1 An analytical solution for two oscillator problem

We wish to construct an analytical solution for a decoupled two-oscillator system. We start by finding two separate solutions to one-oscillator systems and then use them to construct the solution to the two-oscillator case. Consider the Schroödinger equation

$$\dot{\psi} = -iH(t)\psi, \quad \psi(0) = \psi_0. \tag{1}$$

Solution 1: Suppose the Hamiltonian is given by

$$H(t) = H_1(t) = f(t)(a + a^\dagger), \quad a + a^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f(t) = \frac{1}{4}(1 - \cos(\omega t)) \tag{2}$$

An analytical solution is given by

$$\psi_1(t) = \begin{bmatrix} \cos(\phi(t)) \\ -i\sin(\phi(t))] \end{bmatrix}, \qquad \phi(t) = \frac{1}{4}(t - \frac{1}{\omega}\sin(\omega t)) \tag{3}$$

Check:

$$\dot{\psi}_1 = \frac{1}{4}(1 - \cos(\omega t)) \begin{bmatrix} -\sin(\phi) \\ -i\cos(\phi)) \end{bmatrix} \tag{4}$$

$$-iH_1\psi_1 = \frac{-i}{4}(1 - \cos(\omega t)) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi) \\ -i\sin(\phi) \end{bmatrix} = \frac{1}{4}(1 - \cos(\omega t)) \begin{bmatrix} -\sin(\phi) \\ -i\cos(\phi) \end{bmatrix} \tag{5}$$

Solution 2: Suppose the Hamiltonian is given by

$$H(t) = H_2(t) = ig(t)(a - a^\dagger), \quad a - a^\dagger = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g(t) = \frac{1}{4}(1 - \sin(\omega t)) \tag{6}$$

An analytical solution is given by

$$\psi_2(t) = \begin{bmatrix} \cos(\theta(t)) \\ -\sin(\theta(t)) \end{bmatrix}, \qquad \theta(t) = \frac{1}{4}(t + \frac{1}{\omega}\cos(\omega t) - 1) \tag{7}$$

Check:

$$\dot{\psi}_2 = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta)) \end{bmatrix} \tag{8}$$

$$-iH_2\psi_1 = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix} = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta) \end{bmatrix} \tag{9}$$

Constructing the two-oscillator solution: We now show that the solutions 1 and 2 can be used to solve eq (1) for the two-oscillator system given that $\psi = \psi_1 \otimes \psi_2$ and the

Hamiltonian is described as $H(t) = H_1(t) \otimes I_2 + I_2 \otimes H_2(t)$ where I_2 is the 2×2 identity matrix.

$$\dot{\psi} = \dot{\psi_1} \otimes \psi_2 + \psi_1 \otimes \dot{\psi_2} = -i(H_1\psi_1 \otimes \psi_2 + \psi_1 \otimes H_2\psi_2) \tag{10}$$

$$H_1\psi_1 \otimes \psi_2 = f(t) \begin{bmatrix} -i\sin(\phi) \\ \cos(\phi) \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix} = f(t) \begin{bmatrix} -i\sin(\phi)\cos(\theta) \\ i\sin(\phi)\sin(\theta) \\ \cos(\phi)\cos(\theta) \\ -\cos(\phi)\sin(\theta) \end{bmatrix} \tag{11}$$

$$\psi_1 \otimes H_2\psi_2 = g(t) \begin{bmatrix} \cos(\phi) \\ -i\sin(\phi) \end{bmatrix} \otimes \begin{bmatrix} -i\sin(\theta) \\ -i\cos(\theta) \end{bmatrix} = -g(t) \begin{bmatrix} i\cos(\phi)\sin(\theta) \\ i\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) \end{bmatrix} \tag{12}$$

where we have substituted solutions 1 and 2 in the second equality of (10). Now looking at the right hand side:

$$-iH\psi = -i \begin{bmatrix} 0 & ig(t) & f(t) & 0 \\ -ig(t) & 0 & 0 & f(t) \\ f(t) & 0 & 0 & ig(t) \\ 0 & f(t) & -ig(t) & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi)\cos(\theta) \\ -\cos(\phi)\sin(\theta) \\ -i\sin(\phi)\cos(\theta) \\ i\sin(\phi)\sin(\theta) \end{bmatrix} \tag{13}$$

it holds that $-i(H_1\psi_1 \otimes \psi_2 + \psi_1 \otimes H_2\psi_2) = -i(H_1 \otimes I_2 + I_2 \otimes H_2)(\psi_1 \otimes \psi_2)$ and therefore $\dot{\psi} = -iH\psi$.

Density Matrix: From here we can construct the corresponding density matrix

$$\rho(t) = \psi(t)\psi^\dagger(t), \quad \psi = \begin{bmatrix} \cos(\phi)\cos(\theta) \\ -\cos(\phi)\sin(\theta) \\ -i\sin(\phi)\cos(\theta) \\ i\sin(\phi)\sin(\theta) \end{bmatrix} \tag{14}$$

$$
\rho(t) = \begin{bmatrix} \cos(\phi)\cos(\theta) \\ -\cos(\phi)\sin(\theta) \\ -i\sin(\phi)\cos(\theta) \\ i\sin(\phi)\sin(\theta) \end{bmatrix} [\cos(\phi)\cos(\theta), -\cos(\phi)\sin(\theta), i\sin(\phi)\cos(\theta), -i\sin(\phi)\sin(\theta)]
$$

$$
= \begin{bmatrix} \cos^2(\phi)\cos^2(\theta) & * & * & * \\ * & \cos^2(\phi)\sin^2(\theta) & * & * \\ * & * & \sin^2(\phi)\cos^2(\theta) & * \\ * & * & * & \sin^2(\phi)\sin^2(\theta) \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos^2(\phi)\cos^2(\theta) & -\cos^2(\phi)\cos(\theta)\sin(\theta) & i\cos(\phi)\cos^2(\theta)\sin(\phi) & -i\cos(\phi)\cos(\theta)\sin(\phi)\sin(\theta) \\ -\cos^2(\phi)\cos(\theta)\sin(\theta) & \cos^2(\phi)\sin^2(\theta) & -i\cos(\phi)\cos(\theta)\sin(\phi)\sin(\theta) & i\cos(\phi)\sin(\theta^2)\sin(\phi) \\ -i\sin(\phi)\cos^2(\theta)\cos(\phi) & i\cos(\phi)\cos(\theta)\sin(\phi)\sin(\theta) & \sin^2(\phi)\cos^2(\theta) & -\sin^2(\phi)\cos(\theta)\sin(\theta) \\ i\cos(\phi)\cos(\theta)\sin(\phi)\sin(\theta) & -i\sin(\phi)\sin^2(\theta)\cos(\phi) & -\sin^2(\phi)\sin(\theta)\cos(\theta) & \sin^2(\phi)\sin^2(\theta) \end{bmatrix}
$$