ProofNet NL Statements

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chapter 1

section 1

1a: Prove that the the operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$ is not associative.

2a: Prove the the operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$ is not commutative.

3: Prove that the addition of residue classes $\mathbb{Z}/n\mathbb{Z}$ is associative.

4: Prove that the multiplication of residue class $\mathbb{Z}/n\mathbb{Z}$ is associative.

5: Prove that for all $n > 1$ that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication of residue classes.

15: Prove that $(a_1 a_2 \dots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1}$ for all $a_1, a_2, \dots, a_n \in G$.

16: Let x be an element of G. Prove that $x^2 = 1$ if and only if $|x|$ is either 1 or 2.

17: Let x be an element of G. Prove that if $|x| = n$ for some positive integer n then $x^{-1} = x^{n-1}$.

18: Let x and y be elements of G. Prove that $xy = yx$ if and only if $y^{-1}xy = x$ if and only if $x^{-1}y^{-1}xy = 1$.

20: For x an element in G show that x and x^{-1} have the same order.

22a: If x and g are elements of the group G, prove that $|x| = \left|g^{-1}xg\right|$.

22b: Deduce that $|ab| = |ba|$ for all $a, b \in G$.

25: Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

29: Prove that $A \times B$ is an abelian group if and only if both A and B are abelian.

34: If x is an element of infinite order in G, prove that the elements $x^n, n \in \mathbb{Z}$ are all distinct.

section 3

8: Prove that if $\Omega = \{1, 2, 3, \dots\}$ then S_Ω is an infinite group

section 6

4: Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.

11: Let A and B be groups. Prove that $A \times B \cong B \times A$.

17: Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

23: Let G be a finite group which possesses an automorphism σ such that $\sigma(g) = g$ if and only if $g = 1$. If σ^2 is the identity map from G to G, prove that G is abelian.

section 7

5: Prove that the kernel of an action of the group G on a set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

6: Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity.

chapter 2

section 1

5: Prove that G cannot have a subgroup H with $|H| = n - 1$, where $n = |G| > 2$.

13: Let H be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element x of H. Prove that $H = 0$ or $\mathbb{Q}$.

section 2

4: Prove that if H is a subgroup of G then H is generated by the set $H - \{1\}$.

13: Prove that the multiplicative group of positive rational numbers is generated by the set $\left\{\frac{1}{p} \mid p \text{ is a prime}\right\}$.

16a: A subgroup M of a group G is called a maximal subgroup if $M \neq G$ and the only subgroups of G which contain M are M and G. Prove that if H is a proper subgroup of the finite group G then there is a maximal subgroup of G containing H.

16c: Show that if $G = \langle x \rangle$ is a cyclic group of order $n \geq 1$ then a subgroup H is maximal if and only $H = \langle x^p \rangle$ for some prime p dividing n.

Chapter 3

section 1

3a: Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian.

22a: Prove that if H and K are normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G.

22b: Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

section 2

8: Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

11: Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$ (do not assume G is finite).

16: Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$to prove Fermat's Little Theorem: if p is a prime then $a^p \equiv a (\text{mod} p)$ for all $a \in \mathbb{Z}$.

21a: Prove that $\mathbb{Q}$ has no proper subgroups of finite index.

section 3

3: Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $|K : K \cap H| = p$.

section 4

1: Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

4: Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.

5a: Prove that subgroups of a solvable group are solvable.

5b: Prove that quotient groups of a solvable group are solvable.

11: Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrivial subgroup A of H with $A \trianglelefteq G$ and A abelian.

Chapter 4

section 2

8: Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n$!.

9a: Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G.

14: Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

section 3

5: If the center of G is of index n, prove that every conjugacy class has at most n elements.

26: Let G be a transitive permutation group on the finite set A with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element σ is called fixed point free).

27: Let $g_1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

section 4

2: Prove that if G is a n abelian group of order pq, where p and q are distinct primes, then G is cyclic.

6a: Prove that characteristic subgroups are normal.

7: If H is the unique subgroup of a given order in a group G prove H is characteristic in G.

8a: Let G be a group with subgroups H and K with $H \leq K$. Prove that if H is characteristic in K and K is normal in G then H is normal in G.

section 5

1: Prove that if $P \in \mathrm{Syl}_p(G)$ and H is a subgroup of G containing P then $P \in \mathrm{Syl}_p(H)$.

13: Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

14: Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

15: Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

16: Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

17: Prove that if $|G| = 105$ then G has a normal Sylow 5 -subgroup and a normal Sylow 7-subgroup.

18: Prove that a group of order 200 has a normal Sylow 5-subgroup.

19: Prove that if $|G| = 6545$ then G is not simple.

20: Prove that if $|G| = 1365$ then G is not simple.

21: Prove that if $|G| = 2907$ then G is not simple.

22: Prove that if $|G| = 132$ then G is not simple.

23: Prove that if $|G| = 462$ then G is not simple.

28: Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

33: Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Chapter 5

section 4

2: Prove that a subgroup H of G is normal if and only if $[G, H] \leq H$.

Chapter 7

0.0.1 section 1

2: Prove that if u is a unit in R then so is $-u$.

11: Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.

12: Prove that any subring of a field which contains the identity is an integral domain.

15: A ring R is called a Boolean ring if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative.

section 2

2: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be an element of the polynomial ring $R[x]$. Prove that $p(x)$ is a zero divisor in $R[x]$ if and only if there is a nonzero $b \in R$ such that $bp(x) = 0$.

4: Prove that if R is an integral domain then the ring of formal power series $R[[x]]$ is also an integral domain.

12: Let $G = \{g_1, \ldots, g_n\}$ be a finite group. Prove that the element $N = g_1 + g_2 + \ldots + g_n$ is in the center of the group ring RG.

section 3

16: Let $\varphi : R \to S$ be a surjective homomorphism of rings. Prove that the image of the center of R is contained in the center of S.

28: Prove that an integral domain has characteristic p, where p is either a prime or 0

37: An ideal N is called nilpotent if N^n is the zero ideal for some $n \geq 1$. Prove that the ideal $p\mathbb{Z}/p^m\mathbb{Z}$ is a nilpotent ideal in the ring $\mathbb{Z}/p^m\mathbb{Z}$.

section 4

27: Let R be a commutative ring with $1 \neq 0$. Prove that if a is a nilpotent element of R then $1 - ab$ is a unit for all $b \in R$.

Chapter 8

section 1

12: Let N be a positive integer. Let M be an integer relatively prime to N and let d be an integer relatively prime to $\varphi(N)$, where φ denotes Euler's φ-function. Prove that if $M_1 \equiv M^d(\mathrm{mod} N)$ then $M \equiv M_1^{d'}(\mathrm{mod} N)$ where d' is the inverse of d $\mathrm{mod}\varphi(N) : dd' \equiv 1(\mathrm{mod}\varphi(N))$

section 2

4: Let R be an integral domain. Prove that if the following two conditions hold then R is a Principal Ideal Domain: (i) any two nonzero elements a and b in R have a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$, and (ii) if $a_1, a_2, a_3, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

section 3

4: Prove that if an integer is the sum of two rational squares, then it is the sum of two integer squares (for example, $13 = (1/5)^2 + (18/5)^2 = 2^2 + 3^2$).

5a: Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a squarefree integer greater than 3. Prove that $2, \sqrt{-n}$ and $1 + \sqrt{-n}$ are irreducibles in R.

6a: Prove that the quotient ring $\mathbb{Z}[i]/(1+i)$ is a field of order 2 .

6b: Let $q \in \mathbb{Z}$ be a prime with $q \equiv 3 \bmod 4$. Prove that the quotient ring $\mathbb{Z}[i]/(q)$ is a field with q^2 elements.

chapter 9

section 1

6: Prove that (x, y) is not a principal ideal in $\mathbb{Q}[x, y]$.

10: Prove that the ring $\mathbb{Z}\left[x_1, x_2, x_3, \ldots\right] / \left(x_1x_2, x_3x_4, x_5x_6, \ldots\right)$ contains infinitely many minimal prime ideals (cf. Exercise 36 of Section 7.4).

section 3

2: Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.

section 4

2a: Prove that $x^4 - 4x^3 + 6$ is irreducible in $\mathbb{Z}[x]$.

2b: Prove that $x^6 + 30x^5 - 15x^+6x - 120$ is irreducible in $\mathbb{Z}[x]$.

2c: Prove that $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible in $\mathbb{Z}[x]$.

2d: Prove that $\frac{(x+2)^p-2^p}{x}$, where p is an odd prime, is irreducible in $\mathbb{Z}[x]$.

9: Prove that the polynomial $x^2-\sqrt{2}$ is irreducible over $\mathbb{Z}[\sqrt{2}]$. You may assume that $\mathbb{Z}[\sqrt{2}]$ is a U.F.D.

11: Prove that x^2+y^2-1 is irreducible in $\mathbb{Q}[x,y]$.

Rudin

chapter 1

1: If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

2: Prove that there is no rational number whose square is 12.

4: Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

5: Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$

8: Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.

14: If z is a complex number such that $|z|=1$, that is, such that $z\bar{z}=1$, compute $|1+z|^2+|1-z|^2$.

17: Prove that $|\mathbf{x}+\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^2=2|\mathbf{x}|^2+2|\mathbf{y}|^2$ if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$.

18a: If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

25: Prove that every compact metric space K has a countable base.

27a: Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that P is perfect.

27b: Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that at most countably many point of E are not in P.

28: Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable.

29: Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

chapter 2

19a: If A and B are disjoint closed sets in some metric space X, prove that they are separated.

24: Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_J \in X$,

chapter 3

1a: Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

3: If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ $\quad (n=1,2,3,\ldots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1,2,3,\ldots$.

5: For any two real sequences $\{a_n\}, \{b_n\}$, prove that $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.

7: Prove that the convergence of Σa_n implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

8: If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

13: Prove that the Cauchy product of two absolutely convergent series converges absolutely.

20: 20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{nl}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21: If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

22: Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. Hint: Find a shrinking sequence of neighborhoods E_n such that $E_n \subset G_n$.

chapter 4

2a: If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$. ($\overline{E}$ denotes the closure of E).

3: Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

4a: Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$.

5a: If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$.

6: If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

8a: Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

11a: Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in 12: A uniformly continuous function of a uniformly continuous function is uniformly continuous.

14: Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

15: Prove that every continuous open mapping of R^1 into R^1 is monotonic.

19: Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

21a: Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

24: Assume that f is a continuous real function defined in (a, b) such that $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in (a, b)$. Prove that f is convex.

26a: Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$. Prove that f is uniformly continuous if h is uniformly continuous.

chapter 5

1: Let f be defined for all real x, and suppose that $|f(x) - f(y)| \leq (x-y)^2$for all real x and y. Prove that f is constant.

2: Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b)$

3: Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

4: If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1 .

5: Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

6: Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x} \quad (x > 0)$ and prove that g is monotonically increasing.

7: Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

15: Suppose $a \in R^1, f$ is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$.

16: Suppose f is a real, three times differentiable function on $[-1,1]$, such that $f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0$. Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

chapter 5

1: Suppose α increases on $[a,b], a \leq x_0 \leq b, \alpha$ is continuous at $x_0, f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

2: Suppose $f \geq 0, f$ is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

4: If $f(x) = 0$ for all irrational $x, f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

6: Let P be the Cantor set. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0,1]$.

Munkres

chapter 2

section 13

5a: Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

section 16

4: A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

section 17

2: Show that if A is closed in Y and Y is closed in X, then A is closed in X.

3: Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

4: Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

18

8a: Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

8b: Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous. [Hint: Use the pasting lemma.]

13: Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

21

6a: Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0,1]$.

6b: Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence (f_n) does not converge uniformly.

8: Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

22

1: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

2a: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

2b: If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

3: Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\bar{H}$ are topological groups.

23

2: Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

3: Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connectea eubsen of X Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup \mid A_\alpha)$ is connected.

4: Show that if X is an infinite set, it is connected in the finite complement topology.

6: Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects $\mathrm{Bd}A$.

9: Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

11: Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

12: Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

24

2: Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

25

9: Let G be a topological group; let C be the component of G containing the identity element e. Show that C is a normal subgroup of G.

26

9: Theorem. Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that $A \times B \subset U \times V \subset N$.

11: Theorem. Let X be a compact Hausdorff space. Let A be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y = \bigcap_{A \in \mathcal{A}} A$ is connected.

12: Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. (Such a map is called a perfect map.) Show that if Y is compact, then X is compact.

27

4: Show that a connected metric space having more than one point is uncountable.

28

6: Let (X, d) be a metric space. If $f : X \to X$ satisfies the condition $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$, then f is called an isometry of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

29

10: Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

irelandrosen

chapter 1

27: For all odd n show that $8 \mid n^2 - 1$.

30: Prove that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.

31: Show that 2 is divisible by $(1 + i)^2$ in $\mathbb{Z}[i]$.

chapter 2

4: If a is a nonzero integer, then for $n > m$ show that $\left(a^{2^n}+1, a^{2^m}+1\right) = 1$ or 2 depending on whether a is odd or even.

21: Define $\wedge(n) = \log p$ if n is a power of p and zero otherwise. Prove that $\sum_{A|n} \mu(n/d) \log d = \wedge(n)$.

27a: Show that $\sum' 1/n$, the sum being over square free integers, diverges.

chapter 3

1: Show that there are infinitely many primes congruent to -1 modulo 6 .

4: Show that the equation $3x^2 + 2 = y^2$ has no solution in integers.

5: Show that the equation $7x^3 + 2 = y^3$ has no solution in integers.

10: If n is not a prime, show that $(n-1)! \equiv 0(n)$, except when $n = 4$.

14: Let p and q be distinct odd primes such that $p-1$ divides $q-1$. If $(n, pq) = 1$, show that $n^{q-1} \equiv 1(pq)$.

18: Let N be the number of solutions to $f(x) \equiv 0(n)$ and N_i be the number of solutions to $f(x) \equiv 0\,(p_i^{a_i})$. Prove that $N = N_1 N_2 \cdots N_i$.

20: Show that $x^2 \equiv 1\left(2^b\right)$ has one solution if $b = 1$, two solutions if $b = 2$, and four solutions if $b \geq 3$.

chapter 4

4: Consider a prime p of the form $4t + 1$. Show that a is a primitive root modulo p iff - a is a primitive root modulo p.

5: Consider a prime p of the form $4t+3$. Show that a is a primitive root modulo p iff $-a$ has order $(p-1)/2$.

6: If $p = 2^n + 1$ is a Fermat prime, show that 3 is a primitive root modulo p.

8: Let p be an odd prime. Show that a is a primitive root module p iff $a^{(p-1)/q} \not\equiv 1(p)$ for all prime divisors q of $p-1$.

9: Show that the product of all the primitive roots modulo p is congruent to $(-1)^{\phi(p-1)}$ modulo p.

10: Show that the sum of all the primitive roots modulo p is congruent to $\mu(p-1)$ modulo p.

11: Prove that $1^k + 2^k + \cdots + (p-1)^k \equiv 0(p)$ if $p-1 \nmid k$ and $-1(p)$ if $p-1 \mid k$.

22: If a has order 3 modulo p, show that $1 + a$ has order 6 .

24: Show that $ax^m + by^n \equiv c(p)$ has the same number of solutions as $ax^{m'} + by^{n'} \equiv c(p)$, where $m' = (m, p-1)$ and $n' = (n, p-1)$.

chapter 5

2: Show that the number of solutions to $x^2 \equiv a(p)$ is given by $1 + (a/p)$.

3: Suppose that $p \nmid a$. Show that the number of solutions to $ax^2+bx+c \equiv 0(p)$ is given by $1+\left(\left(b^2-4ac\right)/p\right)$.

4: Prove that $\sum_{a=1}^{p-1} (a/p) = 0$.

5: Prove that $\sum_{\substack{p-1 \\ x=0}} ((ax+b)/p) = 0$ provided that $p \nmid a$.

6: Show that the number of solutions to $x^2 - y^2 \equiv a(p)$ is given by $\sum_{y=0}^{p-1} \left(1 + \left(\left(y^2+a\right)/p\right)\right)$.

7: By calculating directly show that the number of solutions to $x^2 - y^2 \equiv a(p)$ is $p-1$ if $p \nmid a$ and $2p-1$ if $p \mid a$.

13: Show that any prime divisor of $x^4 - x^2 + 1$ is congruent to 1 modulo 12 .

27: Suppose that f is such that $b \equiv af(p)$. Show that $f^2 \equiv -1(p)$ and that $2^{(p-1)/4} \equiv f^{ab/2}(p)$

28: Show that $x^4 \equiv 2(p)$ has a solution for $p \equiv 1(4)$ iff p is of the form $A^2 + 64B^2$.

37: Show that if a is negative then $p \equiv q(4a), p \times a$ implies $(a/p) = (a/q)$.

chapter 6

18: Show that there exist algebraic numbers of arbitrarily high degree.

chapter 7

6: Let $K \supset F$ be finite fields with $[K : F] = 3$. Show that if $\alpha \in F$ is not a square in F, it is not a square in K.

24: Suppose that $f(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ has the property that $f(x+y) = f(x) + f(y) \in \mathbb{Z}/p\mathbb{Z}[x,y]$. Show that $f(x)$ must be of the form $a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_mx^{p^m}$.

chapter 12

12: Show that $\sin(\pi/12)$ is an algebraic number.

19: Show that a finite integral domain is a field.

22: Let $F \subset E$ be algebraic number fields. Show that any isomorphism of F into $\mathbb{C}$ extends in exactly $[E : F]$ ways to an isomorphism of E into $\mathbb{C}$.

30: Let p be an odd prime and consider $\mathbb{Q}(\sqrt{p})$. If $q \neq p$ is prime show that $\sigma_q(\sqrt{p}) = (p/q)\sqrt{p}$ where σ_q is the Frobenius automorphism at a prime ideal in $\mathbb{Q}(\sqrt{p})$ lying above q.

chapter 18

1: Show that $165x^2 - 21y^2 = 19$ has no integral solution.

4: Show that 1729 is the smallest positive integer expressible as the sum of two different integral cubes in two ways.

32: Let d be a square-free integer $d \equiv 1$ or 2 modulo 4 . Show that if x and y are integers such that $y^2 = x^3 - d$ then $(x, 2d) = 1$.

steinshakarchi

chapter 1

13: Suppose that f is holomorphic in an open set Ω. Prove that if $|f|$ is constant, then f is constant.

chapter 2

2: Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

9: Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0 \quad$ and $\quad \varphi'(z_0) = 1$ then φ is linear.

chapter 5

1: Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \ldots, z_n, \ldots$ are its zeros ($|z_k| < 1$), then $\sum_n (1 - |z_n|) < \infty$.

3: Show that $\sum \frac{z^n}{(n!)^\alpha}$ is an entire function of order $1/\alpha$.

1 cambridgetripos

2022

IA

1-II-9D-a: Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n} \sum_{k=1}^n a_k$ also converges and $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n a_k = \lim_{n\to\infty} a_n$.

1-II-10D-c: Let a function $g : (0, \infty) \to \mathbb{R}$ be continuous and bounded. Show that for every $T > 0$ there exists a sequence x_n such that $x_n \to \infty$ and $\lim_{n\to\infty} (g(x_n + T) - g(x_n)) = 0$.

4-I-1E-a: By considering numbers of the form $3p_1 \ldots p_k - 1$, show that there are infinitely many primes of the form $3n + 2$ with $n \in \mathbb{N}$.

4-I-2D-a: Prove that $\sqrt[3]{2} + \sqrt[3]{3}$ is irrational.

IB

3-II-11G-b: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $f(x, y) = \left(\frac{\cos x + \cos y - 1}{2}, \cos x - \cos y\right)$. Prove that f has a fixed point.

2018

IA

1-I-3E-b: Let $f : \mathbb{R} \to (0, \infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \to \infty$ as $n \to \infty$.

2 pugh

chapter 2

5: Prove that a set $U \subset M$ is open if and only if none of its points are limits of its complement.

11: Let $\mathcal{T}$ be the collection of open subsets of a metric space M, and $\mathcal{K}$ the collection of closed subsets. Show that there is a bijection from $\mathcal{T}$ onto $\mathcal{K}$.

13a: Show that every subset of $\mathbb{N}$ is clopen.

32a: Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ a bijection. The sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is called a rearrangement of (p_n). Show that if f is an injection, the limit of a sequence is unaffected by rearrangement.

32b: Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ a bijection. The sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is called a rearrangement of (p_n). Show that if f is a surjection, the limit of a sequence is unaffected by rearrangement.

38: Let $\| \ \|$ be any norm on $\mathbb{R}^m$ and let $B = \{x \in \mathbb{R}^m : \|x\| \le 1\}$. Prove that B is compact.

44: Suppose that M is compact and that $\mathcal{U}$ is an open covering of M which is "redundant" in the sense that each $p \in M$ is contained in at least two members of $\mathcal{U}$. Show that $\mathcal{U}$ reduces to a finite subcovering with the same property.

54: Show that if S is connected, it is not true in general that its interior is connected.

79: Prove that if M is nonempty compact, locally path-connected and connected then it is path-connected.

105: A metric on M is an ultrametric if for all $x, y, z \in M$, $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. Show that a metric space with an ultrametric is totally disconnected.

chapter 3

1: Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t-x|^2$ for all t, x. Prove that f is constant.

4: Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

14c-i: Show that the bump function $\beta(x) = e^2 e(1-x) \cdot e(x+1)$ is smooth.

14c-ii: Show that the bump function $\beta(x) = e^2 e(1-x) \cdot e(x+1)$ is identically 0 outside the interval $(-1, 1)$.

putnam

2021

b4: Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

2020

b5: B5 For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that $3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0$.

2018

a5: Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0, f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

b2: Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

b4: Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

b6: Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860 . Prove that the cardinality of S is at most $2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}$.

2017

b3: Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1 . Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

axler

chapter 2

1: Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$obtained by subtracting from each vector (except the last one) the following vector.

2: Prove that if $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ obtained by subtracting from each vector (except the last one) the following vector.

6: Prove that the real vector space consisting of all continuous realvalued functions on the interval $[0, 1]$ is infinite dimensional.

chapter 3

1: Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

8: Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap$ null $T = \{0\}$ and range $T = \{Tu : u \in U\}$.

9: Prove that if T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that null $T = \big\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2$ and $x_3 = 7x_4\big\}$, then T is surjective.

10: Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose mull space equals $\big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2$ and

11: Prove that if there exis ts a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

chapter 4

4: Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

chapter 5

1: Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \ldots, U_m$ are subspaces of V invariant under T, then $U_1 + \cdots + U_m$ is invariant under T.

4: Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null $(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

11: Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

12: Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

13: Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

20: Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

24: Suppose V is a real vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

chapter 6

2: Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if $\|u\| \leq \|u + av\|$for all $a \in \mathbf{F}$.

3: Prove that $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j {a_j}^2\right)\left(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\right)$ for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

7: Prove that if V is a complex inner-product space, then $\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$ for all $u, v \in V$.

13: Suppose $(e_1, \ldots, e_m)$ is an or thonormal list of vectors in V. Let $v \in V$. Prove that $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$ if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

16: Suppose U is a subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$

17: Prove that if $P \in \mathcal{C}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P, then P is an orthogonal projection.

18: Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $\|Pv\| \leq \|v\|$ for every $v \in V$, then P is an or thogonal projection.

19: Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

20: Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

29: Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

chapter 7

4: Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

5: Show that if $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

6: Prove that if $T \in \mathcal{L}(V)$ is normal, then range$T$ = rangeT^*.

8: Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}\left(\mathbf{R}^3\right)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$.

9: Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

10: Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

11: Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V\ )$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

14: Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

15: Suppose U is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that U has a basis consisting of eigenvectors of T if and only if there is an inner product on U that makes T into a self-adjoint operator.

17: Prove that the sum of any two positive operators on V is positive.

18: Prove that if $T \in \mathcal{L}(V)$ is positive, then so is T^k for every positive integer k.