Comparison techniques for $f-$extremal domains in Space Forms

Abstract

Contents

1 Introduction

Asumiremos que $\partial\Omega$ es al menos de clase $\mathcal{C}^2$. De esta forma, las soluciones del problema serán al menos $\mathcal{C}^3$. Basta pedir que $\partial\Omega$ sea regular respecto del operador laplaciano para tener soluciones clásicas, pero creo que en algún momento hace falta más regularidad.

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In [ABM25], the authors study Serrin's equation in the plane, that is, the above equation in the setting $\mathcal{M} = \mathbb{R}^2$ with $f(u) = 2$. The method developed in that work involves associating to each solution (Ω, u, f) of (5.1) a corresponding model solution $(\bar{\Omega}, \bar{u}, f)$ that shares the same *Normalized Wall Shear Stress*—a normalized measure of the maximum normal derivative of the function along the boundary of the domain. A gradient estimate is then obtained by comparing the original solution (Ω, u, f) with its associated model solution $(\bar{\Omega}, \bar{u}, f)$ (see [ABM25, Section 3]).

This same strategy was applied in [EM25] to address a more general semilinear equation on the 2-sphere $\mathbb{S}^2$, and more recently, the comparison results from [ABM25] were extended to the n-dimensional setting in [ABBM25], using the same approach.

In all of these cases, the strategy relies on comparing a given solution to (5.1) with a model solution of the same equation. The key insight in our work is that this restriction is not necessary: the comparison arguments developed in [ABM25] can still be applied even when the model solution comes from a different problem.

1.1 Parrafo de 5.3

In contrast to the case when the dimension is $n = 2$, curvature estimates for the level sets of u within a component $\mathcal{U}$ become less effective in higher dimensions, particularly when $n > 3$.

This limitation arises from the fact that we only have control over the Laplacian of u, which in turn restricts our ability to estimate only the mean curvature of the level sets. Nevertheless, by generalizing the results of [EM25, Proposition 5.1] and [EM25, Proposition 5.2], we are able to derive sharp estimates for the mean curvature of the zero level set and the regular part of the top stratum of u within $\mathrm{cl}(\mathcal{U})$.

1.2 Hot spots

The study of the location of critical points of differentiable functions is a classical topic in geometric analysis, tracing back to the works of Gauss. More recently, Magnanini and Poggesi have obtained estimates for the distance to extremum points of solutions to certain differential equations posed in the Euclidean space.

Among other interesting results, the authors obtain in [MP22, Theorem 1.1] an estimate for the distance to the set of maximum points (the *hot spots*) of a solution to the Dirichlet problem associated with Serrin's equation (that is, (5.1) with $f(u) \equiv 1$) in Euclidean space. The proof of this result relies on a gradient estimate obtained via a suitable P-function (see [MP22, Lemma 2.2]). Subsequently, in [ABM25, Section 5], the authors derive sharp estimates for the distance to the set of maximum points in terms of the expected core radius of a solution. The results in [ABM25, Section 5] also rely on a gradient estimate for solutions to Serrin's problem, which is a particular case of Theorem 5.11 (see [ABM25, Theorem 3.5]). These parallels naturally motivate the formulation of a generalization of [ABM25, Theorem 5.1] in our current setting.

2 Preliminaries

This section fixes notation and recalls the basic geometric background used throughout the paper.

2.1 Geometry of hypersurfaces

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold. For any $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$, we write $g(v, w)$ or $\langle v, w \rangle$ for the scalar product of two vectors and $|v|^2 = \langle v, v \rangle$ for the squared norm of a vector.

Let $\Gamma \subset \mathcal{M}$ be an embedded, two-sided, $\mathcal{C}^2$-hypersurface in $\mathcal{M}$. Write $N : \Gamma \to T\mathcal{M}$ for a Gauss map of Γ and, at a given point $p \in \Gamma$, write

$$\mathrm{II}_p(v, w) = -\langle dN_p(v), w \rangle, \quad \forall v, w \in T_p\Gamma,$$

for the *second fundamental form* of Γ at p with respect to N. Then we define the *mean curvature* of Γ at p as

$$H(p) = \frac{1}{n-1}\mathrm{Trace}(\mathrm{I\!I}_p) = -\frac{1}{n-1}\sum_{i=1}^{n-1}\langle dN_p(e_i), e_i\rangle\,,$$

where $\{e_1,\ldots,e_{n-1}\}$ is an orthonormal basis of $T_p\Gamma$.

For any $l\in\mathbb{R}$, let $\mathcal{H}^l$ denote the l-dimensional Hausdorff measure associated to the metric g. Then, for a given open set $\mathcal{O}\subset\mathcal{M}$, we write $\mathcal{H}^n(\mathcal{O})$ for its volume, while we will write $\mathcal{H}^{n-1}(\Gamma)$ for the area of an hypersurface. Note that we don't ask $\mathcal{O}$ or Γ to be connected.

2.2 Model manifolds

Let $\mathbb{M}^n(k)$ denote the complete, simply connected Riemannian n-manifold of constant sectional curvature k; thus $\mathbb{M}^n(k)=\mathbb{S}^n$ if $k>0$, $\mathbb{R}^n$ if $k=0$, and $\mathbb{H}^n$ if $k<0$. Fix $o\in\mathbb{M}^n(k)$ and consider the exponential map

$$\exp_o : B(0,2\bar{r}_k)\setminus\{0\}\longrightarrow \widetilde{\mathbb{M}}^n(k)\setminus\{o\},$$

where $\bar{r}_k=+\infty$ if $k\le 0$ and $\bar{r}_k=\pi/(2\sqrt{k})$ if $k>0$; here $\widetilde{\mathbb{M}}^n(k)=\mathbb{M}^n(k)$ for $k\le 0$ and $\mathbb{S}^n\setminus\{-o\}$ for $k>0$. Using polar coordinates on $B(0,\bar{r}_k)$, the map

$$X:(0,2\bar{r}_k)\times\mathbb{S}^{n-1}\longrightarrow\mathbb{M}^n(k),\qquad X(r,\theta)=\exp_o(r\theta), \tag{2.1}$$

gives normal coordinates centered at o. In these coordinates, the metric is

$$\bar{g}_k \;=\; dr^2+s_k(r)^2\,g_{\mathbb{S}^{n-1}}. \tag{2.2}$$

Where the function $s_k(r):I_k\mapsto\mathbb{R}$ is defined as follows,

$$s_k(r) := \begin{cases} \dfrac{\sinh(\sqrt{-k}\,r)}{\sqrt{-k}}, & k<0,\\ r, & k=0,\\ \dfrac{\sin(\sqrt{k}\,r)}{\sqrt{k}}, & k>0, \end{cases}\qquad r\in(0,2\bar{r}_k).$$

We now introduce auxiliary functions naturally associated to the warping function, $s_k(r)$,

$$\cot_k(r):=\frac{s_k'(r)}{s_k(r)},\qquad \tan_k(r):=\frac{s_k(r)}{s_k'(r)},\qquad S_k(r):=\int_0^r s_k(x)\,dx.$$

These functions satisfy the relations,

$$s_k''+k\,s_k=0,\qquad s_k(0)=0,\qquad s_k'(0)=1,\qquad \cot_k(r)=(\log s_k(r))',\qquad \cot_k(r)=\tan_k(r)^{-1}.$$

These expressions show that each space form admits a global warped product structure with base an interval I_k and fiber the unit sphere $\mathbb{S}^{n-1}$. The function s_k acts as the warping function, determining the curvature and geometry of the model. Under this expression of the metric of a space form the Laplace–Beltrami operator takes the following form for a $u \in C^2(\mathcal{M}_k^n)$

$$\Delta u \;=\; \partial_r^2 u \;+\; (n-1)\,\cot_k(r)\,\partial_r u \;+\; \frac{1}{s_k(r)^2}\,\Delta_{\mathbb{S}^{n-1}} u. \tag{2.3}$$

Remark 2.1. If a Riemannian manifold $\mathcal{M}$ admits a foliation by totally umbilical hypersurfaces on an open set $U \subset \mathcal{M}$, then U is (locally) isometric to a warped product of the form $dr^2 + \varphi(r)^2 g_F$. In particular, any space form $\mathbb{M}^n(k)$ is locally of this type.

The expression of the metric in (2.2) is a particular case of a warped product manifold. Therefore, the previous construction can be written in a more general set-up. Let (F, g_F) be a smooth Riemannian manifolds and let (I_k, dr^2) be an interval in $\mathbb{R}$ such that $I_k := (0, 2\bar{r}_k)$ with $\bar{r}_k = +\infty$ for $k \leq 0$ and $\bar{r}_k = \pi/(2\sqrt{k})$ for $k > 0$. On the product manifold $\mathcal{M}_k^n(F) = I_k \times F$, the metric $g_k := dr^2 + s_k(r)^2 g_F$ defines a warped product structure. Therefore, the warped product manifold with warping function $s_k(r)$,

$$(\mathcal{M}_k^n(F), g_k) := \big(I_k \times F,\; dr^2 + s_k(r)^2\, g_F\big),$$

will be referred to as a *model manifold*. In this case the Laplace–Beltrami operator for a given $u \in C^2(\mathcal{M}_k^n(F))$ takes the form,

$$\Delta = \partial_r^2 + (n-1)\,\cot_k(r)\,\partial_r + \frac{1}{s_k(r)^2}\,\Delta_F, \tag{2.4}$$

Remark 2.2. When $F = \mathbb{S}^{n-1}$ with the round metric, (2.2) and (2.4) recover the standard expressions on the space forms $\mathbb{M}^n(k)$ for any $n \geq 2$.

2.3 Overdetermined Elliptic Problems

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, and take $\Omega \subset \mathcal{M}$ a domain with $\mathcal{C}^2$-boundary $\partial\Omega$. We consider the classical overdetermined problem for a function $u : \Omega \to \mathbb{R}$, where $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, given by

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, & (2.5.1) \\ u > 0 & \text{in } \Omega, & (2.5.2) \\ u = 0 & \text{on } \partial\Omega, & (2.5.3) \\ \langle \nabla u, \eta \rangle = \alpha_i & \text{on } \Gamma_i \subset \partial\Omega. & (2.5.4) \end{cases}$$

where Δ and ∇ denote the Laplacian and gradient in $\mathcal{M}$, respectively, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, η is the outward unit normal to $\partial\Omega$, and α_i is a constant on each connected component $\Gamma_i \subset \partial\Omega$.

Let u be a solution to (2.5). Observe that since $u \equiv 0$ on $\partial\Omega$ and $u > 0$ inside Ω, it follows that u reach its maximum value inside Ω. Denote $u_{\max} = \max_{p\in\Omega} u$ and its level set as $\Gamma_{u_{\max}} = u^{-1}(u_{\max}) \subset \Sigma$. We say that Γ_M is the *top level set* of u.

In particular, we are interested in characterizing solutions (Ω, u, f) to (2.5) that exhibit certain symmetries. Let $\mathcal{I} \in \mathrm{Iso}(\mathcal{M})$ the triple (Ω, u, f) is said to be $\mathcal{I}$-invariant if

$$\mathcal{I}(\Omega) = \Omega \quad \text{and} \quad u(\mathcal{I}(p)) = u(p) \quad \text{for all } p \in \Omega.$$

In this context, the symmetry properties of solutions play a crucial role in their classification. Let $\mathrm{inj}(\mathcal{M})$ denote the injectivity radius of $\mathcal{M}$ (see [Car76]). We introduce the following notion.

Definition 2.3. A f-extremal domain (Ω, u, f) exhibits *radial symmetry* if there exists a point $p_0 \in \mathcal{M}$ and constants $0 \leq a < b \leq \mathrm{inj}(\mathcal{M})$ such that $\Omega = r_{p_0}^{-1}((a,b))$, and $u = U \circ r_{p_0}$ for some function $U \in \mathcal{C}^2((a,b))$. Here r_{p_0} denote the distance function in $\mathcal{M}$ to the fixed point p_0.

3 Model Solutions

A natural class of solutions to (2.5) arises on model manifolds $(\mathcal{M}^n_k(F), g_k)$, $n \geq 2$, by imposing that the solution only depends on the base projection $r := \pi_B$. When the model manifold is a space form this means radial symmetry, since we would have $r_o(p) = \mathrm{dist}(o,p)$ for a fixed origin $o \in \mathbb{M}^n(k)$.

Given $U \in \mathcal{C}^2(\mathbb{R})$ one checks directly from (2.4) that $u := U \circ r$ solves the PDE in (2.5) if and only if U satisfies

$$U''(r) + (n-1)\, \cot_k(r)\, U'(r) + f\big(U(r)\big) = 0, \tag{3.1}$$

Following [EM25, Section 4], we look for solutions to (3.1) satisfying

$$U(R) = M \quad \text{and} \quad U'(R) = 0, \tag{3.2}$$

for some $R \in [0, 2\bar{r}_k)$ and $M > 0$. Here, we recall that $\bar{r}_k = \pi/2\sqrt{k}$ if $k > 0$ and $\bar{r}_k = +\infty$ otherwise. Since our goal is to define a family of solutions to (2.5), we need solutions to (3.1)-(3.2) that vanish at some bounded interval. In particular, we propose the following:

Definition 3.1. We say that a function $f \in \mathcal{C}(\mathbb{R})$ is admissible if there exist non-empty sets $\mathcal{R}_f \subset [0, 2\bar{r}_k)$ and $\mathcal{I}_f \subset \mathbb{R}_+$ such that the following hold: for any $R \in \mathcal{R}_f$ and $M \in \mathcal{I}_f$ there exists a $\mathcal{C}^2$-solution $U_{R,M,k}$ to (3.1)-(3.2) defined on an interval $[r_-(R,M,k), r_+(R,M,k)]$ with $0 \leq r_-(R,M,k) \leq R < r_+(R,M,k) < 2\bar{r}_k$ such that,

- If $R = 0$, then $0 = r_-(0,M,k) < r_+(0,M,k)$, $U_{0,M,k} > 0$ on the interval $[0, r_+(0,M,k))$ and $U_{0,M,k}(r_+(0,M,k)) = 0$.

- If $R > 0$, then $0 < r_-(R,M,k) < R < r_+(R,M,k)$, $U_{R,M,k} > 0$ on the interval $(r_-(R,M,k), r_+(R,M,k))$ and $U_{R,M,k}(r_\pm(R,M,k)) = 0$.

If $k = 1$ and $n = 2$, then [EM25, Theorem 4.1] implies that each positive Lipschitz function is admissible. In fact, this follows for general $n \geq 2$ and $k > 0$, as we shall see bellow. However, when $k \leq 0$ the situation is very different, as there are functions f for which only positive solutions to (3.1)-(3.2) exists, see [BGGV13, Theorem 2.1]. In this case, we have to impose further conditions on f to ensure that it is an admissible function. For example, we can ask f to satisfy the following:

> **Standard conditions.** A continuous function $f \in \mathcal{C}(\mathbb{R})$ is said to satisfy the *standard conditions* if it is Lipschitz continuous and there exists an open interval $\mathcal{I}_f \subset \mathbb{R}_+$ with $0 \in \partial\mathcal{I}_f$ such that, on this interval, one of the following holds:
>
> - If $k > 0$, then $f > 0$ on $\mathcal{I}_f$;
> - If $k = 0$, then $f \geq C_1 > 0$ on $\mathcal{I}_f$ for some constant $C_1 > 0$;
> - If $k < 0$, then either
> $$f(x) \geq \lambda x \quad \text{for all } x \in \mathcal{I}_f$$
> for some $\lambda > \lambda_1(\mathbb{H}^n(k)) := (n-1)^2k^2/4$, or
> $$f(x) \geq nkx + C_2 \quad \text{for all } x \in \mathcal{I}_f$$
> for some $C_2 > 0$.

The content of the following result, whose proof is deferred to Appendix B, establishes that the set of admissible functions is included in the set of functions satisfying the standard conditions.

Proposition 3.2. *Let $f \in \mathcal{C}(\mathbb{R})$ be a real function and $k \in \mathbb{R}$. Suppose that f satisfies the standard conditions. Then f is admissible. Furthermore, $\mathcal{R}_f = [0, 2\bar{r}_k)$, and the following hold:*

1. *If $R = 0$, then $U'_{0,M,k} < 0$ on $(0, r_+(R,M,k)]$.*
2. *If $R \in (0, 2\bar{r}_k)$, then $U'_{R,M,k} > 0$ on $[r_-(R,M,k), R)$ and $U'_{R,M,k} < 0$ on $(R, r_+(R,M,k)]$. Moreover,*
$$U_{\pi/\sqrt{k}-R,M,k}(\pi/\sqrt{k} - r) = U_{R,M,k}(r), \quad \text{when } k > 0. \tag{3.3}$$

Remark 3.3. We note that the standard conditions are sufficient but not necessary in order to ensure that f is an admissible function.

For $k \in \mathbb{R}$ fixed and f satisfying the conditions asked in Proposition 3.2, we obtain a two-parameter family of solutions that depends solely on the base projection in the model manifold $(\mathcal{M}_k^n(F), g_k)$ and the maximum value of the function.

When $k > 0$, the solutions $U_{R,M,k}$ are determined up to reflection and it is sufficient to restrict to $R \in [0, \bar{r}_k]$ instead of $R \in [0, 2\bar{r}_k)$. Moreover, when $R = \bar{r}_k$, the solution is symmetric with respect to $r \to \bar{r}_k - r$.

Definition 3.4. Let $f \in C(\mathbb{R})$ be an admissible function. Then, for any $k \in \mathbb{R}$, $M \in \mathcal{I}_f$ and $R \in \mathcal{R}_f$, set

$$u_{R,M,k}(p) := U_{R,M,k}(r(p)), \qquad p \in \mathcal{M}_k^n(F),$$

where $U_{R,M,k}$ is the solution to (3.1)–(3.2), and define the domain

$$\Omega_{R,M,k} := \begin{cases} \{\, p \in \mathcal{M}_k^n(F) : r(p) < r_+(0,M,k) \,\}, & R = 0, \\ \{\, p \in \mathcal{M}_k^n(F) : r_-(R,M,k) < r(p) < r_+(R,M,k) \,\}, & R \neq 0. \end{cases}$$

Then we say that the triple $(\Omega_{R,M,k}, u_{R,M,k}, f)$ is a model solution in $\mathcal{M}_k^n(F)$. Moreover, if $R = 0$, we set

$$\Gamma_{0,M,k} = u_{0,M,k}^{-1}(M) \text{ and } \Gamma_{0,M,k}^{+} = \{\, p \in \mathcal{M}_k^n(F) : r_k(p) = r_+(0,M,k) \,\},$$

and we also denote by $\Omega_{0,M,k}^{+} \subset \Omega_{0,M,k} \setminus \Gamma_{0,M,k}$ the subdomain of $\Omega_{0,M,k}$ without maximum points. Analogously, if either $R \in (0, +\infty)$ when $k \leq 0$ or $R \in (0, \bar{r}_k]$ when $k > 0$, we denote

$$\Gamma_{R,M,k} = u_{R,M,k}^{-1}(M) \text{ and } \Gamma_{R,M,k}^{\pm} = \{\, p \in \mathcal{M}_k^n(F) : r_k(p) = r_\pm(R,M,k) \,\},$$

and $\Omega_{R,M,k}^{\pm} \subset \Omega_{R,M,k} \setminus \Gamma_{R,M,k}$ the subdomains of $\Omega_{R,M,k}$ without maximum points such that $\mathrm{cl}(\Omega_{R,M,k}^{\pm}) \cap \partial\Omega_{R,M,k} = \Gamma_{R,M,k}^{\pm}$.

Remark 3.5. The previous construction isolates all positive "radial" solutions dictated by (3.1); it will serve as our reference class when contrasting with the non-radial families. These solutions were used in [EM25] as models to carry out comparison arguments for general solutions to the Dirichlet problem associated with (2.5) when $n = 2$.

From now on, given a model manifold $(\mathcal{M}_k^n(F), g_k)$, an admissible function f and $M \in \mathcal{I}_f$, we denote by $\mathrm{Model}_{M,k,f}^{\pm}$ the family of model triples $(\Omega_{R,M,k}^{\pm}, u_{R,M,k}, f)$, which depends only on $R \in \mathcal{R}_f$. Given any model triple $(\Omega_{R,M,k}^{\pm}, u_{R,M,k}, f)$, following [ABM25], we will say that the parameter R is its *core radius*.

3.1 Model $\overline{\tau}$-function

Given any function f satisfying the standard conditions, one can define the following function:

$$h_k(M) := U'_{0,M,k}(r_+(0,M,k)), \quad \forall M \in \mathcal{I}_f. \tag{3.4}$$

Since $(\Omega_{0,M,k}, u_{0,M,k}, f)$ solves (2.5) with $f \circ u_{0,M,k}$ being positive in $\Omega_{0,M,k}$, Hopf's lemma implies that $h_k < 0$ in $\mathcal{I}_f$. Then, we can define the ratio

$$\overline{\tau}^{\pm}_{M,k}(R) := \frac{U'_{R,M,k}(r_{\pm}(R,M,k))^2}{h_k(M)^2} \quad \forall\, R \in (0, 2\bar{r}_k). \tag{3.5}$$

These functions are important as they parametrizes model solutions, and we call them *model $\overline{\tau}$-functions*.

In this section we prove some properties of the functions $\overline{\tau}^{\pm}_{M,k}$ that will be useful in the sequel. First, we study the case in which $f(x) = f_k(x) := nkx + 1$ and then (2.5) is Serrin's problem. We prove the following

Proposition 3.6. *Define $\overline{\tau}^{-}_{M,k}(R)$ is as in* (3.5) *with the solution to* (3.1)-(3.2) *with $f = f_k$. Therefore,*

$$\lim_{R \to 0^+} \overline{\tau}^{-}_{M,k}(R) = +\infty, \tag{3.6}$$

Proof. Denote by $V_{R,M,k}$ the solution to (3.1)-(3.2) with $f = f_k$. For the case of $k = 0$ the behavior of $\overline{\tau}^{-}_{M,k}(R)$ is proved in [ABBM25; ABM25]. Thus, we are going to focus in the case $k = \pm 1$. In this case, the function $V_{R,M,k}$ can be expressed as follows,

$$V_{R,M,k}(r) = M + c_k(r)\left(A(R,M,k) + B(R,M,k)\left(\frac{-k}{s_k(r)} + G_k(r)\right)\right)$$

for some positive numbers $A(R,M,k)$ and $B(R,M,k)$ and

$$G_k(r) := \begin{cases} \int_0^{\cosh(r)} \frac{1-(1-t^2)^{\frac{n}{2}}}{t^2(1-t^2)^{\frac{n}{2}}} \text{if} & k < 0, \\ \int_0^{\cos(r)} \frac{1-(t^2-1)^{\frac{n}{2}}}{t^2(t^2-1)^{\frac{n}{2}}} \text{ if} & k > 0. \end{cases}$$

Since $G_k(r) \approx r^{1-n}$ near $r = 0$, we deduce that $V'_{R,M,k}(r_{\pm})$ diverges as $R \to 0^+$, and therefore $\overline{\tau}^{-}_{M,k}(R) \to +\infty$. $\square$

Now define the quantities

$$\tau^{-}_k(M) := \inf_{R \in [0,\bar{r}_k)} \overline{\tau}^{-}_{M,k}(R), \quad \tau^{+}_k(M) := \sup_{R \in [0,\bar{r}_k)} \overline{\tau}^{+}_{M,k}(R) \quad \text{and} \quad \tau^{0}_k(M) := \inf_{R \in [0,\bar{r}_k)} \overline{\tau}^{+}_{M,k}(R).$$

We prove the following result by comparing with a solution to Serrin's equation.

Proposition 3.7. *Define $\overline{\tau}^{-}_{M,k}(R)$ as in* (3.5) *with the solution to* (3.1)-(3.2) *with $f(x) \geq nkx + f(0)$, where we assume $f(0) > 0$. Then $\overline{\tau}^{-}_{M,k}$ satisfies the limit* (3.6). *Also, we have that $\tau^{+}_k(M) \leq \tau^{-}_k(M)$ if $k \leq 0$ and $\tau^{+}_k(M) \geq \tau^{-}_k(M)$ if $k > 0$.*

Proof. Up to rescaling, assume that $f(0) = 1$. We first show that $\overline{\tau}^-_{M,k}$ satisfies (3.6). Arguing by contradiction, suppose that

$$\lim_{R \to 0^+} U'_{R,M,k}(r_-(R,M,k)) = \alpha$$

is a positive number. As before, denote by $V_{R,M,k}$ the solution to (3.1)-(3.2) with $f = f_k$. Then, Proposition 3.6 implies that there exists $R_0 > 0$ such that $V'_{R,M,k}(\bar{r}_-(R,M,k)) > \alpha$ for all $R \in (0, R_0)$, where $\bar{r}_-(R,M,k) < R$ is such that $V_{R,M,k}(\bar{r}_-(R,M,k)) = 0$ for all $R \in (0, \bar{r}_k)$.

Now fix $R_1 \leq \bar{r}_{R_0,M,k}$. For any $R > 0$, denote by $(\Omega^-_{R,M,k}, u_{R,M,k}, f)$ and $(\tilde{\Omega}^-_{R,M,k}, v_{R,M,k}, f_k)$ the corresponding model triples in $\mathbb{M}^n(k)$ associated to the equation (3.1)-(3.2) with $f \geq f_k$ and f_k respectively. Then, given M, k fixed and for $R \leq R_0$, we define the function

$$\omega_R = v_{R,M,k} - u_{R_1,M,k} \quad \text{ in } \mathcal{O}_R := \Omega^-_{R_1,M,k} \cap \tilde{\Omega}_{R,M,k}.$$

It follows $\omega_R < 0$ in $\mathcal{O}_R$ if $|R - R_0|$ is small enough, and by decreasing slightly R we reach to a position in which $\omega_R \leq 0$ in $\mathcal{O}_R$ and there exists $x \in \mathrm{cl}(\mathcal{O}_R)$ such that $\omega_R(x) = 0$. But we have that

$$\Delta \omega_R = -f_k(v) + f(u) \geq -nk\omega_R, \quad \text{ in } \mathcal{O}_R,$$

thus, the interior or boundary maximum principles implies that $\omega_R \equiv 0$ in $\mathcal{O}_R$, a contradiction. This proves the first claim of the proposition.

For the second claim, observe that if $k > 0$ then (3.3) implies that $\overline{\tau}^+_{M,k}(\bar{r}_k) = \overline{\tau}^-_{M,k}(\bar{r}_k)$, so it must be $\tau^+_k(M) \geq \tau^-_k(M)$. When $k \leq 0$ note that $\overline{\tau}^+_{M,k}(0) = 1$, so we have that $\overline{\tau}^+_{M,k}(R) < \overline{\tau}^-_{M,k}(R)$ for $R > 0$ small enough. Now, if there exists $R > 0$ such that $\overline{\tau}^+_{M,k}(R) = \overline{\tau}^-_{M,k}(R)$ then $(\Omega_{R,M,k}, u_{R,M,k}, f)$ in $\mathbb{M}^n(k)$ would be a solution to (2.5) with constant Neumann data, so Serrin's Theorem (see [KP98; Ser71]) implies that $\Omega_{R,M,k}$ is a geodesic ball. But this only occurs if $R = 0$, so we conclude that it must be $\overline{\tau}^+_{M,k}(R) < \overline{\tau}^-_{M,k}(R)$ for all $R \in (0, \bar{r}_k)$ and then $\tau^+_k(M) \leq \tau^-_k(M)$. $\square$

Remark 3.8. When $k > 0$, we note that Proposition 3.7 follows with the same arguments assuming $f(0) = 0$, since we can compare with the function $V_{R,M,k} + \frac{1}{nk}$. On the other hand, if $k \leq 0$ and $f(0) = 0$, the condition $f(x) \geq nkx$ does not implies that f is admissible. In fact, if $f(x) = nkx$ with $k \leq 0$ then there exists no model solutions.

Now define the set

$$\mathrm{Adm}_{M,k,f} := \mathrm{Im}(\overline{\tau}^-_{M,k}) \cup \mathrm{Im}(\overline{\tau}^+_{M,k}), \tag{3.7}$$

which we call *admissible set*. Define also the set

$$\mathrm{Gap}_{M,k,f} := [\tau^0_k(M), +\infty) \setminus \mathrm{Adm}_{M,k,f}, \tag{3.8}$$

which we call the *gap*. Note that, if $f \geq f_k$, then Proposition 3.7 implies that $\mathrm{Gap}_{M,k,f} = \emptyset$ if $k > 0$. When $k \leq 0$, it could be $\mathrm{Gap}_{M,k,f} \neq \emptyset$.

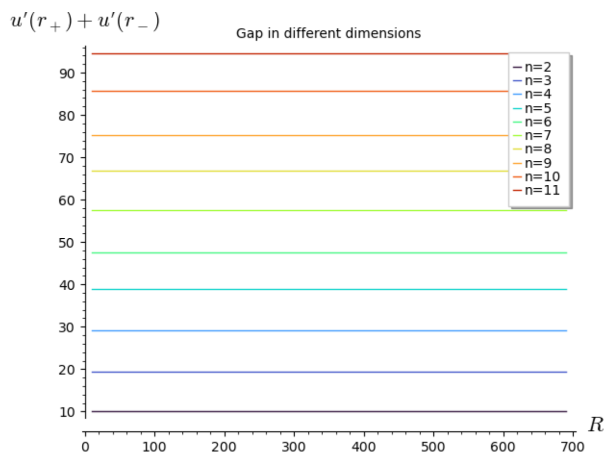

Figure 1: Here we present a numerical approximation of the size of the gap associated to Serrin's equation in hyperbolic space.

Remark 3.9. In the case of Serrin's equation in Euclidean space, the works [ABBM25; ABM25] implies that the Gap_{M,k,f_0} consists of a single point, but this does not have to be the case in general. For example, for Serrin's equation in hyperbolic space the gap is an interval with non-empty interior.

Given any $M \in (0, 1/n)$, write $V_{R,M,-1}(r)$ for the solution to (3.1)-(3.2) with $k = -1$ and $f(x) = f_{-1}(x) = -nx + 1$. Then, by doing the change of variable $r = R + s$, it is straightforward to show that $V_{R,M,-1}$ converge (to first order) to the function

$$V_{\infty,M}(s) = 1 - \frac{1}{(n+1)(M+1)}\left(e^{-ns} + n\, e^{s}\right)$$

as $R \to +\infty$. Thus, if $s_- < 0 < s_+$ denotes the two roots of $V_{\infty,M}$, we can check that

$$\begin{aligned}\lim_{R\to\infty} -V'_{R,M,-1}(r_+) &\approx \frac{n}{(n+1)(M+1)}\left(e^{s_+} - e^{-ns_+}\right)\\ &< \frac{n}{(n+1)(M+1)}\left(e^{-ns_-} - e^{s_-}\right) \approx \lim_{R\to\infty} V'_{R,M,-1}(r_-).\end{aligned}$$

It is not clear, in general, whether the functions $\overline{\tau}^{\pm}_{M,k}$ are monotone. However, we can prove that they are indeed monotone provided that the function f satisfies an additional condition:

Proposition 3.10. *Let $f \in \mathcal{C}^1(\mathbb{R})$ be a function satisfying the standard conditions. Suppose that there exists $\bar{M} > 0$ such that $(0, \bar{M}) \subset \mathcal{I}_f$ and*

$$f(x) \geq x f'(x), \qquad \text{for all } x \in (0, \bar{M}). \tag{3.9}$$

Then, for any $M \in (0, \bar{M})$, we have that $\tau^0_k(M) = 1$ and the functions $\overline{\tau}^-_{M,k}$ and $\overline{\tau}^+_{M,k}$ are strictly decreasing and strictly increasing respectively when restricted to the interval $(0, \bar{r}_k)$.

The proof of this result can be found in Appendix B. Note that if f satisfies the conditions asked in the above proposition, we have that

$$\mathrm{Adm}_{M,k,f} := \begin{cases} [1,+\infty) & \text{if } k > 0, \\ [1,+\infty) \setminus \mathrm{Gap}_{M,k,f} & \text{if } k \leq 0. \end{cases}$$

4 Exotic solutions on $\mathbb{S}^n$ and on quotients

In contrast with space forms of non-positive curvature, the sphere $\mathbb{S}^n$ admits rich isoparametric foliations. This allows non-rotational (*exotic*) solutions of (2.5) whose level sets are isoparametric hypersurfaces (see Appendix A). In this section we construct such solutions on $\mathbb{S}^n$ and show that, whenever the foliation is preserved by a group of isometries, the construction descends to domains in a smooth quotient $\mathbb{S}^n/G$.

4.1 Construction along an isoparametric foliation of $\mathbb{S}^n$

Let $\{\Gamma_r\}_{r\in[-1,1]}$ be an isoparametric family in $\mathbb{S}^n$, given by the restriction of a Cartan-Münzner polynomial $\rho \in \mathcal{C}^\omega(\mathbb{S}^n)$ via $\Gamma_r := \rho^{-1}(r)$. Let $\ell \in \{1,2,3,4,6\}$ be the number of distinct principal curvatures and let m_1, m_2 denote their multiplicities (alternating when ℓ is even, coincident when ℓ is odd). Along a normal geodesic to a focal submanifold, the Cartan–Münzner theory (see e.g. [DV, §4.1]) yields

$$\rho(p) = \cos\big(\ell\, s(p)\big), \qquad s(p) := \mathrm{dist}_{\mathbb{S}^n}(p, \Gamma_1). \tag{4.1}$$

Let $f \in C(\mathbb{R})$ be a continuous real function with $\mathcal{I}_f \neq \emptyset$. Seeking $w = W \circ s$, a direct computation shows that w solves the PDE in (2.5) if and only if W solves

$$W''(s) + \left((n-1)\cot(\ell s) - \frac{c}{\ell \sin(\ell s)}\right) W'(s) + f\big(W(s)\big) = 0, \tag{4.2}$$

where $c = \frac{\ell^2}{2}(m_2 - m_1)$. We prescribe the Cauchy data

$$W(S) = M > 0, \qquad W'(S) = 0, \qquad S \in [0, \pi/\ell]. \tag{4.3}$$

Standard arguments, presented in Appendix B, on ODE and focal geometry allow us to show:

Lemma 4.1. *Let $f \in C(\mathbb{R})$ be such that f is Lipschitz on $\mathcal{I}_f$, the union of the intervals where $f > 0$, and assume that $0 \in \partial\mathcal{I}_f$. Then for any $S \in [0,\pi/\ell]$ and $M \in \mathcal{I}_f$ there exists a unique solution $W_{S,M}$ to* (4.2)–(4.3) *with the following properties:*

i. If $S = 0$ (resp. $S = \pi/l$), there exists $0 < s_+(0, M) < \pi/\ell$ (resp. $0 < s_-(\pi/l, M) < \pi/\ell$) such that $W_{0,M} > 0$ on $[0, s_+(0, M))$, $W_{0,M}(0) = M$, $W_{0,M}(s_+(0, M)) = 0$, and $W'_{0,M} < 0$ on $(0, s_+(0, M)]$ (resp. $W_{\pi/l,M} > 0$ on $(s_-(\pi/l, M), \pi/l]$, $W_{\pi/l,M}(\pi/l) = M$, $W_{\pi/l,M}(s_-(\pi/l, M)) = 0$, and $W'_{\pi/l,M} > 0$ on $[s_-(\pi/l, M), \pi/l)$).

ii. If $S \in (0, \pi/\ell)$, there exist $0 < s_-(S, M) < S < s_+(S, M) < \pi/\ell$ such that $W_{S,M} > 0$ on (s_-, s_+), $W_{S,M}(S) = M$, $W_{S,M}(s_-(S, M)) = W_{S,M}(s_+(S, M)) = 0$, $W'_{S,M} > 0$ on $(s_-(S, M), S)$, and $W'_{S,M} < 0$ on $(S, s_+(S, M))$.

Then, writing $R = \cos(\ell S) \in [-1, 1]$ for $S \in [0, \pi/\ell]$, the function $W_{R,M} := W_{S,M} \circ s$ defined on

$$\Omega_{R,M} := \begin{cases} \{\, p \in \mathbb{S}^n : \ s(p) < s_+(0, M) \,\} & \text{if } R = 1, \\ \{\, p \in \mathbb{S}^n : \ s(p) > s_-(\pi/l, M) \,\} & \text{if } R = -1, \\ \{\, p \in \mathbb{S}^n : \ s_-(S, M) < s(p) < s_+(S, M) \,\} & \text{if } R \in (-1, 1), \end{cases}$$

solves (2.5) in $\Omega_{R,M}$ with $(w_{R,M})_{\max} = M$ and $\mathrm{Max}(w_{R,M}) = \Gamma_R$.

Theorem 4.2 (Exotic solutions on $\mathbb{S}^n$). *Let $f \in C(\mathbb{R})$ be a Lipschitz function positive in $\mathbb{R}^*_+$. For every isoparametric family $\{\Gamma_r\} \subset \mathbb{S}^n$, every $R \in [-1, 1]$ and every $M > 0$, there exists an f-extremal domain $(\Omega_{R,M}, w_{R,M}, f)$ for (2.5) whose level sets are the leaves of $\{\Gamma_r\}$. In particular, when the leaves are not umbilic (e.g. Cartan's families with $\ell = 3$), these are non-rotational (exotic) solutions.*

Remark 4.3. For $f(u) = \lambda u$ or $f \equiv 1$, the existence of extremal domains associated to isoparametric foliations (including cases where the top level set is a focal submanifold) is classical; see Savo [Sav18] and Shklover [Shk00] . Our construction yields the same phenomenon for general positive f.

4.2 Descent to quotients

Let $G \leq \mathrm{Iso}(\mathbb{S}^n)$ be a closed subgroup and $\{\Gamma_r\}_{r\in[-1,1]}$ be an isoparametric family in $\mathbb{S}^n$, $\Gamma_r := \rho^{-1}(r)$. We say that G *preserves* the foliation if $\rho \circ \xi = \rho$ for all $\xi \in G$. Fix an interval of regular leaves $[a, b] \subset (-1, 1)$ and assume the action of G on each Γ_r, $r \in [a, b]$, is free. Then the projection $\pi : \mathbb{S}^n \to \mathbb{S}^n/G$ restricts to smooth embedded hypersurfaces $\pi(\Gamma_r) \subset \mathbb{S}^n/G$, and

$$\Omega^{\Gamma}_{[a,b]} := \bigcup_{r\in[a,b]} \Gamma_r/G \ \subset \ \mathbb{S}^n/G$$

is a smooth domain with smooth boundary $\partial\Omega^{\Gamma}_{[a,b]} = \pi(\Gamma_a) \cup \pi(\Gamma_b)$.

Theorem 4.4 (Exotic solutions on smooth quotients)**.** *Assume G preserves $\{\Gamma_r\}$ and acts freely on the leaves Γ_r for $r \in [a,b]$. Let $(\Omega_{R,M}, w_{R,M}, f)$ be as above with* $\mathrm{Max}(w_{R,M}) = \Gamma_R$ *and $R \in [a,b]$. Then $u := w_{R,M}$ is G-invariant and descends to a function u_G on $\Omega^{\Gamma}_{[a,b]}$ solving* (2.5) *in the quotient, with the same Dirichlet data and a constant Neumann datum on each connected boundary component.*

Proof. Since u depends only on s and $\rho = \cos(\ell s)$ is G-invariant, u is G-invariant. The free action yields a smooth manifold quotient where the projected leaves are embedded and have constant mean curvature inherited from the isoparametric family, hence the Neumann datum is constant along each boundary component. □

Remark 4.5 (Lens spaces and projective spaces). (1) If $n = 2m+1$ and the foliation is invariant under the Hopf $\mathbb{S}^1$-action on $\mathbb{S}^{2m+1}$, then it descends to an isoparametric foliation on $\mathbb{CP}^m$ (and analogously $\mathbb{S}^{4m+3} \to \mathbb{HP}^m$); see [DV, Prop. 8.1]. Hence, for any interval $[a,b]$ of regular leaves in $\mathbb{S}^{2m+1}$, the corresponding exotic solutions descend to $\mathbb{CP}^m$, and further to any lens space $L(p,q) = \mathbb{S}^{2m+1}/\mathbb{Z}_p$ since the $\mathbb{Z}_p$-action is free.

(2) For the antipodal quotient $\mathbb{RP}^n = \mathbb{S}^n/\{\pm 1\}$, Cartan–Münzner families with even ℓ produce $\{\pm 1\}$-invariant ρ and thus descend; odd ℓ do not.

Remark 4.6 (Immersed case and singular quotients). If the G-action has fixed points on some Γ_r, then $\mathbb{S}^n/G$ is an orbifold near the image of the fixed set and $\pi(\Gamma_r)$ is an immersed hypersurface with lower-dimensional singular strata. The function u still descends and solves (2.5) on the regular part; the Neumann datum is constant on each connected piece of $\partial\Omega^{\Gamma}_{[a,b]}$ away from singular strata. For the purposes of a smooth boundary problem, we restrict to free actions on the chosen interval of leaves.

5 Pseudo-radial Functions

In this section, $(\mathcal{M}, g)$ denotes a n-dimensional Riemannian manifold satisfying the curvature bound Ric $\geq (n-1)kg$, for some $k \in \mathbb{R}$. The goal here is to extend the comparison algorithm developed in [ABBM25; ABM25] to the context of general Riemannian manifolds. To do so, given a $\mathcal{C}^2$-domain $\Omega \subset \mathcal{M}$ and a real function f, we consider the general Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

5.1 The comparison triples

Remark 5.1. Throughout this section we fix a solution (Ω, u, f) of problem (5.1), where Ω is a compact domain. We denote by

$$M := \max_{x\in\Omega} u(x)$$

the maximum value of u. We also assume that $M \in \mathcal{I}_f$, so that the model quantities defined in terms of f are well posed. Throughout this section we fix a solution (Ω, u, f) of problem (5.1), where Ω is a compact domain. We denote by

$$\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$$

a connected component of the complement of the set of maxima of u.

We aim to define a canonical association between the triple $(\mathcal{U}, u, f)$ and a model component

$$(\Omega^{\pm}_{R,M,k}, u_{R,M,k}, f) \in \mathrm{Model}^{\pm}_{M,k,f},$$

extending the Definition 5.1 in [EM25].

Definition 5.2. Under the standing assumptions of Remark 5.1, let $\Gamma \in \pi_0(\partial\Omega)$ be a connected component of the boundary. Then we define

$$\overline{\tau}(\Gamma) := \max\left\{ \frac{|\nabla u|^2(p)}{h_k(M)^2} \,:\, p \in \Gamma \right\}. \tag{5.2}$$

where h_k is defined in (3.4). If $\mathcal{U}$ is a connected component of $\Omega \setminus \mathrm{Max}(u)$ such that $\partial\Omega \cap \mathrm{cl}(\mathcal{U}) \neq \emptyset$, we extend the previous definition as

$$\overline{\tau}(\mathcal{U}) := \max\left\{ \overline{\tau}(\Gamma) \,:\, \Gamma \in \pi_0(\partial\Omega \cap \mathrm{cl}(\mathcal{U})) \right\}. \tag{5.3}$$

Otherwise, we set $\overline{\tau}(\mathcal{U}) = 0$.

Note that the quantity (5.3) doesn't depends on the model manifold $(\mathcal{M}^n_k(F), g_k)$ where the model solution is considered, and in fact we have that

$$\overline{\tau}(\Omega^{\pm}_{R,M,k}) = \overline{\tau}^{\pm}_{M,k}(R), \quad \forall R \in \mathcal{R}_f,$$

where $\overline{\tau}^{\pm}_{M,k}$ are the model $\overline{\tau}$-functions defined in Section 3. These observations lead us to define a correspondence between a general solution to (5.1) and our model solutions in this case using the $\overline{\tau}$-map. The following definitions generalizes [EM25, Definition 5.4].

Definition 5.3. Under the standing assumptions of Remark 5.1. Let

$$\Gamma_0 \in \pi_0\big(\overline{\mathcal{U}} \cap \partial\Omega\big)$$

be the boundary component of $\mathcal{U}$ along $\partial\Omega$ at which the quantity $\overline{\tau}$ attains its maximal value, that is,

$$\overline{\tau}(\Gamma_0) = \overline{\tau}(\mathcal{U}).$$

We can consider (Γ_0, g_0) where g_0 is the Riemannian metric induced by g over Γ_0. Let $R \in \mathcal{R}_f$, then,

- If $\overline{\tau}(\mathcal{U}) \leq \overline{\tau}(\Omega^i_{R,M,k})$ for some $i \in \{+,-\}$. Then we can define a comparison triple associated to $(\mathcal{U}, u, f)$ as a triple $(\Omega^i_{R,M,k}, u_{R,M,k}, f)$ in $\mathcal{M}^n_k(\Gamma_0)$.
- If $\overline{\tau}(\mathcal{U}) = \overline{\tau}(\Omega^i_{R,M,k})$. Then $(\Omega^i_{R,M,k}, u_{R,M,k}, f)$ is a model triple associated to $(\mathcal{U}, u, f)$.

In any of the above cases we will say that $u_{R,M,k}$ is a comparison function associated to u inside $\mathcal{U}$.

Remark 5.4 (Non-uniquness and existence of associated model solution).
• The associated model solution in Definition 5.3 is not necessarily unique. Indeed, if the functions $\overline{\tau}^-_{M,k}$ and $\overline{\tau}^+_{M,k}$ are not monotone, there might exist multiple model solutions corresponding to the same triple $(\mathcal{U}, u, f)$. This non-uniqueness does not interfere with the comparison arguments [ABM25], since sharp estimates for $(\mathcal{U}, u, f)$ can still be obtained even when more than one model solution is available for comparison.
• In contrast, the existence of an associated model triple for $(\mathcal{U}, u, f)$ is not guaranteed in general, even if $M \in \mathcal{I}_f$. Such existence would require that,

$$\overline{\tau}(\mathcal{U}) \in \mathrm{Adm}_{M,k,f}$$

, a condition that is not necessarily satisfied in general. Nevertheless, the existence of associated model solutions can be ensured in some cases. The proof will be established in Section 5.3 in 5.3.1.

Lemma 5.5. *Under the standing assumptions of Remark 5.1, assume that f satisfies the standard conditions. Then:*

1. *If $(\mathcal{M}, g) = \mathbb{M}^n(k)$ with $k \leq 0$, then we have that $\overline{\tau}(\mathcal{U}) \notin \mathrm{Gap}_{M,k,f}$.*
2. *If the $(n-1)$-dimensional part of $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ contains a regular point and u is $\mathcal{C}^2$ in a neighborhood of this point, then necessarily $\overline{\tau}(\mathcal{U}) > \tau^0_k(M)$.*

In particular, if $(\mathcal{M}, g) = \mathbb{M}^n(k)$, $f \in \mathcal{C}^\infty(\mathbb{R})$ satisfies

$$f'(x) \geq nk \qquad \text{for all } x \in \mathcal{I}_f. \tag{5.4}$$

and if $M \in (0, \bar{M}) \subset \mathcal{I}_f$. Then the following holds: if $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ contains a smooth hypersurface, then there exists a unique associated model triple to $(\mathcal{U}, u, f)$ with a positive core radius.

Remark 5.6. The condition (5.4) in the particular case $n = 2$ and $k = 1$ reduces to [EM25, Equation (4.7)].

In the remainder of paper, we derive estimates for the gradient of a solution to (5.1), as well as for the mean curvature and volume of its level sets. These estimates will be obtained by comparing with the corresponding quantities associated to the model solutions introduced in Section 3.

5.2 Gradient estimates

Remark 5.7. Throughout this section we work under the standing assumptions of Remark 5.1, and we fix a comparison triple $(\bar{\mathcal{U}}, \bar{u}, f)$. In addition we assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies the standard conditions together with condition (5.4).

Let $\bar{U}$ be the solution of the ODE (3.1) that generates $\bar{u}$, and by $\bar{R}$ its core radius. We also write

$$\bar{r}_\pm := r_\pm(\bar{R}, M, k).$$

Finally, we denote by $(\bar{\Omega}, \bar{u}, f)$ the model solution in which the comparison triple $(\overline{\mathcal{U}}, \bar{u}, f)$ is contained.

We recall that $(\mathcal{M}, g)$ satisfies the curvature condition $\mathrm{Ric} \geq (n-1)kg$.

Remark 5.8. If the function f is such that $f(0) \geq 0$ when $k \leq 0$ and satisfies (5.4), then we have that f satisfies the standard conditions. In particular, there exists $\mathcal{I}_f \subset \mathbb{R}_+^*$ such that for $M \in \mathcal{I}_f$ we have a well-defined family of model solutions with $\mathcal{R}_f = [0, 2\bar{r}_k)$.

We follow [EM25, Subsection 5.41]. Define the function $G : [0, M] \times [\bar{r}_-, \bar{r}_+] \to \mathbb{R}$ as

$$G(u, r) = u - \bar{U}(r).$$

Then we have that $\frac{\partial G}{\partial r}(u, r) = 0$ if, and only if $\bar{U}' = 0$, that is if, and only if $r = \bar{R}$, so the Implicit Function Theorem implies the existence of two $\mathcal{C}^3$-functions

$$\chi_- : [0, M] \to [\bar{r}_-, \bar{R}] \quad \text{and} \quad \chi_+ : [0, M] \to [\bar{R}, \bar{r}_+] \tag{5.5}$$

such that

$$G(u, \chi_\pm(u)) = 0 \quad \text{for all} \quad u \in [0, M].$$

Definition 5.9. Under the standing assumptions of Remark 5.1, let $(\bar{\mathcal{U}}, \bar{u}, f)$ be a comparison triple associated to $(\mathcal{U}, u, f)$. Then we define the pseudo-radial function Ψ relating the triples $(\bar{\mathcal{U}}, \bar{u}, f)$ and $(\mathcal{U}, u, f)$ as

$$\Psi := \begin{cases} \begin{array}{l} \mathcal{U} \to [\bar{r}_-, \bar{R}], \\ \quad p \mapsto \Psi(p) := \chi_-\big(u(p)\big), \end{array} & \text{if } \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M), \\ \begin{array}{l} \mathcal{U} \to [\bar{R}, \bar{r}_+], \\ \quad p \mapsto \Psi(p) := \chi_+\big(u(p)\big), \end{array} & \text{if } \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M). \end{cases}$$

Define the function

$$\bar{W} = |\nabla \bar{u}|^2 \circ \Psi = \bar{U}'(\Psi)^2 \tag{5.6}$$

and set

$$W = |\nabla u|^2 \ \text{in}\, \mathcal{U}. \tag{5.7}$$

We now reach the main result of this section: the gradient comparison $W \leq \bar{W}$ in $\mathcal{U}$. This estimate plays a central role, as it leads to sharp geometric control of the level sets of solutions to (5.1). Its implications are described in the next section, where several statements generalize results from [ABBM25]. For this reason, many arguments are only outlined in the section, although full computations are provided in the appendix whenever necessary.

Lemma 5.10. *Under the standing assumptions of Remark 5.7. We define the function*

$$F_\beta = \left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}} (W - \bar{W}) \quad \text{in } \mathcal{U}.$$

Then we have that F_β satisfies the following inequality in $\mathcal{U}$:

$$\Delta F_\beta - 2\frac{n-1}{n}\lambda(\Psi)\left\langle \nabla F_\beta, \nabla u\right\rangle - 2\frac{n-1}{n}\frac{|\nabla u|^2}{\bar{U}'(\Psi)^2}\mu(\Psi)F_\beta \geq 0, \tag{5.8}$$

where we have that

$$\lambda(r) = \frac{-\bar{U}''(r) + \cot_k(r)\bar{U}'(r)}{\bar{U}'(r)^2}, \quad \forall r \in [\bar{r}_-, \bar{R}) \cup (\bar{R}, \bar{r}_+], \tag{5.9}$$

and also

$$\mu(r) := f'(\bar{U}(r)) - nk + \frac{n+2}{n}\lambda(r)f(\bar{U}(r)) \tag{5.10}$$

$\mu(r)$ is non-negative in $(\bar{r}_-, \bar{R}) \cup (\bar{R}, \bar{r}_+)$.

The proof of the lemma is presented in the Appendix C.For the case in which $(\bar{\mathcal{U}}, \bar{u}, f)$ is an associated model solution to $(\mathcal{U}, u, f)$, we will need the following definition: Then we state the main theorem of the section.

Theorem 5.11. *Let $(\mathcal{M}, g)$ be a Riemannian manifold with $\mathrm{Ric} \geq (n-1)k\, g$. Let $(\mathcal{U}, u, f)$ be a solution of* (5.1) *in $(\mathcal{M}, g)$. Suppose that $M \in \mathcal{I}_f$ and that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies the standard conditions together with* (5.4)*.*

Then

$$W(p) \leq \bar{W}(p) \qquad \text{for all } p \in \mathcal{U}.$$

If equality holds at a point of $\mathcal{U}$, then

$$\mathrm{Ric}(\nabla u, \nabla u) = (n-1)k\,|\nabla u|^2 \quad \text{in } \mathcal{U},$$

and moreover,

$$(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f).$$

Proof of Theorem 5.11. As a consequence of the above lemma and the strong maximum principle, it is straightforward to check that $W \leq \bar{W}$ and $W(p) = \bar{W}(p)$ in one single point if, and only if $W \equiv \bar{W}$ on $\mathcal{U}$. We shall discuss here the rigidity statement.

By looking at the proof of Lemma 5.10 in Appendix C, we get that the equality $W \equiv \bar{W}$ implies that $\mathrm{Ric}(\nabla u, \nabla u) = (n-1)k\,|\nabla u|^2$ and the hessian of u satisfies the identity

$$\nabla^2 u = -\lambda(\Psi) du \otimes du + \frac{1}{n}\big(\lambda(\Psi)|\nabla u|^2 - f(u)\big)\, g_0,$$

where λ is defined in (5.9) and g_0 denotes the restriction of the metric g restricted to some level set of u inside $\mathcal{U}$.

Then, we get that the function $S_k \circ \Psi$ satisfies that $\nabla^2(S_k \circ \Psi) = s_k' \cdot g$, so Brinkmann's Theorem (see for example [Pet, Theorem 4.3.3]) implies that the metric g has the warped product structure

$$g = d\Psi^2 + s_k(\Psi) g_0, \quad \forall p \in \mathcal{U}.$$

$\square$

Remark 5.12. Note that if $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$, then the level sets of u in $\mathcal{U}$ must be compact, embedded hypersurfaces with constant and equal principal curvatures. This, in itself, imposes a restriction on the geometry of the set $\mathcal{U}$.

For example, if $(\mathcal{M}, g)$ is the space form $\mathbb{M}^n(k)$, then by the classification of isoparametric hypersurfaces with $g = 1$, it must be that $\Gamma = \mathbb{S}^{n-1}$ and $g_0 = g_{\mathbb{S}^{n-1}}$, so that $\mathcal{U}$ is an annular domain and u is radially symmetric.

Moreover, in general, if $\overline{\tau}(\mathcal{U}) = 1$, extending Ψ to $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ we have that $\nabla^2 S_k(\Psi) = s_k'(\Psi) g$ with $S_k \circ \Psi = 1$ and $\nabla(S_k \circ \Psi) = 0$ on this set, so it follows again from Brinkmann's TheoremBrinkmann's Theorem that it must be $g_0 = g_{\mathbb{S}^{n-1}}$ and $\Gamma = \mathbb{S}^n$. This, in turn, implies that $\mathrm{Max}(u) = \{o\}$, $(\mathcal{M}, g) = \mathbb{M}^n(k)$, Ω is a ball and u is radially symmetric with respect to o.

5.3 Consequences of the gradient estimates

In this section we gather several consequences of Theorem 5.11. First, we present a generalization—adapted to our setting—of the results established in [ABBM25]. Since the arguments follow very closely those in [ABBM25] (see also [EM25]), we provide brief comments when necessary and omit the full details. We then establish an isoperimetric-type inequality, derived from the co-area formula, which represents a novel contribution of this work. Finally, we discuss the extension of the estimate for the location of hot spots obtained in [ABM25, Section 5].

5.3.1 Curvature and volume estimates

In this subsection we compare the geometry of the level sets of a solution u within a connected component $\mathcal{U} \subset \Omega \setminus \mathrm{Max}(u)$ with the corresponding level sets of a comparison function $\bar{u}$. The results obtained here generalize those in [EM23, Subsection 4.2] to higher dimensions.

Remark 5.13. In concordance with Remark 5.7, we fix (Ω, u, f) be the triple with $f \in \mathcal{C}^1$ satisfying the standard conditions together with the condition (5.4).

Proposition 5.14. *Under the standing assumptions of Remark 5.7. Let $p \in \partial\Omega$ be a point such that*

$$|\nabla u|^2 (p) = \max_{x \in \partial\Omega \cap \mathrm{cl}(\mathcal{U})} |\nabla u|^2 (x). \tag{5.11}$$

Then, if $H(p)$ denotes the mean curvature of $\partial\Omega$ at p with respect the inner orientation to $\mathcal{U}$, it holds

$$H(p) \leq \cot_k(\bar{r}_+) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \quad \text{and} \quad H(p) \leq -\cot_k(\bar{r}_-) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M),$$

where $\bar{r}_-$ and $\bar{r}_+$ are defined as the zeros of the solution to (3.1)-(3.2) *defining* $(\bar{\Omega}, \bar{u}, f)$.

Proof. Let $p \in \partial\Omega \cap \mathrm{cl}(\mathcal{U})$ be a point satisfying (5.11) and define $\gamma : [0, \varepsilon) \to \mathcal{M}^n$ to be the unit speed geodesic with initial data $\gamma(0) = p$ and $\gamma'(0) = \nabla u / |\nabla u| (p)$.

Then, for t close to zero, one has the Taylor expansions proved in Lemma D.1,

$$W(\gamma(t)) = W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2)$$

and

$$\bar{W}(\gamma(t)) = \bar{W}(p) + 2\left(\mp(n-1)\cot_k(\bar{R})\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2).$$

We take $\bar{r}_+$ when $\overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M)$ and $\bar{r}_-$ otherwise. The Equation (5.11) implies that $W(p) = \bar{W}(p)$. Then the result follows from Theorem 5.11 by comparing the first order terms of the previous expansions. $\square$

Remark 5.15. The above result only applies when $(\bar{\mathcal{U}}, \bar{u}, f)$ is an associated model triple to $(\mathcal{U}, u, f)$. This is the only case in which a point $p \in \partial\Omega$ satisfying (5.11) exists.

Take $p \in \mathrm{Max}(u)$ to be a regular point of the top stratum of u. Then we have the following result.

Proposition 5.16. *Under the standing assumptions of Remark 5.7 with $u \in \mathcal{C}^{s+2}$ and $f \in \mathcal{C}^s$ where $s := \max\{n-2, 1\}$. Let $(\bar{\mathcal{U}}, \bar{u}, f)$ be a comparison triple with core radius $\bar{R} > 0$. Let $\Sigma \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$ be a closed set and $p \in \Sigma$ such that Σ is a smooth hypersurface around p. Then, if $H(p)$ denotes the mean curvature of Σ at p with respect the inner orientation to $\mathcal{U}$, it holds*

$$H(p) \leq -\cot_k(\bar{R}) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \quad \text{and} \quad H(p) \leq -\cot(\bar{R}) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M).$$

Proof. First, we note that since $u = 0$ along $\partial\Omega$ and $u > 0$ inside Ω, it follows that $\Sigma \subset \Omega$. Thus, since $p \in \Sigma$ is a regular point, there exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Sigma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Sigma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Sigma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Sigma) & \text{if } x \in \mathcal{V}_-. \end{cases}$$

Then, by Lemma D.2, we have the following expansions around $p \in \Sigma$:

$$W = f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4),$$

$$\bar{W} = f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right) s\right) + O(s^4),$$

where $H(p)$ is defined with respect to the normal pointing to $\mathcal{V}_+$ and, in the expansion for $\bar{W}$, we take the positive sign when $\overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M)$ and the negative one otherwise. Then the result follows as in the previous proposition, taking into account that $W \leq \bar{W}$ inside $\mathcal{V}_+$ by Theorem 5.11. $\square$

As a consequence of the above result and using the maximum principle, we can now give the proof of Lemma 5.5.

[a] Añadir algún dibujo

Proof. [Proof of Lemma 5.5] [a] Assume that there exists a regular point $p \in \mathrm{Max}(u) \cap \overline{\mathcal{U}}$ such that, in a neighborhood of p the function u is of class $\mathcal{C}^2$. We first show the *first claim*,

$$\overline{\tau}(\mathcal{U}) > \tau_k^0(M).$$

Suppose that $\overline{\tau}(\mathcal{U}) \leq 1$ then we argue by contradiction. Then Proposition 5.16 implies that the mean curvature of $\mathrm{Max}(u)$ at p satisfies

$$|H(p)| \geq \cot_k(R) \qquad \text{for all } R \in (0, \bar{r}_k),$$

which implies that $H(p)$ would be unbounded. This contradicts the assumption that p is a regular point of $\mathrm{Max}(u)$. Hence $\overline{\tau}(\mathcal{U}) > \tau_k^0(M)$.

We now prove the *second claim*. Let $(\mathcal{M}, g) = \mathbb{M}^n(k)$ with $k \leq 0$. We show that

$$\overline{\tau}(\mathcal{U}) \notin \mathrm{Gap}_{u_{\max}, k, f},$$

again the result follows by contradiction.

Since f satisfies the standard conditions and $M \in \mathcal{I}_f$, the maximum principle ensures that $\mathrm{cl}(\mathcal{U}) \cap \partial\Omega \neq \emptyset$. Thus, choose a connected component $\Gamma \subset \mathrm{cl}(\mathcal{U}) \cap \partial\Omega$. Because Γ is part of the level set of a $\mathcal{C}^2$ function and Ω is bounded, Γ is a properly embedded, compact $\mathcal{C}^2$-hypersurface in $\mathbb{M}^n(k)$. By the Jordan–Brouwer Separation Theorem, we have $\mathbb{M}^n(k) \setminus \Gamma = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1$ is open and bounded. It follows that either $\mathcal{U} \subset \mathcal{V}_1$ or $\mathcal{U} \subset \mathcal{V}_2$.

Case 1: $\mathcal{U} \subset \mathcal{V}_1$**.** For any point $p \in \mathcal{U}$, let $u_{R,M,k}$ denote the model solution centered at p, depending on r_p, the distance to p. We use the notation $\Omega^{\pm}_{R,M,k}$ introduced in Section 3 for the regions where $u_{R,M,k}$ is positive and does not attain its maximum value. Choose $R_0 > 0$ sufficiently large so that $\Omega^+_{R_0,M,k} \cap \mathcal{V}_1 = \emptyset$. Then, for $R < R_0$, define

$$w = u_{R,M,k} - u \quad \text{in} \quad \Omega^+_{R_0,u_{\max},k} \cap \mathcal{U}.$$

By decreasing R slightly from R_0, we reach a configuration such that,

1. $w \leq 0$ in $\Omega^+_{R,M,k} \cap \mathcal{U}$,
2. there exists a point q either in $\Omega^+_{R,M,k} \cap \mathcal{U}$ or in $\Gamma^+_{R,M,k} \cap \partial\mathcal{U}$ such that $w(q) = 0$.

Since $\overline{\tau}(\Omega^+_{R,M,k}) < \overline{\tau}(\mathcal{U})$, the strong maximum principle implies that there exists a neighborhood of q such that $u_{R,M,k} = u$, which is a contradiction.

Case 2: $\mathcal{U} \subset \mathcal{V}_2$**.** We choose $p \in \mathcal{V}_1$ and $R_0 > 0$ such that $\Omega^-_{R_0,M,k} \subset \mathcal{V}_1$. Define

$$w = u - u_{R,M,k} \quad \text{in} \quad \Omega^-_{R,M,k} \cap \mathcal{U},$$

and, by taking $R > R_0$ sufficiently large, we again obtain a contradiction with the strong maximum principle.

Both cases being impossible, we conclude that

$$\overline{\tau}(\mathcal{U}) \notin \mathrm{Gap}_{M,k,f}.$$

$\square$

The estimates derived below extend [EM25, Proposition 5.4] to arbitrary dimension and to arbitrary level sets, beyond the zero level set considered in that reference.

Proposition 5.17. *Under the standing assumptions of Remark 5.7 with* $u \in \mathcal{C}^n(\Omega)$*. Let* $(\bar{\mathcal{U}}, \bar{u}, f)$ *be a comparison triple with core radius* $\bar{R} > 0$*.*

Let $t \in [0, u_{\max})$ *be a regular value of* u *and set* $\Gamma_t := u^{-1}(t)$*. Assume that the* $(n-1)$*-dimensional part of* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by* $\Gamma_M \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, consists in a (possibly disconnected)* $\mathcal{C}^n$ *hypersurface. Then,*

$$\mathcal{H}^{n-1}(\Gamma_M) \leq \begin{cases} \left(\dfrac{s_k(\bar{R})}{s_k(\chi_+(t))}\right)^{n-1} \mathcal{H}^{n-1}(\Gamma_t) & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \\ \left(\dfrac{s_k(\bar{R})}{s_k(\chi_-(t))}\right)^{n-1} \mathcal{H}^{n-1}(\Gamma_t) & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M), \end{cases} \tag{5.12}$$

where $\chi_\pm$ *is given in* (5.5)*. Furthermore, equality holds in* (5.12) *for some* t *if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$*.*

Proof. The argument presented below is inspired by [EM25, Proposition 5.4], which treats the case $t = 0$ in dimension 2. For clarity, we include a complete proof showing how the method used there for the zero level set Γ_0 extends to an arbitrary level set Γ_t in the present n–dimensional setting.

Let u be a a solution to (5.1). Then we can define the following vector field,

$$\mathcal{X}(p) := \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)} \nabla u \text{ for all } p \in \Omega \setminus \mathrm{Max}(u). \tag{5.13}$$

Since using the properties of u together with that $\bar{U}$ solves (3.1) and $\nabla\Psi = \nabla u/\bar{U}'(\Psi)$. The divergence of $\mathcal{X}$ is given by the following formula,

$$\mathrm{div}(\mathcal{X}) := \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \left(f(u)(W - \bar{W}) - \cot_k(\Psi) W \right).$$

Now, since $u \in \mathcal{C}^2(\Omega)$, Sard's Theorem implies that the set of critical values of u consists of isolated points. Thus, almost all $t \in [0, M)$ are regular values of u. Given $\varepsilon > 0$, define the set

$$\mathcal{U}_{t,\varepsilon} = \{ p \in \mathcal{U} \, : \, t < u(p) < M - \varepsilon \}.$$

By the above observation, we can consider $\varepsilon > 0$ as small as we want and such that $M - \varepsilon$ is a regular value of u. For fixed $\varepsilon > 0$, the set $\partial\mathcal{U}_{t,\varepsilon}$ consists in a family of complete smooth hypersurfaces, and by applying the usual version of Divergence Theorem in $\mathcal{U}_{t,\varepsilon}$ to the vector field $\mathcal{X}$, given by (5.13), we obtain

$$\begin{aligned} \int_{\mathcal{U}_{t,\varepsilon}} \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \Big(f(u)(W - \bar{W}) - \cot_k(\Psi) W \Big) \, d\mu = \\ \int_{\Gamma_t} \frac{-|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)} \, d\sigma + \int_{\Gamma_{M-\varepsilon}} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)} \, d\sigma, \end{aligned} \tag{5.14}$$

where ν is the inner unit normal to $\partial\mathcal{U}_{t,\varepsilon}$ and $d\mu$ and $d\sigma$ denote the n-dimensional and $(n-1)$-dimensional volume elements induced by the metric of $\mathbb{S}^n$, respectively. Note that $\nu = \nabla u / |\nabla u|$ along Γ_t and $\nu = -\nabla u / |\nabla u|$ along $\Gamma_{M-\epsilon}$.

We first analyze the boundary integrals in the previous identity. Along each (regular) level set Γ_t, the pseudo-radial function is constant, that is, $\Psi = \chi_\pm(t)$, where $\chi_\pm$ is given in (5.5). Hence, using that $W \le \bar{W}$ by Theorem 5.11, we conclude

$$\int_{\Gamma_t} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)} \, d\sigma \begin{cases} \geq -\dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{s_k^{n-1}(\chi_+(t))} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq \dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{s_k^{n-1}(\chi_-(t))} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{5.15}$$

On the other hand, observe that taking a sequence $\varepsilon_n \to 0$ such that $M - \varepsilon_n$ is a regular value of u for all n, and using that $\lim_{x\in\mathcal{U}, x\to \mathrm{Max}(u)} \frac{W}{\bar{W}} = 1$ (this follows by the expansions given in Lemma D.2), we obtain

$$\lim_{n\to+\infty} \int_{\Gamma_{M-\varepsilon_n}} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)}\, d\sigma = \begin{cases} -\dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k^{n-1}(\bar{R})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k^{n-1}(\bar{R})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M), \end{cases} \tag{5.16}$$

Finally, observe that $\bar{U}'(\Psi)^3 < 0$ if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and $\bar{U}'(\Psi)^3 > 0$ if $\overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M)$, so we conclude that

$$\int_{\mathcal{U}} \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \Big(f(u)(W - \bar{W}) - \cot_k(\Psi) W^2 \Big)\, d\mu \begin{cases} \geq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{5.17}$$

Making $\varepsilon_n \to 0$ in (5.14), we get (5.12) after substituting (5.15), (5.16) and (5.17).

For the rigidity statement, note that equality in (5.12) implies that the right hand side of (5.14) is equal to zero. But this implies that $W \equiv \bar{W}$ on $\mathcal{U}$, and then $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$ from the rigidity statement of Theorem 5.11. □

5.3.2 An isoperimetric inequality

In this section, we present an isoperimetric inequality for domains $\mathcal{U}$ bounded by an hypersurface of maximum points of a solution to (5.1).

We recall that for a model solution $(\Omega_{R,M,k}, u_{R,M,k}, f)$ with $R \in \mathcal{R}_f$, we decompose

$$\Omega_{R,M,k} \setminus \Gamma_{R,M,k} = \Omega^+_{R,M,k} \cup \Omega^-_{R,M,k},$$

and we denote by $\mathcal{H}^\ell$ the ℓ-dimensional Hausdorff measure associated with the metric of $\mathcal{M}$. Combining the estimate in Proposition 5.17 with the coarea formula yields the following result.

Proposition 5.18. *Under the standing assumptions of Remark 5.1, suppose that $f \in \mathcal{C}^{n-2}(\mathbb{R})$ satisfying the standard conditions and* (5.4)*. Let $(\bar{\mathcal{U}}, \bar{u}, f)$ be a comparison triple with core radius $\bar{R} > 0$. Assume that the $(n-1)$-dimensional part of* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by Γ_M, consists in a (possibly disconnected) $\mathcal{C}^n$ hypersurface. Then we have that*

$$\frac{\mathcal{H}^n(\mathcal{U})}{\mathcal{H}^{n-1}(\Gamma_M)} \geq \begin{cases} \frac{\int_{\bar{R}}^{\bar{r}_+} s_k(r)^{n-1}\, dr}{s_k(\bar{R})^{n-1}} & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M) \\ \frac{\int_{\bar{r}_-}^{\bar{R}} s_k(r)^{n-1}\, dr}{s_k(\bar{R})^{n-1}} & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M). \end{cases} \tag{5.18}$$

Furthermore, equality holds in (5.18) *if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.

Proof. We will only consider the case in which $\overline{\tau}(\mathcal{U}) \leq \tau_k^+(M)$, and thus $\bar{\mathcal{U}} = \Omega^+_{\bar{R},M,k}$. The case of $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M)$ follows with exactly the same argument.

Observe that $u \in \mathcal{C}^n(\mathcal{U})$, thus Sard's Theorem implies that the set of critical values of u is finite, and we can write $\{t_1, \dots, t_k\} \subset (0, M)$ for this set. Note that, for any $t \in (0, M) \setminus \{t_1, \dots, t_k\}$, $\Gamma_t := u^{-1}(t)$ is a (possibly disconnected) $\mathcal{C}^n$ hypersurface. Then, since any level set of u has zero n-dimensional Hausdorff measure, the coarea formula (see [KP08]) implies that

$$\mathcal{H}^n(\mathcal{U}) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} \, d\sigma_t \right) dt,$$

where we set $t_0 = 0$ and $t_{k+1} = M$, and we write $d\sigma_t$ for the $(n-1)$-dimensional area element along each level set Γ_t. Then, recalling that $\bar{W} = (\bar{U}' \circ \Psi)^2$, where $\bar{U}$ is the solution to (3.1) defining the comparison function $\bar{u}$ and Ψ is given in Definition 5.9, by Theorem 5.11 we get that

$$\mathcal{H}^n(\mathcal{U}) \geq \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt = \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt, \tag{5.19}$$

where the last identity follows from the fact that $\bar{W}$ is always positive when restricted to Γ_t with $t < M$.

Now, observe that $\bar{W}$ is constant along any Γ_t and can be expressed in terms of the functions defined in (5.5) as

$$\bar{W} = \bar{U}(\chi_\pm(t))^2, \quad \forall t \in (0, M),$$

where we take χ_+ when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and χ_- in the other case. Hence, we conclude that

$$\begin{aligned} \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}} \, d\sigma_t \right) dt &= \int_0^M \left(\frac{\mathcal{H}^{n-1}(\Gamma_t)}{|\bar{U}'(\chi_+(t))|} \, d\sigma_t \right) dt \\ &\geq \frac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k(\bar{R})^{n-1}} \int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|} \, dt, \end{aligned} \tag{5.20}$$

where we have used the estimate given in Proposition 5.17 for the last inequality.

Finally, noting that $\chi_+ : [0, M] \to [\bar{R}, \bar{r}_+]$ is monotonically decreasing (where $\bar{r}_+ = r_+(\bar{R}, M)$), we can perform the change of variable $t = \chi_+^{-1}(r)$ in the last integral of above to conclude that

$$\int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|} \, dt = \int_{\bar{R}}^{\bar{r}_+} s_k(r)^{n-1} \, dr.$$

Now (5.18) follows from (5.19) and (5.20).

For the rigidity statement, we note that equality in (5.18) implies equality in (5.19), and then we can argue as in the proof of Proposition 5.17. $\square$

Remark 5.19. Note that in the case in which $(\mathcal{M}, g) = \mathbb{M}^n(k)$ we have that the n-dimensional volume of a geodesic ball of radius R, denoted by B_R, and the $(n-1)$-dimensional area of its boundary are given by

$$\mathcal{H}^n(B_R) = \omega_{n-1} \cdot \int_0^R s_k(r)^{n-1}\, dr \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B_R) = \omega_{n-1} \cdot s_k(R)^{n-1}$$

respectively, where we denote by ω_{n-1} the euclidean volume of $\mathbb{S}^{n-1}$.

It follows that in this case (5.18) takes the form

$$\frac{\mathcal{H}^n(\mathcal{U})}{\mathcal{H}^{n-1}(\Gamma_M)} \geq \begin{cases} \frac{\mathcal{H}^n(\Omega^+_{(\bar{R},M,k)})}{\mathcal{H}^{n-1}(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M) \\ \\ \frac{\mathcal{H}^n(\Omega^-_{(\bar{R},M,k)})}{\mathcal{H}^{n-1}(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M). \end{cases}$$

5.3.3 Location of the hot-spots

In this section, we provide estimates for the distance to the hot spots of a solution (Ω, u, f) to (5.1).These results generalizes those contained in [ABM25, Section 5].

Proposition 5.20. *Under the standing assumptions of Remark 5.7. Then, for any point* $p \in \mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$*, the following holds*

$$\mathrm{dist}(p, \partial\Omega \cap \mathrm{cl}(\mathcal{U})) \geq \begin{cases} \bar{r}_+ - \bar{R} & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \\ \\ \bar{R} - \bar{r}_- & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M), \end{cases} \tag{5.21}$$

where $\bar{r}_-$ *and* $\bar{r}_+$ *are defined as the zeros of the solution to* (3.1)-(3.2) *defining* $(\bar{\Omega}, \bar{u}, f)$*.*

Furthermore, equality in (5.21) *holds if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$*.*

The proof of the above result follows exactly as in [ABM25, Theorem 5.1], so we omit it here. The generalization of [ABM25, Theorem E] deserves a bit more of discussion.

First, we note that condition (5.4) implies that we can bound from above the distance from any point $p \in \Omega$ to the boundary of the domain as done in [MP22, Lemma 2.1]. This is the content of the following result:

Lemma 5.21. *Under the standing assumptions of Remark 5.1 with* $f \in \mathcal{C}^1$ *satisfying* (5.4) *and* $f(0) = 1$*. Suppose that* Ω *is contained in a convex domain of* $\mathcal{M}$*, and define the function* $d(x) = \mathrm{dist}(p, \partial\Omega)$ *for all* $p \in \mathrm{cl}(\Omega)$*. Then we have that*

$$u(p) \geq \frac{2}{n} \cdot \frac{s_k\left(\frac{d(p)}{2}\right)^2}{s_k'(d(p))}, \quad \forall p \in \mathrm{cl}(\Omega). \tag{5.22}$$

Moreover, if the identity happens at some point $p \in \Omega$*, then* u *attains its unique maximum at* p*,* Ω *is a geodesic ball centered at* p*,* u *only depends on the distance to* p *and* $f(x) = nkx + 1$*.*

Proof. The idea is to compare with a solution to Serrin's equation, following the approach used in the proof of [MP22, Lemma 2.1].

For any $p \in \Omega$, let $r_p(q) = \mathrm{dist}(p, q)$ denote the distance from p to q for all $q \in \mathrm{cl}(\Omega)$. Since the curvature bound $\mathrm{Ric} \geq (n-1)k\, g$ holds, we have

$$\Delta r_p(q) \leq (n-1)\cot_k\big(r_p(q)\big), \quad \forall\, q \in \mathrm{cl}(\Omega).$$

Let

$$W^R(r) = \frac{2\left[s_k\left(\frac{R}{2}\right)^2 - s_k\left(\frac{r}{2}\right)^2\right]}{n\, s_k'(r)}, \quad \forall r \in [0, R),$$

be the radial function defining the solution to Serrin's equation on a geodesic ball of radius R in $\mathbb{M}^n(k)$.

Since $f(x) \geq nkx+1$, it follows that $w^R := W^R \circ r_x$ is a subsolution to (5.1) on the geodesic ball of radius R centered at x, denoted $B_R(x)$. By the maximum principle, we conclude that

$$u \geq w^{d(p)} \quad \text{in} \quad B_{d(p)}(p),$$

which yields

$$u(p) \geq w^{d(p)}(p) = \frac{2}{n} \cdot \frac{s_k\left(\frac{d(p)}{2}\right)^2}{s_k'(d(p))}.$$

Now, the rigidity statement follows from the strong maximum principle. $\square$

Given a solution (Ω, u, f) to (5.1), following [ABM25, Section 5] we define the $\overline{\tau}$-function associated to the whole domain Ω as

$$\overline{\tau}(\Omega) := \max\left\{\overline{\tau}(\mathcal{U}) \,:\, \mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))\right\}.$$

Let $\Gamma_0 \in \pi_0(\partial\Omega)$ be such that $\overline{\tau}(\Gamma_0) = \overline{\tau}(\Omega)$. Then we extend Definition 5.3 by saying that a model solution $(\bar{\Omega}, \bar{u}, f)$ in $\mathcal{M}_k^n(\Gamma_0)$ is a comparison triple associated to (Ω, u, f) if $u_{\max} = \bar{M}$ and $\overline{\tau}(\Omega) \leq \overline{\tau}(\bar{\Omega})$, and we say that it is an associated model triple if $\overline{\tau}(\Omega) = \overline{\tau}(\bar{\Omega})$. When we have fixed a comparison triple $(\bar{\Omega}, \bar{u}, f)$, we denote its core radius by $\bar{R}$, and we will also write $\bar{r}_\pm = r_\pm(\bar{R}, M, k)$.

We also define the inner radius of Ω as

$$r_\Omega = \max\left\{\mathrm{dist}(p, \partial\Omega) \,:\, p \in \Omega\right\}.$$

Now we are in a position to prove a generalization of [ABM25, Theorem E] to our setting.

Theorem 5.22. *Let* (Ω, u, f) *be a solution of* (5.1)*, where* $f \in \mathcal{C}^1(\mathbb{R})$ *satisfies the standard conditions together with* (5.4) *(i.e.* $f'(x) \geq nk$ *for all* $x \in \mathcal{I}_f$*), and assume additionally that* $f(0) = 1$ *and* $M \in \mathcal{I}_f$*. Let* $(\bar{\Omega}, \bar{u}, f)$ *be an associated model triple associated with* (Ω, u, f)*.*

Assume that Ω is contained in a convex domain of $\mathcal{M}$ and that $\partial\Omega$ is mean convex. If $k > 0$, suppose further that $\bar{r}_+ \leq \bar{\pi}/(2\sqrt{k})$. Then

$$\frac{\operatorname{dist}(p, \partial\Omega)}{\tan_k(r_\Omega)/n} \geq \frac{\bar{r}_+ - \bar{R}}{\sqrt{\frac{2}{n}M + k\, M^2}} \qquad \textit{for all } p \in \Omega. \tag{5.23}$$

Moreover, if equality holds at a point of Ω, then $(\mathcal{M}, g)$ is isometric to the space form $\mathbb{M}^n(k)$, $f(x) = nkx + 1$, and (Ω, u, f) is a ball solution of Serrin's problem.

Proof. First, we note that the mean convexity of $\partial\Omega$, together with the estimates provided in Proposition 5.14 and the fact that $\bar{r}_+ \leq \pi/2\sqrt{k}$ if $k > 0$ implies that it must be $\overline{\tau}(\Omega) \leq \tau_k^+(M)$. Let $o \in \Omega$ be such that $r_\Omega = \operatorname{dist}(o, \partial\Omega)$ Then, Lemma 5.21 implies that

$$\frac{\tan_k(r_\Omega)^2}{n^2} \leq \frac{2}{n}u(o) + ku(o)^2 \leq \frac{2}{n}M + kM^2,$$

so we get (5.23) from this inequality together with Proposition 5.20.

To prove the rigidity statement, note that equality in (5.23) implies equality in (5.22), so get that $f(x) = nkx + 1$, Ω is a geodesic ball centered at o, u is a function depending only on the distance to o and $\operatorname{Max}(u) = \{o\}$. Now, since $\Omega \setminus \operatorname{Max}(u)$ is connected and we must also have equality in (5.21), we get that g admits the warped product decomposition

$$g = d\Psi^2 + s_k(\Psi)^2 g_0$$

in Ω, where here Ψ is defined in Definition 5.9 and g_0 is the metric restricted to Γ, the connected component of $\partial\Omega$ with $\overline{\tau}(\Gamma) = \overline{\tau}(\Omega)$. Now, since it must be $\overline{\tau}(\Omega) = 1$, the rigidity statement follows from Remark 5.12 □

A Appendix: Isoparametric Hypersurfaces of $\mathbb{S}^n$

Isoparametric hypersurfaces have been extensively studied in the literature in space forms and are defined as those whose principal curvatures are all constant (cf. [CR15, Chapter 3]). In $\mathbb{H}^n$ or $\mathbb{R}^n$, isoparametric hypersurfaces are relatively simple: they are totally umbilic submanifolds, spherical cylinders, or equidistant hypersurfaces to totally geodesic submanifolds of codimension greater than one. The latter case only arises in hyperbolic n-space. In $\mathbb{S}^n$, these hypersurfaces are of particular interest because they provide natural foliations of the sphere by level sets of a polynomial.

A fundamental result in the theory of isoparametric hypersurfaces in $\mathbb{S}^n$ states that they can be described as the level sets of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$, known as a *Cartan-Münzner polynomial*, which satisfies the following differential equations:

$$\bar{\Delta}P(x) = c\,|x|^{l-2}, \quad \text{and} \quad |\bar{\nabla}P(x)|^2 = l^2|x|^{2l-2}, \tag{A.1}$$

where $l, m_1, m_2 \in \mathbb{N}$ and $c = l^2(m_2 - m_1)/2$, and $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplacian and gradient in the ambient Euclidean space $\mathbb{R}^{n+1} \supset \mathbb{S}^n$. When restricted to the sphere, the function $\rho := P_{|\mathbb{S}^n}$ satisfies:

Theorem A.1 ([CR15], Theorem 3.32). *Let $\Gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a connected isoparametric hypersurface with l distinct principal curvatures κ_i, with respective multiplicities m_i, $i \in \{1, \dots, l\}$. Then, Γ is an open subset of a level set of the restriction to $\mathbb{S}^n$ of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree l, and $\rho = P_{|\mathbb{S}^n}$ satisfies the differential equations:*

$$\Delta\rho(p) = -l(l + n - 1)\rho(p) + c \quad \text{and} \quad |\nabla\rho|^2 (p) = l^2(1 - \rho^2(p)), \tag{A.2}$$

for each $p \in \mathbb{S}^n$, where $c = l^2(m_2 - m_1)/2$ and $2(n-1) = l(m_1 + m_2)$. Furthermore, we have $m_i = m_{i+2}$ (indices modulo l) for each $i \in \{1, \dots, l\}$. Here, Δ and ∇ denote the Laplacian and gradient on $\mathbb{S}^n$.

It follows from (A.2) that $\rho(\mathbb{S}^n) = [-1, 1]$, which means that $\{\Gamma_r := \rho^{-1}(r)\}_{r \in [-1,1]}$ forms a foliation of $\mathbb{S}^n$, and for each $r \in (-1, 1)$, the level set Γ_r is a connected isoparametric hypersurface in $\mathbb{S}^n$. The level sets Γ_1 and Γ_{-1} are minimal submanifolds of $\mathbb{S}^n$ of codimension $m_1 + 1$ and $m_2 + 1$, respectively, and they serve as the focal submanifolds of each isoparametric hypersurface in the family $\{\Gamma_r\}_{r \in (-1,1)}$.

A.1 Examples of Isoparametric Hypersurfaces

Isoparametric hypersurfaces in $\mathbb{S}^n$ are characterized by the number of distinct principal curvatures l and their multiplicities $0 < m_1, m_2 \leq n - 1$. After a series of papers by Münzner, it was shown that the only possible values for the number of distinct principal curvatures of an isoparametric hypersurface are $l \in \{1, 2, 3, 4, 6\}$ (see [CR15, Theorem 3.49]).

Below, we describe some families of isoparametric hypersurfaces corresponding to each possible value of l.

Umbilic Hypersurfaces ($l = 1$): The simplest example of an isoparametric hypersurface is a *geodesic sphere*, defined as the set of points at a fixed distance r from a given point $p \in \mathbb{S}^n$. These hypersurfaces are totally umbilic, meaning that all their principal curvatures are equal. Up to an isometry, they can be described as level sets of the height function:

$$\rho(x) = x_{n+1}, \quad x = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n.$$

Each geodesic sphere is given by

$$\Gamma_r = \{x \in \mathbb{S}^n \mid x_{n+1} = r\}, \quad r \in (-1, 1).$$

This family foliates $\mathbb{S}^n$ into parallel hypersurfaces, except for the two limiting cases $r = \pm 1$, which correspond to the focal submanifolds—single points (the poles). It is clear that $\Gamma_r \cong \mathbb{S}^{n-1}$, $r \in (-1, 1)$, and this foliation describes cylindrical coordinates in $\mathbb{S}^n$.

Generalized Clifford Tori ($l = 2$): The *generalized Clifford tori* are a natural extension of the classical Clifford torus in $\mathbb{S}^3$. Let $a, b \in \mathbb{N}$ such that $a \leq b$ and $a + b = n + 1$. For $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$, define the Cartan-Münzner polynomial

$$P^{a,b}(x) = (x_1^2 + \cdots + x_a^2) - (x_{a+1}^2 + \cdots + x_{n+1}^2),$$

and write $\rho^{a,b} = P^{a,b}|_{\mathbb{S}^n}$. A generalized Clifford torus is defined as a level set

$$\mathbb{T}_r^{a,b} = \left\{ p \in \mathbb{S}^n \,\middle|\, \rho^{a,b}(p) = r \right\},$$

for some $r \in (-1, 1)$. Each $\mathbb{T}_r^{a,b}$ is homeomorphic to $\mathbb{S}^{a-1} \times \mathbb{S}^{b-1}$, and the multiplicities of the principal curvatures are $m_1 = a - 1$ and $m_2 = b - 1$. As shown in [CR15, Section 3.8.2], each $\mathbb{T}_r^{a,b}$ is a homogeneous hypersurface generated by the subgroup of isometries $SO(a) \times SO(b)$, and its two focal submanifolds are totally geodesic spheres that are polar to each other.

It is evident that the generalized Clifford torus $\mathbb{T}_r^{a,b}$, being a level set of $P^{a,b}$ on $\mathbb{S}^n$, is antipodally symmetric.

Cartan's Isoparametric Hypersurfaces ($l = 3$): É. Cartan classified isoparametric hypersurfaces Γ in $\mathbb{S}^n$ with three distinct principal curvatures ($l = 3$).

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where $\mathbb{C}, \mathbb{H}, \mathbb{O}$ denote the complex numbers, quaternions, and Cayley numbers (or octonions), respectively, and set $m = \dim(\mathbb{F})$. There is a canonical representation of $\mathbb{F}$ as $\mathbb{R}^m$ equipped with a suitable inner product. A point in $\mathbb{R}^{2+3m}$ can then be expressed as (x, y, X, Y, Z), where $x, y \in \mathbb{R}$ and $X, Y, Z \in \mathbb{F}$. Here, $\overline{X}$ denotes the conjugate of $X \in \mathbb{F}$.

An isoparametric hypersurface $\Gamma \subset \mathbb{S}^{1+3m}$ with three distinct principal curvatures is given as a level set of the Cartan-Münzner polynomial

$$\begin{aligned} P(x, y, X, Y, Z) =& x^3 - 3xy^2 + \frac{3}{2}x\left(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}\right) + \frac{3\sqrt{3}}{2}y\left(X\overline{X} - Y\overline{Y}\right) + \\ &+ \frac{3\sqrt{3}}{2}\left(XYZ + \overline{ZYX}\right). \end{aligned}$$

All principal curvatures of Γ have the same multiplicity $m \in \{1, 2, 4, 8\}$. The focal submanifolds of Γ are a pair of antipodal embeddings of $\mathbb{FP}^2$, where $\mathbb{FP}^2$ denotes the projective plane over the division algebra $\mathbb{F}$.

In a seminal work, Cartan showed that Γ is homogeneous, it arises as the orbit of a closed subgroup of $SO(2 + 3m)$ acting on the sphere. The structure of this group depends on the parameter $m = \dim(\mathbb{F})$, as explained in [CR15, Subsection 3.8.3].

More generally, following Cartan's foundational results, subsequent works established that any homogeneous isoparametric hypersurface in a Euclidean sphere arises as a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two. These symmetric spaces have been fully classified. In particular, [DV, Table 1] provides a complete list of all homogeneous isoparametric hypersurfaces in spheres, indicating:

- the number of distinct principal curvatures l,
- the corresponding multiplicities (m_1, m_2),
- and the symmetric space whose isotropy representation generates the associated homogeneous isoparametric hypersurface.

For example, in the case $m = 1$, we have that Γ is a principal orbit of the isotropy representation of $SU(3)/SO(3)$ in $\mathbb{S}^4$.

Exotic Isoparametric Hypersurfaces ($l = 4$): Unlike the previous cases, the family of isoparametric hypersurfaces with four distinct principal curvatures in Euclidean spheres includes both homogeneous and non-homogeneous examples.

Let $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ denote the pair of multiplicities of the principal curvatures of an isoparametric hypersurface. When $l = 4$, all such hypersurfaces are classified by their multiplicities, and—except for two homogeneous examples with multiplicities $(2, 2)$ and $(4, 5)$—all belong to an infinite family constructed by Ferus, Karcher, and Münzner in [KFM81]. This construction is based on representations of *Clifford algebras*, and a detailed description can be found in [CR15, Section 3.9]. We briefly outline how to construct the Cartan-Münzner polynomials associated with this family.

Let $m_1 = m$ and fix $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let $\mathrm{Sym}_k(\mathbb{R})$ denote the space of $k \times k$ symmetric matrices with real entries. Then, an $(m+1)$-tuple $(P_0, \ldots, P_m)$ of elements in $\mathrm{Sym}_{2n+2}(\mathbb{R})$ is called a *Clifford system* if the matrices P_i satisfy:

$$P_i^2 = \mathrm{Id}_{2n+2}, \quad P_i P_j = -P_j P_i, \quad i \neq j, \quad 0 \leq i, j \leq m,$$

where Id_{2n+2} denotes the identity matrix of dimension $(2n+2)$. Then the function $P : \mathbb{R}^{2n+2} \to \mathbb{R}$ defined by

$$P(x) = \langle x, x \rangle^2 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2, \quad \forall x \in \mathbb{R}^{2n+2}, \tag{A.3}$$

is a Cartan-Münzner polynomial defining an isoparametric hypersurface with four distinct principal curvatures and multiplicities $(m, n-m)$. A level set of this polynomial is called a *Clifford isoparametric hypersurface*.

Endowing $\mathrm{Sym}_k(\mathbb{R})$ with the standard inner product for symmetric matrices, given by

$$\langle A, B \rangle = k^{-1} \mathrm{Trace}(A \cdot B), \quad \forall A, B \in \mathrm{Sym}_k(\mathbb{R}),$$

we define the Clifford sphere associated with the Clifford system $(P_0, \ldots, P_m)$ as

$$\mathcal{C}(P_0, \ldots, P_m) = \left\{ A \in \mathrm{span}\left\{P_0, \ldots, P_m\right\} \; : \; |A|^2 = 1 \right\} \subset \mathrm{Sym}_{2n+2}(\mathbb{R}).$$

Two Clifford systems generate the same isoparametric family if and only if they produce the same *Clifford sphere*. It is known that infinitely many Clifford systems with distinct Clifford spheres can be constructed, producing an infinite family of non-congruent Clifford isoparametric hypersurfaces. It follows from the classification of homogeneous examples that most hypersurfaces in this family are inhomogeneous; this can also be verified independently, see [CR15, Theorem 3.77].

The topology of Clifford hypersurfaces is studied in [Wan88]. If $\rho := P_{\mathbb{S}^{2n+1}}$ for some P as in (A.3), then the focal submanifolds of the family $\{\Gamma_r := \rho^{-1}(r)\}_{r\in(-1,1)}$ are given by $\Gamma_{-1} = \rho^{-1}(-1)$ and $\Gamma_1 = \rho^{-1}(1)$. It follows that Γ_{-1} is diffeomorphic to an $\mathbb{S}^{n+1}$-bundle over $\mathbb{S}^m$, while Γ_1 is a Clifford–Stiefel manifold (see [CR15] for a detailed description of these submanifolds). Furthermore, $\Gamma_{\frac{n-2m+1}{n+1}} = \rho^{-1}((n-2m+1)/(n+1))$ is always a minimal hypersurface diffeomorphic to $\Gamma_1 \times \mathbb{S}^m$.

As a particular example, prior to the work of Ferus et al., É. Cartan discovered an isoparametric family of Clifford type, later generalized by Nomizu [Kat74]. Consider $\mathbb{C}^{n+1}$ as the complex vector space of dimension $n+1$, and write $z = (x, y) \in \mathbb{R}^{n+1} \oplus i\mathbb{R}^{n+1}$. Then, using the Euclidean inner product in $\mathbb{R}^{n+1}$,

$$P((x,y)) = 4\,|x|^2\,|y|^2 - 4\,\langle x, y\rangle^2\,, \quad \forall (x,y) = z \in \mathbb{C}^{n+1},$$

is a Cartan-Münzner polynomial that defines an isoparametric family with four distinct principal curvatures in $\mathbb{S}^{2n+1}$. Each hypersurface in this family is an orbit of the isometric action of $SO(2) \times SO(n+1)$ on $\mathbb{S}^{2n+1}$. In particular, all hypersurfaces in this family are homogeneous.

Exotic Isoparametric Hypersurfaces ($l = 6$) There are only two known isoparametric families with six distinct principal curvatures. Both families are homogeneous and consist of hypersurfaces with principal curvatures of equal multiplicity $m_1 = m_2 = m \in \{1, 2\}$.

In the case $m = 1$, Miyaoka proved in [Miy93] that if $\Gamma \subset \mathbb{S}^7$ is an isoparametric hypersurface with $l = 6$ distinct principal curvatures, then Γ arises as the inverse image, under the Hopf fibration $h : \mathbb{S}^7 \to \mathbb{S}^4$, of a Cartan isoparametric hypersurface with three distinct principal curvatures. It is also shown that the two focal submanifolds of Γ are non-congruent minimal embeddings of $\mathbb{RP}^2 \times \mathbb{S}^3$.

When $m = 2$, the isoparametric family arises from the adjoint orbits of the exceptional compact Lie group G_2 (the automorphism group of the octonions $\mathbb{O}$). The group G_2 acts on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^{14}$ by isometries with respect to the bi-invariant metric. Miyaoka showed that these hypersurfaces are fiber bundles over $\mathbb{S}^6$, with fibers given by Cartan's isoparametric hypersurfaces with three distinct principal curvatures of multiplicity two. Thus, isoparametric hypersurfaces with $(l, m) = (6, 2)$ are closely related to those with $(l, m) = (3, 2)$. It follows that an isoparametric hypersurface $\Gamma \subset \mathbb{S}^{13}$ with six distinct principal curvatures of multiplicity two is diffeomorphic to the homogeneous space G_2/T^2, where $T^2 \subset G_2$ is a subgroup isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ as a Lie group. The focal submanifolds of Γ are both diffeomorphic to $G_2/U(2)$, where $U(2)$ denotes the unitary group of dimension two.

B Appendix: ODE toolkit for f-Extremal Domains in model manifolds

Throughout this appendix we collect and prove the ODE statements used in Sections 3 and 4. We first treat the general (isoparametric) reduction on $\mathbb{S}^n$ and then specialize to the equation related to f-extremal domains in model manifolds. Since most of the arguments are standard and already considered in [EM25], we only sketch the proofs, referring to our previous paper when necessary.

B.1 Solutions related to the isoparametric case

Fix a continuous nonlinearity $f \in C(\mathbb{R})$, positive in $\mathbb{R}_+^*$. We study the ODE

$$V''(s) + \Big((n-1)\cot(\ell s) - \frac{c}{\ell \sin(\ell s)}\Big) V'(s) + f\big(V(s)\big) = 0 \quad \text{on } (0, \pi/\ell), \tag{B.1}$$

where $c = \ell^2(m_2 - m_1)/2$ and m_1, m_2 are the principal-curvature multiplicities of the family.

We analyze the Cauchy problem with interior or endpoint critical data

$$V(S) = M > 0, \qquad V'(S) = 0, \qquad S \in [0, \pi/\ell]. \tag{B.2}$$

First we deal with the non-singular case, i.e., when $S \in (0, \pi/l)$.

Proof of Item (ii) from Lemma 4.1. Existence and uniqueness of a solution $V_{S,M}$ to (B.1)-(B.2) defined within an open interval containing S follows from classical theory of differential equations, so we only have to prove the monotonicity properties of the solution.

Writing the equation as $(\mu V')' + \mu f(V) = 0$ with the smooth integrating factor

$$\mu(s) = \exp\left(\int_S^s \Big((n-1)\cot(\ell x) - \frac{c}{\ell \sin(\ell x)}\Big) dx\right),$$

we obtain

$$V_{S,M}'(s) = -\,\mu(s)^{-1} \int_S^s \mu(x)\, f\big(V(x)\big)\, dx < 0 \quad \text{for } s \in (e_1, e_2),$$

being $(e_1, e_2) \subset (0, \pi/l)$ the maximal interval of definition of $V_{S,M}$. Thus, it follows that $V_{S,M}' > 0$ if $s < S$ and $V_{S,M}' < 0$ if $s > S$. Now the existence of $s_-(S, M)$ and $s_+(S, M)$ follows using the same argument of the proof of Item 2 in [EM25, Theorem 4.1]. □

Now we deal with the existence in the singular case, i.e., when $S \in \{0, \pi/l\}$. In this case, we can proceed with a standard fixed point argument, as done in [EM19; EM25].

Proof of Item (i) from Lemma 4.1. We only deal with the case $S = 0$. The other case is completely analogous. As in [EM19, Subsection 3.2.1], for a non-negative integer k and $\varepsilon \in (0, \pi)$, we denote by $\mathcal{C}_e^k(\varepsilon)$ the space of even $\mathcal{C}^k$-functions on $[-\varepsilon, \varepsilon]$, equipped with the norm

$$\|V\|_{k,\varepsilon} = \sum_{i=1}^{k} \sup_{s\in[-\varepsilon,\varepsilon]} \left|V^{(i)}(s)\right|, \quad \forall V \in \mathcal{C}_e^k(\varepsilon).$$

We also define $\mathcal{C}_e(\varepsilon) := \mathcal{C}_e^0(\varepsilon)$. For any $g \in \mathcal{C}_e(\varepsilon)$, define the function given by

$$A(g)(s) = -\int_0^s \mu(y)^{-1}\Big(\int_0^y \mu(x)\, f\big(W(x)\big)\, dx\Big) dy, \quad \forall s \in (-\pi, \pi) \setminus \{0\} \tag{B.3}$$

and $A(g)(0) = 0$. It is straightforward to check that $A(g)$ is $\mathcal{C}^2$ in $(-\pi/l, \pi/l)$.

Now define the operator $\mathcal{A} : \mathcal{C}_e^1(\varepsilon) \times \mathbb{R}_+ \to \mathcal{C}_e^1(\varepsilon)$ given by

$$\mathcal{T}(V) = A(f(M + V)).$$

Then, taking into account that

$$\cot(ls) = \frac{1}{ls} + O(s) \quad \text{and} \quad \sin(ls) = ls + O(s^3)$$

near zero, we check that there exists constants $C_1, C_2 > 0$ such that

$$C_1 s^{lm_1} \leq \mu(s) \leq C_2 s^{lm_1}$$

for s close to zero. In particular, we get that

$$\|\mathcal{T}(V)\|_{0,\varepsilon} \leq C_3 \cdot \varepsilon^2 \cdot \sup_{[M-\|V\|_{0,\varepsilon}, M+\|V\|_{0,\varepsilon}]} |f|,$$

for some $C_3 > 0$. Thus, if ε is small enough, we get that $\mathcal{T}(B) \subsetneq B$, where $B \subset \mathcal{C}_e(\varepsilon)$ is the closed ball of radius 1 centered at 0. Consequently, by Schauder's Fixed Point Theorem, $\mathcal{T}$ has a fixed point $V_{0,M} \in B$, which is a weak solution to (B.1)-(B.2). But $V_{0,M}$ is $\mathcal{C}^2$ on $(-\pi/l, \pi/l)$ by construction, so it is a classical solution to the equation.

Now the existence of $s_+(0, M)$ and the properties of the solution follows as in the other case. □

B.2 Solutions related to the model manifolds

Consider $f \in \mathcal{C}(\mathbb{R})$ satisfying the standard conditions. We study the equation

$$U''(r) + (n-1)\cot_k(r)\, U'(r) + f\big(U(r)\big) = 0, \qquad r \in (0, 2\bar{r}_k), \tag{B.4}$$

together with the initial conditions

$$U(R) = M > 0, \qquad U'(R) = 0, \qquad R \in [0, 2\bar{r}_k), \tag{B.5}$$

with $\bar{r}_k = +\infty$ if $k \leq 0$ and $\bar{r}_k = \pi/\sqrt{k}$ if $k > 0$.

Now we have to check the monotonicity properties of $U_{R,M,k}$. In particular, we prove that f is an admissible function if it satisfies the standard assumptions.

Proof of Proposition 3.2. When $k > 0$, it is a particular case of that of Lemma 4.1, so we assume that $k \leq 0$. Since the argument is the same in both cases, we only show the proof when $k < 0$.

Given any $R \in [0, +\infty)$ and $M \in \mathcal{I}_f$, the existence of a solution $U_{R,M,k}$ to (B.4)-(B.5) is proved by employing standard arguments (as done in the previous section), so we omit the proof here. Let (e_1, e_2) be the maximal interval of definition of $U_{R,M,k}$. Then, from the implicit representation

$$U_{R,M,k}(r) = M - \int_R^r \frac{1}{s_k(y)^{n-1}} \left(\int_R^y s_k(x)^{n-1} f(U_{R,M,k}(x))\, dx \right) dy,$$

we get that, whenever $U_{R,M,k}(r)$ is positive, $U'_{R,M,k} > 0$ if $r < R$ and $U'_{R,M,k} < 0$ if $r > R$.

The existence of $r_-(R, M, k)$ and of $r_+(R, M, k)$ when $e_2 < +\infty$ is immediate by the boundary behaviour of solutions to differential equations with Lipschitz coefficients. Thus, we only prove the existence of $r_+(R, M, k) > R$ when $e_2 = +\infty$. The case if e_1 is the as e_2

Since f satisfies the standard conditions, suppose first that $f(x) \geq \lambda x$ with $\lambda > \lambda_1(\mathbb{H}^n(k))$. Given a fixed point $p \in \mathbb{H}^n(k)$, define the function $r_p = \mathrm{dist}_{\mathbb{H}^n(k)}(p, \cdot)$. Then, define the function $\tilde{u} = U_{R,M,k} \circ r_p$ inside the set $\mathcal{U} := \{r_p(q) > R\}$. Note that $\tilde{u}$ would be a piece of a model solution in $\mathbb{H}^n(k)$ if there exists $r_-(R, M, k) \leq R < r_+(R, M, k)$ as in Definition 3.1. Now, given that $\lambda > \lambda_1(\mathbb{H}^n(k))$, there exists a function v and a geodesic ball $B \subset \mathbb{H}^n(k)$ centered at p such that v is positive and solves $\Delta v + \lambda v = 0$ inside B, and gets the value zero on ∂B. Arguing by contradiction, suppose that $\tilde{u}$ is positive in all $\mathcal{U}$. Then, there exists $\alpha_0 > 0$ such $\alpha \cdot v < \tilde{u}$ inside $\mathcal{U}$. If α is increases then there exists an $\alpha_1 > \alpha_0$ such that the graphs of $\alpha_1 \cdot v$ and $\tilde{u}$ touches tangentially for the first time. But then, since $f(x) \geq \lambda x$, we conclude that it must be $\tilde{u} \equiv \alpha_1 v$ by the maximum principle, which provides the desired contradiction.

If $f \geq g(x) = nkx + C$ with $C > 0$, we can use the same argument by comparing with a solution to the equation $\Delta v + g(v) = 0$ inside some geodesic ball of $\mathbb{H}^n(k)$, provided that $M \in \mathcal{I}_g = (0, -1/nkC)$ as it is proved in [AFM25, Lemma 5.2]. □

Now suppose further that $f \in \mathcal{C}^1(\mathbb{R})$ and there exists $\bar{M} > 0$ such that

$$f(x) \geq f'(x)x \quad \forall x \in (0, \bar{M}) \subset \mathcal{I}_f$$

In this case, we can prove the monotonicity properties of the functions $\overline{\tau}^{\pm}_{M,k}$ defined in (3.5) stated in Proposition 3.10. The key observation is that the function $Z_{R,k} := \partial_R(U_{R,M,k})$ satisfies the differential equation

$$Z''(r) + (n-1)\cot_k(r)\, Z'(r) + f'\big(U(r)\big) Z(r) = 0, \quad \forall r \in [\bar{r}_-, \bar{r}_+],$$

where we write $\bar{r}_\pm = r_\pm(R, M, k)$. Hence, we can follow the proof of Item 4 in [EM25, Theorem 4.1].

Proof of Proposition 3.10. To get the monotonicity properties of the functions $\overline{\tau}^{\pm}_{M,k}$, it is enough to study the sign of the functions

$$\varphi^{\pm}_{M,k}(R) := \partial_R(U_{R,M,k}(\bar{r}_\pm)^2) = -2\left(U''_{R,M,k}(\bar{r}_\pm) Z_{R,k}(\bar{r}_\pm) - Z'_{R,k}(\bar{r}_\pm) U'_{R,M,k}(\bar{r}_\pm)\right).$$

By proceeding as in the proofs of Claims A and B in [EM25, Theorem 4.1], using the Killing vector field

$$\tilde{Y} = \cos\theta_{n-1} \prod_{j=1}^{n-2} \sin\theta_j \partial_r + \cot_k(r) \left(\cos\theta_{n-1} \sum_{i=1}^{n-2} \frac{\prod_{j=1}^{n-2} \sin\theta_j}{\prod_{j=1}^{i-1} \sin\theta_j} \partial_{\theta_i} - \frac{\sin\theta_{n-1}}{\prod_{j=1}^{n-2} \sin\theta_j} \partial_{\theta_{n-1}} \right)$$

in $\mathbb{M}^n(k)$, we get that $\varphi^-_{M,k}$ and $\varphi^+_{M,k}$ are negative and positive respectively in $[0, \bar{r}_k)$. In particular, $\overline{\tau}^-_{M,k}$ and $\overline{\tau}^+_{M,k}$ are strictly decreasing and increasing functions respectively. Note that $\overline{\tau}^+_{M,k}(0) = 1$ by definition, and it is clear that

$$\lim_{R \to 0} \overline{\tau}^-_{M,k}(R) = +\infty.$$

On the other hand, we have that

$$\lim_{R \to \infty} \overline{\tau}^{\pm}_{M,k}(R) = \tau^{\pm}_k(M) > 0,$$

and the Implicit Function Theorem implies that these quantities depends $\mathcal{C}^1$ in M

In the case $k > 0$, the symmetry of the functions $U_{R,M,k}$ with respect to $\bar{r}_k$ implies that $\overline{\tau}^-_{M,k}(\bar{r}_k) = \overline{\tau}^+_{M,k}(\bar{r}_k)$. □

C Appendix: Gradient estimates

In this appendix we present the proof of Lemma 5.10. We follow [EM25, Section 5.2].

Remark C.1. Following the notation of [EM23], we will denote $\Psi_\pm = \Psi$ and $\chi_\pm = \chi$ and we will do the computations considering both possibilities at the same time. We will also denote the derivatives with respect to u with a dot, and the derivatives with respect to Ψ with a $'$.

The proof follows as in [ABM25, Theorem 3.5]. The idea is to find an elliptic inequality involving the functions W and $\bar{W}$ defined in Section 5.2. Here we follow [EM25, Section 5.5].

We recall that

$$W = |\nabla u|^2 \quad \text{and} \quad \bar{W} = |\nabla \bar{u}|^2 \circ \Psi \quad \text{in } \mathcal{U},$$

where $\Psi : \mathcal{U} \to (\bar{r}_-, R) \cup (R, \bar{r}_+)$ is the pseudo-radial function defined in Definition 5.9. We have that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (5.4) and $\bar{U}$ satisfies the differential equation

$$\bar{U}''(r) + (n-1)\cot_k(r)\bar{U}'(r) + f(\bar{U}) = 0. \tag{C.1}$$

From the definition of Ψ, we have then that $u = \bar{U} \circ \Psi$. Write $F_\beta = \beta \cdot (W - \bar{W})$, where

$$\beta(p) = \left(\frac{s_k(\Psi(p))}{\bar{U}'(\Psi(p))}\right)^{2\frac{n-1}{n}}, \quad \forall p \in \mathcal{U}.$$

Then, the Laplacian of F_β is

$$\begin{aligned} \Delta F_\beta =& \Delta\left(\left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}}\right)(W - \bar{W}) + \left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}} \Delta(W - \bar{W}) \\ &+ 2g\left(\nabla\left(\left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}}\right), \nabla(W - \bar{W})\right). \end{aligned} \tag{C.2}$$

Consequently, each term in the right-hand side of (C.2) can be analyzed separately in order to control the sign of ΔF_β. Analysing the function $\beta(\Psi)$, previously defined, a direct computation yields $\frac{\beta'(\Psi)}{\beta(\Psi)} = \frac{2(n-1)}{n}\lambda(\Psi)\bar{U}'(\Psi)$. Consequently, since $u = \bar{U} \circ \Psi$, we obtain

$$\nabla(\beta(\Psi)) = \frac{\beta'(\Psi)}{\bar{U}'(\Psi)}\nabla u. \tag{C.3}$$

Using relation (C.3), the last term in (C.2) can be expressed entirely in terms of u, F_β and the function $\lambda(r)$, defined by (5.9):

$$g(\nabla\beta(\Psi), \nabla(W - \bar{W})) = \frac{2(n-1)}{n}\lambda(\Psi)g(\nabla u, \nabla F_\beta) - F_\beta\left(\frac{2(n-1)}{n}\right)^2 \lambda(\Psi)^2|\nabla u|^2 \tag{C.4}$$

We next compute the Laplacian of $\beta(\Psi)$ using the equation (C.3) and the definiton of the Laplacian operator on manifolds together with the function $\lambda(r)$ it can be simplified,

$$\begin{aligned} \Delta\beta(\Psi) =& \frac{2(n-1)}{n}\beta(\Psi)\Bigg(\Bigg(\frac{f'(\bar{U}) - nk}{\left(\bar{U}'(\Psi)\right)^2} + \frac{n+2}{n}\frac{\lambda(\Psi)g(\bar{U})}{\left(\bar{U}'(\Psi)\right)^2} \\ &+ \frac{3n-4}{n}\lambda(\Psi)^2\Bigg)|\nabla u|^2 - \lambda(\Psi)g(\bar{U})\Bigg) \end{aligned} \tag{C.5}$$

We analyze the second term in (C.2). Using the Bochner formula together with the bounds of the Ricci curvature tensor to obtain a lower bound for the Laplacian of W and using the definition of u we obtain the lower bound for the for the Laplacian of $W - \bar{W}$

$$\begin{aligned}\Delta(W-\bar{W}) \geq & |\nabla^2 u|^2 + \left(k(n-1) - f'(u)\right)|\nabla u|^2 \\ & - 2\bigg(\frac{1}{n}\left((n-1)\lambda(\Psi) + f(\bar{U})\right) f(u) \\ & + (n-1)\left(k + \cot_k(\Psi)\lambda(\Psi)\bar{U}'(\Psi) - \frac{f'(\bar{U})}{n-1}\right)|\nabla u|^2\bigg)\end{aligned} \tag{C.6}$$

Now, we want to give a bound for the norm of the Hessian of u. Using similar arguments as in [ABBM25; ABM25; EM25] we obtain a lower bound for the Hessian,

$$\begin{aligned}|\nabla^2 u|^2 + & \lambda(\Psi) g(\nabla(W-\bar{W}), \nabla u) - \lambda(u)\left[\frac{2(n-1)}{n}\lambda(\Psi)^2\bar{W} + \frac{g(\bar{U})}{n}\right]|\nabla u|^2 \\ & + \frac{n-1}{n}\lambda(\Psi)^2|\nabla u|^4 - \frac{g(\bar{U})^2}{n} + \frac{2}{n}\lambda(\Psi)g(\bar{U})|\nabla u|^2 \geq 0.\end{aligned}$$

From the previous estimate we obtain,

$$\Delta(W-\bar{W}) \geq -2\lambda(\Psi)g(\nabla(W-\bar{W}),\nabla u) + 2\frac{n-1}{n}\lambda(\Psi)\left[f(u) + \lambda(\Psi)|\nabla u|^2\right](W-\bar{W})$$

Writing the previous inequality in terms of F_β and combining it with (C.4) and (C.5), we obtain the following bound for ΔF_β:

$$\begin{aligned}\Delta F_\beta \geq & \frac{2(n-2)}{n}\lambda(\Psi)g(\nabla u, \nabla F_\beta) \\ & + 2\frac{n-1}{n}\frac{|\nabla u|^2}{\left(\bar{U}'(\Psi)\right)^2}F_\beta\left(f'(U) - nk + \frac{n+2}{n}\lambda(\Psi)f(U)\right).\end{aligned} \tag{C.7}$$

Then, if the function

$$\mu(r) := f'(\bar{U}(r)) - nk + \frac{n+2}{n}\lambda(r)f(\bar{U}(r)) \tag{C.8}$$

defined above were non-negative in $\mathcal{U}$, the inequality (5.8) would be elliptic. Now we show that in fact $\mu \geq 0$ when f satisfies hypothesis (5.4).

Lemma C.2. *Let μ be the function defined by* (C.8)*, and suppose that f satisfies condition* (5.4)*. Then $\mu \geq 0$ in $(\bar{r}_-, R) \cup (R, \bar{r}_+)$.*

Proof. First, note that up to take a dilation of the solution and of the metric, we can restrict to the cases in which $k = -1, 0$, or 1.

Observe that when $k = -1, 0$ and $r \in (\bar{r}_-, \bar{R})$ or $k = 1$ and $r \in (\bar{r}_-, \bar{R})$ or $r \in [\bar{r}/2, \bar{r}_+)$ when $\bar{r}_+ > \bar{r}/2$ it is straightforward to check that $\mu(r) > 0$ using equation (3.1). For the other cases, consider the change of variable

$$r(t) = \begin{cases} \operatorname{arccosh}(t) & \text{if} \quad k = -1 \\ \sqrt{2t} & \text{if} \quad k = 0, \\ \arccos(t) & \text{if} \quad k = 1, \end{cases}$$

and define the function $\bar{V}(t) := \bar{U}(r(t))$. Then it is a straightforward computation to check that

$$\mu(t) = \frac{1}{r'(t)^2}\left(\frac{n+2}{n}\bar{V}''(t)^2 - \bar{V}'''(t)\bar{V}'(t)\right), \quad \forall t \in (\bar{R}, \bar{r}_+).$$

The idea now is to argue as in Item 6 in [EM25, Theorem 4.1]. Since the three cases $k = -1, 0, 1$ are very similar, we only write the proof when $k = 0$.

Define $\bar{T} = \bar{R}^2/2$ and $\bar{t}_+ = \bar{r}_+^2/2$. Then, using that $\bar{U}$ solves (3.1), we get that $\bar{V}$ solves

$$2t\bar{V}''(t) + n\bar{V}'(t) + f(\bar{V}) = 0,$$

and thus $\bar{V}'$ solves

$$2t\bar{V}'''(t) + (n+2)\bar{V}''(t) + f'(\bar{V})\bar{V}'(t) = 0.$$

Integrating this equation between $\bar{T}$ and t, we obtain that

$$\bar{V}''(t) = -\frac{1}{(2t)^{\frac{n+2}{2}}}\left((2\bar{T})^{\frac{n}{2}} f(M) + \int_{\bar{T}}^{t} (2s)^{\frac{n}{2}} f'(\bar{V}(s))\bar{V}'(s)\, ds\right),$$

where we write $M = u_{\text{max}}$. Note that $f' \geq 0$ and $\bar{V}'(t) < 0$ if $t \in (\bar{T}, +\infty)$, so the integral in the above formula is negative when t is in this interval. Now take $\bar{T} < a < b$. Then we have that

$$\begin{aligned} \bar{V}''(a) - \bar{V}''(b) = &\left(\frac{1}{(2b)^{\frac{n+2}{2}}} - \frac{1}{(2a)^{\frac{n+2}{2}}}\right)\left((2\bar{T})^{\frac{n}{2}} f(M) - \int_{\bar{T}}^{a} (2s)^{\frac{n}{2}} f'(\bar{V}(s))\bar{V}'(s)\, ds\right) \\ &+ \frac{1}{(2b)^{\frac{n+2}{2}}}\int_a^b (2s)^{\frac{n}{2}} f'(\bar{V}(s))\bar{V}'(s)\, ds \leq 0. \end{aligned}$$

This implies that $\bar{V}(t)''' \geq 0$ in $(\bar{T}, \bar{t}_+)$, so we conclude that $\mu(t) \geq 0$ in this interval. This concludes the proof of the lemma. $\square$

D Appendix: Taylor expansions on level sets

In this appendix, we prove two useful expansions for the functions $W = |\nabla u|^2$ and $\bar{W} = |\nabla u_{\bar{R},M}| \circ \Psi$ near their zero and top level sets. These expansions are employed in Subsection 5.3.1 to derive curvature estimates. We briefly recall the setting.

Let $(\mathcal{M}, g)$ a n-dimensional Riemannian manifold with bounded Ricci curvature $\mathrm{Ric} \geq (n-1)kg$. Let Ω be a compact C^2 domain and $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ and take $p \in \mathrm{cl}(\mathcal{U}) \cap \partial\Omega$. Consider $(\bar{\mathcal{U}}, \bar{u}, f)$ to be a comparison triple associated to $(\mathcal{U}, u, f)$. Write $M = u_{\max}$ and $\bar{R}$ for the core radius of the triple $(\bar{\mathcal{U}}, \bar{u}, f)$.

Define $\gamma : [0, \varepsilon) \to \mathcal{M}$ as the unit speed geodesic with initial data $\gamma(0) = p$ and $\gamma'(0) = \nabla u / |\nabla u| (p)$. We have the following expansions around p.

Lemma D.1. *For any $p \in \mathrm{cl}(\mathcal{U}) \cap \partial\Omega$ there exists $\varepsilon > 0$ such that for any $t \in [0, \varepsilon)$, the following expansions holds:*

$$
\begin{aligned}
W(\gamma(t)) &= W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2),\\
\bar{W}(\gamma(t)) &= \bar{W}(p) + 2\left(\mp(n-1)\cot_k(\Psi)\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2).
\end{aligned}
\tag{D.1}
$$

Proof. First, observe that $W \circ \gamma$ is a real valued function which is $\mathcal{C}^2$ in a neighborhood of $t = 0$. Thus, it exists $\varepsilon > 0$ such that its Taylor expansion is given by

$$W(\gamma(t)) = W(p) + \langle \nabla W, \nabla u / |\nabla u| \rangle (p)t + O(t^2),$$

for all $t \in (0, \varepsilon)$. Now, by [CR15, Theorem 3.3] it follows that

$$H = \frac{1}{(n-1)|\nabla u|^3}\left(\frac{1}{2}\left\langle \nabla |\nabla u|^2, \nabla u\right\rangle - |\nabla u|^2 \Delta u\right),$$

and then, since $\Delta u(p) = -f(0)$ from (5.1), we conclude that

$$W(\gamma(t)) = W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0)\right)\sqrt{W}(p)t + O(t^2),$$

which is the first of the expansions in (D.1).

Now, in order to compute the Taylor expansion of the $\mathcal{C}^2$ function $\bar{W} \circ \gamma$, we remind that

$$\nabla\bar{W} = 2\left((n-1)\cot_k(\Psi)\bar{U}'(\Psi) - f(u)\right)\bar{U}'(\Psi)$$

where $\bar{U}$ is such that $\bar{u} = \bar{U} \circ \Psi$. Then, we conclude that

$$
\begin{aligned}
\bar{W}(\gamma(t)) &= \bar{W}(p) + \left\langle \nabla\bar{W}, \nabla u / |\nabla u| \right\rangle (p) + O(t^2)\\
&= \bar{W}(p) + 2\left((n-1)\cot_k(\Psi)\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2).
\end{aligned}
$$

□

Now take a smooth hypersurface $\Gamma \subset \mathrm{Max}(u)$, and consider $p \in \Gamma$. Note that it exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Gamma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Gamma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_-. \end{cases} \tag{D.2}$$

Observe that s is smooth in $\mathcal{V}_+$ and $\mathcal{V}_-$ by the regularity results of [KP81].

Now suppose that the function f appearing in (5.1) is smooth. Our goal is to extend the expansions from [EM25, Lemma 5.2] to the n-dimensional setting. While the expansions in that lemma are derived under the assumption that the solution to (5.1) is analytic, it is important to note that the arguments remain valid assuming only $\mathcal{C}^4$ regularity. In this appendix, however, we work with smooth functions for the sake of simplicity.

Lemma D.2. *Suppose that the function f appearing in* (5.1) *is smooth. Then for any $x \in \mathcal{V}_+$, the following expansions holds:*

$$\begin{aligned} W &= f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4), \\ \bar{W} &= f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right)s\right) + O(s^4), \end{aligned} \tag{D.3}$$

where $H(p)$ is computed with respect to the inner normal to $\mathcal{U}$.

Proof. First, write $z = \Psi - \bar{R}$, where Ψ is the pseudo-radial function defined in Definition 5.9. Using the equation (B.1), we compute the Taylor expansion of $M - U \circ \Psi$ as a function of the variable Ψ:

$$u - M = \frac{f(M)}{2} z^2 - \frac{(n-1)f(M)\cot_k(R)}{6} z^3 + O(z^4). \tag{D.4}$$

Next, recalling that $\bar{W} = \bar{U}'(\Psi)^2$, where $\bar{U}$ is the solution to (B.1) defining the comparison triple associated to $(\mathcal{U}, u, f)$, we get that

$$\bar{W}''(\bar{R}) = 2f(M)^2 \quad \text{and} \quad \bar{W}'''(R) = -6(n-1)\cot_k(\bar{R})f(M)^2.$$

Thus, we compute the following Taylor expansion in a neighborhood of $z = 0$,

$$\frac{\bar{W}}{M - u} = 2f(M) + \frac{4(n-1)\cot_k(\bar{R})f(M)}{3} z + O(z^2). \tag{D.5}$$

Finally, using (D.4) we get that

$$z = \pm\sqrt{\frac{2}{f(M)}}\sqrt{M - u} + O(M - u),$$

where we take the positive sign if $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and the negative sign if $\overline{\tau}(\mathcal{U}) < \tau_0(M)$, so substituting this into (D.5) we conclude that

$$\bar{W} = 2f(M)(M-u) - \frac{4(n-1)\sqrt{2f^3(M)}}{3}\cot_k(\bar{R})(M-u)^{3/2} + O((M-u)^2). \quad \text{(D.6)}$$

Next, observe that since f is of class $\mathcal{C}^\infty$ standard regularity results implies that u is smooth, and we have a Taylor expansion of u at $p \in \Gamma$ given by [BCM23, Theorem 3.1] (which works also in the smooth setting, as remarked in [BCM23, Remark 3.2]):

$$u = M - \frac{f(M)}{2}s^2 - \frac{(n-1)f(M)}{6}H(p)s^3 + O(s^4),$$

where $s \in \mathcal{C}^\infty(\mathcal{V}_+)$ is defined in (D.2). Then, we get that

$$\nabla u = -sf(M)\left(1 + \frac{(n-1)H(p)}{2}s + O(s^3)\right)\nabla s, \quad \text{(D.7)}$$

so taking into account that $|\nabla s| = 1$, we conclude that

$$W = f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4).$$

In addition, substituting (D.7) into (D.6), we have that

$$\bar{W} = f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right)s\right) + O(s^4),$$

where we recall that the positive sign is taken if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and the negative is taken otherwise. This finishes the proof of the lemma. $\square$

E Top stratum as C^1–limit of regular levels

Let $(\mathcal{M}^n, g)$ be a smooth Riemannian manifold and $\Omega \subset M$ a relatively compact domain with C^2-boundary. Consider u solving (2.5) with $f \in C^{n-2}(\mathbb{R})$ and satisfying the standing structural hypotheses of Section 3 when needed. By classical elliptic regularity, $u \in C^n(\Omega)$ (and $u \in C^{n-1}(\overline{\Omega})$ when $\partial\Omega$ is C^{n-1}). Hence Sard's theorem applies to u and for a.e. $c \in \mathcal{I}_f$ the level Γ_c is a smooth embedded compact hypersurface, possibly disconnected; moreover, at every regular level $\nabla u \neq 0$ and Σ_c admits a C^n local graph representation by the Implicit Function Theorem on manifolds.

E.1 Upgrading the top stratum from C^1 to C^{n-1} via Morse–Bott

We write $M := \max_\Omega u$, $\Gamma_M \subset \mathrm{Max}(u)$, and assume throughout that Γ_M is a $(n-1)$-dimensional connected component which is an embedded C^1 hypersurface (top stratum). We also assume that $f \in C^{n-2}$ so that $u \in C^n(\Omega)$ by Schauder theory, and that u is *Morse–Bott* along Γ_M namely

$$\ker \nabla^2 u|_{\Gamma_M} = T\Gamma_M, \qquad \nabla^2 u \text{ is nondegenerate on the normal bundle of } \Gamma_M.$$

In our setting, the Moser-Bott condition is warantied since u is a solution to (2.5) and hence $\Delta u = -f(M) < 0$ along Γ_M. This means that the lowest eigenvalue of the hessian $\nabla^2 u$ is always negative and, therefore, $\nabla^2 u$ is nondegenerate along the normal direction of Γ_M, which is well-defined since we are assuming Γ_M is C^1.

Let $U \Subset \Omega$ be a geodesic neighborhood of p such that $\Gamma_M \cap U$ is a C^1-hypersurface and U decomposes as $U = U^- \cup (\Gamma_M \cap U) \cup U^+$, where U^+ is the component of $U \cap \{u < M\}$ such that the signed distance s to Γ_M is positive on U^+.

Proposition E.1 (Morse–Bott upgrade and C^1 convergence of regular levels)**.** *Let $p \in \Gamma_M$ and assume Γ_M is an embedded C^1 hypersurface near p, and that u is Morse–Bott along Γ_M. Then:*

1. *(C^{n-1} **upgrade of the top stratum**) There exist a neighborhood $U \Subset \Omega$ of p, a C^{n-1} diffeomorphism*
$$\Phi : (\Gamma_M \cap U) \times (-\varepsilon, \varepsilon) \longrightarrow U,$$
with $\Phi(\,\cdot\,, 0) = \mathrm{id}$ on $\Gamma_M \cap U$, and a function $\lambda \in C^{n-1}(\Gamma_M \cap U)$ with $\lambda > 0$ such that
$$(u \circ \Phi)(y, s) \;=\; M - \frac{\lambda(y)}{2}\, s^2, \qquad (y, s) \in (\Gamma_M \cap U) \times (-\varepsilon, \varepsilon). \tag{E.1}$$
In particular, $\Gamma_M \cap U = \{s = 0\}$ is a C^{n-1} hypersurface (hence the initial C^1 regularity bootstraps to C^{n-1}).

2. *(**Structure and C^1 limit of regular levels**) By Sard's theorem, there exists a sequence of regular values $c_j \uparrow M$. For j large, the level set*
$$\Sigma_{c_j} := u^{-1}(c_j) \cap U$$
is an embedded C^{n-1} hypersurface that can be written as a normal graph *over $\Gamma_M \cap U$:*
$$\Sigma_{c_j} \cap U \;=\; \big\{(y, s) : s = \sigma_{c_j}(y)\big\}, \qquad \sigma_{c_j}(y) \;=\; \sqrt{\frac{2\,(M - c_j)}{\lambda(y)}} \;\in C^{n-1}(\Gamma_M \cap U), \tag{E.2}$$

where the positive branch is chosen towards the side where $u < M$. Moreover, as $j \to \infty$,

$$\sigma_{c_j} \xrightarrow{C^1_{\mathrm{loc}}(\Gamma_M \cap U)} 0,$$

so that $\Sigma_{c_j} \cap U \to \Gamma_M \cap U$ in the C^1 (graphical) sense. In particular,

$$\nu_{\Sigma_{c_j}} \to \nu_{\Gamma_M} \quad \textit{and} \quad \mathrm{II}_{\Sigma_{c_j}} \to \mathrm{II}_{\Gamma_M} \quad \textit{in } C^0_{\mathrm{loc}}(U).$$

Proof. Step 1 (Morse–Bott normal form). Fix $p \in \Gamma_M$ as in the statement and let s be the signed distance to Γ_M in U, positive towards the component $U^+ \subset U$. Since $u \in C^n$ and u is Morse–Bott along Γ_M, the Morse–Bott lemma (in the C^p category), shrinking U is necessary, provides a C^{n-1} change of coordinates straightening u to the quadratic normal form (E.1); see [BH04, Thm 2]. The proof yields that Φ is a C^{n-1} diffeomorphism and $\lambda \in C^{n-1}$, with $\lambda > 0$ along Γ_M (the positivity corresponds to the fact that M is a local maximum in the normal direction). Therefore $\Gamma_M \cap U = \{s = 0\}$ is C^{n-1}.

Step 2 (Explicit form of nearby regular levels). Fix $c \in (M - \varepsilon', M)$ and consider the equation $u \circ \Phi(y, s) = c$. By (E.1), this is equivalent to

$$M - \frac{\lambda(y)}{2}\, s^2 = c, \qquad \text{i.e.} \qquad s = \sigma_c(y) := \sqrt{\frac{2\,(M-c)}{\lambda(y)}}.$$

Since $\lambda \in C^{n-1}$ and is strictly positive on $\Gamma_M \cap U$, the map $y \mapsto \sigma_c(y)$ belongs to $C^{n-1}(\Gamma_M \cap U)$ for each such c. Sard's theorem for u (which holds as $u \in C^n$ with $n = \dim \mathcal{M}$) gives a sequence $c_j \uparrow M$ with c_j regular; for j large the above representation defines an embedded C^{n-1} hypersurface Σ_{c_j} in U.

Step 3 (C^1 convergence). From (E.2) we have

$$\|\sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \;\le\; \sqrt{\frac{2\,(M-c_j)}{\inf_{\Gamma_M \cap U} \lambda}}, \qquad \nabla^\top \sigma_{c_j}(y) \;=\; -\frac{1}{2}\,\sigma_{c_j}(y)\,\nabla^\top\big(\log \lambda(y)\big),$$

hence

$$\|\nabla^\top \sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \;\le\; C\,\|\sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \xrightarrow[j\to\infty]{} 0.$$

Therefore $\sigma_{c_j} \to 0$ in $C^1_{\mathrm{loc}}(\Gamma_M \cap U)$, and the graphical convergence follows. Since (in normal graph coordinates) ν_{Σ_c} and II_{Σ_c} depend smoothly on $\nabla^\top \sigma_c$ and $\nabla^2_y \sigma_c$, the C^1 convergence implies C^0 convergence of normals and second fundamental forms on compact subsets of U.

Moreover, the Taylor expansion proved in Appendix D yields, as $s \downarrow 0$,

$$u = M - \tfrac{f(M)}{2}\, s^2 - \tfrac{(n-1)f(M)}{6}\, H(p)\, s^3 + O(s^4), \qquad |\nabla u|^2 = f(M)^2 s^2 \Big(1 + (n-1) H(p)\, s + O(s^2)\Big),$$

with $H(p)$ the mean curvature of Γ_M at p (inner normal to U^+); see Lemma D2. Consequently, $|\nabla u| \simeq f(M)\, s$ and, for c close to M, the level $\Sigma_c \cap U$ can be written as a normal graph $s = \sigma_c(y)$ over $y \in \Gamma_M \cap U$ with $\sigma_c \to 0$ in C^1 uniformly on compact subsets (invert the expansion $u(y, s) = c$ by the IFT). $\square$

Remark E.2. **Analytic case:** If, in addition, u is real-analytic and Morse–Bott along Γ_M, the Morse–Bott normal form is analytic and Γ_M is a real-analytic hypersurface; the functions σ_c in (E.2) are then analytic in y.

Lower regularity: If the top stratum Γ_M fails to be a C^1 hypersurface, one still has Hausdorff convergence of Σ_c to Γ_M; however, C^1 convergence may fail across singular points.

Remark E.3 (Para nosotros).

- **Morse–Bott lemma (regularity).** If $u \in C^p$, $p \geq 2$, then there exist local coordinates of class C^{p-1} in which u assumes the quadratic normal form along a critical submanifold; see Banyaga–Hurtubise [BH04, Theorem 2]. In our setting $u \in C^n$, hence the chart and the coefficient λ are C^{n-1}.

  `BanyagaHurtubise04`: A. Banyaga and D. E. Hurtubise, "A proof of the Morse–Bott lemma," *Expositiones Mathematicae* 22 (2004), 365–373.

- **Sard and regular level sets.** Since $u \in C^n$ on an n-dimensional manifold, Sard's theorem applies; regular values $c < M$ yield C^{n-1} embedded level hypersurfaces by the Implicit Function Theorem (IFT). Standard references: Lee, *Introduction to Smooth Manifolds* (regular level set theorem and IFT); Krantz–Parks, *The Implicit Function Theorem*.

- **Elliptic bootstrap to C^n.** With $f \in C^{n-2}$ and smooth coefficients for Δ_g, classical Schauder theory implies $u \in C^n$ (interior, and up to the boundary under standard compatibility); see Evans or Gilbarg–Trudinger.

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