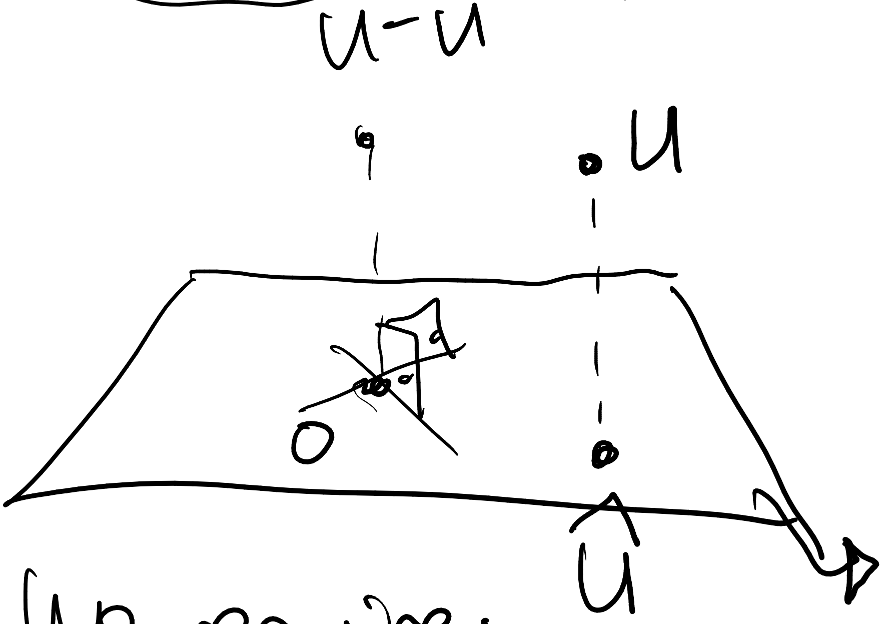
Orthogonal projection to a subspace



in $\mathbb{R}^n$

$H = \text{span}(a_1, \ldots, a_k)$

We require:

(a) $\hat{u} \in H \iff \hat{u} = Ax$ for some $x_{k\times 1}$

where $A = (a_1 \,\vdots\, \cdots \,\vdots\, a_k)_{n\times k}$.

(b) $u - \hat{u} \perp H = \text{col}(A)$

$\iff u - \hat{u} \perp a_i, \forall i.$

$\iff A^T(u - \hat{u}) = 0$

Thus $A^T u = A^T A x$.

Suppose $A^T A$ is invertible.

$x = (A^T A)^{-1} A^T u$ &

$\hat{u} = Ax = \underbrace{A(A^T A)^{-1} A^T}_{P_{n\times n}} u$

What we did is essentially the following:

Given u, we express u as

$$u = \hat{u} + (u - \hat{u})$$

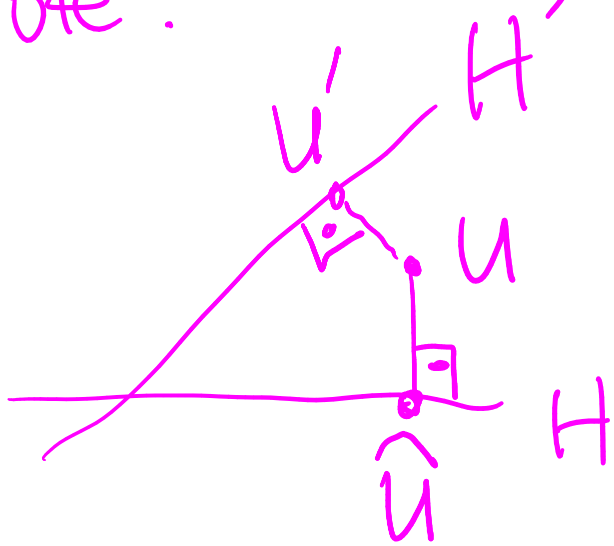
$$= \mathrm{proj}_H u + \mathrm{proj}_{H^\perp} u$$

where $H^\perp$ is the orthogonal complement of H.

- $\hat{u} \perp u - \hat{u}$.
- $\mathrm{proj}_{H^\perp} u = u - \hat{u}$:

  (a) $u - \hat{u} \in H^\perp$; (b) $u - (u - \hat{u}) = \underbrace{\hat{u}}_{\in H} \perp H^\perp$

Note:



If $H' \neq H^\perp$,

$u \neq \hat{u} + u'$

ex: In $\mathbb{R}^4$, $H = \text{span}(a_1, a_2, a_3, a_4)$ with

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix},\ a_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 3 \\ -1 \\ 3 \\ 1 \end{pmatrix},\ a_4 = \begin{pmatrix} 4 \\ -1 \\ 5 \\ 2 \end{pmatrix}.$$

$a_3 = a_1 + a_2$, $a_4 = 2a_1 + a_2$

Find the orthog. proj. of $u = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ on th

$A = (a_1 \vdots \cdots \vdots a_4)_{4\times 4} \to \begin{pmatrix} 1 & & * & \\ 0 & 1 & & \\ - & 0 & - & \\ - & 0 & - & \end{pmatrix}$

rank $A = 2$

$\Leftrightarrow \dim \text{null}\, A = 4 - 2 = 2$

Take $x \in \text{null}(A)$; i.e. $Ax = 0$.

Then $\underset{k\times k}{(A^TA)}\underset{k\times 1}{x} = \vec{0} \Rightarrow x \in \text{Null}\, A^TA$

So $\text{null}(A) \subset \text{null}(A^TA)$.

In particular $2 = \dim \text{null}(A) \leqslant \dim \text{null}(A^TA)$

Hence $\text{rank}(A^TA) \leqslant 2$.

so that A^TA is not invertible.

Instead, define $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}_{4\times 2} = (a_1 \vdots a_2)$

$$B^T B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

$$(B^T B)^{-1} = \frac{1}{20} \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}$$

$$P = B (B^T B)^{-1} B^T$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{20} \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix} B^T$$

$$= \frac{1}{20} \begin{pmatrix} -2 & 8 \\ 4 & -6 \\ 8 & -2 \\ 6 & -4 \end{pmatrix} B^T = \frac{1}{20} \begin{pmatrix} 14 & -8 & 4 & -2 \\ -8 & +6 & 2 & 4 \\ 4 & +2 & 14 & 8 \\ -2 & +4 & 8 & 6 \end{pmatrix}$$

Then $\hat{u} = P . u$.

facts.

① Above P is symmetric. But this is always the case:

$$P^T = \left(A(A^TA)^{-1}A^T\right)^T$$
$$= A\left((A^TA)^{-1}\right)^T A^T$$
$$= A\left((A^TA)^T\right)^{-1} \cdot A^T = A(A^TA)^{-1}A^T = P$$

② Moreover $P^2 = P$:

$$P^2 = \left(A\cancel{(A^TA)^{-1}}\cancel{A^T}\right)\left(\cancel{A}(A^TA)^{-1}A^T\right)$$
$$= A(A^TA)^{-1}A^T = P$$

①&②: P is a projection matrix.

③ If $a_1, \dots, a_k$ are linearly independent then A^TA is invertible

Proof: $null(A)=\{0\}$.

We prove $null(\underbrace{A^TA}_{\text{square}})=\{0\}$ too.

We've shown

$$null(A) \subseteq null(A^TA)$$

If we show the equality then we're done. So ~~take~~ $x \in null(A^TA)$

i.e. $(A^TA)x=\vec{0}$.

Then $x^T((A^TA)x)=0 \Leftrightarrow (x^TA^T)(Ax)=0$

$\Rightarrow (Ax)^T(Ax)=\|Ax\|^2=0$.

Hence $Ax=0$.

Rank-Nullity Thm part III.

Multiplying with $A_{p\times q}$ sends $row(A) \subset \mathbb{R}^q$ to $col(A) \subset \mathbb{R}^p$ in a 1-1 onto fashion.

defn: A 1-1 onto linear transform. is called a <u>linear isomorphism</u>.

<u>proof of part III.</u>

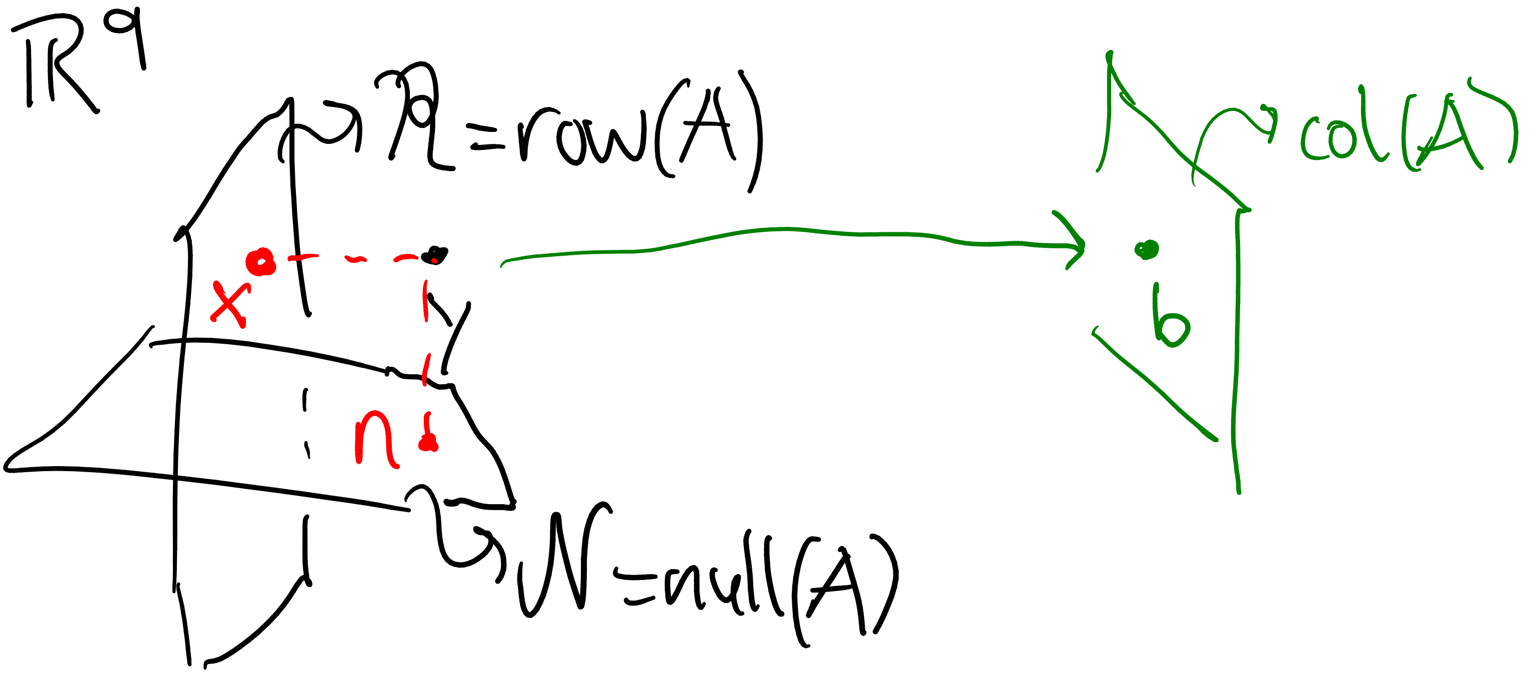
$\boxed{1\text{-}1}$. Let $x, y \in row(A)$ & $Ax = Ay$.

Then $0 = Ax - Ay = A(\underbrace{x-y}_{\in row(A)})$

$\Rightarrow x - y \in null(A)$ too.

$\Rightarrow x - y = 0$.

$\boxed{\text{onto.}}$ Given any $b \in col(A)$, there is $x \in row(A)$ such that $Ax = b$.



Let $x = \mathrm{proj}_{\mathcal{R}} y$ & $n = \mathrm{proj}_{\mathcal{N}} y$.

We know $y = x + n$. Then

$$b = Ay = A(x+n) = Ax + \underbrace{\cancel{An}}_{0} = Ax.$$

RANK-NULLITY THM (final version)

- $\mathrm{row}(A)$ & $\mathrm{null}(A_{p \times q})$ are orthog. complements in $\mathbb{R}^q$.
- $\mathrm{col}(A)$ & $\mathrm{null}(A^T)$ are orthog. complements in $\mathbb{R}^p$.
- $\mathrm{rank}(A) = \dim \mathrm{row}(A) = \dim \mathrm{col}(A)$.
- $\mathrm{row}(A)$ & $\mathrm{col}(A)$ are linearly isomorphic via multiplication with A.