COMP3211 Tutorial 8: Game Theory

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Outline

Game Formulation

Best Response and Nash Equilibria

Sealed-Bid Auctions

Game Formulation

Formulation

Normal-Form Games:

- A set of agents $\mathcal{N} = [1..n]$
- Each agent i holds an action set $\mathcal{A}_i$
- Each agent i holds an utility function $u_i : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathbb{R}$
- A Nash equilibrium $(s_1, \cdots, s_n)$, $s_i \in \Delta(A_i)$: $S_i = (1, 0, 0, 0)$

  $(0.7, 0.3, 0, 0)$

$$s_i = BestResponse(s_{-i}) \triangleq BestResponse(s_1, ..., s_{i-1}, s_{i+1}, \cdots s_n)$$

- Nash's existence theorem: Any game with a finite set of players and finite set of strategies has a Nash equilibrium of mixed strategies.

Maximizing Reward v.s. Minimizing Regret

	Reward
A	100
B	2

	C	D ✓
A	100, 3	$1-\epsilon$, 5
$\boxed{\text{B}}$	2, 3	1, 5

	C	D
A	100, a	$1-\epsilon$, b
B	2, c	1, d

→ A	0	$-\varepsilon$
B	-98	0

Definition[1]:

$Reward(a_i, a_{-i}) \triangleq u_i(a_i, a_{-i})$

$Regret(a_i, a_{-1}) \triangleq \max_{a_i'} u_i(a_i', a_{-i}) - u_i(a_i, a_{-i})$

[1] Shoham, Yoav, and Kevin Leyton-Brown. Multiagent systems: Algorithmic, game-theoretic, and logical foundations. Cambridge University Press, 2008.

Best Response and Nash Equilibria

Example 1: Network Sharing

You and your friend share a network and both of you want to download a movie:

- If both of you do that, the network is jammed, and none of you is happy: say both of you value it 2.

- If only one of you do that, the network works perfectly: the one who did that is very happy (5), the other is very unhappy (0).

- If none of you do that, then none of you is very happy but then you can do something together. So let's assign it a value of 3.

	D	ND
D	(2, 2)	5, 0
ND	0, 5	3, 3

(D, D)

Example 2: Tragedy of The Commons

A generalized continuous version:

- n players. Each has the same strategy of downloading $x_i \in [0, 1]$ units. But the total bandwidth is only 1.

- If $\sum_i x_i > 1$, every one gets zero utility.

- Else, player i gets a utility of $x_i(1 - \sum_j x_j)$

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Solutoin:

- Let $t = \sum_{j \neq i} x_j$, then $u_i = x_i(1 - t - x_i)$, max at $x_i = \frac{1-t}{2}$.

- A unique solution for the equations $\forall i, x_i = \frac{1 - \sum_{j \neq i} x_j}{2} \Rightarrow x_i = \frac{1}{n+1}$.

- Total usage $\sum_i x_i = \frac{n}{1+n}$. Social welfare $= n \cdot \frac{1}{n+1} \frac{1}{n+1} < \frac{1}{n}$.

- But if $\sum_i x_i = 1/2$, social welfare $= \sum_i x_i(1 - \sum_j x_j) = \frac{1}{4}$

- Players are incentivized to overuse the resource.

Example 3: Soccer Penalty Kicks

Kicker/Goalie	Left	Right
Left	0, 1	1, 0
Right	0.75, 0.25	0, 1

- No pure Nash equilibrium, then try to find a mixed Nash equlibrium.

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- The kicker's strategy is to make the goalie indifferent between guarding left and right. Kicker kicks left w.p. q, for the guard,

$$q + 0.25(1 - q) = (1 - q) \Rightarrow q = 3/7$$

Example 4: Battle of Sexes - Pure NE

F

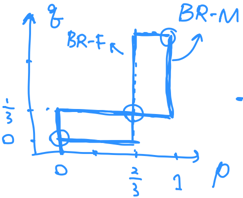
M

	Sports	Movie
Sports	1, 0	-1, -1
Movie	-1, -1	0, 1

Pure-strategy NE: 1. (sports, sports), 2. (movie, movie).

What about mixed-strategy NE?

Example 4 con't: Battle of Sexes - All NE



$q \qquad 1-q$

	Sports	Movie
Sports	1, 0	-1, -1
Movie	-1, -1	0, 1

$1 \times q + -1 \times (1-q)$

$-1 \times q + 0 \times (1-q)$

Best responses:

- Suppose the girl plays [q:S, (1-q):M], for the boy to prefer S

$$q - (1 - q) \geq -q \Rightarrow q \geq \frac{1}{3}$$

- Suppose the boy plays [p:S, (1-p):M], for the girl to prefer S

$$0 - (1 - p) \geq -p + (1 - p) \Rightarrow p \geq \frac{2}{3}$$

Example 3 con't: Battle of Sexes - All NE

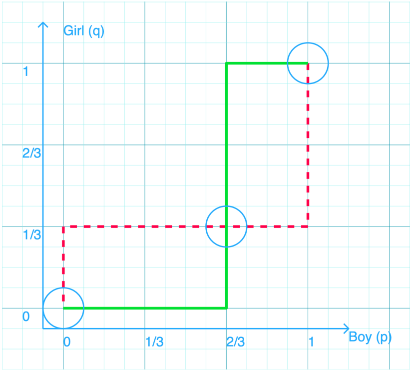


Figure 1: Best response curve.

Thus, mixed NE: [Boy(2/3:S, 1/3:M), Girl(1/3:S, 2/3:M)].

Sealed-Bid Auctions

Sealed-Bid Auctions

In the lecture, we have discussed two implementations of sealed bid actions: first price auction and second price auction.

1. First-price: the winner pays his own bid (the highest)
2. Second-price: the winner pays the second highest bid.

For simplicity, assume there are two bidders *Alice* and *Bob* bidding for one single item. Values that Alice and Bob hold towards the item are a and b, respectively. Those values are private information, i.e. a is unknow to Bob and b is unknown to Alice. Let x denote Alice' bid and y denote Bob's bid. Ties will always be broken in favor of Alice.

Consider first price auction (the highest bidder wins the item and pays the price as she bids).

Assume $a > b$ and Alice somehow knows Bob's valuation b, but Bob still has no idea about Alice's valuation a. Find a pure Nash equilibrium.

(b, b)

$b + \varepsilon$ $\quad a - b - \varepsilon < a - b$

$b + \varepsilon'$ $\quad b - b - \varepsilon' < 0$

Sealed-Bid Auctions: First-Price

Consider first price auction (the highest bidder wins the item and pays the price as she bids).

Assume $a > b$ and Alice somehow knows Bob's valuation b, but Bob still has no idea about Alice's valuation a. Find a pure Nash equilibrium.

(x=b, y=b) is the only pure NE, such that Alice can obtain a maximum utility of $a - b > 0$, while Bob has no way to win the item thus no profitable deviation.

Sealed-Bid Auctions: First-Price con't

Assume a and b are independent random variables, both of which are distributed as $\texttt{uniform}(0, 1)$, which is known as common knowledge. Each one's exact value is unknown to the other one, as the original assumption. Find a pure Nash equilibrium. (Hint: try to find the best response for Alice when Bob bids $y = b/2$.)

Sealed-Bid Auctions: First-Price con't

Assume a and b are independent random variables, both of which are distributed as uniform(0, 1), which is known as common knowledge. Each one's exact value is unknown to the other one, as the original assumption. Find a pure Nash equilibrium. (Hint: try to find the best response for Alice when Bob bids $y = b/2$.) b/α

Let $y = b/2$, Alice's expected utility will be $Pr(Alice\ wins) \times \boxed{(a - x)}$. Given $b \sim \texttt{uniform}[0, 1]$,

$$Pr(Alice\ wins) = Pr(x \geq b/2) = Pr(b \leq 2x) = 2x.$$

Thus, Alice's expected return will be $2x(a - x)$. Take the first derivative and set to 0,

$$\frac{d}{dx} 2x(a - x) = 2a - 4x = 0 \Rightarrow x = a/2.$$

By symmetry, for Alice bidding $a/2$, Bob's best response is also to bid $b/2$. Hence, $(x = a/2, y = b/2)$ is an NE.

Sealed-Bid Auctions: Revenue Equivalence

First-price auction:

One-shot income for the auctioneer: $max(a/2, b/2)$

Let $\alpha = max(a/2, b/2)$,

$$\begin{aligned} Pr[\alpha \leq t] &= Pr[max(a/2, b/2) \leq t] \\ &= Pr[a/2 \leq t] \times Pr[b/2 \leq t] \\ &= 4t^2 \\ f(\alpha = t) &= d/dt(4t^2) = 8t \\ E[\alpha] &= \int_0^{1/2} t \cdot 8t \cdot dt = 1/3 \end{aligned}$$

Sealed-Bid Auctions: Revenue Equivalence

Second-price auction:

It is also proved that (x=a, y=b) is a Nash equilibrium for second price auctions.

One-shot income for the auctioneer: $min(a, b)$

Let $\beta = min(a, b)$,

$$
\begin{aligned}
Pr[\beta \leq t] &= Pr[min(a, b) \leq t] \\
&= 1 - Pr[a > t] \times Pr[b > t] \\
&= 1 - (1 - t)^2 \\
f(\beta = t) &= d/dt[1 - (1 - t)^2] = 2 - 2t \\
E[\beta] &= \int_0^1 t \cdot (2 - 2t) \cdot dt = 1/3
\end{aligned}
$$

Thanks!