Exercises from *Everything* by All Authors

Artin.exercise.10.1.13 If x is nilpotent, then $1 + x$ is a unit.

Artin.exercise.10.2.4 Prove that $\mathrm{span}(2) \cap \mathrm{span}(X) = \mathrm{span}(2X)$.

Artin.exercise.10.4.6 Let R be a commutative ring with no zero divisors. Let I and J be ideals of R. Prove that the ideal $I \cap J$ is nilpotent.

Artin.exercise.10.4.7a If I and J are ideals of a commutative ring R such that $I+J = R$, then $IJ = I \cap J$.

Artin.exercise.10.5.16 If F is a field of characteristic 2, then the quotient of the polynomial ring $F[X]$ by the ideal generated by X^2 is isomorphic to the quotient of $F[X]$ by the ideal generated by $X^2 - 1$.

Artin.exercise.10.6.7 If I is a nonzero ideal of $\mathbb{Z}[i]$, then there is an element $z \in I$ such that z is real.

Artin.exercise.10.7.10 Let M be an ideal of a ring R such that for each $x \in R$, if $x \notin M$ then x is a unit. Prove that M is a maximal ideal and that every maximal ideal of R is equal to M.

Artin.exercise.10.7.6 Prove that the quotient of the polynomial ring $F[X]$ by the ideal generated by $X^2 + X + 1$ is a field if F is a field of order 5.

Artin.exercise.11.12.3 If p is a prime and a is a square root of -5 modulo p, then p is congruent to 1 or 3 modulo 8.

Artin.exercise.11.13.3 Prove that there is a prime p such that $p \geq N$ and $p + 1 \equiv 0 \pmod 4$.

Artin.exercise.11.2.13 If a divides b in the Gaussian integers, then a divides b in the integers.

Artin.exercise.11.3.1 If $p(x)$ is an irreducible polynomial over F, then $p(ax + b)$ is irreducible over F.

Artin.exercise.11.3.4 Prove that $x^3 + 6x + 12$ is irreducible over $\mathbb{Q}$.

Artin.exercise.11.4.1b Prove that $x^3 + 6x + 12$ is irreducible over $\mathbb{F}_2$.

Artin.exercise.11.4.6a Prove that $X^2 + 1$ is irreducible over a field of order 7.

Artin.exercise.11.4.6b Prove that $x^3 - 9$ is irreducible over $\mathbb{F}_{31}$.

Artin.exercise.11.4.6c Prove that $X^3 - 9$ is irreducible over $\mathbb{Z}_{31}$.

Artin.exercise.11.4.8 Prove that $X^n - p$ is irreducible over $\mathbb{Q}$ for any prime p and any positive integer n.

Artin.exercise.13.4.10 Prove that every prime of the form $2^n + 1$ is of the form $2^{2^k} + 1$.

Artin.exercise.13.6.10 Prove that the only element of $\mathbb{F}_2^n$ is -1.

Artin.exercise.2.11.3 If G is a finite group of even order, then G has an element of order 2.

Artin.exercise.2.2.9 Let G be a group and let $a, b \in G$ be such that $ab = ba$. Prove that the closure of the set $\{a, b\}$ is a commutative subgroup of G.

Artin.exercise.2.3.1 The multiplicative group of the reals is isomorphic to the additive group of the reals.

Artin.exercise.2.3.2 Let a and b be elements of a group G. Prove that there exists an element g of G such that $b^{-1}a = gag^{-1}b$.

Artin.exercise.2.4.19 If x is an element of order 2 in G, then x is in the center of G.

Artin.exercise.2.8.6 The center of the direct product of two groups is isomorphic to the direct product of the centers.

Artin.exercise.3.2.7 If F is a field and G is a field extension of F, then the inclusion map $F \to G$ is injective.

Artin.exercise.3.5.6 If S is a countable set of vectors in a vector space V such that $\text{span}(S) = V$, then any set of vectors $\{v_i\}_{i \in I}$ that is linearly independent from S is countable.

Artin.exercise.3.7.2 If V is a vector space over a field K, and $\{W_i\}_{i \in I}$ is a collection of subspaces of V, then $\bigcap_{i \in I} W_i$ is nontrivial.

Artin.exercise.6.1.14 If G is a group whose quotient by the center is cyclic, then the center of G is the whole group.

Artin.exercise.6.4.12 Prove that there is no simple group of order 224.

Artin.exercise.6.4.2 If G is a finite group of order pq where p and q are distinct primes, then G is not simple.

Artin.exercise.6.4.3 Prove that if G is a finite group of order p^2q, where p and q are primes, then G is not simple.

Artin.exercise.6.8.1 Let G be a group and let $a, b \in G$. Prove that the closure of $\{a, b\}$ is equal to the closure of $\{bab^{-1}, bab^{-2}\}$.

Artin.exercise.6.8.4 The closure of the set $\{x, y, z\}$ is the free group generated by x, y, z.

Artin.exercise.6.8.6 If G is cyclic and N is a normal subgroup of G such that G/N is cyclic, then G is generated by two elements.

Axler.exercise.1.2 Prove that $(-1/2 + i\sqrt{3}/2)^3 = -1$.

Axler.exercise.1.3 Prove that $-(-v) = v$.

Axler.exercise.1.4 If v is a vector in a vector space V over a field F, and a is an element of F, then $av = 0$ if and only if $a = 0$ or $v = 0$.

Axler.exercise.1.6 Prove that there exists a subset U of $\mathbb{R}^2$ such that U is not a submodule of $\mathbb{R}^2$.

Axler.exercise.1.7 Prove that there exists a nonempty subset U of $\mathbb{R}^2$ such that $cU = U$ for all $c \in \mathbb{R}$ and U is not a subspace of $\mathbb{R}^2$.

Axler.exercise.1.8 Let V be a vector space over a field F, and let $\{U_i\}_{i \in I}$ be a collection of subspaces of V. Prove that $\bigcap_{i \in I} U_i$ is a subspace of V.

Axler.exercise.1.9 Let U and W be subspaces of a vector space V. Prove that U and W are comparable if and only if $U \cap W$ is a subspace of V.

Axler.exercise.3.1 If T is a linear transformation of a finite-dimensional vector space V over a field F and T has rank 1, then T is a scalar multiple of the identity.

Axler.exercise.3.8 Let $L : V \to W$ be a linear map. Then there is a subspace U of V such that $L(U)$ is a complement of $L(V)$ in W.

Axler.exercise.4.4 If p is a polynomial, then the number of roots of p is equal to the number of roots of p'.

Axler.exercise.5.1 Let V be a vector space over a field F, and let $L : V \to V$ be a linear transformation. Let $U_1, \ldots, U_n$ be subspaces of V such that $L(U_i) = U_i$ for each i. Prove that $L(\sum_{i=1}^n U_i) = \sum_{i=1}^n U_i$.

Axler.exercise.5.11 If S and T are endomorphisms of a vector space V, then the eigenvalues of ST are the same as the eigenvalues of TS.

Axler.exercise.5.12 If S is a linear operator on a finite-dimensional vector space V over a field F such that every vector in V is an eigenvector of S, then S is a scalar multiple of the identity.

Axler.exercise.5.13 Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Suppose that for each subspace U of V of dimension $n - 1$, $T(U) = U$. Prove that T is a scalar multiple of the identity.

Axler.exercise.5.20 Let S and T be linear transformations of a finite-dimensional vector space V over a field F. Prove that if S and T have the same number of distinct eigenvalues, then $ST = TS$.

Axler.exercise.5.24 Let V be a finite-dimensional vector space over $\mathbb{R}$, and let T be a linear operator on V such that $T(x) = cx$ for all $x \in V$ and some $c \in \mathbb{R}$. Prove that the rank of any subspace U of V is even.

Axler.exercise.5.4 Let S and T be linear transformations of a vector space V over a field F such that $ST = TS$. Prove that S maps the kernel of $T - cI$ onto the kernel of $T - cI$.

Axler.exercise.6.13 If $e_1, \ldots, e_n$ is an orthonormal basis for V, then $v \in V$ is in the span of $e_1, \ldots, e_n$ if and only if $|v|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$.

Axler.exercise.6.16 If U is a subspace of a complex inner product space V, then $U^\perp = \{0\}$ if and only if $U = V$.

Axler.exercise.6.2 If u and v are vectors in a complex inner product space, then u is orthogonal to v if and only if $|u| \leq |u + av|$ for all $a \in \mathbb{C}$.

Axler.exercise.6.3 If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$.

Axler.exercise.6.7 If u and v are vectors in a complex inner product space, then $\langle u, v\rangle = (|u+v|^2 - |u-v|^2 + i|u+iv|^2 - i|u-iv|^2)/4$.

Axler.exercise.7.10 Let T be a linear operator on a finite-dimensional inner product space V. Prove that if T is self-adjoint and $T^9 = T^8$, then $T^2 = T$.

Axler.exercise.7.11 Let T be a linear operator on a finite-dimensional inner product space V such that $T^*T = TT^*$. Prove that there exists a linear operator S on V such that $S^2 = T$.

Axler.exercise.7.14 If T is a self-adjoint linear operator on a finite-dimensional inner product space, then for any $\epsilon > 0$ there is an eigenvalue l' of T such that $|l - l'| < \epsilon$.

Axler.exercise.7.5 If V is a finite-dimensional complex inner product space of dimension at least 2, then the set of all operators T such that $T^2 = T^*T$ is a proper subspace of the space of all operators on V.

Axler.exercise.7.6 If T is a linear operator on a finite-dimensional inner product space V such that $T^*T = TT^*$, then T is normal.

Axler.exercise.7.9 Let T be a linear operator on a finite-dimensional inner product space V. Prove that T is self-adjoint if and only if all of its eigenvalues are real.

Dummit-Foote.exercise.1.1.15 If $a_1, \ldots, a_n$ are elements of a group G, then $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$.

Dummit-Foote.exercise.1.1.16 If x is an element of a group G such that $x^2 = 1$, then the order of x is either 1 or 2.

Dummit-Foote.exercise.1.1.17 If x has order n, then $x^{-1} = x^{n-1}$.

Dummit-Foote.exercise.1.1.18 Prove that x and y commute if and only if $y^{-1}xy = x$ and $x^{-1}y^{-1}xy = 1$.

Dummit-Foote.exercise.1.1.20 If x is an element of a group G, then the order of x is the same as the order of x^{-1}.

Dummit-Foote.exercise.1.1.22a If x is an element of a group G, and g is an element of G, then the order of x is equal to the order of $g^{-1}xg$.

Dummit-Foote.exercise.1.1.22b If a and b are elements of a group G, prove that $|ab| = |ba|$.

Dummit-Foote.exercise.1.1.25 If G is a group in which every element has order 2, prove that G is abelian.

Dummit-Foote.exercise.1.1.29 If A and B are groups, then $(A \times B, \cdot)$ is a group if and only if A and B are both commutative.

Dummit-Foote.exercise.1.1.2a Prove that there exist integers a and b such that $a - b \neq b - a$.

Dummit-Foote.exercise.1.1.3 Prove that $a + b + c \equiv a + (b + c) \pmod{n}$.

Dummit-Foote.exercise.1.1.34 If x is an element of infinite order in G, prove that the elements x^n, $n \in \mathbb{Z}$ are all distinct.

Dummit-Foote.exercise.1.1.4 Prove that $(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$ modulo n.

Dummit-Foote.exercise.1.1.5 Prove that the group $\mathbb{Z}_n$ is trivial if n is prime.

Dummit-Foote.exercise.1.3.8 The group of permutations of $\mathbb{N}$ is infinite.

Dummit-Foote.exercise.1.6.11 Prove that the group $A \times B$ is isomorphic to the group $B \times A$.

Dummit-Foote.exercise.1.6.17 Let $f : G \to G$ be defined by $f(x) = x^{-1}$. Prove that f is a homomorphism if and only if G is abelian.

Dummit-Foote.exercise.1.6.23 Let G be a group and let σ be an automorphism of G such that $\sigma(g) = 1$ implies $g = 1$ and $\sigma(\sigma(g)) = g$ for all $g \in G$. Prove that G is abelian.

Dummit-Foote.exercise.1.6.4 Prove that there is no isomorphism between the multiplicative group of real numbers and the multiplicative group of complex numbers.

Dummit-Foote.exercise.2.1.13 Let H be a subgroup of the additive group of rational numbers. Prove that if $x \in H$ implies $1/x \in H$, then H is either $\{0\}$ or the whole group.

Dummit-Foote.exercise.3.1.22a If H and K are normal subgroups of G, then $H \cap K$ is normal in G.

Dummit-Foote.exercise.3.1.22b If H_i is a normal subgroup of G for each $i \in I$, then $\cap_{i \in I} H_i$ is a normal subgroup of G.

Dummit-Foote.exercise.3.1.3a If A is a commutative group and B is a subgroup of A, then the quotient group A/B is abelian.

Dummit-Foote.exercise.3.2.16 If p is prime and a is coprime to p, then $a^p \equiv a \pmod{p}$.

Dummit-Foote.exercise.3.2.21a If H is a nontrivial subgroup of a finite group G, then $H = G$.

Dummit-Foote.exercise.3.2.8 If H and K are finite subgroups of G with coprime orders, then $H \cap K = \{1\}$.

Dummit-Foote.exercise.3.4.1 Prove that a simple group is cyclic and has prime order.

Dummit-Foote.exercise.3.4.4 If G is a finite commutative group, then for any divisor n of $|G|$, there is a subgroup H of G such that $|H| = n$.

Dummit-Foote.exercise.3.4.5a If G is solvable, then so is any subgroup of G.

Dummit-Foote.exercise.3.4.5b If G is solvable and H is a normal subgroup of G, then G/H is solvable.

Dummit-Foote.exercise.4.3.26 If X is a finite set with more than one element, prove that there is a permutation of X that is not the identity.

Dummit-Foote.exercise.4.4.6a A characteristic subgroup of a group is normal.

Dummit-Foote.exercise.4.5.13 If G is a group of order 56, then G has a normal Sylow p-subgroup for some prime p.

Dummit-Foote.exercise.4.5.14 If G is a group of order 312, then G has a normal Sylow p-subgroup for some prime p.

Dummit-Foote.exercise.4.5.1a If P is a p-subgroup of G and H is a subgroup of G containing P, then H is a p-subgroup of G.

Herstein.exercise.2.10.1 If A is a normal subgroup of G and b is an element of prime order, then $A \cap \langle b \rangle = \{1\}$.

Herstein.exercise.2.11.22 If G is a group of order p^n and K is a subgroup of order p^{n-1}, then K is normal in G.

Herstein.exercise.2.11.6 If P is a normal Sylow p-subgroup of G, then P is the only Sylow p-subgroup of G.

Herstein.exercise.2.11.7 If P is a normal Sylow p-subgroup of G, then P is characteristic in G.

Herstein.exercise.2.1.18 If G is a finite group of even order, then G has an element of order 2.

Herstein.exercise.2.1.21 Prove that a group of order 5 is abelian.

Herstein.exercise.2.1.26 If G is a finite group, then every element of G has finite order.

Herstein.exercise.2.1.27 Prove that every finite group has an element of finite order.

Herstein.exercise.2.2.3 Prove that a group G is commutative if and only if there exists an integer n such that $a^n = b^n$ implies $a^n = b^n$ for all $a, b \in G$.

Herstein.exercise.2.2.5 Prove that a group G is commutative if and only if $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$.

Herstein.exercise.2.2.6c If G is a group and $n > 1$, and if $a^n = b^n$ for all $a, b \in G$, then $(abab^{-1}a^{-1})^{n(n-1)} = 1$.

Herstein.exercise.2.3.16 Prove that a group G is cyclic if and only if every subgroup of G is either trivial or the whole group.

Herstein.exercise.2.3.17 If a is an element of G and x is an element of G, then the centralizer of $x^{-1}ax$ is the image of the centralizer of a under the map $g \mapsto x^{-1}gx$.

Herstein.exercise.2.3.19 Let M be a subgroup of G. Prove that M is normal if and only if for each $x \in G$, the set $\{x^{-1}mx : m \in M\}$ is contained in M.

Herstein.exercise.2.4.36 If $a > 1$ and n is a positive integer, prove that n divides $\varphi(a^n - 1)$.

Herstein.exercise.2.5.23 If G is a group in which every subgroup is normal, prove that for any $a, b \in G$, there exists $j \in \mathbb{Z}$ such that $bab^{-1} = a^j$.

Herstein.exercise.2.5.30 If G is a finite group of order pm where p is prime and p does not divide m, then any normal subgroup of order p is characteristic.

Herstein.exercise.2.5.31 If H is a p-subgroup of G and G is a finite group of order $p^n m$, then H is characteristic in G.

Herstein.exercise.2.5.37 If G is a group of order 6 and G is not abelian, then G is isomorphic to S_3.

Herstein.exercise.2.5.43 Prove that a group of order 9 is abelian.

Herstein.exercise.2.5.44 If G is a group of order p^2, where p is prime, then G has a normal subgroup of order p.

Herstein.exercise.2.5.52 If G is a finite group and ϕ is an automorphism of G such that $|\{x \in G : \phi(x) = x^{-1}\}| \geq 3/4|G|$, then $\phi(x) = x^{-1}$ for all $x \in G$ and G is abelian.

Herstein.exercise.2.6.15 If G is a commutative group, m and n are positive integers, and m and n are relatively prime, then there is an element of order mn in G.

Herstein.exercise.2.7.7 If N is a normal subgroup of G and $\phi : G \to G'$ is a homomorphism, then $\phi(N)$ is a normal subgroup of G'.

Herstein.exercise.2.8.12 If G and H are non-abelian groups of order 21, then G is isomorphic to H.

Herstein.exercise.2.8.15 If G and H are groups of order pq where $p > q$ are primes, then G is isomorphic to H.

Herstein.exercise.2.9.2 If G and H are cyclic groups, then $G \times H$ is cyclic if and only if $|G|$ and $|H|$ are coprime.

Herstein.exercise.3.2.21 Let σ and τ be permutations of a finite set X. Prove that if $\sigma(x) = x$ if and only if $\tau(x) \neq x$ for all $x \in X$, and $\tau \circ \sigma = \mathrm{id}_X$, then $\sigma = \mathrm{id}_X$ and $\tau = \mathrm{id}_X$.

Herstein.exercise.4.1.19 Prove that the set of all quaternions x such that $x^2 = -1$ is infinite.

Herstein.exercise.4.1.34 Prove that the group of permutations of $\{1, 2, 3\}$ is isomorphic to the group of 2×2 matrices over $\mathbb{Z}/2\mathbb{Z}$.

Herstein.exercise.4.2.5 Prove that a ring R is commutative if $x^3 = x$ for all $x \in R$.

Herstein.exercise.4.2.6 If $a^2 = 0$ in a ring R, prove that $a(x + xa) = a(x + xa)$.

Herstein.exercise.4.2.9 Prove that if p is an odd prime, then there exist integers a and b such that a/b is equal to the sum of the reciprocals of the integers from 1 to p, and p divides a.

Herstein.exercise.4.3.1 Let R be a commutative ring and a an element of R. Prove that the set of all elements x of R such that $xa = 0$ is an ideal of R.

Herstein.exercise.4.3.25 Prove that every ideal of the ring of 2×2 real matrices is either the zero ideal or the whole ring.

Herstein.exercise.4.4.9 Prove that if p is an odd prime, then there is a subset S of $\mathbb{Z}_p$ such that $|S| = (p-1)/2$ and S contains no element x such that $x^2 = p$.

Herstein.exercise.4.5.16 Let p be a prime and let q be an irreducible polynomial of degree n over $\mathbb{Z}/p\mathbb{Z}$. Prove that the quotient ring $\mathbb{Z}/p\mathbb{Z}[x]/(q)$ is a field of order p^n.

Herstein.exercise.4.5.23 Prove that the polynomials $x^3 - 2$ and $x^3 + 2$ are irreducible over $\mathbb{Z}_7$ and that the quotient rings $\mathbb{Z}_7[x]/(x^3 - 2)$ and $\mathbb{Z}_7[x]/(x^3 + 2)$ are isomorphic.

Herstein.exercise.4.5.25 Prove that $X^p + X^{p-1} + \cdots + X + 1$ is irreducible over $\mathbb{Q}$.

Herstein.exercise.4.6.2 Prove that $X^3 + 3X + 2$ is irreducible over $\mathbb{Q}$.

Herstein.exercise.4.6.3 Prove that the set of integers a such that $X^7 + 15X^2 - 30X + a$ is irreducible is infinite.

Herstein.exercise.5.1.8 If F is a field of characteristic p, then $(a + b)^p = a^p + b^p$.

Herstein.exercise.5.2.20 If V is an infinite-dimensional vector space over a field F, then the union of any collection of proper subspaces of V is a proper subspace of V.

Herstein.exercise.5.3.10 Prove that $\cos(\pi/180)$ is algebraic over $\mathbb{Q}$.

Herstein.exercise.5.3.7 If a is algebraic over F, then a^2 is algebraic over F.

Herstein.exercise.5.4.3 Let $p(x) = x^5 + \sqrt{2}x^3 + \sqrt{5}x^2 + \sqrt{7}x + 11$. Prove that if $p(a) = 0$, then there is a polynomial $q(x)$ of degree less than 80 such that $q(a) = 0$ and the coefficients of $q(x)$ are rational numbers.

Herstein.exercise.5.5.2 Prove that $X^3 - 3X - 1$ is irreducible over $\mathbb{Q}$.

Herstein.exercise.5.6.14 If F is a field of characteristic p, then the number of roots of $x^p - x$ is p.

Munkres.exercise.13.1 Let X be a topological space and let A be a subset of X. Prove that A is open if for each $x \in A$ there is an open set U containing x such that $U \subset A$.

Munkres.exercise.13.3a Prove that the collection of all open sets in a topological space is a topology.

Munkres.exercise.13.3b Prove that the union of an infinite collection of infinite sets need not be infinite.

Munkres.exercise.13.4a1 If T_i is a topology on X for each $i \in I$, then $\bigcap_{i \in I} T_i$ is a topology on X.

Munkres.exercise.13.4a2 There exists a set X and a family of topologies $\{T_i\}_{i \in I}$ on X such that $\bigcap_{i \in I} T_i$ is not a topology on X.

Munkres.exercise.13.4b1 Let T_i, $i \in I$ be a family of topologies on X. Prove that there is a unique topology T on X such that $T_i \subset T$ for all $i \in I$ and T is the smallest topology on X with this property.

Munkres.exercise.13.4b2 Let X be a set and let $\{T_i\}_{i \in I}$ be a family of topologies on X. Prove that there is a unique topology T on X such that $T_i \subset T$ for all $i \in I$ and T is the smallest topology on X with this property.

Munkres.exercise.13.5a Let A be a basis for a topology on X. Prove that the topology generated by A is the intersection of all topologies on X containing A.

Munkres.exercise.13.5b Let X be a set and let A be a collection of subsets of X. Prove that the topology generated by A is the intersection of all topologies on X that contain A.

Munkres.exercise.13.6 The real line with the usual topology is not homeomorphic to the real line with the cofinite topology.

Munkres.exercise.13.8a Prove that the collection of all open intervals (a, b) with rational endpoints is a topological basis for the real line.

Munkres.exercise.13.8b The topology generated by the intervals $[a, b]$ is not the same as the lower limit topology.

Munkres.exercise.16.1 Let X be a topological space, Y a subset of X, and A a subset of Y. Prove that a subset U of A is open in A if and only if U is open in Y.

Munkres.exercise.16.4 Prove that the projections π_1 and π_2 are open maps.

Munkres.exercise.16.6 Let S be the set of all open rectangles in the plane. Prove that S is a basis for the topology of the plane.

Munkres.exercise.17.4 If U is open and A is closed, then $U \setminus A$ is open and $A \setminus U$ is closed.

Munkres.exercise.18.13 Let X be a topological space, Y a T2 space, and A a subset of X. Let f be a continuous mapping of A into Y. Prove that if g is a continuous mapping of $\overline{A}$ into Y such that $g(x) = f(x)$ for all $x \in A$, then g is the only continuous mapping of $\overline{A}$ into Y with this property.

Munkres.exercise.18.8a Let X and Y be topological spaces, and let f and g be continuous functions from X into Y. Prove that the set $\{x \in X : f(x) \leq g(x)\}$ is closed in X.

Munkres.exercise.18.8b If f and g are continuous functions from X into Y, where Y is a linearly ordered topological space, then the function $x \mapsto \min(f(x), g(x))$ is continuous.

Munkres.exercise.19.6a Let $f_1, f_2, \ldots$ be a sequence of functions from X into Y. Prove that f_n converges uniformly to f if and only if $f_{n,x}$ converges to f_x for each x in X.

Munkres.exercise.20.2 Prove that the product topology on $\mathbb{R}^2$ is metrizable.

Munkres.exercise.21.6a Let $f_n : I \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Prove that f_n converges uniformly to f on I.

Munkres.exercise.21.6b Prove that the sequence of functions $f_n(x) = x^n$ does not converge uniformly on any interval I.

Munkres.exercise.21.8 Let X be a topological space, Y a metric space, and $f_n : X \to Y$ a sequence of continuous functions. Let x_n be a sequence of points in X converging to $x_0 \in X$, and let $f_0 : X \to Y$ be a function such that f_n converges uniformly to f_0. Prove that $f_n(x_n)$ converges to $f_0(x_0)$.

Munkres.exercise.22.2a A continuous mapping $p : X \to Y$ is a quotient map if and only if there is a continuous mapping $f : Y \to X$ such that $p \circ f = id_Y$.

Munkres.exercise.22.2b Let X be a topological space, and let A be a subset of X. Let r be a continuous mapping of X into A such that $r(x) = x$ for all $x \in A$. Prove that r is a quotient mapping.

Munkres.exercise.22.5 Let X and Y be topological spaces, and let $p : X \to Y$ be an open mapping. If A is an open subset of X, then the restriction of p to A is an open mapping.

Munkres.exercise.23.11 If X is a topological space, Y is a connected space, and $p : X \to Y$ is a quotient map, then X is connected.

Munkres.exercise.23.2 Let X be a topological space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Prove that $\bigcup_{n=1}^{\infty} A_n$ is connected.

Munkres.exercise.23.3 Let X be a topological space, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of X. Suppose that A_0 is a connected subset of X such that $A_0 \cap A_n$ is nonempty for each n. Prove that $A_0 \cup (\cup_{n=1}^{\infty} A_n)$ is connected.

Munkres.exercise.23.4 If X is a cofinite topological space, then every infinite subset of X is connected.

Munkres.exercise.23.6 If C is a connected set and A is a subset of X such that $C \cap A$ and $C \cap A^c$ are nonempty, then $C \cap \partial A$ is nonempty.

Munkres.exercise.23.9 Let A_1, A_2, B_1, B_2 be connected sets in topological spaces X, Y respectively, with $A_1 \subset A_2$ and $B_1 \subset B_2$. Prove that the set $A_2 \times B_2 - A_1 \times B_1$ is connected.

Munkres.exercise.24.2 Let f be a continuous function on the unit sphere S^n in $\mathbb{R}^{n+1}$. Prove that there exists $x \in S^n$ such that $f(x) = f(-x)$.

Munkres.exercise.24.3a If f is a continuous mapping of a compact space I into itself, prove that f has a fixed point.

Munkres.exercise.25.4 If X is locally path-connected and U is an open connected subset of X, then U is path-connected.

Munkres.exercise.25.9 If G is a topological group and C is the connected component of the identity, then C is a normal subgroup of G.

Munkres.exercise.26.11 Let X be a compact Hausdorff space, and let A be a collection of closed connected subsets of X such that for any two sets $A_1, A_2 \in A$, either $A_1 \subset A_2$ or $A_2 \subset A_1$. Prove that $\bigcap_{A \in A} A$ is connected.

Munkres.exercise.26.12 Suppose X and Y are topological spaces, $p : X \to Y$ is a continuous surjection, and $p^{-1}(y)$ is compact for each $y \in Y$. If Y is compact, then X is compact.

Munkres.exercise.27.4 If X is a connected metric space with at least two points, then X is uncountable.

Munkres.exercise.28.4 A topological space X is countably compact if and only if it is limit point compact.

Munkres.exercise.28.5 A topological space X is countably compact if and only if every countable collection of closed sets with nonempty intersection has a point in common.

Munkres.exercise.28.6 If X is a compact metric space and $f : X \to X$ is an isometry, then f is a bijection.

Munkres.exercise.29.1 Prove that $\mathbb{Q}$ is not locally compact.

Munkres.exercise.29.10 Let X be a T_2 space. If $x \in X$ and U is an open set containing x, then there is an open set V containing x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Munkres.exercise.29.4 Prove that the space $\mathbb{N}^I$ is not locally compact.

Munkres.exercise.30.10 Let X_i be a topological space for each $i \in \mathbb{N}$. Suppose that for each i, there is a countable dense subset S_i of X_i. Prove that there is a countable dense subset of the product space $\prod_{i=1}^{\infty} X_i$.

Munkres.exercise.30.13 Let X be a topological space. If X has a countable dense subset, then the set of all open sets of X is countable.

Munkres.exercise.31.1 If X is a regular space, then for any two points x, y in X there are open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Munkres.exercise.31.2 If A and B are disjoint closed sets in a normal space, then there are open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Munkres.exercise.31.3 Prove that the order topology on a partially ordered set is regular.

Munkres.exercise.32.1 If X is a normal space and A is a closed subset of X, then A is a normal space.

Munkres.exercise.32.2a If X_i is a topological space for each $i \in I$, and if $\prod_{i \in I} X_i$ is a T_2 space, then each X_i is a T_2 space.

Munkres.exercise.32.2b If X_i is a regular space for each i, then $\prod_i X_i$ is regular.

Munkres.exercise.32.2c If X_i is a normal space for each i, then $\prod_i X_i$ is normal.

Munkres.exercise.32.3 Prove that a locally compact Hausdorff space is regular.

Munkres.exercise.33.7 Let X be a locally compact Hausdorff space. Prove that for each closed set A and each point x not in A, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$.

Munkres.exercise.33.8 Let X be a regular space. Suppose that for each $x \in X$ and each closed set A not containing x, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$. Prove that if A and B are disjoint closed sets in X and A is compact, then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Munkres.exercise.34.9 If X is a compact space, and X_1 and X_2 are closed subsets of X such that $X_1 \cup X_2 = X$, and X_1 and X_2 are metrizable, then X is metrizable.

Munkres.exercise.38.4 Let X be a dense subset of a compact Hausdorff space Y. Prove that there is a continuous surjection $g : \beta X \to Y$ such that $g(\beta X)$ is closed in Y and $g(x) = x$ for all $x \in X$.

Munkres.exercise.38.6 Let X be a regular space. Prove that X is connected if and only if the Stone-Čech compactification βX is connected.

Munkres.exercise.43.2 Let X be a metric space, Y a complete metric space, and A a subset of X. Suppose that $f : A \to Y$ is uniformly continuous. Prove that there exists a unique continuous function $g : \overline{A} \to Y$ such that $g(x) = f(x)$ for all $x \in A$.

Pugh.exercise.2.109 Prove that a metric space M is totally disconnected if and only if for all $x, y, z \in M$, $d(x, z) = \max\{d(x, y), d(y, z)\}$.

Pugh.exercise.2.126 If E is an uncountable set of real numbers, then there is a point p which is a cluster point of E.

Pugh.exercise.2.12a Let f be an injective function from $\mathbb{N}$ to $\mathbb{N}$, and let p be a sequence of real numbers such that $p(n) \to a$ as $n \to \infty$. Prove that $p(f(n)) \to a$ as $n \to \infty$.

Pugh.exercise.2.12b Let f be a surjective function from $\mathbb{N}$ to $\mathbb{N}$, and let p be a sequence of real numbers such that $p(n) \to a$ as $n \to \infty$. Prove that $p(f(n)) \to a$ as $n \to \infty$.

Pugh.exercise.2.137 Let M be a separable metric space. If P is a closed subset of M, then for each $x \in P$ there is a neighborhood N of x such that N is uncountable.

Pugh.exercise.2.26 A set U is open if and only if for each $x \in U$, there is a neighborhood of x that does not contain any point of U.

Pugh.exercise.2.29 Prove that there is a bijection between the set of open sets and the set of closed sets in a metric space.

Pugh.exercise.2.32a Prove that the set of all natural numbers is clopen.

Pugh.exercise.2.41 Prove that the closed unit ball in $\mathbb{R}^m$ is compact.

Pugh.exercise.2.46 Let A and B be compact sets in a metric space M such that A and B are disjoint and nonempty. Prove that there exist points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0)$ is less than or equal to $d(a, b)$ for all $a \in A$ and $b \in B$.

Pugh.exercise.2.57 There exists a connected set S such that the interior of S is not connected.

Pugh.exercise.2.79 Prove that a compact connected space is path-connected.

Pugh.exercise.2.85 Let M be a compact metric space and let U be a collection of open sets in M such that for each $p \in M$ there are two distinct sets $U_1, U_2 \in U$ containing p. Prove that there is a finite collection V of open sets in M such that for each $p \in M$ there are two distinct sets $V_1, V_2 \in V$ containing p.

Pugh.exercise.2.92 Let X be a topological space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonempty compact sets such that $A_n \subset A_{n+1}$ for all n. Prove that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

Pugh.exercise.3.1 If f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$, then f is constant.

Pugh.exercise.3.11a Suppose that f is differentiable on the open interval (a, b) and that f' is differentiable at x. Prove that f'' exists at x.

Pugh.exercise.3.18 If L is a closed subset of $\mathbb{R}$, then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if $x \in L$.

Pugh.exercise.3.4 Prove that $\sqrt{n+1} - \sqrt{n}$ tends to 0 as n tends to infinity.

Pugh.exercise.3.63a Prove that the function $f(x) = \frac{1}{x(\log x)^p}$ converges to 0 as x tends to infinity.

Pugh.exercise.3.63b Prove that the function $f(x) = 1/x(\log x)^p$ does not converge for $p \leq 1$.

Pugh.exercise.4.15a Let F be a set of functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that F is uniformly equicontinuous if and only if there exists a function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mu(x) \geq 0$ for all $x \in \mathbb{R}$, $\mu(x) \to 0$ as $x \to 0$, and $|f(s) - f(t)| \leq \mu(|s - t|)$ for all $s, t \in \mathbb{R}$ and $f \in F$.

Pugh.exercise.4.19 Let M be a compact metric space, and let A be a dense subset of M. Then for each $\delta > 0$ there is a finite subset A_δ of A such that for each $x \in M$ there is an $i \in A_\delta$ such that $d(x, i) < \delta$.

Pugh.exercise.5.2 Prove that the space of continuous linear maps from V to W is a normed space.

Rudin.exercise.1.11a Every complex number z can be written in the form $z = r\omega$ where $r \in \mathbb{R}$ and $|\omega| = 1$.

Rudin.exercise.1.12 If $z_1, \ldots, z_n$ are complex, then $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

Rudin.exercise.1.13 If x and y are complex, then $|x| - |y| \leq |x - y|$.

Rudin.exercise.1.14 If z is a complex number of modulus 1, prove that $|1 + z|^2 + |1 - z|^2 = 4$.

Rudin.exercise.1.16a Let x and y be distinct points in $\mathbb{R}^n$, $n \geq 3$. Let $d = |x - y|$ and let r be a positive number such that $2r > d$. Prove that the set of points z in $\mathbb{R}^n$ such that $|z - x| = r$ and $|z - y| = r$ is infinite.

Rudin.exercise.1.17 If x and y are in $\mathbb{R}^n$, then $|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$.

Rudin.exercise.1.18a If x is a nonzero vector in $\mathbb{R}^n$, prove that there is a nonzero vector y in $\mathbb{R}^n$ such that $x \cdot y = 0$.

Rudin.exercise.1.18b Prove that there is no real number x such that for every real number y, $xy = 0$.

Rudin.exercise.1.19 Let a, b, c be three points in the plane, and let r be a positive number. Prove that the following are equivalent: (1) $|x - a| = 2|x - b|$; (2) $|x - c| = r$.

Rudin.exercise.1.1a If x is irrational and y is rational, then $x + y$ is irrational.

Rudin.exercise.1.1b If x is irrational and y is a nonzero rational number, then xy is irrational.

Rudin.exercise.1.2 Prove that there is no rational number x such that $x^2 = 12$.

Rudin.exercise.1.4 If x is a lower bound of a nonempty set S and y is an upper bound of S, then $x \leq y$.

Rudin.exercise.1.5 Let A be a nonempty set of real numbers bounded below. Prove that $\inf A = -\sup(-A)$.

Rudin.exercise.1.8 Prove that there is no linear order on the complex numbers.

Rudin.exercise.2.19a If A and B are disjoint closed sets, then A and B are separated.

Rudin.exercise.2.24 Prove that every metric space X in which every infinite subset has a limit point is separable.

Rudin.exercise.2.25 Prove that every compact metric space has a countable basis.

Rudin.exercise.2.27a Let E be a non-countable subset of $\mathbb{R}^k$ and let P be the set of all points x such that E is not countable in any neighborhood of x. Prove that P is closed and that P is the set of all cluster points of E.

Rudin.exercise.2.27b Let E be a non-countable subset of $\mathbb{R}^n$ and let P be the set of points x such that for every neighborhood U of x, the set $P \cap E$ is non-countable. Prove that $E \setminus P$ is countable.

Rudin.exercise.2.28 Let X be a separable metric space and let A be a closed subset of X. Prove that A can be written as the union of two closed sets P_1 and P_2 such that P_1 is the set of all cluster points of P_1 and P_2 is countable.

Rudin.exercise.2.29 If U is an open set in $\mathbb{R}$, prove that there is a sequence of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$.

Rudin.exercise.3.13 If $\sum_{i=1}^{\infty} |a_i|$ and $\sum_{i=1}^{\infty} |b_i|$ converge, then $\sum_{i=1}^{\infty} \sum_{j=1}^{i+1} a_j b_{i-j}$ converges.

Rudin.exercise.3.1a If f is a sequence of real numbers such that $f(n)$ tends to a limit a, prove that $|f(n)|$ tends to $|a|$.

Rudin.exercise.3.20 Let p be a Cauchy sequence in a metric space X. If $p(lk)$ converges to r for some $l \in \mathbb{N}$, then p converges to r.

Rudin.exercise.3.21 Let X be a complete metric space. Suppose that $\{E_n\}$ is a sequence of nonempty closed sets such that $E_n \supset E_{n+1}$ for all n and $\lim_{n\to\infty} diam(E_n) = 0$. Prove that $\cap_{n=1}^{\infty} E_n$ is a singleton.

Rudin.exercise.3.22 Let X be a complete metric space, and let $\{G_n\}$ be a sequence of open dense subsets of X. Prove that X contains a point x such that $x \in G_n$ for all n.

Rudin.exercise.3.2a Prove that $\lim_{n\to\infty} \sqrt{n^2+n} - n = 1/2$.

Rudin.exercise.3.3 Prove that if f is a sequence of real numbers such that $f(n) < 2$ for all n, then f converges to some real number x.

Rudin.exercise.3.5 If a_n and b_n are bounded sequences, then $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Rudin.exercise.3.6a Prove that the sequence $\sum_{i=1}^{n} g_i$ converges if the sequence g_i converges.

Rudin.exercise.3.7 If $\sum_{i=1}^{n} a_i$ converges, then $\sum_{i=1}^{n} \sqrt{a_i}/n$ converges.

Rudin.exercise.3.8 If a_n is a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and if b_n is a sequence of real numbers such that $b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Rudin.exercise.4.11a If $f : X \to Y$ is uniformly continuous and x_n is a Cauchy sequence in X, then $f(x_n)$ is a Cauchy sequence in Y.

Rudin.exercise.4.12 If $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous, then $g \circ f : X \to Z$ is uniformly continuous.

Rudin.exercise.4.14 Let I be a linearly ordered topological space. Prove that every continuous function $f : I \to I$ has a fixed point.

Rudin.exercise.4.15 If f is continuous and open, then f is monotone.

Rudin.exercise.4.19 Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that for each a, b, c with $a < b$ and $f(a) < c < f(b)$, there is a point x with $a < x < b$ and $f(x) = c$. Prove that f is continuous.

Rudin.exercise.4.1a There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is not continuous but for each $x \in \mathbb{R}$, the function $g_x(y) = f(x + y) - f(x - y)$ is continuous at 0.

Rudin.exercise.4.21a If K and F are disjoint compact and closed subsets of a metric space, then there is a positive number δ such that $d(x, y) \geq \delta$ for all $x \in K$ and $y \in F$.

Rudin.exercise.4.24 Suppose f is continuous on $[a, b]$ and satisfies the condition $f((x + y)/2) \leq (f(x) + f(y))/2$ for all $x, y \in [a, b]$. Prove that f is convex on $[a, b]$.

Rudin.exercise.4.2a Let $f : X \to Y$ be a continuous mapping of a metric space X into a metric space Y. Prove that $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.

Rudin.exercise.4.3 Let f be a continuous function from a metric space X into $\mathbb{R}$. Prove that the set $f^{-1}(0)$ is closed.

Rudin.exercise.4.4a Let f be a continuous mapping of a metric space X into a metric space Y. Prove that if S is a dense subset of X, then $f(X)$ is a subset of the closure of $f(S)$.

Rudin.exercise.4.4b Suppose f and g are continuous functions on a metric space X and S is a dense subset of X. Prove that if $f(x) = g(x)$ for all $x \in S$, then $f(x) = g(x)$ for all $x \in X$.

Rudin.exercise.4.5a Let f be a continuous function on a closed set E. Prove that f is bounded on E.

Rudin.exercise.4.5b There exists a set E and a function $f : E \to \mathbb{R}$ such that f is continuous on E but there is no continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in E$.

Rudin.exercise.4.6 Let E be a compact subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$. Prove that f is continuous on E if and only if the graph of f is compact.

Rudin.exercise.4.8a If f is uniformly continuous on a bounded set E, then $f(E)$ is bounded.

Rudin.exercise.4.8b There exists a uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is unbounded on $\mathbb{R}$.

Rudin.exercise.5.1 Prove that if f is a real-valued function on $\mathbb{R}$ such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$, then f is constant.

Rudin.exercise.5.15 Let f be a function defined on the interval $[a, b]$ and differentiable on (a, b). Suppose that f' and f'' are continuous on $[a, b]$. Prove that $|f'(x)|^2 \leq 4|f(x)||f''(x)|$ for all $x \in [a, b]$.

Rudin.exercise.5.17 Let f be a function defined on $[-1, 1]$ and differentiable on $(-1, 1)$. Suppose that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Prove that there exists a point x in $(-1, 1)$ such that $f'''(x) \geq 3$.

Rudin.exercise.5.2 Let f be a function defined on the interval (a, b) and differentiable on (a, b). Suppose that $f'(x) > 0$ for all $x \in (a, b)$. Prove that the inverse function g of f is differentiable on (a, b) and that $g'(x) = 1/f'(x)$ for all $x \in (a, b)$.

Rudin.exercise.5.3 Suppose g is a continuous function on $\mathbb{R}$ and that g' is bounded. Prove that for each $N > 0$ there is an $\epsilon > 0$ such that the function $x \mapsto x + \epsilon g(x)$ is one-to-one on $\mathbb{R}$.

Rudin.exercise.5.4 Let $C_0, C_1, \ldots, C_n$ be real numbers. Prove that if $\sum_{i=0}^{n} C_i/(i+1) = 0$, then there exists $x \in [0, 1]$ such that $\sum_{i=0}^{n} C_i x^i = 0$.

Rudin.exercise.5.5 Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x)$ tends to 0 as x tends to infinity. Prove that $f(x+1) - f(x)$ tends to 0 as x tends to infinity.

Rudin.exercise.5.6 Suppose that f is a continuous function on $[0, 1]$ such that $f(0) = 0$ and $f'(x)$ is monotone increasing on $[0, 1]$. Prove that the function $g(x) = f(x)/x$ is monotone increasing on $(0, 1]$.

Rudin.exercise.5.7 Suppose that f and g are differentiable at x, that $g'(x) \neq 0$, and that $f(x) = g(x) = 0$. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

Shakarchi.exercise.1.13a Let f be a differentiable function on an open set Ω in $\mathbb{C}$. Suppose that f is constant on the real axis. Prove that f is constant on Ω.

Shakarchi.exercise.1.13b Let f be a differentiable function on an open set Ω in $\mathbb{C}$. Suppose that the imaginary part of f is constant on Ω. Prove that f is constant on Ω.

Shakarchi.exercise.1.13c Let f be a differentiable function on an open set Ω in $\mathbb{C}$. Suppose that f is constant on Ω. Prove that f is constant on any connected subset of Ω.

Shakarchi.exercise.1.19a If z is a complex number of modulus 1, and $s_n = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$, prove that s_n does not converge.

Shakarchi.exercise.1.19b Let z be a complex number of modulus 1. Let $s_n = \sum_{i=1}^{n} iz/i^2$. Prove that s_n converges.

Shakarchi.exercise.1.19c Let z be a complex number of modulus 1 and $z \neq 1$. Let $s_n = \sum_{i=1}^{n} iz^i/i$. Prove that s_n converges.