Summary of last time:

DETERMINANT

definition: A function Δ eating n vectors in $\mathbb{R}^n$ & gives out a real # $\Delta(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) \in \mathbb{R}$ which satisfies

① $\Delta(\vec{e}_1, \dots, \vec{e}_n) = 1$;

② Δ is multilinear;

③ Δ is alternating.

is called a determinant.

We had shown that such a Δ satisfies automatically

④ $\Delta(\dots, \vec{u}, \dots, \vec{u}, \dots) = 0$.

⑤ $\Delta(\vec{u}, a\vec{u}+\vec{v}, \dots) = \Delta(u, v, \dots)$.

⑥ $\Delta(\dots, \vec{0}, \dots) = 0$.

⑦ $\Delta(\text{diag}(a_1, \dots, a_n)) = a_1 \cdot \dots \cdot a_n$.

⑧ $\Delta(\text{lower})$ = product of diagonal entries
$\Delta(\text{upper})$ =

⑨ $\Delta(\text{permutation}) = (-1)^k$ where k is the # of row exchanges in pairs needed to turn the matrix into I.

(10) Given A invertible & $PA = LDU$,

$$\Delta(A) = \underbrace{(-1)^k}_{\text{by } (9)} \underbrace{\Delta(D)}_{\text{by } (7)}$$

So these prove the following

<u>thm:</u> Given $A_{n\times n}$, if rows of A is linearly dependent then by (5) & (6) $\Delta(A) = 0$. Otherwise given P, there is unique $PA = LDU$ and

$$\Delta(A) = (-1)^k \cdot D_{11} \cdot \ldots \cdot D_{nn}.$$

So a fnc that satisfies (1)-(3) is unique.

From this point on we denote this unique function by det. It takes in n vectors and gives out a real #. When we write $\det(A)$, this means it eats the rows of A.

Now several more properties for det:

(11) $\det(\underset{n\times n}{A} \cdot \underset{n\times n}{B}) \overset{(*)}{=} \det(A) \cdot \det(B)$ (A, B square)

<u>proof:</u> If any of A & B is singular then so is $A \cdot B$. In that case both sides are 0 by the thm above. Equality $(*)$ holds.

Now asssume A & B are invertible.

Define a fnc on rows of invertible matrices:

$$f(A) = \det(A.B) / \det(B).$$

If we can show that this f satisfies ①, ②, ③ then f must be det by the theorem above. So that proves:

$$\det(A) = \det(AB) / \det B.$$

This is nothing but Equality $(*)$.

Ok then let's show that f satisfies ①-③:

f satisfies ①: $f(I) = f(e_1, \dots, e_n)$

$$= \det(I \cdot B) / \det(B) = 1.$$

This is exactly property ①.

f satisfies 2: For $\vec{u}, \vec{v}, \vec{u}_2, \ldots$ row vectors.

Observe $\begin{pmatrix} a\vec{u}+b\vec{v} \\ \vec{u}_2 \\ \vdots \end{pmatrix} \cdot B = \begin{pmatrix} a\vec{u}B + b\vec{v}B \\ \vec{u}_2 B \\ \vec{u}_3 B \\ \vdots \end{pmatrix}$

So

$$f\begin{pmatrix} - & a\vec{u}+b\vec{v} & - \\ - & \vec{u}_2 & - \\ - & \vdots & - \\ - & \vdots & - \end{pmatrix} = \frac{1}{\det B} \cdot \det \begin{pmatrix} a\vec{u}B + b\vec{v}B \\ \vec{u}_2 B \\ \vec{u}_3 B \\ \vdots \end{pmatrix}$$

$$= \frac{1}{\det B}\left(a \cdot \det\begin{pmatrix} \vec{u}B \\ \vec{u}_2 B \\ \vdots \end{pmatrix} + b \cdot \det\begin{pmatrix} \vec{v}B \\ \vec{u}_2 B \\ \vdots \end{pmatrix}\right)$$

$$= a \cdot f\begin{pmatrix} - & u & - \\ - & u_2 & - \\ & \vdots & \end{pmatrix} + b \cdot f\begin{pmatrix} - & v & - \\ - & u_2 & - \\ & \vdots & \end{pmatrix}$$

This shows that f satisfies propert 2 too.

f satisfies (3):

$$f\begin{pmatrix} - u_2 - \\ - u_1 - \\ - u_3 - \\ \vdots \end{pmatrix} = \frac{1}{\det B} \det \begin{pmatrix} \vec{u}_2 B \\ \vec{u}_1 B \\ \vec{u}_3 B \\ \vdots \end{pmatrix}$$

$$= \frac{-1}{\det B} \cdot \det \begin{pmatrix} \vec{u}_1 B \\ \vec{u}_2 B \\ \vec{u}_3 B \\ \vdots \end{pmatrix} = (-1) \cdot f\begin{pmatrix} - u_1 - \\ - u_2 - \\ - u_3 - \\ \vdots \end{pmatrix}$$

& this is nothing but property (3).

Last property:

(12) $\det(A^T) = \det(A)$.

This shows that instead of det eating rows of A, it may eat the columns of A & gives out the same number.

Here is a proof for (12):

If A is singular so is A^T and for both $\det = 0$.

Otherwise let $PA = LDU$. Then

$$\det(A) = \det D \,/\, \det P.$$

Meanwhile $A^T P^T = U^T D L^T$ and

$$\det(A^T) = \det D \,/\, \det P^T.$$

Therefore if we make sure that $\det P^T = \det P$ then we're done.

Recall $P = P_1 \cdot \ldots \cdot P_k$ where each P_j swaps two rows and $P_j^T = P_j$. So

$$P^T = P_k^T \cdot \ldots \cdot P_1^T = P_k \cdot \ldots \cdot P_1.$$

Hence $\det P^T = (-1)^k = \det P$. ∎

Thanks to 12, when we write $\det(A)$ we may equally let det eat the columns of A too.

Moreover whatever we've said for det about rows of A is also true for columns of A.

We sometimes write $|A|$ for $\det(A)$.

Computing det.

We know how to compute det using LDU decomp. Here are some other methods.

ex:

Consider $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Observe $(a\ b\ c)$

$= a\cdot(1\ 0\ 0) + b(0\ 1\ 0) + c\,(0\ 0\ 1)$

Then by multilinearity

$$\det A = a\cdot\det\begin{pmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + b\cdot\det\begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$+ c\cdot\det\begin{pmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\underset{\substack{\text{linearity} \\ \text{wrt} \\ 2^{nd}\text{ rows}}}{=} a\cdot\left[d\cdot\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ g & h & i \end{vmatrix} + e\cdot\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & h & i \end{vmatrix} + f\cdot\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ g & h & i \end{vmatrix} \right]$$

$+ \dots$

Let us pause. In the first term the det is zero.

a&d brought $(1\ 0\ 0)$ because they're both at the same (1^{st}) column of A.

Let's use this observation again & again:

$$\det A = \underset{\substack{\text{1st}\\ \text{column}}}{a}\,\underset{\substack{\text{2nd}\\ \text{column}}}{e}\cdot\left[\underset{\text{3rd}}{i}\cdot\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}\right] + \underset{\text{1st}}{a}\,\underset{\text{3rd}}{f}\cdot\left[\underset{\text{2nd}}{h}\cdot\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}\right]$$

$$+\underset{\text{2nd}}{b}\cdot\left[\underset{\text{1st}}{d}\cdot\underset{\text{3rd}}{i}\cdot\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix} + \underset{\text{3rd}}{f}\cdot\underset{\text{1st}}{g}\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}\right]$$

$$+\underset{\text{3rd}}{c}\cdot\left[\underset{\text{1st}}{d}\cdot\underset{\text{2nd}}{h}\cdot\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix} + \underset{\text{2nd}}{e}\cdot\underset{\text{1st}}{g}\cdot\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}\right]$$

See that we chose one entry from each row so that they also belong to different columns, because if any two came from the same column the corresponding matrix has det = 0.

Now all the remaining matrices are permutations.

$$\boxed{\det A = aei - afh + bfg - bdi + cdh - ceg}$$

In general we've a similar picture.
First some new words.
Let $P_{n\times n}$ be a permutation matrix. It shuffles the n rows. Thus it corresponds a shuffling (a permutation) σ of the #s $\{1,\dots,n\}$. $\forall i$ $\sigma(i)$ is a # in $\{1,\dots,n\}$.
Define $\operatorname{sign}(\sigma) \doteq \det P$.

Now we're ready to tell the computation.
<u>thm.</u> det of $A_{n\times n} = (a_{ij})$ is computed as follows. For each permutation σ of numbers $\{1,\dots,n\}$,

$$a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{n\sigma(n)}$$

are n entries of A, which are on different rows & on different columns of A. Then

$$\boxed{\det A = \sum_{\text{all permutations } \sigma} \operatorname{sign}(\sigma) \cdot a_{1\sigma(1)} \cdot \dots \cdot a_{n\sigma(n)}}$$

ex: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d.\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + b.c.\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$
$= ad - bc.$

ex: $\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{M} = aei - afh - bdi + bfg + cdh - ceg$ $(\square)$

$= a(ei-fh) - b(di-fg) + c(dh-eg)$

$= a.\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c.\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ $(*)$

i.e. on the first row pick the jth entry a_{1j}. Delete 1st row & jth column. Denote by M_{1j} the $(n-1)\times(n-1)$ matrix that remains & set

$$C_{1j} = (-1)^{1+j} \det M_{1j}.$$

In this notation

$$(*) = a.C_{11} + b.C_{12} + c.C_{13}$$

Going back to ($\square$):

$$\det M = \underbrace{-bdi + cdh}\underbrace{+aei - ceg}\underbrace{-afh + bfg}$$

$$= -d \cdot \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \cdot \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \cdot \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$= d \cdot C_{21} + e \cdot C_{22} + f \cdot C_{23}.$$

This computes $\det A$ "with respect to the second row".

These examples convince us about:

<u>thm</u>. For $A_{n\times n} = (a_{ij})$, fix a row k.

$$\boxed{\det A = a_{k1} \cdot C_{k1} + \dots + a_{kn} C_{kn}}$$

where

$C_{kj} = (-1)^{k+j} \times \det$ (submatrix of A obtained by deleting its kth row & jth column)

↳ called the k-j <u>cofactor of A</u>

Similarly for some fixed ℓ^{th} column of A:

$$\boxed{\det A = a_{1\ell} C_{1\ell} + a_{2\ell} C_{2\ell} + \cdots + a_{n\ell} \cdot C_{n\ell}}$$