Comparison techniques for $f-$extremal domains in Space Forms

Abstract

1 Introduction

Asumiremos que $\partial\Omega$ es al menos de clase $\mathcal{C}^2$. De esta forma, las soluciones del problema serán al menos $\mathcal{C}^3$. Basta pedir que $\partial\Omega$ sea regular respecto del operador laplaciano para tener soluciones clásicas, pero creo que en algún momento hace falta más regularidad.

2 Preliminaries

This section sets the stage and fixes notation. We begin with space forms $\mathcal{M}^n(k)$ and normal coordinates (r, θ) centered at a point o (see (2.1)), and then recall the basic geometry of hypersurfaces in general Riemannian manifolds.

In the spherical case $\mathbb{S}^n$, we briefly review isoparametric hypersurfaces via the Cartan–Münzner polynomial P. Finally, we introduce the overdetermined problem (2.6) and the symmetry conventions (radial, antipodal, toroidal) that guide our comparison arguments and the construction of exotic solutions.

2.1 Space forms

Let $\mathbb{M}^n(k)$ denote the complete, simply connected Riemannian n-manifold of constant sectional curvature k. That is, $\mathbb{M}^n(k)$ is $\mathbb{S}^n$ if $k = 1$, $\mathbb{R}^n$ if $k = 0$, or $\mathbb{H}^n$ if $k = -1$. From now on, we will work on these manifolds using exponential coordinates.

Given $k \in \mathbb{R}$, fix a point $o \in \mathbb{M}^n(k)$. Also, write $B(0, r) \subset \mathbb{R}^n$ for the Euclidean ball of radius $r > 0$ centered at the origin. We also use the notation $B(0, +\infty) = \mathbb{R}^n$. Consider the exponential map centered at o:

$$\exp_o : B(0, \bar{r}) \longrightarrow \tilde{\mathbb{M}}^n(k),$$

where $\tilde{\mathbb{M}}^n(k) = \mathbb{M}^n(k)$ and $\bar{r} = +\infty$ if $k \leq 0$, and $\tilde{\mathbb{M}}^n(k) = \mathbb{S}^n(k) \setminus \{-o\}$ and $\bar{r} = \pi/\sqrt{k}$ if $k > 0$, with $-o$ being the antipodal point of o. Then, by parametrizing $B(0, \bar{r})$ in polar coordinates, we obtain the polar parametrization of $\mathbb{M}^n(k)$, which is given by the map

$$\begin{aligned} X :& [0, \bar{r}) \times \mathbb{S}^{n-1} \longrightarrow \mathbb{M}^n(k) \\ X&(r, \theta) = \exp_o(r\theta). \end{aligned} \tag{2.1}$$

The coordinates (r, θ) given by X are called *normal coordinates*. It is straightforward to verify that the metric g_k of $\mathbb{M}^n(k)$ and the corresponding Laplace-Beltrami operator can be written in these coordinates as

$$g_k = dr^2 + s_k(r)^2 g_{\mathbb{S}^{n-1}}$$

and

$$\Delta = \partial_r^2 + (n-1)\, ct_k(r)\, \partial_r + \frac{1}{s_k(r)^2} \Delta_{\mathbb{S}^{n-1}}, \tag{2.2}$$

where we have introduced the functions

$$s_k(r) := \begin{cases} \frac{\sinh(\sqrt{-k}r)}{\sqrt{-k}} & \text{if} \quad k < 0 \\ r & \text{if} \quad k = 0, \\ \frac{\sin(\sqrt{k}r)}{\sqrt{k}} & \text{if} \quad k > 0, \end{cases}$$

and

$$\cot_k(r) := \frac{s_k'(r)}{s_k(r)}, \quad \forall r \in (0, \bar{r}).$$

2.2 Geometry of hypersurfaces

Let $(\mathcal{M}, g)$ be a Riemannian manifold. For any $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$, we write $g(v, w)$ or $\langle v, w \rangle$ for the scalar product of two vectors, and $|v|^2 = \langle v, v \rangle$ for the squared norm of a vector.

Let $\Gamma \subset \mathcal{M}$ be an embedded, two-sided, $\mathcal{C}^2$ hypersurface in $\mathcal{M}$. Write $N : \Gamma \to T\mathcal{M}$ for a Gauss map of Γ. Then $N(p) \perp T_p\Gamma$ and $|N(p)| = 1$ for any $p \in \Gamma$. At a given point $p \in \Gamma$, define

$$\mathbb{I}_p(v, w) = -\langle dN_p(v), w \rangle, \quad \forall v, w \in T_p\Gamma,$$

as the *second fundamental form* of Γ at p with respect to N. The *mean curvature* of Γ at p is then defined as

$$H(p) = \frac{1}{n-1} \text{Trace}(\mathbb{I}_p) = -\frac{1}{n-1} \sum_{i=1}^{n-1} \langle dN_p(e_i), e_i \rangle,$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal basis of $T_p\Gamma$.

For any $l \in \mathbb{R}$, let $\mathcal{H}^l$ denote the l-dimensional Hausdorff measure associated with the metric g. Then, for a given open set $\mathcal{O} \subset \mathcal{M}$, we write $\mathcal{H}^n(\mathcal{O})$ for its volume, and $\mathcal{H}^{n-1}(\Gamma)$ for the area of a hypersurface. Note that we do not require $\mathcal{O}$ or Γ to be connected. When $g = g_k$ is the metric of a space form, we write $\mathcal{H}_k^l$ for the associated Hausdorff measures.

2.3 Isoparametric Hypersurfaces of $\mathbb{S}^n$

Isoparametric hypersurfaces have been extensively studied in the literature in space forms and are defined as those whose principal curvatures are all constant (cf. [13, Chapter 3]). In $\mathbb{H}^n$ or $\mathbb{R}^n$, isoparametric hypersurfaces are relatively simple: they are totally umbilic submanifolds, spherical cylinders, or equidistant hypersurfaces to totally geodesic submanifolds of codimension greater than one. The latter case only arises in hyperbolic n-space. In $\mathbb{S}^n$, these hypersurfaces are of particular interest because they provide natural foliations of the sphere by level sets of a polynomial.

A fundamental result in the theory of isoparametric hypersurfaces in $\mathbb{S}^n$ states that they can be described as the level sets of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$, known as a *Cartan-Münzner polynomial*, which satisfies the following differential equations:

$$\bar{\Delta}P(x) = c\,|x|^{l-2}, \quad \text{and} \quad |\bar{\nabla}P(x)|^2 = l^2|x|^{2l-2}, \tag{2.3}$$

where $l, m_1, m_2 \in \mathbb{N}$ and $c = l^2(m_2 - m_1)/2$, and $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplacian and gradient in the ambient Euclidean space $\mathbb{R}^{n+1} \supset \mathbb{S}^n$. When restricted to the sphere, the function $\rho := P_{|_{\mathbb{S}^n}}$ satisfies:

Theorem 2.1 ([13], Theorem 3.32)**.** *Let $\Gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a connected isoparametric hypersurface with l distinct principal curvatures κ_i, with respective multiplicities m_i, $i \in \{1, \ldots, l\}$. Then, Γ is an open subset of a level set of the restriction to $\mathbb{S}^n$ of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree l, and $\rho = P_{|_{\mathbb{S}^n}}$ satisfies the differential equations:*

$$\Delta\rho(p) = -l(l+n-1)\rho(p) + c \quad \text{and} \quad |\nabla\rho|^2(p) = l^2(1-\rho^2(p)), \tag{2.4}$$

for each $p \in \mathbb{S}^n$, where $c = l^2(m_2 - m_1)/2$ and $2(n-1) = l(m_1 + m_2)$. Furthermore, we have $m_i = m_{i+2}$ (indices modulo l) for each $i \in \{1, \ldots, l\}$. Here, Δ and ∇ denote the Laplacian and gradient on $\mathbb{S}^n$.

It follows from (2.4) that $\rho(\mathbb{S}^n) = [-1, 1]$, which means that $\{\Gamma_r := \rho^{-1}(r)\}_{r\in[-1,1]}$ forms a foliation of $\mathbb{S}^n$, and for each $r \in (-1, 1)$, the level set Γ_r is a connected isoparametric hypersurface in $\mathbb{S}^n$. The level sets Γ_1 and Γ_{-1} are minimal submanifolds of $\mathbb{S}^n$ of codimension $m_1 + 1$ and $m_2 + 1$, respectively, and they serve as the focal submanifolds of each isoparametric hypersurface in the family $\{\Gamma_r\}_{r\in(-1,1)}$.

2.3.1 Examples of Isoparametric Hypersurfaces

Isoparametric hypersurfaces in $\mathbb{S}^n$ are characterized by the number of distinct principal curvatures l and their multiplicities $0 < m_1, m_2 \leq n - 1$. After a series of papers by Münzner, it was shown that the only possible values for the number of distinct principal curvatures of an isoparametric hypersurface are $l \in \{1, 2, 3, 4, 6\}$ (see [13, Theorem 3.49]).

Below, we describe some families of isoparametric hypersurfaces corresponding to each possible value of l.

Umbilic Hypersurfaces ($l = 1$): The simplest example of an isoparametric hypersurface is a *geodesic sphere*, defined as the set of points at a fixed distance r from a given point $p \in \mathbb{S}^n$. These hypersurfaces are totally umbilic, meaning that all their principal curvatures are equal. Up to an isometry, they can be described as level sets of the height function:

$$\rho(x) = x_{n+1}, \quad x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n.$$

Each geodesic sphere is given by

$$\Gamma_r = \{x \in \mathbb{S}^n \mid x_{n+1} = r\}, \quad r \in (-1, 1).$$

This family foliates $\mathbb{S}^n$ into parallel hypersurfaces, except for the two limiting cases $r = \pm 1$, which correspond to the focal submanifolds—single points (the poles). It is clear that $\Gamma_r \cong \mathbb{S}^{n-1}$, $r \in (-1, 1)$, and this foliation describes cylindrical coordinates in $\mathbb{S}^n$.

Generalized Clifford Tori ($l = 2$): The *generalized Clifford tori* are a natural extension of the classical Clifford torus in $\mathbb{S}^3$. Let $a, b \in \mathbb{N}$ such that $a \leq b$ and $a + b = n + 1$. For $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$, define the Cartan-Münzner polynomial

$$P^{a,b}(x) = (x_1^2 + \cdots + x_a^2) - (x_{a+1}^2 + \cdots + x_{n+1}^2),$$

and write $\rho^{a,b} = P^{a,b}|_{\mathbb{S}^n}$. A generalized Clifford torus is defined as a level set

$$\mathbb{T}_r^{a,b} = \left\{p \in \mathbb{S}^n \,\middle|\, \rho^{a,b}(p) = r\right\},$$

for some $r \in (-1, 1)$. Each $\mathbb{T}_r^{a,b}$ is homeomorphic to $\mathbb{S}^{a-1} \times \mathbb{S}^{b-1}$, and the multiplicities of the principal curvatures are $m_1 = a - 1$ and $m_2 = b - 1$. As shown in [13, Section 3.8.2], each $\mathbb{T}_r^{a,b}$ is a homogeneous hypersurface generated by the subgroup of isometries $SO(a) \times SO(b)$, and its two focal submanifolds are totally geodesic spheres that are polar to each other.

It is evident that the generalized Clifford torus $\mathbb{T}_r^{a,b}$, being a level set of $P^{a,b}$ on $\mathbb{S}^n$, is antipodally symmetric.

Cartan's Isoparametric Hypersurfaces ($l = 3$): É. Cartan classified isoparametric hypersurfaces Γ in $\mathbb{S}^n$ with three distinct principal curvatures ($l = 3$).

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where $\mathbb{C}, \mathbb{H}, \mathbb{O}$ denote the complex numbers, quaternions, and Cayley numbers (or octonions), respectively, and set $m = \dim(\mathbb{F})$. There is a canonical representation of $\mathbb{F}$ as $\mathbb{R}^m$ equipped with a suitable inner product. A point in $\mathbb{R}^{2+3m}$ can then be expressed as (x, y, X, Y, Z), where $x, y \in \mathbb{R}$ and $X, Y, Z \in \mathbb{F}$. Here, $\overline{X}$ denotes the conjugate of $X \in \mathbb{F}$.

An isoparametric hypersurface $\Gamma \subset \mathbb{S}^{1+3m}$ with three distinct principal curvatures is given as a level set of the Cartan-Münzner polynomial

$$\begin{aligned} P(x,y,X,Y,Z) = & x^3 - 3xy^2 + \frac{3}{2}x\left(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}\right) + \frac{3\sqrt{3}}{2}y\left(X\overline{X} - Y\overline{Y}\right) + \\ & + \frac{3\sqrt{3}}{2}\left(XYZ + \overline{ZYX}\right). \end{aligned}$$

All principal curvatures of Γ have the same multiplicity $m \in \{1, 2, 4, 8\}$. The focal submanifolds of Γ are a pair of antipodal embeddings of $\mathbb{FP}^2$, where $\mathbb{FP}^2$ denotes the projective plane over the division algebra $\mathbb{F}$.

In a seminal work, Cartan showed that Γ is homogeneous, meaning it arises as the orbit of a closed subgroup of $SO(2 + 3m)$ acting on the sphere. The structure of this group depends on the parameter $m = \dim(\mathbb{F})$, as explained in [13, Subsection 3.8.3].

More generally, following Cartan's foundational results, subsequent works established that any homogeneous isoparametric hypersurface in a Euclidean sphere arises as a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two. These symmetric spaces have been fully classified. In particular, [18, Table 1] provides a complete list of all homogeneous isoparametric hypersurfaces in spheres, indicating:

- the number of distinct principal curvatures l,
- the corresponding multiplicities (m_1, m_2),
- and the symmetric space whose isotropy representation generates the associated homogeneous isoparametric hypersurface.

For example, in the case $m = 1$, we have that Γ is a principal orbit of the isotropy representation of $SU(3)/SO(3)$ in $\mathbb{S}^4$.

Exotic Isoparametric Hypersurfaces ($l = 4$): Unlike the previous cases, the family of isoparametric hypersurfaces with four distinct principal curvatures in Euclidean spheres includes both homogeneous and non-homogeneous examples.

Let $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ denote the pair of multiplicities of the principal curvatures of an isoparametric hypersurface. When $l = 4$, all such hypersurfaces are classified by their multiplicities, and—except for two homogeneous examples with multiplicities $(2, 2)$ and $(4, 5)$—all belong to an infinite family constructed by Ferus, Karcher, and Münzner in a 1981 paper. This construction is based on representations of *Clifford algebras*, and a detailed description can be found in [13, Section 3.9]. We briefly outline how to construct the Cartan-Münzner polynomials associated with this family.

Let $m_1 = m$ and fix $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let $H(k, \mathbb{R})$ denote the space of $k \times k$ symmetric matrices with real entries. Then, an $(m + 1)$-tuple $(P_0, \dots, P_m)$ of elements in $H(2n + 2, \mathbb{R})$ is called a *Clifford system* if the matrices P_i satisfy:

$$P_i^2 = \mathrm{Id}_{2n+2}, \quad P_i P_j = -P_j P_i, \quad i \neq j, \quad 0 \leq i, j \leq m,$$

where Id_{2n+2} denotes the identity matrix of dimension $(2n+2)$. Then the function $P : \mathbb{R}^{2n+2} \to \mathbb{R}$ defined by

$$P(x) = \langle x, x \rangle^2 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2, \quad \forall x \in \mathbb{R}^{2n+2}, \tag{2.5}$$

is a Cartan-Münzner polynomial defining an isoparametric hypersurface with four distinct principal curvatures and multiplicities $(m, n-m)$. A level set of this polynomial is called a *Clifford isoparametric hypersurface*.

Two Clifford systems generate the same isoparametric family if and only if they produce the same *Clifford sphere*. Endowing $H(k, \mathbb{R})$ with the standard inner product for symmetric matrices, given by

$$\langle A, B\rangle = k^{-1}\text{Trace}(A \cdot B), \quad \forall A, B \in H(k, \mathbb{R}),$$

we define the Clifford sphere associated with the Clifford system $(P_0, \ldots, P_m)$ as

$$\mathcal{C}(P_0, \ldots, P_m) = \left\{A \in \text{span}\left\{P_0, \ldots, P_m\right\} \ : \ |A|^2 = 1\right\} \subset H(2n+2, \mathbb{R}).$$

It is known that infinitely many Clifford systems with distinct Clifford spheres can be constructed, producing an infinite family of non-congruent Clifford isoparametric hypersurfaces. It follows from the classification of homogeneous examples that most hypersurfaces in this family are inhomogeneous; this can also be verified independently, see [13, Theorem 3.77].

The topology of Clifford hypersurfaces is studied in [77]. If $\rho := P_{\mathbb{S}^{2n+1}}$ for some P as in (2.5), then the focal submanifolds of the family $\left\{\Gamma_r := \rho^{-1}(r)\right\}_{r\in(-1,1)}$ are given by $\Gamma_{-1} = \rho^{-1}(-1)$ and $\Gamma_1 = \rho^{-1}(1)$. It follows that Γ_{-1} is diffeomorphic to an $\mathbb{S}^{n+1}$-bundle over $\mathbb{S}^m$, while Γ_1 is a Clifford–Stiefel manifold (see [13] for a detailed description of these submanifolds). Furthermore, $\Gamma_{\frac{n-2m+1}{n+1}} = \rho^{-1}((n-2m+1)/(n+1))$ is always a minimal hypersurface diffeomorphic to $\Gamma_1 \times \mathbb{S}^m$.

As a particular example, prior to the work of Ferus et al., É. Cartan discovered an isoparametric family of Clifford type, later generalized by Nomizu. Consider $\mathbb{C}^{n+1}$ as the complex vector space of dimension $n+1$, and write $z = (x, y) \in \mathbb{R}^{n+1} \oplus i\mathbb{R}^{n+1}$. Then, using the Euclidean inner product in $\mathbb{R}^{n+1}$,

$$P((x, y)) = 4\,|x|^2\,|y|^2 - 4\,\langle x, y\rangle^2\,, \quad \forall (x, y) = z \in \mathbb{C}^{n+1},$$

is a Cartan-Münzner polynomial that defines an isoparametric family with four distinct principal curvatures in $\mathbb{S}^{2n+1}$. Each hypersurface in this family is an orbit of the isometric action of $SO(2) \times SO(n+1)$ on $\mathbb{S}^{2n+1}$. In particular, all hypersurfaces in this family are homogeneous.

Exotic Isoparametric Hypersurfaces ($l = 6$) There are only two known isoparametric families with six distinct principal curvatures. We follow [13] in their description. Both families are homogeneous and consist of hypersurfaces with principal curvatures of equal multiplicity $m_1 = m_2 = m \in \{1, 2\}$.

In the case $m = 1$, Miyaoka proved in 1991 that if $\Gamma \subset \mathbb{S}^7$ is an isoparametric hypersurface with $l = 6$ distinct principal curvatures, then Γ arises as the inverse image, under the Hopf fibration $h : \mathbb{S}^7 \to \mathbb{S}^4$, of a Cartan isoparametric hypersurface with three distinct principal curvatures. It is also shown that the two focal submanifolds of Γ are non-congruent minimal embeddings of $\mathbb{RP}^2 \times \mathbb{S}^3$.

When $m = 2$, the isoparametric family arises from the adjoint orbits of the exceptional compact Lie group G_2 (the automorphism group of the octonions $\mathbb{O}$). The group G_2 acts on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^{14}$ by isometries with respect to the bi-invariant metric. Miyaoka showed that these hypersurfaces are fiber bundles over $\mathbb{S}^6$, with fibers given by Cartan's isoparametric hypersurfaces with three distinct principal curvatures of multiplicity two. Thus, isoparametric hypersurfaces with $(l, m) = (6, 2)$ are closely related to those with $(l, m) = (3, 2)$. It follows that an isoparametric hypersurface $\Gamma \subset \mathbb{S}^{13}$ with six distinct principal curvatures of multiplicity two is diffeomorphic to the homogeneous space G_2/T^2, where $T^2 \subset G_2$ is a subgroup isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ as a Lie group. The focal submanifolds of Γ are both diffeomorphic to $G_2/U(2)$, where $U(2)$ denotes the unitary group of dimension two.

2.4 Overdetermined Elliptic Problems

Let $(\mathcal{M}, g)$ be a $\mathcal{C}^2$ Riemannian manifold, and let $\Omega \subset \mathcal{M}$ be a domain with $\mathcal{C}^2$-boundary $\partial\Omega$. We consider the classical overdetermined problem for a function $u : \Omega \to \mathbb{R}$, where $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, given by

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \eta \rangle = \alpha_i & \text{on } \Gamma_i \subset \partial\Omega, \end{cases} \tag{2.6}$$

where Δ and ∇ denote the Laplacian and gradient in $\mathcal{M}$, respectively, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, η is the outward unit normal to $\partial\Omega$, and α_i is a constant on each connected component $\Gamma_i \subset \partial\Omega$.

Let u be a solution to (2.6). Since $u \equiv 0$ on $\partial\Omega$ and $u > 0$ in the interior, it follows that u attains its maximum value at some point in the interior of Ω. Denote $M := u_{\text{max}} = \max_{p\in\Omega} u$, and define its level set as $\Gamma_M = u^{-1}(M) \subset \Omega$. We refer to Γ_M as the *top level set* of u.

Following the notation introduced in [22], we denote a solution to (2.6) by (Ω, u, f). If (Ω, u, f) is a solution and $\mathcal{I} \in \text{Iso}(\mathcal{M})$, then $(\mathcal{I}(\Omega), u \circ \mathcal{I}^{-1}, f)$ is also a solution to (2.6). This follows immediately from the fact that the first PDE in (2.6) is invariant under isometries. Domains that admit such solutions are called *f-extremal domains* (cf. [63]).

In particular, we are interested in characterizing solutions (Ω, u, f) to (2.6) that exhibit specific symmetry properties. We say that the triple (Ω, u, f) is $\mathcal{I}$-invariant if $\mathcal{I}(\Omega) = \Omega$ and $u(\mathcal{I}(p)) = u(p)$ for all $p \in \Omega$. In this context, the symmetry of solutions plays a crucial role in their classification. Let inj$(\mathcal{M})$ denote the injectivity radius of $\mathcal{M}$ (see [15]). We introduce the following notions:

- *Radial symmetry*: We say that (Ω, u, f) exhibits radial symmetry if there exists a point $p_0 \in \mathcal{M}$ and constants $0 \leq a < b \leq \text{inj}(\mathcal{M})$ such that $\Omega = d_{p_0}^{-1}((a, b))$, and $u = U \circ d_{p_0}$

for some function $U \in \mathcal{C}((a,b))$. Here, d_{p_0} denotes the distance function in $\mathcal{M}$ to the fixed point p_0.

In the case $\mathcal{M} = \mathbb{S}^n$, we also have:

- *Antipodal symmetry*: The triple (Ω, u, f) is said to have antipodal symmetry if $\mathcal{A}(\Omega) = \Omega$ and $u(p) = u(\mathcal{A}(p))$ for all $p \in \Omega$, where $\mathcal{A} : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map.

- *Toroidal symmetry*: We say that (Ω, u, f) has toroidal symmetry if it is invariant under the group of symmetries of a generalized Clifford torus, that is, if $\mathcal{I}(\Omega) = \Omega$ and $u(p) = u(\mathcal{I}(p))$ for all $p \in \Omega$ and every $\mathcal{I} \in SO(a) \times SO(b)$, for some $a + b = n - 1$.

We say that two triples (Ω, u, f) and $(\tilde{\Omega}, \tilde{u}, f)$ are *congruent* if there exists an isometry $\Phi : \mathcal{M} \to \mathcal{M}$ such that $\Phi(\Omega) = \tilde{\Omega}$ and $u(\Phi(p)) = \tilde{u}(p)$ for all $p \in \Omega$. In this case, we write $(\Omega, u, f) \equiv (\tilde{\Omega}, \tilde{u}, f)$.

3 Model and Exotic Solutions

This section develops two complementary classes of solutions to (2.6). First, in space forms we reduce (2.6) to the radial ODE (3.1) with Cauchy data (3.2), establish existence–uniqueness and monotonicity, and introduce ball/annulus model solutions (Def. 3.1). Then, on the sphere $\mathbb{S}^n$ we exploit isoparametric foliations, recast the equation as (3.8)–(3.9), and construct solutions whose top level sets are isoparametric hypersurfaces.

3.1 Model Solutions in Space Forms

A natural class of solutions to the overdetermined problem (2.6) arises in space forms of dimension $n \geq 2$; these solutions will be referred to as *model solutions*.

Fix an origin $o \in \mathcal{M}^n(k)$ and define the distance function

$$r(x) := \text{dist}(o, x), \quad \forall x \in \mathcal{M}^n(k).$$

Then, given any real-valued $\mathcal{C}^2$ function U, it follows from (2.2) that $u = U \circ r$ is a solution to the first equation of (2.6) if and only if U solves the differential equation

$$U''(r) + (n-1)\cot_k(r)U'(r) + f(U(r)) = 0. \tag{3.1}$$

Following [22, Section 4], we look for solutions to (3.1) satisfying

$$U(R) = M \quad \text{and} \quad U'(R) = 0, \tag{3.2}$$

for some $R \in [0, \bar{r})$ and $M \in \mathcal{I}_f$. Here $\mathcal{I}_f \subset \mathbb{R}_+^*$ denotes the set of admissible maxima of (3.1), i.e. $\mathcal{I}_f = (0, \bar{M})$ with $\bar{M}$ the maximal positive number such that f is positive on $(0, \bar{M})$. We have the following result, whose proof follows that of [22, Theorem 4.1].

Proposition 3.1. *Let $f \in \mathcal{C}(\mathbb{R})$ be a continuous function with non-empty $\mathcal{I}_f$. Then, for any $R \in [0, \bar{r})$ and $M \in \mathcal{I}_f$, there is a solution $U_{R,M}$ to* (3.1)-(3.2) *with the following properties:*

1. *$U_{R,M}$ is unique. Moreover, when $k > 0$, it follows that $U_{R,M}$ can be extended to a solution on $[0, \pi/\sqrt{k}]$ (in particular, $U_{\pi/2\sqrt{k},M}$ is symmetric with respect to $r = \pi/2\sqrt{k}$).*

2. *When $R \in (0, \bar{r})$, there exist numbers $0 < r_-(R, M) < R < r_+(R, M)$ such that $U'_{R,M} > 0$ on $[r_-(R, M), R)$ and $U'_{R,M} < 0$ on $(R, r_+(R, M)]$.*

   Also, when $R = 0 =: r_-(0, M)$, there exists a number $r_+(0, M)$ such that $U_{0,M}(r_+(0, M)) = 0$ and $U'_{0,M} < 0$ on $(0, r_+(0, M)]$.

3. *When $R \in (0, \bar{r})$, both functions $r_-(R, M)$ and $r_+(R, M)$ depend on R and M in a $\mathcal{C}^1$ manner. Moreover, both functions are nondecreasing in R.*

   Also, the functions $r_-(1, M)$ and $r_+(-1, M)$ depend $\mathcal{C}^1$ on $M > 0$.

If we suppose furthermore that f is $\mathcal{C}^1$ and:

$$f(x) \geq x f'(x), \quad \text{for all } x \in \mathcal{I}_f, \tag{3.3}$$

then we also have

4. *Fix $M \in \mathcal{I}_f$ and consider the real-valued functions*

$$g_\pm^M(R) := |\nabla u_{R,M}|^2 (r_\pm(R, M)), \quad \forall R \in [0, \bar{r}). \tag{3.4}$$

   Then, for any $M \in \mathcal{I}_f$, the functions g_-^M and g_+^M are decreasing and increasing on $[0, \bar{r})$, respectively.

For any $M \in \mathcal{I}_f$ and $R \in [0, \bar{r})$, it follows from items 1 and 2 of the above proposition that $U_{R,M} \circ r$ defines a radially symmetric solution to (2.6). Therefore, for any positive function f, we obtain a two-parameter family of radially symmetric solutions. These solutions were used in [22] as models to carry out comparison arguments for general solutions to the Dirichlet problem associated with (2.6). Since, in the next section, we will compare with radially symmetric solutions to Serrin's equation, this family will serve as our class of model solutions.

SI hacemos esto, la única solución modelo es la de Serrin, se pierde potencia para poder comparar con otras funciones, solo se puede caracterizar Serrin. Faltaria el punto 5 del caso de dim 2 en la porposicion anterior

Definition 3.1. *Let $k \in \mathbb{R}$, $R \in [0, \bar{r})$ and $M > 0$ with $M < -1/nk$ if $k < 0$. Using normal coordinates, define*

$$u_{R,M,k}(r, y) = U_{R,M,k}(r), \quad \forall y \in \mathbb{S}^{n-1},\ r \in [r_-(R, M), r_+(R, M)]$$

and

$$\Omega_{R,M,k} = \begin{cases} \{X(r,y) \in \mathcal{M}^n(k) \,:\, r \in (r_-(R,M), r_+(R,M))\}, & \textit{if} \quad R \in (0,\bar{r}) \\ \{X(r,y) \in \mathcal{M}^n(k) \,:\, r \in [0, r_+(R,M))\}, & \textit{if} \quad R = 0, \end{cases}$$

where X is the normal map defined in (2.1) *and $U^k_{R,M}$ is the solution to* (3.1)-(3.2) *with $f_k(x) = nkx + 1$. Then we say that the triple $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ is a* model solution.

We observe that, from item 1 of Proposition 3.1, it follows that when $k > 0$ the model solutions $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ are determined up to a reflection with respect to the hyperplane $\{x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$. Therefore, it is sufficient to restrict to $R \in [0, \pi/2\sqrt{k}]$ instead of $R \in [0, \pi/\sqrt{k}]$ in that case.

We observe that, from item 1 of Proposition 3.1, it follows that when $k > 0$ the model solutions $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ are determined up to a reflection with respect to the hyperplane $\{x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$. Therefore, it is sufficient to restrict to $R \in [0, \pi/2\sqrt{k}]$ instead of $R \in [0, \pi/\sqrt{k}]$ in that case.

A fundamental question is whether there exist *exotic solutions*, that is, solutions which are not the classical radially symmetric *model solutions*, but instead are organized around isoparametric hypersurfaces. In this section, we construct f-extremal domains (Ω, u, f) in $\mathbb{S}^n$, $n \geq 3$, for the overdetermined problem (2.6) such that the level sets of u are isoparametric hypersurfaces of $\mathbb{S}^n$, for a general positive function $f \in \mathcal{C}(\mathbb{R})$.

To construct such solutions, we consider a general foliation of $\mathbb{S}^n$ by isoparametric hypersurfaces $\{\Gamma_r\}_{r\in[-1,1]}$, where

$$\Gamma_r := \rho^{-1}(r),$$

with $\rho \in \mathcal{C}^\omega(\mathbb{S}^n)$ denoting the isoparametric function that defines the family $\{\Gamma_r\}_{r\in[-1,1]}$. Let $l \in \{1,2,3,4,6\}$ denote the number of distinct principal curvatures of a hypersurface in the family, and let m_1, m_2 denote their multiplicities. Recall that, by Theorem 2.1, if l is even, then a principal curvature k_i has multiplicity m_1 if and only if i is odd, and has multiplicity m_2 if and only if i is even. On the other hand, if l is odd, then all principal curvatures have the same multiplicity, i.e., $m_1 = m_2$.

We construct solutions to (2.6) that depend solely on the distance to one of the focal submanifolds Γ_1 or Γ_{-1}. Without loss of generality, we discuss only the case of Γ_1. Thus, define the analytic function $s(p) := \mathrm{dist}_{\mathbb{S}^n}(p, \Gamma_1)$, and let $V \in \mathcal{C}^2([0,\pi))$ be such that $v = V \circ s$ satisfies the first equation in (4.1). Since the isoparametric function ρ can be written as (cf. [13])

$$\rho(p) = \cos\big(l \cdot s(p)\big) := r, \tag{3.5}$$

it follows that v satisfies the PDE if and only if V solves the ordinary differential equation

$$V''(s) + \left((n-1)\cot(ls) - \frac{c}{l\sin(ls)}\right)V'(s) + f(V(s)) = 0, \tag{3.6}$$

where $c = l^2(m_2 - m_1)/2$.

Through a detailed analysis, similar to the methods developed in [22] and [24], we can establish the existence of solutions to the above equation that satisfy specific Cauchy data. This, in turn, guarantees the existence of f-extremal domains in $\mathbb{S}^n$ that are foliated by isoparametric hypersurfaces. We will state the main results without proof, deferring the technical details to the appendix.

When $R \in (0, \bar{r})$ the domain $\Omega_{R,M,k} \subset \mathcal{M}^n(k)$ has the topology of an annulus, and we denote its boundary components as

$$\Gamma^{\pm}_{R,M,k} = \left\{ X(r, y) \in \mathcal{M}^n(k) : \ r = r_{\pm}(R, M), \, y \in \mathbb{S}^{n-1} \right\}.$$

Write $\Gamma_{R,M,k} = \mathrm{Max}(u_{R,M,k})$ for the hypersurface of maximum points. We also denote by $\Omega^{\pm}_{R,M,k} \subset \Omega_{R,M,k} \setminus \Gamma_{R,M,k}$ the subdomains of $\Omega_{R,M,k}$ containing no maximum points, with $\mathrm{cl}(\Omega^{\pm}_{R,M,k}) \cap \partial\Omega_{R,M,k} = \Gamma^{\pm}_{R,M,k}$.

3.2 Exotic solutions in $\mathbb{S}^n$

Unlike the Euclidean or hyperbolic settings, the existence of nontrivial isoparametric hypersurfaces in $\mathbb{S}^n$ suggests that solutions may arise in compact domains beyond those with rotational symmetry. In particular, if Ω is foliated by isoparametric hypersurfaces, their symmetry can be used to construct solutions, as shown in [66, 70].

A fundamental question is whether there exist *exotic solutions*, that is, solutions that are not the classical radially symmetric *model solutions*, but instead are organized around isoparametric hypersurfaces. In this section, we construct f-extremal domains (Ω, u, f) in $\mathbb{S}^n$, with $n \geq 3$, for the overdetermined problem (2.6) such that the level sets of u are isoparametric hypersurfaces of $\mathbb{S}^n$, for a general positive function $f \in \mathcal{C}(\mathbb{R})$.

To construct such solutions, we consider a general foliation of $\mathbb{S}^n$ by isoparametric hypersurfaces $\{\Gamma_r\}_{r\in[-1,1]}$, where

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with $\rho \in \mathcal{C}^\omega(\mathbb{S}^n)$ denoting the isoparametric function that defines the family $\{\Gamma_r\}_{r\in[-1,1]}$. Let $l \in \{1, 2, 3, 4, 6\}$ denote the number of distinct principal curvatures of a hypersurface in the family, and let m_1, m_2 denote their multiplicities. Recall that, by Theorem 2.1, if l is even, then a principal curvature k_i has multiplicity m_1 if and only if i is odd, and has multiplicity m_2 if and only if i is even. On the other hand, if l is odd, then all principal curvatures have the same multiplicity, i.e., $m_1 = m_2$.

We construct solutions to (2.6) that depend solely on the distance to one of the focal submanifolds Γ_1 or Γ_{-1}. Without loss of generality, we discuss only the case of Γ_1. Thus, define the analytic function $s(p) := \mathrm{dist}_{\mathbb{S}^n}(p, \Gamma_1)$, and let $V \in \mathcal{C}^2([0, \pi))$ be such that $v = V \circ s$ satisfies the first equation of (4.1). Since the isoparametric function ρ can be written as (cf. [13])

$$\rho(p) = \cos\big(l \cdot s(p)\big) := r, \tag{3.7}$$

it follows that v satisfies the PDE if and only if V solves the ordinary differential equation

$$V''(s) + \left((n-1)\cot(ls) - \frac{c}{l\sin(ls)}\right)V'(s) + f(V(s)) = 0, \tag{3.8}$$

where $c = l^2(m_2 - m_1)/2$.

By a detailed analysis, similar to the methods developed in [22] and [24], we can establish the existence of solutions to the above equation that satisfy specific Cauchy data. This, in turn, guarantees the existence of f-extremal domains in $\mathbb{S}^n$ that are foliated by isoparametric hypersurfaces. We will state the main results without proof, deferring the technical details to the appendix.

Thus, we solve (3.8) with the Cauchy data

$$V(S) = M, \quad V'(S) = 0, \tag{3.9}$$

for given $M > 0$ and $S \in [0, \pi/l]$. By solving (3.8)-(3.9) we obtain a solution to the first equation of (2.6) with a local maximum on the set $\Gamma_{\cos(lS)} \subset \mathbb{S}^n$. To obtain a solution to (2.6), we need to show that a solution to (3.8)-(3.9) is positive on some interval containing S and vanishes at the extrema of this interval. We can show that such an interval exists, independently of l, m_1, m_2, S, and M.

Proposition 3.2. *Let $f \in \mathcal{C}(\mathbb{R})$ be a continuous function positive on $\mathbb{R}_+^*$. Then, for any $s \in [0, \pi/l]$ and $M > 0$, there is a unique solution $V_{S,M}$ to problem* (3.8)-(3.9)*.*

Furthermore, when $S \in (0, \pi/l)$, there exist numbers $0 < s_1(S,M) < S < s_2(S,M) < \pi/l$ such that $V_{S,M}(s_1(S,M)) = V_{S,M}(s_2(S,M)) = 0$, $V_{S,M} > 0$ on $(s_1(S,M), s_2(S,M))$, $V'_{S,M} > 0$ on $[s_1(S,M), S)$ and $V'_{S,M} < 0$ on $(S, s_2(S,M)]$.

Also, when $S = 0$ (resp. $S = \pi/l$) there is a number $s_2(0,M)$ (resp. $s_1(\pi/l, M)$) such that $V_{0,M}(s_2(0,M)) = 0$ and $V'_{0,M} < 0$ on $(0, s_2(0,M)]$ (resp. $V_{\pi/l,M}(s_1(\pi/l,M)) = 0$ and $V'_{\pi/l,M} > 0$ on $[s_1(\pi/l,M), \pi/l)$).

We leave the proof of this result to Appendix A. Observe that the solution $V_{S,M}$ to (3.8)-(3.9) given in the above proposition attains its maximum value at S. Thus, it follows that we can construct solutions to (2.6) whose top level set is one of the submanifolds of the family $\{\Gamma_r\}_{r\in[-1,1]}$.

Corollary 3.1. *Let $\{\Gamma\}_{r\in[-1,1]}$ be an isoparametric family contained in $\mathbb{S}^n$, let $\rho \in \mathcal{C}^\omega(\mathbb{S}^n)$ be the isoparametric function that defines it and take $f \in \mathcal{C}(\mathbb{R})$ positive on $\mathbb{R}_+^*$. Then, for each $R \in [-1,1]$ and $M \in \mathcal{I}_f$ there exists a solution $(\Omega_{R,M}, v_{R,M}, f)$ to* (2.6)*. If we write $S = (1/l)\arccos(R)$, then $v_{R,M}$ is defined as*

$$v_{R,M} = V_{S,M} \circ s(p), \quad \forall p \in \Omega_{R,M},$$

where

$$\Omega_{R,M} = \begin{cases} \{p \in \mathbb{S}^n : \; s_1(S,M) < s(p) < s_2(S,M)\}\,, & \textit{if} \quad R \in (0,1) \\ \{p \in \mathbb{S}^n : \; s(p) < s_2(0,M)\}\,, & \textit{if} \quad R = 0, \end{cases}.$$

Furthermore, we have that $\mathrm{Max}(v_{R,M}) = \Gamma_R$ *and* $(v_{R,M})_{\max} = M$.

Remark 3.1. *In [70], Shklover showed that for* $f(u) = \lambda u$ *with* $\lambda \in \mathbb{R}^+$ *or* $f(u) = 1$*, and given an isoparametric foliation* $\{\Gamma_r\}_{r\in[-1,1]} \subset \mathbb{S}^n$*, there exist* f*-extremal domains whose top-level set is one of the focal submanifolds of the family. This is stated in [70, Theorem 2] (see also [66] for a different proof of this result). Hence, Corollary 3.1 generalizes this result by constructing solutions to* (2.6) *for a general* $f \in \mathcal{C}(\mathbb{R})$*, positive on* $\mathbb{R}_+^*$*, whose top-level set is either an isoparametric hypersurface or a focal submanifold of an isoparametric family.*

Remark 3.2. *It follows from Corollary 3.1 and Subsection 2.3.1 that on* n*-dimensional spheres, there exist* f*-extremal domains with a wide variety of symmetries and topologies. In particular, model solutions are not the only solutions to* (2.6) *with a nontrivial top level set. Thus, [1, Theorem 2.3] does not generalize to the spherical setting.*

On the other hand, it also follows that the dichotomy proved in [22, Theorem A] does not hold in higher dimensions. In fact, there exist f*-extremal domains with a non-antipodally symmetric top level set that are not rotationally symmetric. For example, in* $\mathbb{S}^4$*, take* $\{\Gamma_r\}_{r\in[-1,1]}$ *to be the isoparametric foliation of Cartan's hypersurfaces (with* $l = 3$ *distinct principal curvatures). Then, for any* $r \in (-1,1)$*, Corollary 3.1 implies the existence of an* f*-extremal domain* (Ω, u, f) *with* $\mathrm{Max}(u) = \Gamma_r$*. As noted in Subsection 2.3.1,* Γ_r *is not rotationally symmetric (it is an orbit of the isotropy representation of* $SU(3)/SO(3)$ *in* $\mathbb{S}^4$*), and it is also not antipodally symmetric, since it is a level set of a polynomial of degree three.*

Remark 3.3. *As observed in [70], by constructing Cartan–Münzner polynomials that are invariant under one of the isometry groups* $K \subset \mathrm{SO}(n)$ *acting on the sphere, one can obtain examples of isoparametric foliations in the quotient space* $\mathbb{S}^n/K$*. In particular, when* $K = \{\pm 1\}$ *acting on* $\mathbb{S}^n$*,* $K = \mathbb{S}^1$ *on* $\mathbb{S}^{2n+1}$*, or* $K = \mathbb{S}^3$ *on* $\mathbb{S}^{4n+3}$*, it is possible to obtain examples of isoparametric foliations in certain real, complex, and quaternionic projective spaces, namely* $\mathbb{RP}^n$*,* $\mathbb{CP}^n$*, and* $\mathbb{HP}^n$*. In [70, Corollary 3], Shklover employs this construction to produce* λ*-extremal domains with non-homogeneous boundaries in some of these spaces. Proposition 3.2 shows that this result extends to the setting of* f*-extremal domains.*

4 Pseudo-radial Functions

In this section, $(\mathcal{M}, g)$ denotes a n-dimensional riemannian manifold satisfying the curvature bound $\mathrm{Ric} \geq (n-1)kg$, for some $k \in \mathbb{R}$. The goal here is to extend the comparison algorithm developed in [2] to the context of general Riemannian manifolds. To do so, given a $\mathcal{C}^2$-domain

$\Omega \subset \mathcal{M}$ and a real function f, we consider the general Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In [2], the authors study Serrin's equation in the plane, that is, the above equation in the setting $\mathcal{M} = \mathbb{R}^2$ with $f(u) = 2$. The method developed in that work involves associating to each solution (Ω, u, f) of (4.1) a corresponding model solution $(\bar{\Omega}, \bar{u}, f)$ that shares the same *Normalized Wall Shear Stress*—a normalized measure of the maximum normal derivative of the function along the boundary of the domain. A gradient estimate is then obtained by comparing the original solution (Ω, u, f) with its associated model solution $(\bar{\Omega}, \bar{u}, f)$ (see [2, Section 3]).

This same strategy was applied in [22] to address a more general semilinear equation on the 2-sphere $\mathbb{S}^2$, and more recently, the comparison results from [2] were extended to the n-dimensional setting in [1], using the same approach.

In all of these cases, the strategy relies on comparing a given solution to (4.1) with a model solution of the same equation. The key insight in our work is that this restriction is not necessary: the comparison arguments developed in [2] can still be applied even when the model solution comes from a different problem. In fact, our comparison family here consists of model solutions defined in space forms.

From this point onward, we fix $k \in \mathbb{R}$ and consider an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ satisfying the Ricci curvature lower bound Ric $\geq (n-1)kg$. We begin by extending the concept of the expected core radius to this setting. Then, we establish a gradient estimate for solutions to (4.1), from which various comparison results follow—some of which generalize the results presented in [1].

4.1 The expected core radius

Consider a solution (Ω, u, f) to (4.1), where f is a positive function on $\mathbb{R}_+$, and let $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$ be an open connected component. We aim to define a canonical association between the triple $(\mathcal{U}, u, f)$ and a component $(\Omega^{\pm}_{R,M,k}, u_{R,M,k}, f_k)$ of a model solution in $\mathbb{M}^n(k)$. The following definition extends [**?**, Definition 4.1] (see also [**?**]).

Definition 4.1. *Let* (Ω, u, f) *be a solution to* (4.1)*. We define the* Generalized Normalized Wall Shear Stress *(GNWSS) map* $\overline{\tau} : \pi_0(\Omega \setminus \text{Max}(u)) \to \mathbb{R}_+$ *as:*

$$\overline{\tau}(\mathcal{U}) := \max \left\{ \frac{\max_{\Gamma} |\nabla u|^2}{\frac{2}{n} \int_0^{u_{\max}} f(t)\, dt} \ : \ \Gamma \in \pi_0(\partial\Omega \cap \text{cl}(\mathcal{U})) \right\}. \tag{4.2}$$

If $\text{cl}(\mathcal{U}) \cap \partial\Omega = \emptyset$*, we set* $\overline{\tau}(\mathcal{U}) = 0$*.*

Note that
$$P(x) = |\nabla u|^2(x) + \frac{2}{n}\int_0^{u(x)} f(t)\,dt \tag{4.3}$$
is the P-function associated to problem (4.1) when $f' \leq f_k' = nk$, see [**?**, Lemma 1]. This explains the denominator of the fraction appearing in (4.2).

Although this normalization factor is not important from a technical point of view, it is worth noting that it reduces to
$$\frac{2}{n}\int_0^{u_{\max}} f_k(t)\,dt = ku_{\max}^2 + \frac{2}{n}u_{\max} = |\nabla u_{0,u_{\max}}|_{|\partial\Omega_{0,M}^k}$$
when (4.1) is a model solution; this follows from [**?**, Theorem B]. Thus, it follows that the above definition coincides with [22, Definition 5.1] when dealing with Serrin's equation in a space form.

When restricted to the connected components of a model solution, the GNWSS map depends solely on the two parameters defining the model solution. Accordingly, we define the functions
$$\overline{\tau}_{M,k}^{\pm}(R) := \overline{\tau}(\Gamma_{R,M,k}^{\pm}), \quad \forall R \in [0,\bar{r}), \tag{4.4}$$
and we set $\tau_k^0(M) := \overline{\tau}_{M,k}^{+}(0)$, for all $M \in \mathcal{I}_{f_k}$.

We recall that, when $k > 0$, we have that $\bar{r} = \pi/\sqrt{k}$, and the model solution $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ is symmetric with respect to the hyperplane $\{x_{n+1} = 0\}$. In this case, we define $\tau_k^{\infty}(M) := \overline{\tau}_{M,k}^{+}(\pi/2\sqrt{k}) = \overline{\tau}_{M,k}^{-}(\pi/2\sqrt{k})$ for all $M \in \mathbb{R}_+^*$. Then Item 4 of Proposition 3.1 implies that

- $\overline{\tau}_{M,k}^{-} : (0, \pi/2\sqrt{k}] \to [\tau_k^{\infty}(M), +\infty)$ is decreasing,
- $\overline{\tau}_{M,k}^{+} : [0, \pi/2\sqrt{k}] \to [\tau_k^{0}(M), \tau_k^{\infty}(M)]$ is increasing.

When $k \leq 0$ we have that $\bar{r} = +\infty$, and we require the following result:

Lemma 4.1. *Let $k \leq 0$ and $M \in \mathcal{I}_{f_k}$ be given. Then we have that*
$$\lim_{R\to+\infty} U'_{R,M,k}(r_-(R,M)) = \lim_{R\to+\infty} -U'_{R,M,k}(r_+(R,M)) < +\infty, \tag{4.5}$$
where $U_{R,M,k}$ is given in Definition 3.1 and $r_\pm(R,M)$ in item 2 from Proposition 3.1.

Proof. En proceso... □

For $k \leq 0$, define $\tau_\infty^k(M)$ as the limit (4.5). Then, again from Proposition 3.1, we have that

- $\overline{\tau}_{k,-}^{M} : (0, +\infty) \to (\tau_\infty^k(M), +\infty)$ is decreasing,
- $\overline{\tau}_{k,+}^{M} : [0, +\infty) \to [\tau_0^k(M), \tau_\infty^k(M))$ is increasing.

These observations lead us to define a correspondence between a general solution to (4.1) and our model solutions using the GNWSS map, following the approach in [2] (see also [22, Definition 5.2]).

Definition 4.2. *Let* (Ω, u, f) *be a solution to* (4.1) *and* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. *Then we define the* expected core radius map $\bar{R} : \pi_0(\Omega \setminus \mathrm{Max}(u)) \to [\tau_0^k(M), +\infty)$ *as follows:*

$$\bar{R}(\mathcal{U}) := \begin{cases} (\overline{\tau}^{+}_{u_{\max},k})^{-1}\,(\overline{\tau}(\mathcal{U})) & \text{if} \quad \overline{\tau}(\mathcal{U}) > \tau_k^{\infty}(u_{\max}) \\ (\overline{\tau}^{-}_{u_{\max},k})^{-1}\,(\overline{\tau}(\mathcal{U})) & \text{if} \quad \overline{\tau}(\mathcal{U}) \in (\tau_k^0(u_{\max}), \tau_k^{\infty}(u_{\max}))\,, \\ 0 & \text{if} \quad \overline{\tau}(\mathcal{U}) \le \tau_k^0(u_{\max}), \\ \bar{r}/2 & \text{if} \quad \overline{\tau}(\mathcal{U}) = \tau_k^{\infty}(u_{\max}). \end{cases} \tag{4.6}$$

We make some remarks regarding the case $\bar{R}(\mathcal{U}) = 0$ in the above definition, in comparison with [1, Definition 1.4]. Assume that $(\mathcal{M}, g)$ is a real analytic Riemannian manifold. When $f = f_k$, we have $\tau_k^0(M) \equiv 1$, and if $\mathcal{U} \in \Omega \setminus \mathrm{Max}(u)$ satisfies $\bar{R}(\mathcal{U}) = 0$, then $\overline{\tau}(\mathcal{U}) \le 1$. It then follows from [**?**, Theorem B] that $\bar{R}(\mathcal{U}) = 0$ if, and only if $(\mathcal{M}, g) = \mathbb{M}^n(k)$ and (Ω, u, f) is a ball solution to Serrin's problem.

When f is a general function, we do not expect $\tau_k^0(M) \equiv 1$. Indeed, if we assume that

$$f'(x) \le nk, \quad \forall x \in \mathbb{R}_+, \tag{4.7}$$

with strict inequality, then necessarily $\tau_k^0(M) > 1$. An interesting question is whether there exist solutions (Ω, u, f) to (4.1) containing a component $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ such that $\overline{\tau}(\mathcal{U}) \in (0, \tau_k^0(M))$. The following result—generalizing [**?**, Theorem B]—shows that the answer is negative when $\overline{\tau}(\mathcal{U}) \in (0, 1)$.

Theorem 4.1. *Let* $(\mathcal{M}, g)$ *be a real analytic manifold of dimension* n *with* $\mathrm{Ric} \ge (n-1)kg$, *and let* (Ω, u, f) *be a solution to* (4.1) *with* f *positive and satisfying* (4.7). *Suppose that there exists* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *such that* $\overline{\tau}(\mathcal{U}) \le 1$. *Then it must be* $\overline{\tau}(\mathcal{U}) = 1$, *which occurs if, and only if* $(\mathcal{M}, g) = \mathcal{M}^n(k)$, $f(x) = nkx + f(0)$, Ω *is a ball and* u *is radially symmetric.*

Proof. The proof follows the same arguments of [2, Section 2] (see also [1, Section 3]). Here, we adopt the approach of [**?**, Section 5].

If $\mathrm{cl}(\mathcal{U}) = \mathrm{cl}(\Omega)$, the result follows directly from [**?**, Theorem 5]. If $\mathcal{U} \subsetneq \Omega \setminus \mathrm{Max}(u)$, we will derive a contradiction. For any regular value $t \in (0, u_{\max})$, define the energy function

$$E(t) = \frac{1}{\left(\int_t^{u_{\max}} f(s)\,ds\right)^{\frac{n}{2}}} \int_{\{u=t\}\cap \mathrm{cl}(\mathcal{U})} |\nabla u|\, d\sigma_t,$$

where $d\sigma_t$ denotes the $(n-1)$-dimensional area element along $\{u = t\} \cap \mathrm{cl}(\mathcal{U})$. Arguing as in the proof of [**?**, Proposition 5.3], and using the P-function defined in (4.3), we conclude that $E(t)$ is non-increasing.

Moreover, from (4.7) we have

$$\int_t^{u_{\max}} f(s)\,ds \leq f(0)\,(u_{\max}-t) + \frac{nk}{2}\,(u_{\max}^2 - t^2), \quad \forall t \in (0, u_{\max}).$$

Therefore, following the arguments in the proof of [**?**, Theorem B], we deduce that

$$E(0) \geq \lim_{t \to u_{\max}} E(t) = +\infty,$$

which yields the desired contradiction. □

In the remainder of this section, we derive estimates for the gradient of a solution to (4.1), as well as for the mean curvature and volume of its level sets. These estimates will be obtained by comparing with the corresponding quantities associated to the model solutions introduced in Subsection 3.1. To formulate these estimates in full generality, we introduce the following definition, adapted from [22, Definition 5.4].

Definition 4.3. *Let* (Ω, u, f) *be a solution to* (4.1)*, let* $\mathcal{U} \in \pi_0\,(\Omega \setminus \mathrm{Max}(u))$ *and set* $M = u_{\max}$*. Let* $\bar{R} \in [0, 1]$ *be such that* $\overline{\tau}(\mathcal{U}) \leq \overline{\tau}(\Omega^i_{\bar{R},M,k})$ *for some* $i \in \{+, -\}$*. Then, we say that* $(\Omega^i_{\bar{R},M,k}, u_{\bar{R},M,k}, f_k)$ *is a* comparison triple *associated to* $(\mathcal{U}, u, f)$*. We will also say that* $u_{\bar{R},M,k}$ *is a* comparison function *associated to* u *inside* $\mathcal{U}$*.*

If $\bar{R} := R(\mathcal{U}) < +\infty$ *is the expected expected core radius of* $\mathcal{U}$ *and* $\Omega^i_{\bar{R},M,k} \subset \Omega_{\bar{R},M,k}$ *is such that* $\overline{\tau}(\Omega^i_{\bar{R},M,k}) = \overline{\tau}(\mathcal{U})$*, we will say that* $(\Omega^i_{\bar{R},M,k}, u_{\bar{R},M,k}, f_k)$ *is the* associated model triple *to* $(\mathcal{U}, u, f)$*. We will also say that* $u_{\bar{R},M,k}$ *is the* model function *associated to* u *inside* $\mathcal{U}$*.*

Note that when $k \leq 0$ and $\overline{\tau}(\mathcal{U}) = \tau_k^\infty(u_{\max})$, there is no model triple to associate to $(\mathcal{U}, u, f)$. Nevertheless, it is also possible to obtain sharp estimates for the triple $(\mathcal{U}, u, f)$ by comparing with model solutions via a limiting argument, as done in [2].

4.2 Gradient estimates

In this section we will consider solutions to (4.1) with $f \in \mathcal{C}^1(\mathbb{R})$ positive in $(0, +\infty)$, satisfying condition (4.7) and the normalization $f(0) = 1$. We recall that $(\mathcal{M}, g)$ satisfies the curvature condition $\mathrm{Ric} \geq (n-1)kg$. Let (Ω, u, f) be a solution to (4.1) and take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. Fix a comparison triple $(\bar{\mathcal{U}}, \bar{u}, f_k)$.

Remark 4.1. *From now on, we write* $M = u_{\max}$ *and* $\bar{R}(\mathcal{U}) = \bar{R}$*. Furthermore, we denote* $\bar{U} = U_{\bar{R},M,k}$*, which is the solution to the O.D.E.* (3.1) *defining* $\bar{u}$*, and denote also* $r_\pm(\bar{R}, M) = \bar{r}_\pm$*.*

We follow [22, Subsection 5.2]. Define the function $G : [0, M] \times [\bar{r}_-, \bar{r}_+] \to \mathbb{R}$ as

$$G(u, r) = u - \bar{U}(r).$$

Then we have that $\frac{\partial G}{\partial r}(u,r) = 0$ if, and only if $\bar{U}' = 0$, that is if, and only if $r = \bar{R}$, so the Implicit Function Theorem implies the existence of two $\mathcal{C}^3$-functions

$$\chi_- : [0,M] \to [\bar{r}_-, \bar{R}] \quad \text{and} \quad \chi_+ : [0,M] \to [\bar{R}, \bar{r}_+] \tag{4.8}$$

such that

$$G(u, \chi_\pm(u)) = 0 \quad \text{for all} \quad u \in [0,M].$$

Definition 4.4. *Let (Ω, u, f) be a solution to* (4.1) *and let $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. Let $(\bar{\mathcal{U}}, \bar{u}, f_k)$ be a comparison triple associated to $(\mathcal{U}, u, f)$. Then we define the pseudo-radial function Ψ relating the triples $(\bar{\mathcal{U}}, \bar{u}, f_k)$ and $(\mathcal{U}, u, f)$ as*

$$\Psi := \begin{cases} \mathcal{U} \to [\bar{r}_-, \bar{R}] & \\ \quad p \mapsto \Psi(p) := \chi_-(u(p)) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_\infty^k(u_{\max}), \\ & \\ \mathcal{U} \to [\bar{R}, \bar{r}_+] & \\ \quad p \mapsto \Psi(p) := \chi_+(u(p)) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) > \tau_\infty^k(u_{\max}). \end{cases}$$

Define the function

$$\bar{W} = |\nabla \bar{u}|^2 \circ \Psi = \bar{U}'(\Psi)^2 \tag{4.9}$$

and set

$$W = |\nabla u|^2 \ \text{in}\, \mathcal{U}. \tag{4.10}$$

We are now ready to state the main result of this section. Although the proof is omitted here, as the computations closely follow [1, Section 4] and [22, Subsection 5.5], we include it in Appendix B for the reader's convenience, particularly for those less familiar with the arguments in [2].

Theorem 4.2. *Let (Ω, u, f) be a solution to* (4.1) *with $f \in \mathcal{C}^1$, positive in $\mathbb{R}_+^*$, and satisfying condition* (4.7)*. Take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$, and consider $(\bar{\mathcal{U}}, \bar{u}, f_k)$ to be a comparison triple associated to $(\mathcal{U}, u, f)$. Then, it holds*

$$W(p) \leq \bar{W}(p) \text{ for all } p \in \mathcal{U}.$$

Moreover, if the equality holds at one single point of $\mathcal{U}$, then $(\mathcal{M}, g)$ is the space form $\mathcal{M}^n(k)$, $f = f_k$ and $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f_k)$.

This result is significant as it yields sharp estimates on the geometry of the level sets of a solution to (4.1). The consequences of Theorem 4.2 are presented in the following section. Many of the results stated there are generalizations of those in [1], and therefore, many proofs are only sketched. In some instances, however, we include the full computations in the appendix for completeness.

Remark 4.2. *It is clear from the proof of Theorem 4.2 that the rigidity statement can occur if, and only if $(\bar{\mathcal{U}}, \bar{u}, f)$ is the associated model triple to $(\mathcal{U}, u, f)$. Furthermore, it follows by analyticity that it must be $(\bar{\Omega}, \bar{u}, f_k) \equiv (\Omega, u, f)$, where $\bar{\Omega}$ is the domain of which $\bar{\mathcal{U}}$ is a connected component.*

4.3 Consequences of the gradient estimates

In this section, (Ω, u, f) will always denote a solution to (4.1) with f satisfying condition (4.7) and $f(0) = 1$, and $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ a connected component containing no maximum points.

We gather here several consequences of Theorem 4.2. First, we present a generalization—adapted to our setting—of the results established in [1]. Since the proofs follow very closely those in [1] (see also [22]), we omit the detailed arguments, providing brief comments when appropriate. We then establish an isoperimetric-type inequality, derived from the co-area formula, which represents a novel contribution of this work. Finally, we discuss the extension of the estimate for the location of hot spots obtained in [2, Section 5].

4.3.1 Curvature and volume estimates

Hay que reescribir esto en nuestro contexto

Here, we compare the curvature of the level sets of a general solution u within a connected component $\mathcal{U} \subset \Omega \setminus \mathrm{Max}(u)$, with those of a corresponding comparison function $\bar{u}$. The results obtained here constitute generalizations of those presented in [21, Subsection 4.2] to the higher-dimensional setting.

In contrast to the case when the dimension is $n = 2$, curvature estimates for the level sets of u within a component $\mathcal{U}$ become less effective in higher dimensions, particularly when $n > 3$. This limitation arises from the fact that we only have control over the Laplacian of u, which in turn restricts our ability to estimate only the mean curvature of the level sets. Nevertheless, by generalizing the results of [22, Proposition 5.1] and [22, Proposition 5.2], we are able to derive sharp estimates for the mean curvature of the zero level set and the regular part of the top stratum of u within $\mathrm{cl}(\mathcal{U})$.

Proposition 4.1. *Let (Ω, u, f) be a solution to (4.1) with $f \in \mathcal{C}^1$, positive in $\mathbb{R}^*_+$, and satisfying conditions (3.3) and (4.7). Write $u_{\max} = M$, take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ a connected component and $(\bar{\mathcal{U}}, \bar{u}, f)$ the associated model triple to $(\mathcal{U}, u, f)$. Let $p \in \partial\Omega$ be a point such that*

$$|\nabla u|^2(p) = \max_{\partial\Omega \cap \mathrm{cl}(\mathcal{U})} |\nabla u|^2 . \tag{4.11}$$

Then, if $H(p)$ denotes the mean curvature of $\partial\Omega$ at p with respect the inner orientation to $\mathcal{U}$, it holds

$$H(p) \leq -\frac{\bar{r}_+}{\sqrt{1-\bar{r}_+^2}} \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M) \quad \text{and} \quad H(p) \leq \frac{\bar{r}_-}{\sqrt{1-\bar{r}_-^2}} \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M),$$

where $\bar{r}_-$ and $\bar{r}_+$ are defined as the zeros of the solution to (**??**)-(3.2) *defining $(\bar{\Omega}, \bar{u}, f)$.*

Proof. Let $p \in \partial\Omega \cap \mathrm{cl}(\mathcal{U})$ be a point satisfying (4.11) and define $\gamma : [0, 2\pi) \to \mathbb{S}^n$ to be the unit speed geodesic with initial data $\gamma(0) = p$ and $\gamma'(0) = \nabla u / |\nabla u|(p)$. Then, for t close to zero, one has the Taylor expansions

$$W(\gamma(t)) = W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2)$$

and

$$\bar{W}(\gamma(t)) = \bar{W}(p) + 2\left(\mp(n-1)\frac{\bar{r}_\pm}{\sqrt{1-\bar{r}_\pm^2}}\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2),$$

proved in Lemma C.1, where we take $\bar{r}_+$ when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and $\bar{r}_-$ otherwise. Since (4.11) implies that $W(p) = \bar{W}(p)$, the result follows from Theorem 4.2 by comparing the first order terms of the previous expansions. □

Remark 4.3. *Observe that the above result only applies when $(\bar{\mathcal{U}}, \bar{u}, f)$ is the associated model triple to $(\mathcal{U}, u, f)$, because this is the only case in which a point $p \in \partial\Omega$ satisfying* (4.11) *exists.*

Now we derive an estimate that works for smooth solutions to (4.1). Although the regularity can certainly be relaxed, we only deal with smooth functions for the sake of simplicity.

Take $p \in \mathrm{Max}(u)$ to be a regular point of the top stratum of u. Then we have the following result.

Proposition 4.2. *Let (Ω, u, f) be a solution to* (4.1) *with $f \in \mathcal{C}^\infty$, positive in $\mathbb{R}_+^*$, and satisfying conditions* (3.3) *and* (4.7)*. Write $u_{\max} = M$, take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ a connected component and $(\bar{\mathcal{U}}, \bar{u}, f)$ a comparison triple with expected critical height $\bar{R} < 1$. Let $\Gamma \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$ be a smooth hypersurface (possibly with boundary) and $p \in \Gamma$. Then, if $H(p)$ denotes the mean curvature of Γ at p with respect the inner orientation to $\mathcal{U}$, it holds*

$$H(p) \leq \frac{\bar{R}}{\sqrt{1-\bar{R}^2}} \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M) \quad \text{and} \quad H(p) \leq -\frac{\bar{R}}{\sqrt{1-\bar{R}^2}} \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M).$$

Proof. First, we note that since $u = 0$ along $\partial\Omega$ and $u > 0$ inside Ω, it follows that $\Gamma \subset \Omega$. Thus, since $p \in \Gamma$ is a regular point, it exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Gamma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Gamma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_-. \end{cases}$$

Then, by Lemma C.2, we have the following expansions around $p \in \Gamma$:

$$W = f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4),$$

$$\bar{W} = f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\bar{R}}{3\sqrt{1-\bar{R}^2}}\right) s\right) + O(s^4),$$

where $H(p)$ is defined with respect to the normal pointing to $\mathcal{V}_+$ and in the expansion for $\bar{W}$ we take the positive sign when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and the negative one otherwise. Then the result follows as in the previous proposition, taking into account that $W \leq \bar{W}$ inside $\mathcal{V}_+$ by Theorem 4.2. □

As a consequence of the above result, we can extract a bound for the $\overline{\tau}$-function of smooth solutions to (4.1), as it is done in [22, Proposition 5.3].

Proposition 4.3. *Let* (Ω, u, f) *be a solution to* (4.1) *with* $f \in \mathcal{C}^\infty$*, positive in* $\mathbb{R}^*_+$*, and satisfying conditions* (3.3) *and* (4.7)*. Take* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$*, and suppose that* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$ *contains an* $(n-1)$*-dimensional smooth manifold* Γ*. Then it must be* $\overline{\tau}(\mathcal{U}) > 1$*.*

Proof. By contradiction, let us suppose that $\overline{\tau}(\mathcal{U}) \leq 1$. Then if we write $M = u_{\max}$, it follows that $(\Omega^-_{R,M}, u_{R,M}, f)$ is an admissible comparison triple for each $R \in [0,1]$. Then, for $R < 1$, Proposition 4.2 ensures that for each point $p \in \Gamma$ we have that

$$|H(p)| \geq \frac{R}{\sqrt{1-R^2}}.$$

Since this is true for each $R \in [0,1)$, making $R \to 1$ we conclude that the mean curvature of Σ is unbounded in p. But this is a contradiction with the fact that Γ is smooth. □

In this section we shall present some estimates for the area of the level sets of a given smooth solution (Ω, u, f) to (4.1), and also for the volume of a given domain $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$.

The first result is a generalization of the estimate established in [22, Proposition 5.4], extending it to the n-dimensional setting and to more general level sets. The second result, which constitutes a novelty of this work, is a consequence of the co-area formula. Both estimates are stated for smooth solutions to (4.1) whose top level set contains a hypersurface.

We begin with an estimate that relates the area of the $(n-1)$-dimensional part of the top-level set of a solution to (4.1) with the area of a given regular level set.

Proposition 4.4. *Let* (Ω, u, f) *be a solution to* (4.1)*, where* $f \in \mathcal{C}^\infty(\mathbb{R})$ *is a function positive in* $\mathbb{R}^*_+$ *and satisfying* (3.3) *and* (4.7)*. Let* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *be a connected component and* $(\bar{\mathcal{U}}, \bar{u}, f)$ *a comparison triple with critical height* $\bar{R} < 1$*.*

Let $t \in [0, M)$ *be a regular value of* u *and set* $\Gamma_t := u^{-1}(t)$*. Assume that the* $(n-1)$*-dimensional part of* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by* $\Gamma_M \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, consists in a (possibly disconnected) smooth hypersurface. Then,*

$$\mathcal{H}^{n-1}(\Gamma_M) \leq \begin{cases} \left(\dfrac{1-\bar{R}^2}{1-\chi_+(t)^2}\right)^{\frac{n-1}{2}} \mathcal{H}^{n-1}(\Gamma_t) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0 \\ \left(\dfrac{1-\bar{R}^2}{1-\chi_-(t)^2}\right)^{\frac{n-1}{2}} \mathcal{H}^{n-1}(\Gamma_t) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0, \end{cases} \tag{4.12}$$

where $\chi_\pm$ is given in (4.8). *Furthermore, equality holds in* (4.12) *if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.

Proof. The proof of this result follows closely that of [22, Proposition 5.4], where it was proved only for the case $t = 0$ in dimension 2. However, we present here a detailed proof to show how the arguments made for the zero level sets Γ_0 in [22, Proposition 5.4] work when a general level set Γ_t is considered.

First, define the vector field

$$\mathcal{X}(p) := \frac{1}{(1-\Psi^2)^{\frac{n}{2}}\bar{U}'(\Psi)}\nabla u \text{ for all } p \in \Omega \setminus \mathrm{Max}(u). \tag{4.13}$$

Then, using that u is a solution to (4.1), $\bar{U}$ solves (**??**) and $\nabla\Psi = \nabla u/\bar{U}'(\Psi)$, one can check that its divergence is given by

$$\mathrm{div}\mathcal{X} := \frac{f(u)}{(1-\Psi^2)^{\frac{n}{2}+1}\bar{U}'(\Psi)^3}(W - \bar{W}).$$

Now, since $u \in \mathcal{C}^\infty(\Omega)$, Sard's Theorem implies that the set of critical values of u consists of isolated points. Thus, almost all $t \in [0, M)$ is a regular value of u. Given $\varepsilon > 0$, define the set

$$\mathcal{U}_{t,\varepsilon} = \{p \in \mathcal{U} \,:\, t < u(p) < M - \varepsilon\},$$

where, as always, we denote $M = u_{\max}$. By the above observation, we can consider $\varepsilon > 0$ as small as we want and such that $M - \varepsilon$ is a regular value of u. For fixed $\varepsilon > 0$, the set $\partial\mathcal{U}_{t,\varepsilon}$ consists in a family of complete smooth hypersurfaces, and by applying the usual version of Divergence Theorem in $\mathcal{U}_{t,\varepsilon}$ to the vector field $\mathcal{X}$, given by (4.13), we obtain

$$\int_{\mathcal{U}_{t,\varepsilon}} \frac{f(u)}{(1-\Psi^2)^{\frac{n}{2}+1}\bar{U}'(\Psi)^3}(W - \bar{W})\, d\mu = \int_{\Gamma_t} \frac{-|\nabla u|}{(1-\Psi^2)^{\frac{n}{2}}\bar{U}'(\Psi)}\, d\sigma + \int_{\Gamma_{M-\varepsilon}} \frac{|\nabla u|}{(1-\Psi^2)^{\frac{n}{2}}\bar{U}'(\Psi)}\, d\sigma, \tag{4.14}$$

where ν is the inner unit normal to $\partial\mathcal{U}_{t,\varepsilon}$ and $d\mu$ and $d\sigma$ denote the n-dimensional and $(n-1)$-dimensional volume elements induced by the metric of $\mathbb{S}^n$, respectively. Note that $\nu = \nabla u/|\nabla u|$ along Γ_t and $\nu = -\nabla u/|\nabla u|$ along $\Gamma_{M-\epsilon}$.

We first analyze the boundary integrals in the previous identity. Along each (regular) level set Γ_t, the pseudo-radial function is constant, that is, $\Psi = \chi_\pm(t)$, where $\chi_\pm$ is given in (4.8). Hence, using that $W \leq \bar{W}$ by Theorem 4.2, we conclude

$$\int_{\Gamma_t} \frac{|\nabla u|}{(1-\Psi^2)^{\frac{n}{2}}\bar{U}'(\Psi)}\, d\sigma \begin{cases} \geq -\dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{(1-\chi_+(t)^2)^{\frac{n-1}{2}}} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq \dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{(1-\chi_-(t)^2)^{\frac{n-1}{2}}} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{4.15}$$

On the other hand, observe that taking a sequence $\varepsilon_n \to 0$ such that $M - \varepsilon_n$ is a regular value of u for all n, and using that $\lim_{x \in \mathcal{U}, x \to \mathrm{Max}(u)} \frac{W}{\bar{W}} = 1$ (this follows by the expansions given in Lemma C.2), we obtain

$$\lim_{n \to +\infty} \int_{\Gamma_{M-\varepsilon_n}} \frac{|\nabla u|}{(1-\Psi^2)^{\frac{n}{2}} \bar{U}'(\Psi)}\, d\sigma = \begin{cases} -\dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{(1-\bar{R}^2)^{\frac{n-1}{2}}} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{(1-\bar{R}^2)^{\frac{n-1}{2}}} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M), \end{cases} \tag{4.16}$$

Finally, observe that $\bar{U}'(\Psi)^3 < 0$ if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and $\bar{U}'(\Psi)^3 > 0$ if $\overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M)$, so we conclude that

$$\int_{\mathcal{U}} \frac{f(u)}{(1-\Psi^2)^{\frac{n}{2}+1} \bar{U}'(\Psi)^3} (W - \bar{W})\, d\mu \begin{cases} \geq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{4.17}$$

Making $\varepsilon_n \to 0$ in (4.14), we get (4.12) after substituting (4.15), (4.16) and (4.17).

For the rigidity statement, note that equality in (4.12) implies that the right hand side of (4.14) is equal to zero. But this implies that $W \equiv \bar{W}$ on $\mathcal{U}$, and then $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$ from the rigidity statement of Theorem 4.2. □

4.3.2 An isoperimetric inequality

In this section, we present an isoperimetric inequality for domains $\mathcal{U}$ bounded by an hypersurface of maximum points of a solution to (4.1).

We recall that, for a model solution $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ with $R \in [0, \bar{r})$, we write $\Omega_{R,M,k} \setminus \Gamma_{R,M,k} = \Omega^+_{R,M,k} \cup \Omega^-_{R,M,k}$, and we write $\mathcal{H}^l_k$ for the l-dimensional Hausdorff measure associated to the metric of the space form $\mathcal{M}^n(k)$. Then, as consequence of the estimate given in Proposition 4.4 and the coarea formula we obtain the following result.

Proposition 4.5. *Let* (Ω, u, f) *be a solution to* (4.1)*, where* $f \in \mathcal{C}^\infty(\mathbb{R})$ *is a function positive in* $\mathbb{R}^*_+$ *and satisfying* (4.7) *and such that* $f(0) = 1$*. Write* $M = u_{\max}$*, let* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *be a connected component and* $(\bar{\mathcal{U}}, \bar{u}, f_k)$ *a comparison triple with core radius* $\bar{R} > 0$*. Assume that the* $(n-1)$*-dimensional part of* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by* Γ_M*, consists in a (possibly disconnected)* $\mathcal{C}^3$*-hypersurface. Then we have that*

$$\frac{\mathcal{H}^n(\mathcal{U})}{\mathcal{H}^{n-1}(\Gamma_M)} \geq \begin{cases} \frac{\mathcal{H}^n_k(\Omega^+_{\bar{R},M,k})}{\mathcal{H}^{n-1}_k(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0 \\ \frac{\mathcal{H}^n_k(\Omega^-_{\bar{R},M,k})}{\mathcal{H}^{n-1}_k(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0. \end{cases} \tag{4.18}$$

Furthermore, equality holds in (4.18) *if, and only if* $(\mathcal{M}, g) = \mathcal{M}^n(k)$*,* $f = f_k$ *and* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$*.*

Proof. We will only consider the case in which $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$, and thus $\bar{\mathcal{U}} = \Omega^+_{\bar{R},M,k}$. The other case follows with exactly the same argument.

Observe that $u \in \mathcal{C}^\infty(\mathcal{U})$, thus Sard's Theorem implies that the set of critical values of u is finite, and we can write $\{t_1, \ldots, t_k\} \subset (0, M)$ for this set. Note that, for any $t \in (0, M) \setminus \{t_1, \ldots, t_k\}$, $\Gamma_t := u^{-1}(t)$ is a (possibly disconnected) smooth hypersurface. Then, since any level set of u has zero n-dimensional Hausdorff measure, the coarea formula (see [44]) implies that

$$\mathcal{H}^n(\mathcal{U}) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} \, d\sigma_t \right) dt,$$

where we set $t_0 = 0$ and $t_{k+1} = M$, and we write $d\sigma_t$ for the $(n-1)$-dimensional area element along each level set Γ_t. Then, recalling that $\bar{W} = (\bar{U}' \circ \Psi)^2$, where $\bar{U}$ is the solution to (3.1) defining the comparison function $\bar{u}$ and Ψ is given in Definition 4.4, by Theorem 4.2 we get that

$$\mathcal{H}^n(\mathcal{U}) \geq \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt = \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt, \tag{4.19}$$

where the last identity follows from the fact that $\bar{W}$ is always positive when restricted to Γ_t with $t < M$.

Now, observe that $\bar{W}$ is constant along any Γ_t and can be expressed in terms of the functions defined in (4.8) as

$$\bar{W} = \bar{U}(\chi_\pm(t))^2, \quad \forall t \in (0, M),$$

where we take χ_+ when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and χ_- in the other case. Hence, we conclude that

$$\begin{aligned} \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt &= \int_0^M \left(\frac{\mathcal{H}^{n-1}(\Gamma_t)}{|\bar{U}'(\chi_+(t))|} \, d\sigma_t \right) dt \\ &\geq \frac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k(\bar{R})^{n-1}} \int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|} \, dt, \end{aligned} \tag{4.20}$$

where we have used the estimate given in Proposition 4.4 for the last inequality.

Finally, noting that $\chi_+ : [0, M] \to [\bar{R}, \bar{r}_+]$ is monotonically decreasing (where $\bar{r}_+ = r_+(\bar{R}, M)$), we can perform the change of variable $t = \chi_+^{-1}(r)$ in the last integral of above to conclude that

$$\int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|} \, dt = \int_{\bar{R}}^{\bar{r}_+} s_k(r)^{n-1} \, dr = \frac{\mathcal{H}^n_k(\Omega^+_{\bar{R},M})}{\omega_{n-1}},$$

where $\omega_{n-1} = \mathrm{Vol}(\mathbb{S}^{n-1})$. Now (4.18) follows from (4.19) and (4.20), taking into account that $s_k(\bar{R})^{n-1} \cdot \omega_{n-1} = \mathcal{H}^{n-1}_k(\Gamma_{\bar{R},M,k})$.

For the rigidity statement, we note that equality in (4.18) implies equality in (4.19), and then we can argue as in the proof of Proposition 4.4. $\square$

4.3.3 Location of the hot-spots

In this section, we provide estimates for the distance to the hot spots of a solution (Ω, u, f) to (4.1) in terms of the expected core radius of the solution. These results generalizes those contained in [2, Section 5].

The study of the location of critical points of differentiable functions is a classical topic in geometric analysis, tracing back to the works of Gauss. More recently, Magnanini and Poggesi have obtained estimates for the distance to extremum points of solutions to certain differential equations posed in the Euclidean space.

Among other interesting results, the authors obtain in [**?**, Theorem 1.1] an estimate for the distance to the set of maximum points (the *hot spots*) of a solution to the Dirichlet problem associated with Serrin's equation (that is, (4.1) with $f(u) \equiv 1$) in Euclidean space. The proof of this result relies on a gradient estimate obtained via a suitable P-function (see [**?**, Lemma 2.2]). Subsequently, in [2, Section 5], the authors derive sharp estimates for the distance to the set of maximum points in terms of the expected core radius of a solution. The results in [2, Section 5] also rely on a gradient estimate for solutions to Serrin's problem, which is a particular case of Theorem 4.2 (see [2, Theorem 3.5]). These parallels naturally motivate the formulation of a generalization of [2, Theorem 5.1] in our current setting.

Proposition 4.6. *Let* (Ω, u, f) *be a solution to* (4.1) *with* $f \in \mathcal{C}^1$*, positive in* $\mathbb{R}^*_+$*, and satisfying conditions* (4.7) *and* $f(0) = 1$*. Take* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$*, and consider* $(\bar{\mathcal{U}}, \bar{u}, f_k)$ *to be a comparison triple associated to* $(\mathcal{U}, u, f)$*. Then, for any point* $p \in \mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$*, the following holds*

$$\mathrm{dist}(p, \partial\Omega \cap \mathrm{cl}(\mathcal{U})) \geq \begin{cases} \bar{R} - \bar{r}_+ & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0 \\ \\ \bar{r}_- - \bar{R} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0, \end{cases} \tag{4.21}$$

where $\bar{r}_-$ *and* $\bar{r}_+$ *are defined as the zeros of the solution to* (3.1)-(3.2) *defining* $(\bar{\Omega}, \bar{u}, f)$*.*

Furthermore, equality in (4.21) *holds if, and only if* $(\mathcal{M}, g) = \mathcal{M}^n(k)$*,* $f = f_k$ *and* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f_k)$*.*

The proof of the above result follows exactly as in [2, Theorem 5.1], so we omit it here. The generalization of [2, Theorem E] deserves a bit more of discussion.

First, we note that we can bound from above the distance from any point $x \in \Omega$ to the boundary of the domain as done in [**?**, Lemma 2.1]. This is the content of the following result:

Lemma 4.2. *Let* (Ω, u, f) *be a solution to* (4.1) *with* $f \in \mathcal{C}^1$*, positive in* $\mathbb{R}^*_+$*, and satisfying conditions* (4.7) *and* $f(0) = 1$*. Suppose that* Ω *is contained in a convex domain of* $\mathcal{M}$*, and define the function* $d(x) = \mathrm{dist}(x, \partial\Omega)$ *for all* $x \in \mathrm{cl}(\Omega)$*. Then we have that*

$$u(x) \geq \frac{2}{n} \cdot \frac{s_k\left(\frac{d(x)}{2}\right)^2}{s_k'(d(x))}, \quad \forall x \in \mathrm{cl}(\Omega).$$

Proof. The idea is to compare with a solution to Serrin's equation, following the approach used in the proof of [**?**, Lemma 2.1].

For any $x \in \Omega$, let $r_x(y) = \mathrm{dist}(x, y)$ denote the distance from x to y for all $y \in \mathrm{cl}(\Omega)$. Since the curvature bound $\mathrm{Ric} \geq (n-1)k\, g$ holds, we have

$$\Delta r_x(y) \leq (n-1)\cot_k\big(r_x(y)\big), \quad \forall\, y \in \mathrm{cl}(\Omega).$$

Let

$$W^R(r) = \frac{2\left[s_k\left(\frac{R}{2}\right)^2 - s_k\left(\frac{r}{2}\right)^2\right]}{n\, s_k'(r)}, \quad \forall r \in [0, R),$$

be the radial function defining the solution to Serrin's equation on a geodesic ball of radius R in $\mathbb{M}^n(k)$.

Since $f(x) \leq f_k(x) = nkx + 1$, it follows that $w^R := W^R \circ r_x$ is a subsolution to (4.1) on the geodesic ball of radius R centered at x, denoted $B_R(x)$. By the maximum principle, we conclude that

$$u \geq w^{d(x)} \quad \text{in} \quad B_{d(x)}(x),$$

which yields

$$u(x) \geq w^{d(x)}(x) = \frac{2}{n} \cdot \frac{s_k\left(\frac{d(x)}{2}\right)^2}{s_k'(d(x))}.$$

□

Para que la generalización del Theorem 5.2 tenga sentido, hace falta tener alguna normalización de las soluciones modelo como en [2].

A Appendix: Study of the solutions to (3.8)

This section is devoted to the proof of Proposition 3.2.

We recall the notation introduced in Section 3. Here, $\rho \in \mathcal{C}^\omega(\mathbb{S}^n)$ is an isoparametric function, and $f \in \mathcal{C}(\mathbb{R})$ is a continuous function that is positive in $(0, +\infty)$. We denote by $l \in \{1, 2, 3, 4, 6\}$ the number of distinct principal curvatures of the isoparametric foliation associated with ρ and by $0 < m_1, m_2 \leq n-1$ their respective multiplicities. Given $\Gamma \in \{\rho^{-1}(-1), \rho^{-1}(1)\}$, one of the focal submanifolds of the foliation, we define $s(p) = \mathrm{dist}_{\mathbb{S}^n}(p, \Gamma)$ for all $p \in \mathbb{S}^n$. Then, if V is a $\mathcal{C}^2$ real function such that $v = V \circ s$ satisfy the equation $\Delta w + f(w) = 0$ on $\mathbb{S}^n$, then the function V satisfy the equation

$$V''(s) + \left(l(\beta - 1)\cot(ls) - \frac{\alpha}{\sin(ls)}\right)V'(s) + f(V(s)) = 0, \tag{A.1}$$

where $\alpha = (m_2 - m_1)/2$ and $\beta = (n-1)/l$. Here we will study this equation for general positive numbers β, l and a real number α, assuming that $\alpha/l < \beta$.

We begin by analyzing the existence of solutions to the equation (A.1) with Cauchy data (3.9) when $S = 0$. Following [24, Section 3], we seek even solutions to this problem. The key observation is that any even solution to this equation, defined in a neighborhood of $s = 0$ and satisfying $V(0) = M > 0$, must fulfill the relation

$$V(s) = M - \int_0^s \frac{1}{\sin(ly)^{\beta-1}} \left(\int_0^y \sin(lx)^{\beta-1} \left(f(V(x)) - \frac{\alpha V'(x)}{\sin(lx)} \right) dx \right) dy, \qquad \text{(A.2)}$$

for all s in a neighborhood of 0.

As in [24, Subsection 3.2.1], for a non-negative integer k and $\varepsilon \in (0, \pi)$, we denote by $\mathcal{C}_e^k(\varepsilon)$ the space of even $\mathcal{C}^k$-functions on $[-\varepsilon, \varepsilon]$, equipped with the norm

$$\|V\|_{k,\varepsilon} = \sum_{i=1}^{k} \sup_{s\in[-\varepsilon,\varepsilon]} \left|V^{(i)}(s)\right|, \quad \forall V \in \mathcal{C}_e^k(\varepsilon).$$

We also define $\mathcal{C}_e(\varepsilon) := \mathcal{C}_e^0(\varepsilon)$. If $V \in \mathcal{C}_e^1(\varepsilon)$ satisfies the relation given in (A.2), we say that V is a $\mathcal{C}_e^1$-weak solution to (A.1)-(3.9). The existence of such solutions follows from a fixed-point argument as in [24, Lemma 3.5].

Lemma A.1. *For any $M > 0$ and $f \in \mathcal{C}_e(\varepsilon)$, there exists an unique solution $V_{0,M}$ to* (A.1) *with Cauchy data*

$$V_{0,M}(0) = M, \quad V_{0,M}'(0) = 0,$$

defined on $(-\pi, \pi)$. Moreover, the solution depends $\mathcal{C}^1$ on $M \in \mathbb{R}_+^*$.

Proof. Take $g, h \in \mathcal{C}_e(\varepsilon)$ with g Lipschitz and h such that $h(0) = 0$, and define the function

$$A(g,h)(s) = -\int_0^s \frac{1}{\sin(ly)^{\beta-1}} \left(\int_0^y \sin(lx)^{\beta-1} \left(g(x) - \frac{\alpha h(x)}{\sin(lx)} \right) dx \right) dy \qquad \text{(A.3)}$$

for all $s \in (-\pi, \pi) \setminus \{0\}$, and $A(g,h)(0) = 0$.

Claim A: We have that $A(g,h)$ is $\mathcal{C}^1$ and even in $(-\pi, \pi)$.

Proof of Claim A. First we note that

$$\int_0^s \frac{1}{\sin(ly)^{\beta-1}} \left(\int_0^y \sin(lx)^{\beta-1}\, dx \right) dy = \frac{s^2}{n} + O(s^3)$$

in a neighborhood of $s = 0$, and that

$$\lim_{s\to 0} \frac{\int_0^s \sin(lx)^{\beta-2}\, dx}{\sin(ls)^{\beta-1}} = \frac{1}{l(\beta-1)},$$

so we have that there exists $C_1 < 1$ such that

$$|A(f,g)(s)| \leq C_1 \left(s^2 \|g\|_{0,\pi} + s \|h\|_{0,\pi} \right), \quad \forall s \in (-\pi, \pi), \tag{A.4}$$

and $A(f,g)$ is well defined and continuous in $s = 0$. It is also straightforward that

$$|A(g,h)'(s)| \leq C_1 \left(s \|g\|_{0,\pi} + \|h\|_{0,\pi} \right), \quad \forall s \in (-\pi, \pi), \tag{A.5}$$

so it is clear that $A(g,h)$ is $\mathcal{C}^1$ in $(-\pi,\pi) \setminus \{0\}$. To check that $A(g,h)$ is $\mathcal{C}^1$ and even, we compute the limit of $A(g,h)'$ as s approaches to zero. Using l'Hoppital's rule we conclude that

$$\lim_{s \to 0} \frac{\int_0^s \sin(lx)^{\beta-1} \left(g(x) - \frac{\alpha h(x)}{\sin(lx)} \right) \, dx}{\sin(lx)^{\beta-1}} = -\frac{\alpha h(0)}{l(\beta-1)} = 0,$$

because $h(0) = 0$. This concludes the proof of the claim. □

Now define $\mathcal{A}$ as an operator $\mathcal{A} : \mathcal{C}_e^1(\varepsilon) \to \mathcal{C}_e^1(\varepsilon)$ given by

$$\mathcal{A}(V) = A(f(M+V), V'),$$

where $A(f(M+V), V')$ is given by (A.3). By Claim A, the operator $\mathcal{A}$ is well-defined. Moreover, from (A.4) and (A.5), we deduce that

$$\|\mathcal{A}(V)\|_{1,\varepsilon} \leq C_1 \left(\varepsilon \cdot \sup_{[M - \|V\|_{0,\varepsilon}, M + \|V\|_{0,\varepsilon}]} |f| + \|V'\|_{0,\varepsilon} \right), \tag{A.6}$$

for all $V \in \mathcal{C}_e^1(\varepsilon)$.

Let $B \subset \mathcal{C}_e^1(\varepsilon)$ be the closed ball of radius 1 centered at 0. Since $C_1 < 1$, it follows from (A.6) that $\mathcal{A}(B) \subsetneq B$ for sufficiently small ε. Consequently, by Schauder's Fixed Point Theorem, $\mathcal{A}$ has a fixed point $V_{0,M} \in B$. Furthermore, by Claim A, $V_{0,M}$ can be extended to $(-\pi,\pi)$. This establishes the existence of a $\mathcal{C}^1$-weak solution to (A.1)-(3.9). Now, since

$$\begin{aligned} V_{0,M}''(s) =& \frac{(\beta-1) l \cos(ls)}{\sin(ls)^\beta} \int_0^s \sin(lx)^{\beta-1} \left(f(V_{0,M}(x)) - \frac{\alpha V_{0,M}'(x)}{\sin(lx)} \right) \, dx \\ & - \left(f(V_{0,M}(s)) - \frac{\alpha V_{0,M}'(s)}{\sin(ls)} \right), \end{aligned} \tag{A.7}$$

for all s, it is clear that $V_{0,M}''$ is well defined and continuous in $(-\pi,\pi) \setminus \{0\}$. To see that $V_{0,M}$ is in fact a classical solution, we check that $V_{0,M}''$ is continuous in $s = 0$. Note that since $f(M) > 0$ and $V_{0,M}'(0) = 0$, it follows from

$$V_{0,M}'(s) = -\frac{1}{\sin(ls)^{\beta-1}} \int_0^s \sin(lx)^{\beta-1} \left(f(V(x)) - \frac{\alpha V_{0,M}'(x)}{\sin(lx)} \right) \, dx, \quad \forall s \in (0, \pi), \tag{A.8}$$

that $V_{0,M}$ ha a local maximum at $s = 0$, and then there exists $s_0 > 0$ such that $V'_{0,M} < 0$ in $(0, s_0)$. Hence, we have that it must be $\lim_{s\to 0} V'_{0,M}(s)/\sin(ls) = 0$ and by (A.7) we can compute using l'Hoppital's rule that

$$\lim_{s\to 0} V''_{0,M}(s) = -\frac{f(M)}{\beta},$$

and we thus conclude that $V''_{0,M}$ is continuous in $(-\pi, \pi)$.

Finally, the fact that $V_{0,M}$ depends $\mathcal{C}^1$ on M follows using the Implicit Function Theorem as in the proof of [24, Lemma 3.5].

□

Next, we prove that the solution $V_{0,M}$ is monotone within some interval, for each $M > 0$.

Lemma A.2. *Given any $M > 0$, let $V_{0,M}$ be the unique solution to* (A.1)-(3.9)*. Then there exists $s(M) \in (0, \pi)$ such that $V_{0,M}(s(M)) = 0$ and $V'_{0,M} < 0$ on $(0, s(M)]$.*

Proof. We follow the proof of [24, Lemma 3.6]. From (A.8), there exists $s_0 > 0$ such that $V'_{0,M}(s) < 0$ for all $s \in (0, s_0)$. Now, suppose that $V_{0,M} > 0$ on $(0, \pi)$. Then, we must have $V'_{0,M} < 0$ on $(0, \pi)$, because if there existed $s_1 \in (0, \pi)$ such that $V'_{0,M}(s_1) = 0$, then s_1 would be a local maximum by (A.1), which contradicts our assumption.

Thus, $V_{0,M}$ is a positive, monotonically decreasing function on $(0, \pi)$, therefore it must have a limit as $s \to \pi$. However, since $V'_{0,M} < 0$ on $(0, \pi)$, it follows that

$$\int_0^s \sin(x)^{n-1} \left(f(V(x)) - \frac{\alpha V'_{0,M}(x)}{\sin(x)} \right) dx > 0, \quad \forall s \in (0, \pi). \tag{A.9}$$

In particular, $V'_{0,M}$ is not integrable, which leads to a contradiction. Therefore, there must exist $s(M) > 0$ such that $V_{0,M}(s(M)) = 0$, and consequently, $V'_{0,M} < 0$ on $(0, s(M)]$. □

Now we analyze the Cauchy problem (A.1)-(3.2) with $S > 0$. In this case, the existence of a solution follows directly from classical differential equation theory (cf. [25]), so we only have to prove the monotonicity properties of the solution. The computations follow from [22, Section 4].

Note that a solution to (A.1)-(3.2) must satisfy the implicit relation

$$V(s) = M - \int_S^s \frac{1}{\sin(ly)^{\beta-1}} \left(\int_S^y \sin(lx)^{\beta-1} \left(f(V(x)) - \frac{\alpha V'(x)}{\sin(lx)} \right) dx \right) dy, \tag{A.10}$$

for all s in a neighborhood of S where the function is defined (compare with [22, (4.4)]). Using this relation, we can prove a the next result.

Lemma A.3. *If $S \in (0, \pi/l)$, then there exists points $0 < s_-(S,M) < S < s_+(S,M) < \pi/l$ such that $V_{S,M}(s_-(S,M)) = V_{S,M}(s_+(S,M)) = 0$, $V_{S,M} > 0$ inside $(s_-(S,M), s_+(S,M))$, $V'_{S,M} > 0$ inside $[s_-(S,M), S)$ and $V'_{S,M} < 0$ in $(S, s_+(S,M)]$. Also, if f is of class $\mathcal{C}^1$ then both functions $s_-(S,M)$ and $s_+(S,M)$ depend $\mathcal{C}^1$ on the parameters S and M.*

Proof. We only prove the existence and properties of $S < s_+(S,M) < \pi/l$; the other part is analogous. Denote by $(e_1, e_2) \subset (0, \pi/l)$ the maximal interval of definition of $V_{S,M}$. Note that

$$V'(s) = \frac{1}{\sin(ls)^{\beta-1}} \int_S^s \sin(lx)^{\beta-1} \left(f(V(x)) - \frac{\alpha V'(x)}{\sin(lx)} \right) dx, \tag{A.11}$$

for all $s \in (e_1, e_2)$, so in particular we have that $V'_{S,M}(s) < 0$ for $s > S$ such that $S - s$ is small enough. By contradiction, suppose that $V_{S,M}$ never vanishes on (S, e_2). Then, reasoning as in the proof of Lemma A.2, we have that it must be $V'_{S,M} < 0$ on (S, e_2). Suppose that $e_2 < 1$. Then, by the boundary behavior of differential equations, we have that $V_{S,M}$ or $V'_{S,M}$ goes to minus infinity as s approaches to e_2. But if $V'_{S,M} \to -\infty$ when $s \to e_2$, then from (A.11) it must be $V_{S,M} \to -\infty$, a contradiction. Then the existence of $s_+(S,M) \in (S, e_2)$ is ensured, and it is clear that $V'_{S,M} < 0$ in $(S, s_+(S,M))$. To prove that it must be $V'_{S,M}(s_+(S,M)) < 0$ for all S, M, we note that if $V'_{S,M}(s_+(S,M)) = 0$ then from (A.1) it would be $V''_{S,M}(s_+(S,M)) = -f(0) < 0$, a contradiction.

Finally, the derivatives of s_+ are well defined and continuous from the Implicit Function Theorem. □

B Appendix: Gradient estimates

In this appendix we present the proof of Theorem 4.2. We follow [22, Section 5.2].

Remark B.1. *Following the notation of [21], we will denote $\Psi_\pm = \Psi$ and $\chi_\pm = \chi$ and we will do the computations considering both possibilities at the same time. We will also denote the derivatives with respect to u with a dot, and the derivatives with respect to Ψ with a $'$.*

Along this section we will consider the function

$$\varphi(r) = \frac{f(u) - nr\bar{U}'(r)}{(1-r^2)\bar{U}'(r)^2}, \quad \forall r \in (\bar{r}_1, \bar{R}) \cup (\bar{R}, \bar{r}_2), \tag{B.1}$$

in order to make the computations easier to follow. Here, $\bar{U}$ is the solution to (**??**) defining the comparison function $\bar{u}$.

Straightforward computations using the definition of χ shows that

$$\dot{\chi}(u) = \frac{1}{\bar{U}'(\chi(u))} \tag{B.2}$$

and

$$\ddot{\chi}(u) = -\frac{\bar{U}''(\chi(u))}{\bar{U}'(\chi(u))^3} = \frac{\varphi(\chi(u))}{\bar{U}'(\chi(u))}. \tag{B.3}$$

Define the functions W and $\bar{W}$ as in (4.10) and (4.9) respectively. Then, using Bochner's identity, we get that En la siguiente formula, hay que pensar con una cota $Ric \geq k(n-1)$, que va bien con el sentido de las desigualdades

$$\Delta W = 2\left((n-1) - f'(u)\right)|\nabla u|^2 + 2\left|\nabla^2 u\right|^2.$$

On the other hand, using the identities (B.2) and (B.3), we get that Comprobar como cambia esta igualdad; da la impresion que sigue igual

$$\Delta \bar{W} = 2\left(\Psi(n-1)\bar{U}'(\Psi) - 2f(u)\right) + 2(n-1)\left(1 + \frac{n\Psi^2}{1-\Psi^2} - \frac{f(u)\Psi}{(1-\Psi^2)\bar{U}'(\Psi)} - \frac{f'(u)}{n-1}\right)|\nabla u|^2,$$

so we conclude that

$$\begin{aligned}\Delta(W - \bar{W}) =& 2\left|\nabla^2 u\right|^2 - 2\left(\Psi(n-1)\bar{U}'(\Psi) - 2f(u)\right) \\ & - \frac{2(n-1)}{1-\Psi^2}\left(n\Psi^2 - \frac{f(u)\Psi}{\bar{U}'(\Psi)}\right)|\nabla u|^2.\end{aligned} \tag{B.4}$$

Now we want to find an estimate for the term $|\nabla^2 u|^2$. The idea is to proceed as in the proof of [2, Theorem 3.5]. Noting that it must be

$$\left|\nabla^2 u + \varphi(\Psi) du \otimes du + \frac{1}{n}\left(f(u) - \varphi(\Psi)|\nabla u|^2\right) g_{\mathbb{S}^n}\right|^2 \geq 0, \tag{B.5}$$

we can isolate $|\nabla^2 u|^2$ to get the desired bound:

$$\begin{aligned}\left|\nabla^2 u\right|^2 \geq & -\varphi(\Psi)\left\langle \nabla(W - \bar{W}), \nabla u\right\rangle - \frac{n-1}{n}\varphi(\Psi)^2|\nabla u|^4 + 2\frac{n-1}{n}\varphi(\Psi)^2\bar{W}|\nabla u|^2 \\ & + \frac{1}{n}f(u)^2.\end{aligned} \tag{B.6}$$

Thus, by substituting (B.6) in (B.4), we get that

$$\begin{aligned}\Delta(W - \bar{W}) \geq & -2\varphi(\Psi)\left\langle \nabla(W - \bar{W}), \nabla u\right\rangle \\ & + 2\frac{n-1}{n}\varphi(\Psi)\left(f(u) - \varphi(\Psi)|\nabla u|^2\right)(W - \bar{W}).\end{aligned} \tag{B.7}$$

Now we want to get an elliptic inequality in $W - \bar{W}$ using the previous expression. To do this, we introduce the function $F_\beta = \beta(\Psi) \cdot (W - \bar{W})$ for a certain function $\beta > 0$, which will be determined later. Following the computations preformed in [22, Section 5.2], we get that

$$
\begin{aligned}
\Delta F_\beta \geq & \frac{2}{\bar{U}'(\Psi)}\left(\frac{\beta'(\Psi)}{\beta(\Psi)}-\varphi(\Psi)\bar{U}'(\Psi)\right)\langle \nabla F_\beta, \nabla u\rangle \\
& -\frac{f(u)}{\bar{U}'(\Psi)}\left(\frac{\beta'(\Psi)}{\beta(\Psi)}-2\frac{n-1}{n}\varphi(\Psi)\bar{U}'(\Psi)\right)F_\beta \\
& \frac{|\nabla u|^2}{\bar{U}'(\Psi)^2}\left(\left(\frac{\beta'(\Psi)}{\beta(\Psi)}\right)'-\left(\frac{\beta'(\Psi)}{\beta(\Psi)}\right)^2+3\varphi(\Psi)\bar{U}'(\Psi)\frac{\beta'(\Psi)}{\beta(\Psi)}-2\frac{n-1}{n}\varphi(\Psi)^2\bar{U}'(\Psi)^2\right)F_\beta.
\end{aligned} \tag{B.8}
$$

To conclude the proof of Theorem 4.2, it only remains to study the asymptotic behavior of $\bar{W}$ near $\mathrm{Max}(u)$. This is contained in the following lemma:

Lemma B.1. *Let $(\mathcal{U}, u)$ and $(\bar{\mathcal{U}}, \bar{u})$ be as in Theorem 4.2 and consider $\bar{W}$ defined in* (4.9)*. Then, if $p \in \mathrm{Max}(u)$, it holds:*

$$
\lim_{x\in\mathcal{U}, x\to p}\frac{\bar{W}}{u_{\max}-u}=2f(u_{\max}).
$$

Proof. The proof is a straightforward computation using l'Hoppital's rule. Since $u(p) = \bar{U}(\Psi(p))$ and $\bar{W}(p) = (1-\Psi^2(p))\bar{U}'(\Psi(p))^2$ for all $p \in \mathcal{U}$, writing $M = u_{\max}$ we have that

$$
\begin{aligned}
\lim_{x\in\mathcal{U}, x\to p}\frac{\bar{W}}{M-u} &= \lim_{\Psi\to\bar{R}}\frac{(1-\Psi^2)\bar{U}'(\Psi)^2}{M-\bar{U}(\Psi)} = \lim_{\Psi\to\bar{R}}\frac{-2\Psi\bar{U}'(\Psi)^2+(1-\Psi^2)\bar{U}'(\Psi)\bar{U}''(\Psi)}{-\bar{U}'(\Psi)} \\
&= \lim_{\Psi\to\bar{R}} 2(\Psi-n)\bar{U}'(\Psi)+2f(\bar{U}(\Psi)) = 2f(M),
\end{aligned}
$$

where in third identity we have used that $\bar{U}$ is a solution to (**??**). $\square$

Now define the function β as

$$
\beta(\Psi) := \frac{1}{\left|\bar{U}'(\Psi)\right|^{2\frac{n-1}{n}}}, \tag{B.9}
$$

which satisfies the relation

$$
\frac{\beta'(\Psi)}{\beta(\Psi)} = 2\frac{n-1}{n}\varphi(\Psi)\bar{U}'(\Psi)
$$

on $\mathcal{U}$. Then, by substituting (B.9) in (B.8), we get the elliptic inequality

$$
\Delta F_\beta - \frac{2(n-2)}{n\bar{U}'(\Psi)}\varphi(\Psi)\bar{U}'(\Psi)\langle \nabla F_\beta, \nabla u\rangle - \frac{2(n-1)}{n^2\bar{U}'(\Psi)^2}\varrho(\Psi)F_\beta \geq 0,
$$

where

$$\begin{aligned}\varrho(\Psi) &= \frac{\bar{U}'(\Psi)^2}{1-\Psi^2}(f'(u)-n) - \frac{(n+2)f(u)}{1-\Psi^2}\bar{U}''(\Psi)\\ &= (n+2)\bar{U}''(\Psi)^2 - n\bar{U}'''(\Psi)\bar{U}'(\Psi).\end{aligned}$$

Since ϱ is positive on $\mathcal{U}$ THIS NEEDS TO BE CHECKED, we conclude that F_β satisfies the maximum principle inside $\mathcal{U}$.

Now we study the value of

$$F_\beta = \frac{1}{\left|\bar{U}'(\Psi)\right|^{2\frac{n-1}{n}}}(W-\bar{W}) = \frac{|\nabla u|^2}{\left|\bar{U}'(\Psi)\right|^{2\frac{n-1}{n}}} - (1-\Psi^2)\left|\bar{U}'(\Psi)\right|^{\frac{2}{n}}$$

along $\partial\mathcal{U}$. Observe that, since $(\bar{\mathcal{U}}, \bar{u}, f)$ is a comparison triple, it is clear that $W(p) \leq \bar{W}(p)$ for each $p \in \partial\Omega \cap \mathrm{cl}(\mathcal{U})$. Hence, it follows that $F_\beta \leq 0$ along $\partial\Omega \cap \mathrm{cl}(\mathcal{U})$. On the other hand, observe that Lemma B.1 implies that

$$\begin{aligned}\lim_{p\in\mathcal{U}, p\to \mathrm{Max}(u)} \frac{|\nabla u|^2}{\left|\bar{U}'(\Psi)\right|^{2\frac{n-1}{n}}} &= \lim_{p\in\mathcal{U}, p\to \mathrm{Max}(u)} \frac{(1-\Psi^2)^{\frac{n-1}{n}}|\nabla u|^2}{\bar{W}^{\frac{n-1}{n}}}\\ &= \lim_{p\in\mathcal{U}, p\to \mathrm{Max}(u)} \frac{(1-\Psi^2)^{\frac{n-1}{n}}|\nabla u|^2}{(2f(u_{\max}))^{\frac{n-1}{n}}(u_{\max}-u)^{\frac{n-1}{n}}} = 0,\end{aligned}$$

where the last identity follows from [9, Corollary 2.3] (note that the proof of this result works for $\mathcal{C}^3$-functions, although it stated for smooth functions). Thus, we conclude that

$$\lim_{p\in\mathcal{U}, p\to \mathrm{Max}(u)} F_\beta(p) = 0,$$

so it follows that $F_\beta \leq 0$ along $\partial\mathcal{U}$. But then the maximum principle implies that $F_\beta \leq 0$ inside $\mathcal{U}$, and hence $W \leq \bar{W}$ in $\mathcal{U}$.

Now, if $W(p) = \bar{W}(p)$ for one single point $p \in \mathcal{U}$, then it follows that $W \equiv \bar{W}$ inside $\mathcal{U}$ by the strong maximum principle. Then, the inequality (B.7) must be an identity, so the norm in (B.5) must be identically zero and this implies that

$$\nabla^2 u = \frac{1}{n}\left(\varphi(\Psi)|\nabla u|^2 - f(u)\right) g_{\mathbb{S}^n} - \varphi(\Psi) du \otimes du.$$

But then it follows that the level sets of u inside $\mathcal{U}$ are totally umbilical, so they must be geodesic spheres centered in some point in $\mathbb{S}^n$ and we conclude that $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.

C Appendix: Taylor expansions on level sets

In this appendix, we prove two useful expansions for the functions $W = |\nabla u|^2$ and $\bar{W} = |\nabla u_{\bar{R},M}| \circ \Psi$ around their zero and top level sets. These expansions are employed in Subsection 4.3.1 to derive curvature estimates. We briefly recall the setting.

Let $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ and take $p \in \mathrm{cl}(\mathcal{U}) \cap \partial\Omega$, and consider $(\bar{U}, \bar{u}, f)$ to be a comparison triple associated to $(\mathcal{U}, u, f)$. Write $M = u_{\max}$.

Define $\gamma : [0, 2\pi) \to \mathbb{S}^n$ as the unit speed geodesic with initial data $\gamma(0) = p$ and $\gamma'(0) = \nabla u / |\nabla u| (p)$. First, we suppose that the function f appearing in (4.1) is only $\mathcal{C}^1$. Then we have the following expansions around p.

Lemma C.1. *Suppose that the function f appearing in* (4.1) *is of class $\mathcal{C}^1$, positive in $\mathbb{R}^*_+$ and satisfies condition* (3.3)*. Then, there exists $\varepsilon > 0$ such that for any $t \in [0, \varepsilon)$, the following expansions holds:*

$$\begin{aligned} W(\gamma(t)) &= W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2), \\ \bar{W}(\gamma(t)) &= \bar{W}(p) + 2\left(\mp(n-1)\frac{\bar{r}_\pm}{\sqrt{1-\bar{r}_\pm^2}}\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2), \end{aligned} \tag{C.1}$$

where we take $\bar{r}_+$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and $\bar{r}_-$ if $\overline{\tau}(\mathcal{U}) < \tau_0(M)$.

Proof. First, observe that $W \circ \gamma$ is a real valued function which is $\mathcal{C}^2$ in a neighborhood of $t = 0$. Thus, it exists $\varepsilon > 0$ such that its Taylor expansion is given by

$$W(\gamma(t)) = W(p) + \langle \nabla W, \nabla u / |\nabla u| \rangle (p)t + O(t^2),$$

for all $t \in (0, \varepsilon)$. Now, by [13, Theorem 3.3] it follows that

$$H = \frac{1}{(n-1)|\nabla u|^3}\left(\frac{1}{2}\left\langle \nabla |\nabla u|^2, \nabla u \right\rangle - |\nabla u|^2 \Delta u\right),$$

and then, since $\Delta u(p) = -f(0)$ from (4.1), we conclude that

$$W(\gamma(t)) = W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2),$$

which is the first of the expansions in (C.1).

Now, in order to compute the Taylor expansion of the $\mathcal{C}^2$ function $\bar{W} \circ \gamma$, we remind that

$$\nabla \bar{W} = 2\left((n-1)\Psi \bar{U}'(\Psi) - f(u)\right)\nabla u$$

and

$$\sqrt{\bar{W}} = \sqrt{1 - \Psi^2}\left|\bar{U}'(\Psi)\right|,$$

where $\bar{U}$ is such that $\bar{u} = \bar{U} \circ \Psi$. Then, we conclude that

$$\begin{aligned}\bar{W}(\gamma(t)) &= \bar{W}(p) + \left\langle \nabla \bar{W}, \nabla u / |\nabla u| \right\rangle (p) + O(t^2) \\ &= \bar{W}(p) + 2\left(\mp (n-1) \frac{\bar{r}_\pm}{\sqrt{1-\bar{r}_\pm^2}} \sqrt{\bar{W}}(p) - f(0) \right) \sqrt{\bar{W}}(p) t + O(t^2),\end{aligned}$$

where we take r_- if $\overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M)$ and r_+ if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$. The lemma is proved. □

Now take a smooth hypersurface $\Gamma \subset \mathrm{Max}(u)$, and consider $p \in \Gamma$. Note that it exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Gamma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Gamma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_-. \end{cases} \tag{C.2}$$

Observe that s is smooth in $\mathcal{V}_+$ and $\mathcal{V}_-$ by the regularity results of [43].

Now suppose that the function f appearing in (4.1) is smooth. Our goal is to extend the expansions from [22, Lemma 5.2] to the n-dimensional setting. While the expansions in that lemma are derived under the assumption that the solution to (4.1) is analytic, it is important to note that the arguments remain valid assuming only $\mathcal{C}^4$ regularity. In this appendix, however, we work with smooth functions for the sake of simplicity.

Lemma C.2. *Suppose that the function f appearing in* (4.1) *is smooth, positive in $\mathbb{R}_+^*$ and satisfies condition* (3.3). *Then for any $x \in \mathcal{V}_+$, the following expansions holds:*

$$\begin{aligned} W &= f(M)^2 s^2 \left(1 + (n-1) H(p) s\right) + O(s^4), \\ \bar{W} &= f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\bar{R}}{3\sqrt{1-\bar{R}^2}}\right) s\right) + O(s^4), \end{aligned} \tag{C.3}$$

where $H(p)$ is computed with respect to the inner normal to $\mathcal{U}$ and in the second expansion we take the positive sign when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and the negative sign otherwise.

Proof. First, write $z = \Psi - \bar{R}$, where Ψ is the pseudo-radial function defined in Definition 4.4. Using the equation (**??**), we compute the Taylor expansion of $M - u \circ \Psi$ as a function of the variable Ψ:

$$M - u = \frac{f(M)}{2(1-\bar{R}^2)} z^2 + \frac{(n+2)\bar{R} f(M)}{6(1-\bar{R}^2)^2} z^3 + O(z^4). \tag{C.4}$$

Next, recalling that $\bar{W} = (1-\Psi^2)\bar{U}'(\Psi)^2$, where $\bar{U}$ is the solution to (**??**) defining the comparison triple associated to $(\mathcal{U}, u, f)$, we get that

$$\bar{W}''(\bar{R}) = \frac{2f(M)^2}{1-\bar{R}^2} \quad \text{and} \quad \bar{W}'''(R) = \frac{6n\bar{R} f(M)^2}{(1-\bar{R}^2)^2}.$$

Thus, we compute the following Taylor expansion in a neighborhood of $z = 0$,

$$\frac{\bar{W}}{M-u} = 2f(M) + \frac{4(n-1)\bar{R}f(M)}{3(1-\bar{R}^2)}z + O(z^2). \tag{C.5}$$

Finally, using (C.4) we get that

$$z = \pm\sqrt{\frac{2(1-\bar{R}^2)}{f(M)}}\sqrt{M-u} + O(M-u),$$

where we take the positive sign if $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and the negative sign if $\overline{\tau}(\mathcal{U}) < \tau_0(M)$, so substituting this into (C.5) we conclude that

$$\bar{W} = 2f(M)(M-u) \pm \frac{4(n-1)\bar{R}\sqrt{2f(M)}}{3\sqrt{1-\bar{R}^2}}(M-u)^{3/2} + O((M-u)^2). \tag{C.6}$$

Next, observe that since f is of class $\mathcal{C}^\infty$ standard regularity results implies that u is smooth, and we have a Taylor expansion of u at $p \in \Gamma$ given by [9, Theorem 3.1] (which works also in the smooth setting, as remarked in [9, Remark 3.2]):

$$u = M - \frac{f(M)}{2}s^2 - \frac{(n-1)f(M)}{6}H(p)s^3 + O(s^4),$$

where $s \in \mathcal{C}^\infty(\mathcal{V}_+)$ is defined in (C.2). Then, we get that

$$\nabla u = -sf(M)\left(1 + \frac{(n-1)H(p)}{2}s + O(s^3)\right)\nabla s, \tag{C.7}$$

so taking into account that $|\nabla s| = 1$, we conclude that

$$W = f(M)^2 s^2\left(1 + (n-1)H(p)s\right) + O(s^4).$$

In addition, substituting (C.7) into (C.6), we have that

$$\bar{W} = f(M)^2 s^2\left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\bar{R}}{3\sqrt{1-\bar{R}^2}}\right)s\right) + O(s^4),$$

where we recall that the positive sign is taken if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and the negative is taken otherwise. This finishes the proof of the lemma. $\square$

References

[1] V. Agostiniani, C. Bernardini, S. Borghini, L. Mazzieri, *On the Serrin problem for ring-shaped domains: the n-dimensional case*, arXiv:2505.04458v1.

[2] V. Agostiniani, S. Borghini, L. Mazzieri, *On the Serrin problem for ring-shaped domains*, to appear on J. Eur. Math. Soc.

[3] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large,* Amer. Math. Soc. Transl., Vol. 21 no. 2 (1962), 412-416.

[4] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners*, Springer (2011).

[5] H. Berestycki, L. Caffarelli, L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Communications on Pure and Applied Mathematics, Vol. 50 (1997), 1089-1111.

[6] H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*. Bol. Soc. Bras. Mat., Vol. 22 (1991), 1-37.

[7] S. Borghini, *Static Black Hole Uniqueness for nonpositive masses*, Nonlinear Analysis, Vol. 220 (2022), paper no. 112843, 1-12.

[8] S. Borghini, *Symmetry results for Serrin-type problems in doubly connected domains*, Math. Eng., Vol. 5 no. 2 (2023), paper no. 027, 1-16.

[9] S. Borghini, P. T. Chrusciel, L. Mazzieri, *On the uniqueness of Schwarzschild-de Sitter spacetime*, to appear on Archive for Rational Mechanics and Analysis, (2023).

[10] A. Carlotto, *Free boundary minimal surfaces: a survey of recent results*, Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche, Serie 4, Vol. 86 no. 1 (2019), 103-121.

[11] E. Cartan, *Sur quelque familles remarquables d'hypersurfaces*, C.R. Congrès Math. Liège (1939), pp. 30-41.

[12] G. Ciraolo, L. Vezzoni, *On Serrin's overdetermined problem in space forms*, Manuscripta Math., Vol. 159 (2019), 445-452.

[13] T. E. Cecil, P. J. Ryan, *Geometry of Hypersurfaces*, Springer Monographs in Mathematics (2015).

[14] Quo-Shin Chi, *The isoparametric story, a heritage of Élie Cartan*, Proceedings of the International Consortium of Chinese Mathematicians (2018), 197-260, International Press, 2021.

[15] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Englewood Cliffs, New Jersey (1976).

[16] M. del Pino, F. Pacard, J. Wei, *Serrin's overdetermined problem and constant mean curvature surfaces*, Duke Math. J., Vol. 164 (2015), 2643-2722.

[17] B. Devyver, *Index of the critical catenoid* Geom. Dedicata, Vol. 199 (2019), 355-371.

[18] M. Domínguez-Vázquez, *Introduction to Isoparametric foliations*, notes from a minicourse at Universidade de São Paulo in June 2014.

[19] M. Domínguez-Vázquez, A. Enciso, D. Peralta-Salas, *Overdetermined boundary problems with nonconstant Dirichlet and Neumann data*, Anal. P.D.E., in press.

[20] J.M. Espinar, A. Farina, L. Mazet, *f-extremal domains in hyperbolic space*, Calc. Var., Vol. 60 (2021), paper no. 112, 1-19.

[21] J.M. Espinar, Diego A. Marín, *An overdetermined eigenvalue problem and the Critical Catenoid conjecture*, arXiv:2310.06705.

[22] J.M. Espinar, Diego A. Marín, *Overdermined problems over annular domains in the n-sphere*, en proceso.

[23] J.M. Espinar, J. Mao, *Extremal domains on Hadamard manifolds*, Journal of Differential Equations, Vol. 265 (2018), 2671-2707.

[24] J. M. Espinar, L. Mazet, *Characterization of f-extremal disks*, Journal of Differential Equations, Vol. 266 (2019), 2052-2077.

[25] J.M. Espinar, M. Vaiana, *Differential Equations: A Dynamical Systems Approach to Theory and Practice*, American Mathematical Society (2022).

[26] M. M. Fall, I. A. Minlend, T. Weth, *Serrin's overdetermined problem on the sphere* Calc. Var., Vol. 57 no. 3 (2018), paper no. 3, 1-24.

[27] A. Farina, A. Roncoroni, *Serrin's type problems in warped product manifolds*, Communications in Contemporary Mathematics, Vol. 24 no. 4 (2022), paper no. 2150020, 1-21.

[28] A. Fraser, R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*, Advances in Mathematics, Vol. 226 no. 5 (2011), 4011-4030.

[29] A. Fraser, R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math., Vol. 203 no. 3 (2016), 823-890.

[30] D. Gilbarg and N.-S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd Edition. *Springer Verlag*, New York, 1983.

[31] R. Goldman, *Curvature formulas for implicit curves and surfaces*, Computer Aided Geometric Design, Vol. 22 no. 7 (2005), 632-658.

[32] K. Grove, S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, Journal of Differential Geometry, Vol. 26 (1987), 429-459.

[33] P. Hartman, A. Winter, *On the Third Fundamental Form of a Surface*, American Journal of Mathematics, Vol. 75 no. 2 (1953), 298-334.

[34] F. Hélein, L. Hauswirth, F. Pacard, *A note on some overdetermined elliptic problem*, hal-00444828 (2010).

[35] H. Hopf, *Differential Geometry in the large*, Springer, Berlin, Lecture Notes in Math., Vol. 1000 (1983).

[36] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, Vol. 33 (1994).

[37] N. Kamburov, L. Sciaraffia , *Nontrivial solutions to Serrin's problem in annular domains,* Ann. I. H. Poincaré AN, Vol. 38 no. 1 (2021), 1-22.

[38] H. Karcher, *Umkreise und Inkreise konvexer Kurven in der spharischen und der hyperbolischen Geometrie*, Math. Ann., Vol. 177 (1968), 122-132.

[39] M. A. Karlovitz, *Some solutions to overdetermined b oundary value problems on sub domains of spheres*, Ph. D. Thesis. Univ. of Maryland, 1990.

[40] D. Khavinson, A.Y. Solynin, D. Vassilev, *Overdetermined Boundary Value Problems, Quadrature Domains and Applications*, Computational Methods and Function Theory, Vol. 5 no. 1 (2005), 19-48.

[41] D. Kinderlherer, L. Nierenberg, *Regularity in free boundary value problems*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e series, Vol. 4 no. 2 (1977), 373-391.

[42] S. G. Krantz, H. R. Parks, *A primer of real analytic functions.* Birkhäuser Advanced Texts: Basler-Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition (2002).

[43] S. G. Krantz, H. R. Parks, *Distance to $\mathcal{C}^k$ hypersurfaces*, Journal of Differential Equations, Vol. 40 (1981), 116-120.

[44] S. G. Krantz, H. R. Parks, *Geometric Integration Theory*, Birkhäuser Boston, MA, (2008).

[45] R. Kusner and P. McGrath, *On Steklov Eigenspaces for Free Boundary Minimal Surfaces in the Unit Ball*, arXiv:2011.06884.

[46] S. Kumaresan, J. Prajapat, *Serrin's result for hyperbolic space and sphere* Duke Mathematical Journal, Vol. 91 no. 1 (1998), 17-28.

[47] M. Li, *Free boundary minimal surfaces in the unit ball: recent advances and open questions*, Proceedings of the International Consortium of Chinese Mathematicians, 2017 (First Annual Meeting), 401-436.

[48] V. Lima, A. Menezes, *A two-piece property for free boundary minimal surfaces in the ball*, Transactions of the American Mathematical Society, Vol. 374 no. 3 (2021), 1661-1686.

[49] P. McGrath, *A Characterization of the Critical Catenoid*, Indiana University Mathematics Journal, Vol. 67 no. 2 (2018), 889-897.

[50] P. Mira, *Overdetermined elliptic problems in topological disks*, Journal of Differential Equations, Vol 264 (2018), 6994-7005.

[51] R. Miyaoka, *Isoparametric hypersurfaces with $(g, m) = (6, 2)$*, Annals of Mathematics, Vol 177 (2013), 53-110.

[52] S. Montiel, A. Ros, *Cuves and Surfaces*, Graduate Studies in Mathematics, Vol. 69 (2009).

[53] C. Nistch, C. Trombetti, *The classical overdetermined Serrin problem*, Complex Variables and Elliptic Equations, Vol. 63 no. 7-8 (2018), 1107-1122.

[54] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal., Vol. 89 no. 1 (1985), 1-19.

[55] F. Pacard, P. Sicbaldi, *Extremal domains for the first eigenvalue of the Laplace-Beltrami operator*, Ann. Inst. Fourier, Grenoble, Vol. 59 no. 2 (2009) 515-542.

[56] F. Pacella, *Symmetry Results for Solutions of Semilinear Elliptic Equations with Convex Nonlinearities*, Journal of Functional Analysis, Vol. 192 (2002), 271-282.

[57] F. Pacella, T. Weth, *Symmetry of solutions to semilinear elliptic equations via Morse index*, Proceedings of the American Mathematical Society, Vol. 135 no. 6 (2007), 1753-1762.

[58] P. Pucci, J. Serrin, *The Maximum Principle*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 73 (2007).

[59] G. Qiu, C. Xia, *Overdetermined Boundary Value Problems in $\mathbb{S}^n$,* J. Math.Study, Vol. 50 no. 2 (2017), 165-173.

[60] W. Reichel, *Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains*, Elliptic and Parabolic equations (1994), 164-182.

[61] W. Reichel, *Radial Symmetry for Elliptic Boundary-Value Problems on Exterior Domains*, Arch. Rational Mech. Anal., Vol. 137 no. 4 (1997), 381-394.

[62] A. G. Reznikov, *Linearization and explicit solutions of the minimal surface equation*, Publicacions Matematiques, Vol. 36 (1992), 39-46.

[63] A. Ros, P. Sicbaldi, *Geometry and topology of some overdetermined elliptic problems*, Journal of Differential Equations, Vol. 255 no. 5 (2013), 951-977.

[64] D. Ruiz, *Nonsymmetric sign-changing solutions to overdetermined elliptic problems in bounded domains*, arXiv:2211.14014.

[65] D. Ruiz, P. Sicbaldi, J. Wu, *Overdetermined elliptic problems in nontrivial contractible domains of the sphere*, to appear in J. Math. Pures Appliquées.

[66] A. Savo, *Geometric rigidity of constant heat flow*, Calc. Var., Vol. 57 no. 156 (2018).

[67] F. Schlenk, P. Sicbaldi, *Bifurcating extremal domains for the first eigenvalue of the Laplacian*, Advances in Mathematics, Vol. 229 (2012), 602-632.

[68] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, Vol. 44 (1993).

[69] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., Vol. 43 (1971), 304-318.

[70] V. E. Shklover, *Schiffer problem and isoparametric hypersurfaces*, Revista Matemática Iberoamericana, Vol. 16 no. 3 (2000).

[71] R. Souam, *Schiffer's problem and an isoperimetric inequality for the first buckling eigenvalue of domains on* $\mathbb{S}^2$, Ann. Global Anal. Geom., Vol. 27 no. 4 (2005), 341-354.

[72] J. Soucek, V. Soucek, *Morse-Sard theorem for real-analytic functions*, Commentationes Mathematicae Universitatis Carolinae, Vol. 13 no. 1 (1972), 45-51.

[73] P. Sicbaldi, *New extremal domains for the first eigenvalue of the Laplacian in flat tori*, Calc. Var., Vol. 37 (2010), 329-344.

[74] B. Sirakov, *Symmetry for exterior elliptic problems and two conjectures in potential theory*, Ann. Inst. Henri Poincaré, Anal. non linéaire, Vol. 18 no. 2 (2001), 135-156.

[75] M. Struwe, *Variational Methods*, A Series of Modern Surveys in Mathematics, Stringer, fourth edition (2008).

[76] M. Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane*, Geom. Funct. Anal. Vol. 24 (2014), 690-720.

[77] Q.-M. Wang, *On the topology of Clifford isoparametric hypersurfaces*, Journal of Differential Geometry, Vol. 27 (1988), 55-66.

[78] H. F. Weinberger, Remark on the preceding paper of Serrin. Arch. Rational Mech. Anal., Vol. 43 (1971), 319-320.