# Exercises from *Linear Algebra Done Right* by Sheldon Axler

**Exercise 1.2** Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

**Exercise 1.3** Prove that $-(-v) = v$ for every $v \in V$.

**Exercise 1.4** Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

**Exercise 1.6** Give an example of a nonempty subset $U$ of $\mathbf{R}^2$ such that $U$ is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but $U$ is not a subspace of $\mathbf{R}^2$.

**Exercise 1.7** Give an example of a nonempty subset $U$ of $\mathbf{R}^2$ such that $U$ is closed under scalar multiplication, but $U$ is not a subspace of $\mathbf{R}^2$.

**Exercise 1.8** Prove that the intersection of any collection of subspaces of $V$ is a subspace of $V$.

**Exercise 1.9** Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.

**Exercise 2.1** Prove that if $(v_1, \ldots, v_n)$ spans $V$, then so does the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ obtained by subtracting from each vector (except the last one) the following vector.

**Exercise 2.2** Prove that if $(v_1, \ldots, v_n)$ is linearly independent in $V$, then so is the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ obtained by subtracting from each vector (except the last one) the following vector.

**Exercise 2.6** Prove that the real vector space consisting of all continuous realvalued functions on the interval $[0, 1]$ is infinite dimensional.

**Exercise 3.1** Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

**Exercise 3.8** Suppose that $V$ is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace $U$ of $V$ such that $U \cap \operatorname{null} T = \{0\}$ and range $T = \{Tu : u \in U\}$.

**Exercise 3.9** Prove that if $T$ is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that $\operatorname{null} T = \big\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\big\}$, then $T$ is surjective.

**Exercise 3.10** Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\big\}$.

**Exercise 3.11** Prove that if there exists a linear map on $V$ whose null space and range are both finite dimensional, then $V$ is finite dimensional.

**Exercise 4.4** Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree $m$. Prove that $p$ has $m$ distinct roots if and only if $p$ and its derivative $p'$ have no roots in common.

**Exercise 5.1** Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \ldots, U_m$ are subspaces of $V$ invariant under $T$, then $U_1 + \cdots + U_m$ is invariant under $T$.

**Exercise 5.4** Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null}(T - \lambda I)$ is invariant under $S$ for every $\lambda \in \mathbf{F}$.

**Exercise 5.11** Suppose $S, T \in \mathcal{L}(V)$. Prove that $ST$ and $TS$ have the same eigenvalues.

**Exercise 5.12** Suppose $T \in \mathcal{L}(V)$ is such that every vector in $V$ is an eigenvector of $T$. Prove that $T$ is a scalar multiple of the identity operator.

**Exercise 5.13** Suppose $T \in \mathcal{L}(V)$ is such that every subspace of $V$ with dimension $\dim V - 1$ is invariant under $T$. Prove that $T$ is a scalar multiple of the identity operator.

**Exercise 5.20** Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as $T$ (not necessarily with the same eigenvalues). Prove that $ST = TS$.

**Exercise 5.24** Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of $V$ invariant under $T$ has even dimension.

**Exercise 6.2** Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$.

**Exercise 6.3** Prove that $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j a_j{}^2\right)\left(\sum_{j=1}^{n} \frac{b_j{}^2}{j}\right)$ for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

**Exercise 6.7** Prove that if $V$ is a complex inner-product space, then $\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$ for all $u, v \in V$.

**Exercise 6.13** Suppose $(e_1, \ldots, e_m)$ is an or thonormal list of vectors in $V$. Let $v \in V$. Prove that $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$ if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

**Exercise 6.16** Suppose $U$ is a subspace of $V$. Prove that $U^{\perp} = \{0\}$ if and only if $U = V$

**Exercise 6.17** Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$, then $P$ is an orthogonal projection.

**Exercise 6.18** Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $\|Pv\| \leq \|v\|$ for every $v \in V$, then $P$ is an orthogonal projection.

**Exercise 6.19** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove that $U$ is invariant under $T$ if and only if $P_U T P_U = T P_U$.

**Exercise 6.20** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove that $U$ and $U^{\perp}$ are both invariant under $T$ if and only if $P_U T = T P_U$.

**Exercise 6.29** Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove that $U$ is invariant under $T$ if and only if $U^{\perp}$ is invariant under $T^*$.

**Exercise 7.4** Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that $P$ is an orthogonal projection if and only if $P$ is self-adjoint.

**Exercise 7.5** Show that if $\dim V \geq 2$, then the set of normal operators on $V$ is not a subspace of $\mathcal{L}(V)$.

**Exercise 7.6** Prove that if $T \in \mathcal{L}(V)$ is normal, then $\operatorname{range} T = \operatorname{range} T^*$.

**Exercise 7.8** Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}\left(\mathbf{R}^3\right)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$.

**Exercise 7.9** Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

**Exercise 7.10** Suppose $V$ is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that $T$ is self-adjoint and $T^2 = T$.

**Exercise 7.11** Suppose $V$ is a complex inner-product space. Prove that every normal operator on $V$ has a square root. (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

**Exercise 7.14** Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$, then $T$ has an eigenvalue $\lambda'$ such that $|\lambda - \lambda'| < \epsilon$.

**Exercise 7.15** Suppose $U$ is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that $U$ has a basis consisting of eigenvectors of $T$ if and only if there is an inner product on $U$ that makes $T$ into a self-adjoint operator.