Spherical Harmonics

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December 15, 2016

I will be attempting to derive the spherical harmonics in this pdf file.

$$\check{L}_{\pm} \xrightarrow{pos.} \frac{\hbar}{i} e^{\pm i\phi}\left(\pm\, i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right), \quad \check{L}_{-}\left|l,\lambda\right\rangle = \sqrt{(l+\lambda)(l-\lambda+1)}\hbar\left|l,\lambda-1\right\rangle$$

$$Y_{l,l}(\theta,\phi) = \langle\theta,\phi|l,l\rangle = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l\theta$$

If I apply $\check{L}_{-}$ an $l-m$ times, we will eventually reach an arbitrary eigenstate $|l,m\rangle$. That will be the choice to derive $Y_{l,m}(\theta,\phi) = \langle\theta,\phi|l,m\rangle$.

So the first step is to, in a sense, use the operator $\check{L}_{-}$ and its eigenvalues. We need to find K first before we can do anything else.

$$\langle\theta,\phi|\,\check{L}_{-}^{l-m}\,|l,l\rangle = K\,\langle\theta,\phi|l,m\rangle \xrightarrow{pos.} \frac{\hbar^{l-m}}{i^{l-m}}\left[e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\right]^{l-m}\langle\theta,\phi|l,l\rangle$$

To start it off, lets apply the operator $l-m$ times.

$$\check{L}_{-}^{l-m}\,|l,l\rangle = K\,|l,m\rangle$$
$$\sqrt{(2l)(1)}\sqrt{(2l-1)(2)}\sqrt{(2l-2)(3)}...\sqrt{(l+m+2)(l-m-1)}\sqrt{(l+m+1)(l-m)}\hbar^{l-m}\,|l,m\rangle$$

Sorting the left terms in each radical and aligning them together we can see that there are two series to sort out.

$1: \quad \longrightarrow (2l)(2l-1)(2l-2)(2l-3) \quad . \quad . \quad . \quad (l+m+2)(l+m+1).$

$2: \quad \longrightarrow (1)(2)(3) \quad . \quad . \quad . \quad (l-m-1)(l-m).$

Well, the second line is easy. That is just a factorial of the form $(l-m)!$. The first one, on the otherhand, will require some sort of modification to account for those missing terms. It sort of takes the form $(2l)!$.

$$(2l)! = (2l)(2l-1)(2l-2)...(l+m+2)(l+m+1) \bowtie (l+m)(l+m-1)...(3)(2)(1)$$

$1: \quad \longrightarrow \dfrac{(2l)!}{(l+m)!}$

$2: \quad \longrightarrow (l-m)!$

So what we can do next is replace that really long term up top now.

$$\check{L}_{-}^{l-m}|l,l\rangle = K|l,m\rangle = \sqrt{\frac{(2l)!(l-m)!}{(l+m)!}}\hbar^{l-m}|l,m\rangle$$

Hey! We have our constant now!

$$Y_{l,m}(\theta,\phi) = \langle\theta,\phi|l,m\rangle \xrightarrow{pos.} \frac{1}{K}\frac{\hbar^{l-m}}{i^{l-m}}\left[e^{-i\phi}\Bigg(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]^{l-m}\langle\theta,\phi|l,l\rangle$$

$$\langle\theta,\phi|l,m\rangle = \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}}\frac{1}{\hbar^{l-m}}\frac{\hbar^{l-m}}{i^{l-m}}\left[e^{-i\phi}\Bigg(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]^{l-m}\langle\theta,\phi|l,l\rangle$$

Don't bitch, it cleaned up a little...

$$\sqrt{\frac{(l+m)!}{(2l)!(l-m)!}}\frac{1}{\hbar^{l-m}}\frac{\hbar^{l-m}}{i^{l-m}}\left[e^{-i\phi}\Bigg(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]^{l-m}\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l\theta$$

So lets change some stuff and re-arrange.

$$\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{(2l)!}}\sqrt{\frac{(l+m)!}{4\pi(l-m)!}}\frac{1}{\hbar^{l-m}}\frac{\hbar^{l-m}}{i^{l-m}}i^{l-m}\left[e^{-i\phi}\Bigg(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]^{l-m}e^{il\phi}\sin^l\theta$$

Look! Shit cancels!

$$\langle\theta,\phi|l,m\rangle = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}}\left[e^{-i\phi}\Bigg(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]^{l-m}e^{il\phi}\sin^l\theta$$

Let's take a closer look at the following derivatives.

$$...\left[e^{-i\phi}\Bigg(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]\cdot\left[e^{-i\phi}\Bigg(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Bigg)\right]e^{il\phi}\sin^l\theta$$

So operate with each term with only respect to ϕ.

$$e^{im\phi}\Bigg(-\frac{\partial}{\partial\theta} - (m+1)\cot\theta\Bigg)\cdot\Bigg(-\frac{\partial}{\partial\theta} - (m+2)\cot\theta\Bigg)...$$

$$...\Bigg(-\frac{\partial}{\partial\theta} - (l-1)\cot\theta\Bigg)\cdot\Bigg(-\frac{\partial}{\partial\theta} - (l)\cot\theta\Bigg)\sin^l\theta$$

Here is a random equation!

$$\frac{1}{\sin^k\theta}\frac{d}{d\theta}\Big(f(\theta)\sin^k\theta\Big) = \frac{1}{\sin^k\theta}\Big(\frac{df}{d\theta}\sin^k\theta + kf(\theta)\cos\theta\sin^{k-1}\theta\Big)$$
$$= \frac{df}{d\theta} + kf(\theta)\cot\theta = \Big(\frac{d}{d\theta} + k\cot\theta\Big)f(\theta)$$

So plug that shit right in!

$$e^{im\phi}\Big(-\frac{d}{d\theta} - (m+1)\cot\theta\Big)\cdot\Big(-\frac{d}{d\theta} - (m+2)\cot\theta\Big)...$$
$$...\Big(-\frac{d}{d\theta} - (l-1)\cot\theta\Big)\cdot\Big(-\frac{d}{d\theta} - (l)\cot\theta\Big)\sin^l\theta$$

Substituted into above yields.

$$e^{im\phi}\Big(\frac{-1}{\sin^{m+1}\theta}\frac{d}{d\theta}\sin^{m+1}\theta\Big)\cdot\Big(\frac{-1}{\sin^{m+2}\theta}\frac{d}{d\theta}\sin^{m+2}\theta\Big)...$$
$$...\Big(\frac{-1}{\sin^{l-1}\theta}\frac{d}{d\theta}\sin^{l-1}\theta\Big)\cdot\Big(\frac{-1}{\sin^{l}\theta}\frac{d}{d\theta}\sin^{l}\theta\Big)\sin^l\theta$$

As you can see some terms cancel out leaving the dimensionalizing fucntion.

$$e^{im\phi}\frac{1}{\sin^m\theta}\left(\frac{-1}{\sin\theta}\frac{d}{d\theta}\right)^{l-m}\sin^{2l}\theta$$

Which can be simplified even further unto $x = \cos\theta \rightarrow dx = d(\cos\theta) = -\sin\theta d\theta$.

$$e^{im\phi}\frac{1}{\sin^m\theta}\left(\frac{d}{d(\cos\theta)}\right)^{l-m}\sin^{2l}\theta$$

Somewhat more usable, but whatever...

$$\langle\theta,\phi|l,m\rangle = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}\sin^{2l}\theta$$

So now I will put it in terms of the associated Legendre polynomial. So $x = \cos\theta$

$$\frac{(-1)^l}{2^l l!}\frac{1}{\sin^m\theta}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}\sin^{2l}\theta = \frac{1}{2^l l!}\frac{1}{(1-x^2)^{m/2}}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l$$
$$\star\ \frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l = (-1)^m\frac{(l-m)!}{(l+m)!}(1-x^2)^m\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$
$$\because P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

Substitute it into the original and modify.

$$= \frac{1}{2^l l!} \frac{1}{(1-x^2)^{m/2}} (-1)^m \frac{(l-m)!}{(l+m)!} (1-x^2)^m \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$
$$= \frac{(l-m)!}{(l+m)!} \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$
$$= \frac{(l-m)!}{(l+m)!} P_l^m(x)$$
$$\langle \theta, \phi | l, m \rangle = \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} e^{im\phi} \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

So... Imma do a Townsend and just give you an equation. So here is the Spherical Harmonic equations related to the associated Legendre polynomials.

$$Y_{l,m}(\theta, \phi) = \langle \theta, \phi | l, m \rangle = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi} P_l^m(\cos\theta)$$
$$Y_{l,-m}(\theta, \phi) = \langle \theta, \phi | l, -m \rangle = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{-im\phi} P_l^m(\cos\theta)$$

...and that is how you derive it, EZPZ. (pg. 69durmumlastnite)

$$R_{n,l}(r) = \sqrt{\frac{4Z(n-l-1)!}{n^2 a_0 (n+l)!}} \left(\frac{2Z}{na_0}\right)^l r^l L_{n_r}^{(2l+1)}\left(\frac{2Z}{na_0} r\right) e^{-\frac{Zr}{na_0}}$$

That is the radial wave function in terms of the associated Laguerre polynomials. So now let's see if there is enough room to place the full wave function for the electron of the hydrogen atom.

$$\psi(\vec{r}) = \langle r, \theta, \phi | n, l, m \rangle = R_{n,l}(r) Y_{l,m}(\theta, \phi) =$$
$$\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi} P_l^m(\cos\theta) \sqrt{\frac{4Z(n_r!)}{n^2 a_0 (n+l)!}} \rho^l e^{-\rho/2} L_{n_r}^{(2l+1)}(\rho)$$
$$n_r = n - l - 1 \quad \rho = \frac{2Z}{na_0} r$$

Isn't elegant? Fuck me, I chose the wrong major... Well there you go, a half assed derivation that took me about four hours to type out. Someone shoot me. Good night, I hate you all, and don't even bother for fucking time evolution.