

# 1 An analytical solution for two oscillator problem

We wish to construct an analytical solution for a decoupled two-oscillator system. We start by finding two separate solutions to one-oscillator systems and then use them to construct the solution to the two-oscillator case. Consider the Schrödinger equation

$$\dot{\psi} = -iH(t)\psi, \quad \psi(0) = \psi_0. \quad (1)$$

**Solution 1:** Suppose the Hamiltonian is given by

$$H(t) = H_1(t) = f(t)(a + a^\dagger), \quad a + a^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f(t) = \frac{1}{4}(1 - \cos(\omega t)) \quad (2)$$

An analytical solution is given by

$$\psi_1(t) = \begin{bmatrix} \cos(\phi(t)) \\ -i \sin(\phi(t)) \end{bmatrix}, \quad \phi(t) = \frac{1}{4}\left(t - \frac{1}{\omega} \sin(\omega t)\right) \quad (3)$$

**Check:**

$$\dot{\psi}_1 = \frac{1}{4}(1 - \cos(\omega t)) \begin{bmatrix} -\sin(\phi) \\ -i \cos(\phi) \end{bmatrix} \quad (4)$$

$$-iH_1\psi_1 = \frac{-i}{4}(1 - \cos(\omega t)) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi) \\ -i \sin(\phi) \end{bmatrix} = \frac{1}{4}(1 - \cos(\omega t)) \begin{bmatrix} -\sin(\phi) \\ -i \cos(\phi) \end{bmatrix} \quad (5)$$

**Solution 2:** Suppose the Hamiltonian is given by

$$H(t) = H_2(t) = ig(t)(a - a^\dagger), \quad a - a^\dagger = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g(t) = \frac{1}{4}(1 - \sin(\omega t)) \quad (6)$$

An analytical solution is given by

$$\psi_2(t) = \begin{bmatrix} \cos(\theta(t)) \\ -\sin(\theta(t)) \end{bmatrix}, \quad \theta(t) = \frac{1}{4}\left(t + \frac{1}{\omega} \cos(\omega t) - 1\right) \quad (7)$$

**Check:**

$$\dot{\psi}_2 = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta) \end{bmatrix} \quad (8)$$

$$-iH_2\psi_2 = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix} = \frac{1}{4}(1 - \sin(\omega t)) \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta) \end{bmatrix} \quad (9)$$

**Constructing the two-oscillator solution:** We now show that the solutions 1 and 2 can be used to solve eq (1) for the two-oscillator system given that  $\psi = \psi_1 \otimes \psi_2$  and the

Hamiltonian is described as  $H(t) = H_1(t) \otimes I_2 + I_2 \otimes H_2(t)$  where  $I_2$  is the  $2 \times 2$  identity matrix.

$$\dot{\psi} = \dot{\psi}_1 \otimes \psi_2 + \psi_1 \otimes \dot{\psi}_2 = -i(H_1\psi_1 \otimes \psi_2 + \psi_1 \otimes H_2\psi_2) \quad (10)$$

$$H_1\psi_1 \otimes \psi_2 = f(t) \begin{bmatrix} -i \sin(\phi) \\ \cos(\phi) \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix} = f(t) \begin{bmatrix} -i \sin(\phi) \cos(\theta) \\ i \sin(\phi) \sin(\theta) \\ \cos(\phi) \cos(\theta) \\ -\cos(\phi) \sin(\theta) \end{bmatrix} \quad (11)$$

$$\psi_1 \otimes H_2\psi_2 = g(t) \begin{bmatrix} \cos(\phi) \\ -i \sin(\phi) \end{bmatrix} \otimes \begin{bmatrix} -i \sin(\theta) \\ -i \cos(\theta) \end{bmatrix} = -g(t) \begin{bmatrix} i \cos(\phi) \sin(\theta) \\ i \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) \\ \sin(\phi) \cos(\theta) \end{bmatrix} \quad (12)$$

where we have substituted solutions 1 and 2 in the second equality of (10). Now looking at the right hand side:

$$-iH\psi = -i \begin{bmatrix} 0 & ig(t) & f(t) & 0 \\ -ig(t) & 0 & 0 & f(t) \\ f(t) & 0 & 0 & ig(t) \\ 0 & f(t) & -ig(t) & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi) \cos(\theta) \\ -\cos(\phi) \sin(\theta) \\ -i \sin(\phi) \cos(\theta) \\ i \sin(\phi) \sin(\theta) \end{bmatrix} \quad (13)$$

it holds that  $-i(H_1\psi_1 \otimes \psi_2 + \psi_1 \otimes H_2\psi_2) = -i(H_1 \otimes I_2 + I_2 \otimes H_2)(\psi_1 \otimes \psi_2)$  and therefore  $\dot{\psi} = -iH\psi$ .

**Density Matrix:** From here we can construct the corresponding density matrix

$$\rho(t) = \psi(t)\psi^\dagger(t), \quad \psi = \begin{bmatrix} \cos(\phi) \cos(\theta) \\ -\cos(\phi) \sin(\theta) \\ -i \sin(\phi) \cos(\theta) \\ i \sin(\phi) \sin(\theta) \end{bmatrix} \quad (14)$$

$$\begin{aligned}
\rho(t) &= \begin{bmatrix} \cos(\phi) \cos(\theta) \\ -\cos(\phi) \sin(\theta) \\ -i \sin(\phi) \cos(\theta) \\ i \sin(\phi) \sin(\theta) \end{bmatrix} [\cos(\phi) \cos(\theta), -\cos(\phi) \sin(\theta), i \sin(\phi) \cos(\theta), -i \sin(\phi) \sin(\theta)] \\
&= \begin{bmatrix} \cos^2(\phi) \cos^2(\theta) & * & * & * \\ * & \cos^2(\phi) \sin^2(\theta) & * & * \\ * & * & \sin^2(\phi) \cos^2(\theta) & * \\ * & * & * & \sin^2(\phi) \sin^2(\theta) \end{bmatrix} \quad (15)
\end{aligned}$$