COMP3211 Tutorial 9: First-Order Logic

Fengming ZHU

May 6&9, 2024

Department of CSE
HKUST

Operator Precedence

Precedence of logical connectives:

$$(\forall \sim \exists) \succ \neg \succ \wedge \succ \vee \succ \implies \succ \iff$$

E.g.,

$$\neg A \vee B \iff C \wedge D \implies E \equiv [(\neg A) \vee B] \iff [(C \wedge D) \implies E]$$

And all connectives are right-associative (grouped from the right to the left),

$$A \implies B \implies C \equiv A \implies (B \implies C)$$

And with quantifiers,

$$\forall x, p(x) \implies \forall x, q(x) \equiv [\forall x, p(x)] \implies [\forall y, q(y)]$$

Representation - warm up

1. Mary loves everyone.

Representation - warm up

1. Mary loves everyone.

$$\forall x, love(Mary, x)$$

2. Everyone loves herself.

Representation - warm up

1. Mary loves everyone.

$$\forall x, love(Mary, x)$$

2. Everyone loves herself.

$$\forall x, love(x, x)$$

3. Everyone loves everyone.

Representation - warm up

1. Mary loves everyone.

$$\forall x, love(Mary, x)$$

2. Everyone loves herself.

$$\forall x, love(x, x)$$

3. Everyone loves everyone.

$$\forall x \forall y, love(x, y)$$

4. Everyone loves everyone except herself.

Representation - warm up

1. Mary loves everyone.

$$\forall x, love(Mary, x)$$

2. Everyone loves herself.

$$\forall x, love(x, x)$$

3. Everyone loves everyone.

$$\forall x \forall y, love(x, y)$$

4. Everyone loves everyone except herself.

$$\forall x \forall y, x \neq y \implies love(x, y)$$

Representation - quantifiers

1. For every person, there is someone whom she loves.

Representation - quantifiers

1. For every person, there is someone whom she loves.

$$\forall x \exists y, \mathit{love}(x, y)$$

2. There is someone whom everyone loves.

Representation - quantifiers

1. For every person, there is someone whom she loves.

$$\forall x \exists y, love(x, y)$$

2. There is someone whom everyone loves.

$$\exists x \forall y, love(y, x)$$

3. There is someone who loves everyone.

Representation - quantifiers

1. For every person, there is someone whom she loves.

$$\forall x \exists y, love(x, y)$$

2. There is someone whom everyone loves.

$$\exists x \forall y, love(y, x)$$

3. There is someone who loves everyone.

$$\exists x \forall y, love(x, y)$$

4. For everyone, there is someone who loves her.

Representation - quantifiers

1. For every person, there is someone whom she loves.

$$\forall x \exists y, love(x, y)$$

2. There is someone whom everyone loves.

$$\exists x \forall y, love(y, x)$$

3. There is someone who loves everyone.

$$\exists x \forall y, love(x, y)$$

4. For everyone, there is someone who loves her.

$$\forall x \exists y, love(y, x)$$

Representation - more complex

1. Every student who loves Mary is happy.

Representation - more complex

1. Every student who loves Mary is happy.

$$\forall x, student(x) \land love(x, Mary) \implies happy(x)$$

2. Every boy who loves Mary hates every boy whom Mary loves.

Representation - more complex

1. Every student who loves Mary is happy.

$$\forall x, student(x) \wedge love(x, Mary) \implies happy(x)$$

2. Every boy who loves Mary hates every boy whom Mary loves.

$$\begin{aligned}&\forall x,\\ &\{boy(x) \wedge love(x, Mary) \implies \forall y,\\ &[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}\end{aligned}$$

3. Every boy who loves Mary hates every other boy whom Mary loves.

Representation - more complex

1. Every student who loves Mary is happy.

$$\forall x, student(x) \land love(x, Mary) \implies happy(x)$$

2. Every boy who loves Mary hates every boy whom Mary loves.

$$\begin{aligned} &\forall x, \\ &\{boy(x) \land love(x, Mary) \implies \forall y, \\ &[boy(y) \land love(Mary, y) \implies hate(x, y)]\} \end{aligned}$$

3. Every boy who loves Mary hates every other boy whom Mary loves.

$$\begin{aligned} &\forall x, \\ &\{boy(x) \land love(x, Mary) \implies \forall y, \\ &[boy(y) \land love(Mary, y) \land x \neq y \implies hate(x, y)]\} \end{aligned}$$

Typical mistake

Every boy who loves Mary hates every boy whom Mary loves.

$$\forall x,$$
$$\{boy(x) \wedge love(x, Mary) \wedge \forall y,$$
$$[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}$$

Typical mistake

Every boy who loves Mary hates every boy whom Mary loves.

$$\begin{aligned}&\forall x,\\&\{boy(x) \wedge love(x, Mary) \wedge \forall y,\\&[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}\end{aligned}$$

Correct:

$$\begin{aligned}&\forall x,\\&\{boy(x) \wedge love(x, Mary) \implies \forall y,\\&[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}\end{aligned}$$

Underlying subtlety

When we say $\forall x, Precondition(x) \implies Conclusion(x)$, it means

For every $x_i \in Domain(x)$,

$$
\begin{aligned}
&[Precondition(x_1) \implies Conclusion(x_1)] \wedge \\
&[Precondition(x_2) \implies Conclusion(x_2)] \wedge \\
&\cdots \wedge \\
&[Precondition(x_n) \implies Conclusion(x_n)] \wedge
\end{aligned}
$$

E.g., Every boy loves Mary:

$$\forall x, boy(x) \implies love(x, Mary)$$

Underlying subtlety (con't)

When we say $\exists x, Precondition(x) \wedge Conclusion(x)$, it means

For every $x_i \in Domain(x)$,

$$
\begin{aligned}
&[Precondition(x_1) \wedge Conclusion(x_1)] \vee \\
&[Precondition(x_2) \wedge Conclusion(x_2)] \vee \\
&\cdots \vee \\
&[Precondition(x_n) \wedge Conclusion(x_n)] \vee
\end{aligned}
$$

E.g., One boy loves Mary:

$$\exists x, boy(x) \wedge love(x, Mary)$$

Transformation

Every boy who loves Mary hates every boy whom Mary

$$\begin{aligned}&\forall x,\\&\{boy(x) \land love(x, Mary) \implies \forall y,\\&[boy(y) \land love(Mary, y) \implies hate(x, y)]\}\end{aligned}$$

How about this one:

$$\begin{aligned}&\forall x \forall y,\\&[boy(x) \land love(x, Mary)] \land [boy(y) \land love(Mary, y)]\\&\implies hate(x, y)\end{aligned}$$

Transformation

Every boy who loves Mary hates every boy whom Mary

$$\forall x,$$
$$\{boy(x) \wedge love(x, Mary) \implies \forall y,$$
$$[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}$$

How about this one:

$$\forall x \forall y,$$
$$[boy(x) \wedge love(x, Mary)] \wedge [boy(y) \wedge love(Mary, y)]$$
$$\implies hate(x, y)$$

Yes, equivalent!

(every such boy x that love(x,Mary)) hates
(every such boy y that love(Mary, y))

Transformation (cont'd)

$$
\begin{aligned}
&\forall x \forall y, \\
&[boy(x) \wedge love(x, Mary)] \wedge [boy(y) \wedge love(Mary, y)] \\
&\ \implies hate(x, y) \\
\equiv\ &\forall x \forall y, \\
&\neg[boy(x) \wedge love(x, Mary)] \vee \\
&\neg[boy(y) \wedge love(Mary, y)] \vee hate(x, y) \\
\equiv\ &\forall x, \\
&\neg[boy(x) \wedge love(x, Mary)] \vee \\
&\{\forall y, \neg[boy(y) \wedge love(Mary, y)] \vee hate(x, y)\} \\
\equiv\ &\forall x, \\
&\{boy(x) \wedge love(x, Mary) \implies \forall y, \\
&[boy(y) \wedge love(Mary, y) \implies hate(x, y)]\}
\end{aligned}
$$

Transformation (cont'd)

The two control flows have the same effects:

```
for x:
    if x:
        for y:
            if y:
```

```
for x:
    for y:
        if x:
            if y:
```

References

1. First-Order Logic, Stanford, https://web.stanford.edu/class/archive/cs/cs103/cs103.1232/lectures/04/Condensed%20Slides.pdf
2. Practice in Logic and HW 1, UMass, https://people.umass.edu/partee/NZ_2006/More%20Answers%20for%20Practice%20in%20Logic%20and%20HW%201.pdf

Thanks!