Exercises from *Everything* by All Authors

Artin.exercise.2.3.2 Let a, b be elements of a group G. Prove that there exists an element $g \in G$ such that $bab^{-1} = gag^{-1}$.

Artin.exercise.2.8.6 The center of the direct product of two groups is isomorphic to the direct product of the centers.

Artin.exercise.3.2.7 If F is a field and G is a field extension of F, then the inclusion map $F \to G$ is injective.

Artin.exercise.3.7.2 Let V be a vector space over a field K, and let $\{W_i\}_{i \in I}$ be a family of subspaces of V. Prove that $\bigcap_{i \in I} W_i$ is nontrivial.

Artin.exercise.6.4.2 Prove that if G is a finite group of order pq where p and q are distinct primes, then G is not simple.

Artin.exercise.6.4.12 Prove that there is no simple group of order 224.

Artin.exercise.10.1.13 If x is nilpotent, then $1 + x$ is a unit.

Artin.exercise.10.4.7a Let R be a commutative ring with identity. Prove that if I and J are ideals of R such that $I + J = R$, then $IJ = I \cap J$.

Artin.exercise.10.6.7 If I is a nonzero ideal of $\mathbb{Z}[i]$, then there exists $z \in I$ such that z is real.

Artin.exercise.11.2.13 Prove that if a divides b in $\mathbb{Z}[i]$, then a divides b in $\mathbb{Z}$.

Artin.exercise.11.4.6a Prove that $X^2 + 1$ is irreducible over $\mathbb{F}_7$.

Artin.exercise.11.4.6c Prove that $X^3 - 9$ is irreducible over $\mathbb{Z}_{31}$.

Artin.exercise.11.13.3 Prove that there exists a prime p such that $p \geq N$ and $p + 1 \equiv 0 \pmod 4$.

Artin.exercise.13.6.10 Prove that the only element of the multiplicative group of a field is -1.

Axler.exercise.1.2 Prove that $(-1/2 + i\sqrt{3}/2)^3 = -1$.

Axler.exercise.1.4 If v is a vector in a vector space V over a field F, and a is an element of F, then $av = 0$ if and only if $a = 0$ or $v = 0$.

Axler.exercise.1.7 Prove that there exists a nonempty subset U of $\mathbb{R}^2$ such that $cU = U$ for all $c \in \mathbb{R}$ and U is not a subspace of $\mathbb{R}^2$.

Axler.exercise.1.9 Let U and W be subspaces of a vector space V. Prove that $U \cap W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

Axler.exercise.3.8 Let $L : V \to W$ be a linear map. Prove that there exists a subspace U of V such that $U \cap \ker L = \{0\}$ and $\operatorname{im} L = \operatorname{im}(L|_U)$.

Axler.exercise.5.1 Let V be a vector space over a field F, and let $L : V \to V$ be a linear transformation. Let $U_1, \ldots, U_n$ be subspaces of V such that $L(U_i) = U_i$ for each i. Prove that $L(\sum_{i=1}^n U_i) = \sum_{i=1}^n U_i$.

Axler.exercise.5.11 If S and T are endomorphisms of a vector space V, then the eigenvalues of ST are the same as the eigenvalues of TS.

Axler.exercise.5.13 Let T be a linear transformation of a finite-dimensional vector space V over a field F. Prove that if T fixes every subspace of V of codimension 1, then T is a scalar multiple of the identity.

Axler.exercise.5.24 Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $T : V \to V$ be a linear transformation such that $T(x) = cx$ for all $x \in V$ and some $c \in \mathbb{R}$. Prove that the rank of any subspace U of V is even.

Axler.exercise.6.3 If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$.

Axler.exercise.6.13 Let V be a complex inner product space, and let $e_1, \ldots, e_n$ be an orthonormal basis for V. Prove that $v \in V$ is in the span of $e_1, \ldots, e_n$ if and only if $\|v\|^2 = \sum_{i=1}^n |\langle v, e_i\rangle|^2$.

Axler.exercise.7.5 If V is a finite-dimensional complex inner product space of dimension at least 2, then the set of all operators T on V such that $T^*T = TT^*$ is not a subspace of the space of all operators on V.

Axler.exercise.7.9 Let T be a linear operator on a finite-dimensional inner product space V. Prove that T is self-adjoint if and only if all of its eigenvalues are real.

Axler.exercise.7.11 Let T be a linear operator on a finite-dimensional inner product space V such that $T^*T = TT^*$. Prove that there exists a linear operator S on V such that $S^2 = T$.

Dummit-Foote.exercise.1.1.2a Find two integers a and b such that $a - b \neq b - a$.

Dummit-Foote.exercise.1.1.4 Prove that $(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$ in $\mathbb{Z}/n\mathbb{Z}$.

Dummit-Foote.exercise.1.1.15 If $a_1, \ldots, a_n$ are elements of a group G, then $(a_1a_2 \ldots a_n)^{-1} = a_n^{-1}a_{n-1}^{-1} \ldots a_1^{-1}$.

Dummit-Foote.exercise.1.1.17 If x has order n, then $x^{-1} = x^{n-1}$.

Dummit-Foote.exercise.1.1.20 Prove that the order of an element x of a group G is equal to the order of x^{-1}.

Dummit-Foote.exercise.1.1.22b Prove that the order of ab is equal to the order of ba.

Dummit-Foote.exercise.1.1.29 Prove that a group G is abelian if and only if $G \times G$ is abelian.

Dummit-Foote.exercise.1.3.8 Prove that the set of all permutations of $\mathbb{N}$ is infinite.

Dummit-Foote.exercise.1.6.11 Prove that the group $A \times B$ is isomorphic to the group $B \times A$.

Dummit-Foote.exercise.1.6.23 Let G be a group and let σ be an automorphism of G such that $\sigma(g) = 1$ implies $g = 1$ and $\sigma(\sigma(g)) = g$ for all $g \in G$. Prove that G is abelian.

Dummit-Foote.exercise.2.1.13 Let H be a subgroup of $\mathbb{Q}$. Prove that H is either $\{0\}$ or $\mathbb{Q}$.

Dummit-Foote.exercise.2.4.16a Let H be a proper subgroup of G. Then there exists a maximal subgroup M of G containing H.

Dummit-Foote.exercise.2.4.16c Let H be a subgroup of $(\mathbb{Z}/n\mathbb{Z}, +)$. Prove that H is a maximal subgroup if and only if H is nontrivial and H is the subgroup generated by a prime p.

Dummit-Foote.exercise.3.1.22a If H and K are normal subgroups of G, then $H \cap K$ is a normal subgroup of G.

Dummit-Foote.exercise.3.2.8 If H and K are finite subgroups of G with coprime orders, then $H \cap K = \{1\}$.

Dummit-Foote.exercise.3.2.16 If p is prime and a is coprime to p, then $a^p \equiv a \pmod{p}$.

Dummit-Foote.exercise.3.3.3 Let H be a normal subgroup of G of index p. Prove that if K is a subgroup of G, then either $K \leq H$, or $H \cap K$ has index p in K, or $H \cup K = G$.

Dummit-Foote.exercise.3.4.4 If G is a finite commutative group, then for any divisor n of $|G|$, there is a subgroup H of G such that $|H| = n$.

Dummit-Foote.exercise.3.4.5b If G is solvable and H is a normal subgroup of G, then G/H is solvable.

Dummit-Foote.exercise.4.2.8 If H is a subgroup of G of index n, then there is a normal subgroup K of H of index at most $n!$.

Dummit-Foote.exercise.4.2.9a Let G be a finite group of order p^n, where p is prime. Prove that every subgroup of index p is normal.

Dummit-Foote.exercise.4.4.2 Prove that a group of order pq is cyclic, where p and q are distinct primes.

Dummit-Foote.exercise.4.4.6b Prove that there exists a characteristic subgroup of a group that is not normal.

Dummit-Foote.exercise.4.4.8a If H is a normal subgroup of K and K is a normal subgroup of G, then H is a normal subgroup of G.

Dummit-Foote.exercise.4.5.13 If G is a group of order 56, then G has a normal Sylow p-subgroup for some prime p.

Dummit-Foote.exercise.4.5.15 If G is a group of order 351, then G has a normal Sylow p-subgroup for some prime p.

Dummit-Foote.exercise.4.5.17 If G is a group of order 105, then G has a Sylow 5-subgroup and a Sylow 7-subgroup.

Dummit-Foote.exercise.4.5.19 Prove that a group of order 6545 is not simple.

Dummit-Foote.exercise.4.5.21 Prove that there is no simple group of order 2907.

Dummit-Foote.exercise.4.5.23 Prove that there is no simple group of order 462.

Dummit-Foote.exercise.4.5.33 Let G be a finite group, P a Sylow p-subgroup of G, and H a subgroup of G. Prove that the Sylow p-subgroups of H are precisely the subgroups of H which are conjugate to $H \cap P$.

Dummit-Foote.exercise.7.1.2 Prove that if u is a unit in a ring R, then $-u$ is a unit in R.

Dummit-Foote.exercise.7.1.12 If K is a subring of a field F and $1 \in K$, then K is a domain.

Dummit-Foote.exercise.7.2.2 A polynomial p is divisible by 0 if and only if there is a nonzero scalar b such that $bp = 0$.

Dummit-Foote.exercise.7.3.16 If R is a ring and $\phi : R \to S$ is a surjective ring homomorphism, then $\phi(Z(R)) \subset Z(S)$.

Dummit-Foote.exercise.7.4.27 Let R be a commutative ring with $1 \neq 0$. If a is nilpotent and b is an element of R, then $1 - ab$ is a unit.

Dummit-Foote.exercise.8.2.4 Prove that a ring R is a principal ideal ring if and only if for any two nonzero elements $a, b \in R$ there exist $r, s \in R$ such that $gcd(a, b) = ra + sb$.

Dummit-Foote.exercise.8.3.5a Prove that if n is a squarefree integer greater than 3, then $2, 1+\sqrt{-n}, \sqrt{-n}$ are irreducible in $\mathbb{Z}[\sqrt{-n}]$.

Dummit-Foote.exercise.8.3.6b Let q be a prime congruent to 3 modulo 4. Prove that the quotient ring $\mathbb{Z}[i]/\langle q \rangle$ is a field of order q^2.

Dummit-Foote.exercise.9.1.10 Let $f_i(x) = x_i x_{i+1}$ for $i = 1, 2, \ldots$. Prove that the set of minimal primes of the ideal generated by $f_1, f_2, \ldots$ is infinite.

Dummit-Foote.exercise.9.4.2a Prove that $X^4 - 4X^3 + 6$ is irreducible over $\mathbb{Z}$.

Dummit-Foote.exercise.9.4.2c Prove that $X^4 + 4X^3 + 6X^2 + 2X + 1$ is irreducible over $\mathbb{Z}$.

Dummit-Foote.exercise.9.4.9 Prove that $X^2 - C\sqrt{2}$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

Dummit-Foote.exercise.11.1.13 Prove that $\mathbb{R}^n$ is isomorphic to $\mathbb{R}$ as a $\mathbb{Q}$-vector space.

Herstein.exercise.2.1.18 If G is a finite group of even order, then G has an element of order 2.

Herstein.exercise.2.1.26 If G is a finite group, then every element of G has finite order.

Herstein.exercise.2.2.3 Let G be a group. Suppose that for some n, the following three statements hold:

1. $a^n b^n = b^n a^n$ for all $a, b \in G$.
2. $a^{n+1} b^{n+1} = b^{n+1} a^{n+1}$ for all $a, b \in G$.
3. $a^{n+2} b^{n+2} = b^{n+2} a^{n+2}$ for all $a, b \in G$.

Prove that G is abelian.

Herstein.exercise.2.2.6c Let G be a group and let n be an integer greater than 1. Suppose that for all $a, b \in G$, $(ab)^n = a^n b^n$. Prove that for all $a, b \in G$, $(aba^{-1}b^{-1})^{n(n-1)} = 1$.

Herstein.exercise.2.3.16 Prove that a group G is cyclic if and only if every subgroup of G is either trivial or the whole group.

Herstein.exercise.2.5.23 Let G be a group in which every subgroup is normal. Prove that for any $a, b \in G$, there exists $j \in \mathbb{Z}$ such that $b * a = a^j * b$.

Herstein.exercise.2.5.31 If G is a finite group of order $p^n m$ where p is prime and p does not divide m, then the subgroup of order p^n is characteristic.

Herstein.exercise.2.5.43 Prove that a group of order 9 is abelian.

Herstein.exercise.2.5.52 Let G be a finite group and let ϕ be an automorphism of G. If $|I| > 3/4|G|$ and $\phi(x) = x^{-1}$ for all $x \in I$, then $\phi(x) = x^{-1}$ for all $x \in G$ and G is abelian.

Herstein.exercise.2.7.7 If N is a normal subgroup of G and $\phi : G \to G'$ is a homomorphism, then $\phi(N)$ is a normal subgroup of G'.

Herstein.exercise.2.8.15 Let G and H be finite groups of order pq, where p and q are distinct primes with q dividing $p - 1$. Prove that G and H are isomorphic.

Herstein.exercise.2.10.1 Let G be a group, A a normal subgroup of G, and b an element of G of prime order. Prove that $A \cap \langle b \rangle = \{1\}$.

Herstein.exercise.2.11.7 If P is a normal Sylow p-subgroup of G, then P is characteristic.

Herstein.exercise.3.2.21 Let σ and τ be permutations of a finite set X. Prove that if $\sigma(x) = x$ if and only if $\tau(x) \neq x$ for all $x \in X$, and $\tau \circ \sigma = 1$, then $\sigma = 1$ and $\tau = 1$.

Herstein.exercise.4.1.34 Prove that the group of permutations of $\{1, 2, 3\}$ is isomorphic to the group of 2×2 matrices over $\mathbb{Z}_2$.

Herstein.exercise.4.2.6 Prove that if $a^2 = 0$ then $a(ax + xa) = x + xa^2$.

Herstein.exercise.4.3.1 Let R be a commutative ring and a an element of R. Prove that the set of all elements x of R such that $xa = 0$ is an ideal of R.

Herstein.exercise.4.4.9 If p is an odd prime, then there exists a subset S of $\mathbb{Z}_p$ of size $(p-1)/2$ such that S is the set of squares modulo p.

Herstein.exercise.4.5.23 Prove that the polynomials $x^3 - 2$ and $x^3 + 2$ are irreducible over $\mathbb{Z}_7$ and that the quotient rings $\mathbb{Z}_7[x]/(x^3 - 2)$ and $\mathbb{Z}_7[x]/(x^3 + 2)$ are isomorphic.

Herstein.exercise.4.6.2 Prove that $X^3 + 3X + 2$ is irreducible over $\mathbb{Q}$.

Herstein.exercise.5.1.8 If F is a field of characteristic p, then $(a + b)^p = a^p + b^p$.

Herstein.exercise.5.3.7 If a is algebraic over F, then a^2 is algebraic over F.

Herstein.exercise.5.4.3 Let $p(x) = x^5 + \sqrt{2}x^3 + \sqrt{5}x^2 + \sqrt{7}x + 11$. Prove that if $p(a) = 0$, then a is a root of a polynomial of degree less than 80 with integer coefficients.

Herstein.exercise.5.6.14 If F is a field of characteristic p, then the number of roots of $x^p - x$ is p.

Ireland-Rosen.exercise.1.30 Prove that $\sum_{i=1}^{n} 1/(n+2)$ is not an integer.

Ireland-Rosen.exercise.2.4 Let a be a nonzero integer. Prove that the function $f_a(x) = ax$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$.

Ireland-Rosen.exercise.2.27a Prove that the series $\sum_{p \in \mathbb{Z}, p \text{ squarefree}} 1/p$ diverges.

Ireland-Rosen.exercise.3.4 Prove that there are no integers x and y such that $3x^2 + 2 = y^2$.

Ireland-Rosen.exercise.3.10 Prove that if n is not prime and $n \neq 4$, then n divides $(n-1)!$.

Ireland-Rosen.exercise.4.4 If $p = 4t + 1$ is prime, then a is a primitive root modulo p if and only if $-a$ is a primitive root modulo p.

Ireland-Rosen.exercise.4.6 Prove that 3 is a primitive root modulo p if $p = 2^n + 1$.

Ireland-Rosen.exercise.4.11 Let p be a prime and let k be a positive integer. Prove that p^k divides $p^{k+s} - 1$ if and only if p divides $p^s - 1$.

Ireland-Rosen.exercise.5.28 If p is a prime congruent to 1 modulo 4, then there is an x such that $x^4 \equiv 2 \pmod{p}$ if and only if there are integers A and B such that $p = A^2 + 64B^2$.

Ireland-Rosen.exercise.12.12 Prove that $\sin(\pi/12)$ is algebraic over $\mathbb{Q}$.

Munkres.exercise.13.3b Prove that the following statement is false: If X is any set, and if S is any collection of subsets of X such that each member of S is either finite or empty or equal to X, then the union of the members of S is either finite or empty or equal to X.

Munkres.exercise.13.4a2 There exists a family of topologies $\{T_i\}_{i \in I}$ on a set X such that $\bigcap_{i \in I} T_i$ is not a topology on X.

Munkres.exercise.13.4b2 Let X be a set and let $\{T_i\}_{i \in I}$ be a family of topologies on X. Prove that there is a unique topology T on X such that $T_i \subset T$ for all $i \in I$ and T is the smallest topology on X with this property.

Munkres.exercise.13.5b Let X be a set and let A be a collection of subsets of X. Prove that the topology generated by A is the intersection of all topologies on X that contain A.

Munkres.exercise.13.8a Prove that the collection of all open intervals with rational endpoints is a topological basis for the real line.

Munkres.exercise.16.1 Let X be a topological space, Y a subset of X, and A a subset of Y. Prove that a subset U of A is open in A if and only if U is open in Y.

Munkres.exercise.16.6 Prove that the collection of all open rectangles in the plane is a basis for the Euclidean topology.

Munkres.exercise.18.8a Let X and Y be topological spaces, and let f and g be continuous functions from X into Y. Prove that the set $\{x \in X : f(x) \leq g(x)\}$ is closed in X.

Munkres.exercise.18.13 Let X be a topological space, Y a T_2 space, and A a subset of X. Let f be a continuous mapping of A into Y. Prove that if g is a continuous mapping of $\overline{A}$ into Y such that $g(x) = f(x)$ for all $x \in A$, then g is the only continuous mapping of $\overline{A}$ into Y with this property.

Munkres.exercise.20.2 Prove that the product topology on $\mathbb{R}^2$ is metrizable.

Munkres.exercise.21.6b Prove that the sequence of functions $f_n(x) = x^n$ does not converge uniformly on any interval I.

Munkres.exercise.22.2a Let $p : X \to Y$ be a continuous map. Then p is a quotient map if and only if there is a continuous map $f : Y \to X$ such that $p \circ f = id_Y$.

Munkres.exercise.22.5 Let X and Y be topological spaces, and let $p : X \to Y$ be an open mapping. Let A be an open subset of X. Prove that the restriction of p to A is an open mapping.

Munkres.exercise.23.3 Let X be a topological space, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of X. Suppose that A_0 is a connected subset of X such that $A_0 \cap A_n \neq \emptyset$ for all n. Prove that $A_0 \cup (\cup_{n=1}^{\infty} A_n)$ is connected.

Munkres.exercise.23.6 Let X be a topological space, and let A and C be subsets of X. Suppose that C is connected, that $C \cap A$ is nonempty, and that $C \cap A^c$ is nonempty. Prove that $C \cap \partial A$ is nonempty.

Munkres.exercise.23.11 Let X and Y be topological spaces, and let $p : X \to Y$ be a quotient map. If Y is connected and each fiber $p^{-1}(y)$ is connected, then X is connected.

Munkres.exercise.24.3a If $f : I \to I$ is continuous, then f has a fixed point.

Munkres.exercise.25.9 If G is a topological group and C is the connected component of the identity, then C is a normal subgroup of G.

Munkres.exercise.26.12 Suppose X and Y are topological spaces, $p : X \to Y$ is a continuous surjection, and $p^{-1}(y)$ is compact for each $y \in Y$. If Y is compact, then X is compact.

Munkres.exercise.28.4 A topological space X is countably compact if and only if it is limit point compact.

Munkres.exercise.28.6 If X is a compact metric space and $f : X \to X$ is an isometry, then f is a bijection.

Munkres.exercise.29.4 Prove that the space $\mathbb{N}^I$ is not locally compact.

Munkres.exercise.30.10 Let $X_1, X_2, \ldots$ be topological spaces. Suppose that for each i, there is a countable dense subset S_i of X_i. Prove that there is a countable dense subset of the product space $X_1 \times X_2 \times \ldots$.

Munkres.exercise.31.1 Let X be a regular space. Prove that for any two points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Munkres.exercise.31.3 Prove that the order topology on a partially ordered set is regular.

Munkres.exercise.32.2a If X_i is a topological space for each $i \in I$, and if $\prod_{i \in I} X_i$ is a T_2 space, then each X_i is a T_2 space.

Munkres.exercise.32.2c Prove that if X_i is a normal space for each $i \in I$, then $\prod_{i \in I} X_i$ is normal.

Munkres.exercise.33.7 Let X be a locally compact Hausdorff space. Prove that for each closed set A and each point x not in A, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$.

Munkres.exercise.34.9 If X is a compact space, and X_1 and X_2 are closed subsets of X such that $X_1 \cup X_2 = X$, and X_1 and X_2 are metrizable, then X is metrizable.

Munkres.exercise.43.2 Let X be a metric space, Y a complete metric space, and A a subset of X. Suppose that $f : A \to Y$ is uniformly continuous. Prove that there exists a unique continuous function $g : \overline{A} \to Y$ such that $g(x) = f(x)$ for all $x \in A$.

Pugh.exercise.2.26 A set U is open if and only if for each $x \in U$, x is not a cluster point of U^c.

Pugh.exercise.2.32a Prove that the set of all natural numbers is clopen.

Pugh.exercise.2.46 Let A and B be compact sets in a metric space M such that A and B are disjoint and nonempty. Prove that there exist points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0)$ is less than or equal to $d(a, b)$ for all $a \in A$ and $b \in B$.

Pugh.exercise.2.92 Let X be a topological space, and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of nonempty compact subsets of X such that $S_n \subset S_{n+1}$ for all n. Prove that $\bigcap_{n=1}^{\infty} S_n$ is nonempty.

Pugh.exercise.3.1 Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Pugh.exercise.3.63a Prove that the function $f(x) = \frac{1}{x(\log x)^p}$ converges to 0 as x tends to infinity, for any $p > 1$.

Pugh.exercise.4.15a Let F be a set of functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that F is equicontinuous at x if and only if there exists a function μ such that $\mu(x) \geq 0$ for all x, $\mu(x) \to 0$ as $x \to 0$, and $|f(s)-f(t)| \leq \mu(|s-t|)$ for all $f \in F$.

Putnam.exercise.1998.b6 Prove that there exists a positive integer n such that the equation $x^2 = n^3 + an^2 + bn + c$ has no integer solution.

Putnam.exercise.1999.b4 Let f be a function of class C^3 on $\mathbb{R}$ such that $f^{(n)}(x) > 0$ for $n = 0, 1, 2, 3$ and $f^{(3)}(x) \leq f(x)$ for all x. Prove that $f'(x) < 2f(x)$ for all x.

Putnam.exercise.2001.a5 Prove that there is a unique pair of positive integers (a, n) such that $a^{n+1} - (a+1)^n = 2001$.

Putnam.exercise.2014.a5 Let $P_n(x) = (n+1)x^n + (n+1)x^{n-1} + \cdots + (n+1)x + (n+1)$. Prove that P_j and P_k are coprime for $j \neq k$.

Putnam.exercise.2018.a5 Let f be a continuous function on $\mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all x. Prove that there exists a positive integer n and a point x such that $f^{(n)}(x) = 0$.

Putnam.exercise.2018.b4 Let $x_0, x_1, x_2, \ldots$ be a sequence of real numbers defined by $x_0 = a$, $x_1 = a$, and $x_n = 2x_{n-1}x_{n-2} - x_{n-3}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then $x_n = x_{n+k}$ for all $k \geq 0$.

Rudin.exercise.1.1b If x is irrational and y is rational and nonzero, then xy is irrational.

Rudin.exercise.1.4 Let S be a nonempty set of real numbers which is bounded above. Prove that there is a number b such that $b \leq s$ for every $s \in S$.

Rudin.exercise.1.8 Prove that there is no linear order on $\mathbb{C}$.

Rudin.exercise.1.12 If $z_1, \ldots, z_n$ are complex, then $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

Rudin.exercise.1.14 If z is a complex number of modulus 1, prove that $(1+z)^2 + (1-z)^2 = 4$.

Rudin.exercise.1.17 If x and y are vectors in $\mathbb{R}^n$, prove that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Rudin.exercise.1.18b Prove that the statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y \neq 0 \wedge xy = 0$ is false.

Rudin.exercise.2.19a If A and B are disjoint closed sets, then A and B are separated.

Rudin.exercise.2.25 Prove that every compact metric space has a countable basis.

Rudin.exercise.2.27b Let E be a non-countable subset of $\mathbb{R}^k$ and let P be the set of points x such that for every neighborhood U of x, the set $P \cap E$ is non-countable. Prove that $E \setminus P$ is countable.

Rudin.exercise.2.29 Let U be an open set in $\mathbb{R}$. Prove that there exists a sequence of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$.

Rudin.exercise.3.2a Prove that $\lim_{n\to\infty} \sqrt{n^2+n} - n = 1/2$.

Rudin.exercise.3.5 If a_n and b_n are bounded sequences, then $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Rudin.exercise.3.7 Prove that if $\sum_{i=1}^{\infty} a_i$ converges, then $\sum_{i=1}^{\infty} \frac{\sqrt{a_i}}{n}$ converges.

Rudin.exercise.3.13 If $\sum_{i=1}^{\infty} |a_i|$ and $\sum_{i=1}^{\infty} |b_i|$ converge, then $\sum_{i=1}^{\infty} \sum_{j=1}^{i+1} a_j b_{i-j}$ converges.

Rudin.exercise.3.21 Let X be a complete metric space. Suppose that $\{E_n\}_{n=1}^{\infty}$ is a sequence of nonempty closed subsets of X such that $E_n \supset E_{n+1}$ for all n and $\lim_{n\to\infty} diam(E_n) = 0$. Prove that $\bigcap_{n=1}^{\infty} E_n$ is a singleton.

Rudin.exercise.4.1a There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is not continuous but for each $x \in \mathbb{R}$, the function $y \mapsto f(x+y) - f(x-y)$ is continuous at 0.

Rudin.exercise.4.3 Let f be a continuous function from a metric space X into $\mathbb{R}$. Prove that the set $f^{-1}(0)$ is closed.

Rudin.exercise.4.4b Suppose f and g are continuous functions on a metric space X and that S is a dense subset of X. Prove that if $f(x) = g(x)$ for all $x \in S$, then $f(x) = g(x)$ for all $x \in X$.

Rudin.exercise.4.5b There exists a set E and a function $f : E \to \mathbb{R}$ such that f is continuous on E but there is no continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in E$.

Rudin.exercise.4.8a If f is uniformly continuous on a bounded set E, then $f(E)$ is bounded.

Rudin.exercise.4.11a If $f : X \to Y$ is uniformly continuous and x_n is a Cauchy sequence in X, then $f(x_n)$ is a Cauchy sequence in Y.

Rudin.exercise.4.15 If f is continuous and open, then f is monotone.

Rudin.exercise.4.21a Suppose K and F are disjoint compact and closed subsets of a metric space X. Prove that there is a positive number δ such that $d(p,q) \geq \delta$ for all $p \in K$ and $q \in F$.

Rudin.exercise.5.1 Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Rudin.exercise.5.3 Suppose g is continuous and g' is bounded. Prove that for each $\epsilon > 0$ there is a $\delta > 0$ such that the function $x \mapsto x + \epsilon g(x)$ is one-to-one on the interval $(-\delta, \delta)$.

Rudin.exercise.5.5 Suppose that f is differentiable on $\mathbb{R}$ and that $\lim_{x\to\infty} f'(x) = 0$. Prove that $\lim_{x\to\infty} f(x+1) - f(x) = 0$.

Rudin.exercise.5.7 Suppose that f and g are differentiable at x, that $g'(x) \neq 0$, and that $f(x) = g(x) = 0$. Prove that $\lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

Rudin.exercise.5.17 Let f be a function defined on $[-1, 1]$ and differentiable on $(-1, 1)$. Suppose that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Prove that there exists a point x in $(-1, 1)$ such that $f'''(x) \geq 3$.

Shakarchi.exercise.1.13b Let f be a differentiable function on an open set Ω in $\mathbb{C}$. Suppose that the imaginary part of f is constant on Ω. Prove that f is constant on Ω.

Shakarchi.exercise.1.19a Let z be a complex number of modulus 1. Let $s_n = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. Prove that s_n does not converge.

Shakarchi.exercise.1.19c Let z be a complex number of modulus 1 and $z \neq 1$. Let $s_n = \sum_{i=1}^{n} iz^i/i$. Prove that s_n converges.

Shakarchi.exercise.2.2 Prove that $\lim_{y\to\infty} \int_0^y \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Shakarchi.exercise.2.13 Let f be a function from $\mathbb{C}$ to $\mathbb{C}$. Suppose that for each $z_0 \in \mathbb{C}$ there is an open set S containing z_0 and a sequence of complex numbers $\{c_n\}$ such that for each $z \in S$ the series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges to $f(z)$ and $c_n = 0$ for all but finitely many n. Prove that f is a polynomial.

Shakarchi.exercise.3.4 Prove that $\lim_{y\to\infty} \int_{-y}^{y} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}$.

Shakarchi.exercise.3.14 If f is a differentiable function from $\mathbb{C}$ to $\mathbb{C}$ which is one-to-one, then f is of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$.

Shakarchi.exercise.5.1 Let f be a non-zero complex-valued function defined on the unit disk D and differentiable on D. Suppose that f has only finitely many zeros in D. Prove that the series $\sum_{n=1}^{\infty}(1 - z_n)$ converges, where z_n is the nth zero of f in D.