Quantum harminic oscillator

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Notation and lowering operator

Ground state eigenfunction $\psi_0(x)$ and first excited state $\psi_1(x)$ using vector and bra-ket notation

$$\psi_0(x) = |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \psi_1(x) = |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix},$$

The n^{th} excited state eigenfunction: $\psi_n(x) = |n\rangle$.
The lowering operator satisfies $a\psi_n(x) = \sqrt{n}\psi_{n-1}(x)$. In matrix form,

$$a = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & \ddots & \ddots \end{bmatrix}, \quad a|n\rangle = \sqrt{n}|n-1\rangle.$$

Raising and number operators

The raising operator satisfies $a^\dagger \psi_n(x) = \sqrt{n+1}\psi_{n+1}(x)$,

$$a^\dagger = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

The number operator, N,

$$N = a^\dagger a = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{bmatrix}, \quad N|n\rangle = n|n\rangle.$$

also, $aa^\dagger = N + I$.

Hamiltonian and energy

Hamiltonian of a quantum harmonic oscillator in operator form,

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) = \hbar\omega \left(N + \frac{1}{2} I \right)$$

and in matrix form

$$H = \frac{\hbar\omega}{2} \begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix}$$

The Hamiltonian of and eigenfunction $\psi_n(x)$,

$$H\psi_n = \hbar\omega \left(N + \frac{1}{2} I \right) \psi_n = \hbar\omega \left(n + \frac{1}{2} \right) \psi_n,$$

gives the energy level, $E_n = \hbar\omega \left(n + \frac{1}{2} \right)$.

Coupled oscillators

The system Hamiltonian for Q qudits is

$$H_0 = \sum_{j=0}^{Q-1} \left(\omega_j a_j^\dagger a_j - \chi_{jj} a_j^\dagger a_j^\dagger a_j a_j - \sum_{k \neq j} \chi_{jk} a_j^\dagger a_j a_k^\dagger a_k \right)$$

If we retain L levels in each qudit, the lowering operators are defined in terms of the L by L identity matrix I_L. If $Q = 2$,

$$a_0 = a \otimes I_L, \quad a_1 = I_L \otimes a.$$

Thus,

$$a_0^\dagger a_0 = (a^\dagger \otimes I_L)(a \otimes I_L) = (a^\dagger a) \otimes (I_L I_L) = N \otimes I_L,$$
$$a_1^\dagger a_1 = (I_L \otimes a^\dagger)(I_L \otimes a) = (I_L I_L) \otimes (a^\dagger a) = I_L \otimes N.$$

The matrices $a_0^\dagger$ and $a_1^\dagger$ have size L^2 by L^2.

The system Hamiltonian

The matrices

$$a_0^\dagger a_0 = N \otimes I_L =: N_0,$$
$$a_1^\dagger a_1 = I_L \otimes N =: N_1,$$

are both diagonal. Furthermore,

$$a_0^\dagger a_0^\dagger a_0 a_0 = N_0 N_0 - N_0,$$
$$a_1^\dagger a_1^\dagger a_1 a_1 = N_1 N_1 - N_1,$$

are also diagonal. Thus all terms in the system Hamiltonian

$$H_0 = \sum_{j=0}^{Q-1} \left(\omega_j N_j - \chi_{jj}(N_j^2 - N_j) - \sum_{k \neq j} \chi_{jk} N_j N_k \right)$$

are diagonal.

The control Hamiltonian

The control Hamiltonian satisfies

$$H_c(t) = \sum_{k=0}^{Q-1} F(t)\left(a_k^\dagger + a_k\right) + i \sum_{k=0}^{Q-1} G(t)\left(a_k^\dagger - a_k\right),$$

where F and G are real-valued functions. These matrices are not diagonal, but block-diagonal. For example,

$$a_0^\dagger + a_0 = (a \otimes I_L) + (a^\dagger \otimes I_L) =$$

$$\begin{bmatrix} 0 & I_L & & & \\ I_L & 0 & \sqrt{2} I_L & & \\ & \sqrt{2} I_L & 0 & \sqrt{3} I_L & \\ & & \sqrt{3} I_L & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

The control Hamiltonian 2

Similarly,

$$a_1^\dagger + a_1 = (I_L \otimes a) + (I_L \otimes a^\dagger) =$$

$$\begin{bmatrix} a^\dagger + a & 0 & & & \\ 0 & a^\dagger + a & 0 & & \\ & 0 & a^\dagger + a & 0 & \\ & & 0 & a^\dagger + a & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

is a block-diagonal matrix.