Last time. Given $A_{p\times q}$,

- $\text{null}(A) = \{x \in \mathbb{R}^q \mid Ax = 0\}$ is a subspace of $\mathbb{R}^q$.
- $\text{leftnull}(A) = \{x \in \mathbb{R}^p \mid A^T x = 0\}$ is a subspace of $\mathbb{R}^p$.
- We have the theorem to be developed:

thm: The following cases are equivalent:

1. There is a permutation of rows of $A_{p\times p}$ so that G-J is successful; i.e. there is no missing pivot.
2. A has an inverse
3. For every $b \in \mathbb{R}^p$ the system $Ax = b$ has a unique solution.
4. $\text{null}\, A = \{0\}$
5. $\#$ of pivots $= p$.

⋮ more to come

Recall the example of last time:

$$A=\begin{bmatrix}1&2&3&4&5\\6&12&7&8&9\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3&4&5\\0&0&-11&-16&-21\end{bmatrix}$$

null(A)

$$=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=x_2\underbrace{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}}_{u_1}+x_4\underbrace{\begin{pmatrix}-4/11\\0\\-16/11\\1\\0\end{pmatrix}}_{u_2/11}+x_5\underbrace{\begin{pmatrix}-8/11\\0\\-21/11\\0\\1\end{pmatrix}}_{u_3/11}\ \&\ x_2,x_4,x_5\in\mathbb{R}\right\}$$

$=\text{span}(u_1,u_2,u_3)$

<u>defn</u>: Consider $u_1,\dots,u_k\in\mathbb{R}^n$.

- A <u>linear combination</u> of $u_1,\dots,u_k$ is $a_1u_1+\dots+a_ku_k\in\mathbb{R}^n$ for some real #s $a_1,\dots,a_k\in\mathbb{R}$.
- The <u>span</u> of $u_1,\dots,u_k$ is the set of all their linear combinations:

$\mathrm{span}(u_1, \dots, u_k) \overset{\text{defn}}{=} \{ a_1 u_1 + \dots + a_k u_k \mid a_1, \dots, a_k \in \mathbb{R} \}$

<u>thm</u>: Span is a subspace.

<u>proof</u>: We claim $S := \mathrm{span}(u_1, \dots, u_k)$ is a subspace of $\mathbb{R}^n$; i.e.

(i) $v, w \in S$ then $v + w \in S$;

(ii) $v \in S$, $k \in \mathbb{R}$: $kv \in S$.

Indeed:

(i) For $v = a_1 u_1 + \dots + a_k u_k \in S$
$w = b_1 u_1 + \dots + b_k u_k \in S$
with $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{R}$.

Then $v + w = \left(\sum a_i u_i\right) + \left(\sum b_i u_i\right)$
$= a_1 u_1 + b_1 u_1 + \dots + a_k u_k + b_k u_k$
$= (a_1 + b_1) u_1 + \dots + (a_k + b_k) u_k$
$\in S$

(ii) For $v \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$tv = t(a_1 u_1 + \dots + a_k u_k)$$
$$= t(a_1 u_1) + \dots + t(a_k u_k)$$
$$= (ta_1) u_1 + \dots + (ta_k) u_k \in S$$

ex: • $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$

$\operatorname{span}(v_1, v_2, v_3) = \mathbb{R}^2$.

Observe $\quad = \operatorname{span}(v_1, v_2)$

$\overset{\text{OR}}{=} \operatorname{span}(v_2, v_3)$

I.e. there is an unnecessary vector without which the span is the same:

$v_3 = 2v_1 + 5v_2$ OR $v_1 = \frac{5}{2} v_2 - \frac{1}{2} v_3$

Q: $\operatorname{span}(v_1, v_2) \overset{?}{=} \operatorname{span}(v_1)$

No, because $v_2 \notin \operatorname{span}(v_1)$.

• back to ex.

$$u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -4 \\ 0 \\ -16 \\ 11 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -8 \\ 0 \\ -21 \\ 0 \\ 11 \end{pmatrix}$$

These span null(A) = span(u_1, u_2, u_3)

However we cannot drop any of them & preserve the span because of the marked entries.

Column & row spaces

Given $A_{p\times q}$, its column space is the span of its columns. It's a subspace of $\mathbb{R}^p$: col$(A) \subset \mathbb{R}^p$.

Similarly row space is the span of its rows; it's a subspace of $\mathbb{R}^q$:

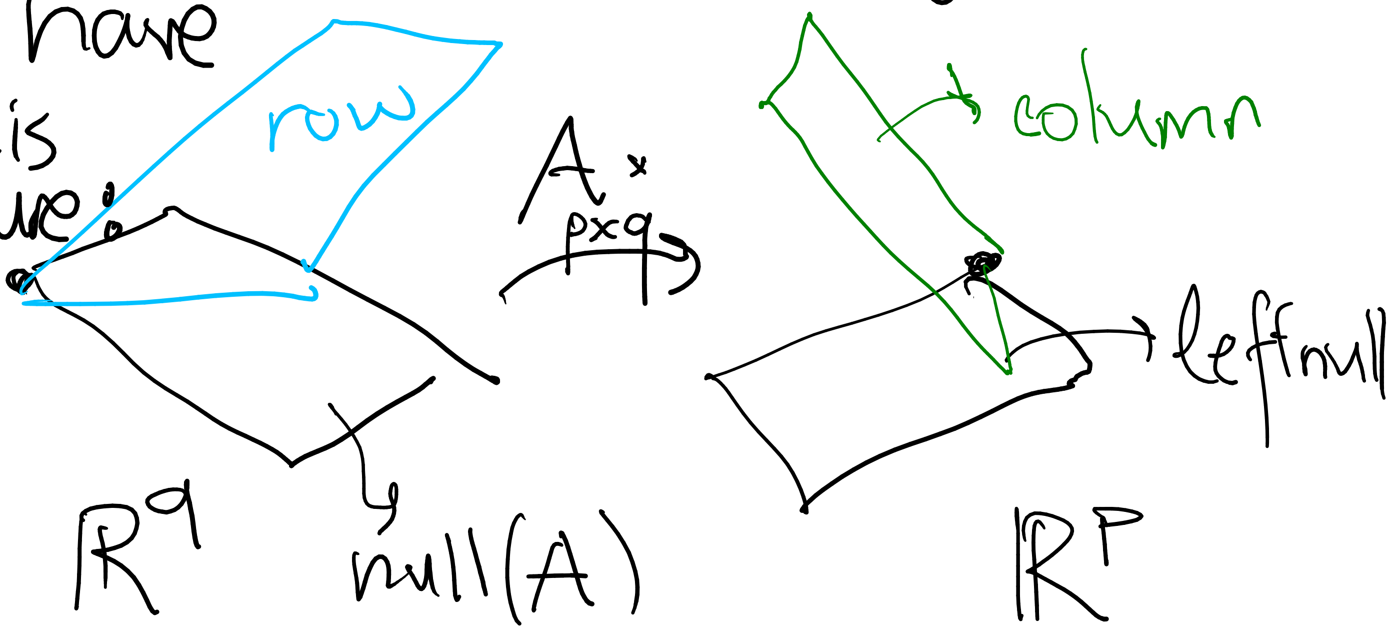
$$\text{row}(A) \subset \mathbb{R}^q.$$

In the example:

$$\mathrm{col}(\underset{2\times 5}{A}) = \mathrm{span}\left(\begin{pmatrix}1\\6\end{pmatrix}, \begin{pmatrix}2\\12\end{pmatrix}, \begin{pmatrix}3\\7\end{pmatrix}, \begin{pmatrix}4\\8\end{pmatrix}, \begin{pmatrix}5\\9\end{pmatrix}\right) \subset \mathbb{R}^2.$$

$$\mathrm{row}(A) = \mathrm{span}\left(\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}6\\12\\7\\8\\9\end{pmatrix}\right) \subset \mathbb{R}^5$$

We have this picture:



<u>defn</u>: Consider $u_1, \ldots, u_k \in \mathbb{R}^n$.

- They are called <u>linearly dependent</u> if there are $a_1, \ldots, a_k \in \mathbb{R}$, not all zero, such that

$$a_1 u_1 + \cdots + a_k u_k = 0$$

ex: ☆ $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}$ are linearly dependent because

$$2\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix} - 1\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

☆ $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$ are lin. dependent because $0\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

- If they're not linearly dependent, they're called linearly independent; i.e. if $a_1 u_1 + \dots + a_k u_k = 0$ then $a_1, \dots, a_k \in \mathbb{R}$ are forced to be 0.

thm 1 The u_1, u_2, u_3 of the example are linearly independent.

proof: We prove: if $a_1 u_1 + \dots + a_3 u_3 = 0$ then $a_1 = 0$, $a_2 = 0$, $a_3 = 0$:

Assume

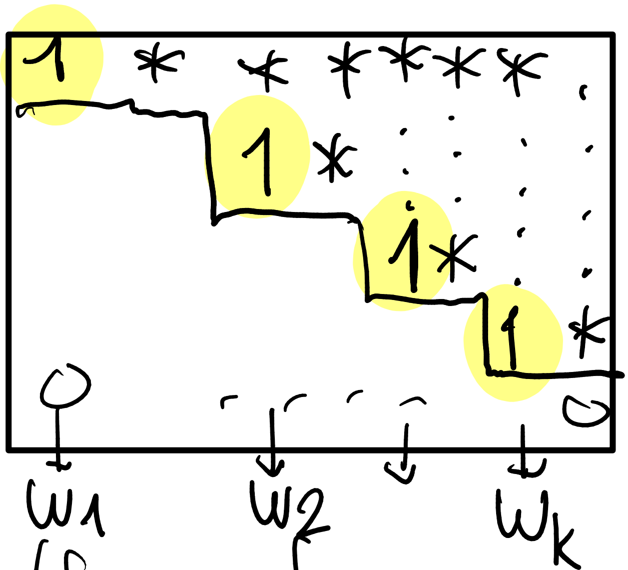
$$a_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} -4 \\ 0 \\ -16 \\ 11 \\ 0 \end{pmatrix} + a_3 \cdot \begin{pmatrix} -8 \\ 0 \\ -21 \\ 0 \\ 11 \end{pmatrix} = 0$$

Then $11a_3 = 0 \Rightarrow a_3 = 0$

$11a_2 = 0 \Rightarrow a_2 = 0$

& $a_1 = 0$

② Let



be in row reduced echelon form.

Then the columns with pivots are a linearly indep. collection.

<u>proof</u>. Suppose $a_1 w_1 + \cdots + a_k w_k = 0$

Then $a_k = 0$ from the last row.

Similarly in the 2^{nd} row from bottom we have $a_{k-1} + * \cdot a_k = 0$

$\Rightarrow a_{k-1} = 0$.

Similarly $a_{k-2} = \cdots = a_2 = a_1 = 0$.

Basis & dimension.

defn: $u_1, \ldots, u_k \in V$ is said to constitute a basis for V if

(a) $V = \text{span}(u_1, \ldots, u_k)$;

and

(b) $u_1, \ldots, u_k$ are linearly indep.

The # of vectors in a basis will be called the dimension of V.

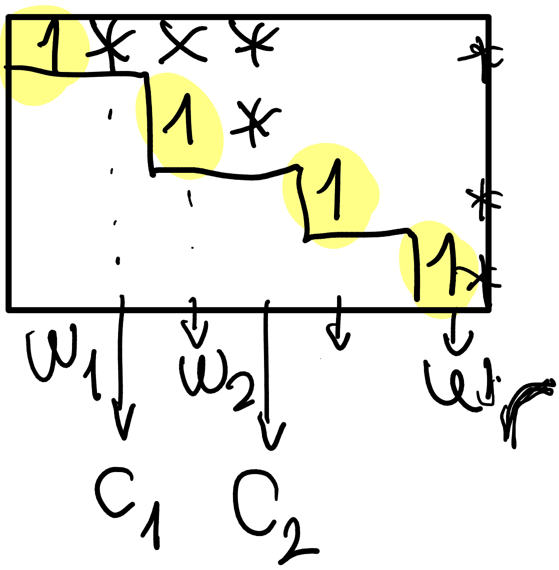
ex: • $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is not a basis for $\mathbb{R}^2$ because (b) is not satisfied.

• $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not a basis for $\mathbb{R}^2$ because (a) is not satisfied.

back to ex: null(A) = span(u_1, u_2, u_3)
& we saw in thm ① that they're lin. indep. So u_1, u_2, u_3 constitute a basis for null(A).

More generally, given any $A_{p\times q}$ compute the null(A) as in the example. You have $u_1, \dots, u_k$ corresponding to the free variables of A. span$(u_1, \dots, u_k)$ = null(A) & thm ② says they're lin. indep. So $u_1, \dots, u_k$ constitute a basis for null(A). Here k = # of free variables.

ex: $U =$



in row reduced echelon form.

$\mathrm{col}(U) =$ span of all columns

$= \mathrm{span}(w_1, w_2, \ldots, w_r)$

claim: because $c_1 = * \cdot w_1$

$c_2 = * w_1 + * w_2$ etc

Moreover by thm 2 $w_1, \ldots, w_r$ are linearly independent.

So the columns with pivots constitute a basis for $\mathrm{col}(U)$.

r here $=$ # of pivots.