Last time:

- ☐ Any system of equations is in the form: $A_{p\times q}\,x_{q\times 1} = b_{p\times 1}$.
- ☐ $\boxed{\text{If}}$ $p=q$ & A has inverse then the system has a unique soln:

$$x = A^{-1}\cdot b$$

- ☐ Every square matrix can be written as a product: $A = LU$ (U → upper triang, L → lower triang.) after possibly a number of row exchanges.

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ex: $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[-R1+R3\to R3]{-R1+R2\to R2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$?!

The process is stuck! One has to exchange, say, 2^{nd} & 3^{rd} rows.

Observe: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot B = A$

This "elementary matrix" is responsible of exchanging 2^{nd} & 3^{rd} rows. That & any product of such matrices is called a <u>permutation matrix</u>. So:

<u>THM</u>: Every square matrix A is:

$$\underset{\text{permutation}}{P} \cdot A = \underset{\text{lower}}{L} \cdot \underset{\text{upper}}{U}$$

<u>ex</u>:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{-R1+R2 \to R2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}]{-R1+R3 \to R3}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = U \ ; \quad E_2 \cdot E_1 \cdot A = U$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

L: ↳ lower   ↳ upper: U

Why do we like the LU-decomposition:

Solve: $Ax = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = b$ with A as above.

$$\Leftrightarrow L(\underbrace{Ux}) = b$$

Step I: $L \cdot y = b$ solve this.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} y_1 = 1 \\ y_2 = 1 \\ y_3 = -2 \end{matrix}$$

Step II: Solve $Ux = y = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} x_3 = -1, \; x_2 = 2 \\ x_1 = -1 \end{matrix}}$$

the unique solution for x

If the coeff matrix is triangular, solution becomes very easy.

THM: Any square matrix A satisfies:

shuffles the rows if necessary $\rightarrow$ $P \cdot A = L \cdot D \cdot U$ $\rightarrow$ upper

L: lower, D: diagonal

with the diagonal entries of L & U being all 1.

ex: $$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

So in the prev ex:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe $C^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$

Some more observations.

thm 1. Let $A_{p\times p}, B_{p\times p}$ be invertible matrices.
Then $A \cdot B$ is invertible too with inverse $B^{-1} \cdot A^{-1}$.

proof. Just multiply: $(B^{-1}A^{-1})(AB)$
$= B^{-1}(A^{-1}A)B = B^{-1}I_{p\times p}B = B^{-1}B = I.$

thm 2. Diagonal matrices with nonzero diagonal entries are invertible.
$(\mathrm{diag}(a_1, \ldots, a_n))^{-1} = \mathrm{diag}(a_1^{-1}, \ldots, a_n^{-1})$

defn: For an $A_{p\times q}$, the <u>transpose</u> of A, denoted by A^T, is the $q\times p$ matrix with $(A^T)_{ij} = A_{ji}$.

<u>ex:</u> $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

thm 3. $(\underset{p\times q}{A}\,\underset{q\times r}{B})^T = \underset{r\times q}{B^T}\,\underset{q\times p}{A^T}$.

proof. $\left((AB)^T\right)_{ij} = (\underbrace{AB}_{p\times r})_{ji} = \sum_{k=1}^{q} a_{jk} b_{ki}$

Meanwhile:

$$\left(\underset{r\times q}{B^T}\cdot\underset{q\times p}{A^T}\right)_{ij} = \sum_{k=1}^{q} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{q} b_{ki}\cdot a_{jk}$$

thm 4. Let $\underset{p\times p}{A}$ be invertible. Then A^T is invertible too with $(A^T)^{-1} = (A^{-1})^T$

proof. Try: $A^T\cdot(A^{-1})^T \underset{\text{thm3}}{=} (A^{-1}\cdot A)^T$

$$= I_{p\times p}^T = I$$

thm 5. The product of two upper tri. matrices is upper tri.

Proof. Let $\underset{p\times p}{U}, \underset{p\times p}{V}$ be upper. Then

$$(U\cdot V)^T \underset{\text{thm3}}{=} V^T \cdot U^T \underset{\text{last week}}{=} \text{lower}$$

V^T ↳ lower, U^T ↳ lower

Hence UV is upper.

How do we find inverses?

Gauss-Jordan algorithm gives the inverse of a matrix, if exists.

ex:
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[E_1]{-R1+R2\to R2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[E_2]{-R1+R3\to R3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[D_1]{\frac{R3}{2}\to R3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow[F_1]{-R3+R2\to R2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

$$\xrightarrow[F_2]{-R2+R1\to R1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

B

Stop when you obtain $I_{3\times3}$ in the first block.

Claim: $A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$.

Try $AA^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = I_{3\times3}$

Getting $I_{3\times3}$ here means

$$(F_2F_1D_1E_2E_1)A = I.$$

Then the expression in paranthesis is the inverse of A.

Meanwhile what happened in the second block:

$$F_2 F_1 D_1 E_2 E_1 \cdot I_{3\times 3} = B$$

Hence B is exactly the expression in the paranthesis: the inverse of A.

Gauss-Jordan: If I is obtained at the left block then the right block is the inverse of A.

The algorithm is stuck if there are 0 entries on the diagonal of the row reduced echelon form. (Just at the moment where F_1 starts.)