Notes on a general form for the Quantum optimal control Hamiltonian

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Abstract

Notes on the form of Hamiltonian and rules for constructing the general matrix form of H. Some additional comments on notation.

1 Hamiltonian

A fairly general form of the system Hamiltonian we are interested in looks like this:

$$H_0 = \sum_i \omega_i a_i^\dagger a_i - \chi_{ii} a_i^\dagger a_i^\dagger a_i a_i - \sum_{i \neq j} \chi_{ij} a_i^\dagger a_i a_j^\dagger a_j$$

where a_i are "annihilation operators" that are operating on a subspace of the full Hilbert space (In a way I'll expand on below) $a|n\rangle = \sqrt{n}|n-1\rangle$,

$$a = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{n_a - 1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{1}$$

a is a matrix of size $n_a \times n_a$. $|n\rangle$ is a column vector where only the nth entry is nonzero

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

The lowering operator a_i is constructed using Kronecker products, More detail on this flaky notation

$$\langle A|B\rangle \doteq A_1^*B_1 + A_2^*B_2 + \cdots + A_N^*B_N = \begin{pmatrix} A_1^* & A_2^* & \cdots & A_N^* \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{pmatrix}$$

and

$$|\phi\rangle\langle\psi| \doteq \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \begin{pmatrix} \psi_1^* & \psi_2^* & \cdots & \psi_N^* \end{pmatrix} = \begin{pmatrix} \phi_1\psi_1^* & \phi_1\psi_2^* & \cdots & \phi_1\psi_N^* \\ \phi_2\psi_1^* & \phi_2\psi_2^* & \cdots & \phi_2\psi_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N\psi_1^* & \phi_N\psi_2^* & \cdots & \phi_N\psi_N^* \end{pmatrix}$$

The full Hamiltonian is built up from tensor products of a and $a^\dagger$ operators acting on tensored subspaces. For example, in the case of qubits, each a acts on a two level subsystem.

$$a = \begin{pmatrix} 0.0 & 1.0 \\ 0.0 & 0.0 \end{pmatrix}$$

the full Hamiltonian matrix H for a two qubit system would then be made up of products of operators $a_0 = a \otimes I_2$ and $a_1 = I_2 \otimes a$ that look like

$$a_0 = a \otimes I_2 = \begin{pmatrix} 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

$$a_1 = I_2 \otimes a = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

Here, I_2 is the 2x2 identity.

So the full Hamiltonian for a two qubit system would look like

$$\begin{aligned} H = \quad & \omega_0(a \otimes I_2)^\dagger(a \otimes I_2) + \omega_1(I_2 \otimes a)^\dagger(I_2 \otimes a) + \\ & \chi_{00}(a \otimes I_2)^\dagger(a \otimes I_2)^\dagger(a \otimes I_2)(a \otimes I_2) + \\ & \chi_{11}(I_2 \otimes a)^\dagger(I_2 \otimes a)^\dagger(I_2 \otimes a)(I_2 \otimes a) + \\ & \chi_{01}(a \otimes I_2)^\dagger(a \otimes I_2)(I_2 \otimes a)^\dagger(I_2 \otimes a) \end{aligned}$$

2 Control terms

$$H_c = \sum_k \mathcal{F}(t, \underline{\alpha}_k)(a_k + a_k^\dagger) + \sum_k i\mathcal{G}(t, \underline{\alpha}_k)(a_k^\dagger - a_k) \tag{2}$$

3 example problem

3.1 state to state transfer

The objective is to solve for the control signals that will take the system from some predefined initial state ρ_0 to a target final state ρ_1

I this example we'll look at a problem suggested by Eric Holland that is fairly simple but physically relevant. The idea is that the system starts in a thermal state rather than the ideal ground state initial condition and our goal is to find a set of drives that quickly takes the system from this thermal state to the ground state. Here is

$$\rho_0 = \begin{pmatrix} 0.909 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.083 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.008 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 6.830 \times 10^{-04} & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 6.209 \times 10^{-05} & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 5.645 \times 10^{-06} \end{pmatrix}$$