Construction of general 2-oscillator Hamiltonian

Consider 2 oscillators system with n levels. Let $\rho(t) \in \mathbb{C}^{n^2 \times n^2}$ denote the density matrix. The Hamiltonian $H(t) \in \mathbb{C}^{n^2 \times n^2}$ is given by

$$\begin{aligned} H(t) = H_s &+ f_1(t)(a + a^\dagger) + f_2(t)(b + b^\dagger) \\ &+ i \left(g_1(t)(a - a^\dagger) + g_2(t)(b - b^\dagger) \right) \end{aligned} \tag{1}$$

and $a = a_1 \otimes I_n, b = I_n \otimes a_1$ and $a_1 \in \mathbb{R}^{n \times n}$ is the lowering operator

$$a_1 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \sqrt{n-1} \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

We solve the Liouville-van-Neumann equation

$$\begin{aligned} & \dot{\rho} = -i \left(H(t)\rho - \rho H(t) \right) && \in \mathbb{C}^{n^2 \times n^2} \\ \Leftrightarrow \quad & \mathrm{vec}\dot{\rho} = \underbrace{-i \left(I_N \otimes H(t) - H(t)^T \otimes I_N \right)}_{=: A(t) + iB(t)} \mathrm{vec}\rho && \in \mathbb{C}^{n^4} \end{aligned}$$

where $\mathrm{vec}\rho(t) \in \mathbb{C}^{n^4}$ denotes the vectorized density matrix. Splitting the vectorized density into real and imaginary part $\mathrm{vec}\rho =: u + iv$ we solve the following system for $u, v \in \mathbb{R}^{n^4}$:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t) \\ B(t) & A(t) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{2}$$

The real and imaginary part of the system matrix are given by

$$\begin{aligned} A(t) &= g_1(t) \left(I_{n^2} \otimes (a - a^\dagger) - (a - a^\dagger)^T \otimes I_{n^2} \right) \\ &\quad + g_2(t) \left(I_{n^2} \otimes (b - b^\dagger) - (b - b^\dagger)^T \otimes I_{n^2} \right) \\ B(t) &= -I_N \otimes H_s + H_s^T \otimes I_N \\ &\quad + f_1(t) \left(-I_{n^2} \otimes (a + a^\dagger) + (a + a^\dagger)^T \otimes I_{n^2} \right) \\ &\quad + f_2(t) \left(-I_{n^2} \otimes (b + b^\dagger) + (b + b^\dagger)^T \otimes I_{n^2} \right). \end{aligned}$$

Note that $A^\dagger = -A$ and $B^\dagger = B$ with $A, B \in \mathbb{R}^{n^4 \times n^4}$.

In order to construct A and B, we define the building matrices

$$\begin{aligned} C^+(k,m) &:= I_k \otimes (a_1 + a_1^\dagger) \otimes I_m \quad &\in \mathbb{R}^{nkm \times nkm} \\ C^-(k,m) &:= I_k \otimes (a_1 - a_1^\dagger) \otimes I_m \quad &\in \mathbb{R}^{nkm \times nkm} \end{aligned}$$

for given $k, m \in \mathbb{N}$. k is the number of repetitions of the blocks $(a_1 + a_1^\dagger) \otimes I_m \in \mathbb{R}^{nm}$. m is the number of repetitions of each $1, \sqrt{2}, \ldots,$ entry within the blocks. Note that $a_1 \pm a_1^\dagger = C^\pm(1,1) \in \mathbb{R}^{n \times n}$.

Implementing a function that takes k, m as input and returns $C^\pm(k,m)$, we can constructe $A(t)$ and $B(t)$ from

$$\begin{aligned} A(t) &= g_1(t)\left(C^-(n^2,n) - C^-(1,n^3)^T\right) + g_2(t)\left(C^-(n^3,1) - C^-(n,n^2)^T\right) \\ B(t) &= f_1(t)\left(C^+(1,n^3)^T - C^+(n^2,n)\right) + f_2(t)\left(C^+(n,n^2)^T - C^+(n^3,1)\right) \\ &\quad - I_N \otimes H_s + H_s^T \otimes I_N \end{aligned}$$