ProofNet NL Statements

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chapter 1

section 1

1a: Prove that the the operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$ is not associative.

2a: Prove the the operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$ is not commutative.

3: Prove that the addition of residue classes $\mathbb{Z}/n\mathbb{Z}$ is associative.

4: Prove that the multiplication of residue class $\mathbb{Z}/n\mathbb{Z}$ is associative.

5: Prove that for all $n > 1$ that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication of residue classes.

15: Prove that $(a_1 a_2 \ldots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \ldots a_1^{-1}$ for all $a_1, a_2, \ldots, a_n \in G$.

16: Let x be an element of G. Prove that $x^2 = 1$ if and only if $|x|$ is either 1 or 2.

17: Let x be an element of G. Prove that if $|x| = n$ for some positive integer n then $x^{-1} = x^{n-1}$.

18: Let x and y be elements of G. Prove that $xy = yx$ if and only if $y^{-1}xy = x$ if and only if $x^{-1}y^{-1}xy = 1$.

20: For x an element in G show that x and x^{-1} have the same order.

22a: If x and g are elements of the group G, prove that $|x| = \left|g^{-1}xg\right|$.

22b: Deduce that $|ab| = |ba|$ for all $a, b \in G$.

25: Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

29: Prove that $A \times B$ is an abelian group if and only if both A and B are abelian.

34: If x is an element of infinite order in G, prove that the elements $x^n, n \in \mathbb{Z}$ are all distinct.

section 3

8: Prove that if $\Omega = \{1, 2, 3, \ldots\}$ then S_Ω is an infinite group

section 6

4: Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.

11: Let A and B be groups. Prove that $A \times B \cong B \times A$.

17: Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

23: Let G be a finite group which possesses an automorphism σ such that $\sigma(g) = g$ if and only if $g = 1$. If σ^2 is the identity map from G to G, prove that G is abelian.

section 7

5: Prove that the kernel of an action of the group G on a set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

6: Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity.

chapter 2

section 1

5: Prove that G cannot have a subgroup H with $|H| = n - 1$, where $n = |G| > 2$.

13: Let H be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element x of H. Prove that $H = 0$ or $\mathbb{Q}$.

section 2

4: Prove that if H is a subgroup of G then H is generated by the set $H - \{1\}$.

13: Prove that the multiplicative group of positive rational numbers is generated by the set $\left\{\frac{1}{p} \mid p \text{ is a prime}\right\}$.

16a: A subgroup M of a group G is called a maximal subgroup if $M \neq G$ and the only subgroups of G which contain M are M and G. Prove that if H is a proper subgroup of the finite group G then there is a maximal subgroup of G containing H.

16c: Show that if $G = \langle x \rangle$ is a cyclic group of order $n \geq 1$ then a subgroup H is maximal if and only $H = \langle x^p \rangle$ for some prime p dividing n.

Chapter 3

section 1

3a: Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian.

22a: Prove that if H and K are normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G.

22b: Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

section 2

8: Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

11: Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$ (do not assume G is finite).

16: Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$to prove Fermat's Little Theorem: if p is a prime then $a^p \equiv a (\text{mod} p)$ for all $a \in \mathbb{Z}$.

21a: Prove that $\mathbb{Q}$ has no proper subgroups of finite index.

section 3

3: Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $|K : K \cap H| = p$.

section 4

1: Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

4: Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.

5a: Prove that subgroups of a solvable group are solvable.

5b: Prove that quotient groups of a solvable group are solvable.

11: Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrivial subgroup A of H with $A \trianglelefteq G$ and A abelian.

Chapter 4

section 2

8: Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n$!.

9a: Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G.

14: Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

section 3

5: If the center of G is of index n, prove that every conjugacy class has at most n elements.

26: Let G be a transitive permutation group on the finite set A with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element σ is called fixed point free).

27: Let $g_1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

section 4

2: Prove that if G is a n abelian group of order pq, where p and q are distinct primes, then G is cyclic.

6a: Prove that characteristic subgroups are normal.

7: If H is the unique subgroup of a given order in a group G prove H is characteristic in G.

8a: Let G be a group with subgroups H and K with $H \leq K$. Prove that if H is characteristic in K and K is normal in G then H is normal in G.

section 5

1: Prove that if $P \in \mathrm{Syl}_p(G)$ and H is a subgroup of G containing P then $P \in \mathrm{Syl}_p(H)$.

13: Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

14: Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

15: Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

16: Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r.

17: Prove that if $|G| = 105$ then G has a normal Sylow 5 -subgroup and a normal Sylow 7-subgroup.

18: Prove that a group of order 200 has a normal Sylow 5-subgroup.

19: Prove that if $|G| = 6545$ then G is not simple.

20: Prove that if $|G| = 1365$ then G is not simple.

21: Prove that if $|G| = 2907$ then G is not simple.

22: Prove that if $|G| = 132$ then G is not simple.

23: Prove that if $|G| = 462$ then G is not simple.

28: Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

33: Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Chapter 5

section 4

2: Prove that a subgroup H of G is normal if and only if $[G, H] \leq H$.

Chapter 7

0.0.1 section 1

2: Prove that if u is a unit in R then so is $-u$.

11: Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.

12: Prove that any subring of a field which contains the identity is an integral domain.

15: A ring R is called a Boolean ring if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative.

section 2

2: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an element of the polynomial ring $R[x]$. Prove that $p(x)$ is a zero divisor in $R[x]$ if and only if there is a nonzero $b \in R$ such that $bp(x) = 0$.

4: Prove that if R is an integral domain then the ring of formal power series $R[[x]]$ is also an integral domain.

12: Let $G = \{g_1, \ldots, g_n\}$ be a finite group. Prove that the element $N = g_1 + g_2 + \ldots + g_n$ is in the center of the group ring RG.

section 3

16: Let $\varphi : R \to S$ be a surjective homomorphism of rings. Prove that the image of the center of R is contained in the center of S.

28: Prove that an integral domain has characteristic p, where p is either a prime or 0

37: An ideal N is called nilpotent if N^n is the zero ideal for some $n \geq 1$. Prove that the ideal $p\mathbb{Z}/p^m\mathbb{Z}$ is a nilpotent ideal in the ring $\mathbb{Z}/p^m\mathbb{Z}$.

section 4

27: Let R be a commutative ring with $1 \neq 0$. Prove that if a is a nilpotent element of R then $1 - ab$ is a unit for all $b \in R$.

Chapter 8

section 1

12: Let N be a positive integer. Let M be an integer relatively prime to N and let d be an integer relatively prime to $\varphi(N)$, where φ denotes Euler's φ-function. Prove that if $M_1 \equiv M^d(\mathrm{mod} N)$ then $M \equiv M_1^{d'}(\mathrm{mod} N)$ where d' is the inverse of d $\mathrm{mod}\varphi(N) : dd' \equiv 1(\mathrm{mod}\varphi(N))$

section 2

4: Let R be an integral domain. Prove that if the following two conditions hold then R is a Principal Ideal Domain: (i) any two nonzero elements a and b in R have a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$, and (ii) if $a_1, a_2, a_3, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

section 3

4: Prove that if an integer is the sum of two rational squares, then it is the sum of two integer squares (for example, $13 = (1/5)^2 + (18/5)^2 = 2^2 + 3^2$).

5a: Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a squarefree integer greater than 3. Prove that $2, \sqrt{-n}$ and $1 + \sqrt{-n}$ are irreducibles in R.

6a: Prove that the quotient ring $\mathbb{Z}[i]/(1 + i)$ is a field of order 2 .

6b: Let $q \in \mathbb{Z}$ be a prime with $q \equiv 3 \bmod 4$. Prove that the quotient ring $\mathbb{Z}[i]/(q)$ is a field with q^2 elements.

chapter 9

section 1

6: Prove that (x, y) is not a principal ideal in $\mathbb{Q}[x, y]$.

10: Prove that the ring $\mathbb{Z}\left[x_1, x_2, x_3, \ldots\right] / \left(x_1x_2, x_3x_4, x_5x_6, \ldots\right)$ contains infinitely many minimal prime ideals (cf. Exercise 36 of Section 7.4).

section 3

2: Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.

section 4

2a: Prove that $x^4 - 4x^3 + 6$ is irreducible in $\mathbb{Z}[x]$.

2b: Prove that $x^6 + 30x^5 - 15x^+6x - 120$ is irreducible in $\mathbb{Z}[x]$.

2c: Prove that $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible in $\mathbb{Z}[x]$.

2d: Prove that $\frac{(x+2)^p-2^p}{x}$, where p is an odd prime, is irreducible in $\mathbb{Z}[x]$.

9: Prove that the polynomial $x^2-\sqrt{2}$ is irreducible over $\mathbb{Z}[\sqrt{2}]$. You may assume that $\mathbb{Z}[\sqrt{2}]$ is a U.F.D.

11: Prove that x^2+y^2-1 is irreducible in $\mathbb{Q}[x,y]$.

Rudin

chapter 1

1: If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

2: Prove that there is no rational number whose square is 12.

5: Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$

14: If z is a complex number such that $|z|=1$, that is, such that $z\bar{z}=1$, compute $|1+z|^2+|1-z|^2$.

18a: If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$

chapter 3

1a: Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

3: If $s_1=\sqrt{2}$, and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$ $\quad(n=1,2,3,\ldots)$, prove that $\{s_n\}$ converges, and that $s_n<2$ for $n=1,2,3,\ldots$.

5: For any two real sequences $\{a_n\},\{b_n\}$, prove that $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty-\infty$.

7: Prove that the convergence of Σa_n implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

8: If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

13: Prove that the Cauchy product of two absolutely convergent series converges absolutely.

20: 20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{nl}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21: If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

22: Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. Hint: Find a shrinking sequence of neighborhoods E_n such that $E_n \subset G_n$.

Munkres

chapter 2

section 13

5a: Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

section 16

4: A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

section 17

2: Show that if A is closed in Y and Y is closed in X, then A is closed in X.

3: Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

4: Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

18

8a: Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

8b: Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous. [Hint: Use the pasting lemma.]

13: Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

21

6a: Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$.

6b: Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence (f_n) does not converge uniformly.

8: Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

22

1: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

2a: Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

2b: If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

3: Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\bar{H}$ are topological groups.