COMP3211 Tutorial 6: MDP/RL

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Outline

MDP V.S. Search

Policy Iteration/Value Iteration

Coding Examples

- Grid Worlds - via Policy Iteration
- Gambler's Problem - via Value Iteration
- Maximization Bias Problem - via Q Learning

MDP V.S. Search

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Search:

- A set of states $\mathcal{S}$, initial state I, goal state G
- A set of actions $\mathcal{A}$
- Deterministic transitions $T : \mathcal{S} \times \mathcal{A} \mapsto \mathcal{S}$
- cost function $c : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$
- Objective: a path p from I to G that minimizes $c(p)$

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① $|S|^2$

② $|A|^2$

✓ ③ $|S| \times |A|$

MDP:

Q_1 $T(S_j | S_i, a_i)$

- A set of states $\mathcal{S}$, a terminating condition $End(s)$
- A set of actions $\mathcal{A}$
- Stochastic transitions $T : \mathcal{S} \times \mathcal{A} \mapsto \boxed{\Delta}(\mathcal{S})$
- Reward function $r : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \mapsto \mathbb{R}$, with a discount factor γ
- Objective: maximize $\sum_t \gamma^t r_t$

$|S| \times |S| \times |A|$

Solution Concept: Policy

Question:
Given a sequential and stochastic decision making problem, in order to come up with an optimal solution, you'd like to know:

- (A) Only your current state
- (B) All the history from the beginning up until now
- (C) Need to know more

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Markov property:
A state S_t is Markovian iff $P[S_{t+1}|S_t, \cdots, S_0] = P[S_{t+1}|S_t]$. That is, your current state is already a "sufficient statistic" that well summarizes the history, also known as the information state.

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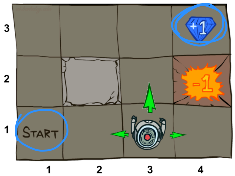
Policy in MDP:
A solution is a policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$

Follow-up Question: Maze

Question:
Given a large maze, you (with deterministic actions U/D/L/R) are supposed to find a nice way from the entrance to the exit, which agent you'd like to choose

- (A) State machines with infite memory
- (B) Agents that can do A^* search
- (C) Agents that can compute policies
- (D) None of them



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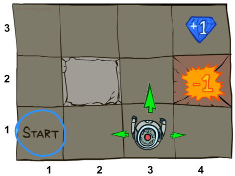
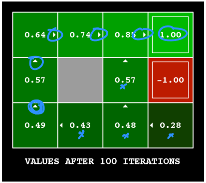
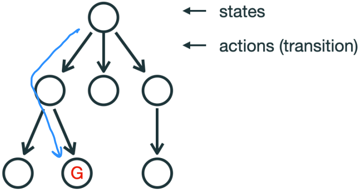
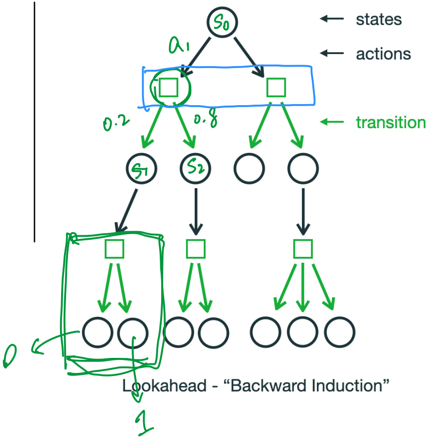




Tree Search: A Conceptual Bridge



Lookahead - Backtrack



Lookahead - "Backward Induction"

A policy is nothing but a conditional plan!

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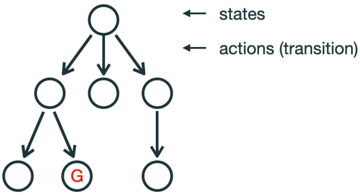

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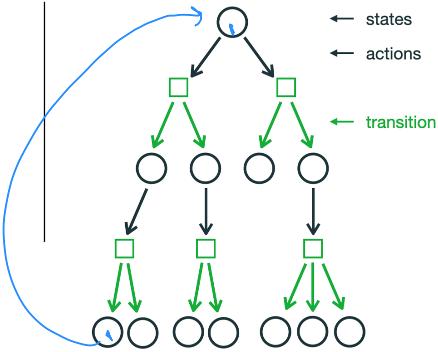


Lookahead - “Backward Induction”

A policy is nothing but a conditional plan!

What if (1) repeat states; (2) infinite horizon?

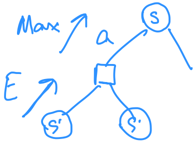
Bellman (Optimality) Equation

$v_1 \leftarrow f_B(v_0)$

$v_2 \leftarrow f_B(v_1)$

$\vdots$

$v_\infty \leftarrow f_B(v_\infty)$



- For state-value function,

$$v_*(s) = \boxed{\max_a} \sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma \boxed{v_*(s')}]$$

- For action-value function,

$$q_*(s,a) = \max_a \sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma \max_{a'} q_*(s',a')]$$

- However, non-linear thus no closed form solution in general!

Policy Iteration/Value Iteration

Policy Interation

Policy iteration:

- Initialize $\pi \leftarrow \pi_0, v(\cdot) \leftarrow \vec{0}$
- While π still changing:
  - Policy evaluation: iterate **until convergence**
    $\forall s, v_\pi^t(s) \leftarrow \boxed{\sum_a \pi(a|s)} \sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma v_\pi^{t-1}(s')]$
  - Policy improvement: greedy update
    $\forall s, \pi_+(s) \leftarrow \arg\max_{a \in A(s)} q_\pi(s,a)$
  - $\pi \leftarrow \pi_+$

Value Interation

Value iteration:

- Initialize $\pi \leftarrow \pi_0, v(\cdot) \leftarrow \vec{0}$
- While V still changing:

$$\forall s, v_*^{t+1}(s) \leftarrow \max_a \sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma v_*^t(s')]$$

- Extract π_*

$$\forall s, \pi_*(s) \leftarrow \arg\max_a \sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma v_*(s')]$$

But note that, intermediate value functions might not correspond to any underlying policy π.

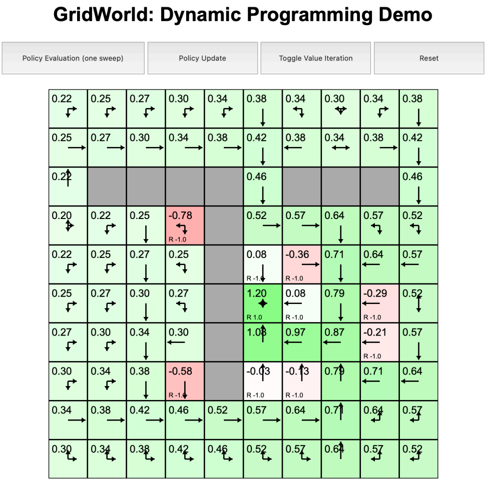
Value Interation

Value iteration (another perspective):

- Initialize $\pi \leftarrow \pi_0, v(\cdot) \leftarrow \vec{0}$
- While V still changing:
  - Policy evaluation (one sweep): iterate **only once**
    $\forall s, v(s) \leftarrow \sum_a \pi(a|s) \sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma v(s')]$
  - Policy improvement: greedy update
    $\forall s, \pi_+(s) \leftarrow \arg\max_{a \in A(s)} q(s, a)$
  - $\pi \leftarrow \pi_+$ (every π will be deterministic except for π_0)

Still, intermediate value functions might not correspond to any underlying policy π.

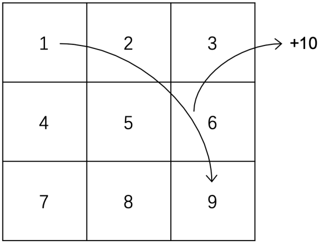
REINFORCEjs: Dynamic Programming Demo



https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

Coding Examples

3×3 Grid World



The agent can be in one of the nine cells at any starting time. It can then move in one of four directions: {E,S,W,N}. If the agent hits a wall, it remains in its current cell and gets a reward 1. When the agent moves to cell 1, it then immediately moves to cell 9 and gets a reward of 10. The discount factor $\gamma = 0.9$.

Evaluate the uniform policy - Iterative procedures

Initialize $V_\pi^0(s) = 0$ for all s.

For each t=1,...,Max:

For each state s:

$$V_\pi^t(s) = \sum_{s'} T(s, \pi(s), s')[Reward(s, \pi(s), s') + \gamma V_\pi^{t-1}(s')].$$

```
V:
[[ 8.69914207  2.42431251 -0.11319971]
 [ 2.42431251  0.87545285 -0.47892324]
 [-0.11319971 -0.47892324 -1.30088793]]
```

Evaluate the uniform policy - Matrix form

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

```
P:
[[0.    0.    0.    0.    0.    0.    0.    0.    1.   ]
 [0.225 0.225 0.225 0.    0.225 0.    0.    0.    0.   ]
 [0.    0.225 0.45  0.    0.    0.225 0.    0.    0.   ]
 [0.225 0.    0.    0.225 0.225 0.    0.225 0.    0.   ]
 [0.    0.225 0.    0.225 0.    0.225 0.    0.225 0.   ]
 [0.    0.    0.225 0.    0.225 0.225 0.    0.    0.225]
 [0.    0.    0.    0.225 0.    0.    0.45  0.225 0.   ]
 [0.    0.    0.    0.    0.225 0.    0.225 0.225 0.225]
 [0.    0.    0.    0.    0.    0.225 0.    0.225 0.45 ]]
R:
[10.   -0.25 -0.5  -0.25  0.   -0.25 -0.5  -0.25 -0.5 ]
```

```
V:
[[ 8.69883122  2.42401952 -0.1134855 ]
 [ 2.42401952  0.87516596 -0.47920629]
 [-0.1134855  -0.47920629 -1.30116878]]
```

Find the optimal policy - Analytically

Analytically, the optimal policy for every state is to get state 1 as soon as possible. For state 9, the minimal number of steps to reach state 1 is 4 and then agent jumps back to state 9. Thus,

$$V_1 = V_9 + 10, V_9 = 0.9^4 V_1 \Rightarrow V_1 = 29.08, V_9 = 19.08$$

The value of rest states are

$$V_2 = 0.9V_1 = 26.17, V_3 = 0.9V_2 = 23.55, V_4 = 0.9V_1 = 26.17,$$

$$V_5 = 0.9V_2 = 23.55, V_6 = 0.9V_3 = 21.20, V_7 = 0.9V_4 = 23.55,$$

$$V_8 = 0.9V_7 = 21.20$$

The optimal policy will be

NULL	←	←
↑	←↑	←↑
↑	←↑	←↑

Find the optimal policy - Policy iteration

See `mdp_example.ipynb`

Optimal state-values:

```
[[29.078 26.17  23.553]
 [26.17  23.553 21.198]
 [23.553 21.198 19.078]]
```

Extracted policy (random tie-breaking):

NULL	$\leftarrow$	$\leftarrow$
$\uparrow$	$\leftarrow$	$\leftarrow$
$\uparrow$	$\leftarrow$	$\leftarrow$

Gambler's Problem

(Sutton's book, example 4.3)

A gambler has the opportunity to make bets on the outcomes of a sequence of coin flips. If the coin comes up heads, he wins as many dollars as he has staked on that flip; if it lands with tails, he loses his stake. The game ends when the gambler wins by reaching his goal of $100, or loses by running out of money. On each flip, the gambler must decide what portion of his capital to stake, in integer numbers of dollars.

Gambler's Problem- Formulation

This problem can be formulated as an undiscounted, episodic, finite MDP. The state is the gambler's capital, $s \in \{1, 2, ..., 99\}$and the actions are stakes, $a \in \{0, 1, ..., \min(s, 100 - s)\}$. The reward is zero on all transitions except those on which the gambler reaches his goal, when it is $+1$. The state-value function then gives the probability of winning from each state. A policy is a mapping from levels of capital to stakes. The optimal policy maximizes the probability of reaching the goal. Let p_h denote the probability of the coin coming up heads. If p_h is known, then the entire problem is known and it can be solved, for instance, by value iteration.

Gambler's Problem - Value iteration

See `mdp_example.ipynb`

Gambler's Problem - Results

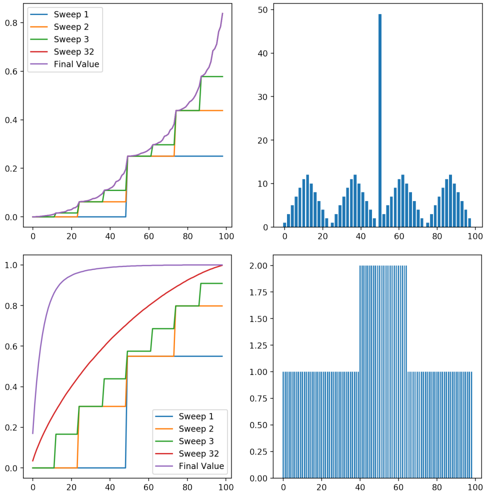


Figure 1: $p_h = 0.25(top), p_h = 0.55(bottom)$

Maximization Bias Problem

(Sutton's book, example 6.7)

Consider a single state s where there are many actions a whose true values, q(s,a), are all zero but whose estimated values, Q(s,a), are uncertain and thus distributed some above and some below zero. The maximum of the true values is zero, but the maximum of the estimates is positive, a positive bias. We call this maximization bias.

Maximization Bias Problem

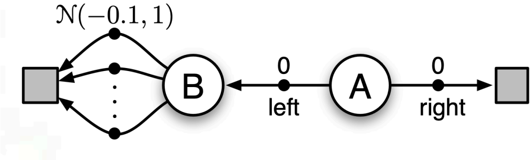


Figure 2: Example for maximization bias.

The MDP has two non-terminal states A and B. Episodes always start in A with a choice between two actions, left and right. The right action transitions immediately to the terminal state with a reward and return of zero. The left action transitions to B, also with a reward of zero, from which there are many possible actions all of which cause immediate termination with a reward drawn from a normal distribution with mean 0.1 and variance 1.0.

Maximization Bias Problem - Q Learning

While observing (S, A, R, S')...

Q Learning
ϵ-greedy according to Q and update it
$Q(S, A) \leftarrow (1 - \alpha) Q(S, A) + \alpha(R + \gamma \max_{a \in A} Q(S', a))$

Double Q Learning
ϵ-greedy according to $Q_1 + Q_2$ and update them
$Q_1(S, A) \leftarrow (1 - \alpha) Q_1(S, A) + \alpha(R + \gamma Q_2(S', \arg\max_{a \in A} Q_1(S', a)))$
$Q_2(S, A) \leftarrow (1 - \alpha) Q_2(S, A) + \alpha(R + \gamma Q_1(S', \arg\max_{a \in A} Q_2(S', a)))$

See `mdp_example.ipynb`

Maximization Bias Problem - Results

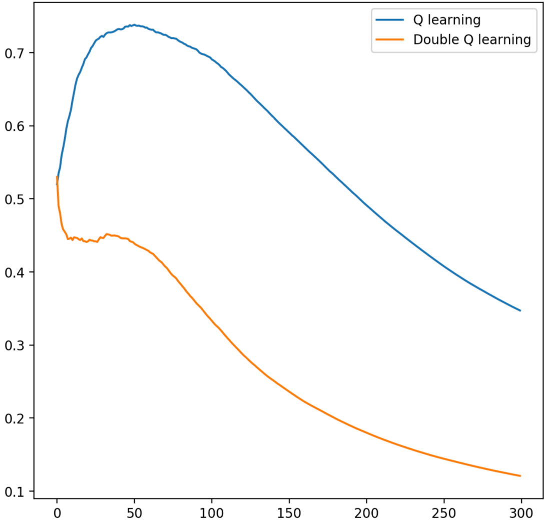


Figure 3: Ratio of selecting the wrong actions.

Thanks!