# ProofNet NL Statements

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### chapter 1

#### section 1

15: Prove that $(a_1 a_2 \dots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1}$ for all $a_1, a_2, \dots, a_n \in G$.

16: Let $x$ be an element of $G$. Prove that $x^2 = 1$ if and only if $x|x|$ is either 1 or 2.

17: Let $x$ be an element of $G$. Prove that if $|x| = n$ for some positive integer $n$ then $x^{-1} = x^{n-1}$.

18: Let $x$ and $y$ be elements of $G$. Prove that $xy = yx$ if and only if $y^{-1}xy = x$ if and only if $x^{-1}y^{-1}xy = 1$.

20: For $x$ an element in $G$ show that $x$ and $x^{-1}$ have the same order.

22a: If $x$ and $g$ are elements of the group $G$, prove that $|x| = \left|g^{-1}xg\right|$.

22b: Deduce that $|ab| = |ba|$ for all $a, b \in G$.

25: Prove that if $x^2 = 1$ for all $x \in G$ then $G$ is abelian.

29: Prove that $A \times B$ is an abelian group if and only if both $A$ and $B$ are abelian.

34: If $x$ is an element of infinite order in $G$, prove that the elements $x^n, n \in \mathbb{Z}$ are all distinct.

#### section 3

8: Prove that if $\Omega = \{1, 2, 3, \dots\}$ then $S_\Omega$ is an infinite group

#### section 6

4: Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.

11: Let $A$ and $B$ be groups. Prove that $A \times B \cong B \times A$.

12: Let $A$ and $B$ be groups. Prove that $A \times B \cong B \times A$.

17: Let $G$ be any group. Prove that the map from $G$ to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if $G$ is abelian.

23: Let $G$ be a finite group which possesses an automorphism $\sigma$ such that $\sigma(g) = g$ if and only if $g = 1$. If $\sigma^2$ is the identity map from $G$ to $G$, prove that $G$ is abelian.

### section 7

5: Prove that the kernel of an action of the group $G$ on a set $A$ is the same as the kernel of the corresponding permutation representation $G \to S_A$.

6: Prove that a group $G$ acts faithfully on a set $A$ if and only if the kernel of the action is the set consisting only of the identity.

## chapter 2

### section 1

5: Prove that $G$ cannot have a subgroup $H$ with $|H| = n-1$, where $n = |G| > 2$.

13: Let $H$ be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element $x$ of $H$. Prove that $H = 0$ or $\mathbb{Q}$.

### section 2

4: Prove that if $H$ is a subgroup of $G$ then $H$ is generated by the set $H - \{1\}$.

13: Prove that the multiplicative group of positive rational numbers is generated by the set $\left\{\frac{1}{p} \mid p \text{ is a prime}\right\}$.

16a: A subgroup $M$ of a group $G$ is called a maximal subgroup if $M \neq G$ and the only subgroups of $G$ which contain $M$ are $M$ and $G$. Prove that if $H$ is a proper subgroup of the finite group $G$ then there is a maximal subgroup of $G$ containing $H$.

16c: Show that if $G = \langle x \rangle$ is a cyclic group of order $n \geq 1$ then a subgroup $H$ is maximal if and only $H = \langle x^p \rangle$ for some prime $p$ dividing $n$.

## Chapter 3

### section 1

3a: Let $A$ be an abelian group and let $B$ be a subgroup of $A$. Prove that $A/B$ is abelian.

22a: Prove that if $H$ and $K$ are normal subgroups of a group $G$ then their intersection $H \cap K$ is also a normal subgroup of $G$.

22b: Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

**section 2**

8: Prove that if $H$ and $K$ are finite subgroups of $G$ whose orders are relatively prime then $H \cap K = 1$.

11: Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$ (do not assume $G$ is finite).

16: Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^{\times}$ to prove Fermat's Little Theorem: if $p$ is a prime then $a^p \equiv a(\mathrm{mod} p)$ for all $a \in \mathbb{Z}$.

21a: Prove that $\mathbb{Q}$ has no proper subgroups of finite index.

**section 3**

3: Prove that if $H$ is a normal subgroup of $G$ of prime index $p$ then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $|K : K \cap H| = p$.

# Rudin

## chapter 1

1: If $r$ is rational ($r \neq 0$) and $x$ is irrational, prove that $r + x$ and $rx$ are irrational.

2: Prove that there is no rational number whose square is 12.

5: Let $A$ be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$

14: If $z$ is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute $|1 + z|^2 + |1 - z|^2$.

18a: If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

## chapter 3

1a: Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

3: If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ $(n = 1, 2, 3, \ldots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

5: For any two real sequences $\{a_n\}, \{b_n\}$, prove that $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.

7: Prove that the convergence of $\Sigma a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

8: If $\Sigma a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

13: Prove that the Cauchy product of two absolutely convergent series converges absolutely.

20: 20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space $X$, and some subsequence $\{p_{nl}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to $p$.

21: If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space $X$, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

22: Suppose $X$ is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of $X$. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. Hint: Find a shrinking sequence of neighborhoods $E_n$ such that $E_n \subset G_n$.

# Munkres

## chapter 2

### section 13

5a: Show that if $\mathcal{A}$ is a basis for a topology on $X$, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on $X$ that contain $\mathcal{A}$.

### section 16

4: A map $f : X \to Y$ is said to be an open map if for every open set $U$ of $X$, the set $f(U)$ is open in $Y$. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

### section 17

2: Show that if $A$ is closed in $Y$ and $Y$ is closed in $X$, then $A$ is closed in $X$.

3: Show that if $A$ is closed in $X$ and $B$ is closed in $Y$, then $A \times B$ is closed in $X \times Y$.

4: Show that if $U$ is open in $X$ and $A$ is closed in $X$, then $U - A$ is open in $X$, and $A - U$ is closed in $X$.

**18**

8a: Let $Y$ be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in $X$.

8b: Let $Y$ be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that $h$ is continuous. [Hint: Use the pasting lemma.]

13: Let $A \subset X$; let $f : A \to Y$ be continuous; let $Y$ be Hausdorff. Show that if $f$ may be extended to a continuous function $g : \bar{A} \to Y$, then $g$ is uniquely determined by $f$.

**21**

6a: Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0,1]$.

6b: Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n)$ does not converge uniformly.

8: Let $X$ be a topological space and let $Y$ be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let $x_n$ be a sequence of points of $X$ converging to $x$. Show that if the sequence $(f_n)$ converges uniformly to $f$, then $(f_n(x_n))$ converges to $f(x)$.

**22**

1: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of $Y$, then $p$ is a quotient map.

2a: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of $Y$, then $p$ is a quotient map.

2b: If $A \subset X$, a retraction of $X$ onto $A$ is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

3: Let $H$ be a subspace of $G$. Show that if $H$ is also a subgroup of $G$, then both $H$ and $\bar{H}$ are topological groups.