Exercises from *Algebra* by Michael Artin

Exercise 2.2.9 Let H be the subgroup generated by two elements a, b of a group G. Prove that if $ab = ba$, then H is an abelian group.

Exercise 2.3.1 Prove that the additive group $\mathbb{R}^+$ of real numbers is isomorphic to the multiplicative group P of positive reals.

Exercise 2.3.2 Prove that the products ab and ba are conjugate elements in a group.

Exercise 2.4.19 Prove that if a group contains exactly one element of order 2 , then that element is in the center of the group.

Exercise 2.8.6 Prove that the center of the product of two groups is the product of their centers.

Exercise 2.10.11 Prove that the groups $\mathbb{R}^+/\mathbb{Z}^+$and $\mathbb{R}^+/2\pi\mathbb{Z}^+$are isomorphic.

Exercise 2.11.3 Prove that a group of even order contains an element of order 2.

Exercise 3.2.7 Prove that every homomorphism of fields is injective.

Exercise 3.5.6 Let V be a vector space which is spanned by a countably infinite set. Prove that every linearly independent subset of V is finite or countably infinite.

Exercise 3.7.2 Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

Exercise 6.1.14 Let Z be the center of a group G. Prove that if G/Z is a cyclic group, then G is abelian and hence $G = Z$.

Exercise 6.4.2 Prove that no group of order pq, where p and q are prime, is simple.

Exercise 6.4.3 Prove that no group of order p^2q, where p and q are prime, is simple.

Exercise 6.4.12 Prove that no group of order 224 is simple.

Exercise 6.8.1 Prove that two elements a, b of a group generate the same subgroup as bab^2, bab^3.

Exercise 6.8.4 Prove that the group generated by x, y, z with the single relation $yxyz^{-2} = 1$ is actually a free group.

Exercise 6.8.6 Let G be a group with a normal subgroup N. Assume that G and G/N are both cyclic groups. Prove that G can be generated by two elements.

Exercise 10.1.13 An element x of a ring R is called nilpotent if some power of x is zero. Prove that if x is nilpotent, then $1 + x$ is a unit in R.

Exercise 10.2.4 Prove that in the ring $\mathbb{Z}[x], (2) \cap (x) = (2x)$.

Exercise 10.6.7 Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

Exercise 10.6.16 Prove that a polynomial $f(x) = \sum a_i x^i$ can be expanded in powers of $x - a :\ f(x) = \Sigma c_i (x-a)^i$, and that the coefficients c_i are polynomials in the coefficients a_i, with integer coefficients.

Exercise 10.3.24a Let I, J be ideals of a ring R. Show by example that $I \cup J$ need not be an ideal.

Exercise 10.4.6 Let I, J be ideals in a ring R. Prove that the residue of any element of $I \cap J$ in R/IJ is nilpotent.

Exercise 10.4.7a Let I, J be ideals of a ring R such that $I + J = R$. Prove that $IJ = I \cap J$.

Exercise 10.5.16 Let F be a field. Prove that the rings $F[x]/\left(x^2\right)$ and $F[x]/\left(x^2 - 1\right)$ are isomorphic if and only if F has characteristic 2.

Exercise 10.7.6 Prove that the ring $\mathbb{F}_5[x]/\left(x^2 + x + 1\right)$ is a field.

Exercise 10.7.10 Let R be a ring, with M an ideal of R. Suppose that every element of R which is not in M is a unit of R. Prove that M is a maximal ideal and that moreover it is the only maximal ideal of R.

Exercise 11.2.13 If a, b are integers and if a divides b in the ring of Gauss integers, then a divides b in $\mathbb{Z}$.

Exercise 11.3.1 Let a, b be elements of a field F, with $a \neq 0$. Prove that a polynomial $f(x) \in F[x]$ is irreducible if and only if $f(ax+b)$ is irreducible.

Exercise 11.3.2 Let $F = \mathbb{C}(x)$, and let $f, g \in \mathbb{C}[x, y]$. Prove that if f and g have a common factor in $F[y]$, then they also have a common factor in $\mathbb{C}[x, y]$.

Exercise 11.3.4 Prove that two integer polynomials are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains an integer.

Exercise 11.4.1a Prove that $x^2 + 27x + 213$ is irreducible in $\mathbb{Q}$.

Exercise 11.4.1b Prove that $x^3 + 6x + 12$ is irreducible in $\mathbb{Q}$.

Exercise 11.4.6a Prove that $x^2 + x + 1$ is irreducible in the field $\mathbb{F}_2$.

Exercise 11.4.6b Prove that $x^2 + 1$ is irreducible in $\mathbb{F}_7$

Exercise 11.4.6c Prove that $x^3 - 9$ is irreducible in $\mathbb{F}_31$.

Exercise 11.4.8 Let p be a prime integer. Prove that the polynomial $x^n - p$ is irreducible in $\mathbb{Q}[x]$.

Exercise 11.4.10 Let p be a prime integer, and let $f \in \mathbb{Z}[x]$ be a polynomial of degree $2n+1$, say $f(x) = a_{2n+1}x^{2n+1} + \cdots + a_1x + a_0$. Suppose that $a_{2n+1} \not\equiv 0$ (modulo p), $a_0, a_1, \ldots, a_n \equiv 0$ (modulo p^2), $a_{n+1}, \ldots, a_{2n} \equiv 0$ (modulo p), $a_0 \not\equiv 0$ (modulo p^3). Prove that f is irreducible in $\mathbb{Q}[x]$.

Exercise 11.9.4 Let p be a prime which splits in R, say $(p) = P\bar{P}$, and let $\alpha \in P$ be any element which is not divisible by p. Prove that P is generated as an ideal by (p, α).

Exercise 11.12.3 Prove that if $x^2 \equiv -5$ (modulo p) has a solution, then there is an integer point on one of the two ellipses $x^2 + 5y^2 = p$ or $2x^2 + 2xy + 3y^2 = p$.

Exercise 11.13.3 Prove that there are infinitely many primes congruent to -1 (modulo 4).

Exercise 13.1.3 Let R be an integral domain containing a field F as subring and which is finite-dimensional when viewed as vector space over F. Prove that R is a field.

Exercise 13.3.1 Let F be a field, and let α be an element which generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

Exercise 13.3.8 Let K be a field generated over F by two elements α, β of relatively prime degrees m, n respectively. Prove that $[K : F] = mn$.

Exercise 13.4.10 Prove that if a prime integer p has the form $2^r + 1$, then it actually has the form $2^{2^k} + 1$.

Exercise 13.6.10 Let K be a finite field. Prove that the product of the nonzero elements of K is -1.