:

$$\underbrace{\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & 0 & & t_{nn} \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & & \vdots \\ \vdots & * & & * \\ \bar{t}_{1n} & & & \bar{t}_{nn} \end{pmatrix}}_{T^H}$$

has 1-1 entry: $\|t_{11}\|^2 + \|t_{12}\|^2 + \cdots + \|t_{1n}\|^2$

while $T^H \cdot T$ has 1-1 entry equal to $\|t_{11}\|^2$

$TT^H = T^H T \Rightarrow t_{12} = \cdots = t_{1n} = 0$

Similarly one can show all off-diagonal entries of T are 0. $\blacksquare$

CAYLEY-HAMILTON THEOREM.

An application of Schur's lemma

thm: A square matrix satisfies its characteristic polynomial.

ex: $A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$. char. polyn. $= \lambda^2 + \lambda - 2$

$$A^2 + A - 2I = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} + A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

proof of thm: Use Schur's lemma:

$A = UTU^H$ and insert in the char. polyn. of A: $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$

$$f(A) = (UTU^H)^n + a_{n-1}(UTU^H)^{n-1} + \dots + a_0 UU^H$$

$$= U \cdot f(T) \cdot U^H$$

So we show $f(T) = $ zero matrix.

- Assume $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots$

Then $f(T) = (\lambda_1 I - T)(\lambda_2 I - T) \cdots$

$$= \begin{pmatrix} \lambda_1 - \lambda_1 {\color{cyan}=0} & & & * \\ & \lambda_1 - \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_1 - \lambda_n \end{pmatrix} \cdot \begin{pmatrix} \lambda_2 - \lambda_1 & & * \\ & \lambda_2 - \lambda_2 {\color{cyan}=0} & \\ 0 & & \ddots \end{pmatrix} \cdots$$

$$= \begin{pmatrix} 0 & 0 & \\ \vdots & \vdots & * \\ 0 & 0 & \end{pmatrix} \cdot (\lambda_3 I - T) \cdots$$

One can show inductively that the product is the zero matrix.