COMP3211 Tutorial 7: Markov Decision Process

Fengming ZHU

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Department of CSE
HKUST

Outline

MDP V.S. Search

Value Functions

Bellman Expectation Equation

Bellman Optimality Equation

MDP V.S. Search

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Search:

- A set of states $\mathcal{S}$, initial state I, goal state G
- A set of actions $\mathcal{A}$
- Deterministic transitions $T : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$
- cost function $c : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$
- Objective: a path p from I to G that minimizes $c(p)$

MDP V.S. Search

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MDP:

- A set of states $\mathcal{S}$, a terminating condition $End(s)$
- A set of actions $\mathcal{A}$
- Stochastic transitions $T : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$
- Reward function $r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, with a discount factor γ
- Objective: maximize $\sum_t \gamma^t r_t$

Solution Concept: Policy

Question:
To make sure you can come up with an optimal solution, you'd like to know:

$S_0 \xrightarrow{a_0} S_1 \xrightarrow{a_1} \boxed{S_2} \xrightarrow{a_2}$

- (A) Only your current state S_2
- (B) All the history from the beginning up until now S_0, a_0, S_1, a_1, S_2
- (C) Need to know more

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Theorem:
Markov property holds: $P[S_{t+1}|S_t, a_t, \boxed{\cdots, S_0, a_0}] = P[S_{t+1}|S_t, a_t]$.
That is, your current state is already a "sufficient statistic", also known as the information state.

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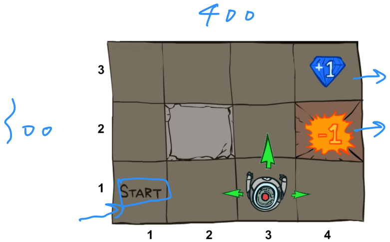
Policy:
A solution is a policy $\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$

Follow-up Question: Maze

Question:
Given a large maze, you (with deterministic actions U/D/L/R) are supposed to find a nice way from the entrance to the exit, which agent you'd like to choose

- (A) State machines with infite memory
- (B) Agents that can A^* search
- (C) Agents that can compute policies
- (D) None of them



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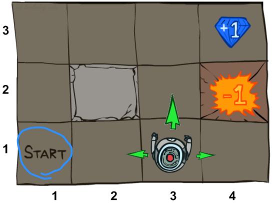
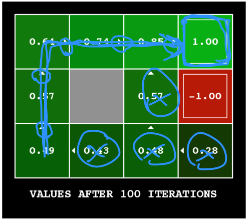




Value Functions

Notations with Time Index

$\downarrow t \times$ $t \downarrow$

- Transition: $T^a_{s,s'} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$
- Reward: $R^a_s = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$ $R_{t+1} \sim \mathbb{P}(S, a)$
- Stationary policy: $\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$
- Return: The return G_t is the total discounted reward from time t,

$G_t = R_{t+1} + \gamma G_{t+1}$

Time-dependent

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Policy: $\pi_t(a|s) = \mathbb{P}[A_t = a | S_t = s]$

$v(s)$ and $q(s, a)$

$\pi_1 \rightarrow \pi_2 \rightarrow \cdots \rightarrow \pi_*$

State-value function:
The state-value function $v_\pi(s)$ for an MDP is the expected return starting from state s, and then following policy π,

$$\boxed{v_\pi(s)} = \boxed{\mathbb{E}_\pi}[G_t | S_t = s]$$

$\pi(a_1|s)$ s_1'

s

$\pi(a_2|s)$ s_2'

Action-value function:
The action-value function $q_\pi(s, a)$ for an MDP is the expected return starting from state s, taking action a, and then following policy π,

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

π v, q

Bellman Expectation Equation

Adam's Law:
For any random variables X and Y,

$$E[E[Y|X]] = E[Y]$$

Adam's Law with Extra Conditioning:
For any random variables X, Y and Z,

$$E[E[Y|X, Z]|Z] = E[Y|Z]$$

$$\hat{E}[\hat{E}[Y|X]] = \hat{E}[Y] = E[Y|Z]$$

$$\hat{E}(\cdot) = E(\cdot|Z)$$

Bellman Expectation Equation

- For state-value function,

$$v_\pi(s) = E_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s]$$

- For action-value function,

$$q_\pi(s,a) = E_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1})|S_t = s, A_t = a]$$

Bellman Expectation Equation for $v^{\pi}(s)$

We first prove the Bellman equation for state-value function.

$$
\begin{aligned}
v_\pi(S_t = s) &= E_\pi[\boxed{G_t}|S_t = s] \\
&= E_\pi[R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \cdots)|S_t = s] \\
&= E_\pi[R_{t+1} + \gamma G_{t+1}|S_t = s]
\end{aligned}
$$

Linearity

Since

$Y \quad X \quad Z \quad Z$

$$
\begin{aligned}
E_\pi[G_{t+1}|S_t] &= E_\pi[E_\pi[G_{t+1}|S_{t+1}, S_t]|S_t] \\
&= E_\pi[E_\pi[G_{t+1}|S_{t+1}]|S_t] \\
&= E_\pi[v_\pi(S_{t+1})|S_t]
\end{aligned}
$$

Adam Law.

Markov

Thus,

$$
\begin{aligned}
v(S_t = s) &= E_\pi[G_t|S_t = s] \\
&= E_\pi[R_{t+1} + \gamma G_{t+1}|S_t = s] \\
&= E_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s]
\end{aligned}
$$

$= E_\pi[R_{t+1}] + \gamma E(G_{t+1}|S_t = s)$

Bellman Expectation Equation for $q^\pi(s, a)$

We then prove the Bellman equation for action-state function.

$$\begin{aligned} q_\pi(S_t = s, A_t = a) &= E_\pi[G_t | S_t = s, A_t = a] \\ &= E_\pi[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \end{aligned}$$

Since

$$\begin{aligned} E[G_{t+1} | S_t, A_t] &= E[E[\overset{Y}{G_{t+1}} | (\overset{X}{S_{t+1}, A_{t+1}}), \cancel{(\overset{Z}{S_t, A_t})}] | (\overset{Z}{S_t, A_t})] \quad \text{Adam} \\ &= E[E[G_{t+1} | (S_{t+1}, A_{t+1})] | (S_t, A_t)] \quad \text{Markov} \end{aligned}$$

Under policy π, we have

$$\begin{aligned} E_\pi[G_{t+1} | S_t, A_t] &= E_\pi[E_\pi[G_{t+1} | (S_{t+1}, A_{t+1})] | (S_t, A_t)] \\ &= E_\pi[q_\pi(S_{t+1}, A_{t+1}) | (S_t, A_t)] \end{aligned}$$

Thus,

$$\begin{aligned} q_\pi(S_t = s, A_t = a) &= E_\pi[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \\ &= E_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | S_t = s, A_t = a] \end{aligned}$$

$\pi_0 \rightarrow \underline{v \cdot q} \rightarrow \underline{v \cdot q} \rightarrow \underline{?}$

definition Equation Solve

$\underline{?} \downarrow$

π_1

Bellman Optimality Equation

$\boxed{\underline{? \ \pi_*}} \rightarrow \underline{v \cdot q} \rightarrow \underline{v \cdot q} \rightarrow \underline{?}$

definition Equation solve

Optimal Value Function

?

π_1: S_1, S_2 $\quad$ π_2: S_3, S_4

$V_* = V_{\pi_1}$ $\quad$ $V_* = V_{\pi_2}$

- For state-value function,

$V_*(S_1) = V_{\pi_1}(S_1)$

$V_*(S_2) = V_{\pi_3}(S_3)$

$$v_*(s) = \max_\pi \boxed{v_\pi(s)}$$

- For action-value function,

$$q_*(s, a) = \max_\pi \boxed{q_\pi(s, a)}$$

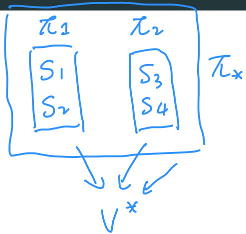
- The optimal value function specifies the best possible performance in the MDP.
- An MDP is “solved” once we know the optimal values.

Optimal Policy

Define a partial ordering over policies:

$$\pi \geq \pi', \text{ if } v_\pi(s) \geq v'_\pi(s), \text{ for all } s.$$

Optimal Policy



Define a partial ordering over policies:

$$\pi \geq \pi', \text{ if } v_\pi(s) \geq v'_\pi(s), \text{ for all } s.$$

For any MDP:

- There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi$, for all π.
- All optimal policies achieve the optimal state-value function, $v_{\pi_*}(s) = v_*(s)$, for all s.
- All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$, for all s, a.

Finding Optimal Policy

$V/q_* \rightarrow \pi_* \rightarrow \pi_*^{deterministic}$!

Theorem:

- An optmial policy can be found by maximizing over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{, if } a = argmax_{a \in A} q_*(s, a) \\ 0 & \text{, otherwise} \end{cases}$$

- There is always a deterministic optimal policy for any MDP.
- If we know $q_*(s, a)$, we immediately have the optimal policy.

$V_*(s) = \max_a q_*(s, a)$

Bellman Optimality Equation

- For state-value function,

$$v_*(s) = \max_a E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a]$$

- For action-value function,

$$q_*(s, a) = E[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a]$$

$q \sim G_t \mid s, a$

Bellman Optimality Equation for v_* and q_*

Under the optimal policy, we first show the relation of v^* and q^*.

- v_* in terms of q_*,

$$\boxed{v_*(s) = \max_a q_*(s, a)}$$

Bellman Optimality Equation for v_* and q_* (cont'd)

- q_* in terms of v_*,

$$
\begin{aligned}
q_*(s,a) &= \max_{\pi} q_\pi(s,a) \\
&= R_s^a + \gamma \sum_{s'} T_{ss'}^a \max_{\pi} v_\pi(s') \\
&= R_s^a + \gamma \sum_{s'} T_{ss'}^a v_*(s') \\
&= E[R_{t+1}|S_t = s, A_t = a] \\
&+ \gamma \sum_{s'} \{P(S_{t+1} = s'|S_t = s, A_t = a) \\
&\cdot E[v_*(S_{t+1})|S_{t+1} = s', S_t = s, A_t = a]\} \\
&= E[R_{t+1}|S_t = s, A_t = a] + \gamma E[v_*(S_{t+1})|S_t = s, A_t = a] \\
&= E[R_{t+1} + \gamma v_*(S_{t+1})|S_t = s, A_t = a]
\end{aligned}
$$

Bellman Optimality Equation for V_* and Q_*

Then we show the Bellman optimal equation for v^*,

$$
\begin{aligned}
v_*(s) &= \max_a \boxed{q_*(s, a)} \quad \text{obvious} \\
&= \max_a E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a]
\end{aligned}
$$

Finally, we show the Bellman equation for q^*,

$$
\begin{aligned}
q_*(s, a) &= E[R_{t+1} + \gamma \underline{v_*(S_{t+1})} | S_t = s, A_t = a] \\
&= E[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a]
\end{aligned}
$$

$\pi_0 \to$ BEE $\to$ $\underline{V, q}$

solve.

PI ✓

VI ✓

Thanks!

$\pi_* \to$ BOE $\to$ $\underline{V_*, q_*}$

solve