Exercises from
Principles of Mathematical Analysis
by Walter Rudin

Exercise 1.1a   If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ is irrational.

Proof. If r and $r + x$ were both rational, then $x = r + x - r$ would also be rational. $\square$

Exercise 1.1b   If r is rational ($r \neq 0$) and x is irrational, prove that rx is irrational.

Proof. If rx were rational, then $x = \frac{rx}{r}$ would also be rational. $\square$

Exercise 1.2   Prove that there is no rational number whose square is 12.

Proof. Suppose $m^2 = 12n^2$, where m and n have no common factor. It follows that m must be even, and therefore n must be odd. Let $m = 2r$. Then we have $r^2 = 3n^2$, so that r is also odd. Let $r = 2s + 1$ and $n = 2t + 1$. Then

$$4s^2 + 4s + 1 = 3\left(4t^2 + 4t + 1\right) = 12t^2 + 12t + 3,$$

so that

$$4\left(s^2 + s - 3t^2 - 3t\right) = 2.$$

But this is absurd, since 2 cannot be a multiple of 4 . $\square$

Exercise 1.4   Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since E is nonempty, there exists $x \in E$. Then by definition of lower and upper bounds we have $\alpha \leq x \leq \beta$, and hence by property *ii* in the definition of an ordering, we have $\alpha < \beta$ unless $\alpha = x = \beta$. $\square$

Exercise 1.5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

Proof. We need to prove that $-\sup(-A)$ is the greatest lower bound of A. For brevity, let $\alpha = -\sup(-A)$. We need to show that $\alpha \le x$ for all $x \in A$ and $\alpha \ge \beta$ if β is any lower bound of A.

Suppose $x \in A$. Then, $-x \in -A$, and, hence $-x \le \sup(-A)$. It follows that $x \ge -\sup(-A)$, i.e., $\alpha \le x$. Thus α is a lower bound of A.

Now let β be any lower bound of A. This means $\beta \le x$ for all x in A. Hence $-x \le -\beta$ for all $x \in A$, which says $y \le -\beta$ for all $y \in -A$. This means $-\beta$ is an upper bound of $-A$. Hence $-\beta \ge \sup(-A)$ by definition of sup, i.e., $\beta \le -\sup(-A)$, and so $-\sup(-A)$ is the greatest lower bound of A. $\square$

Exercise 1.8 Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. By Part (a) of Proposition 1.18, either i or $-i$ must be positive. Hence $-1 = i^2 = (-i)^2$ must be positive. But then $1 = (-1)^2$, must also be positive, and this contradicts Part (a) of Proposition 1.18, since 1 and -1 cannot both be positive. $\square$

Exercise 1.11a If z is a complex number, prove that there exists an $r \ge 0$ and a complex number w with $|w| = 1$ such that $z = rw$.

Proof. If $z = 0$, we take $r = 0, w = 1$. (In this case w is not unique.) Otherwise we take $r = |z|$ and $w = z/|z|$, and these choices are unique, since if $z = rw$, we must have $r = r|w| = |rw| = |z|, z/r$ $\square$

Exercise 1.12 If $z_1, \ldots, z_n$ are complex, prove that $|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$.

Proof. We can apply the case $n = 2$ and induction on n to get

$$\begin{aligned}
|z_1 + z_2 + \cdots z_n| &= |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\
&\le |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\
&\le |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|
\end{aligned}$$

$\square$

Exercise 1.13 If x, y are complex, prove that $||x| - |y|| \le |x - y|$.

Proof. Since $x = x - y + y$, the triangle inequality gives

$$|x| \le |x - y| + |y|$$

so that $|x| - |y| \le |x - y|$. Similarly $|y| - |x| \le |x - y|$. Since $|x| - |y|$ is a real number we have either $——x| - |y|| = |x| - |y|$ or $——x| - |y|| = |y| - |x|$. In either case, we have shown that $——x| - |y|| \le |x - y|$. $\square$

Exercise 1.14 If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute $|1 + z|^2 + |1 - z|^2$.

Proof. $|1 + z|^2 = (1 + z)(1 + \bar{z}) = 1 + \bar{z} + z + z\bar{z} = 2 + z + \bar{z}$. Similarly $|1 - z|^2 = (1 - z)(1 - \bar{z}) = 1 - z - \bar{z} + z\bar{z} = 2 - z - \bar{z}$. Hence

$$|1 + z|^2 + |1 - z|^2 = 4.$$

$\square$

Exercise 1.16a Suppose $k \geq 3, x, y \in \mathbb{R}^k, |x - y| = d > 0$, and $r > 0$. Prove that if $2r > d$, there are infinitely many $z \in \mathbb{R}^k$ such that $|z - x| = |z - y| = r$.

Proof. (a) Let w be any vector satisfying the following two equations:

$$\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0,$$

$$|\mathbf{w}|^2 = r^2 - \frac{d^2}{4}.$$

From linear algebra it is known that all but one of the components of a solution **w** of the first equation can be arbitrary. The remaining component is then uniquely determined. Also, if w is any non-zero solution of the first equation, there is a unique positive number t such that t w satisfies both equations. (For example, if $x_1 \neq y_1$, the first equation is satisfied whenever

$$z_1 = \frac{z_2 (x_2 - y_2) + \cdots + z_k (x_k - y_k)}{y_1 - x_1}.$$

If $(z_1, z_2, \ldots, z_k)$ satisfies this equation, so does $(tz_1, tz_2, \ldots, tz_k)$ for any real number t.) Since at least two of these components can vary independently, we can find a solution with these components having any prescribed ratio. This ratio does not change when we multiply by the positive number t to obtain a solution of both equations. Since there are infinitely many ratios, there are infinitely many distinct solutions. For each such solution **w** the vector $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} + \mathbf{w}$ is a solution of the required equation. For

$$\begin{aligned} |\mathbf{z} - \mathbf{x}|^2 &= \left| \frac{\mathbf{y} - \mathbf{x}}{2} + \mathbf{w} \right|^2 \\ &= \left| \frac{\mathbf{y} - \mathbf{x}}{2} \right|^2 + 2\mathbf{w} \cdot \frac{\mathbf{x} - \mathbf{y}}{2} + |\mathbf{w}|^2 \\ &= \frac{d^2}{4} + 0 + r^2 - \frac{d^2}{4} \\ &= r^2 \end{aligned}$$

and a similar relation holds for $|z - y|^2$. $\square$

Exercise 1.17   Prove that $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$ if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$.

Proof. The proof is a routine computation, using the relation

$$|x \pm y|^2 = (x \pm y) \cdot (x \pm y) = |x|^2 \pm 2x \cdot y + |y|^2.$$

If x and y are the sides of a parallelogram, then $x + y$ and $\mathbf{x} - y$ are its diagonals. Hence this result says that the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides. □

Exercise 1.18a   If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

Proof. If $\mathbf{x}$ has any components equal to 0 , then $\mathbf{y}$ can be taken to have the corresponding components equal to 1 and all others equal to 0 . If all the components of $\mathbf{x}$ are nonzero, $\mathbf{y}$ can be taken as $(-x_2, x_1, 0, \ldots, 0)$. This is, of course, not true when $k = 1$, since the product of two nonzero real numbers is nonzero. □

Exercise 1.18b   If $k = 1$ and $\mathbf{x} \in R^k$, prove that there does not exist $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

Proof. Not true when $k = 1$, since the product of two nonzero real numbers is nonzero. □

Exercise 1.19   Suppose $a, b \in R^k$. Find $c \in R^k$ and $r > 0$ such that $|x - a| = 2|x - b|$ if and only if $|x - c| = r$. Prove that $3c = 4b - a$ and $3r = 2|b - a|$.

Proof. Since the solution is given to us, all we have to do is verify it, i.e., we need to show that the equation

$$|\mathrm{x} - \mathrm{a}| = 2|\mathrm{x} - \mathrm{b}|$$

is equivalent to $|\mathrm{x} - \mathbf{c}| = r$, which says

$$\left| \mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a} \right| = \frac{2}{3}|\mathbf{b} - \mathbf{a}|.$$

If we square both sides of both equations, we an equivalent pair of equations, the first of which reduces to

$$3|\mathbf{x}|^2 + 2\mathbf{a} \cdot \mathbf{x} - 8\mathbf{b} \cdot \mathbf{x} - |\mathbf{a}|^2 + 4|\mathbf{b}|^2 = 0,$$

and the second of which reduces to this equation divided by 3 . Hence these equations are indeed equivalent. □

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Exercise 2.19a If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof. We are given that $A \cap B = \varnothing$. Since A and B are closed, this means $A \cap \bar{B} = \varnothing = \bar{A} \cap B$, which says that A and B are separated. $\square$

Exercise 2.24 Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Exercise 2.25 Prove that every compact metric space K has a countable base.

Proof. It is easier simply to refer to the previous problem. The hint shows that K can be covered by a finite union of neighborhoods of radius $1/n$, and the previous problem shows that this implies that K is separable.

It is not entirely obvious that a metric space with a countable base is separable. To prove this, let $\{V_n\}_{n=1}^{\infty}$ be a countable base, and let $x_n \in V_n$. The points V_n must be dense in X. For if G is any non-empty open set, then G contains V_n for some n, and hence $x_n \in G$. (Thus for a metric space, having a countable base and being separable are equivalent.) $\square$

Exercise 2.27a Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that P is perfect.

Proof. We see that $E \cap W$ is at most countable, being a countable union of at-most-countable sets. It remains to show that $P = W^c$, and that P is perfect. $\square$

Exercise 2.27b Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of condensation points of E. Prove that at most countably many points of E are not in P.

Proof. If $x \in W^c$, and O is any neighborhood of x, then $x \in V_n \subseteq O$ for some n. Since $x \notin W$, $V_n \cap E$ is uncountable. Hence O contains uncountably many points of E, and so x is a condensation point of E. Thus $x \in P$, i.e., $W^c \subseteq P$. Conversely if $x \in W$, then $x \in V_n$ for some V_n such that $V_n \cap E$ is countable. Hence x has a neighborhood (any neighborhood contained in V_n) containing at most a countable set of points of E, and so $x \notin P$, i.e., $W \subseteq P^c$. Hence $P = W^c$. It is clear that P is closed (since its complement W is open), so that we need only show that $P \subseteq P'$. Hence suppose $x \in P$, and O is any neighborhood of x. (By definition of P this means $O \cap E$ is uncountable.) We need to show that there is a point $y \in P \cap (O \backslash \{x\})$. If this is not the case, i.e., if every point y in $O \backslash \{x\}$ is in P^c, then for each such point y there is a set V_n containing y such that $V_n \cap E$ is at most countable. That would mean that $y \in W$, i.e., that $O \backslash \{x\}$ is contained in W. It would follow that $O \cap E \subseteq \{x\} \cup (W \cap E)$, and so $O \cap E$ contains at most a countable set of points, contrary to the hypothesis that $x \in P$. Hence O contains a point of P different from x, and so $P \subseteq P'$. Thus P is perfect. $\square$

Exercise 2.28 Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable.

Proof. If E is closed, it contains all its limit points, and hence certainly all its condensation points. Thus $E = P \cup (E \backslash P)$, where P is perfect (the set of all condensation points of E), and $E \backslash P$ is at most countable.

Since a perfect set in a separable metric space has the same cardinality as the real numbers, the set P must be empty if E is countable. The at-mostcountable set $E \backslash P$ cannot be perfect, hence must have isolated points if it is nonempty. □

Exercise 2.29 Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Proof. Let O be open. For each pair of points $x \in O, y \in O$, we define an equivalence relation $x \sim y$ by saying $x \sim y$ if and only if $[\min(x, y), \max(x, y)] \subset 0$. This is an equivalence relation, since $x \sim x([x, x] \subset O$ if $x \in O$); if $x \sim y$, then $y \sim x$ (since $\min(x, y) = \min(y, x)$ and $\max(x, y) = \max(y, x)$); and if $x \sim y$ and $y \sim z$, then $x \sim z([\min(x, z), \max(x, z)] \subseteq [\min(x, y), \max(x, y)] \cup [\min(y, z), \max(y, z)] \subseteq O)$. In fact it is easy to prove that

$$\min(x, z) \geq \min(\min(x, y), \min(y, z))$$

and

$$\max(x, z) \leq \max(\max(x, y), \max(y, z))$$

It follows that O can be written as a disjoint union of pairwise disjoint equivalence classes. We claim that each equivalence class is an open interval.

To show this, for each $x \in O$; let $A = \{z : [z, x] \subseteq O\}$ and $B = \{z : [x, z] \subseteq O\}$, and let $a = \inf A, b = \sup B$. We claim that $(a, b) \subset O$. Indeed if $a < z < b$, there exists $c \in A$ with $c < z$ and $d \in B$ with $d > z$. Then $z \in [c, x] \cup [x, d] \subseteq O$. We now claim that (a, b) is the equivalence class containing x. It is clear that each element of (a, b) is equivalent to x by the way in which a and b were chosen. We need to show that if $z \notin (a, b)$, then z is not equivalent to x. Suppose that $z < a$. If z were equivalent to x, then $[z, x]$ would be contained in O, and so we would have $z \in A$. Hence a would not be a lower bound for A. Similarly if $z > b$ and $z \sim x$, then b could not be an upper bound for B.

We have now established that O is a union of pairwise disjoint open intervals. Such a union must be at most countable, since each open interval contains a rational number not in any other interval. □

Exercise 3.1a Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

Proof. Let $\varepsilon > 0$. Since the sequence $\{s_n\}$ is a Cauchy sequence, there exists N such that $|s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. We then have $||s_m| - |s_n|| \leq |s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. Hence the sequence $\{|s_n|\}$ is also a Cauchy sequence, and therefore must converge. □

Exercise 3.2a Prove that $\lim_{n\to\infty}\sqrt{n^2+n}-n=1/2$.

Proof. Multiplying and dividing by $\sqrt{n^2+n}+n$ yields

$$\sqrt{n^2+n}-n=\frac{n}{\sqrt{n^2+n}+n}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

It follows that the limit is $\frac{1}{2}$. $\qquad\square$

Exercise 3.3 If $s_1=\sqrt{2}$, and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$ $(n=1,2,3,\ldots)$, prove that $\{s_n\}$ converges, and that $s_n<2$ for $n=1,2,3,\ldots$.

Proof. Since $\sqrt{2}<2$, it is manifest that if $s_n<2$, then $s_{n+1}<\sqrt{2+2}=2$. Hence it follows by induction that $\sqrt{2}<s_n<2$ for all n. In view of this fact,it also follows that $(s_n-2)(s_n+1)<0$ for all $n>1$, i.e., $s_n>s_n^2-2=s_{n-1}$. Hence the sequence is an increasing sequence that is bounded above (by 2) and so converges. Since the limit s satisfies $s^2-s-2=0$, it follows that the limit is 2. $\qquad\square$

Exercise 3.5 For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that $\limsup_{n\to\infty}(a_n+b_n)\le\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$, provided the sum on the right is not of the form $\infty-\infty$.

Proof. Since the case when $\limsup_{n\to\infty}a_n=+\infty$ and $\limsup_{n\to\infty}b_n=-\infty$ has been excluded from consideration, we note that the inequality is obvious if $\limsup_{n\to\infty}a_n=+\infty$. Hence we shall assume that $\{a_n\}$ is bounded above.

Let $\{n_k\}$ be a subsequence of the positive integers such that $\lim_{k\to\infty}(a_{n_k}+b_{n_k})=\limsup_{n\to\infty}(a_n+b_n)$. Then choose a subsequence of the positive integers $\{k_m\}$ such that

$$\lim_{m\to\infty}a_{n_{k_m}}=\limsup_{k\to\infty}a_{n_k}.$$

The subsequence $a_{n_{k_m}}+b_{n_{k_m}}$ still converges to the same limit as $a_{n_k}+b_{n_k}$, i.e., to $\limsup_{n\to\infty}(a_n+b_n)$. Hence, since a_{n_k} is bounded above (so that $\limsup_{k\to\infty}a_{n_k}$ is finite), it follows that $b_{n_{k_m}}$ converges to the difference

$$\lim_{m\to\infty}b_{n_{k_m}}=\lim_{m\to\infty}\left(a_{n_{k_m}}+b_{n_{k_m}}\right)-\lim_{m\to\infty}a_{n_{k_m}}.$$

Thus we have proved that there exist subsequences $\{a_{n_{k_m}}\}$ and $\{b_{n_{k_m}}\}$ which converge to limits a and b respectively such that $a+b=\limsup_{n\to\infty}(a_n+b_n^*)$. Since a is the limit of a subsequence of $\{a_n\}$ and b is the limit of a subsequence of $\{b_n\}$, it follows that $a\le\limsup_{n\to\infty}a_n$ and $b\le\limsup_{n\to\infty}b_n$, from which the desired inequality follows. $\qquad\square$

Exercise 3.6a Prove that $\lim_{n\to\infty}\sum_{i<n}a_i=\infty$, where $a_i=\sqrt{i+1}-\sqrt{i}$.

Proof. (a) Multiplying and dividing a_n by $\sqrt{n+1}+\sqrt{n}$, we find that $a_n=\frac{1}{\sqrt{n+1}+\sqrt{n}}$, which is larger than $\frac{1}{2\sqrt{n+1}}$. The series $\sum a_n$ therefore diverges by comparison with the p series $\left(p=\frac{1}{2}\right)$. $\qquad\square$

Exercise 3.7 Prove that the convergence of Σa_n implies the convergence of $\Sigma \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

Proof. Since $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$, it follows that

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right).$$

Now Σa_n^2 converges by comparison with Σa_n (since Σa_n converges, we have $a_n < 1$ for large n, and hence $a_n^2 < a_n$). Since $\Sigma \frac{1}{n^2}$ also converges (p series, $p = 2$), it follows that $\Sigma \frac{\sqrt{a_n}}{n}$ converges. $\qquad\square$

Exercise 3.8 If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

Proof. We shall show that the partial sums of this series form a Cauchy sequence, i.e., given $\varepsilon > 0$ there exists N such that $\left|\sum_{k=m+1}^{n} a_k b_k\right| \langle \varepsilon$ if $n\rangle$ $m \geq N$. To do this, let $S_n = \sum_{k=1}^{n} a_k$ $(S_0 = 0)$, so that $a_k = S_k - S_{k-1}$ for $k = 1, 2, \ldots$ Let M be an uppper bound for both $|b_n|$ and $|S_n|$, and let $S = \sum a_n$ and $b = \lim b_n$. Choose N so large that the following three inequalities hold for all $m > N$ and $n > N$:

$$|b_n S_n - bS| < \frac{\varepsilon}{3}; \quad |b_m S_m - bS| < \frac{\varepsilon}{3}; \quad |b_m - b_n| < \frac{\varepsilon}{3M}.$$

Then if $n > m > N$, we have, from the formula for summation by parts,

$$\sum_{k=m+1}^{n} a_n b_n = b_n S_n - b_m S_m + \sum_{k=m}^{n-1} \left(b_k - b_{k+1}\right) S_k$$

Our assumptions yield immediately that $|b_n S_n - b_m S_m| < \frac{2\varepsilon}{3}$, and

$$\left|\sum_{k=m}^{n-1} \left(b_k - b_{k+1}\right) S_k\right| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}|.$$

Since the sequence $\{b_n\}$ is monotonic, we have

$$\sum_{k=m}^{n-1} |b_k - b_{k+1}| = \left|\sum_{k=m}^{n-1} \left(b_k - b_{k+1}\right)\right| = |b_m - b_n| < \frac{\varepsilon}{3M},$$

from which the desired inequality follows. $\qquad\square$

Exercise 3.13 Prove that the Cauchy product of two absolutely convergent series converges absolutely.

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Proof. Since both the hypothesis and conclusion refer to absolute convergence, we may assume both series consist of nonnegative terms. We let $S_n = \sum_{k=0}^n a_n, T_n = \sum_{k=0}^n b_n$, and $U_n = \sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l}$. We need to show that U_n remains bounded, given that S_n and T_n are bounded. To do this we make the convention that $a_{-1} = T_{-1} = 0$, in order to save ourselves from having to separate off the first and last terms when we sum by parts. We then have

$$U_n = \sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l}$$

$$= \sum_{k=0}^n \sum_{l=0}^k a_l \left(T_{k-l} - T_{k-l-1} \right)$$

$$= \sum_{k=0}^n \sum_{j=0}^k a_{k-j} \left(T_j - T_{j-1} \right)$$

$$= \sum_{k=0}^n \sum_{j=0}^k \left(a_{k-j} - a_{k-j-1} \right) T_j$$

$$= \sum_{j=0}^n \sum_{k=j}^n \left(a_{k-j} - a_{k-j-1} \right) T_j \quad = \sum_{j=0}^n a_{n-j} T_j$$

$$\leq T \sum_{m=0}^n a_m$$

$$= T S_n$$

$$\leq ST.$$

Thus U_n is bounded, and hence approaches a finite limit.

$\square$

Exercise 3.20 Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some sequence $\{p_{nl}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Proof. Let $\varepsilon > 0$. Choose N_1 so large that $d(p_m, p_n) < \frac{\varepsilon}{2}$ if $m > N_1$ and $n > N_1$. Then choose $N \geq N_1$ so large that $d(p_{n_k}, p) < \frac{\varepsilon}{2}$ if $k > N$. Then if $n > N$, we have

$$d(p_n, p) \leq d\left(p_n, p_{n_{N+1}}\right) + d\left(p_{n_{N+1}}, p\right) < \varepsilon$$

For the first term on the right is less than $\frac{\varepsilon}{2}$ since $n > N_1$ and $n_{N+1} > N+1 > N_1$. The second term is less than $\frac{\varepsilon}{2}$ by the choice of N. $\square$

Exercise 3.21 If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n \to \infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

Proof. Choose $x_n \in E_n$. (We use the axiom of choice here.) The sequence $\{x_n\}$ is a Cauchy sequence, since the diameter of E_n tends to zero as n tends to infinity and E_n contains E_{n+1}. Since the metric space X is complete, the sequence x_n converges to a point x, which must belong to E_n for all n, since E_n is closed and contains x_m for all $m \geq n$. There cannot be a second point y in all of the E_n, since for any point $y \neq x$ the diameter of E_n is less than $d(x, y)$ for large n. $\qquad\square$

Exercise 3.22 Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open sets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty.

Proof. Let F_n be the complement of G_n, so that F_n is closed and contains no open sets. We shall prove that any nonempty open set U contains a point not in any F_n, hence in all G_n. To this end, we note that U is not contained in F_1, so that there is a point $x_1 \in U \backslash F_1$. Since $U \backslash F_1$ is open, there exists $r_1 > 0$ such that B_1, defined as the open ball of radius r_1 about x_1, is contained in $U \backslash F_1$. Let E_1 be the open ball of radius $\frac{r_1}{2}$ about x_1, so that the closure of E_1 is contained in B_1. Now F_2 does not contain E_1, and so we can find a point $x_2 \in E_1 \backslash F_2$. Since $E_1 \backslash F_2$ is an open set, there exists a positive number r_2 such that B_2, the open ball of radius R_2 about x_2, is contained in $E_1 \backslash F_2$, which in turn is contained in $U \backslash (F_1 \cup F_2)$. We let E_2 be the open ball of radius $\frac{r_2}{2}$ about x_2, so that $\bar{E}_2 \subseteq B_2$. Proceeding in this way, we construct a sequence of open balls E_j, such that $E_j \supseteq \bar{E}_{j+1}$, and the diameter of E_j tends to zero. By the previous exercise, there is a point x belonging to all the sets $\bar{E}_j$, hence to all the sets $U \backslash (F_1 \cup F_2 \cup \cdots \cup F_n)$. Thus the point x belongs to $U \cap (\cap_1^\infty G_n)$. $\quad\square$

Exercise 4.1a Suppose f is a real function defined on $\mathbb{R}$ which satisfies $\lim_{h\to0} f(x+h) - f(x-h) = 0$ for every $x \in \mathbb{R}$. Show that f does not need to be continuous.

Proof.
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

(If x is an integer, then $f(x+h) - f(x-h) \equiv 0$ for all h; while if x is not an integer, $f(x+h) - f(x-h) = 0$ for $|h| < \min(x - [x], 1 + [x] - x)$. $\qquad\square$

Exercise 4.2a If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\bar{E}) \subset \overline{f(E)}$ for every set $E \subset X$. ($\bar{E}$ denotes the closure of E).

Proof. Let $x \in \bar{E}$. We need to show that $f(x) \in \overline{f(E)}$. To this end, let O be any neighborhood of $f(x)$. Since f is continuous, $f^{-1}(O)$ contains (is) a neighborhood of x. Since $x \in \bar{E}$, there is a point u of E in $f^{-1}(O)$. Hence

$\frac{f(u)}{f(E)} \in O \cap f(E)$. Since O was any neighborhood of $f(x)$, it follows that $f(x) \in \overline{f(E)}$ $\qquad \square$

Exercise 4.3 Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof. $Z(f) = f^{-1}(\{0\})$, which is the inverse image of a closed set. Hence $Z(f)$ is closed. $\qquad \square$

Exercise 4.4a Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$.

Proof. To prove that $f(E)$ is dense in $f(X)$, simply use that $f(X) = f(\bar{E}) \subseteq \overline{f(E)}$. $\qquad \square$

Exercise 4.4b Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that if $g(p) = f(p)$ for all $p \in P$ then $g(p) = f(p)$ for all $p \in X$.

Proof. The function $\varphi : X \to R^1$ given by

$$\varphi(p) = d_Y(f(p), g(p))$$

is continuous, since

$$|d_Y(f(p), g(p)) - d_Y(f(q), g(q))| \leq d_Y(f(p), f(q)) + d_Y(g(p), g(q))$$

(This inequality follows from the triangle inequality, since

$$d_Y(f(p), g(p)) \leq d_Y(f(p), f(q)) + d_Y(f(q), g(q)) + d_Y(g(q), g(p)),$$

and the same inequality holds with p and q interchanged. The absolute value $|d_Y(f(p), g(p)) - d_Y(f(q), g(q))|$ must be either $d_Y(f(p), g(p)) - d_Y(f(q), g(q))$ or $d_Y(f(q), g(q)) - d_Y(f(p), g(p))$, and the triangle inequality shows that both of these numbers are at most $d_Y(f(p), f(q)) + d_Y(g(p), g(q))$.) By the previous problem, the zero set of φ is closed. But by definition

$$Z(\varphi) = \{p : f(p) = g(p)\}.$$

Hence the set of p for which $f(p) = g(p)$ is closed. Since by hypothesis it is dense, it must be X. $\qquad \square$

Exercise 4.5a If f is a real continuous function defined on a closed set $E \subset \mathbb{R}$, prove that there exist continuous real functions g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$.

Exercise 4.5b Show that there exist a set $E \subset \mathbb{R}$ and a real continuous function f defined on E, such that there does not exist a continuous real function g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$.

Exercise 4.6 If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof. Let Y be the co-domain of the function f. We invent a new metric space $E \times Y$ as the set of pairs of points $(x, y), x \in E, y \in Y$, with the metric $\rho\left((x_1, y_1), (x_2, y_2)\right) = d_E(x_1, x_2) + d_Y(y_1, y_2)$. The function $\varphi(x) = (x, f(x))$ is then a mapping of E into $E \times Y$.

    We claim that the mapping φ is continuous if f is continuous. Indeed, let $x \in X$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $d_Y(f(x), f(u)) < \frac{\varepsilon}{2}$ if $d_E(x, y) < \eta$. Then let $\delta = \min\left(\eta, \frac{\varepsilon}{2}\right)$. It is easy to see that $\rho(\varphi(x), \varphi(u)) < \varepsilon$ if $d_E(x, u) < \delta$. Conversely if φ is continuous, it is obvious from the inequality $\rho(\varphi(x), \varphi(u)) \geq d_Y(f(x), f(u))$ that f is continuous.

    From these facts we deduce immediately that the graph of a continuous function f on a compact set E is compact, being the image of E under the continuous mapping φ. Conversely, if f is not continuous at some point x, there is a sequence of points x_n converging to x such that $f(x_n)$ does not converge to $f(x)$. If no subsequence of $f(x_n)$ converges, then the sequence $\{(x_n, f(x_n)\}_{n=1}^{\infty}$ has no convergent subsequence, and so the graph is not compact. If some subsequence of $f(x_n)$ converges, say $f(x_{n_k}) \to z$, but $z \neq f(x)$, then the graph of f fails to contain the limit point (x, z), and hence is not closed. A fortiori it is not compact. $\qquad\square$

Exercise 4.8a Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

Exercise 4.8b Let E be a bounded set in R^1. Prove that there exists a real function f such that f is uniformly continuous and is not bounded on E.

Exercise 4.11a Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

Exercise 4.12 A uniformly continuous function of a uniformly continuous function is uniformly continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous. Then $g \circ f : X \to Z$ is uniformly continuous, where $g \circ f(x) = g(f(x))$ for all $x \in X$. To prove this fact, let $\varepsilon > 0$ be given. Then, since g is uniformly continuous, there exists $\eta > 0$ such that $d_Z(g(u), g(v)) < \varepsilon$ if $d_Y(u, v) < \eta$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \eta$ if $d_X(x, y) < \delta$

It is then obvious that $d_Z(g(f(x)), g(f(y))) < \varepsilon$ if $d_X(x, y) < \delta$, so that $g \circ f$ is uniformly continuous. $\square$

Exercise 4.14 Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof. If $f(0) = 0$ or $f(1) = 1$, we are done. If not, then $0 < f(0)$ and $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the intermediate value theorem, there must be a point $x \in (0, 1)$ where $g(x) = 0$ $\square$

Exercise 4.15 Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof. Suppose f is continuous and not monotonic, say there exist points $a < b < c$ with $f(a) < f(b)$, and $f(c) < f(b)$. Then the maximum value of f on the closed interval $[a, c]$ is assumed at a point u in the open interval (a, c). If there is also a point v in the open interval (a, c) where f assumes its minimum value on $[a, c]$, then $f(a, c) = [f(v), f(u)]$. If no such point v exists, then $f(a, c) = (d, f(u)]$, where $d = \min(f(a), f(c))$. In either case, the image of (a, c) is not open. $\square$

Exercise 4.19 Suppose f is a real function with domain R^1 which has the intermediate value property: if $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Proof. The contradiction is evidently that x_0 is a limit point of the set of t such that $f(t) = r$, yet, x_0 does not belong to this set. This contradicts the hypothesis that the set is closed. $\square$

Exercise 4.21a Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

Exercise 4.24 Assume that f is a continuous real function defined in (a, b) such that $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in (a, b)$. Prove that f is convex.

Proof. We shall prove that

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

for all "dyadic rational" numbers, i.e., all numbers of the form $\lambda = \frac{k}{2^n}$, where k is a nonnegative integer not larger than 2^n. We do this by induction on n. The case $n = 0$ is trivial (since $\lambda = 0$ or $\lambda = 1$). In the case $n = 1$ we have $\lambda = 0$ or $\lambda = 1$ or $\lambda = \frac{1}{2}$. The first two cases are again trivial, and the third is precisely the hypothesis of the theorem. Suppose the result is proved for $n \le r$, and

consider $\lambda = \frac{k}{2^{r+1}}$. If k is even, say $k = 2l$, then $\frac{k}{2^{r+1}} = \frac{l}{2^r}$, and we can appeal to the induction hypothesis. Now suppose k is odd. Then $1 \le k \le 2^{r+1} - 1$, and so the numbers $l = \frac{k-1}{2}$ and $m = \frac{k+1}{2}$ are integers with $0 \le l < m \le 2^r$. We can now write

$$\lambda = \frac{s+t}{2},$$

where $s = \frac{k-1}{2^{r+1}} = \frac{l}{2^r}$ and $t = \frac{k+1}{2^{r+1}} = \frac{m}{2^r}$. We then have

$$\lambda x + (1 - \lambda)y = \frac{[sx + (1-s)y] + [tx + (1-t)y]}{2}$$

Hence by the hypothesis of the theorem and the induction hypothesis we have

$$\begin{aligned}
f(\lambda x + (1 - \lambda)y) &\le \frac{f(sx + (1-s)y) + f(tx + (1-t)y)}{2} \\
&\le \frac{sf(x) + (1-s)f(y) + tf(x) + (1-t)f(y)}{2} \\
&= \left(\frac{s+t}{2}\right)f(x) + \left(1 - \frac{s+t}{2}\right)f(y) \\
&= \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

This completes the induction. Now for each fixed x and y both sides of the inequality

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

are continuous functions of λ. Hence the set on which this inequality holds (the inverse image of the closed set $[0, \infty)$ under the mapping $\lambda \mapsto \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1 - \lambda)y))$ is a closed set. Since it contains all the points $\frac{k}{2^n}$, $0 \le k \le n, n = 1, 2, \ldots$, it must contain the closure of this set of points, i.e., it must contain all of $[0, 1]$. Thus f is convex. $\qquad\square$

Exercise 5.1 Let f be defined for all real x, and suppose that $|f(x) - f(y)| \le (x - y)^2$ for all real x and y. Prove that f is constant.

Proof. Dividing by $x - y$, and letting $x \to y$, we find that $f'(y) = 0$ for all y. Hence f is constant. $\qquad\square$

Exercise 5.2 Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$ $(a < x < b)$.

Proof. For any c, d with $a < c < d < b$ there exists a point $p \in (c, d)$ such that $f(d) - f(c) = f'(p)(d - c) > 0$. Hence $f(c) < f(d)$

We know from Theorem 4.17 that the inverse function g is continuous. (Its restriction to each closed subinterval $[c, d]$ is continuous, and that is sufficient.) Now observe that if $f(x) = y$ and $f(x + h) = y + k$, we have

$$\frac{g(y+k) - g(y)}{k} - \frac{1}{f'(x)} = \frac{1}{\frac{f(x+h) - f(x)}{h}} - \frac{1}{f'(x)}$$

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Since we know $\lim \frac{1}{\varphi(t)} = \frac{1}{\lim \varphi(t)}$ provided $\lim \varphi(t) \neq 0$, it follows that for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left| \frac{1}{\frac{f(x+h)-f(x)}{h}} - \frac{1}{f'(x)} \right| < \varepsilon$$

if $0 < |h| < \eta$. Since $h = g(y+k) - g(y)$, there exists $\delta > 0$ such that $0 < |h| < \eta$ if $0 < |k| < \delta$. The proof is now complete. $\qquad\square$

Exercise 5.3   Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

Proof. If $0 < \varepsilon < \frac{1}{M}$, we certainly have

$$f'(x) \geq 1 - \varepsilon M > 0,$$

and this implies that $f(x)$ is one-to-one, by the preceding problem. $\qquad\square$

Exercise 5.4   If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$p(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1},$$

whose derivative is

$$p'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n.$$

It is obvious that $p(0) = 0$, and the hypothesis of the problem is that $p(1) = 0$. Hence Rolle's theorem implies that $p'(x) = 0$ for some x between 0 and 1 . $\quad\square$

Exercise 5.5   Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. Let $\varepsilon > 0$. Choose x_0 such that $|f'(x)| < \varepsilon$ if $x > x_0$. Then for any $x \geq x_0$ there exists $x_1 \in (x, x+1)$ such that

$$f(x+1) - f(x) = f'(x_1).$$

Since $|f'(x_1)| < \varepsilon$, it follows that $|f(x+1) - f(x)| < \varepsilon$, as required. $\qquad\square$

Exercise 5.6   Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$ $(x > 0)$ and prove that g is monotonically increasing.

Proof. Put

$$g(x) = \frac{f(x)}{x}   (x > 0)$$

and prove that g is monotonically increasing. By the mean-value theorem

$$f(x) = f(x) - f(0) = f'(c)x$$

for some $c \in (0, x)$. Since f' is monotonically increasing, this result implies that $f(x) < xf'(x)$. It therefore follows that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0,$$

so that g is also monotonically increasing. □

Exercise 5.7   Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

Proof. Since $f(x) = g(x) = 0$, we have

$$
\begin{aligned}
\lim_{t \to x} \frac{f(t)}{g(t)} &= \lim_{t \to x} \frac{\frac{f(t) - f(x)}{t - x}}{\frac{g(t) - g(x)}{t - x}} \\
&= \frac{\lim_{t \to x} \frac{f(t) - f(x)}{t - x}}{\lim_{t \to x} \frac{g(t) - g(x)}{t - x}} \\
&= \frac{f'(x)}{g'(x)}
\end{aligned}
$$

□

Exercise 5.15   Suppose $a \in R^1, f$ is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \leq 4M_0 M_2$.

Proof. The inequality is obvious if $M_0 = +\infty$ or $M_2 = +\infty$, so we shall assume that M_0 and M_2 are both finite. We need to show that

$$|f'(x)| \leq 2\sqrt{M_0 M_2}$$

for all $x > a$. We note that this is obvious if $M_2 = 0$, since in that case $f'(x)$ is constant, $f(x)$ is a linear function, and the only bounded linear function is a constant, whose derivative is zero. Hence we shall assume from now on that

$0 < M_2 < +\infty$ and $0 < M_0 < +\infty$. Following the hint, we need only choose $h = \sqrt{\frac{M_0}{M_2}}$, and we obtain

$$|f'(x)| \leq 2\sqrt{M_0 M_2},$$

which is precisely the desired inequality. The case of equality follows, since the example proposed satisfies

$$f(x) = 1 - \frac{2}{x^2 + 1}$$

for $x \geq 0$. We see easily that $|f(x)| \leq 1$ for all $x > -1$. Now $f'(x) = \frac{4x}{(x^2+1)^2}$ for $x > 0$ and $f'(x) = 4x$ for $x < 0$. It thus follows from Exercise 9 above that $f'(0) = 0$, and that $f'(x)$ is continuous. Likewise $f''(x) = 4$ for $x < 0$ and $f''(x) = \frac{4-4x^2}{(x^2+1)^3} = -4\frac{x^2-1}{(x^2+1)^3}$. This shows that $|f''(x)| < 4$ for $x > 0$ and also that $\lim_{x \to 0} f''(x) = 4$. Hence Exercise 9 again implies that $f''(x)$ is continuous and $f''(0) = 4$.

On n-dimensional space let $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))$, $M_0 = \sup|\mathbf{f}(x)|$, $M_1 = \sup|\mathbf{f}'(x)|$, and $M_2 = \sup|\mathbf{f}''(x)|$. Just as in the numerical case, there is nothing to prove if $M_2 = 0$ or $M_0 = +\infty$ or $M_2 = +\infty$, and so we assume $0 < M_0 < +\infty$ and $0 < M_2 < \infty$. Let a be any positive number less than M_1, let x_0 be such that $|\mathbf{f}'(x_0)| > a$, and let $\mathbf{u} = \frac{1}{|\mathbf{f}'(x_0)|}\mathbf{f}'(x_0)$. Consider the real-valued function $\varphi(x) = \mathbf{u} \cdot \mathbf{f}(x)$. Let N_0, N_1, and N_2 be the suprema of $|\varphi(x)|, |\varphi'(x)|$, and $|\varphi''(x)|$ respectively. By the Schwarz inequality we have (since $|\mathbf{u}| = 1$) $N_0 \leq M_0$ and $N_2 \leq M_2$, while $N_1 \geq \varphi(x_0) = |\mathbf{f}'(x_0)| > a$. We therefore have $a^2 < 4N_0N_2 \leq 4M_0M_2$. Since a was any positive number less than M_1, we have $M_1^2 \leq 4M_0M_2$, i.e., the result holds also for vector-valued functions.

Equality can hold on any R^n, as we see by taking $\mathbf{f}(x) = (f(x), 0, \ldots, 0)$ or $\mathbf{f}(x) = (f(x), f(x), \ldots, f(x))$, where $f(x)$ is a real-valued function for which equality holds. $\qquad\square$

Exercise 5.17 Suppose f is a real, three times differentiable function on $[-1, 1]$, such that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$. Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Proof. Following the hint, we observe that Theorem 5.15 (Taylor's formula with remainder) implies that

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s)$$

$$f(-1) = f(0) - f'(0) + \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t)$$

for some $s \in (0, 1), t \in (-1, 0)$. By subtracting the second equation from the first and using the given values of $f(1), f(-1)$, and $f'(0)$, we obtain

$$1 = \frac{1}{6}\left(f^{(3)}(s) + f^{(3)}(t)\right),$$

which is the desired result. Note that we made no use of the hypothesis $f(0) = 0$. $\qquad \square$

Exercise 6.1 Suppose α increases on $[a, b], a \leq x_0 \leq b, \alpha$ is continuous at $x_0, f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Proof. Let $\varepsilon > 0$, and let δ be such that $|\alpha(x) - \alpha(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. Now consider any partiion $a = t_0 < t_1 < \cdots < t_n = b$ with $n \geq 2$ such that $|t_i - t_{i-1}| < \frac{\delta}{2}$. There exists an index i such that $t_{i-1} < x_0 < t_{i+1}$ (there may possibly be 2 such indices). We then have, for any choice of $t_0^*, t_1^*, \ldots, t_n^*$,

$$\left| \sum_{j=1}^{n} f\left(t_j^*\right) \left(\alpha\left(t_j\right) - \alpha\left(t_{j-1}\right)\right) \right| \leq |f\left(t_i^*\right)| \left[\alpha\left(t_i\right) - \alpha\left(t_{i-1}\right)\right] +$$
$$+ \left|f\left(t_{i+1}^*\right)\right| \left[\alpha\left(t_{i+1}\right) - \alpha\left(t_i\right)\right]$$
$$\leq \alpha\left(t_{i+1}\right) - \alpha\left(t_{i-1}\right) < \varepsilon$$

By definition of the Riemann-Stieltjes integral, this means that $f \in \mathcal{R}(\alpha)$ and $\int f d\alpha = 0$ $\qquad \square$

Exercise 6.2 Suppose $f \geq 0, f$ is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. Since $f(x)$ is continuous on $[a, b]$ and $\frac{f(x_0)}{2} > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$ for all $x \in [a, b]$ such that $|x - x_0| < \delta$. Let $\eta = \min\left(\delta, \max\left(x_0 - a, b - x_0\right)\right)$, so that $\eta > 0$. Let I be the interval $[x_0 - \eta, x_0]$ if it is contained in $[a, b]$; otherwise let $I = [x_0, x_0 + \eta]$. Whichever is the case, $I \subseteq [a, b]$ and $f(x) =$ $\qquad \square$

Exercise 6.4 If $f(x) = 0$ for all irrational $x, f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Every upper Riemann sum equals $b - a$, and every lower Riemann sum equals 0. Hence the set of upper sums and the set of lower sums do not have a common bound. $\qquad \square$

Exercise 6.6 Let P be the Cantor set. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on $[0, 1]$.

Proof. Let $M = \sup\{|f(x)| : a \leq x \leq b\}$, and let $\varepsilon > 0$ be given. Cover P by a finite collection of open intervals $O = \bigcup_{i=1}^{k} (a_i, b_i)$ such that $\sum (b_i - a_i) < \frac{\varepsilon}{4M}$. Let $\theta = \inf\{|x - y| : x \in P, y \in [a, b] \backslash O\}$. Since x and y range over disjoint compact sets, θ is a positive number. On the compact set $E = \left\{x : d(x, P) \geq \frac{1}{2}\theta\right\}$ the function f is uniformly continuous. Let $\delta > 0$ be such that $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$ if $x, y \in E$ and $|x - y| < \delta$. Then consider any partition $\{t_j\}$ of $[a, b]$ with $\max\left(t_j - t_{j-1}\right) < \min\left(\delta, \frac{1}{2}\theta\right)$. The difference

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between the upper and lower Riemann sums for this partition can be expressed as two sums:

$$\sum (M_j - m_j)(t_j - t_{j-1}) = \Sigma_1 + \Sigma_2,$$

where Σ_1 contains all the terms for which $[t_{j-1}, t_j]$ is contained in E and Σ_2 all the other terms. It is then obvious that

$$\Sigma_1 < \frac{\varepsilon}{2(b-a)} \sum (t_j - t_{j-1}) \leq \frac{\varepsilon}{2}$$

and, since each interval $[t_{j-1}, t_j]$ that occurs in Σ_2 is contained in O,

$$\Sigma_2 < 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

Therefore the upper and lower Riemann sums for any such partition differ by less than ε, and so f is Riemann integrable. $\qquad\square$