Torsional Rigidity Under Geometric Flows

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Set up

- (M^n, g) is a Riemannian manifold.
- $\Omega \subset M^n$ is a precompact domain with smooth boundary.

Dirichlet Problem

Given $\Omega \subset M^n$ there exists a function $u \in C_c^\infty(\Omega)$ such that

$$\begin{cases} \Delta u = -1 \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega. \end{cases}$$

Then we can consider $\mathcal{T}(\Omega) = \int_\Omega u \, dV_g$.

Understanding $\mathcal{T}(\Omega)$

• **Torsional rigidity** $\mathcal{T}(\Omega)$ give us the torque required when twisting an elastic beam with the domain of study as uniform cross section.

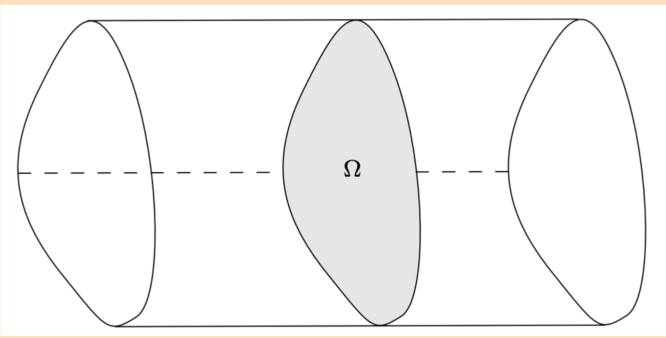

Figure: Sketch of a beam with cross section Ω

• **Mean exit time** expected time of a particle starting at a point $x \in \Omega$ will reach the boundary of the domain following a Brownian Motion.

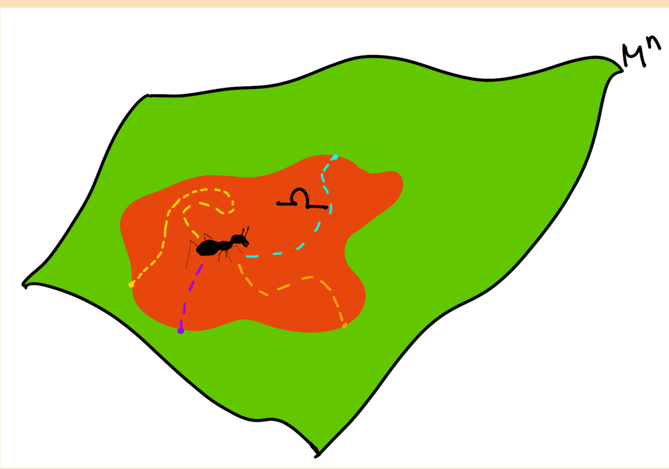

Figure: Example of a brownian ant.

Variational approach

$$\mathcal{T}(\Omega) = \sup \left\{ \frac{\left(\int_\Omega u \, dV\right)^2}{\int_\Omega \|\nabla u\|^2 \, dV} : \begin{array}{c} u \in C_0^\infty(\Omega) \text{ and} \\ \displaystyle\int_\Omega \|\nabla u\| \, dV \neq 0 \end{array} \right\}.$$

$$\mathcal{T}(\Omega) = \inf \left\{ \int_\Omega \|X\|^2 \, dV : X \in \mathfrak{X}(\Omega) \text{ with } \operatorname{div}_g X = -1 \right\}.$$

Intrinsic flows

Ricci flow, $g(t)$, are smooth paths in the space of metrics, $\mathcal{M}(M^n)$. The Ricci flow is a path governed by

$$\frac{\partial g_t}{\partial t} = -2Ric_g(t) \in T_{g_t}\mathcal{M}(M).$$

Extrinsic flow

Inverse Mean Curvature Flow let $\varphi : M^n \times [0, T_{max}] \mapsto \mathbb{R}^{n+1}$ a family of isometric immersions such that

$$\partial_t \varphi(p,t) = -\vec{H}_t(p) \left\| \vec{H}_t(p) \right\|^{-2}.$$

Then the evolution equations under this two flows are,

Ricci flow	IMCF
$\partial_t g_{ij} = -2\,\mathrm{Ric}_{ij}$	$\partial_t g_{ij} = \dfrac{2\,h_{ij}}{H}$
$\partial_t d\mu_t = -R\, d\mu_t$	$\partial_t d\mu_t = d\mu_t$
$\partial_t(\operatorname{div}_{g_t} X) = -X(R)$	Preserved

Theorem for Ricci flow

Let $(M^n, (g_t)_{t\in[0,T_{\max})})$ be a Ricci flow solution. Suppose that for any $t \in [0, T_{\max})$ and any $p \in M$ $\mathrm{Scal}_{g_t}(p) = b(t)$. Suppose moreover that for any $t \in [0, T_{\max})$, any $p \in M$ and any $v \in T_pM$, there exists $A, B : \mathbb{R} \to \mathbb{R}$ such that $A(t)g_t(v,v) \leq \mathrm{Ric}_{g_t}(v,v) \leq B(t)g_t(v,v)$. Then for any precompact domain $\Omega \subset M$ with smooth boundary

$$e^{\int_0^t(-b(s)-2B(s))ds}\mathcal{T}(\Omega_0) \leq \mathcal{T}(\Omega_t) \leq e^{\int_0^t(-2A(s)-b(s))ds}\mathcal{T}(\Omega_0).$$

Which implies

$$e^{-2\int_0^t B(s)ds}\frac{\mathcal{T}(\Omega_0)}{\mathrm{V}(\Omega_0)} \leq \frac{\mathcal{T}(\Omega_t)}{\mathrm{V}(\Omega_t)} \leq e^{-2\int_0^t A(s)ds}\frac{\mathcal{T}(\Omega_0)}{\mathrm{V}(\Omega_0)}.$$

- Let $(SU(2), g = \epsilon A_0 \Theta^1 \otimes \Theta^1 + B_0 \Theta^2 \otimes \Theta^2 + C_0 \Theta^3 \otimes \Theta^3)$ such that $\delta := \frac{B_0 - \epsilon A_0}{\epsilon A_0} < 1$.

$$\left[1 - \frac{4(1+6\delta)t}{B_0}\right]^{\frac{5(1+\delta)^3}{2(1+6\delta)}} \leq \frac{\mathcal{T}(\Omega_t)}{\mathcal{T}(\Omega_0)} \leq \left[1 - \frac{4t}{B_0}\right]^{\frac{5(1-\delta)}{2(1+\delta)}}$$

Theorem for IMCF

Let $\varphi : M \times [0, T_{\max}) \to \mathbb{R}^{n+1}$ by an inverse mean curvature flow from M to $\mathbb{R}^{n+1}$ such that $\varphi_t(M) = \Sigma$ is strictly convex hypersurface of $\mathbb{R}^{n+1}$.Thence the function

$$t \mapsto (\mathrm{V}(\Omega_t))^{-3} \cdot \mathcal{T}(\Omega_t)$$

is non-increasing for $t \in [0, T_{\max})$, and the function

$$t \mapsto (\mathrm{V}(\Omega_t))^{-1} \cdot \mathcal{T}(\Omega_t)$$

is non-decreasing for $t \in [0, T_{\max})$.

- Let Σ be a strictly convex free boundary disk-type hypersurface in the ball $\mathbb{B} \subset \mathbb{R}^{n+1}$. Then we have

$$\frac{\mathcal{T}(\Sigma)}{\mathrm{V}^3(\Sigma)} \geq \frac{\mathcal{T}(\mathbb{D})}{\mathrm{V}^3(\mathbb{D})}, \quad \frac{\mathcal{T}(\Sigma)}{\mathrm{V}(\Sigma)} \leq \frac{\mathcal{T}(\mathbb{D})}{\mathrm{V}(\mathbb{D})}$$

Bibliography

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