COMP3211 Tutorial 6: Markov Decision Process

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Outline

MDP V.S. Search

Value Functions

Bellman Expectation Equation

Bellman Optimality Equation

MDP V.S. Search

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Search:

- A set of states $\mathcal{S}$, initial state I, goal state G
- A set of actions $\mathcal{A}$
- Deterministic transitions $T : \mathcal{S} \times \mathcal{A} \rightarrow \mathcal{S}$
- cost function $c : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$
- Objective: a path p from I to G that minimizes $c(p)$

MDP V.S. Search

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MDP:

- A set of states $\mathcal{S}$, a terminating condition $End(s)$
- A set of actions $\mathcal{A}$
- Stochastic transitions $T : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$
- Reward function $r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$, with a discount factor γ
- Objective: maximize $\sum_t \gamma^t r_t$

Solution Concept: Policy

Question:
To make sure you can come up with an optimal solution, you'd like to know:

- (A) Only your current state
- (B) All the history from the beginning up until now
- (C) Need to know more

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Markov property:
A state S_t is Markovian iff $P[S_{t+1}|S_t, \cdots, S_0] = P[S_{t+1}|S_t]$. That is, your current state is already a "sufficient statistic", also known as the information state.

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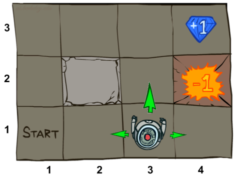
Policy:
A solution is a policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$

Follow-up Question: Maze

Question:
Given a large maze, you (with deterministic actions U/D/L/R) are supposed to find a nice way from the entrance to the exit, which agent you'd like to choose

- (A) State machines with infite memory
- (B) Agents that can A^* search
- (C) Agents that can compute policies
- (D) None of them



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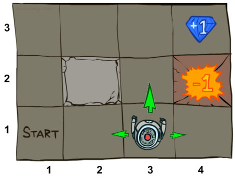

Value Functions

Notations with Time Index

- Transition: $T^a_{s,s'} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$
- Reward: $R^a_s = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$
- Stationary policy: $\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$
- Return: The return G_t is the total discounted reward from time t,

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

$v(s)$ and $q(s,a)$

State-value function:
The state-value function $v_\pi(s)$ for an MDP is the expected return starting from state s, and then following policy π,

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$

Action-value function:
The action-value function $q_\pi(s,a)$ for an MDP is the expected return starting from state s, taking action a, and then following policy π,

$$q_\pi(s,a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

Policy Interation

Policy iteration:

- Initialize $\pi \leftarrow \pi_0$
- While π still changing:
  - Policy evaluation: iterate until convergence
    $\forall s, v_\pi^t(s) = \sum_{s' \in S} T(s, \pi(s), s')[R(s, \pi(a), s') + \gamma v_\pi^{t-1}(s')]$
  - Policy improvement: greedy update
    $\forall s, \pi_{new}(s) = \arg\max_{a \in A(s)} q_\pi(s, a)$
  - $\pi \leftarrow \pi_{new}$

Bellman Expectation Equation

Bellman Expectation Equation

- For policy evaluation, we desire

$$v_\pi(s) = \sum_{s' \in S} T(s, \pi(s), s')[R(s, \pi(a), s') + \gamma v_\pi(s')]$$

- For state-value function,

$$v_\pi(s) = E_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$$

- For action-value function,

$$q_\pi(s, a) = E_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | S_t = s, A_t = a]$$

Bellman Expectation Equation

Warming-up: Adam's Law

Adam's Law:
For any random variables X and Y,

$$E[E[X|Y]] = E[X]$$

Adam's Law with Extra Conditioning:
For any random variables X, Y and Z,

$$E[E[X|Y,Z]|Z] = E[X|Z]$$

Bellman Expectation Equation for $v^{\pi}(s)$

We first prove the Bellman equation for state-value function.

$$
\begin{aligned}
v_{\pi}(S_t = s) &= E_{\pi}[G_t | S_t = s] \\
&= E_{\pi}[R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \cdots) | S_t = s] \\
&= E_{\pi}[R_{t+1} + \gamma G_{t+1} | S_t = s]
\end{aligned}
$$

Since

$$
\begin{aligned}
E_{\pi}[G_{t+1} | S_t] &= E_{\pi}[E_{\pi}[G_{t+1} | S_{t+1}, S_t] | S_t] \\
&= E_{\pi}[E_{\pi}[G_{t+1} | S_{t+1}] | S_t] \\
&= E_{\pi}[v_{\pi}(S_{t+1}) | S_t]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v(S_t = s) &= E_{\pi}[G_t | S_t = s] \\
&= E_{\pi}[R_{t+1} + \gamma G_{t+1} | S_t = s] \\
&= E_{\pi}[R_{t+1} + \gamma v_{\pi}(S_{t+1}) | S_t = s]
\end{aligned}
$$

Bellman Expectation Equation for $q^{\pi}(s, a)$

We then prove the Bellman equation for action-state function.

$$\begin{aligned} q_{\pi}(S_t = s, A_t = a) &= E_{\pi}[G_t | S_t = s, A_t = a] \\ &= E_{\pi}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \end{aligned}$$

Since

$$\begin{aligned} E[G_{t+1} | S_t, A_t] &= E[E[G_{t+1} | (S_{t+1}, A_{t+1}), (S_t, A_t)] | (S_t, A_t)] \\ &= E[E[G_{t+1} | (S_{t+1}, A_{t+1})] | (S_t, A_t)] \end{aligned}$$

Under policy π, we have

$$\begin{aligned} E_{\pi}[G_{t+1} | S_t, A_t] &= E_{\pi}[E_{\pi}[G_{t+1} | (S_{t+1}, A_{t+1})] | (S_t, A_t)] \\ &= E_{\pi}[q_{\pi}(S_{t+1}, A_{t+1}) | (S_t, A_t)] \end{aligned}$$

Thus,

$$\begin{aligned} q_{\pi}(S_t = s, A_t = a) &= E_{\pi}[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] \\ &= E_{\pi}[R_{t+1} + \gamma q_{\pi}(S_{t+1}, A_{t+1}) | S_t = s, A_t = a] \end{aligned}$$

Bellman Optimality Equation

Optimal Value Function

- For state-value function,

$$v_*(s) = \max_\pi v_\pi(s)$$

- For action-value function,

$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" once we know the optimal values.

Optimal Policy

Define a partial ordering over policies:

$$\pi \geq \pi', \text{ if } v_\pi(s) \geq v'_\pi(s), \text{ for all } s.$$

Optimal Policy

Define a partial ordering over policies:

$$\pi \geq \pi', \text{ if } v_\pi(s) \geq v'_\pi(s), \text{ for all } s.$$

For any MDP:

- There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi$, for all π.
- All optimal policies achieve the optimal state-value function, $v_{\pi_*}(s) = v_*(s)$, for all s.
- All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$, for all s, a.

Finding Optimal Policy

Theorem:

- An optmial policy can be found by maximizing over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{, if } a = argmax_{a \in A} q_*(s, a) \\ 0 & \text{, otherwise} \end{cases}$$

- There is always a deterministic optimal policy for any MDP.
- If we know $q_*(s, a)$, we immediately have the optimal policy.

Bellman Optimality Equation

- For state-value function,

$$v_*(s) = \max_a E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a]$$

- For action-value function,

$$q_*(s, a) = E[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a]$$

- However, non-linear thus no closed form solution in general.

Bellman Optimality Equation for v_* and q_*

Under the optimal policy, we first show the relation of v^* and q^*.

- v_* in terms of q_*,

$$v_*(s) = \max_a q_*(s, a)$$

Bellman Optimality Equation for v_* and q_* (cont'd)

- q_* in terms of v_*,

$$
\begin{aligned}
q_*(s,a) &= \max_\pi q_\pi(s,a) \\
&= R_s^a + \gamma \sum_{s'} T_{ss'}^a \max_\pi v_\pi(s') \\
&= R_s^a + \gamma \sum_{s'} T_{ss'}^a v_*(s') \\
&= E[R_{t+1} | S_t = s, A_t = a] \\
&+ \gamma \sum_{s'} \{ P(S_{t+1} = s' | S_t = s, A_t = a) \\
&\cdot E[v_*(S_{t+1}) | S_{t+1} = s', S_t = s, A_t = a] \} \\
&= E[R_{t+1} | S_t = s, A_t = a] + \gamma E[v_*(S_{t+1}) | S_t = s, A_t = a] \\
&= E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a]
\end{aligned}
$$

Bellman Optimality Equation for V_* and Q_*

Then we show the Bellman optimal equation for v^*,

$$\begin{aligned} v_*(s) &= \max_a q_*(s,a) \\ &= \max_a E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a] \end{aligned}$$

Finally, we show the Bellman equation for q^*,

$$\begin{aligned} q_*(s,a) &= E[R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a] \\ &= E[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a] \end{aligned}$$

One Last Remark on PI v.s. VI

Thanks!