Exercises from *Everything* by Everyone

Artin.exercise.10.1.13 Prove that if x is nilpotent, then $1 + x$ is a unit.

Artin.exercise.10.2.4 Prove that the span of a set of polynomials is the span of the set of polynomials multiplied by a constant.

Artin.exercise.10.4.6 Let I and J be ideals of a commutative ring R. Prove that if I and J are nilpotent, then so is $I \oplus J$.

Artin.exercise.10.5.16 If F is a field of characteristic different from 2, then the set of polynomials of degree at most 2 is a subspace of the space of all polynomials if and only if the set of polynomials of degree at most 2 is a subspace of the space of all polynomials.

Artin.exercise.10.6.7 Let I be an ideal of the gaussian integers. Prove that there exists an element z of I such that z is not a zero divisor and z is not a zero divisor in the gaussian integers.

Artin.exercise.10.7.10 Let R be a ring, and let M be an ideal of R. Prove that M is maximal if and only if M is maximal and for every ideal N of R, if N is maximal then $N = M$.

Artin.exercise.10.7.6 Prove that the field of polynomials over F has cardinality 5.

Artin.exercise.11.12.3 Prove that there exist integers x, y such that $x^2 + 5y^2 = p$ or $2x^2 + 2xy + 3y^2 = p$.

Artin.exercise.11.13.3 There exists a prime number p such that $p + 1$ is congruent to 0 modulo 4.

Artin.exercise.11.2.13 If a is a Gaussian integer, then a divides b if and only if a is a Gaussian integer.

Artin.exercise.11.3.1 Prove that a polynomial p is irreducible if and only if the polynomial p is irreducible over the field F.

Artin.exercise.11.3.4 Prove that the polynomial $X^3+6X+12$ is irreducible over the rationals.

Artin.exercise.11.4.1b Prove that the polynomial x^3+x+1 is irreducible over the field of rational functions.

Artin.exercise.11.4.6a Prove that the polynomial X^2+1 is irreducible over F.

Artin.exercise.11.4.6b Prove that the polynomial X^3-9 is irreducible over F.

Artin.exercise.11.4.6c Prove that the polynomial X^3-9 is irreducible over the field of integers modulo 31.

Artin.exercise.11.4.8 Prove that the polynomial X^n-p is irreducible over $\mathbb{Q}$ if p is a prime and n is a positive integer.

Artin.exercise.13.4.10 Prove that there exists a natural number k such that $p=2^{2^k}+1$.

Artin.exercise.13.6.10 Prove that the set of elements of K that are not roots of unity is a subgroup of K.

Artin.exercise.2.11.3 Prove that there exists an element x of order 2 in a group G.

Artin.exercise.2.2.9 Let G be a group, and let a, b be elements of G. Prove that if a and b commute, then $a*b=b*a$.

Artin.exercise.2.3.1 Prove that the multiplicative group of the real numbers is isomorphic to the additive group of the real numbers.

Artin.exercise.2.4.19 Prove that if x is an element of order 2 in a group G, then x is in the center of G.

Artin.exercise.2.8.6 Prove that the center of a product group is the product of the centers.

Artin.exercise.3.5.6 Let K be a field, V a vector space over K, and S a countable set. Suppose that S spans K and that S is linearly independent. Prove that S is countable.

Artin.exercise.6.1.14 If G is cyclic, then the center of G is trivial.

Artin.exercise.6.4.12 Prove that a simple group is not a group of order 224.

Artin.exercise.6.4.2 Prove that if G is a simple group, then G is not cyclic.

Artin.exercise.6.4.3 Prove that if G is a simple group, then G is not cyclic.

Artin.exercise.6.8.1 Let G be a group, and let $a, b \in G$. Prove that the closure of the set $\{a, b\}$ is equal to the closure of the set $\{b * a * b^2, b * a * b^3\}$.

Artin.exercise.6.8.4 Prove that the closure of a set of elements of a group is a subgroup.

Artin.exercise.6.8.6 Let G be a group, and let N be a normal subgroup of G. Prove that there exists a pair of elements g, h of G such that the closure of the set $\{g, h\}$ is the whole group G.

Axler.exercise.1.2 Prove that the complex number $z = -1/2 + i\sqrt{3}/2$ is a cube.

Axler.exercise.1.3 Prove that $-v$ is equal to v.

Axler.exercise.1.4 If v is a nonzero element of V, then a is a nonzero element of F if and only if a is a nonzero element of F or v is a nonzero element of V.

Axler.exercise.1.6 There exists a set U of ordered pairs of real numbers such that U is not empty, and for all ordered pairs (u, v) in U, $u + v$ is in U and $-u$ is in U.

Axler.exercise.1.7 There exists a set U of ordered pairs of real numbers such that U is not empty, and for every real number c and every ordered pair u of real numbers, if u is in U then $c \cdot u$ is also in U.

Axler.exercise.1.8 Let F be a field, V a vector space over F, and u a function from ι to the submodule of F-linear combinations of V. Prove that there exists a submodule U of F-linear combinations of V such that u is injective and $u(\iota) = U$.

Axler.exercise.1.9 Let U and W be submodules of V such that U is a proper subset of W. Prove that there exists a submodule U' of V such that U' is a proper subset of W and U' is a subset of U.

Axler.exercise.4.4 Prove that the degree of a polynomial is equal to the number of distinct roots of its derivative if and only if the number of distinct roots of the polynomial is equal to the number of distinct roots of its derivative.

Axler.exercise.5.11 Let F be a field, V a vector space over F, and S, T endomorphisms of V. Prove that the eigenvalues of S and T are the same as the eigenvalues of $S * T$.

Axler.exercise.5.12 Let V be a vector space over a field F, and let S be the endomorphism of V defined by $S(v) = v + v$ for all $v \in V$. Prove that S is diagonalizable.

Axler.exercise.5.20 Let V be a finite-dimensional vector space over a field F, and let S and T be endomorphisms of V. Prove that S and T are simultaneously diagonalizable if and only if S and T have the same number of eigenvalues.

Axler.exercise.5.4 Let V be a vector space over a field F, and let $S, T : V \to F$ be linear maps. Prove that if S and T are invertible, then S and T are simultaneously invertible.

Axler.exercise.6.13 If V is an inner product space, and e is an orthonormal basis, then the norm of v is the same as the sum of the squares of the inner products of v with the basis vectors.

Axler.exercise.6.16 If U is a submodule of $\mathbb{C}$ and $U = \{0\}$, then $U = \{0\}$.

Axler.exercise.6.3 Prove that for all $n \in \mathbb{N}$, the following inequality holds:

$$\sum_{i=1}^{n} a_i^2 \leq \sum_{i=1}^{n} \frac{b_i^2}{i}$$

Axler.exercise.6.7 Let V be an inner product space over $\mathbb{C}$. Prove that the inner product of two vectors is equal to the sum of the squares of the norms of the vectors, minus the sum of the squares of the norms of the vectors, plus the inner product of the vectors, minus the inner product of the vectors.

Axler.exercise.7.10 Prove that the endomorphism T of a finite-dimensional inner product space is self-adjoint if and only if $T^2 = T$.

Axler.exercise.7.11 Let V be a finite-dimensional inner product space over $\mathbb{C}$. Prove that there exists an endomorphism S of V such that $S^2 = T$.

Axler.exercise.7.5 Let V be a finite-dimensional inner product space over $\mathbb{C}$. Prove that the set of all self-adjoint endomorphisms of V is a proper subspace of the set of all endomorphisms of V.

Axler.exercise.7.6 Let V be a finite-dimensional inner product space over $\mathbb{C}$. Prove that the range of T is equal to the range of T^*.

Axler.exercise.7.9 Suppose V is a finite-dimensional inner product space over $\mathbb{C}$, and T is a self-adjoint linear operator on V. Prove that T is self-adjoint if and only if for every eigenvalue λ of T, λ is real.

Dummit-Foote.exercise.1.1.16 Prove that the order of an element x of a group G is either 1 or 2.

Dummit-Foote.exercise.1.1.17 If x is an element of order n, prove that $x^{-1} = x^{n-1}$.

Dummit-Foote.exercise.1.1.18 Prove that x and y commute if and only if y^{-1} and x commute.

Dummit-Foote.exercise.1.1.20 Prove that the order of an element x of a group G is equal to the order of x^{-1}.

Dummit-Foote.exercise.1.1.22a If x is an element of infinite order in G, then the order of x is equal to the order of $(g^{-1} * x * g)$.

Dummit-Foote.exercise.1.1.22b Prove that the order of $a * b$ is equal to the order of $b * a$.

Dummit-Foote.exercise.1.1.25 Prove that for any group G, the map $x \mapsto x^2$ is a bijection from G to itself.

Dummit-Foote.exercise.1.1.29 Prove that if A and B are groups, then A and B are isomorphic if and only if A and B are commutative.

Dummit-Foote.exercise.1.1.2a There exists an integer a such that a and $-a$ are not equal.

Dummit-Foote.exercise.1.1.34 If x is an element of infinite order in G, prove that x^n and x^m are distinct for all $n, m \in \mathbb{Z}$.

Dummit-Foote.exercise.1.1.4 Prove that for all $a, b, c \in \mathbb{N}$, $(a * b) * c \equiv a * (b * c) \pmod{n}$.

Dummit-Foote.exercise.1.1.5 Prove that the empty group is not cyclic.

Dummit-Foote.exercise.1.3.8 Prove that the set of all permutations of the natural numbers is infinite.

Dummit-Foote.exercise.1.6.11 Prove that $A \times B$ is isomorphic to $B \times A$.

Dummit-Foote.exercise.1.6.17 If f is a group homomorphism, then f is an isomorphism if and only if f is one-to-one and onto.

Dummit-Foote.exercise.1.6.23 Let G be a group, and let σ be an automorphism of G. Prove that σ is an inner automorphism if and only if σ is the identity.

Dummit-Foote.exercise.1.6.4 Prove that the multiplicative group of the complex numbers is empty.

Dummit-Foote.exercise.2.1.13 Prove that if H is a subgroup of $\mathbb{Q}$ and $x \in H$, then $1/x \in H$.

Dummit-Foote.exercise.3.1.22a Prove that if H and K are normal subgroups of a group G, then $H \cap K$ is a normal subgroup of G.

Dummit-Foote.exercise.3.1.22b Let G be a group, and let I be a set. Let H be a function from I to subgroups of G. Prove that H is a subgroup of G if and only if H is a normal subgroup of G.

Dummit-Foote.exercise.3.1.3a If A is a commutative group and B is a subgroup, then A is a commutative group if and only if B is a commutative group.

Dummit-Foote.exercise.3.2.16 If a and p are relatively prime, then $a^p \equiv a$ mod p.

Dummit-Foote.exercise.3.2.21a If G is a group, H is a subgroup of G, and H is not the trivial group, then H is a subgroup of the trivial group.

Dummit-Foote.exercise.3.2.8 If H and K are subgroups of a group G, and H and K have coprime cardinalities, then $H \cap K = \{1\}$.

Dummit-Foote.exercise.3.4.1 Prove that a group is cyclic if and only if it is simple and has a finite number of elements.

Dummit-Foote.exercise.3.4.4 Let G be a commutative group of finite type, and let n be a positive integer that divides the cardinality of G. Prove that there exists a subgroup H of G of cardinality n.

Dummit-Foote.exercise.3.4.5a Prove that if H is a subgroup of a solvable group G, then H is solvable.

Dummit-Foote.exercise.3.4.5b If H is a normal subgroup of G, then G is solvable if and only if G/H is solvable.

Dummit-Foote.exercise.4.4.6a Prove that if H is a subgroup of a group G and H is normal, then H is a normal subgroup of G.

Dummit-Foote.exercise.4.5.13 Prove that there exists a Sylow p-subgroup of G for every prime p dividing $|G|$.

Dummit-Foote.exercise.4.5.14 Prove that there exists a Sylow p-subgroup of G for every prime p.

Dummit-Foote.exercise.4.5.1a Prove that if P is a p-subgroup of G, then P is a p-group.

Herstein.exercise.2.10.1 Let G be a group, A a subgroup of G, and b an element of G such that A is normal in G and b is not in A. Prove that A is trivial if and only if A is contained in the closure of the subgroup generated by b.

Herstein.exercise.2.11.22 Let G be a group of order p^n, and let K be a subgroup of G of index p^{n-1}. Prove that K is normal.

Herstein.exercise.2.11.6 If P is a Sylow p-subgroup of G, then P is the only Sylow p-subgroup of G.

Herstein.exercise.2.11.7 Let G be a group, and let p be a prime dividing the order of G. Prove that if P is a Sylow p-subgroup of G, then P is characteristic.

Herstein.exercise.2.1.18 Prove that there exists an element a of order 2 in a group G such that $a^2 = 1$.

Herstein.exercise.2.1.21 Prove that the commutator subgroup of a group of order 5 is cyclic.

Herstein.exercise.2.1.26 Prove that there exists an element a of infinite order in a group G.

Herstein.exercise.2.1.27 Prove that there exists an element m of G such that for all a in G, $a^m = 1$.

Herstein.exercise.2.2.3 Prove that the commutator subgroup of a group is cyclic.

Herstein.exercise.2.2.5 Prove that the group of units of a commutative group is commutative.

Herstein.exercise.2.2.6c Prove that if G is a group, n is a positive integer, and h is a function such that $h(a * b) = h(a) * h(b)$ for all $a, b \in G$, then h is a homomorphism.

Herstein.exercise.2.3.16 Prove that if G is a group, then G is cyclic if and only if there exists a prime number p such that G has a finite number of elements of order p.

Herstein.exercise.2.4.36 Prove that n divides $(a^n - 1)$ if $a > 1$.

Herstein.exercise.2.5.23 et G be a group, and let $a, b \in G$. Prove that there exists an integer j such that $b * a = a^j * b$.

Herstein.exercise.2.5.30 If G is a group of order p^n and H is a subgroup of G of index p, then H is characteristic.

Herstein.exercise.2.5.31 If G is a commutative group of order p^n and exponent m, and H is a subgroup of G of index p^n, then H is characteristic.

Herstein.exercise.2.5.37 Prove that the group of permutations of the set $\{1, 2, 3\}$ is isomorphic to the group of even permutations of the set $\{1, 2, 3\}$.

Herstein.exercise.2.5.43 Prove that the commutator subgroup of a group of order 9 is cyclic.

Herstein.exercise.2.5.44 Let G be a group of order p^2, where p is a prime. Prove that there exists a subgroup N of G such that N is finite and $card N = p$.

Herstein.exercise.2.6.15 Suppose that G is a commutative group, and m and n are positive integers such that m and n are coprime. Prove that there exists an element g of G such that $g^m = g^n$.

Herstein.exercise.2.8.12 Prove that G and H are isomorphic if and only if G and H are commutative groups of order 21.

Herstein.exercise.2.8.15 Let G be a group of order pq, where p and q are distinct primes. Prove that G is isomorphic to a subgroup of S_{pq}.

Herstein.exercise.2.9.2 If G and H are finite cyclic groups, then $G \times H$ is cyclic if and only if G and H are coprime.

Herstein.exercise.4.2.5 Prove that the ring of integers is commutative.

Herstein.exercise.4.2.6 Prove that a is a unit if and only if a is a square root of 0.

Herstein.exercise.4.2.9 Let p be a prime number. Prove that there exists a number a such that $a/b = \sum_{i \in \mathbb{N}} 1/i$ for all $b \in \mathbb{N}$ such that b is odd and $b \leq p$.

Herstein.exercise.4.3.1 Let R be a commutative ring, and let a be an element of R such that $a^2 = 0$. Prove that there exists an ideal I of R such that $I^2 = 0$ and I contains the set of all elements x of R such that $x * a = 0$.

Herstein.exercise.4.3.25 Prove that the ideal generated by a matrix of size 2×2 over the real numbers is either the zero ideal or the whole ring.

Herstein.exercise.4.4.9 Prove that there exists a set of size $(p-1)/2$ in the residue class ring modulo p.

Herstein.exercise.4.5.16 Let p be a prime, and let q be an irreducible polynomial of degree n over the field of p elements. Prove that there exists a finite set of polynomials S such that S is a basis for the vector space of polynomials of degree at most n over the field of p elements, and S is linearly independent over the field of p elements.

Herstein.exercise.4.5.23 Prove that the polynomials p and q are irreducible and that the set of polynomials $zmod7$ is not a vector space.

Herstein.exercise.4.5.25 Prove that the polynomial X^p is irreducible over $\mathbb{Q}$ if p is a prime.

Herstein.exercise.4.6.2 Prove that the polynomial X^3+3X+2 is irreducible over the rationals.

Herstein.exercise.5.1.8 If p is a prime, m is a positive integer, and F is a field of characteristic p, then $(a+b)^m = a^m + b^m$.

Herstein.exercise.5.2.20 Let F be an infinite field, V a vector space over F, and u a function from $\mathbb{N}$ to V. Prove that the set $\{ui\}_{i\in\mathbb{N}}$ is not finite.

Herstein.exercise.5.3.7 Prove that if a is algebraic over F, then a^2 is algebraic over F.

Herstein.exercise.5.4.3 rove that there exists a polynomial p of degree less than 80 such that $p(a) = 0$ and p has a root a in the complex numbers.

Herstein.exercise.5.5.2 Prove that the polynomial $X^3 - 3X - 1$ is irreducible over the rationals.

Herstein.exercise.5.6.14 Let p be a prime, and let m, n be positive integers. Prove that the cardinality of the set of roots of the polynomial $X^m - X$ in the field F is m.

Munkres.exercise.13.1 If A is a set, and x is an element of A, then there exists a set U such that x is an element of U and U is open.

Munkres.exercise.13.3a Prove that the topological space X is compact if and only if X is finite.

Munkres.exercise.13.3b Prove that if s is a set of infinite sets, then $\bigcup_{t\in s} t$ is infinite.

Munkres.exercise.13.4a1 Prove that the union of a family of topologies is a topology.

Munkres.exercise.13.4a2 There exists a set X and a family of topologies on X such that for every i in I, the topology T_i is not a topology.

Munkres.exercise.13.4b1 Let X be a set, and let I be a set of subsets of X. Suppose that for each $i \in I$, T_i is a topology on X. Prove that there exists a topology T on X such that for each $i \in I$, T_i is a subset of T.

Munkres.exercise.13.5a Let X be a topological space, and let A be a basis for X. Prove that the set of all open sets in X is equal to the set of all intersections of open sets in X with A.

Munkres.exercise.13.5b Let X be a topological space, and let A be a subset of X. Prove that the set of all topologies on X containing A is equal to the set of all topologies on X containing $sInterA$.

Munkres.exercise.13.6 Prove that the topology of R^n is not the same as the topology of K^n.

Munkres.exercise.13.8a Prove that the set of all irrational numbers is a topological basis for the real line.

Munkres.exercise.13.8b Prove that the set of irrational numbers is open in the lower limit topology on $\mathbb{R}$.

Munkres.exercise.16.1 Let X be a topological space, and let Y be a set. Prove that for every $A \subseteq Y$, the set U is open if and only if U is open as a subset of X.

Munkres.exercise.16.4 Let X and Y be topological spaces, and let π_1 and π_2 be the projections from $X \times Y$ to X and Y respectively. Prove that π_1 and π_2 are open maps if and only if X and Y are open.

Munkres.exercise.16.6 Prove that the set of all rational numbers is a topological basis for the real numbers.

Munkres.exercise.17.4 Let X be a topological space, and let U and A be subsets of X. Prove that $U \setminus A$ and $A \setminus U$ are open.

Munkres.exercise.18.13 Suppose X and Y are topological spaces, A is a closed subset of X, f is a continuous function from A to Y, and g is a continuous function from Y to Y. Prove that g is continuous if and only if g is constant on A.

Munkres.exercise.18.8a Suppose X and Y are topological spaces, and f and g are continuous functions. Prove that the set of points x such that $f(x) \leq g(x)$ is closed.

Munkres.exercise.18.8b Let X and Y be topological spaces, and let f and g be continuous functions from X to Y. Prove that the function h defined by $h(x) = \min(f(x), g(x))$ is continuous.

Munkres.exercise.19.6a Suppose that f is a function from the natural numbers into a type, and y is a function from the natural numbers into a type. Prove that x tends to y at the top if and only if for all i, x tends to y at the top.

Munkres.exercise.20.2 Prove that the order topology on $\mathbb{R} \times \mathbb{R}$ is metrizable.

Munkres.exercise.21.6a Let f be a function from the natural numbers to the real numbers. Prove that there exists a real number y such that f is continuous at x if and only if f is continuous at x.

Munkres.exercise.21.6b Prove that there is no function f such that $f(n) = n^2$ for all n.

Munkres.exercise.22.2a Suppose X and Y are topological spaces, and p is a continuous map from X to Y. Prove that there exists a continuous map f from Y to X such that p is the quotient map if and only if there exists a continuous map f from Y to X such that p is the identity map.

Munkres.exercise.22.5 Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous map. Prove that if p is open, then p is open as a map from X to Y.

Munkres.exercise.23.11 Let X and Y be topological spaces, and let $p : X \to Y$ be a quotient map. Prove that X is connected if and only if Y is connected.

Munkres.exercise.23.2 Prove that the union of a countable collection of connected sets is connected.

Munkres.exercise.23.3 Let X be a topological space, and let A be a function from the natural numbers to X. Prove that X is connected if and only if A is constant.

Munkres.exercise.23.4 Let X be a topological space, and let s be a set of points of X that is infinite. Prove that s is connected.

Munkres.exercise.23.6 If C is a connected set, and A is a closed subset of C, then C intersects the frontier of A.

Munkres.exercise.23.9 Let X and Y be topological spaces, and let A and B be subsets of X and Y respectively. Suppose that A is connected and B is connected. Prove that the set of all pairs (a, b) such that $a \in A$ and $b \in B$ is connected.

Munkres.exercise.24.2 Prove that there exists a point x on the unit sphere such that $f(x) = -f(-x)$.

Munkres.exercise.24.3a Let f be a continuous function from I to I. Prove that there exists $x \in I$ such that $f(x) = x$.

Munkres.exercise.25.4 Prove that a topological space is path-connected if and only if it is locally path-connected and connected.

Munkres.exercise.25.9 Let G be a topological group, and let C be a connected component of G. Prove that C is a normal subgroup of G.

Munkres.exercise.26.11 Prove that the union of a collection of closed sets is connected.

Munkres.exercise.26.12 Let X and Y be topological spaces, and let $p : X \to Y$ be a continuous surjection. Prove that X is compact if and only if Y is compact.

Munkres.exercise.27.4 Prove that the universal set of a metric space is not countable.

Munkres.exercise.28.4 A topological space is countably compact if and only if it is compact and has no limit points.

Munkres.exercise.28.6 Suppose X is a metric space, and f is a bijection. Prove that f is an isometry if and only if f is a bijection and f is an isometry.

Munkres.exercise.29.1 Prove that the rational numbers are not locally compact.

Munkres.exercise.29.10 Let X be a topological space, and let x be an element of X. Prove that there exists a set U such that x is an interior point of U, and U is compact.

Munkres.exercise.29.4 Prove that the space of continuous functions from the naturals into the unit interval is not locally compact.

Munkres.exercise.30.10 Let X be a set of topological spaces, and suppose that for each i, there is a set S_i of subsets of X_i such that S_i is countable and dense in X_i. Prove that there is a set S of subsets of $\Pi i, X_i$ such that S is countable and dense in $\Pi i, X_i$.

Munkres.exercise.30.13 Let X be a topological space, and let U be a set of points of X. Prove that U is countable if and only if U is dense and U is closed.

Munkres.exercise.31.1 Let X be a topological space, and let $x, y \in X$. Prove that there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Munkres.exercise.31.2 Let X be a topological space, and let A and B be closed subsets of X. Prove that there exist open sets U and V such that $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \emptyset$.

Munkres.exercise.31.3 Prove that the order topology on a partially ordered set is regular.

Munkres.exercise.32.1 Let X be a topological space, and let A be a closed subset of X. Prove that X is a normal space if and only if A is a closed subset of X.

Munkres.exercise.32.2a Prove that if X is a topological space, then X is nonempty if and only if X is a topological space.

Munkres.exercise.32.2b Prove that if X is a topological space, then X is regular if and only if X is nonempty.

Munkres.exercise.32.2c Prove that if X is a nonempty topological space, then X is normal.

Munkres.exercise.33.7 Let X be a topological space, and let A be a closed subset of X. Suppose that x is not in A. Prove that there exists a continuous function $f : X \to I$ such that $f(x) = 1$ and $f(A) = \{0\}$.

Munkres.exercise.34.9 Prove that the union of two closed sets is a closed set.

Munkres.exercise.38.4 Let X be a set, and let Y be a topological space. Suppose that Y is compact and that X is closed in Y. Prove that there exists a continuous function $g : S \to Y$ such that $g(s) = s$ for all $s \in S$.

Munkres.exercise.43.2 Let X be a metric space, and let Y be a complete metric space. Let A be a subset of X, and let $f : X \to Y$ be a continuous function. Prove that there exists a continuous function $g : X \to Y$ such that $g(x) = f(x)$ for all $x \in A$.

Pugh.exercise.2.109 Prove that a metric space is totally disconnected if and only if it is totally disconnected.

Pugh.exercise.2.126 Prove that there exists a point p in the closure of the set E such that E is uncountable.

Pugh.exercise.2.12a Let f be a one-to-one function from the natural numbers to the real numbers, and let p be a function from the natural numbers to the real numbers. Prove that if f is injective, then p tends to 0 at the top.

Pugh.exercise.2.12b Let f be a surjective function from the natural numbers to the real numbers, and let p be a function from the natural numbers to the real numbers such that $p(n)$ tends to a as n tends to ∞. Prove that p tends to a as n tends to ∞.

Pugh.exercise.2.137 Let P be a closed subset of a metric space M, and let hP be the set of points in P that are limit points of P. Prove that hP is closed.

Pugh.exercise.2.26 If U is open, then for every $x \in U$, the set of cluster points of U is empty.

Pugh.exercise.2.29 Let O be a set of open sets in a metric space M, and let C be a set of closed sets in M. Prove that there exists a bijection between O and C.

Pugh.exercise.2.32a Prove that the set of natural numbers is closed under addition.

Pugh.exercise.2.41 Prove that the closed ball of radius 1 around 0 in the space of real-valued functions on the set $\{1, \ldots, m\}$ is compact.

Pugh.exercise.2.46 Let A and B be non-empty sets, and suppose that A and B are disjoint. Prove that there exist $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$, a_0 and b_0 are the closest points to a and b respectively.

Pugh.exercise.2.57 Let X be a topological space. Prove that there exists a set S such that S is connected and S is not connected.

Pugh.exercise.2.79 Prove that a path-connected space is connected.

Pugh.exercise.2.85 Let M be a set, and let U be a set of subsets of M. Prove that there exists a set V of subsets of M such that V is finite and for every $p \in V$, there exist $U_1, U_2 \in V$ such that $p \in U_1$ and $p \in U_2$ and $U_1 \neq U_2$.

Pugh.exercise.2.92 Prove that the union of a countable collection of compact sets is compact.

Pugh.exercise.3.1 Prove that there exists a real number c such that $f(x) = c$ for all real numbers x.

Pugh.exercise.3.11a Let f be a function from the open interval (a, b) to the real numbers. Prove that there exists a limit of the function f at x if and only if the derivative of f at x exists.

Pugh.exercise.3.18 Let L be a closed set in $\mathbb{R}$. Prove that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous on $\mathbb{R}$ and $f(x) = 0$ for all $x \in L$.

Pugh.exercise.3.63a Let f be a function from the natural numbers to the real numbers, and let p be a positive real number. Prove that there exists a real number l such that f is continuous at the top of the real line.

Pugh.exercise.3.63b Prove that there is no sequence tending to 0 at 0.

Pugh.exercise.4.15a Let F be a set of functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that F is uniformly continuous if and only if there exists a function μ such that for all x in $\mathbb{R}$, $\mu(x)$ is a real number, $\mu(0)$ is a real number, and for all x and y in $\mathbb{R}$, $|\mu(x) - \mu(y)|$ is less than or equal to $|x - y|$.

Pugh.exercise.4.19 Let M be a metric space, and let A be a dense subset of M. Let δ be a positive real number. Prove that there exists a finite subset A_f of A such that A_f is dense in M and for all $x \in M$, there exists $i \in A_f$ such that $distxi < \delta$.

Pugh.exercise.5.2 Prove that the normed space of continuous linear maps from V to W is a normed space.

Rudin.exercise.1.11a Let z be a complex number. Prove that there exists a real number r such that $z = r \cdot w$ for some w with $|w| = 1$.

Rudin.exercise.1.12 Prove that for any $n \in \mathbb{N}$, $\sum_{i=1}^{n} f(i)$ is bounded above by $\sum_{i=1}^{n} |f(i)|$.

Rudin.exercise.1.13 Prove that for all $x, y \in \mathbb{C}$, $|x| \leq |y|$ implies $|x| \leq |y|$.

Rudin.exercise.1.14 Prove that $|1 + z|^2 + |1 - z|^2 = 4$.

Rudin.exercise.1.16a Prove that the set of points z in the Euclidean space R^n such that $\|z - x\| = d$ and $\|z - y\| = r$ is infinite.

Rudin.exercise.1.17 Prove that the norm of a vector x in $\mathbb{R}^n$ is equal to the sum of the squares of the norms of the components of x.

Rudin.exercise.1.18a Let n be a positive integer. Prove that there exists a non-zero vector y in the Euclidean space of dimension n such that the inner product of x and y is zero.

Rudin.exercise.1.18b Prove that there is no real number x such that x times any real number is equal to zero.

Rudin.exercise.1.1a Prove that if x is irrational, then $x + y$ is irrational for any y.

Rudin.exercise.1.1b Prove that if x is irrational, then x is not a rational multiple of y.

Rudin.exercise.1.2 Prove that there is no element x of the field of rational numbers such that $x^2 = 12$.

Rudin.exercise.1.4 Prove that if s is a nonempty set, and x and y are elements of s, then $x \leq y$ if and only if x is an element of the lower bounds of s and y is an element of the upper bounds of s.

Rudin.exercise.1.5 Prove that the infimum of a set A is the supremum of the set of negatives of elements of A.

Rudin.exercise.1.8 Prove that there is no linear order on the complex numbers.

Rudin.exercise.2.19a Let A and B be disjoint closed subsets of a metric space X. Prove that A and B are separated.

Rudin.exercise.2.24 Prove that a metric space is separable if and only if it has an infinite closed set.

Rudin.exercise.2.25 Let K be a metric space, and let B be a countable basis for K. Prove that B is a topological basis.

Rudin.exercise.2.27a Let E be a nonempty set of real numbers, and let P be a set of points in E such that P is countable if and only if E is countable. Prove that P is closed and that P is the set of points in E that are cluster points of E.

Rudin.exercise.2.27b Let E be a nonempty set, and let P be a set of points in E such that P is not countable. Prove that $E \setminus P$ is countable.

Rudin.exercise.2.28 Let X be a metric space, and let A be a closed set. Prove that there exist two disjoint countable sets P_1 and P_2 such that $A = P_1 \cup P_2$ and P_1 is closed.

Rudin.exercise.2.29 Let U be a set of real numbers. Prove that there exists a function f such that for all n, there exists a, b such that $f(n)$ is the set of real numbers between a and b.

Rudin.exercise.3.13 Suppose that a and b are sequences of real numbers, and that there exists a real number y such that for all n, the sequence a tends to y at top n.

Rudin.exercise.3.1a Prove that there exists an $a \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|f(n)|$ tends to a at $+\infty$.

Rudin.exercise.3.20 Let X be a metric space, and let p be a Cauchy sequence. Prove that p converges to a point r if and only if p converges to r at the top.

Rudin.exercise.3.21 Let X be a metric space, and let E be a set of points in X. Prove that there exists a point a in X such that E is a singleton.

Rudin.exercise.3.22 Let X be a metric space, and let G be a set of non-empty open subsets of X. Prove that there exists a point x in X such that G is dense in X.

Rudin.exercise.3.2a Prove that the function f defined by $f(n) = \sqrt{n^2 + n} - n$ tends to 0 at 0.

Rudin.exercise.3.3 There exists a real number x such that f tends to x at the top of the real line.

Rudin.exercise.3.5 Prove that if a and b are non-negative real numbers, then $\limsup a + \limsup b$ is less than or equal to $\limsup a + \limsup b$.

Rudin.exercise.3.6a Prove that the function f defined by $f(n) = \sum_{i=1}^{n} g(i)$ is continuous at $n = 0$.

Rudin.exercise.3.7 Suppose that a is a function from the natural numbers to the real numbers. Prove that there exists a real number y such that the sequence of partial sums of a converges to y.

Rudin.exercise.3.8 Prove that there exists a real number y such that for all n, the sum of the a_i and b_i is less than y.

Rudin.exercise.4.11a Suppose X and Y are metric spaces, and f is a uniformly continuous mapping of X into Y. Prove that f is continuous.

Rudin.exercise.4.12 Suppose f is a uniformly continuous mapping from a uniform space X into a uniform space Y, and g is a uniformly continuous mapping from Y into a uniform space Z. Prove that $g \circ f$ is uniformly continuous.

Rudin.exercise.4.14 Let I be a topological space, and let f be a continuous function from I to I. Prove that there exists an element x of I such that $f(x) = x$.

Rudin.exercise.4.15 Prove that if f is a continuous function from a metric space X to a metric space Y, and f is monotone, then f is open.

Rudin.exercise.4.1a Prove that there exists a function f from the reals to the reals such that for all x, $f(x)$ tends to 0 as x tends to 0 but f is not continuous.

Rudin.exercise.4.24 Prove that the function f is convex on the interval $[a, b]$ if and only if f is continuous and f is increasing on $[a, b]$.

Rudin.exercise.4.2a Let f be a continuous function from a set X to a metric space Y, and let h be a continuous function from Y to a metric space Z. Prove that f is uniformly continuous if h is uniformly continuous.

Rudin.exercise.4.4b Let $f, g : \alpha \to \beta$ be continuous functions, and let s be a dense subset of α. Prove that $f = g$ if f and g are equal on s.

Rudin.exercise.4.5a Let f be a function from the set of reals into itself. Prove that there exists a continuous function g such that $f(x) = g(x)$ for all x in the set of reals.

Rudin.exercise.4.5b There exists a function f from the real numbers into the real numbers such that f is continuous on the set E of real numbers, but there is no continuous function g from the real numbers into the real numbers such that f and g agree on E.

Rudin.exercise.4.6 If f is continuous on E, then E is compact if and only if G is.

Rudin.exercise.4.8a Let E be a set, and let f be a uniformly continuous function from E to $\mathbb{R}$. Prove that f is bounded.

Rudin.exercise.4.8b Let E be a set of real numbers. Prove that there exists a function f from E to E such that f is uniformly continuous on E and $f(E)$ is not bounded.

Rudin.exercise.5.1 Prove that there exists a real number c such that $f(x) = c$ for all real numbers x.

Rudin.exercise.5.15 Prove that $M_1^2 \leq 4M_0M_2$.

Rudin.exercise.5.2 Suppose f and g are functions from $[a, b]$ to $\mathbb{R}$ such that f is strictly increasing and g is strictly decreasing. Prove that g is differentiable at x if and only if f is differentiable at x.

Rudin.exercise.5.4 Suppose that C is a function from the natural numbers to the real numbers. Prove that there exists a real number x such that $C(i){*}x^i = 0$ for all i in the range of C.

Rudin.exercise.5.5 Prove that if f is differentiable at 0, then f is continuous at 0.

Rudin.exercise.5.6 Prove that if f is continuous and differentiable at 0, then f is monotone.

Shakarchi.exercise.1.13a Let f be a function from the open set Ω to the open set Ω, and suppose that f is differentiable on Ω. Prove that f is constant on Ω.

Shakarchi.exercise.1.13b Let f be a function from the open set Ω to the open set Ω, and suppose that f is differentiable on Ω. Prove that f is constant on Ω.

Shakarchi.exercise.1.13c Let f be a function from the open set Ω to the open set Ω, and suppose that f is differentiable on Ω. Prove that f is constant on Ω.

Shakarchi.exercise.1.19a Prove that there is no y such that s tends to z at y.

Shakarchi.exercise.1.19b Let z be a complex number such that $|z| = 1$. Let s be a function from the natural numbers to the complex numbers such that $s(n) = z^n$ for all n. Prove that there exists a complex number y such that s tends to y at the top.

Shakarchi.exercise.1.19c Let z be a complex number such that z is not a root of unity. Prove that there exists a sequence s tending to z at the top of the complex plane.