The quantum control problem in a rotating frame of reference

N. Anders Petersson*

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1 The Schrödinger equations

Consider the Schrödinger equation in a laboratory frame of reference,

$$\dot{\psi} = -iH(t)\psi, \quad 0 \le t \le T, \quad \psi(0) = \psi_0, \quad H(t) = H_s + f(t)(a + a^\dagger). \tag{1}$$

Here, $\psi(t) \in [0, T] \to \mathbb{C}^N$ is the wave function, $f(t) \in [0, T] \to \mathbb{R}$ is the control function and $H_s = H_s^\dagger$ is the $N \times N$ system Hamiltonian matrix, which we assume to be real and independent of time. The lowering and raising matrices are denoted a and $a^\dagger$, respectively. These real matrices satisfy

$$a = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & 0 & \sqrt{3} & & \\ & & & 0 & \sqrt{4} & \\ & & & & \ddots & \ddots \end{bmatrix}, \quad a^\dagger = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \tag{2}$$

The basic model for the system Hamiltonian of a quantum oscillator is

$$H_s = \omega_a a^\dagger a - \pi \xi_a a^\dagger a^\dagger a a, \quad a^\dagger a =: N = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{bmatrix}. \tag{3}$$

*Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, L-561, PO Box 808, Livermore CA 94551.

Here, N is called the number operator. Clearly, $aa^\dagger = a^\dagger a + I$, so

$$a^\dagger a^\dagger aa = a^\dagger (aa^\dagger - I)a = N^2 - N = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 2 & & \\ & & & 6 & \\ & & & & \ddots \end{bmatrix}.$$

We conclude that H_s is a diagonal matrix. Note that the units of ω_a is [rad/s].

1.1 Time-dependent unitary transformations

Consider the unitary tranformation

$$\psi(t) = R^\dagger(t)v(t), \quad R^\dagger R = I.$$

We have

$$\dot{\psi} = \dot{R}^\dagger v + R^\dagger \dot{v},$$
$$H\psi = HR^\dagger v$$

Thus, (1) gives

$$\dot{R}^\dagger v + R^\dagger \dot{v} = -iHR^\dagger v,$$

By using the identity $R\dot{R}^\dagger = -\dot{R}R^\dagger$ and reorganizing the terms,

$$\dot{v} = -iRHR^\dagger v + \dot{R}R^\dagger v = -i\left(RHR^\dagger + i\dot{R}R^\dagger\right)v.$$

Thus, the transformed problem becomes

$$\dot{v} = -i\tilde{H}(t)v, \quad \tilde{H}(t) = R(t)H(t)R(t)^\dagger + i\dot{R}(t)R(t)^\dagger. \tag{4}$$

1.2 Rotating frame of reference

In the first term on the right hand side of (3), the difference between consequtive diagonal elements is constant. This structure suggests the unitary transformation

$$R(t) = \exp(i\omega_a Nt), \quad \dot{R}R^\dagger = i\omega_a N.$$

Both N and N^2 commute with $R(t)$. From (4), the first term in the system Hamiltonian (3) cancels and the transformed Hamiltonian becomes

$$\widetilde{H}(t) = -\pi\xi_a\left(N^2 - N\right) + f(t)\left(RaR^\dagger + Ra^\dagger R^\dagger\right). \tag{5}$$

We have

$$
RaR^\dagger = \begin{bmatrix} 1 & & & \\ & e^{i\omega_a t} & & \\ & & e^{2i\omega_a t} & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & \\ & e^{-i\omega_a t} & & \\ & & e^{-2i\omega_a t} & \\ & & & \ddots \end{bmatrix} =
$$

$$
\begin{bmatrix} 0 & e^{-i\omega_a t} & & & \\ & 0 & \sqrt{2}e^{-i\omega_a t} & & \\ & & 0 & \sqrt{3}e^{-i\omega_a t} & \\ & & & \ddots & \ddots \end{bmatrix} =: e^{-i\omega_a t} a.
$$

Taking the conjugate transpose gives $Ra^\dagger R = e^{i\omega_a t} a^\dagger$. Thus, (5) becomes

$$
\widetilde{H}(t) = -\pi\xi_a \left(N^2 - N\right) + f(t)\left(e^{-i\omega_a t} a + e^{i\omega_a t} a^\dagger\right). \tag{6}
$$

We would like to absorb the highly oscillatory factors $\exp(\pm i\omega_a t)$ into $f(t)$. Because the control function $f(t)$ is real-valued, this can only be done in an approximate fashion. We make the ansatz,

$$
f(t) = 2g_1(t)\cos(\omega_a t) - 2g_2(t)\sin(\omega_a t) = \\ (g_1 + ig_2)\exp(i\omega_a t) + (g_1 - ig_2)\exp(-i\omega_a t), \tag{7}
$$

where $g_1(t)$ and $g_2(t)$ are real-valued function. Thus,

$$
f(t)\exp(-i\omega_a t) = (g_1 + ig_2) + (g_1 - ig_2)\exp(-2i\omega_a t), \tag{8}
$$

$$
f(t)\exp(i\omega_a t) = (g_1 + ig_2)\exp(2i\omega_a t) + (g_1 - ig_2). \tag{9}
$$

The transformed Hamiltonian (6) becomes

$$
\begin{aligned}
\widetilde{H}(t) = &-\pi\xi_a \left(N^2 - N\right) \\
&+ (g_1 + ig_2)\, a + (g_1 - ig_2)\exp(-2i\omega_a t) a + (g_1 + ig_2)\exp(2i\omega_a t) a^\dagger + (g_1 - ig_2)\, a^\dagger \\
= &-\pi\xi_a \left(N^2 - N\right) + g_1\left(a + a^\dagger\right) + ig_2\left(a - a^\dagger\right) \\
&+ (g_1 - ig_2)\exp(-2i\omega_a t) a + (g_1 + ig_2)\exp(2i\omega_a t) a^\dagger.
\end{aligned}
$$

The rotating frame approximation follows by ignoring the terms that oscillate with twice the frequency, $\exp(\pm 2i\omega_a t)$, resulting in the system

$$
\dot{v} = -\widetilde{H}_r(t) v, \quad v(0) = R(0)\psi_0 = \psi_0, \tag{10}
$$

$$
\widetilde{H}_r(t) = H_d + g_1(t)\left(a + a^\dagger\right) + ig_2(t)\left(a - a^\dagger\right), \quad H_d = -\pi\xi_a\left(N^2 - N\right). \tag{11}
$$

Here, H_d is called the drift Hamiltonian. When $\xi_a \ll \omega_a$, the solutions of this system varies on a significantly longer time scale than (1).

After the control functions $g_1(t)$ and $g_2(t)$ have been determined, the corresponding control function in the laboratory frame becomes

$$f(t) = 2g_1(t)\cos(\omega_a t) - 2g_2(t)\sin(\omega_a t). \tag{12}$$

2 Analytical solutions

The 2×2 Schrödinger system in the rotating frame with constant control functions $g_1 = \Omega_r$ and $g_2 = \Omega_i$ satisfies

$$\dot{v} = -i\widetilde{H}v, \quad \widetilde{H} = \Omega_r(a + a^\dagger) + i\Omega_i(a - a^\dagger) = \begin{bmatrix} 0 & \Omega \\ \bar{\Omega} & 0 \end{bmatrix}, \quad \Omega = \Omega_r + i\Omega_i. \tag{13}$$

The complex matrix $\widetilde{H}$ can be diagonalized by the unitary matrix X,

$$X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -\Omega/|\Omega| \\ \bar{\Omega}/|\Omega| & 1 \end{bmatrix}, \quad X^\dagger \widetilde{H} X = \begin{bmatrix} |\Omega| & 0 \\ 0 & |\Omega| \end{bmatrix}.$$

By the variable transformation

$$\widetilde{v} = X^\dagger v,$$

the system (13) becomes and is solved by

$$\dot{\widetilde{v}} = -i\begin{bmatrix} |\Omega| & 0 \\ 0 & |\Omega| \end{bmatrix}\widetilde{v}, \quad \widetilde{v}(t) = \begin{bmatrix} \alpha\exp(-i|\Omega|t) \\ \beta\exp(i|\Omega|t) \end{bmatrix},$$

where α and β are constants. Transforming back to the original variables, $v = X\widetilde{v}(t)$, gives

$$v(t) = X\widetilde{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha\exp(-i|\Omega|t) - \frac{\Omega}{|\Omega|}\beta\exp(i|\Omega|t) \\ \frac{\bar{\Omega}}{|\Omega|}\alpha\exp(-i|\Omega|t) + \beta\exp(i|\Omega|t) \end{bmatrix}$$

To form a basis for all initial data, we first consider

$$v^I(0) := \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha_1 - \frac{\Omega}{|\Omega|}\beta_1 \\ \frac{\bar{\Omega}}{|\Omega|}\alpha_1 + \beta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\frac{\bar{\Omega}}{|\Omega|} \end{bmatrix}.$$

The second initial data is

$$v^{II}(0) := \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha_2 - \frac{\Omega}{|\Omega|}\beta_2 \\ \frac{\bar{\Omega}}{|\Omega|}\alpha_2 + \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\Omega}{|\Omega|} \\ 1 \end{bmatrix}.$$

After some algebra, the two fundamental solutions can be written,

$$v^I(t) = \begin{bmatrix} \cos(|\Omega|t) \\ -i\frac{\bar{\Omega}}{|\Omega|}\sin(|\Omega t) \end{bmatrix}, \quad v^{II}(t) = \begin{bmatrix} -i\frac{\Omega}{|\Omega|}\sin(|\Omega t) \\ \cos(|\Omega|t) \end{bmatrix}.$$

Thus, the evolution from the canonical basis is

$$V(t) = \begin{bmatrix} \cos(|\Omega|t) & -i\frac{\Omega}{|\Omega|}\sin(|\Omega t) \\ -i\frac{\bar{\Omega}}{|\Omega|}\sin(|\Omega t) & \cos(|\Omega|t) \end{bmatrix}.$$

It is instructive to set

$$\Omega := \Omega_r + i\Omega_i = |\Omega|(\cos(\theta) + i\sin(\theta)). \tag{14}$$

and

$$V(t) = \begin{bmatrix} \cos(|\Omega|t) & (\sin(\theta) - i\cos(\theta))\sin(|\Omega t) \\ -(\sin(\theta) + i\cos(\theta))\sin(|\Omega t) & \cos(|\Omega|t) \end{bmatrix}.$$

The control function in the laboratory frame follows by inserting $g_1 = \Omega_r$ and $g_2 = \Omega_i$ in (12),

$$f(t) = 2\Omega_r\cos(\omega_a t) - 2\Omega_i\sin(\omega_a t) = 2|\Omega|\cos(\omega_a t + \theta). \tag{15}$$

In this case, the amplitude of the control function equals the angular frequency of $V(t)$. Hence, the period of the oscillation is

$$T = \frac{2\pi}{|\Omega|}.$$

The phase of $V(t)$ is controlled by the phase of the control function, θ.