Matrix multiplication (cont'd)

$$\boxed{(\underset{p\times q}{A}\cdot \underset{q\times r}{B})_{ij} = \sum_{k=1}^{q} a_{ik}\cdot b_{kj}}$$

Observations.

1 G1 axiom: associativity?

If $A\cdot(B\cdot C)$ & $(A\cdot B)\cdot C$ are defined, are they equal?

theorem 1. For $A_{p\times q}, B_{q\times r}, C_{r\times s}$,

$$\underset{p\times q}{A}\cdot\underset{q\times s}{(B\cdot C)} = \underset{p\times r}{(A\cdot B)}\cdot \underset{r\times s}{C}$$

proof.

$$\begin{aligned}
(A\cdot(B\cdot C))_{ij} &\overset{\text{by defn}}{=} \sum_{k=1}^{q} A_{ik}(B\cdot C)_{kj} \\
&\overset{\text{by defn}}{=} \sum_{k=1}^{q} A_{ik} \sum_{\ell=1}^{r} B_{k\ell}C_{\ell j} \\
&= \sum_{\ell=1}^{r} \underbrace{\sum_{k=1}^{q} A_{ik}B_{k\ell}}\, C_{\ell j} \\
&\overset{\text{by defn}}{=} \sum_{\ell=1}^{r} (AB)_{i\ell}\, C_{\ell j} \\
&\overset{\text{by defn}}{=} ((AB)\cdot C)_{ij}
\end{aligned}$$

i.e. matrix multiplication is associative for appropriately sized matrices.

② G4? Is matrix multiplication commutative? $AB \overset{?}{=} BA$

ex: For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = BA$$

No: matrix mult. is not commutative!

③ G3: Is there an identity element?

ex: $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}}_{= I_{2\times 2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 1 \end{pmatrix}_{2\times 4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 1 \end{pmatrix}_{2\times 4}$

defn: • A square matrix D is diagonal if $D_{ij} = 0$ when $i \neq j$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Notation: $\text{diag}(d_1, \ldots, d_p)$

• The $p \times p$ identity matrix is $I_{p\times p} = \text{diag}(1, \ldots, 1)$

claim: (I) $I_{p\times p}$ is a left identity for $p\times n$ matrices.

(II) $I_{p\times p}$ identity matrix is a right identity for $n\times p$ matrices.

(III) $I_{p\times p}$ is the identity element for matrix multiplication between $p\times p$ square matrices.

(4) Does every square matrix A have an inverse? If A is invertible, we denote the inverse by A^{-1};

$$A \cdot A^{-1} = I_{p\times p} = A^{-1} \cdot A$$

ex: • $(0)_{1\times 1}$ has no inverse.

• $O_{p\times p}$ has no inverse.

• $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ +1 & 1 \end{pmatrix}$

Some square matrices have inverse, some don't.

Back to systems.

ex:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ x_1 + x_2 + 2x_3 &= -1 \end{aligned}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}_{3\times 1}}_{b}$$

$$\Leftrightarrow (A\,x)_{3\times 1} = b_{3\times 1}$$

If A has an inverse:

$$A^{-1}(Ax) = A^{-1}b \quad \text{// soln! :)}$$

by G1 $\parallel$

$$(A^{-1}A)x = I_{3\times 3} \cdot x = x$$

Thus A has inverse $\Rightarrow$ there is a unique solution.

ex:

$$x_1 + x_2 + x_4 = 1$$
$$x_1 + 2x_2 + x_3 - x_4 = 2$$
$$x_1 + x_2 + 2x_3 + x_4 = -1$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}}_{A=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\downarrow$ $-R1 + R2 \to R2$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & 1 & 2 & 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}}^{E_1} \cdot A_{3\times 4}$$

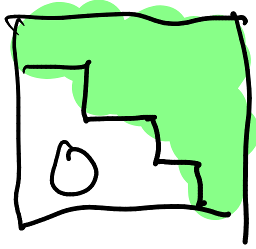
$\downarrow$ $-R1 + R3 \to R3$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}^{E_2} \cdot (E_1 A)$$

Thus

$$\underset{3\times 3}{E_2} \, \underset{3\times 3}{E_1} A = \text{row reduced echelon form!}$$

If A is square then

(*) $E_m \cdot E_{m-1} \cdot \ldots \cdot E_1 A =$ 

<u>defn</u>: • A square matrix L is <u>lower triangular</u> if $L_{ij} = 0$ when $i < j$.

• A square matrix U is <u>upper triangular</u> if $U_{ij} = 0$ when $i > j$.

<u>ex</u>: • A diagonal matrix is both upper & lower triangular.

• A square row reduced echelon matrix is upper triangular.

PLAN: In (*) above, we will show:

I) E_j's are lower.

II) Their product is lower, denote E.

III) Each has an inverse; E^{-1} exists.

In that case we will have:

$$\underset{\text{lower}}{E}A = \underset{\text{upper}}{U}$$

so that

$$A = E^{-1}U$$

If also we can show:

IV E^{-1} is lower too.

Then I–IV will lead us: given A square,

$$A = \text{lower} \times \text{upper}.$$

So let us elaborate I–IV:

I Recall: E_k is in charge of $tR_i + R_j \longrightarrow R_j$ $(i<j)$ & is of the form

called an elementary matrix

$$\begin{array}{c} \\ j\text{th} \rightarrow \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & \\ & \ddots\ 1 & & 0 \\ 0 & \boxed{k} \cdots 1 & & \\ 0 & 0 & \ddots & 1 \end{pmatrix}_{p\times p}$$

$$\uparrow$$
$$i\text{th}$$

$(\square)$

Therefore E_k is lower.

II thm 2. The product of two lower triangular matrices is lower triangular too.

proof. Let K & L be $p \times p$ lower.

For $i<j$, $(KL)_{ij} = \sum_{a=1}^{p} K_{ia} L_{aj}$

$$= \sum_{a=1}^{j-1} K_{ia} \underbrace{L_{aj}}_{0} + \sum_{a=j}^{p} \underbrace{K_{ia}}_{0} L_{aj} = 0$$

$(a \geq j > i)$

III & IV Is $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ invertible?

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1^{-1}} = I_{3\times 3}$$

$$E_1^{-1} \cdot E_1 = I_{3\times 3}$$

thm 3. (□) has inverse

$$\begin{array}{cc} j\text{th} \rightarrow & \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ 0 & & \ddots & & \\ 0 & -k & & \ddots & \\ 0 & 0 & & 0 & 1 \end{pmatrix} \\ & \uparrow \\ & i\text{th} \end{array}$$

Then

$$E_m \cdot E_{m-1} \cdot \ldots \cdot E_1 \cdot \underbrace{A}_{\text{square}} = \underbrace{U}_{\text{upper}}$$

↳ square, invertible, lower

So that

$$\Rightarrow \left(E_m^{-1} \cdot (E_m) \cdots E_1\right) A = \underbrace{E_m^{-1}}_{\text{lower}} U$$

$\vdots$

$$\Rightarrow A = \underbrace{E_1^{-1} \cdots E_m^{-1}}_{\text{lower}} U$$

thm2

$$\Rightarrow A = \underbrace{L}_{\text{lower}} \cdot \underbrace{U}_{\text{upper}}$$

THM: Any square matrix can be written as a product of a lower and an upper.

This is called the LU-decomposition of A (after row exchanges).