COMP3211 Tutorial 7: Game Theory

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Outline

Game Formulation

Best Response and Nash Equilibria

Sealed-Bid Auctions

Game Formulation

Formulation

Normal-Form Games:

- A set of agents $\mathcal{N} = [1..n]$
- Each agent i holds an action set $\mathcal{A}_i$
- Each agent i holds an utility function $u_i : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathbb{R}$
- A Nash equilibrium $(s_1, \cdots, s_n)$, $s_i \in \Delta(A_i)$:

$$s_i = BestResponse(s_{-i}) \triangleq BestResponse(s_1, ..., s_{i-1}, s_{i+1}, \cdots s_n)$$

- Nash's existence theorem: Any game with a finite set of players and finite set of strategies has a Nash equilibrium of mixed strategies.

Maximizing Reward v.s. Minimizing Regret

	Reward
A	100
B	2

	C	D
A	100, 3	$1-\epsilon$, 5
B	2, 3	1, 5

	C	D
A	100, a	$1-\epsilon$, b
B	2, c	1, d

Definition[1]**:**
$Reward(a_i, a_{-i}) \triangleq u_i(a_i, a_{-i})$
$Regret(a_i, a_{-1}) \triangleq \max_{a_i'} u_i(a_i', a_{-i}) - u_i(a_i, a_{-i})$

[1] Shoham, Yoav, and Kevin Leyton-Brown. Multiagent systems: Algorithmic, game-theoretic, and logical foundations. Cambridge University Press, 2008.

Best Response and Nash Equilibria

Example 1: Network Sharing

You and your friend share a network and both of you want to download a movie:

- If both of you do that, the network is jammed, and none of you is happy: say both of you value it 2.

- If only one of you do that, the network works perfectly: the one who did that is very happy (5), the other is very unhappy (0).

- If none of you do that, then none of you is very happy but then you can do something together. So let's assign it a value of 3.

	D	ND
D	(2, 2)	5, 0
ND	0, 5	3, 3

Example 2: Tragedy of The Commons

A generalized continuous version:

- n players. Each has the same strategy of downloading $x_i \in [0, 1]$ units. But the total bandwidth is only 1.

- If $\sum_i x_i > 1$, every one gets zero utility.

- Else, player i gets a utility of $x_i(1 - \sum_j x_j)$

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Solutoin:
- Let $t = \sum_{j \neq i} x_i$, then $u_i = x_i(1 - t - x_i)$, max at $x_i = \frac{1-t}{2}$.

- A unique solution for the equations $\forall i, x_i = \frac{1-\sum_{j \neq i} x_j}{2} \Rightarrow x_i = \frac{1}{n+1}$.

- Total usage $\sum_i x_i = \frac{n}{1+n}$. Social welfare $= n \cdot \frac{1}{n+1} \frac{1}{n+1} < \frac{1}{n}$.

- But if $\sum_i x_i = 1/2$, social welfare $= \sum_i x_i(1 - \sum_j x_j) = \frac{1}{4}$

- Players are incentivized to overuse the resource.

Example 3: Soccer Penalty Kicks

Kicker/Goalie	Left	Right
Left	0, 1	1, 0
Right	0.75, 0.25	0, 1

- No pure Nash equilibrium, then try to find a mixed Nash equlibrium.

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- The kicker's strategy is to make the goalie indifferent between guarding left and right. Kicker kicks left w.p. q, for the guard,

$$q + 0.25(1-q) = (1-q) \Rightarrow q = 3/7$$

Example 4: Battle of Sexes - Pure NE

	Sports	Movie
Sports	1, 0	-1, -1
Movie	-1, -1	0, 1

Pure-strategy NE: 1. (sports, sports), 2. (movie, movie).

What about mixed-strategy NE?

Example 4 con't: Battle of Sexes - All NE

	Sports	Movie
Sports	1, 0	-1, -1
Movie	-1, -1	0, 1

Best responses:

- Suppose the girl plays [q:S, (1-q):M], for the boy to prefer S

$$q - (1 - q) \geq -q \Rightarrow q \geq \frac{1}{3}$$

- Suppose the boy plays [p:S, (1-p):M], for the girl to prefer S

$$0 - (1 - p) \geq -p + (1 - p) \Rightarrow p \geq \frac{2}{3}$$

Example 3 con't: Battle of Sexes - All NE

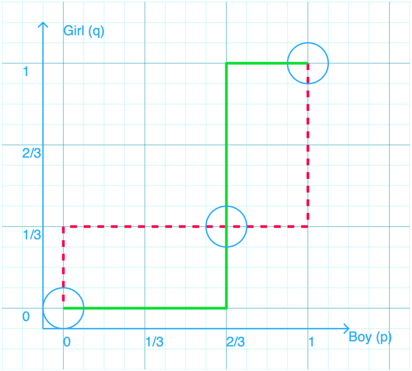

Figure 1: Best response curve.

Thus, mixed NE: [Boy(2/3:S, 1/3:M), Girl(1/3:S, 2/3:M)].

Sealed-Bid Auctions

Sealed-Bid Auctions

In the lecture, we have discussed two implementations of sealed bid actions: first price auction and second price auction.

1. First-price: the winner pays his own bid (the highest)
2. Second-price: the winner pays the second highest bid.

For simplicity, assume there are two bidders *Alice* and *Bob* bidding for one single item. Values that Alice and Bob hold towards the item are a and b, respectively. Those values are private information, i.e. a is unknow to Bob and b is unknown to Alice. Let x denote Alice' bid and y denote Bob's bid. Ties will always be broken in favor of Alice.

Sealed-Bid Auctions: First-Price

Consider first price auction (the highest bidder wins the item and pays the price as she bids).

Assume $a > b$ and Alice somehow knows Bob's valuation b, but Bob still has no idea about Alice's valuation a. Find a pure Nash equilibrium.

Sealed-Bid Auctions: First-Price

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Assume $a > b$ and Alice somehow knows Bob's valuation b, but Bob still has no idea about Alice's valuation a. Find a pure Nash equilibrium.

(x=b, y=b) is the only pure NE, such that Alice can obtain a maximum utility of $a - b > 0$, while Bob has no way to win the item thus no profitable deviation.

Sealed-Bid Auctions: First-Price con't

Assume a and b are independent random variables, both of which are distributed as `uniform`(0, 1), which is known as common knowledge. Each one's exact value is unknown to the other one, as the original assumption. Find a pure Nash equilibrium. (Hint: try to find the best response for Alice when Bob bids $y = b/2$.)

Assume a and b are independent random variables, both of which are distributed as `uniform`(0, 1), which is known as common knowledge. Each one's exact value is unknown to the other one, as the original assumption. Find a pure Nash equilibrium. (Hint: try to find the best response for Alice when Bob bids $y = b/2$.)

Let $y = b/2$, Alice's expected utility will be $Pr(Alice\ wins) \times (a - x)$. Given $b \sim \texttt{uniform}[0, 1]$,

$$Pr(Alice\ wins) = Pr(x \geq b/2) = Pr(b \leq 2x) = 2x.$$

Thus, Alice's expected return will be $2x(a - x)$. Take the first derivative and set to 0,

$$\frac{d}{dx} 2x(a - x) = 2a - 4x = 0 \Rightarrow x = a/2.$$

By symmetry, for Alice bidding $a/2$, Bob's best response is also to bid $b/2$. Hence, $(x = a/2, y = b/2)$ is an NE.

Sealed-Bid Auctions: Revenue Equivalence

First-price auction:

One-shot income for the auctioneer: $max(a/2, b/2)$

Let $\alpha = max(a/2, b/2)$,

$$\begin{aligned}
Pr[\alpha \leq t] &= Pr[max(a/2, b/2) \leq t] \\
&= Pr[a/2 \leq t] \times Pr[b/2 \leq t] \\
&= 4t^2 \\
f(\alpha = t) &= d/dt(4t^2) = 8t \\
E[\alpha] &= \int_0^{1/2} t \cdot 8t \cdot dt = 1/3
\end{aligned}$$

Sealed-Bid Auctions: Revenue Equivalence

Second-price auction:

It is also proved that (x=a, y=b) is a Nash equilibrium for second price auctions.

One-shot income for the auctioneer: $min(a, b)$

Let $\beta = min(a, b)$,

$$\begin{aligned} Pr[\beta \leq t] &= Pr[min(a, b) \leq t] \\ &= 1 - Pr[a > t] \times Pr[b > t] \\ &= 1 - (1 - t)^2 \\ f(\beta = t) &= d/dt[1 - (1 - t)^2] = 2 - 2t \\ E[\beta] &= \int_0^1 t \cdot (2 - 2t) \cdot dt = 1/3 \end{aligned}$$

Thanks!