Last time:

Gaussian eimination is an algorithm with three operations:

- multiply rows with real #s
- add rows to other (below)
- swap the rows

ex: $x_1+x_2+x_3+x_4+x_5=0$
$2x_1+x_2-x_3+3x_4-x_5=0$
$x_2+3x_3+x_4-3x_5=0$

homogeneous system

$-2\cdot R1+R2 \to R2$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ & -1 & -3 & 1 & -3 & 0 \\ & 1 & 3 & +1 & -3 & 0 \end{array}\right)$$

$R2+R3 \to R3$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 2 & -6 & 0 \end{array}\right) \to$$

the row reduced echelon form

Up to here: forward elimination

Pivot: the 1^{st} nonzero entry of a row

The corresponding system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ -x_2 - 3x_3 + x_4 - 3x_5 &= 0 \\ 2x_4 - 6x_5 &= 0 \end{aligned}$$

free variables

R3: $x_4 = 3x_5$

R2: $x_2 = -3x_3 + x_4 - 3x_5$

$= -3x_3 + 3x_5 - 3x_5 = -3x_3$

backward substitution

R1: $x_1 = -x_2 - x_3 - x_4 - x_5$

$= +3x_3 - x_3 - 3x_5 - x_5$

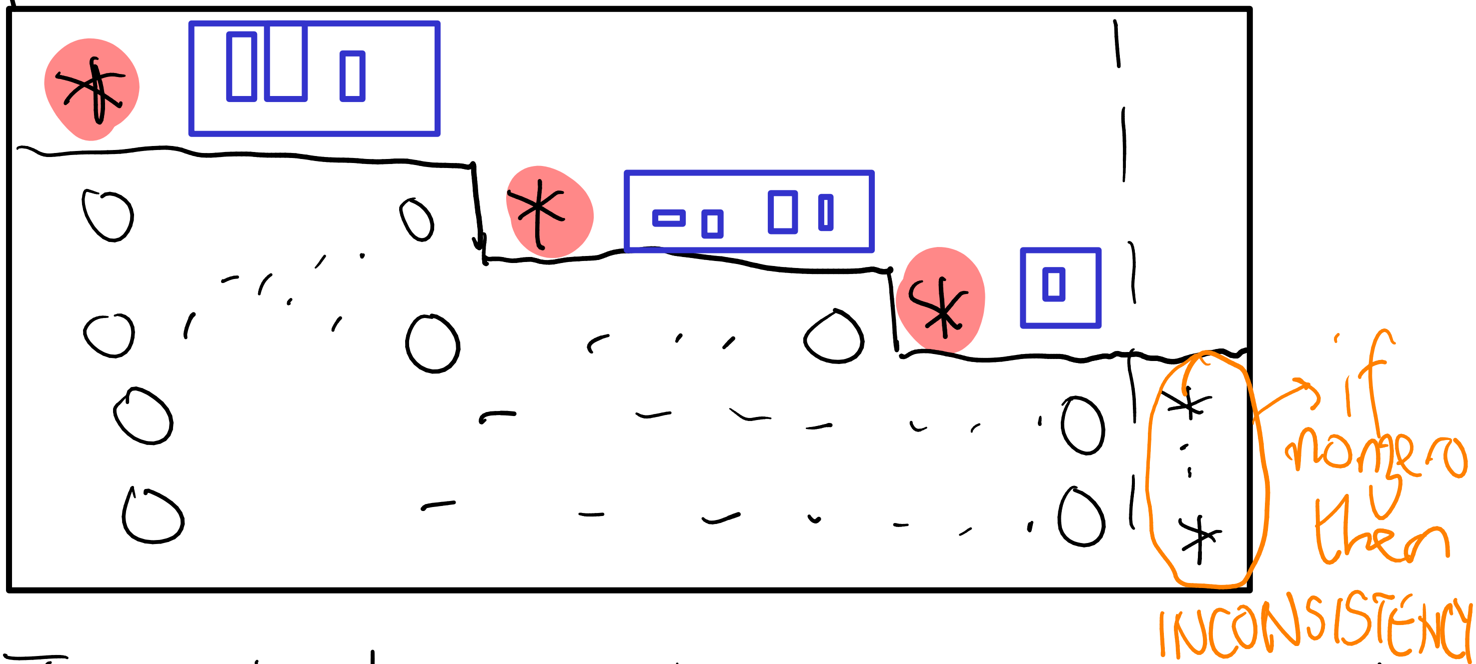
$= 2x_3 - 4x_5$

Soln set $= \{x_3, x_5 \in \mathbb{R};$

$x_4 = 3x_5, x_2 = -3x_3,$

$x_1 = 2x_3 - 4x_5 \}$

Summary: By forward elimination we come to the row reduced echelon form:



The pivot variables are dependent on the nonpivot variables: the ones in the blue boxes are free.

Remark. Of course you're free to choose which variables are free. However numbers must match:
dependent ones = # of pivots
The choice we made above is the most reasonable.

<u>Today</u>: some algebra

Let V be a set, with two operations:

- $+$: $u, v \in V \rightsquigarrow u+v \in V$
- given $k \in \mathbb{R}$, $u \in V$: $ku \in V$.

We require they satisfy:

$(k, l \in \mathbb{R};\ u, v, w \in V)$

$\boxed{G1}$: $u+(v+w)=(u+v)+w$: associativity

$\boxed{G2}$: There is some $a \in V$ which satisfies

$$a+u=u=u+a \quad \forall u.$$

a is called the ~~identity~~ zero element for $+$ & is denoted by $0 \in V$.

$\boxed{G3}$: $\forall u$, there's some v such that

$$u+v=0=v+u.$$

~~inverse~~ OR opposite of u.

$\boxed{G4}$: $u+v=v+u$: commutativity

$\boxed{S1}$: $k(lu) = (k \cdot l)u$

$\boxed{S2}$: $(k \underset{\mathbb{R}}{+} l)u = ku + lu$

$\boxed{S3}$: $\underset{\in \mathbb{R}}{0}u = 0 \in V$.

If $(V, +, \cdot)$ satisfies G1–G4, S1–S3 then it's called a <u>vector space</u>. Its elements are called <u>vectors</u>. The 2^{nd} operation is called <u>scalar product</u>.

<u>ex:</u> Take $V = \mathbb{R}^{\cancel{3}\; n \geq 1}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Define

$$u + v \underset{\text{defn}}{=} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}, \quad ku \underset{\text{defn}}{=} \begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \end{pmatrix}.$$

Then we get a vector space.

(HW: check G1–G4, S1–S3 hold!)

ex: A matrix is a grid of integers

$$A_{p\times q} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ a_{p1} & a_{p2} & & a_{pq} \end{pmatrix}$$ is a p by q matrix.

a_{p2} → an entry

$p \times q$: p → # of rows, q → # of columns

Define, for A, B $p \times q$ matrices,

$A+B$ as follows:

$$(A+B)_{ij} \overset{\text{defn}}{=} a_{ij} + b_{ij}$$

kA as follows:

$$(kA)_{ij} \overset{\text{defn}}{=} k \cdot a_{ij}.$$

Claim: The set of all $p\times q$ matrices with these operations is a vector space

Let's check:

G2. the zero element for this + is the zero matrix: $\begin{pmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}_{p\times q}$

G3: The opposite of $A = (a_{ij})_{p\times q}$ is $(-a_{ij})_{p\times q}$.

G1-G4, S1-S3 are satisfied.

ex: F = set of all cont fncs $[0,1] \to \mathbb{R}$, with standard $f+g$, kf operations, is a vector space.

Multiplying matrices

Let $A_{p\times q}$ & $B_{q\times r}$ be two matrices.

ex:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}_{2\times 3} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}_{3\times 2}$$

$$\overset{\text{defn}}{=} \begin{pmatrix} 1\cdot(-1)+1\cdot 0+1\cdot 1 & \underset{1,2}{1\cdot 0+1\cdot 2+1\cdot 3} \\ \underset{2,1}{1\cdot(-1)+2\cdot 0+3\cdot 1} & 1\cdot 0+2\cdot 2+3\cdot 3 \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} 0 & 5 \\ 2 & 13 \end{pmatrix}_{2\times 2}$$

defn: Given $A_{p\times q}$ & $B_{q\times r}$, $A\cdot B$ is a $p\times r$ matrix given as:

$$\boxed{(A\cdot B)_{ij} = \sum_{k=1}^{q} a_{ik}\cdot b_{kj}}$$

Rmks. • You cannot multiply any pair of matrices!

• Let $A_{1\times p}$ (a row vector) & $B_{p\times 1}$ (a column vector)

$A \cdot B = a_1 b_1 + \cdots + a_p b_p$ } a 1x1 matrix

$A \to (a_1, \ldots, a_p)$, $B \to \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}$ (dot product in MATH102)

• $A_{p\times 1}$, $B_{1\times q}$: AB is $p\times q$.

• Is this operation associative? That is, does it satisfy G1?

ex: $(A_{p\times q} \cdot B_{q\times s}) \cdot C_{q\times s}$ is not even defined!

claim: Using appropriate sizes, multiplication is associative.