Last time.

- Gram-Schmidt process.
- Given $A_{n\times n}$ of full rank,

$$A_{n\times n} = Q_{n\times n} \cdot R_{n\times n}$$

R: upper triang

Q: orthogonal matrix, i.e. $Q^TQ = I$

ex: Let P be the permutation matrix associated with $A_{n\times m}$ $(n \geq m)$, with lin. indep columns.

$$P = A(A^TA)^{-1}A^T$$

$$= Q_{n\times n}R_{n\times m} \cdot (R^TQ^TQR)^{-1} \cdot R^TQ^T$$

$$= Q\left(R(R^TR)^{-1}R^T\right)Q^T$$

DETERMINANT

<u>ex:</u> Find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if exists.

If both a & c are zero then $\operatorname{rank} A < 2$ so A is not invertible.

Assume, without loss of generality, that $a \neq 0$:

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow{R2 - \frac{c}{a}R1 \to R2} \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\overbrace{\qquad\qquad}^{B_{2\times 2}}} & & \\ a & b & 1 & 0 \\ 0 & \underbrace{\frac{ad-bc}{a}}_{\Delta/a} & -\frac{c}{a} & 1 \end{array}\right)$$

Let $\Delta \neq 0$:

$$\xrightarrow{R1 - \frac{ab}{\Delta}R2 \to R1} \left(\begin{array}{cc|cc} a & 0 & 1 + \frac{bc}{\Delta} & -\frac{ab}{\Delta} \\ 0 & \Delta/a & -\frac{c}{a} & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & d/\Delta & -b/\Delta \\ 0 & 1 & -c/\Delta & a/\Delta \end{array}\right)$$

with $\underbrace{\begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}}_{A^{-1}}$

Hence $A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\Delta = 0$: B has a 0 row and hence A has no inverse.

What we proved:

<u>thm</u>: A 2x2 matrix is invertible if and only if $\Delta \neq 0$.

This Δ is gonna be the "determinant" of a 2x2 matrix.

(Fix n)

<u>definition</u>: A function Δ eating n vectors in $\mathbb{R}^n$ & gives out a real #, i.e.

$$\Delta : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\Delta(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \in \mathbb{R}$$

which satisfies 1 – 3 below

is called a <u>determinant</u>:

1. $\Delta(\vec{e}_1, \dots, \vec{e}_n) = 1$.
2. Δ is <u>multilinear</u>, i.e. it is linear at each of its parameters:

for $a, b \in \mathbb{R}$, $u, v, v_2, \dots, v_n \in \mathbb{R}^n$,

$$\Delta(au+bv, v_2, \dots, v_n)$$
$$= \Delta(au, v_2, \dots, v_n) + \Delta(bv, v_2, \dots, v_n)$$
$$= a\Delta(u, v_2, \dots, v_n) + b\Delta(v, v_2, \dots, v_n)$$

Similarly for $2^{nd}, 3^{rd}, \dots, n^{th}$ parameters.

<u>ex</u>: $n=2$. 1. $\Delta(a\vec{u}, \vec{v}) = a\Delta(u, v)$
$$= \Delta(u, av)$$

2. $\Delta(au+bv, cw+d\lambda)$
$$= c\cdot\Delta(au+bv, w) + d\cdot\Delta(au+bv, \lambda)$$
↳ linearity wrt 2^{nd}

$$= c\left[a\Delta(u,w) + b\Delta(v,w)\right] + d\cdot\left[a\Delta(u,\lambda) + b\Delta(v,\lambda)\right]$$

linearity wrt the 1st

③ Δ is <u>alternating</u>: if you change the places of two vectors Δ earns a $-$.

$$\Delta(v_2, v_1, v_3, \dots, v_n) = -\Delta(v_1, v_2, v_3, \dots, v_n)$$

A Δ satisfying ①, ②, ③ is called a <u>determinant</u>.

<u>ex</u>: $\Delta\left(\begin{matrix}(a,b),\\(c,d)\end{matrix}\right) = \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies:

① $\Delta(e_1, e_2) = 1$; ② exercise.

③ $\Delta\left(\begin{matrix}(c,d),\\(a,b)\end{matrix}\right) = bc - ad = -\Delta\left(\begin{matrix}(a,b),\\(c,d)\end{matrix}\right)$.

When we write $\Delta(A_{n\times n})$ we mean Δ eats the rows of A.

What follows from ①–③:

④ If two vectors are the same then $\Delta = 0$:

$$\Delta(v_1, u, v_3, \dots, u, \dots, v_n)$$

$$\overset{③}{=} -\Delta(v_1, u, v_3, \dots, u, \dots, v_n)$$

So $\Delta(\dots) = 0$.

⑤ If $c \times$ a vector is added to some other then Δ is unchanged:

$$\Delta(v_1, v_2, cv_1 + v_3, v_4, \dots, v_n) \quad \text{by ④}$$

$$= \Delta(v_1, v_2, v_3, \dots, v_n) + c \cdot \underbrace{\Delta(v_1, v_2, v_1, v_4, \dots)}_{0}$$

↳ linearity wrt 3rd

⑥ If a vector is 0 then $\Delta = 0$:

$$\Delta(v_1, 0, v_3, \ldots, v_n) \overset{5}{=} \Delta(v_1, v_1, v_3, \ldots, v_n) \overset{4}{=} 0$$

$\overset{2}{=} 0 \cdot \Delta(v_1, 0, v_3, \ldots) = 0$

④: If two rows are the same in A then $\Delta(A) = 0$.

⑤: Row operations do not change Δ

⑥: If a row is 0 then $\Delta(A) = 0$.

⑦
$$\Delta \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots d_n \end{pmatrix} =$$

$$= \Delta((d_1, 0, \ldots, 0), (0, d_2, 0 \ldots 0), \ldots, (0, \ldots, 0, d_n))$$

$$\overset{2}{=} d_1 d_2 \cdots d_n \cdot \Delta(e_1, e_2, \ldots, e_n)$$

$$\overset{1}{=} d_1 \cdot d_2 \cdot \ldots \cdot d_n$$

(8) Let L be lower: $L = \begin{pmatrix} l_1 & & & 0 \\ & l_2 & & \\ & * & \ddots & \\ & & & l_n \end{pmatrix}$

Then $\Delta(L) = l_1 \cdots l_n$.

<u>proof</u>. If all l_i's are nonzero then $\Delta(L) = \Delta(\operatorname{diag}(l_1, \dots, l_n)) =$

$$= l_1 \cdot \ldots \cdot l_n.$$

Otherwise suppose, say $l_3 = 0$.

Then $\Delta \begin{pmatrix} l_1 & & & 0 \\ & l_2 & & \\ & & 0 & \\ & * & & l_4 \\ & & & & \ddots \end{pmatrix} \overset{(5)}{=} \Delta \begin{pmatrix} l_1 & & & 0 \\ 0 & l_2 & 0 & \\ 0 & 0 & 0 & \\ \vdots & \vdots & & \\ 0 & 0 & * & l_4 \; \ddots \end{pmatrix}$

zero vector

$\overset{(6)}{=} 0$

(8') Same for upper triangular.

(9) A permutation matrix is obtained from $I_{n \times n}$ via K exchanges of pairs of rows.

$$\Delta(P) \overset{3}{=} (-1)^k \cdot \Delta(e_1, \dots, e_n) \overset{1}{=} (-1)^k$$

10 Given $A_{n \times n}$: $PA = LDU$

$$\Delta(PA) \overset{3}{=} (-1)^k \Delta(A)$$

& meaning of P

$$\Delta(LDU) \overset{5}{=} \Delta(DU) = \Delta\begin{pmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

↳ is responsible of row opns on DU

$$= d_1 \cdot \ldots \cdot d_n .$$

Hence $\boxed{\Delta(A) = (-1)^k d_1 \cdots d_n}$

For this to be well-defined, we prove:

thm: Given $A_{n\times n}$ (and P) the invertible LDU-decomposition is unique.

proof. $A = L_1 D_1 U_1 = L_2 D_2 U_2$

Then $L_1, D_1, U_1, L_2, D_2, U_2$ are all invertible.

$$\underbrace{L_2^{-1} L_1}_{\text{lower}} = \underbrace{(D_2 U_2)(D_1 U_1)^{-1}}_{\text{upper}}$$

L_2^{-1} lower, L_1 lower

$L_2^{-1} L_1$ is diagonal with diagonal entries are all 1.

I.e. $L_2^{-1} L_1 = I \Leftrightarrow L_1 = L_2$.

So $\underbrace{D_2^{-1} D_1}_{\text{diagonal}} = \underbrace{U_2 U_1^{-1}}_{\text{upper with diag 1}} = I$