Seminar in Cobordism: Sculpting positivity: The artistry of connected sum & surgery of positive scalar curvature manifolds.

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1 Introduction

As we have seen in the previous talks we can perform some modifications on our manifolds, such as glue handles, perform surgery and connect manifolds. One question can arise during the last talks: What happen with the curvature? Can we preserve the scalar curvature during these process? The answer for this questions was given by Gromov and Lawson in the first theorem of the paper *Classification of simply connected manifolds.* It is stated as this theorem:

Main Theorem. *Let M^n be a compact manifold which carries a Riemannian metric with positive scalar curvature, from now and so on P.S.C. metric. Then:*

- *Any manifold which can be obtained from M^n by performing surgeries in codimension ≥ 3 also carries a metric with positive scalar curvature.*
- *Let M_1 and M_2 two compact manifolds with dimension greater or equal to 3 with P.S.C. metric, then their connected sum will also carry positive scalar curvature. The same is true of the connected sum along embedded spheres with trivial normal bundles in codimension ≥ 3.*

Therefore the presentation will follow the inverse structure of the theorem. The first section will show the proof of the case of the connected sum and the connected sum along embedded spheres, showing in detail the computations and examples for the connected sum. At the end of this talk we will present a sketch of the proof of the first part of the theorem. To begin the presentation, we will prove one technical lemma and show basic concepts about operators on manifolds and normal coordinates.

Definition 1.1. [3] Let $(M^n, \hat{g})$ be a Riemannian manifold and $N \subseteq M$ a submanifold, (N, g). Then we define the following operators:

- **Second fundamental form,** let $X, Y \in \mathfrak{X}(N)$ and $\hat{Y} \in \mathfrak{X}(M)$ be a extension of Y. Then

$$\begin{aligned}\mathbb{I} : \mathfrak{X}(N) \times \mathfrak{X}(N) &\to \Gamma(\nu N)\\ (X, Y) &\to pr_{\nu N}(\hat{\nabla}_X \hat{Y}).\end{aligned}$$

  If the submanifold N is of dimension $n-1$ we will have that $\mathbb{I} : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathbb{R}$.

- **Shape operator,** let $\eta \in \Gamma(\nu N)$ then,

$$\begin{aligned}L_\eta : \mathfrak{X}(N) &\to \mathfrak{X}(N)\\ X &\to L_\eta(X) = pr_{TN}(\hat{\nabla}_X \eta).\end{aligned}$$

This two operators are relate in the following way:

$$g(L_\eta X, Y) = -g(\nabla^\eta_X, Y) = g(\eta, \nabla_X Y) = \mathbb{I}(X, Y).$$

Proposition 1.2. *[3] Let (M, g) be a Riemannian manifold, $p \in M$ and consider $(x_1, x_2, \dots, x_n)$ normal coordinates of the neighbourhood U centered on p. Therefore the expression of the metric tensor, the Christoffel symbols and the curvature tensor respect these chart are:*

$$\begin{aligned}g_{ij} =& \delta_{ij} - \frac{1}{3} R_{kijl}(0) x^k x^l + \mathcal{O}(||x||^3)\\ \Gamma^k_{ij} =& -\frac{1}{3}\bigg(R_{kijl}(0) + R_{kijl}(0)x^l + \mathcal{O}(|x|^2)\bigg)\\ R_{ijkl} =& \frac{1}{2}\left[\frac{\partial^2 g_{jl}}{\partial x^l \partial x^k} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j}\right].\end{aligned}$$

Lemma 1.3. *[1][2][4] Let D be the glued disk in M^n and g_ε the induced metric on $\mathbb{S}^{n-1}(\varepsilon)$ and $g_{0,\varepsilon}$ be the standard euclidean metric of curvature $\frac{1}{\epsilon^2}$.*

- *The principal curvatures of the hypersurface $\mathbb{S}^{n-1} \subset D$ are each of the form $\frac{-1}{\varepsilon} + \mathcal{O}(\varepsilon)$ for a small ε.*
- *As $\varepsilon \to 0$ then $\frac{1}{\varepsilon^2} g_\varepsilon \to \frac{1}{\varepsilon^2} g_{0,\varepsilon} = g_{0,1}$ in the C^2-topology.*
- *The terms $\mathcal{O}(\varepsilon)$ only depends on the metric and it's derivatives.*

Proof. We are going to show first that the principal curvatures of $\mathbb{S}^{n-1}(\varepsilon)$ will be of the form of $\frac{-1}{\varepsilon} + \mathcal{O}(\varepsilon)$. First of all we will see that the Christoffel symbols are of $\mathcal{O}(||x||)$. By the definition this symbols:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{k.l}\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Then we can compute the derivative of any of the terms:

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^l} =& \frac{\partial}{\partial x^l}\left(\delta_{ij} - \frac{1}{3}R_{kijl}(0)x^k x^l + \mathcal{O}(||x||^3)\right)\\ =& 0 - \frac{1}{3}R_{kijl}(0)x^k + \mathcal{O}(||x||^3) = \mathcal{O}(||x||).\end{aligned}$$

For the term of g^{kl} we have that that we can write it as $g^{kl} = Id + A$ where A is the matrix with entrances $-\frac{1}{3}R_{kijl}(0)x^k x^l + \mathcal{O}(||x||^3)$. Using some elementary analysis:

$$\frac{1}{1+a} = 1 + \sum_{i=1}^{\infty}(-1)^i a^i.$$

Hence for g^{kl} we obtain that:

$$g^{kl} = Id - A + A^2 - A^3 \ldots = \delta_{ij} + \mathcal{O}(||x||^2).$$

Finally we obtain that,

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l (\delta_{ij} + \mathcal{O}(||x||^2))(\mathcal{O}(||x||) + \mathcal{O}(||x||) + \mathcal{O}(||x||)) = \mathcal{O}(||x||).$$

Now we define the curve that join the points $(0, \varepsilon, \ldots, 0)$ and $(\varepsilon, 0, \ldots, 0)$:

$$\begin{aligned}\alpha : [0, \varepsilon\frac{\pi}{2}] &\to \mathbb{S}^{n-1}(\varepsilon)\\ t &\to (\varepsilon \sin\frac{t}{\varepsilon}, \varepsilon\cos\frac{t}{\varepsilon}, 0, \ldots, 0).\end{aligned}$$

Hence:

$$\left(\frac{D}{ds}\dot{\alpha}(s)\right) = \sum_k \left(\alpha''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{d\alpha_i(t)}{dt}\frac{d\alpha_j(t)}{dt}\right)e_k$$

Therefore:

$$\begin{aligned}\left(\frac{D}{ds}\dot{\alpha}(s)\right)\Big|_{t=0} =& \Bigg[\left(-\frac{1}{\varepsilon}\sin\frac{t}{\varepsilon} + \sum_{i,j=1}^{2}\Gamma_{ij}^1(\alpha(t))\frac{d\alpha_i(t)}{dt}\frac{d\alpha_j(t)}{dt}\right)e_1\\ &+ \left(-\frac{1}{\varepsilon}\cos\frac{t}{\varepsilon} + \sum_{i,j=1}^{2}\Gamma_{ij}^2(\alpha(t))\frac{d\alpha_i(t)}{dt}\frac{d\alpha_j(t)}{dt}\right)e_2\Bigg]\Bigg|_{t=0}\\ =& \left(-\frac{1}{\varepsilon} + \sum_{i,j=1}^{2}\Gamma_{ij}^2(0, \varepsilon, 0 \ldots, 0)\frac{d\alpha_i(t)}{dt}\frac{d\alpha_j(t)}{dt}\right)\\ =& \left(-\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\right)e_2.\end{aligned}$$

Then using the equivalence between the second fundamental form and the shape operator:

$$\begin{aligned}\mathbb{I}(X(0), X(0)) =& - g(L(X(0)), X(0)) \\ =& - X(0)g(\eta, X(0)) + g(e_2, \nabla_{X(0)}X(0)) \\ =& - \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon).\end{aligned}$$

Then we can normalise $\dot{\alpha}(0)$ as $v = \frac{\dot{\alpha}(0)}{||\alpha||}$ therefore we obtain that:

$$\mathbb{I}(\dot{\alpha}(0)), \dot{\alpha}(0)) = ||\dot{\alpha}(0)||\mathbb{I}(v, v).$$

Hence we can compute the norm of $\dot{\alpha}(0)$:

$$\begin{aligned}g(\dot{\alpha}(0), \dot{\alpha}(0))_{\alpha(0)} =& g(e_1, e_1)_{\alpha(0)} \\ =& g_{11}(\alpha(0)) \\ =& \delta_{11} - \frac{1}{3}R_{k11l}(0)x^k x^l + \mathcal{O}(||x||^3) \\ =& 1 + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Therefore, we obtain that:

$$\begin{aligned}\Big(- \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\Big) =& \Big(1 + \mathcal{O}(\varepsilon^2)\Big)\mathbb{I}(v, v) \\ \Big(- \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\Big) + \varepsilon =& \mathbb{I}(v, v) \\ \mathbb{I}(v, v) =& \Big(- \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\Big)\end{aligned}$$

By an orthogonal change of coordinates we obtain that all of the principal curvatures of the submanifold $\mathbb{S}^{n-1}(\varepsilon)$ are of the form $\frac{-1}{\varepsilon} + \mathcal{O}(\varepsilon)$.

Since the space of metrics is a Frechet manifold we can define the C^2-norm that is given by

$$|| \cdot ||_{C^2} = || \cdot ||_{C^0} + \sum_{1 \le \alpha \le 2} ||\partial^\alpha \cdot ||_{C^0}$$

where $\alpha = (\alpha_0, \alpha_1)$ is a multi-index. We can define the contraction map, f_ε, between $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}(\varepsilon)$ given by $x \to \varepsilon x$. Thus,

$$\begin{aligned}\frac{1}{\varepsilon^2} f_\varepsilon^* g_\varepsilon(p) =& \sum_{i,j=1}^n g_{\varepsilon i,j}(\varepsilon p) dx_i dx_j \\ =& \sum_{i,j=1}^n (\delta_{ij} - \frac{\varepsilon^2}{3}R_{kijl}(0)x^k x^l + \mathcal{O}(||\varepsilon x||)) dx_i dx_j.\end{aligned}$$

Hence, when $\varepsilon \to 0$ therefore $\frac{1}{\varepsilon^2} f_\varepsilon^* g_\varepsilon(p) = \sum_i dx_i dx_i$. Moreover, as we have seen during this proof the terms depending of $\mathcal{O}(||x||)$ only depend on the metric and it's derivatives. $\square$

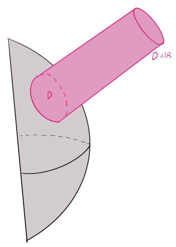

Figure 1: Example of the cylinder.

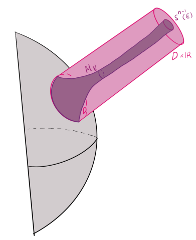

Figure 2: Example of the submanifold M_γ.

2 Connected sum

The case of the connected sum is the same as performing surgery in a 0-sphere. The rough idea of the connected sum is join two manifolds by a "tube". Since this operation is only defined up to diffeomorphisms we can play with the "tube" and find submanifolds inside of it with positive scalar curvature such that when they are far from the original manifold are isometric to a sphere with small radii.

The proof of this part of the theorem will start by computing the scalar curvature of a submanifold, M_γ, generated by a curve $\gamma(r)$ in $[0, \infty[$, where r is the distance between the point and the centre of the disk. Then we will construct this curve that satisfies the following properties:

- At at the origin is a vertical line between $\bar{r}$ and r_0, i.e. M_γ is isometric to (M, g) with $\bar{r} > r_0$.
- When the time is big enough γ reaches a plateau with radius ε_0, i.e. (t_0, ε_0), is isometric to $S^{n-1}(\varepsilon) \times \mathbb{R}$.
- The curvature of γ is positive.

Therefore, the generated submanifold:

$$M_\gamma = \{(x, t) \in D \times \mathbb{R} : (t, ||x||) \in graph(\gamma)\}$$

will generate a torpedo metric on our original manifold (M^n, g), i.e. g_τ is a metric of positive scalar curvature on D such that $Scal(g_\tau)\big|_{\partial D} = Scal(g)\big|_{\partial D}$ and is the standard round metric of $\mathbb{S}^{n-1} \times \mathbb{R}$ in the centre of D. To sum up this procedure, first we will compute the scalar curvature for this submanifold and then we will construct it having the obstruction that $\kappa^{M_\gamma} > 0$.

First of all we choose a point $p \in M$ and we define a normal chart around it using the disk of radius $r < inj(M)$, small enough, $D = \{v = \sum_{i=1}^{n} v_i e_i \in T_pM : ||v|| \leq r\}$ where e_i is an orthonormal basis of T_pM and the $\mathbb{S}^{n-1}(r) = \{v \in D : ||v|| = r\}$. Therefore in this setup we have that we can work nicely with normal coordinates and the principal curvatures of $\mathbb{S}^{n-1}(r)$ are of the form $-\frac{1}{r} + \mathcal{O}(r)$.

2.1 Scalar curvature of the submanifold

Now we are going to construct the curve with more detail. First we will compute the principal curvatures of a submanifold M_γ, where γ is a generic positive curvature curve. Once we have it we can compute the sectional curvatures using the Gauß equation. Finally, compute the scalar curvature:

$$\begin{aligned}\kappa^{M_\gamma} =&\kappa^{D\times\mathbb{R}} - 2\mathcal{R}ic\big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\big)\sin\theta^2 \\ &- 2(n-1)\big(\frac{1}{r} + \mathcal{O}(r)\big)\kappa\sin\theta \\ &+ (n-1)(n-2)\big(\frac{1}{r^2} + \mathcal{O}(r^2)\big)\sin\theta^2.\end{aligned} \tag{1}$$

2.1.1 The principal curvatures of M_γ

To compute the principal curvatures of the manifold first we will find a principal curve in M_γ. Let $l : [0,\infty) \to D\times\mathbb{R}$ be a geodesic ray emanating from the origin, i.e. a geodesic such that $d(l(a), l(b)) = |b-a|$ and $l(0) = o$. Therefore, $l\times\mathbb{R} \subseteq D\times\mathbb{R}$.

Definition 2.1. [3] Let (M^n, g) be a Riemannian manifold and $N \subseteq M$ a submanifold. We will say that N is totally geodesic if for every $p \in M^n$ and $v \in T_pM^n$ the geodesic $\gamma_v(s)$, such that $\dot{\gamma(0)} = v$ and $\gamma(0) = p$, is a geodesic in N.

Lemma 2.2. *Let $N \subseteq (M, g)$ a submanifold. N is totally geodesic then the second fundamental form of M vanish everywhere.*

Proof. Let $p \in M^n$ and $v \in T_pM^n$ the geodesic $\gamma_v(s)$, such that $\dot{\gamma(0)} = v$ and $\gamma(0) = p$, will satisfy that $D_t^M\dot{\gamma}_v = 0$. Since N is totally geodesic $D_t^N\dot{\gamma}_v = 0$, hence:

$$\mathbb{I}(v,v) = \mathbb{I}(v,\dot{\gamma}_v) = D_t^M\dot{\gamma}_v - D_t^N\dot{\gamma}_v = 0.$$

Then for any $v, w \in T_pM^n$:

$$\mathbb{I}(v,w) = \frac{1}{2}\big(\mathbb{I}(v+w, v+w) - \mathbb{I}(v,v) - \mathbb{I}(w,w)\big) = 0.$$

□

Then, we will use the previous lemma to show that $l\times\mathbb{R}$ is a totally geodesic submanifold.

Proposition 2.3. *$l\times\mathbb{R}$ is a totally geodesic submanifold of $D\times\mathbb{R}$. The curve $\gamma_l = M_\gamma \cap (l\times\mathbb{R})$ is a principal curve, as in the figure 3.*

Proof. First we will show that is a totally geodesic manifold. This comes from the fact that any geodesic in $l\times\mathbb{R}$ projects onto geodesics in l and $\mathbb{R}$. But $D\times\mathbb{R}$ is product of two Riemannian manifolds and so the curve is a geodesic in $l\times\mathbb{R}$.

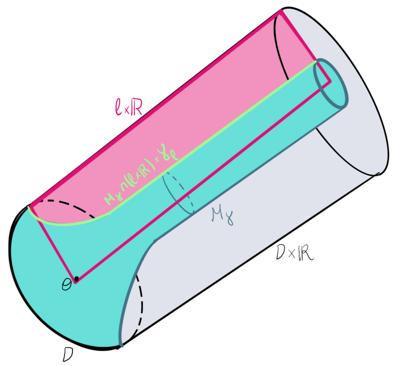

Figure 3: Sketch of the curve γ_l

Now we will show that γ_l is a principal curve of M_γ, i.e. the direction tangent to the curve γ is a principal direction of the manifold M_γ. The key point is to show that the shape operator on M_γ and on $l \times \mathbb{R}$ are equal. Since $T\gamma_l$ is a one dimensional tangent bundle we obtain that $\dot{\gamma}_l$ is a principal curve.

Let $\eta \in T_p(l \times \mathbb{R})$, such that $\eta = \eta_r + \eta_t$, where $\eta_r \in T_p l$ and $\eta_t \in T_p\mathbb{R}$. By the Gauß lemma we have that the η_l will be perpendicular to the level set $\gamma^{-1}(t_i)$. Thus η is a normal vector to γ_l and to M_γ.

Then we can define the shape operator respect this vector, η, as L^l and L^{M_γ},

$$\begin{aligned} L_\eta^{M_\gamma} \dot{\gamma}_l &= - \nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta \\ &= - \left(\nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta^T + \nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta^\perp \right) \\ &= - \nabla_{\dot{\gamma}_l}^{l\times\mathbb{R}} \eta = L_\eta^{l\times\mathbb{R}}(\dot{\gamma}_l) \end{aligned}$$

We have that $\nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta^\perp$ is part of $\nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta$ that lies in the tangent space of $l \times \mathbb{R}$. Since $l \times \mathbb{R}$ is a totally geodesic manifold then we will have that $\nabla_{\dot{\gamma}_l}^{D\times\mathbb{R}} \eta^\perp = 0$. Then we can see γ_l as a submanifold of $l \times \mathbb{R}$. That is one dimensional hence $\dot{\gamma}_l$ will be an eigenvector of $L_\eta^{\gamma_l}$. Therefore, by definition γ_l is a principal curve of M_γ. $\square$

We can define a basis of principal directions of M_γ in $T_{\gamma_l(t)} M_\gamma$ $\{e_1, e_2, \ldots, e_n\}$ such that:

- $e_1 = \dot{\gamma}_l(t)$.
- $e_2, e_3, \ldots, e_n$ are tangent to $\partial D = \mathbb{S}^{n-1}$.

To simplify the computations we will take $\eta = \cos\theta \partial_t + \sin\theta \partial_r$ where θ is the angle between the tangent to $\dot{\gamma}_l$ and the vertical vector. First we will compute the eigenvalue for any $j \in \{1, \ldots, n\}$:

$$\begin{aligned}\lambda_j =&\mathbb{I}(e_j, e_j) = g(L_\eta^{D\times\mathbb{R}}(e_j), e_j)\\ =&-g(\nabla_{e_j}^{D\times\mathbb{R}} \cos\theta\partial_t + \sin\theta\partial_r, e_j).\\ =&-(g(\nabla_{e_j}^{D\times\mathbb{R}} \cos\theta\partial_t, e_j) + g(\nabla_{e_j}^{D\times\mathbb{R}} \sin\theta\partial_r, e_j)).\end{aligned}$$

Hence, we will compute each of the e_j for $j \in \{2, \ldots, n\}$:

- $\nabla_{e_j}^{D\times\mathbb{R}} \cos\theta\partial_t = \cos\theta\nabla_{e_j}^{D\times\mathbb{R}}\partial_t + e_j(\cos\theta)\partial_t = 0.$

- $\nabla_{e_j}^{D\times\mathbb{R}} \sin\theta\partial_r = \sin\theta\nabla_{e_j}^{D\times\mathbb{R}}\partial_r + e_j(\sin\theta)\partial_r = \sin\theta\lambda_j^{\mathbb{S}^{n-1}(||x||)\times\mathbb{R}}.$

Where the eigenvalue $\lambda_j^{\mathbb{S}^{n-1}(||x||\times\mathbb{R})} = \bigg(-\frac{1}{||x||} + \mathcal{O}(||x||)\bigg)$, by the lemma 1.3. Now we are going to compute for each of the elements of the basis:

- $j = 1$ then $\lambda_1 = -\mathbb{I}(e_1, e_1) = g(L_\eta\dot{\gamma}_l, e_1) = \Bbbk$. Where $\Bbbk$ is the curvature of γ_l.

- $j \in \{2, 3, \ldots, n\}$ then $\lambda_j = \mathbb{I}(e_j, e_j) = \sin\theta\bigg(\frac{1}{||x||} + \mathcal{O}(||x||)\bigg).$

For future computations we will perform this one:

$$\begin{aligned}\sum_{i<j}\lambda_i\lambda_j =&\sum_j \lambda_1\lambda_j + \sum_{1<i<j}\lambda_i\lambda_j\\ =&-(n-1)\Bbbk\sin\theta\bigg(\frac{1}{||x||} + \mathcal{O}(||x||)\bigg)\\ &+\frac{(n-1)(n-2)}{2}\sin\theta^2\bigg(\frac{1}{||x||}^2 + \mathcal{O}(||x||)\bigg).\end{aligned} \tag{2}$$

2.1.2 The sectional curvature of M_γ

We will use the Gauß equation for the curvature:

$$g(R_{XY}^{M_\gamma}Z, W) = g(R_{XY}^{D\times\mathbb{R}}Z, W) + g(\mathbb{I}(X, W), \mathbb{I}(Y, Z)) - g(\mathbb{I}(X, Z), \mathbb{I}(Y, W))$$

Then for our basis we can compute the sectional curvature respect each of the components of the basis:

$$\begin{aligned}g(R_{e_ie_j}^{M_\gamma}e_j, e_i) =&g(R_{e_ie_j}^{D\times\mathbb{R}}e_i, e_j) + g(\mathbb{I}(e_i, e_i), \mathbb{I}(e_j, e_j)) - g(\mathbb{I}(e_i, e_j), \mathbb{I}(e_j, e_i))\\ =&R_{e_ie_je_je_i}^{D\times\mathbb{R}} + \lambda_i\lambda_j.\end{aligned}$$

First we will compute the curvature tensor terms that involve ∂_t and ∂_r:

1. $g(R_{\partial_t, e_j}e_j, \partial_t) = g(R_{\partial_t, e_j}\partial_t, e_j) = g(\nabla_{\partial_t}\nabla_{e_j}\partial_t - \nabla_{e_j}\nabla_{\partial_t}\partial_t - \nabla_{[\partial_t, e_j]}\partial_t) = 0$, since $[e_j, \partial_t] = 0$ and $\nabla_{e_j}\partial_t = \nabla_{\partial_t}\partial_t = 0$.

2. $g(R_{\partial_t,e_j}e_j,\partial_r) = g(R_{e_j,\partial_t}\partial_r, e_j) = g(R_{\partial_r,e_j}e_j,\partial_t)$ by the symmetries of the curvature tensor.

3. $g(R_{\partial_r,e_j}e_j,\partial_t) = 0$ since $R_{\partial_r,e_j}e_j \perp \partial_t$.

Now we will compute the sectional curvature in each of the cases:

- For $i = 1$ then we can write $e_1 = \cos\theta\partial_r + \sin\theta\partial_t$:

$$
\begin{aligned}
K_{1,j}^{M_\gamma} =& R^{D\times\mathbb{R}}_{e_1e_je_je_1} + k\lambda_j \\
=& g(R^{D\times\mathbb{R}}_{\cos\theta\partial_r+\sin\theta\partial_t,e_j}e_j, \cos\theta\partial_r + \sin\theta\partial_t) + k\lambda_j \\
=& g(R^{D\times\mathbb{R}}_{\cos\theta\partial_r,e_j}e_j, \cos\theta\partial_r) + g(R^{D\times\mathbb{R}}_{\cos\theta\partial_r,e_j}e_j, \sin\theta\partial_t) \\
&+ g(R^{D\times\mathbb{R}}_{\sin\theta\partial_t,e_j}e_j, \cos\theta\partial_r) + g(R^{D\times\mathbb{R}}_{\sin\theta\partial_t,e_j}e_j, \sin\theta\partial_t) + k\lambda_j \\
=& \cos\theta^2 g(R^{D\times\mathbb{R}}_{\partial_r,e_j}e_j,\partial_r) + \cos\theta\sin\theta g(R^{D\times\mathbb{R}}_{\partial_r,e_j}e_j,\partial_t) \\
&+ \cos\theta\sin\theta g(R^{D\times\mathbb{R}}_{\partial_t,e_j}e_j,\partial_r) + \sin\theta^2 g(R^{D\times\mathbb{R}}_{\partial_t,e_j}e_j,\partial_t) + k\lambda_j \\
=& \cos\theta^2 g(R^{D}_{\partial_r,e_j}e_j,\partial_r) + k\lambda_j
\end{aligned}
$$

  We can just substitute $\cos\theta^2 = 1 - \sin\theta$ then we have that $K_{1,j}^{M_\gamma} = g(R^D_{\partial_R,e_j}e_j,\partial_R) + k\lambda_j - \sin\theta^2 g(R^D_{\partial_R,e_j}e_j,\partial_R)$.

- For $i \neq 1$ then we have that $R^{M_\gamma}_{e_ie_je_je_i} = R^{\mathbb{R}\times D}_{e_ie_je_je_i} + \lambda_i\lambda_j$.

2.1.3 Compute the scalar curvature of M_γ

We will use the equivalence between the scalar curvature and the sectional curvature:

$$\kappa^{M_\gamma} = tr(Ric) = 2\sum_{i<j} K_{ij}^{M_\gamma}$$

Therefore respect our basis $\{e_1, e_2, \dots, e_n\}$:

$$
\begin{aligned}
\kappa^{M_\gamma} =& 2\sum_{i<j} K_{ij}^{M_\gamma} \\
=& 2\sum_{i<j} K_{ij}^{D\times\mathbb{R}} + \lambda_i\lambda_j \\
=& 2\sum_{j} K_{1j}^{D\times\mathbb{R}} + 2\sum_{i<j} K_{ij}^{D\times\mathbb{R}} + \lambda_i\lambda_j \\
=& -2\sum_{j} \sin\theta^2 K_{1j}^{D\times\mathbb{R}} + 2\sum_{i<j} K_{ij}^{D} \\
&- 2(n-1)\Bbbk\sin\theta\left(\frac{1}{||x||} + \mathcal{O}(||x||)\right) \\
&+ (n-1)(n-2)\sin\theta^2\left(\frac{1}{||x||}^2 + \mathcal{O}(||x||)\right) \\
=& -2Ric\left(\partial_r,\partial_r\right)\sin\theta^2 + \kappa^D.
\end{aligned}
$$

$$
\begin{aligned}
&- 2(n-1)\Bbbk \sin\theta\left(\frac{1}{||x||} + \mathcal{O}(||x||)\right) \\
&+ (n-1)(n-2)\sin\theta^2\left(\frac{1}{||x||}^2 + \mathcal{O}(||x||)\right).
\end{aligned}
$$

2.2 The construction of γ: The bending argument.

In this last part we will construct the curve γ that satisfies the conditions mentioned at the beginning of the section. Therefore, our curve γ will have the following structure:

1. The beginning will be a vertical curve that joins $\bar{r}$ with r_0, small:

$$\gamma_0(t) = (0, (1-t)\bar{r} + tr_0)$$

2. The curve γ_1 will perform a small bending and continue with a straight line and ending with a final bending joining the points $(0, r_0)$ to (t_0, ε_0).

3. The last part of γ will be an horizontal line:

$$\gamma_2(t) = (t, \varepsilon_0)$$

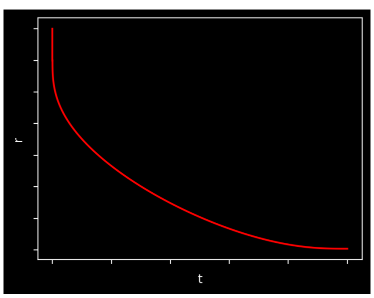

Figure 4: Example of the curve γ.

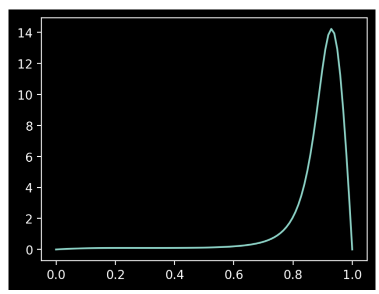

Figure 5: Example of the curvature of γ.

We will explain in detail the second part of the structure of γ. Since we want to control the curvature of γ we will work with r_0 and ε_0 small enough. To perform the first bend we will make that the scalar curvature during this bending will be a bump function, as in figure 6. Since all the terms in the formula of the scalar curvature of M_γ depends on $\Bbbk$ and θ we can control it, such that $\kappa^{M_\gamma} > 0$.

Once that we perform this bending we continue the curve with a straight line with a fixed angle θ_0 such that makes $\kappa^{M_\gamma} > 0$. When $r = ||x||$ is small enough and it is near ε_0 we perform the last bend. In this last bend we must

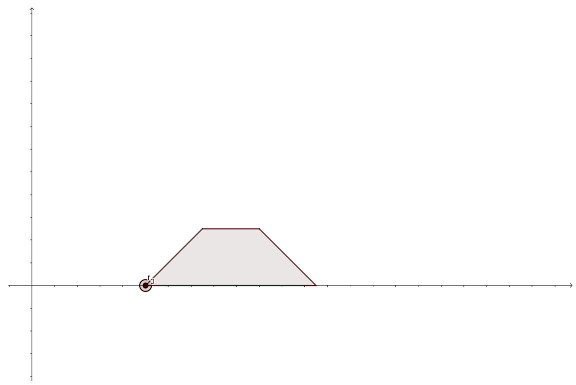

Figure 6: Example of the bump function for the bending argument.

take care about the angle of the curve and the curvature as we will see in the next derivation:

$$
\begin{aligned}
0 <& \kappa^{M_\gamma} \\
0 <& -2Ric\left(\partial_R, \partial_R\right)\sin\theta^2 + \kappa^D \\
& - 2(n-1)\Bbbk \sin\theta\left(\frac{1}{||x||} + \mathcal{O}(||x||)\right) \\
& + (n-1)(n-2)\sin\theta^2\left(\frac{1}{||x||}^{2} + \mathcal{O}(||x||)\right) \\
2(n-1)\Bbbk\sin\theta\left(\frac{1}{||x||} + \mathcal{O}(||x||)\right) <& -2Ric\left(\partial_R, \partial_R\right)\sin\theta^2 + \kappa^D \\
& + (n-1)(n-2)\sin\theta^2\left(\frac{1}{||x||}^{2} + \mathcal{O}(||x||)\right) \\
\Bbbk\left(1 + \mathcal{O}(||x||^2)\right) <& \frac{-2Ric\left(\partial_R, \partial_R\right)\sin\theta^2||x||}{2(n-1)\sin\theta} + \frac{||x||}{2(n-1)\sin\theta}\kappa^D \\
& + \frac{(n-1)(n-2)\sin\theta^2||x||}{2(n-1)\sin\theta||x||^2}\left(1 + \mathcal{O}(||x||^3)\right) \\
\Bbbk\left(1 + \mathcal{O}(||x||^2)\right) <& \frac{-Ric\left(\partial_R, \partial_R\right)\sin\theta||x||}{(n-1)} + \frac{||x||}{2(n-1)\sin\theta}\kappa^D \\
& + \frac{(n-2)\sin\theta}{2||x||}\left(1 + \mathcal{O}(||x||^3)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Bbbk\left(1+\mathcal{O}(||x||^2)\right) < &+\frac{||x||}{2(n-1)\sin\theta}\kappa^D \\
&\frac{\sin\theta}{2||x||}\left(-Ric\left(\partial_R,\partial_R\right)\frac{2||x||^2}{(n-1)(n-2)}\right. \\
&\left. +(n-2)(1+\mathcal{O}(||x||^3)\right)
\end{aligned}
$$

Therefore for $||x||$ small enough:

$$
\Bbbk \leq \frac{\sin\theta}{2||x||}.(n-2)
$$

Therefore if we want to find a curve of the form $\gamma(t) = (t, f(t))$ we can get a solution for this problem. First we are going to use the following equalities:

$$
\begin{aligned}
\sin\theta = &\frac{1}{\sqrt{1+\dot{f}^2}} \\
\Bbbk = &\frac{\ddot{f}}{\left(\sqrt{1+\dot{f}^2}\right)^3}
\end{aligned}
$$

Finally, if we get the equality we have a solution for the non-linear O.D.E.:

$$
\frac{\ddot{f}}{\left(\sqrt{1+\dot{f}^2}\right)^3} = \frac{1}{2\sqrt{1+\dot{f}^2}f(t)} \Rightarrow f(t) = \frac{c_1^2+1}{4c_2}x^2 + c_1x + c_2. \tag{3}
$$

Thus, we have describe detailed the construction of the curve γ that will create our manifold M_γ with positive scalar curvature. Hence we can end the section showing the part of the theorem about the connected sum of p.s.c. manifolds.

Theorem I. *Let (M, g_M) and (N, g_N) two manifolds with positive scalar curvature. Therefore, the connected sum of M and N, $M\sharp N$, carries positive scalar curvature.*

Proof. First of all we will glue to both manifolds (M, g_M) and (N, g_N) two submanifolds of $D\times\mathbb{R}$, M_γ and N^γ, with positive scalar curvature and both of them for really big $t \in \mathbb{R}$ will be isometric to $\mathbb{S}^{n-1}(\varepsilon)$. Therefore, we can glue both manifolds by identifying the tops of both submanifolds, that are $\mathbb{S}^{n-1}(\varepsilon_0)$,as in figure 7. □

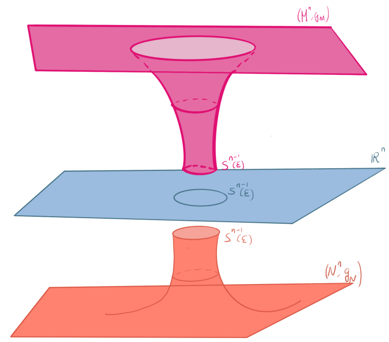

Figure 7: Example of the connected sum.

3 Surgery in codimension greater or equal to 3.

Theorem II. *Let (M^n, g_M) a compact and p.s.c. manifold. Any manifold (N, g_N) get by surgery in M in* ***codimension greater or equal to 3*** *will carry positive scalar curvature.*

The importance of the codimension 3 is because we can easily find a counterexample. We can obtain the 2-dimension torus via surgery in codimension 2 on the sphere. We have that the sphere has P.S.C. and the 2 dimensonal torus cannot get P.S.C..

The set up in this occasion is the embedded $\mathbb{S}^p \times D^q \hookrightarrow M^n$, such that $q = n - p \geq 3$. The proof of this theorem can be divided in two the first one is really similar to the connected sum, where we will create a submanifold $M_\gamma \subset \mathbb{S}^p \times D^q \times \mathbb{R}$ in a similar fashion as previously, and in the second part is related to the space of metrics with positive scalar curvature, where we will find a homotopy between our original metric and $D^{p+1} \times D^{q-1}$.

As in the case of the connected sum we will need a submanifold that carries p.s.c. and we will generate it in a similar fashion:

$$M_\gamma = \{(y, x, t) \in \mathbb{S}^p \times D^q \times \mathbb{R} : (||x||, t) \in graph(\gamma)\}$$

So in this case we will have some differences with the case of the connected sum, following the same structure as previously. The principal curvatures will be of the form:

$$\lambda_j = \begin{cases} k & \text{if } j = 1 \\ \sin\theta \cdot \left(-\frac{1}{r} + O(r)\right) & \text{if } 2 \leq j \leq q \\ \sin\theta \cdot O(1) & \text{if } q + 1 \leq j \leq n \end{cases}$$

Since the curvature of the p-dimensional sphere is bounded then $\lambda_j^{\mathbb{S}^p \times \mathbb{S}^{q-1}} = \mathcal{O}(1)$, for $j \in \{q, \ldots, n\}$. Therefore the product of the principals curvatures will be different:

$$
\begin{aligned}
\sum_{i<j}\lambda_i\lambda_j =& \sum_{j=2}^{q}\lambda_1\lambda_j + \sum_{j=q}^{n}\lambda_1\lambda_j + \sum_{1<i<j\leq q}\lambda_i\lambda_j + \sum_{q\leq i<j}\lambda_i\lambda_j + \sum_{1<i<q\leq j}\lambda_i\lambda_j \\
=& k\sum_{j=2}^{q}\sin\theta\cdot\left(-\frac{1}{r}+O(r)\right) + k\sum_{j=q}^{n}\sin\theta\cdot O(1) \\
&+ \sum_{1<i<j\leq q}\sin\theta^2\cdot\left(-\frac{1}{r}+O(r)\right)^2 + \sum_{q\leq i<j}\sin\theta^2\cdot O(1) \\
&+ \sum_{1<i<q\leq j}\sin\theta^2\cdot\left(-\frac{1}{r}+O(r)\right) \\
=& k(q-1)\sin\theta\cdot\left(-\frac{1}{r}+O(r)\right) + k(n-q)\sin\theta\cdot O(1) \\
&+ \frac{(q-1)(q-2)}{2}\sin\theta^2\cdot\left(-\frac{1}{r}+O(r)\right)^2 \\
&+ \frac{(n-q-1)(n-q-2)}{2}\sin\theta^2\cdot O(1) \\
&+ (q-1)\sin\theta^2\cdot\left(-\frac{1}{r}+O(r)\right)
\end{aligned}
$$

Thus the formula of the scalar curvature of M_γ will have the following structure:

$$
\begin{aligned}
\kappa^{M_\gamma} =& 2\sum_{i<j}K_{i,j} \\
=& 2\sum_{i<j}\left(K_{ij}^{\mathbb{S}^p\times D^q\times\mathbb{R}} + \lambda_i\lambda_j\right) \\
=& 2\sum_{i<j}\left(K_{ij}^{\mathbb{S}^p\times D^q\times\mathbb{R}} + \lambda_i\lambda_j\right) - 2\sin\theta^2\Bbbk\sum_{1<i<j}K_{1j}^{\mathbb{S}^p\times D^q\times\mathbb{R}} \\
=& \kappa^{\mathbb{S}^p\times D^q} + 2\sum\lambda_i\lambda_j + \sin\theta^2\mathcal{O}(1) \\
=& \kappa^{S^P\times D^q} + \sin^2\theta\cdot O(1) + k\cdot\sin\theta\cdot\left(-\frac{2\cdot(q-1)}{r}+O(1)\right) \\
&+ \sin^2\theta\cdot\left(\frac{(q-1)(q-2)}{r^2}+\frac{2\cdot(q-1)}{r}\cdot O(1)\right) + \sin\theta\cdot k\cdot(q-1)\cdot O(r).
\end{aligned}
$$

Then the construction of the curve γ is done in a similar way to the case of the connected sum. Therefore, we get our submanifold M_γ with p.s.c.. Now the objective is to homotope the metric from our $\mathbb{S}^p\times\mathbb{S}^{q-1}(\varepsilon)$ to the objective $D^{p+1}\times\mathbb{S}^{q-1}$ with p.s.c.. We can see the submanifold M_γ as the sum of two parts, the bottom part that contains the bending and the upper part that is of the form $M\times[0,1]$.

To find a homotopy between the two metrics first we need to understand why is difficult to construct this homotopy. We know that in two dimensions the connected component of the metrics of p.s.c. in $\mathcal{R}^+(\mathbb{S}^2)$ is constractible but if the dimension is greater than 7 we have infinite path components. Thus we have this lemma who can give us a radius for our top part of M_γ.

Lemma 3.1. *[2] Let M be a compact manifold. If $g_n \to g \in \mathcal{R}^+(M)$in the C^r topology, for $r \geq 1$. Then for some n we will have that g_n lies in the same connected components as g.*

Once that we have that both metrics lies in the same connected component then need to prove that we can go from g_ε to $g_{0,1}$ with p.s.c. on $[0,1] \times S^{p-1}(\varepsilon)$.

Lemma 3.2. *[1] Let $\{g_t\}_{t\in[0,1]}$ a continuous family of metrics on a compact manifold M. If for all t we have that $g_t \in \mathcal{R}^+(M)$ then there exists an $a_0 >$ 0such that for all $a > a_0$ the metric*

$$h^a = dt^2 + g_{\frac{t}{a}}$$

on $[0,a] \times M$ has P.S.C., where dt^2 is the usual metric in $\mathbb{R}$.

Proof. The main idea of this proof is to compute the Christoffel symbols. Then we will use it to compute the second fundamental form and the sectional curvature of the h^t.

Te set up of the problem is the following:

- Let $(M, g_t, \triangledown)$ the compact manifold equipped with the Levi-Civita connection.
- $(M \times [0,a], h^a, \tilde{\triangledown})$ the product manifold equipped with the product metric and the Levi-Civita connection on it.
- Let $(t,p) \in [0,a] \times M$ and $(x_0, x_1, \ldots, x_n)$ a chart of $U \in \mathcal{N}((t,p))$ where the x_0 is the time coordinate. Such that $\left(\frac{\partial}{\partial x^i}\right)_{i=0}^{n}$ is an orthonormal basis.

Respect to this chart this metrics looks like the following equations:

$$\begin{aligned}
g_{\frac{t}{a}}(x_0, x_1, \ldots, x_n) &= \sum_{i,j=1}^{n} \left(g_{\frac{t}{a}}\right)_{ij}(x_0, \ldots, x_n) dx^i dx^j \\
&= \sum_{i,j} g_{ij}(\frac{t}{a}, x_1, \ldots, x_n) \\
h^a(x_0, \ldots, x_n) &= \sum_{i_j=0}^{n} h^a_{ij}\big(\frac{t}{a}, x_1, \ldots, x_n\big) dx^i dx^j \\
&= \sum_{i,j} g_{ij}(\frac{t}{a}, x_1, \ldots, x_n) + dt^2
\end{aligned}$$

Therefore, the matrix h^a is the matrix way is:

$$h^a(\frac{t}{a}, x_1, \dots, x_n) = \begin{bmatrix} 1 & 0 \\ 0 & g_{ij}(\frac{t}{a}, \dots, x_n) \end{bmatrix}$$

$$(h^a)^{-1}(\frac{t}{a}, x_1, \dots, x_n) = \begin{bmatrix} 1 & 0 \\ 0 & g^{ij}(\frac{t}{a}, \dots, x_n) \end{bmatrix}$$

We will use the expression of the Christoffel symbols respect to the metric:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=0}^{n} h^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

Hence we will compute the partial derivatives of the metric:

$$\frac{\partial h^a_i j}{\partial x^l} = \begin{cases} \frac{1}{a}\frac{\partial g_{ij}}{\partial x^0}(\frac{t}{a}, x_0, \dots, x_n) & \text{if } l = 0 \wedge i, j \in \{1, \dots, n\} \\ 0 & \text{if } i \text{ or } j = 0 \\ \frac{\partial g_{ij}}{\partial x^0}(\frac{t}{a}, x_0, \dots, x_n) & \text{if } l, i, j \in \{1, \dots, n\} \end{cases}$$

Since M is compact and g_t is a family of continuous functions then $\frac{\partial h^a_{ij}}{\partial x^0}$ is bounded then we can bound this derivative with $\mathcal{O}(1)$. Now we can compute the Christoffel symbols:

- if $k = 0$ then:

$$\Gamma^0_{ij} = \frac{1}{2}h^{00}\left(\frac{\partial g_{i0}}{\partial x^j} + \frac{\partial g_{j0}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^0}\right).$$

  Thus we have two cases:

  - $i, j \in \{1, \dots, n\}$ then $\Gamma^0_{ij} = -\frac{\partial g_{ij}}{\partial x^0} = \mathcal{O}\big(\frac{1}{a}\big)$
  - $i = 0$ or $j = 0$ then $\Gamma^0_{i0} = \Gamma^0_{0i} = 0$

- if $i = 0$ or $j = 0$ then:

$$\begin{aligned} \Gamma^k_{0j} =& \frac{1}{2}\sum_{l=0}^{n} h^{kl}\left(\frac{\partial g_{0l}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^0} - \frac{\partial g_{0j}}{\partial x^l}\right) \\ =& \frac{1}{2}\sum_{l=0}^{n} h^{kl}\frac{\partial g_{jl}}{\partial x^0} \end{aligned}$$

For the next computations we will compute the derivative of this symbols:

$$\frac{\partial \Gamma^0_{ij}}{\partial x^0} = -\frac{1}{2}\frac{1}{a^2}\frac{\partial^2 g_{ij}}{\partial {x^0}^2} = \mathcal{O}\big(\frac{1}{a^2}\big).$$

Then the computation the curvature tensor respect to the Christoffel symbols is:

$$\begin{aligned}
R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} =& \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} \\
=& \nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^m_{jk}\frac{\partial}{\partial x^m}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^m_{ik}\frac{\partial}{\partial x^m}\right) \\
=& \left(\frac{\partial}{\partial x^i}\Gamma^m_{jk} - \frac{\partial}{\partial x^j}\Gamma^m_{ik}\right)\frac{\partial}{\partial x^m} + \left(\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right)\frac{\partial}{\partial x^l} \\
=& \left(\frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right)\frac{\partial}{\partial x^l}
\end{aligned}$$

Therefore the Riemannian curvature formula is:

$$R_{ijkl} = \left(\frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^w_{jk}\Gamma^l_{iw} - \Gamma^w_{ik}\Gamma^l_{jw}\right)g_{wl}$$

Now we will compute the second fundamental form of M as a submanifold of $[0,a]\times M$. Then the normal vector to M is $\frac{\partial}{\partial x^0}$. Hence,

$$\begin{aligned}
\mathbb{I}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) =& pr_{\nu TM}\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right) \\
=& pr_{\nu TM}\left(\sum_k \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right) \\
=& \Gamma^0_{ij}\frac{\partial}{\partial x^0} = \mathcal{O}\left(\frac{1}{a}\right)\frac{\partial}{\partial x^0}
\end{aligned}$$

Therefore each section curvature is:

$$K_{ij} = R_{ijji} = \left(\frac{\partial \Gamma^l_{jj}}{\partial x^i} - \frac{\partial \Gamma^i_{ij}}{\partial x^j} + \Gamma^w_{jj}\Gamma^i_{iw} - \Gamma^w_{ij}\Gamma^i_{jw}\right)g_{wi}.$$

Now we can compute the sectional curvature for each of the components:

- If $i,j\in\{1,\dots,n\}$ then,

$$\begin{aligned}
K^{[0,a]\times M}_{ij} =& K^M_{ij} + \mathbb{I}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
=& K_{ij} + \mathcal{O}\left(\frac{1}{a^2}\right)
\end{aligned}$$

- If $i=0$ or $j=0$ then,

$$\begin{aligned}
K^{[0,a]\times M} =& K^M_{0j} + \mathbb{I}\left(\frac{\partial}{\partial x^0},\frac{\partial}{\partial x^j}\right) \\
=& K_{0j} + 0 \\
=& \left(\frac{\partial \Gamma^l_{00}}{\partial x^i} - \frac{\partial \Gamma^i_{i0}}{\partial x^0} + \Gamma^w_{00}\Gamma^i_{iw} - \Gamma^w_{i0}\Gamma^i_{0w}\right) \\
=& \mathcal{O}\left(\frac{1}{a^2}\right).
\end{aligned}$$

Finally, we can compute the scalar curvature:

$$\begin{aligned}
\kappa^{M\times[0,a]} =& 2\sum_{i<j} K_{ij} \\
=& 2\Big[\sum_{j} K^M_{0j} + \sum_{1<i<j} K^M_{ij}\Big] \\
=& \mathcal{O}(\frac{1}{a^2}) + 2\sum_{1<i<j} K^M_{ij} \\
=& \kappa^M + \mathcal{O}(\frac{1}{a^2})
\end{aligned}$$

Thus, in the neighbourhood of (t, p) we have that the scalar curvature is positive for a big enough and doesn't depend on the t. Hence we can cover $[0, a] \times M$ with neighbourhoods of p.s.c. applying the same process. □

Therefore, this theorem what it states is that we can find an ε_0 small enough such that for all $\varepsilon \in [0, \varepsilon_0]$ will satisfy that the metric $h^a = \varepsilon^0 g_t + dt^2$ can carry positive scalar curvature on $[0, 1] \times M$.

Theorem II. *[1] Let (M^n, g_M) a compact and p.s.c. manifold. Any manifold (N, g_N) get by surgery in M in* ***codimension greater or equal to 3*** *will carry positive scalar curvature.*

Proof.

- First, we will construct the submanifold of $\mathbb{S}^p \times D^q \times \mathbb{R}$ with p.s.c. M_γ as we have done previously. Choosing an ε_0 such that g_{ε_0} of $\mathbb{S}^{n-1}(\varepsilon_0)$ lies in the same connected component as $g_{0,1}$ of $\mathbb{S}^{n-1}$, as in lemma 3.1.
- Since we have a create of a manifold with p.s.c. M_γ such that g_{ε_0} has p.s.c. and lies in the same connected component as $g_{0,1}$ therefore we have a family of metrics with p.s.c. generated by the homotopy,

  $$g^t = (1-t)g_{\varepsilon_0} + tg_{0,1}.$$

  As in the lemma 3.2 we have that g^t is continuous family of p.s.c. metrics hence $[0, 1] \times M$ has p.s.c.. In the top of it we will have the standard metric of $\mathbb{S}^p \times \mathbb{S}^{q-1}$.
- Finally, we can glue $D^{p+1} \times \mathbb{S}^{q-1}$, where we have that D^{p+1} is the upper hemisphere of $\mathbb{S}^{p+1}$ and we have the isometric inclusion of $\mathbb{S}^p \subset \mathbb{S}^{p+1}$. On the top of the cylinder we get the metric $D^{p+1} \times \mathbb{S}^{q-1}$.

□

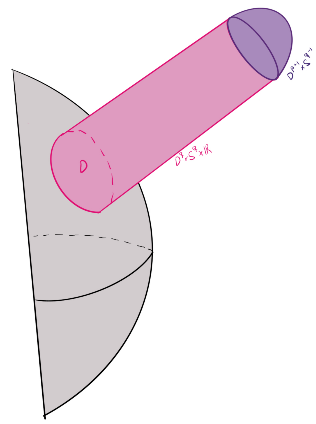

Figure 8: Example of the result of the surgery.

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