Exercises from
Putnam Competition

Exercise 2021.b4 Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

Exercise 2020.b5 For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that $3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0$.

Exercise 2018.a5 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0, f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Exercise 2018.b2 Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Exercise 2018.b4 Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Exercise 2018.b6 Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860 . Prove that the cardinality of S is at most $2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}$.

Exercise 2017.b3 Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1 . Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Exercise 2016.a6 Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form $g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$ with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{-1, 1\}$. (Here $|G|$ is the number of elements of G.)

Exercise 2015.b1 Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

Exercise 2014.a5 Let $P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$. Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \neq k$.

Exercise 2014.a3 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy $\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0$. Prove that there exists a real number y with $0 < y < 1$ such that $a_0 + a_1 y + \cdots + a_n y^n = 0$.

Exercise 2010.a4 Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Exercise 2008.a1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

Exercise 2007.b1 Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

Exercise 2005.a3 Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Exercise 2001.a5 Prove that there are unique positive integers a, n such that $a^{n+1} - (a + 1)^n = 2001$.

Exercise 2000.a2 Prove that there exist infinitely many integers n such that $n, n + 1, n + 2$ are each the sum of the squares of two integers.

Exercise 1999.b4 Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Exercise 1998.a3 Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0$.

Exercise 1998.b6 Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.