Exercises from
Principles of Mathematical Analysis
by Walter Rudin

Exercise 1.1a If r is rational $(r \neq 0)$ and x is irrational, prove that $r + x$ is irrational.

Proof. If r and $r + x$ were both rational, then $x = r + x - r$ would also be rational. □

Exercise 1.1b If r is rational $(r \neq 0)$ and x is irrational, prove that rx is irrational.

Proof. If rx were rational, then $x = \frac{rx}{r}$ would also be rational. □

Exercise 1.2 Prove that there is no rational number whose square is 12.

Proof. Suppose $m^2 = 12n^2$, where m and n have no common factor. It follows that m must be even, and therefore n must be odd. Let $m = 2r$. Then we have $r^2 = 3n^2$, so that r is also odd. Let $r = 2s + 1$ and $n = 2t + 1$. Then

$$4s^2 + 4s + 1 = 3\left(4t^2 + 4t + 1\right) = 12t^2 + 12t + 3,$$

so that

$$4\left(s^2 + s - 3t^2 - 3t\right) = 2.$$

But this is absurd, since 2 cannot be a multiple of 4 . □

Exercise 1.4 Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since E is nonempty, there exists $x \in E$. Then by definition of lower and upper bounds we have $\alpha \leq x \leq \beta$, and hence by property *ii* in the definition of an ordering, we have $\alpha < \beta$ unless $\alpha = x = \beta$. □

Exercise 1.5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

Proof. We need to prove that $-\sup(-A)$ is the greatest lower bound of A. For brevity, let $\alpha = -\sup(-A)$. We need to show that $\alpha \leq x$ for all $x \in A$ and $\alpha \geq \beta$ if β is any lower bound of A.

Suppose $x \in A$. Then, $-x \in -A$, and, hence $-x \leq \sup(-A)$. It follows that $x \geq -\sup(-A)$, i.e., $\alpha \leq x$. Thus α is a lower bound of A.

Now let β be any lower bound of A. This means $\beta \leq x$ for all x in A. Hence $-x \leq -\beta$ for all $x \in A$, which says $y \leq -\beta$ for all $y \in -A$. This means $-\beta$ is an upper bound of $-A$. Hence $-\beta \geq \sup(-A)$ by definition of sup, i.e., $\beta \leq -\sup(-A)$, and so $-\sup(-A)$ is the greatest lower bound of A. □

Exercise 1.8 Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. By Part (a) of Proposition 1.18, either i or $-i$ must be positive. Hence $-1 = i^2 = (-i)^2$ must be positive. But then $1 = (-1)^2$, must also be positive, and this contradicts Part (a) of Proposition 1.18, since 1 and -1 cannot both be positive. □

Exercise 1.11a If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$.

Proof. If $z = 0$, we take $r = 0, w = 1$. (In this case w is not unique.) Otherwise we take $r = |z|$ and $w = z/|z|$, and these choices are unique, since if $z = rw$, we must have $r = r|w| = |rw| = |z|, z/r$ □

Exercise 1.12 If $z_1, \ldots, z_n$ are complex, prove that $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

Proof. We can apply the case $n = 2$ and induction on n to get

$$\begin{aligned} |z_1 + z_2 + \cdots z_n| &= |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\ &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n| \end{aligned}$$

□

Exercise 1.13 If x, y are complex, prove that $||x| - |y|| \leq |x - y|$.

Proof. Since $x = x - y + y$, the triangle inequality gives

$$|x| \leq |x - y| + |y|$$

so that $|x| - |y| \leq |x - y|$. Similarly $|y| - |x| \leq |x - y|$. Since $|x| - |y|$ is a real number we have either ——$x| - |y|| = |x| - |y|$ or ——$x| - |y|| = |y| - |x|$. In either case, we have shown that ——$x| - |y|| \leq |x - y|$. □

Exercise 1.14 If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute $|1+z|^2 + |1-z|^2$.

Proof. $|1+z|^2 = (1+z)(1+\bar{z}) = 1+\bar{z}+z+z\bar{z} = 2+z+\bar{z}$. Similarly $|1-z|^2 = (1-z)(1-\bar{z}) = 1-z-\bar{z}+z\bar{z} = 2-z-\bar{z}$. Hence

$$|1+z|^2 + |1-z|^2 = 4.$$

□

Exercise 1.16a Suppose $k \geq 3, x, y \in \mathbb{R}^k, |x-y| = d > 0$, and $r > 0$. Prove that if $2r > d$, there are infinitely many $z \in \mathbb{R}^k$ such that $|z-x| = |z-y| = r$.

Proof. (a) Let w be any vector satisfying the following two equations:

$$\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0,$$
$$|\mathbf{w}|^2 = r^2 - \frac{d^2}{4}.$$

From linear algebra it is known that all but one of the components of a solution $\mathbf{w}$ of the first equation can be arbitrary. The remaining component is then uniquely determined. Also, if w is any non-zero solution of the first equation, there is a unique positive number t such that t w satisfies both equations. (For example, if $x_1 \neq y_1$, the first equation is satisfied whenever

$$z_1 = \frac{z_2\,(x_2-y_2) + \cdots + z_k\,(x_k-y_k)}{y_1-x_1}.$$

If $(z_1, z_2, \ldots, z_k)$ satisfies this equation, so does $(tz_1, tz_2, \ldots, tz_k)$ for any real number t.) Since at least two of these components can vary independently, we can find a solution with these components having any prescribed ratio. This ratio does not change when we multiply by the positive number t to obtain a solution of both equations. Since there are infinitely many ratios, there are infinitely many distinct solutions. For each such solution $\mathbf{w}$ the vector $\mathbf{z} = \frac{1}{2}\mathrm{x} + \frac{1}{2}\mathrm{y} + \mathrm{w}$ is a solution of the required equation. For

$$\begin{aligned}
|\mathrm{z} - \mathbf{x}|^2 &= \left|\frac{\mathbf{y}-\mathbf{x}}{2} + \mathbf{w}\right|^2 \\
&= \left|\frac{\mathbf{y}-\mathbf{x}}{2}\right|^2 + 2\mathbf{w}\cdot\frac{\mathbf{x}-\mathbf{y}}{2} + |\mathbf{w}|^2 \\
&= \frac{d^2}{4} + 0 + r^2 - \frac{d^2}{4} \\
&= r^2
\end{aligned}$$

and a similar relation holds for $|z-y|^2$. □

Exercise 1.17 Prove that $|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$ if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$.

Proof. The proof is a routine computation, using the relation

$$|x \pm y|^2 = (x \pm y) \cdot (x \pm y) = |x|^2 \pm 2x \cdot y + |y|^2.$$

If x and y are the sides of a parallelogram, then $\mathrm{x}+\mathrm{y}$ and $\mathbf{x}-\mathrm{y}$ are its diagonals. Hence this result says that the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides. □

Exercise 1.18a If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

Proof. If $\mathbf{x}$ has any components equal to 0 , then $\mathbf{y}$ can be taken to have the corresponding components equal to 1 and all others equal to 0 . If all the components of $\mathbf{x}$ are nonzero, $\mathbf{y}$ can be taken as $(-x_2, x_1, 0, \ldots, 0)$. This is, of course, not true when $k=1$, since the product of two nonzero real numbers is nonzero. □

Exercise 1.18b If $k=1$ and $\mathbf{x} \in R^k$, prove that there does not exist $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

Proof. Not true when $k=1$, since the product of two nonzero real numbers is nonzero. □

Exercise 1.19 Suppose $a, b \in R^k$. Find $c \in R^k$ and $r > 0$ such that $|x-a| = 2|x-b|$ if and only if $|x-c| = r$. Prove that $3c = 4b - a$ and $3r = 2|b-a|$.

Proof. Since the solution is given to us, all we have to do is verify it, i.e., we need to show that the equation

$$|\mathrm{x}-\mathrm{a}| = 2|\mathrm{x}-\mathrm{b}|$$

is equivalent to $|\mathrm{x}-\mathbf{c}| = r$, which says

$$\left|\mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\right| = \frac{2}{3}|\mathbf{b}-\mathbf{a}|.$$

If we square both sides of both equations, we an equivalent pair of equations, the first of which reduces to

$$3|\mathbf{x}|^2 + 2\mathbf{a} \cdot \mathbf{x} - 8\mathbf{b} \cdot \mathbf{x} - |\mathbf{a}|^2 + 4|\mathbf{b}|^2 = 0,$$

and the second of which reduces to this equation divided by 3 . Hence these equations are indeed equivalent. □

Exercise 2.19a If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof. We are given that $A \cap B = \varnothing$. Since A and B are closed, this means $A \cap \bar{B} = \varnothing = \bar{A} \cap B$, which says that A and B are separated. □

Exercise 2.24 Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Exercise 2.25 Prove that every compact metric space K has a countable base.

Proof. It is easier simply to refer to the previous problem. The hint shows that K can be covered by a finite union of neighborhoods of radius $1/n$, and the previous problem shows that this implies that K is separable.

It is not entirely obvious that a metric space with a countable base is separable. To prove this, let $\{V_n\}_{n=1}^{\infty}$ be a countable base, and let $x_n \in V_n$. The points V_n must be dense in X. For if G is any non-empty open set, then G contains V_n for some n, and hence $x_n \in G$. (Thus for a metric space, having a countable base and being separable are equivalent.) □

Exercise 2.27a Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that P is perfect.

Proof. We see that $E \cap W$ is at most countable, being a countable union of at-most-countable sets. It remains to show that $P = W^c$, and that P is perfect. □

Exercise 2.27b Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that at most countably many points of E are not in P.

Proof. If $x \in W^c$, and O is any neighborhood of x, then $x \in V_n \subseteq O$ for some n. Since $x \notin W, V_n \cap E$ is uncountable. Hence O contains uncountably many points of E, and so x is a condensation point of E. Thus $x \in P$, i.e., $W^c \subseteq P$. Conversely if $x \in W$, then $x \in V_n$ for some V_n such that $V_n \cap E$ is countable. Hence x has a neighborhood (any neighborhood contained in V_n) containing at most a countable set of points of E, and so $x \notin P$, i.e., $W \subseteq P^c$. Hence $P = W^c$. It is clear that P is closed (since its complement W is open), so that we need only show that $P \subseteq P'$. Hence suppose $x \in P$, and O is any neighborhood of x. (By definition of P this means $O \cap E$ is uncountable.) We need to show that there is a point $y \in P \cap (O \backslash \{x\})$. If this is not the case, i.e., if every point y in $O \backslash \{x\}$ is in P^c, then for each such point y there is a set V_n containing y such that $V_n \cap E$ is at most countable. That would mean that $y \in W$, i.e., that $O \backslash \{x\}$ is contained in W. It would follow that $O \cap E \subseteq \{x\} \cup (W \cap E)$, and so $O \cap E$ contains at most a countable set of points, contrary to the hypothesis that $x \in P$. Hence O contains a point of P different from x, and so $P \subseteq P'$. Thus P is perfect. □

Exercise 2.28 Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable.

Proof. If E is closed, it contains all its limit points, and hence certainly all its condensation points. Thus $E = P \cup (E \backslash P)$, where P is perfect (the set of all condensation points of E), and $E \backslash P$ is at most countable.

Since a perfect set in a separable metric space has the same cardinality as the real numbers, the set P must be empty if E is countable. The at-mostcountable set $E \backslash P$ cannot be perfect, hence must have isolated points if it is nonempty. $\square$

Exercise 2.29 Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Proof. Let O be open. For each pair of points $x \in O, y \in O$, we define an equivalence relation $x \sim y$ by saying $x \sim y$ if and only if $[\min(x,y), \max(x,y)] \subset 0$. This is an equivalence relation, since $x \sim x([x,x] \subset O$ if $x \in O)$; if $x \sim y$, then $y \sim x$ (since $\min(x,y) = \min(y,x)$ and $\max(x,y) = \max(y,x)$); and if $x \sim y$ and $y \sim z$, then $x \sim z([\min(x,z), \max(x,z)] \subseteq [\min(x,y), \max(x,y)] \cup [\min(y,z), \max(y,z)] \subseteq O)$. In fact it is easy to prove that

$$\min(x,z) \geq \min(\min(x,y), \min(y,z))$$

and

$$\max(x,z) \leq \max(\max(x,y), \max(y,z))$$

It follows that O can be written as a disjoint union of pairwise disjoint equivalence classes. We claim that each equivalence class is an open interval.

To show this, for each $x \in O$; let $A = \{z : [z,x] \subseteq O\}$ and $B = \{z : [x,z] \subseteq O\}$, and let $a = \inf A, b = \sup B$. We claim that $(a,b) \subset O$. Indeed if $a < z < b$, there exists $c \in A$ with $c < z$ and $d \in B$ with $d > z$. Then $z \in [c,x] \cup [x,d] \subseteq O$. We now claim that (a,b) is the equivalence class containing x. It is clear that each element of (a,b) is equivalent to x by the way in which a and b were chosen. We need to show that if $z \notin (a,b)$, then z is not equivalent to x. Suppose that $z < a$. If z were equivalent to x, then $[z,x]$ would be contained in O, and so we would have $z \in A$. Hence a would not be a lower bound for A. Similarly if $z > b$ and $z \sim x$, then b could not be an upper bound for B.

We have now established that O is a union of pairwise disjoint open intervals. Such a union must be at most countable, since each open interval contains a rational number not in any other interval. $\square$

Exercise 3.1a Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

Proof. Let $\varepsilon > 0$. Since the sequence $\{s_n\}$ is a Cauchy sequence, there exists N such that $|s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. We then have $||s_m| - |s_n|| \leq |s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. Hence the sequence $\{|s_n|\}$ is also a Cauchy sequence, and therefore must converge. $\square$

Exercise 3.2a Prove that $\lim_{n\to\infty} \sqrt{n^2+n} - n = 1/2$.

Proof. Multiplying and dividing by $\sqrt{n^2+n}+n$ yields

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

It follows that the limit is $\frac{1}{2}$. $\square$

Exercise 3.3 If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ $(n = 1, 2, 3, \ldots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

Proof. Since $\sqrt{2} < 2$, it is manifest that if $s_n < 2$, then $s_{n+1} < \sqrt{2+2} = 2$. Hence it follows by induction that $\sqrt{2} < s_n < 2$ for all n. In view of this fact,it also follows that $(s_n - 2)(s_n + 1) < 0$ for all $n > 1$, i.e., $s_n > s_n^2 - 2 = s_{n-1}$. Hence the sequence is an increasing sequence that is bounded above (by 2) and so converges. Since the limit s satisfies $s^2 - s - 2 = 0$, it follows that the limit is 2. $\square$

Exercise 3.5 For any two real sequences $\{a_n\}, \{b_n\}$, prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.

Proof. Since the case when $\limsup_{n\to\infty} a_n = +\infty$ and $\limsup_{n\to\infty} b_n = -\infty$ has been excluded from consideration, we note that the inequality is obvious if $\limsup_{n\to\infty} a_n = +\infty$. Hence we shall assume that $\{a_n\}$ is bounded above.

Let $\{n_k\}$ be a subsequence of the positive integers such that $\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \limsup_{n\to\infty}(a_n + b_n)$. Then choose a subsequence of the positive integers $\{k_m\}$ such that

$$\lim_{m\to\infty} a_{n_{k_m}} = \limsup_{k\to\infty} a_{n_k}.$$

The subsequence $a_{n_{k_m}} + b_{n_{k_m}}$ still converges to the same limit as $a_{n_k} + b_{n_k}$, i.e., to $\limsup_{n\to\infty}(a_n + b_n)$. Hence, since a_{n_k} is bounded above (so that $\limsup_{k\to\infty} a_{n_k}$ is finite), it follows that $b_{n_{k_m}}$ converges to the difference

$$\lim_{m\to\infty} b_{n_{k_m}} = \lim_{m\to\infty}\left(a_{n_{k_m}} + b_{n_{k_m}}\right) - \lim_{m\to\infty} a_{n_{k_m}}.$$

Thus we have proved that there exist subsequences $\left\{a_{n_{k_m}}\right\}$ and $\left\{b_{n_{k_m}}\right\}$ which converge to limits a and b respectively such that $a + b = \limsup_{n\to\infty}(a_n + b_n^*)$. Since a is the limit of a subsequence of $\{a_n\}$ and b is the limit of a subsequence of $\{b_n\}$, it follows that $a \leq \limsup_{n\to\infty} a_n$ and $b \leq \limsup_{n\to\infty} b_n$, from which the desired inequality follows. $\square$

Exercise 3.6a Prove that $\lim_{n\to\infty} \sum_{i<n} a_i = \infty$, where $a_i = \sqrt{i+1} - \sqrt{i}$.

Proof. (a) Multiplying and dividing a_n by $\sqrt{n+1} + \sqrt{n}$, we find that $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which is larger than $\frac{1}{2\sqrt{n+1}}$. The series $\sum a_n$ therefore diverges by comparison with the p series $\left(p = \frac{1}{2}\right)$. $\square$

Exercise 3.7 Prove that the convergence of Σa_n implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

Proof. Since $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$, it follows that

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right).$$

Now Σa_n^2 converges by comparison with Σa_n (since Σa_n converges, we have $a_n < 1$ for large n, and hence $a_n^2 < a_n$). Since $\Sigma \frac{1}{n^2}$ also converges (p series, $p = 2$), it follows that $\Sigma \frac{\sqrt{a_n}}{n}$ converges. $\square$

Exercise 3.8 If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

Proof. We shall show that the partial sums of this series form a Cauchy sequence, i.e., given $\varepsilon > 0$ there exists N such that $\left|\sum_{k=m+1}^{n} a_k b_k\right| \langle \varepsilon \text{ if } n\rangle\ m \geq N$. To do this, let $S_n = \sum_{k=1}^{n} a_k\ (S_0 = 0)$, so that $a_k = S_k - S_{k-1}$ for $k = 1, 2, \ldots$ Let M be an uppper bound for both $|b_n|$ and $|S_n|$, and let $S = \sum a_n$ and $b = \lim b_n$. Choose N so large that the following three inequalities hold for all $m > N$ and $n > N$:

$$|b_n S_n - bS| < \frac{\varepsilon}{3}; \quad |b_m S_m - bS| < \frac{\varepsilon}{3}; \quad |b_m - b_n| < \frac{\varepsilon}{3M}.$$

Then if $n > m > N$, we have, from the formula for summation by parts,

$$\sum_{k=m+1}^{n} a_n b_n = b_n S_n - b_m S_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k$$

Our assumptions yield immediately that $|b_n S_n - b_m S_m| < \frac{2\varepsilon}{3}$, and

$$\left|\sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k\right| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}|.$$

Since the sequence $\{b_n\}$ is monotonic, we have

$$\sum_{k=m}^{n-1} |b_k - b_{k+1}| = \left|\sum_{k=m}^{n-1} (b_k - b_{k+1})\right| = |b_m - b_n| < \frac{\varepsilon}{3M},$$

from which the desired inequality follows. $\square$

Exercise 3.13 Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Proof. Since both the hypothesis and conclusion refer to absolute convergence, we may assume both series consist of nonnegative terms. We let $S_n = \sum_{k=0}^{n} a_n, T_n = \sum_{k=0}^{n} b_n$, and $U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l}$. We need to show that U_n remains bounded, given that S_n and T_n are bounded. To do this we make the convention that $a_{-1} = T_{-1} = 0$, in order to save ourselves from having to separate off the first and last terms when we sum by parts. We then have

$$\begin{aligned}
U_n &= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l} \\
&= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l \left(T_{k-l} - T_{k-l-1}\right) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} a_{k-j} \left(T_j - T_{j-1}\right) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} \left(a_{k-j} - a_{k-j-1}\right) T_j \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} \left(a_{k-j} - a_{k-j-1}\right) T_j \quad = \sum_{j=0}^{n} a_{n-j} T_j \\
&\leq T \sum_{m=0}^{n} a_m \\
&= T S_n \\
&\leq ST.
\end{aligned}$$

Thus U_n is bounded, and hence approaches a finite limit.

□

Exercise 3.20 Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some sequence $\{p_{nl}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Proof. Let $\varepsilon > 0$. Choose N_1 so large that $d\left(p_m, p_n\right) < \frac{\varepsilon}{2}$ if $m > N_1$ and $n > N_1$. Then choose $N \geq N_1$ so large that $d\left(p_{n_k}, p\right) < \frac{\varepsilon}{2}$ if $k > N$. Then if $n > N$, we have

$$d\left(p_n, p\right) \leq d\left(p_n, p_{n_{N+1}}\right) + d\left(p_{n_{N+1}}, p\right) < \varepsilon$$

For the first term on the right is less than $\frac{\varepsilon}{2}$ since $n > N_1$ and $n_{N+1} > N + 1 > N_1$. The second term is less than $\frac{\varepsilon}{2}$ by the choice of N. □

Exercise 3.21 If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^{\infty} E_n$ consists of exactly one point.

Proof. Choose $x_n \in E_n$. (We use the axiom of choice here.) The sequence $\{x_n\}$ is a Cauchy sequence, since the diameter of E_n tends to zero as n tends to infinity and E_n contains E_{n+1}. Since the metric space X is complete, the sequence x_n converges to a point x, which must belong to E_n for all n, since E_n is closed and contains x_m for all $m \geq n$. There cannot be a second point y in all of the E_n, since for any point $y \neq x$ the diameter of E_n is less than $d(x, y)$ for large n. $\square$

Exercise 3.22 Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open sets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty.

Proof. Let F_n be the complement of G_n, so that F_n is closed and contains no open sets. We shall prove that any nonempty open set U contains a point not in any F_n, hence in all G_n. To this end, we note that U is not contained in F_1, so that there is a point $x_1 \in U \backslash F_1$. Since $U \backslash F_1$ is open, there exists $r_1 > 0$ such that B_1, defined as the open ball of radius r_1 about x_1, is contained in $U \backslash F_1$. Let E_1 be the open ball of radius $\frac{r_1}{2}$ about x_1, so that the closure of E_1 is contained in B_1. Now F_2 does not contain E_1, and so we can find a point $x_2 \in E_1 \backslash F_2$. Since $E_1 \backslash F_2$ is an open set, there exists a positive number r_2 such that B_2, the open ball of radius R_2 about x_2, is contained in $E_1 \backslash F_2$, which in turn is contained in $U \backslash (F_1 \cup F_2)$. We let E_2 be the open ball of radius $\frac{r_2}{2}$ about x_2, so that $\bar{E}_2 \subseteq B_2$. Proceeding in this way, we construct a sequence of open balls E_j, such that $E_j \supseteq \bar{E}_{j+1}$, and the diameter of E_j tends to zero. By the previous exercise, there is a point x belonging to all the sets $\bar{E}_j$, hence to all the sets $U \backslash (F_1 \cup F_2 \cup \cdots \cup F_n)$. Thus the point x belongs to $U \cap (\cap_1^\infty G_n)$. $\square$

Exercise 4.1a Suppose f is a real function defined on $\mathbb{R}$ which satisfies $\lim_{h \to 0} f(x+h) - f(x-h) = 0$ for every $x \in \mathbb{R}$. Show that f does not need to be continuous.

Proof.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

(If x is an integer, then $f(x+h) - f(x-h) \equiv 0$ for all h; while if x is not an integer, $f(x+h) - f(x-h) = 0$ for $|h| < \min(x - [x], 1 + [x] - x)$. $\square$

Exercise 4.2a If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$. ($\overline{E}$ denotes the closure of E).

Proof. Let $x \in \bar{E}$. We need to show that $f(x) \in \overline{f(E)}$. To this end, let O be any neighborhood of $f(x)$. Since f is continuous, $f^{-1}(O)$ contains (is) a neighborhood of x. Since $x \in \bar{E}$, there is a point u of E in $f^{-1}(O)$. Hence

$\frac{f(u)}{f(E)} \in O \cap f(E)$. Since O was any neighborhood of $f(x)$, it follows that $f(x) \in \overline{f(E)}$ ☐

Exercise 4.3 Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof. $Z(f) = f^{-1}(\{0\})$, which is the inverse image of a closed set. Hence $Z(f)$ is closed. ☐

Exercise 4.4a Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$.

Proof. To prove that $f(E)$ is dense in $f(X)$, simply use that $f(X) = f(\bar{E}) \subseteq \overline{f(E)}$. ☐

Exercise 4.4b Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that if $g(p) = f(p)$ for all $p \in P$ then $g(p) = f(p)$ for all $p \in X$.

Proof. The function $\varphi : X \to R^1$ given by

$$\varphi(p) = d_Y(f(p), g(p))$$

is continuous, since

$$|d_Y(f(p), g(p)) - d_Y(f(q), g(q))| \leq d_Y(f(p), f(q)) + d_Y(g(p), g(q))$$

(This inequality follows from the triangle inequality, since

$$d_Y(f(p), g(p)) \leq d_Y(f(p), f(q)) + d_Y(f(q), g(q)) + d_Y(g(q), g(p)),$$

and the same inequality holds with p and q interchanged. The absolute value $|d_Y(f(p), g(p)) - d_Y(f(q), g(q))|$ must be either $d_Y(f(p), g(p)) - d_Y(f(q), g(q))$ or $d_Y(f(q), g(q)) - d_Y(f(p), g(p))$, and the triangle inequality shows that both of these numbers are at most $d_Y(f(p), f(q)) + d_Y(g(p), g(q))$.) By the previous problem, the zero set of φ is closed. But by definition

$$Z(\varphi) = \{p : f(p) = g(p)\}.$$

Hence the set of p for which $f(p) = g(p)$ is closed. Since by hypothesis it is dense, it must be X. ☐

Exercise 4.5a If f is a real continuous function defined on a closed set $E \subset \mathbb{R}$, prove that there exist continuous real functions g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$.

Exercise 4.5b Show that there exist a set $E \subset \mathbb{R}$ and a real continuous function f defined on E, such that there does not exist a continuous real function g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$.

Exercise 4.6 If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof. Let Y be the co-domain of the function f. We invent a new metric space $E \times Y$ as the set of pairs of points $(x, y), x \in E, y \in Y$, with the metric $\rho\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = d_E\left(x_1, x_2\right) + d_Y\left(y_1, y_2\right)$. The function $\varphi(x) = (x, f(x))$ is then a mapping of E into $E \times Y$.

We claim that the mapping φ is continuous if f is continuous. Indeed, let $x \in X$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $d_Y(f(x), f(u)) < \frac{\varepsilon}{2}$ if $d_E(x, y) < \eta$. Then let $\delta = \min\left(\eta, \frac{\varepsilon}{2}\right)$. It is easy to see that $\rho(\varphi(x), \varphi(u)) < \varepsilon$ if $d_E(x, u) < \delta$. Conversely if φ is continuous, it is obvious from the inequality $\rho(\varphi(x), \varphi(u)) \geq d_Y(f(x), f(u))$ that f is continuous.

From these facts we deduce immediately that the graph of a continuous function f on a compact set E is compact, being the image of E under the continuous mapping φ. Conversely, if f is not continuous at some point x, there is a sequence of points x_n converging to x such that $f\left(x_n\right)$ does not converge to $f(x)$. If no subsequence of $f\left(x_n\right)$ converges, then the sequence $\left\{\left(x_n, f\left(x_n\right)\right\}_{n=1}^{\infty}$ has no convergent subsequence, and so the graph is not compact. If some subsequence of $f\left(x_n\right)$ converges, say $f\left(x_{n_k}\right) \to z$, but $z \neq f(x)$, then the graph of f fails to contain the limit point (x, z), and hence is not closed. A fortiori it is not compact. □

Exercise 4.8a Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

Exercise 4.8b Let E be a bounded set in R^1. Prove that there exists a real function f such that f is uniformly continuous and is not bounded on E.

Exercise 4.11a Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\left\{f\left(x_n\right)\right\}$ is a Cauchy sequence in Y for every Cauchy sequence $\left\{x_n\right\}$ in X.

Exercise 4.12 A uniformly continuous function of a uniformly continuous function is uniformly continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous. Then $g \circ f : X \to Z$ is uniformly continuous, where $g \circ f(x) = g(f(x))$ for all $x \in X$. To prove this fact, let $\varepsilon > 0$ be given. Then, since g is uniformly continuous, there exists $\eta > 0$ such that $d_Z(g(u), g(v)) < \varepsilon$ if $d_Y(u, v) < \eta$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \eta$ if $d_X(x, y) < \delta$

It is then obvious that $d_Z(g(f(x)), g(f(y))) < \varepsilon$ if $d_X(x, y) < \delta$, so that $g \circ f$ is uniformly continuous. $\square$

Exercise 4.14 Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof. If $f(0) = 0$ or $f(1) = 1$, we are done. If not, then $0 < f(0)$ and $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the intermediate value theorem, there must be a point $x \in (0, 1)$ where $g(x) = 0$ $\square$

Exercise 4.15 Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof. Suppose f is continuous and not monotonic, say there exist points $a < b < c$ with $f(a) < f(b)$, and $f(c) < f(b)$. Then the maximum value of f on the closed interval $[a, c]$ is assumed at a point u in the open interval (a, c). If there is also a point v in the open interval (a, c) where f assumes its minimum value on $[a, c]$, then $f(a, c) = [f(v), f(u)]$. If no such point v exists, then $f(a, c) = (d, f(u)]$, where $d = \min(f(a), f(c))$. In either case, the image of (a, c) is not open. $\square$

Exercise 4.19 Suppose f is a real function with domain R^1 which has the intermediate value property: if $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Proof. The contradiction is evidently that x_0 is a limit point of the set of t such that $f(t) = r$, yet, x_0 does not belong to this set. This contradicts the hypothesis that the set is closed. $\square$

Exercise 4.21a Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

Exercise 4.24 Assume that f is a continuous real function defined in (a, b) such that $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in (a, b)$. Prove that f is convex.

Proof. We shall prove that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all "dyadic rational" numbers, i.e., all numbers of the form $\lambda = \frac{k}{2^n}$, where k is a nonnegative integer not larger than 2^n. We do this by induction on n. The case $n = 0$ is trivial (since $\lambda = 0$ or $\lambda = 1$). In the case $n = 1$ we have $\lambda = 0$ or $\lambda = 1$ or $\lambda = \frac{1}{2}$. The first two cases are again trivial, and the third is precisely the hypothesis of the theorem. Suppose the result is proved for $n \leq r$, and consider $\lambda = \frac{k}{2^{r+1}}$. If k is even, say $k = 2l$, then $\frac{k}{2^{r+1}} = \frac{l}{2^r}$, and we can appeal to the induction hypothesis. Now suppose k is odd. Then $1 \leq k \leq 2^{r+1} - 1$, and so the numbers $l = \frac{k-1}{2}$ and $m = \frac{k+1}{2}$ are integers with $0 \leq l < m \leq 2^r$. We can now write

$$\lambda = \frac{s+t}{2},$$

where $s = \frac{k-1}{2^{r+1}} = \frac{l}{2^r}$ and $t = \frac{k+1}{2^{r+1}} = \frac{m}{2^r}$. We then have

$$\lambda x + (1-\lambda)y = \frac{[sx + (1-s)y] + [tx + (1-t)y]}{2}$$

Hence by the hypothesis of the theorem and the induction hypothesis we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \frac{f(sx + (1-s)y) + f(tx + (1-t)y)}{2} \\ &\leq \frac{sf(x) + (1-s)f(y) + tf(x) + (1-t)f(y)}{2} \\ &= \left(\frac{s+t}{2}\right) f(x) + \left(1 - \frac{s+t}{2}\right) f(y) \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

This completes the induction. Now for each fixed x and y both sides of the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

are continuous functions of λ. Hence the set on which this inequality holds (the inverse image of the closed set $[0, \infty)$ under the mapping $\lambda \mapsto \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)$) is a closed set. Since it contains all the points $\frac{k}{2^n}$, $0 \leq k \leq n, n = 1, 2, \ldots$, it must contain the closure of this set of points, i.e., it must contain all of $[0, 1]$. Thus f is convex. □

Exercise 5.1 Let f be defined for all real x, and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all real x and y. Prove that f is constant.

Proof. Dividing by $x - y$, and letting $x \to y$, we find that $f'(y) = 0$ for all y. Hence f is constant. □

Exercise 5.2 Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b)$.

Proof. For any c, d with $a < c < d < b$ there exists a point $p \in (c, d)$ such that $f(d) - f(c) = f'(p)(d - c) > 0$. Hence $f(c) < f(d)$

We know from Theorem 4.17 that the inverse function g is continuous. (Its restriction to each closed subinterval $[c, d]$ is continuous, and that is sufficient.) Now observe that if $f(x) = y$ and $f(x + h) = y + k$, we have

$$\frac{g(y+k) - g(y)}{k} - \frac{1}{f'(x)} = \frac{1}{\frac{f(x+h) - f(x)}{h}} - \frac{1}{f'(x)}$$

Since we know $\lim \frac{1}{\varphi(t)} = \frac{1}{\lim \varphi(t)}$ provided $\lim \varphi(t) \neq 0$, it follows that for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left| \frac{1}{\frac{f(x+h)-f(x)}{h}} - \frac{1}{f'(x)} \right| < \varepsilon$$

if $0 < |h| < \eta$. Since $h = g(y+k) - g(y)$, there exists $\delta > 0$ such that $0 < |h| < \eta$ if $0 < |k| < \delta$. The proof is now complete. □

Exercise 5.3 Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

Proof. If $0 < \varepsilon < \frac{1}{M}$, we certainly have

$$f'(x) \geq 1 - \varepsilon M > 0,$$

and this implies that $f(x)$ is one-to-one, by the preceding problem. □

Exercise 5.4 If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$p(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1},$$

whose derivative is

$$p'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n.$$

It is obvious that $p(0) = 0$, and the hypothesis of the problem is that $p(1) = 0$. Hence Rolle's theorem implies that $p'(x) = 0$ for some x between 0 and 1 . □

Exercise 5.5 Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. Let $\varepsilon > 0$. Choose x_0 such that $|f'(x)| < \varepsilon$ if $x > x_0$. Then for any $x \geq x_0$ there exists $x_1 \in (x, x+1)$ such that

$$f(x+1) - f(x) = f'(x_1).$$

Since $|f'(x_1)| < \varepsilon$, it follows that $|f(x+1) - f(x)| < \varepsilon$, as required. □

Exercise 5.6 Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$ $(x > 0)$ and prove that g is monotonically increasing.

Proof. Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing. By the mean-value theorem

$$f(x) = f(x) - f(0) = f'(c)x$$

for some $c \in (0, x)$. Since f' is monotonically increasing, this result implies that $f(x) < xf'(x)$. It therefore follows that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0,$$

so that g is also monotonically increasing. □

Exercise 5.7 Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

Proof. Since $f(x) = g(x) = 0$, we have

$$\begin{aligned}
\lim_{t \to x} \frac{f(t)}{g(t)} &= \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} \\
&= \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} \\
&= \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

Exercise 5.15 Suppose $a \in R^1, f$ is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$.

Proof. The inequality is obvious if $M_0 = +\infty$ or $M_2 = +\infty$, so we shall assume that M_0 and M_2 are both finite. We need to show that

$$|f'(x)| \leq 2\sqrt{M_0M_2}$$

for all $x > a$. We note that this is obvious if $M_2 = 0$, since in that case $f'(x)$ is constant, $f(x)$ is a linear function, and the only bounded linear function is a constant, whose derivative is zero. Hence we shall assume from now on that

$0 < M_2 < +\infty$ and $0 < M_0 < +\infty$. Following the hint, we need only choose $h = \sqrt{\frac{M_0}{M_2}}$, and we obtain

$$|f'(x)| \leq 2\sqrt{M_0 M_2},$$

which is precisely the desired inequality. The case of equality follows, since the example proposed satisfies

$$f(x) = 1 - \frac{2}{x^2+1}$$

for $x \geq 0$. We see easily that $|f(x)| \leq 1$ for all $x > -1$. Now $f'(x) = \frac{4x}{(x^2+1)^2}$ for $x > 0$ and $f'(x) = 4x$ for $x < 0$. It thus follows from Exercise 9 above that $f'(0) = 0$, and that $f'(x)$ is continuous. Likewise $f''(x) = 4$ for $x < 0$ and $f''(x) = \frac{4-4x^2}{(x^2+1)^3} = -4\frac{x^2-1}{(x^2+1)^3}$. This shows that $|f''(x)| < 4$ for $x > 0$ and also that $\lim_{x\to 0} f''(x) = 4$. Hence Exercise 9 again implies that $f''(x)$ is continuous and $f''(0) = 4$.

On n-dimensional space let $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))$, $M_0 = \sup|\mathbf{f}(x)|$, $M_1 = \sup|\mathbf{f}'(x)|$, and $M_2 = \sup|\mathbf{f}''(x)|$. Just as in the numerical case, there is nothing to prove if $M_2 = 0$ or $M_0 = +\infty$ or $M_2 = +\infty$, and so we assume $0 < M_0 < +\infty$ and $0 < M_2 < \infty$. Let a be any positive number less than M_1, let x_0 be such that $|\mathbf{f}'(x_0)| > a$, and let $\mathbf{u} = \frac{1}{|\mathbf{f}'(x_0)|}\mathbf{f}'(x_0)$. Consider the real-valued function $\varphi(x) = \mathrm{u}\cdot\mathrm{f}(x)$. Let N_0, N_1, and N_2 be the suprema of $|\varphi(x)|$, $|\varphi'(x)|$, and $|\varphi''(x)|$ respectively. By the Schwarz inequality we have (since $|\mathbf{u}| = 1$)$N_0 \leq M_0$ and $N_2 \leq M_2$, while $N_1 \geq \varphi(x_0) = |\mathbf{f}'(x_0)| > a$. We therefore have $a^2 < 4N_0N_2 \leq 4M_0M_2$. Since a was any positive number less than M_1, we have $M_1^2 \leq 4M_0M_2$, i.e., the result holds also for vector-valued functions.

Equality can hold on any R^n, as we see by taking $\mathbf{f}(x) = (f(x), 0, \ldots, 0)$ or $\mathbf{f}(x) = (f(x), f(x), \ldots, f(x))$, where $f(x)$ is a real-valued function for which equality holds.

□

Exercise 5.17 Suppose f is a real, three times differentiable function on $[-1,1]$, such that $f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0$. Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

Proof. Following the hint, we observe that Theorem 5.15 (Taylor's formula with remainder) implies that

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s)$$
$$f(-1) = f(0) - f'(0) + \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t)$$

for some $s \in (0,1), t \in (-1,0)$. By subtracting the second equation from the first and using the given values of $f(1), f(-1)$, and $f'(0)$, we obtain

$$1 = \frac{1}{6}\left(f^{(3)}(s) + f^{(3)}(t)\right),$$

which is the desired result. Note that we made no use of the hypothesis $f(0) = 0$. $\square$