AI engine: https://deepai.org/chat/mathematics

PROMPT: given integers p, r, i, k, where p is prime and greater than 2, none of (i+k), r, i, k equal to zero, can the following expressions simultaneously be integers: (r+i+k)^p/(2r+i+k), (i^p + k^p)/(r+i+k)

RESPONSE:

Here is a sketch of the simplest "size–and–divisibility" argument showing that, as soon as p>2 and r,i,k are all nonzero with i+k≠0, you cannot make both

A = (r + i + k)^p / (2r + i + k)

B = (i^p + k^p) / (r + i + k)

integers at the same time.

1. Notation. Write

s = i + k, d = r + s (so 2r + s = r + d).

Then

A = d^p/(r + d),

B = (i^p + k^p)/s.

2. A-divisibility forces d^p ≡ 0 mod (r + d).

Since gcd(d, r+d) = gcd(r+s, 2r+s) = gcd(d, 2r+s − 2(r+s))

= gcd(d, −s) = gcd(d,s),

write g = gcd(d,s). Then r + d = r + s + (d−s) = s + (d) so one checks

r + d = g · M, d = g · D, s = g · S,

with gcd(D,S)=1. The condition d^p ≡ 0 mod (r + d) becomes

(g·D)^p is divisible by g·M ⇒ g^{p−1}·D^p is divisible by M.

But M > D whenever r ≠ 0 and i+k≠0 (easy to check by signs), so the only way a larger integer M can divide the smaller power g^{p−1}D^p is if M = 1. Hence

$r + d = 1$ (or -1).

3. But $r + d = 1$ means $r + (r + s) = 1 \Rightarrow 2r + s = 1 \Rightarrow$ the denominator of A is ± 1. Plugging back one finds $r = (1 - s)/2$ must be an integer, so $s = i+k$ is odd. In particular $i+k$ is $\pm 1, \pm 3, \ldots$

4. Now look at

$$B = (i^p + k^p)/s.$$

Since p is odd, $i^p + k^p = (i + k)(i^{p-1} - i^{p-2}k + \ldots + k^{p-1})$

$$= s \cdot T,$$

so $B = T$ is an integer automatically. So the only way to make A an integer is to force its denominator to ± 1, i.e. $2r+s = \pm 1$. But that in turn forces $r = (\pm 1 - s)/2$ to be an integer, which in our setup forces $s = i+k$ to be odd.

5. Finally one checks that if you really do set $2r + (i+k) = \pm 1$ then

$$A = (r + i + k)^p/(2r + i + k) = (d)^p/\pm 1 = \pm d^p$$

is an integer and

$$B = T$$

is also an integer—but at the price of forcing $r = (\pm 1-(i+k))/2$. In other words the only simultaneous-integral solutions occur exactly in the "denominator $= \pm 1$" case, which when you unwind it is equivalent to

$$2r + i + k = \pm 1.$$

Conclusion. Except for the very special (and easily described) choice $2r + i + k = \pm 1$, there is no way to make both

$$(r+i+k)^p/(2r+i+k) \quad \text{and} \quad (i^p + k^p)/(r+i+k)$$

simultaneously integers once $p>2$ and none of r,i,k or $i+k$ vanishes.