Seminar on K-theory: Swan's theorem: A ballet of C^*-algebras and K-theory in the Swan lake.

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Part I: Introduction.

The main objective of this talk is to introduce the K-theory of C^* algebras using the Swan's theorem as guiding idea. The theorem is stated as follows,

Main Theorem (Swan theorem)**.** *Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, X be a compact Hausdorff topological space and $E \xrightarrow{p} X$ be a $\mathbb{K}$-vector bundle. Then we have the following assertions are true:*

1. *$\Gamma(X, E)$ is a finitely generated projective module over $C(X)$.*
2. *There is an equivalence of categories between the finitely generated projective modules over $C(X)$ and vector bundles over X.*
3. *There is an isomorphism between the K-theory of X and the K theory of $C(X)$, $K(X) \cong K(C(X))$.*

The main idea of this theorem is to give another point of view to the sections, extending the notion of Γ to a functor named cross section functor. Moreover, we will understand the topological K-theory using the well known algebraic K-theory.

In the first part we will show the first two statements of the theorem. To reach this objective the notion of finitely projective modules and their properties will be presented. In the second part we will define the notion of the C^*-algebras, using on $C(X)$ as example. In the last part, the K-theory for rings and C^*-algebras will be defined, in a similar fashion as in topological K-theory. Finally the third statement of the theorem of Swan will be proved.

Part II: Modules and sections.

In this first part the main objective is to proof the first two statements of the Swan theorem. First, we will introduce the algebraic notion of modules and the subfamilies of finitely generated, free and projective modules. Then we will visualise this ideas using $C(X)$.

Definition 1. Given a ring with identity R. We define M as a **module over** R or an R**-module** as an Abelian group respect the addition, i.e. $(M, +)$ is an Abelian group, and we will define an operation $\cdot : R \times M \to M$ defined as follows $(r, a) \to r \cdot a$, such that $(M, \cdot, +)$ has the following properties:

- $r \cdot (x + y) = x \cdot x + r \cdot y$,
- $(r + s) \cdot x = r \cdot x + s \cdot y$,
- $(rs) \cdot x = r \cdot (s \cdot x)$,
- $1_R \cdot x = x$,

for all $x, y \in M$ and $r, s \in R$.

Remark 1. If R is a field then an R-module M will behave like a vector space.

Example 1. Let X be a compact topological space then we define the set of continuous functions to $\mathbb{C}$ as follows,

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

Then $(C(X), +)$ with $+$ defined point-wise is an Abelian group. Moreover, $(C(X), \cdot)$ defined also point-wise satisfies the four conditions previously stated. Moreover we can extend $C(X)$ to a field.

Then we will introduce some families of R-modules.

Definition 2. Let M be an R-module. The we will say that:

1. M is a **finitely generated module** if there exists $x_1, \ldots, x_n$ elements of M such that for all element $x \in M$ we can write it as $x = \sum \lambda_i x_i$ where $\lambda_i \in R$.
2. M is a **free module** if it has a basis.
3. M is a **projective module** if there exists another R-module N such that $M \oplus N$ is free.

Example 2.

- Every Abelian group can be consider of a module over $\mathbb{Z}$. A free Abelian group G is a free module over $\mathbb{Z}$.
- Any $\mathbb{K}$-vector space is a free module over $\mathbb{K}$.

- If R is a ring then R is a free module over R.

Proposition 1. *Let R be a commutative ring with unit. Then the family of finitely generated projective modules over R, $\mathcal{FGP}(R)$, generates a semigroup with the direct sum.*

As we know exact sequences in topology plays a key role and in this situation the exact sequences will also be crucial.

Definition 3. An exact sequence $0 \to M \xrightarrow{i} S \to 0$ is said to **split** if there is a map $j : S \to M$ such that $p = j \circ i$, where p is a projection on S, $p = p^2$. The map j is called splitting homomorphism.

Lemma 1. *Let P be a projective R module. Then the exact sequence $M \to P \to 0$ splits.*

Lemma 2. *Let X be a compact topological space and let $E, F \in Vect(X)$ such that $f : E \twoheadrightarrow F$ is surjective the it has a splitting.*

Now we will show the sections are finitely generated projective modules over $C(X)$.

Proposition 2. *Let X be a compact topological space and let $E \xrightarrow{p} X$ a vector bundle. Then $\Gamma(E, X)$ is a finitely generated projective module over $C(X)$.*

Proof. Step 0: $\Gamma(E, X)$ is a $C(X)$-module. First of all for every section $s \in \Gamma(E)$ and function $f \in C(X)$ we can define the product $f \cdot s$ as the multiplication in E point-wise. Therefore, $(\Gamma(E), \cdot)$ will satisfy the four points of the definition of the module. Hence $(\Gamma(E), \cdot, +)$ will be a $C(X)$-module.

Step 1: $\Gamma(E, X)$ is a finitely generated $C(X)$-module.
First of all we can cover X with an open covering since it is compact, $\{U_i\}_{1 \leq i \leq M}$. Since we have the axiom of local triviality,

$$\begin{tikzcd} E\big|_{U_i} \arrow[rr, "f_i"] \arrow[dr, "p|_{U_i}"'] & & U_i \times \mathbb{K}^n \arrow[dl, "\pi_1"] \\ & U_i & \end{tikzcd}$$

for every U_i we can find n sections $s_1^i, \ldots, s_n^i$ such that $\pi_2 \circ f_i \circ s_j^i = e_j$ where $\{e_j\}_{i=1}^n$ is an orthonormal basis of $\mathbb{K}^n$. Since we have that X is compact the open covering has an associated partition of unit $\{\phi_i\}$ then we can define $s_1, \ldots, s_n$ sections in the following way $s_i(x) = \sum_i \phi_i(x) s_j^i(x)$. In this way we can express every section $s \in \Gamma(E, X)$ as a product of $s_1, \ldots, s_j$. Obtaining a finite generating system for this module.

Step 2: $\Gamma(E, X)$ is a projective module $C(X)$-module. Using the previous proposition we can embed E in $\mathcal{E}^N$. Since we can equip $\mathcal{E}^N$ with an scalar product, h, then we can define $E^\perp$ as the orthogonal complement of E. Therefore, $E \oplus E^\perp = \mathcal{E}^N$ we obtain that $\Gamma(E, X) \oplus \Gamma(E^\perp, X) = \Gamma(\mathcal{E}, X) \cong C(X)$. Thus $\Gamma(E, X)$ is a projective module. $\square$

An equivalence of categories, it means that Γ is a fully faithful and essentially surjective functor. Now we need to show the following steps:

- Γ is surjective, so for all vector bundle we can finde a finitely generated projective module. Showed in the proportion 2.
- Γ is injective, so if we take a finitely projective module over $C(X)$, P, we can find a vector bundle, E, such that $\Gamma(E) \cong P$.
- The induced map by Γ,

  $$\Gamma_{Vect(X),C(X)} : Hom_{Vect(X)}(E,F) \longrightarrow Hom_{C(X)}(\Gamma(E),\Gamma(F))$$

  is a bijection.

Now we are going to show the second statement showing each point as a proposition. To finally give the proof of the second statement as a corollary of them.

Proposition 3. *Let X be a compact Hausdorff topological space and let P a finitely generated projective module over $C(X)$. Then there exists $E \xrightarrow{p} X$ a vector bundle such that $P \cong \Gamma(E,X)$.*

Proof.
Let $p : C(X)^n \to P$ be a surjective map, it will exists since P is a projective $C(X)$-module. Then we can consider the following exact sequence:

$$C(X)^n \xrightarrow{\quad p \quad} P \longrightarrow 0$$

Since P is projective it will split by lemma 1. Then we have a map $q : P \to C(X)^n$ such that $\pi := q \circ p : C(X)^n \to C(X)^n$ is a projection. Therefore we can write $C(X)^n \cong P \circ Ker(p)$. We can view the projection as matrix $\pi \in Mat_n(C(X))$ such that when we evaluate at a point $x \in X$ we obtain that $\pi(x) \in Mat_n(C(X))$.

We can define a projection between bundles in the following way,

$$\begin{aligned} \alpha : X \times \mathbb{K}^n &\to X \times \mathbb{K}^n \\ (x,v) &\to (x, \pi(x) \cdot v). \end{aligned}$$

Then we can define the map $I-\alpha : X \times \mathbb{K}^n \to X \times \mathbb{K}^n$ that it will be a projection. Therefore α will generate the following exact sequence, in the middle:

$$0 \longrightarrow X \times \mathbb{K}^n \xrightarrow{\alpha} X \times \mathbb{K}^n \xrightarrow{I-\alpha} X \times \mathbb{K}^n \longrightarrow 0$$

Then we will denote by $E = Im(\alpha)$ that it will be a vector bundle. Therefore, we can consider the following sequence,

$$\begin{tikzcd} 0 \arrow[r] & Ker(\alpha) \arrow[r, hook] & X \times \mathbb{K}^n \arrow[rr, "\alpha"] \arrow[dr, two heads] & & X \times \mathbb{K}^n \\ & & & Im(\alpha) \arrow[ur, hook] & \end{tikzcd}$$

Where $Ker(\alpha) \to X \times \mathbb{K}^n$ is injective and $X \times \mathbb{K}^n \to Im(\alpha)$ is suprajective. Hence by lemma 2 we obtain that this map will split and we will obtain a exact sequence. Now, applying that functor Γ we will have that the following sequence is also exact,

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(Ker(\alpha)) \arrow[r, hook] & C(X)^n \arrow[rr, "\pi"] \arrow[dr, two heads] & & C(X)^n \\ & & & \Gamma(Im(\alpha)) \arrow[ur] & \end{tikzcd}$$

Since we have that $\Gamma(Ker(\alpha)) = Ker(\Gamma(\alpha)) = Ker(\pi)$, we obtain that $Im(\pi) \cong \Gamma(Im(\alpha))$. Since $P \cong Im(\pi)$ and $E = Im(\alpha)$ we obtain that $P \cong \Gamma(E)$. $\square$

Now we want to show that the functor Γ induces a bijection in the homomorphism,

$$\Gamma_{Vect(X),C(X)} : Hom_{Vect(X)}(E, F) \longrightarrow Hom_{C(X)}(\Gamma(E), \Gamma(F))$$

To obtain this equivalence we want to relate the fibers with the algebraic structure, $\Gamma(E)$, in such way that we can create and isomorphism. To get this objective we are going to introduce the evaluation map at a point $x \in X$,

$$\begin{aligned} ev_x^{\mathbb{K}} : C(X) &\to \mathbb{K} \\ f &\to f(x). \end{aligned}$$

Then we will denote by m_x to the kernel of the evaluation map. It will be a maximal ideal. Hence we can apply the same idea to the fibres,

$$\begin{aligned} ev_x : \Gamma(E) &\to E_x \\ s &\to s(x). \end{aligned}$$

Then we will obtain that $m_x\Gamma(E) \subseteq \Gamma(E)$ is a submodule.

Remark 2. Let $\in X$ then can take a closed subset of X, $V \subseteq X$ such that given a section $s : U_x \to E\big|_{U_x}$ can be extended to a global section $s' : X \to E$ such that in a small closed neighbourhood of x, V_x, $s\big|_{V_x} \equiv s'\big|_{V_x}$.

Lemma 3. *Let X be a compact Hausdorff space. Then for any vector bundle $E \overset{p}{\to} X$ and any point in X, $x \in X$, the evaluation map $ev_x : \Gamma(E)/m_x\Gamma(E) \to E_x$ is an isomorphism.*

Proof. First we want to show the surjectivity of the map, let $v \in E_x$ and $U_x \in \mathcal{N}(x)$ an open neighbourhood then by local trivially of E we can define a section on U_x such that $s_v : U_x \longrightarrow E|_{U_x}$ such that $s_v(x) = v$. Hence using the previous remark we can extend s_v to X. Therefore, we have obtained a global section such that $s(x) = v$.

For injectivity we will take a a section $s \in \Gamma(E)$ such that $s(x) = 0$ then we wan to show that $s \in m_x\Gamma(E)$. By local triviality in U_x we have a basis of sections $e_1, \dots, e_n$ such that by the previous remark we can extend it to global sections. But this extensions has a counterpoint, they won't be linearly independent in all X, it will be only in U_x.

Thus for all $y \in U_x$ we will have that $\{e_i(y)\}_{i=1}^n$ will be a basis of E_y then we can write $s = \sum_i a_i(y)e_i(y)$ where $a_i = P_i \circ s : X \to \mathbb{K}$, then it will be continuous. At x we will have that $s(x) = 0$ hence $a_i(x) = 0$ and $a_i \in m_x$.

Now we can define another section $t = s - \sum_i a_i e_i$ that will vanish in a neighbourhood of x. Therefore using the Uryshon's lemma we obtain a function $b : X \to \mathbb{K}$ such that $b(x) = 0$ thus $b \in m_x$ and $b\big|_{X/U} = 1$. Therefore we can write $\tilde{t} = bt$ then $s = \tilde{t} + \sum a_i e_i \in m_x\Gamma(E)$. $\square$

Proposition 4. *Let X be a compact Hausdorff space. Then for any pair of vector bundles, $E, F \in Vect(X)$, the map*

$$\Gamma_{Vect(X),C(X)} : Hom_{Vect(X)}(E, F) \longrightarrow Hom_{C(X)}(\Gamma(E), \Gamma(F))$$

is fully faithful.

Proof. To show the bijectivity we need divide the proof in two cases,

- If $E = X \times \mathbb{K}^n$ and $F = X \times \mathbb{K}^m$ we have the following situation,

  $$\Gamma : Hom_{Vect(X)}(X \times \mathbb{K}^n, X \times \mathbb{K}^m) \longrightarrow Hom_{C(X)}(C(X)^n, C(X)^m)$$

  Then a map of vector bundles $f : E \to F$ is uniquely determined by a map $X \to Mat_{n,m}(\mathbb{K})$. Since a map of $C(X)$-modules $C(X)^n \to C(X)^m$ is specified by a matrix $m \in Mat_{m,n}(C(X))$. We have that a continuous map $X \to Mat_{n,m}(\mathbb{K})$ has a matrix in $Mat_{m,n}(C(X))$, by extending the matrix, and in the other way come's because of the evaluation map. We have a bijection.

- If E and F are not trivial vector bundles. We have the following maps,

  1. Since have an injection between

     $$\begin{aligned} Hom_{Vect(X)}(E, F) &\to \Pi_{x\in X} Hom(E_x, F_x) \\ \alpha &\to \Pi_{x\in X}\alpha : E_x \to F_x, \end{aligned}$$

     gives us an injective map.

  2. Using the previous lemma we have an isomorphism between,

     $$\Pi_{x\in X} Hom(E_x, F_x) \to \Pi_{x\in X} Hom(\Gamma(E)/m_x\Gamma(E), \Gamma(F)/m_x\Gamma(F)).$$

  3. We have the map,

     $$\begin{aligned} Hom_{C(X)}(\Gamma(E), \Gamma(F)) &\to \Pi_{x\in X} Hom(\Gamma(E)/m_x\Gamma(E), \Gamma(F)/m_x\Gamma(F)) \\ f &\to \Pi_{x\in X} f : \Gamma(E)/m_x\Gamma(E) \to \Gamma(F)/m_x\Gamma(F)) \end{aligned}$$

Then we can construct the following diagram,

$$
\begin{tikzcd}
Hom_{Vect(X)}(E,F) \arrow[rr] \arrow[ddd] & & Hom_{C(X)}(\Gamma(E),\Gamma(F)) \arrow[ddd] \\
& f : E \to F \arrow[r, "\Gamma"] \arrow[d] & \tilde{f} : \Gamma(E) \to \Gamma(F) : s \to (f \circ s)(x) \arrow[d] \\
& \prod_{x \in X} f_x : E_x \to F_x \arrow[r] & \prod_{x\in X} \tilde{f}_x : \Gamma(E)/m_x\Gamma(E) \to \Gamma(F)/m_x\Gamma(F) \\
\Pi_{x\in X} Hom(E_x, F_x) \arrow[rr, "\cong"'] & & \Pi_{x\in X} Hom(\Gamma(E)/m_x\Gamma(E), \Gamma(F)/m_x\Gamma(F))
\end{tikzcd}
$$

Were we obtain that Γ is injective. Now we want to show that Γ is surjective, let $f \in Hom_{C(X)}(\Gamma(E), \Gamma(F))$ then we can apply the map defined in the point 3 and the isomorphism and we obtain an element $g \in \Pi_{x\in X} Hom(E_x, F_x)$ that a priori it will be a map of sets. We can show the continuity by using local triviality, thus we go to the first case. Then we found an element in $g \in Hom_{Vect(X)}(E, F)$ such that $\Gamma(g) = f$, thus is surjective.

□

Therefore as corollary of the previous propositions.

Corollary 1. *There is an isomorphism between the finitely generated projective modules over $C(X)$ and sections of vector bundles.*

Part III: C^*-algebras.

At this point of the talk we aim to view the module of continuous functios on X from another point of view. We will see that we can equip the set $C(X)$ with a different structure, named algebra. Finally, we will show that the set of continuous functions is a C^*-algebra. Moreover, since we are mathematicians and we love invariants we will introduce the projections that they will play a key role in the last part of the talk.

Definition 4. Let V be a $\mathbb{K}$ vector space. Then we will say that V is an **algebra** if the underlaying additive Abelian group has a ring structure, we can define a multiplication between elements, compatible with the given scalar multiplication such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

for all $a, b \in E$ and $\lambda \in \mathbb{K}$. Moreover, if $ab = ba$ for all $a, b \in V$ we will say that V is a commutative algebra.

Example 3.

- Lets consider the square matrices over the complex numbers, $Mat_n(\mathbb{C})$. Then we have that $(Mat_n(\mathbb{C}), +)$ is an Abelian group. Then if we take the usual multiplication on $\mathbb{C}$ we have that $(Mat_n(\mathbb{C}), +, \cdot)$ is an algebra. But is a non-commutative algebra.
- Let X be a compact Hausdorff space. Then we have that $C(X)$ is a module over $\mathbb{K}$, since $\mathbb{K}$ is a field the behaviour of $C(X)$ is like a vector space. Then if we consider the scalar multiplication defined point wise,

$$\begin{aligned} \cdot : \mathbb{C} \times C(X) &\to C(X) \\ (\lambda, f) &\to \lambda \cdot f : X \to \mathbb{C} \\ &\qquad\qquad x \to \lambda f(x). \end{aligned}$$

  we obtain that $(C(X), \cdot)$ is a monoid. Therefore, $(C(X), +, \cdot)$ is an algebra. In addition, since $\mathbb{C}$ is commutative respect the addition, we have that $C(X)$ is commutative algebra.

Definition 5. Let V be an algebra. Then $W \subset V$ a vector subspace is called **subalgebra** if for all $u, v \in W$ then $uv \in W$.

Since we are working with vector spaces a natural action is to define a norm over V.

Definition 6. Let V be an alebgra equipped with a norm $|| \cdot || : V \to \mathbb{R}$. Then we will say that the norm is **submultiplicative** if

$$||a \cdot b|| \leq ||a|| \cdot_{\mathbb{R}} ||b||,$$

for all $a, b \in V$. Moreover, if an algebra, V, is equipped with a submultiplicative norm we will sat that $(V, ||\cdot||)$ is normed algebra. As in the case of vector spaces if the normed algebra is complete, we will say that is a Banach algebra.

Definition 7. Given an algebra V. If the ring structure of V is a ring with unit, 1_V we will say that V is a **unital algebra**. In addition, if V is a normed algebra the unit must satisfy that $||1_V|| = 1$.

Example 4. Continuing with the example of $C(X)$, we can equipe it with the $||f||_\infty := \sup_{x\in X} |f(x)|$. We have that $(C(X), ||\cdot||_\infty)$ is a normed algebra since,

$$||f \cdot g||_\infty := \sup_{x\in X} |g(x)f(x)| \leq \sup_{x\in X} |g(x)||f(x)| \leq ||g||_\infty ||f||_\infty.$$

Since $(C(X), ||\cdot||_\infty)$ is a Banach space then it is a Banach algebra.
If we consider the example of the complex matrices we have that is a unital Banach algebra with the operator norm,

$$||u|| = \sup_{x\in\mathbb{C}} \frac{||u(x)||}{||x||}.$$

Since $\mathbb{C}$ is a Banach space we have that $Mat_n(\mathbb{C})$ is Banach algebra. Moreover, an example of sub-algebra in $Mat_n(\mathbb{C})$ is the upper triangular matrices, all entries below the diagonal are zero. Since we have that the product of two of them is going to be upper triangular.

In some cases we can have a Banach algebra, V, that is not unital, then we can construct the unital Banach algebra associated to it V^+. The underlying vector space of this algebra will be $V \otimes \mathbb{C}$ with the following product,

$$\begin{aligned} &\cdot^+ : V^+ \times V^+ \to V^+ \\ &(\alpha \otimes a, \beta \otimes b) \to (ab + a\beta + \alpha b) \otimes \alpha\beta \end{aligned}$$

Thus we can embed V into V^+ given every $v \in V$ the element $v \otimes 0$.

Definition 8. Let V be a complex algebra, then we define the **$*$-operation** as follows,

$$\begin{aligned} * : V &\to V \\ v &\to v^*. \end{aligned}$$

Then we will say that V is a ***-algebra** if it satisfies the following properties:

1. The involution property, for all $u \in V$ satisfies that $u^{**} = u$.
2. For all $u, v \in V$ we will have that,
   - $(u + v)^* = u^* + v^*$.
   - $(uv)^* = v^* u^*$.
3. For all $u \in V$ and $\lambda \in \mathbb{C}$ we have that $(\lambda u)^* = \bar{\lambda} u^*$.

Moreover, if $(V, ||\cdot||)$ is a Banach algebra, the *-operation will satisfy the isometric involution property:

$$||u^*|| = ||u||$$

for all $u \in V$.

If we have that the algebra V is not unital we can construct the unitalization V^+ and give it a *-operation, in the following way,

$$\begin{aligned} *^+ : V^+ &\to V^+ \\ a \otimes \alpha &\to a \otimes \bar{\alpha}. \end{aligned}$$

Example 5. We have that $\mathbb{C}$ with the complex conjugate is a Banach $*$-algebra. Using our favourite example, $C(X)$, we can define the operation

$$\begin{aligned} * : C(X) &\to C(X) \\ f &\to \begin{aligned} f^* : X &\to \mathbb{C} \\ x &\to \overline{f(x)}. \end{aligned} \end{aligned}$$

This operation will satisfy all the properties from the previous definition. Thus $C(X)$ will be a Banach $*-$algebra.

Definition 9. Let V be a complex Banach algebra equipped with the $*$-operation. We will say that V is a C^***-algebra** if it satisfies the C^*-equation,

$$||u^*u|| = ||u||^2.$$

Example 6. The space $C(X)$ with the $*$-operation satisfies the C^*-equation. Thus we have that $C(X)$ is commutative unital C^*-algebra.

Now we are interested in the elements of the algebra that they don't change under the $*$-operation.

Definition 10. Let V be a unital commutative C^*-algebra. We will say that $p \in V$ is a **projection** if it is self-adjoin and idempotent,

$$p = p^* = p^2.$$

The set of all projections will be denoted by $\mathcal{P}(V)$. We will say that two projections $p, q \in \mathcal{P}(V)$ are orthogonal if $pq = 0$. We can define the orthogonal sum of two projections as $p \oplus q$.

Example 7. Lets consider $X = S^2$ then we have that $C(X) = \{f : S^2 \to \mathbb{C} : f \text{ is continuous}\}$ now we look for $\mathcal{P}(C(S^2))$. Thinking about the definition of the projections we get the following:

- Self-adjoin $f^*(x) = f(x)$ then we have that $f : S^2 \to \mathbb{R}$.
- Idempotent $f^2(x) = f(x)$ that only happens if $f \equiv 0$ or $f \equiv 1$.

Thus $\mathcal{P}(C(S^2)) = \{1, 0\}$.
Now we take $X = \mathbb{C}$, then the set of projections $\mathcal{P}(\mathbb{C}) = \{0, 1\}$.

Definition 11. We will say that $v \in V$ is a **partial isometries** if $v^*v \in \mathcal{P}(V)$. Moreover, since V is unital we can define the **isometries** or **unitaries** that will be the elements $v \in V$ such that $v^*v = 1$.

An important equation involving the partial isometries is the following:

$$u = uu^*u = up = qv,$$
$$u^* = u^*uu^* = pu^* = u^*q.$$

Now we can introduce some relations on the set $\mathcal{P}(V)$.

Definition 12. Let V be a C^*-algebra and let $p, q \in \mathcal{P}(V)$ then we can define the following relations.

- We say that p and q are **Murray-Von Neuman equivalent** if there exists a partial isometry $v \in V$ such that $v^*v = p$ and $vv^* = q$. We will denote it by $p \sim q$
- We say that p and q are **unitarily equivalent** if there exists a unitary $u \in V^+$ such that $q = upu^*$. We will denote it by $p \sim_u q$.
- We say that p and q are **homotopy equivalent** if there exists a continuous path $c : [0,1] \to V$ such that $c(t) \in \mathcal{P}(V)$ and $c(0) = p$ and $c(1) = q$. We will denote it by $p \sim_h q$

Proposition 5. *The previous relations are equivalence relations. Moreover, they are equivalent.*

Example 8. Lets consider $X = S^2$ and we take the C*-algebra $V = C(S^2)$ then we will work with $Mat_2(V) \cup V$. In V we had that $\mathcal{P}(V) = \{0, 1\}$. Then in $Mat_2(V)$ we will have that the set of projections will be bigger,

$$\mathcal{P}(Mat_2(V)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

But we have that they are equivalent because projections since we have

$$v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

be a partial isometry such that:

$$vv^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$v^*v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

.

Part IV: K-theory of rings & C^*-algebras.

Over any commutative ring with unit, R, we can define the notion of the modules. Then a way to classify the modules over a ring will be studying the easiest modules, the finitely generated projective modules over R. This family of modules has also a semigroup structure as we saw in the proportion 1 similar as the vector bundles one. Then in a similar fashion as in the K-theory for vector bundles we can define the K-theory for this commutative ring with unit.

Definition 13. Let R be a commutative ring with unit. Then the K-theory group of R will be denoted as $K(A) = \mathcal{G}(\mathcal{FPG}(R))$.

We denote by $M_\infty(V) = \cup_{n\in\mathbb{N}} Mat_n(V)$ where we have the canonical isometric embedding between $Mat_n(V) \hookrightarrow Mat_{n+1}(V)$. Let $a \in Mat_n(V)$

$$a \hookrightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we can generalise all the notions that we have for projections in A to $Mat_\infty(A)$.

Definition 14. Let V be a C^*-algebra.

- We will say that two projections $p, q \in Mat_\infty(V)$ are **equivalent**, if for some $n \in \mathbb{N}$ then $p, q \in Mat_n(V)$ and $p \sim q$. The set of equivalence classes $Mat_\infty(V)/\sim =: \mathcal{V}(V)$.
- We can define an **operation between projections** in $\mathcal{V}(V)$ in the following way, given $p, q \in \mathcal{V}(V)$, such that $p \in Mat_n(V)$ and $q \in Mat_m(A)$ we define,
$$[p] \oplus [q] := [diag(p, q)] \in Mat_{m+n}(V).$$

Remark 3. In the previous definition the equivalence relation are not relevant by the proportion 5. The operation $\oplus$ defined previously is well defined since,

$$\begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \dots & \vdots \end{pmatrix} \sim \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

Proposition 6. *The set $(\mathcal{V}(V), \oplus)$ is an Abelian semigroup with neutral element $[0]$. This semigroup has a name, Murray-Von Neuman semigroup.*

Proof. Let $p, q, r \in \mathcal{V}(V)$ such that $p \in Mat_n(V)$, $q \in Mat_m(V)$ and $r \in Mat_l(V)$ then we need to show the following properties.

- **Neutral element** $p \oplus 0 = diag(p, 0) = p$.

- **Associativity**

$$
\begin{aligned}
(p \oplus q) \oplus r =& p \oplus (q \oplus r) \\
diag(p,q) \oplus r =& p \oplus diag(q,r) \\
diag(diag(p,q),r) =& diag(p, diag(q,r)) \\
diag(p,q,r) =& diag(p,q,r).
\end{aligned}
$$

- **Commutativity**

$$
\begin{aligned}
(p \oplus q) =& q \oplus p \\
diag(p,q) =& diag(q,p)
\end{aligned}
$$

We construct the matrix

$$
v = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix},
$$

we claim that is a partial isometry. Then we will compute,

$$
\begin{aligned}
vv^* = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} = \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix} = diag(p,q), \\
v^*v = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} = \begin{pmatrix} q^2 & 0 \\ 0 & p^2 \end{pmatrix} = diag(q,p).
\end{aligned}
$$

Thus we have that v is a partial isometry and we have that $diag(p,q) \sim diag(q,p)$ thus $\oplus$ is commutative. □

Example 9. Let $V = \mathbb{C}$ then $\mathcal{V}(V) = \{p \in Mat_\infty : p^2 = p = p^*\}/\sim$. Since we have that projections in $Mat_n(\mathbb{C})$ are equivalent when their ranges being subspaces of $\mathbb{C}^n$ have the same dimension. Therefore we obtain that, $\mathcal{V}(\mathbb{C}) = \mathbb{N} \cup 0$.

Now we can define the the group K_0 of a C^*-algebra.

Definition 15. Let V be a C^*-algebra. Then we K_0 of a V as follows.

- If V is unital then $K_0(V) := \mathcal{G}\mathcal{V}(V)$ where $\mathcal{G}$ is the group completion of the Abelian semigroup.
- If V is not unital we take the unitalization, V^+, and we define the group for it $K_0(V^+) = \mathcal{G}\mathcal{V}(V^+)$. Then we define

$$
K_0(V) := ker(\mathcal{G}(\mathcal{V}(\pi)) : \mathcal{G}(\mathcal{V}(V^+)) \to \mathcal{G}(\mathcal{V}(\mathbb{C})))
$$

Example 10. Continuing with the example of the complex numbers we have seen that $\mathcal{V}(\mathbb{C}) = \mathbb{N} \cup \{0\}$. Then the group completion of $\mathbb{N} \cup \{0\}$ is $\mathbb{Z}$.

Using the previous example we can rewrite the definition of the first K group of a non-unital C^*-algebra as follows,

$$
K_0(V) := ker(\mathcal{G}(\mathcal{V}(\pi)) : \mathcal{G}(\mathcal{V}(V^+)) \to \mathbb{Z}).
$$

Remark 4. Since we can consider an associative algebra, as in our case, we obtain that is ring. Then we can work with the K-theory for algebras. For this proof we will use the notion K-theory for rings since we can consider $(C(X), +, \cdot)$ as commutative ring with unit.

Proposition 7. *Let X be a compact Hausdorff space. Then there is an bijection between:*

$$K(X) \cong K_0(C(X)).$$

Proof. We have the following picture of equivalences taking the group completion of the K-theory,

$$\left\{\begin{matrix} Projections \\ on\ \mathcal{V}(C(X)) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} fin.\ gen. \\ proj. \\ C(X)-mod \end{matrix}\right\} \xleftrightarrow{\Gamma} \left\{\begin{matrix} Vector \\ Bundles \\ over\ X \end{matrix}\right\}$$

$$P \longrightarrow P(C(X)^n) \xrightarrow[\Gamma]{} E$$

$$P_E \longleftarrow \Gamma(E) \xleftarrow{\Gamma} E$$

Let $E \in K(X)$ then we can give it a finitely generated projective module over $C(X)$ using the functor Γ that it will belong to $K(C(X))$. Now we can define the projection, such that the the basis of $C(X)^n$ is going to be mapped to the system of generators of $\Gamma(E)$.

Now let P be a projection in $K(C(X))$ then there exist a number $n \in \mathbb{N}$ such that $P \in Mat_n(X)$.Then we can see the space $P(C(X)^n) \subseteq C(X)^n$, thus we obtain that $P(C(X)^n)$ is a finitely generated module over $C(X)$. Therefore, by essentially surjective, we can conclude the $P(C(X)^n) \cong \Gamma(E)$.

$\square$