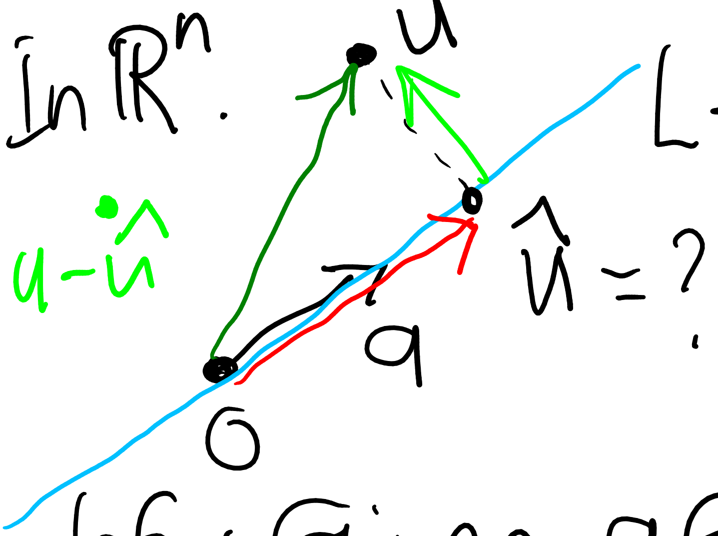
Orthogonal Projection (to a line)

In $\mathbb{R}^n$:



$L = \mathrm{span}(a)$

$(a \neq 0)$

<u>defn</u>: Given $a \in \mathbb{R}^n$ & $L = \mathrm{span}(a)$.

The <u>orthogonal projection</u> $\hat{u}$ of $u \in \mathbb{R}^n$ to $L = \mathrm{span}(a)$ satisfies:

(i) $\hat{u} \in L$

(ii) $u - \hat{u} \perp L$.

So (i) $\hat{u} = ka$ for some $k \in \mathbb{R}$,

(ii) $u - \hat{u} \perp a$

$$a^T(u - \hat{u}) = 0 \iff a^T u = a^T \hat{u}$$

$$\underset{(i)}{\Longrightarrow} \underbrace{a^T u}_{\in \mathbb{R}} = a^T \cdot ka$$

So $\boxed{k = \frac{a^T u}{a^T a}} \in \mathbb{R}$ & hence

$$\boxed{\hat{u} = \mathrm{proj}_a u = ka = \underbrace{\frac{a^T u}{a^T a}}_{\in \mathbb{R}} a_{n \times 1}}$$

Observe

$$\hat{u} = a_{n\times 1} \cdot \left(\frac{a^T u}{a^T a}\right)_{1\times 1}$$

$$= \frac{1}{a^T a} \cdot \left(a(a^T u)\right) = \frac{1}{a^T a} (aa^T)_{n\times n} u_{n\times 1}$$

$$= \underbrace{\frac{aa^T}{a^T a}}_{\text{this } n\times n \text{ matrix } P \text{ depends only on } a!} u$$

this $n\times n$ matrix P depends only on a!

Conclusion & observations.

$\boxed{A}$ P is responsible with orthog. proj. Given u, Pu is one & only one.

$\boxed{B}$ P is symmetric; i.e. $P^T = P$:

$$P^T = \left(\frac{aa^T}{a^Ta}\right)^T = \frac{1}{a^Ta}(aa^T)^T = \frac{aa^T}{a^Ta} = P$$

$\boxed{C}$ $P^2 = P$:

$$P^2 = \left(\frac{aa^T}{a^Ta}\right)^2 = \frac{1}{(a^Ta)^2}(aa^T)(aa^T)$$

$$= \frac{1}{(a^Ta)^2}\, a\underbrace{(a^Ta)}_{\in\mathbb{R}}a^T = \frac{\cancel{a^Ta}}{(a^Ta)^{\cancel{2}}} \cdot aa^T = P$$

<u>defn</u>: A square matrix P which is symmetric & satisfies $P^2 = P$ is called a <u>projection matrix</u>.

$\boxed{D}$ In $\mathbb{R}^4$ given $a = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, it determines a line $L = \text{span}(a)$.

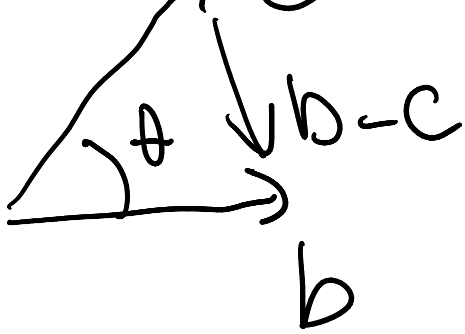
$$P = \frac{1}{a^T a} a a^T = \frac{1}{30}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 6 & 8\\3 & 6 & 9 & 12\\4 & 8 & 12 & 16\end{pmatrix}_{4\times 4}. \text{ Observe } \operatorname{rank} P = 1.$$

Given $u = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}$,

$$\operatorname{proj}_a u = Pu = \frac{1}{30}\begin{pmatrix}8\\16\\24\\32\end{pmatrix} = \frac{4}{15}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$\boxed{E}$ Recall the Cosine Theorem:



$$\|b-c\|^2 = \|b\|^2 + \|c\|^2 - 2\cdot\|b\|\cdot\|c\|\cdot\cos\theta$$

F Compute:

$$\|b-c\|^2 = (b-c)^T(b-c)$$

$$= b^Tb + c^Tc - b^Tc - c^Tb$$

$$= \|b\|^2 + \|c\|^2 - 2\,b^Tc$$

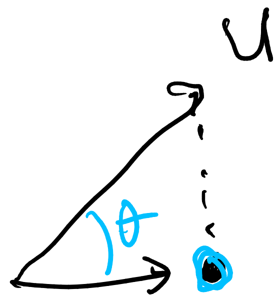
Hence

$$\boxed{b^Tc = \|b\|\cdot\|c\|\cdot\cos\theta_{b,c}}$$

G Now: $\mathrm{proj}_a u = \dfrac{a^Tu}{a^Ta}\,a$

$$= \frac{1}{\|a\|^{\cancel{2}}} \cdot \cancel{\|a\|}\cdot\|u\|\cdot\cos\theta_{a,u} \cdot a$$

$$= \underbrace{\|u\|\cdot\cos\theta_{a,u}}_{\text{the length of projection}} \cdot \underbrace{\frac{a}{\|a\|}}_{\text{the unit direction along } a.}$$



$\boxed{H}$ Rotations & reflections are linear transformations; i.e. $f: \mathbb{R}^n \to \mathbb{R}$ is a rotation then $f(x+y)=f(x)+f(y)$

Assume a rotation (and a reflection) preserves lengths & angles by definition (whatever it is). Then

$$\|f(x+y)-f(x)-f(y)\|^2$$

$$=(f(x+y)-f(x)-f(y))^T(f(x+y)-f(x)-f(y))$$

$$=\overbrace{f(x+y)^T f(x+y)}^{\|f(x+y)\|^2}+f(x)^T f(x)+f(y)^T f(y)$$

$$-2f(x+y)^T f(x)-2f(x+y)^T f(y)$$

$$+2f(x)^T f(y)$$

$$=(x+y)^T(x+y)+x^T\cancel{x}+y^T y$$

$$-2(x+y)^T x-2(\cancel{x}+y)^T y+\cancel{2x^T y}$$

Because

$$f(x+y)^T f(y) = \|f(x+y)\| \cdot \|f(y)\| \cos\theta_{f(y), f(x+y)}$$

$$= \|x+y\| \cdot \|y\| \cdot \cos\theta_{x+y, y}$$

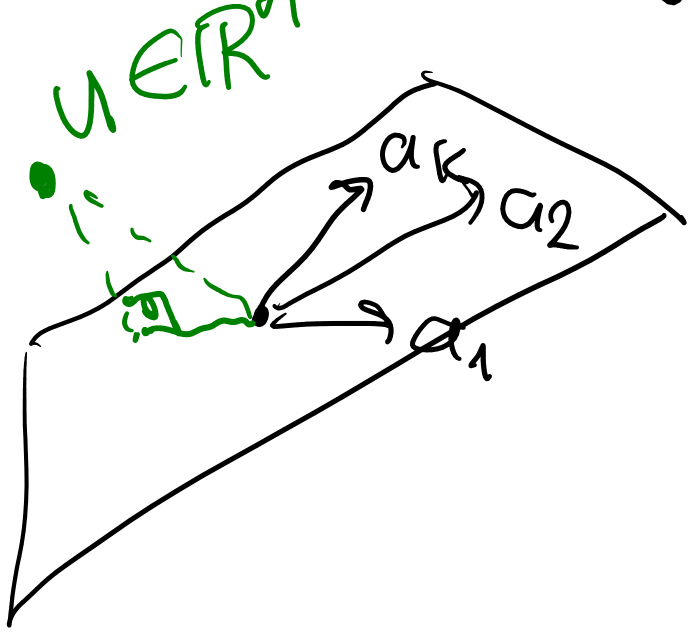
$$= (x+y)^T \cdot y$$

$$= \|x+y\|^2 - \|x\|^2 - \|y\|^2 - 2x^T y$$

$$= 0 \quad \text{by Cosine Theorem} \quad \blacksquare$$

Conclusion: Any function on $\mathbb{R}^n$ that preserves lengths & angles is a linear transformation.

Orthogonal Projection (to a subspace)



Given $H = \operatorname{span}(a_1, \dots, a_k)$

The <u>orthogonal projection</u> $\hat{u}$ of u on the subspace $H = \text{span}(a_1, \dots, a_k)$ satisfies:

(i) $\hat{u} \in H$

(ii) $u - \hat{u} \perp H$

Let $\underset{n \times k}{A} = (a_1 \vdots \cdots \vdots a_k)$. Then

(i) $\hat{u} \in \text{col} A \iff \hat{u} = Ax$ for some $x \in \mathbb{R}^k$

(ii) $u - \hat{u} \perp a_j, \forall j$.

$\iff u - \hat{u} \perp \text{col}(A)$

$\iff u - \hat{u} \in \text{leftnull}(A)$

$\iff A^T(u - \hat{u}) = 0$

Thus: $A^T u = A^T \hat{u} = A^T A x$

Solve x here. Then $\hat{u} = Ax$.

Let us suppose $(A^TA)_{k\times k}$ is invertible

Then $\boxed{x = (A^TA)^{-1}A^Tu}$ so that

$$\boxed{\hat{u} = Ax = \underbrace{A(A^TA)^{-1}A^T}_{P}u}.$$

Recall:

for $k=1$: $\hat{u} = \dfrac{aa^T}{a^Ta}u = \left[a(a^Ta)^{-1}a^T\right]u$