Comparison techniques for $f-$extremal domains in Space Forms

Abstract

Contents

1 Introduction

Asumiremos que $\partial\Omega$ es al menos de clase $\mathcal{C}^2$. De esta forma, las soluciones del problema serán al menos $\mathcal{C}^3$. Basta pedir que $\partial\Omega$ sea regular respecto del operador laplaciano para tener soluciones clásicas, pero creo que en algún momento hace falta más regularidad.

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In [1], the authors study Serrin's equation in the plane, that is, the above equation in the setting $\mathcal{M} = \mathbb{R}^2$ with $f(u) = 2$. The method developed in that work involves associating to each solution (Ω, u, f) of (5.1) a corresponding model solution $(\bar{\Omega}, \bar{u}, f)$ that shares the same *Normalized Wall Shear Stress*—a normalized measure of the maximum normal derivative of the function along the boundary of the domain. A gradient estimate is then obtained by comparing the original solution (Ω, u, f) with its associated model solution $(\bar{\Omega}, \bar{u}, f)$ (see [1, Section 3]).

This same strategy was applied in [12] to address a more general semilinear equation on the 2-sphere $\mathbb{S}^2$, and more recently, the comparison results from [1] were extended to the n-dimensional setting in [2], using the same approach.

In all of these cases, the strategy relies on comparing a given solution to (5.1) with a model solution of the same equation. The key insight in our work is that this restriction is not necessary: the comparison arguments developed in [1] can still be applied even when the model solution comes from a different problem.

2 Preliminaries

This section fixes notation and recalls the basic geometric background used throughout the paper.

2.1 Geometry of hypersurfaces

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold. For any $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$, we write $g(v, w)$ or $\langle v, w\rangle$ for the scalar product of two vectors and $|v|^2 = \langle v, v\rangle$ for the squared norm of a vector.

Let $\Gamma \subset \mathcal{M}$ be an embedded, two-sided, $\mathcal{C}^2$-hypersurface in $\mathcal{M}$. Write $N : \Gamma \to T\mathcal{M}$ for a Gauss map of Γ and, at a given point $p \in \Gamma$, write

$$\mathbb{I}_p(v, w) = -\langle dN_p(v), w\rangle, \quad \forall v, w \in T_p\Gamma,$$

for the *second fundamental form* of Γ at p with respect to N. Then we define the *mean curvature* of Γ at p as

$$H(p) = \frac{1}{n-1}\mathrm{Trace}(\mathbb{I}_p) = -\frac{1}{n-1}\sum_{i=1}^{n-1}\langle dN_p(e_i), e_i\rangle,$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal basis of $T_p\Gamma$.

For any $l \in \mathbb{R}$, let $\mathcal{H}^l$ denote the l-dimensional Hausdorff measure associated to the metric g. Then, for a given open set $\mathcal{O} \subset \mathcal{M}$, we write $\mathcal{H}^n(\mathcal{O})$ for its volume, while we will write $\mathcal{H}^{n-1}(\Gamma)$ for the area of an hypersurface. Note that we don't ask $\mathcal{O}$ or Γ to be connected.

2.2 Model manifolds

Let $\mathbb{M}^n(k)$ denote the complete, simply connected Riemannian n-manifold of constant sectional curvature k; thus $\mathbb{M}^n(k) = \mathbb{S}^n$ if $k > 0$, $\mathbb{R}^n$ if $k = 0$, and $\mathbb{H}^n$ if $k < 0$. Fix $o \in \mathbb{M}^n(k)$ and consider the exponential map

$$\exp_o : B(0, 2\bar{r}_k) \setminus \{0\} \longrightarrow \widetilde{\mathbb{M}}^n(k) \setminus \{o\},$$

where $\bar{r}_k = +\infty$ if $k \leq 0$ and $\bar{r}_k = \pi/(2\sqrt{k})$ if $k > 0$; here $\widetilde{\mathbb{M}}^n(k) = \mathbb{M}^n(k)$ for $k \leq 0$ and $\mathbb{S}^n \setminus \{-o\}$ for $k > 0$. Using polar coordinates on $B(0, \bar{r}_k)$, the map

$$X : (0, 2\bar{r}_k) \times \mathbb{S}^{n-1} \longrightarrow \mathbb{M}^n(k), \qquad X(r, \theta) = \exp_o(r\theta), \tag{2.1}$$

gives normal coordinates centered at o. In these coordinates, the metric is

$$\bar{g}_k \;=\; dr^2 + s_k(r)^2\, g_{\mathbb{S}^{n-1}}. \tag{2.2}$$

Where the function $s_k(r) : I_k \mapsto \mathbb{R}$ is defined as follows,

$$s_k(r) := \begin{cases} \dfrac{\sinh(\sqrt{-k}\, r)}{\sqrt{-k}}, & k < 0, \\ r, & k = 0, \\ \dfrac{\sin(\sqrt{k}\, r)}{\sqrt{k}}, & k > 0, \end{cases} \qquad r \in (0, 2\bar{r}_k).$$

We now introduce auxiliary functions naturally associated to the warping function, $s_k(r)$,

$$\cot_k(r) := \frac{s_k'(r)}{s_k(r)}, \qquad \tan_k(r) := \frac{s_k(r)}{s_k'(r)}, \qquad S_k(r) := \int_0^r s_k(x)\,dx.$$

These functions satisfy the relations,

$$s_k'' + k\,s_k = 0, \qquad s_k(0) = 0, \qquad s_k'(0) = 1, \qquad \cot_k(r) = (\log s_k(r))', \qquad \cot_k(r) = \tan_k(r)^{-1}.$$

These expressions show that each space form admits a global warped product structure with base an interval I_k and fiber the unit sphere $\mathbb{S}^{n-1}$. The function s_k acts as the warping function, determining the curvature and geometry of the model. Under this expression of the metric of a space form the Laplace–Beltrami operator takes the following form for a $u \in C^2(\mathcal{M}_k^n)$

$$\Delta u \;=\; \partial_r^2 u \;+\; (n-1)\,\cot_k(r)\,\partial_r u \;+\; \frac{1}{s_k(r)^2}\,\Delta_{\mathbb{S}^{n-1}} u. \tag{2.3}$$

Remark 2.1. *If a Riemannian manifold $\mathcal{M}$ admits a foliation by totally umbilical hypersurfaces on an open set $U \subset \mathcal{M}$, then U is (locally) isometric to a warped product of the form $dr^2 + \varphi(r)^2 g_F$. In particular, any space form $\mathbb{M}^n(k)$ is locally of this type.*

The expression of the metric in (2.2) is a particular case of a warped product manifold. Therefore, the previous construction can be written in a more general set-up. Let (F, g_F) be a smooth Riemannian manifolds and let (I_k, dr^2) be an interval in $\mathbb{R}$ such that $I_k := (0, 2\bar{r}_k)$ with $\bar{r}_k = +\infty$ for $k \leq 0$ and $\bar{r}_k = \pi/(2\sqrt{k})$ for $k > 0$. On the product manifold $\mathcal{M}_k^n(F) = I_k \times F$, the metric $g_k := dr^2 + s_k(r)^2 g_F$ defines a warped product structure. Therefore, the warped product manifold with warping function $s_k(r)$,

$$(\mathcal{M}_k^n(F), g_k) := \big(I_k \times F,\; dr^2 + s_k(r)^2\, g_F\big),$$

will be referred to as a *model manifold*. In this case the Laplace–Beltrami operator for a given $u \in C^2(\mathcal{M}_k^n(F))$ takes the form,

$$\Delta = \partial_r^2 + (n-1)\;\cot_k(r)\,\partial_r + \frac{1}{s_k(r)^2}\,\Delta_F, \tag{2.4}$$

Remark 2.2. *When $F = \mathbb{S}^{n-1}$ with the round metric,* (2.2) *and* (2.4) *recover the standard expressions on the space forms $\mathbb{M}^n(k)$ for any $n \geq 2$.*

2.3 Overdetermined Elliptic Problems

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, and take $\Omega \subset \mathcal{M}$ a domain with $\mathcal{C}^2$-boundary $\partial\Omega$. We consider the classical overdetermined problem for a function $u : \Omega \to \mathbb{R}$, where $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, given by

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, & (2.5.1) \\ u > 0 & \text{in } \Omega, & (2.5.2) \\ u = 0 & \text{on } \partial\Omega, & (2.5.3) \\ \langle \nabla u, \eta \rangle = \alpha_i & \text{on } \Gamma_i \subset \partial\Omega. & (2.5.4) \end{cases}$$

where Δ and ∇ denote the Laplacian and gradient in $\mathcal{M}$, respectively, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, η is the outward unit normal to $\partial\Omega$, and α_i is a constant on each connected component $\Gamma_i \subset \partial\Omega$.

Let u be a solution to (2.5). Observe that since $u \equiv 0$ on $\partial\Omega$ and $u > 0$ inside Ω, it follows that u reach its maximum value inside Ω. Denote $M := u_{\max} = \max_{p\in\Omega} u$ and its level set as $\Gamma_M = u^{-1}(M) \subset \Sigma$. We say that Γ_M is the *top level set* of u.

Following the notation introduced in [12], a solution to (2.5) is denoted by (Ω, u, f). If (Ω, u, f) is a solution to (2.5) and $\mathcal{I} \in \mathrm{Iso}(\mathcal{M})$, then $(\mathcal{I}(\Omega), u \circ \mathcal{I}^{-1}, f)$ is also a solution to (2.5) since the differential equation (2.5.1) is invariant under isometries. We say that two triples (Ω, u, f) and $(\tilde{\Omega}, \tilde{u}, f)$ are *congruent* if there exists an isometry $\mathcal{I} : \mathcal{M} \to \mathcal{M}$ such that $\mathcal{I}(\Omega) = \tilde{\Omega}$ and $u(\mathcal{I}(p)) = \tilde{u}(p)$ for all $p \in \Omega$. Domains that admit such solutions are referred to as *f-extremal domains* (cf. [21]).

In particular, we are interested in characterizing solutions (Ω, u, f) to (2.5) that exhibit certain symmetries. We say that the triple (Ω, u, f) is $\mathcal{I}$-invariant if $\mathcal{I}(\Omega) = \Omega$ and $u(\mathcal{I}(p)) = u(p)$ for all $p \in \Omega$. In this context, the symmetry properties of solutions play a crucial role in their classification. Let $\mathrm{inj}(\mathcal{M})$ denote the injectivity radius of $\mathcal{M}$ (see [7]). We introduce the following notion.

Definition 2.1. *A f-extremal domain (Ω, u, f) exhibits* radial symmetry *if there exists a point $p_0 \in \mathcal{M}$ and constants $0 \le a < b \le \mathrm{inj}(\mathcal{M})$ such that $\Omega = r_{p_0}^{-1}((a, b))$, and $u = U \circ r_{p_0}$ for some function $U \in \mathcal{C}^2((a, b))$. Here r_{p_0} denote the distance function in $\mathcal{M}$ to the fixed point p_0.*

3 Model Solutions

A natural class of solutions to (2.5) arises on model manifolds $(\mathcal{M}^n_k(F), g_k)$, $n \ge 2$, by imposing that the solution only depends on the base projection $r := \pi_B$. When the model manifold is a space form this means radial symmetry, since we would have $r_o(p) = \mathrm{dist}(o, p)$ for a fixed origin $o \in \mathbb{M}^n(k)$.

Given $U \in \mathcal{C}^2(\mathbb{R})$ one checks directly from (2.4) that $u := U \circ r$ solves the PDE in (2.5) if and only if U satisfies

$$U''(r) + (n-1)\,\cot_k(r)\,U'(r) + f\big(U(r)\big) = 0, \tag{3.1}$$

Following [12, Section 4], we look for solutions to (3.1) satisfying

$$U(R) = M \quad \text{and} \quad U'(R) = 0, \tag{3.2}$$

for some $R \in [0, 2\bar{r}_k)$ and $M > 0$. Here, we recall that $\bar{r}_k = \pi/2\sqrt{k}$ if $k > 0$ and $\bar{r}_k = +\infty$ otherwise. Since our goal is to define a family of solutions to (2.5), we need solutions to (3.1)-(3.2) that vanish at some bounded interval. In particular, we propose the following:

Definition 3.1. *We say that a function $f \in \mathcal{C}(\mathbb{R})$ is* admissible *if there exist non-empty sets $\mathcal{R}_f \subset [0, 2\bar{r}_k)$ and $\mathcal{I}_f \subset \mathbb{R}_+$ such that the following hold: for any $R \in \mathcal{R}_f$ and $M \in \mathcal{I}_f$ there exists a $\mathcal{C}^2$-solution $U_{R,M,k}$ to* (3.1)-(3.2) *defined on the interval $[r_-(R, M, k), r_+(R, M, k)]$ with $0 \le r_-(R, M, k) \le R < r_+(R, M, k) < 2\bar{r}_k$ such that,*

- *If $R = 0$, then $0 = r_-(0, M, k) < r_+(0, M, k)$, $U_{0,M,k} > 0$ on the interval $[0, r_+(0, M, k))$ and $U_{0,M,k}(r_+(0, M, k)) = 0$.*
- *If $R > 0$, then $0 < r_-(R, M, k) < R < r_+(R, M, k)$, $U_{R,M,k} > 0$ on the interval $(r_-(R, M, k), r_+(R, M, k))$ and $U_{R,M,k}(r_\pm(R, M, k)) = 0$.*

If $k = 1$ and $n = 2$, then [12, Theorem 4.1] implies that each positive Lipschitz function $f \in \mathcal{C}(\mathbb{R})$ is admissible. In fact, this follows for general $n \ge 2$ and $k > 0$, as we shall see bellow. However, when $k \le 0$ the situation is very different, as there are functions f for which only positive solutions to (3.1)-(3.2) exists, see [[5], Theorem 2.1]. In this case, we have to impose further conditions on f to ensure that it is an admissible function. For example, we can ask f to satisfy the following:

> **Standard Conditions:** We will say that a continuous function $f \in \mathcal{C}(\mathbb{R})$ satisfy the *standard conditions* if it is Lipschitz continuous and there is an open interval containing zero at it's boundary$\mathcal{I}_f \subset \mathbb{R}_+$ such that within this interval and one of the following conditions holds
>
> - f is positive if $k > 0$,
> - $f \ge C_1 > 0$ for some $C_1 > 0$ if $k = 0$,
> - $f \ge \lambda x$ for some $\lambda > \lambda_1(\mathbb{H}^n(k)) := (n-1)^2 k^2/4$ or $f \ge nkx + C_2$ for some $C_2 > 0$ and it is positive if $k < 0$.

The content of the following result, whose proof is deferred to Appendix B, establishes that the set of admissible functions is included in the set of functions satisfying the standard conditions.

Proposition 3.1. *Let $f \in \mathcal{C}(\mathbb{R})$ be a real function and $k \in \mathbb{R}$. Suppose that f satisfies the standard conditions. Then f is admissible. Furthermore, $\mathcal{R}_f = [0, 2\bar{r}_k)$, and the following hold:*

1. *If $R = 0$, then $U'_{0,M,k} < 0$ on $(0, r_+(R, M, k)]$.*

2. *If $R \in (0, 2\bar{r}_k)$, then $U'_{R,M,k} > 0$ on $[r_-(R,M,k), R)$ and $U'_{R,M,k} < 0$ on $(R, r_+(R,M,k)]$. Moreover,*
$$U_{\pi/\sqrt{k}-R,M,k}(\pi/\sqrt{k} - r) = U_{R,M,k}(r), \quad \text{when } k > 0. \tag{3.3}$$

Remark 3.1. *We note that the standard conditions are sufficient but not necessary in order to ensure that f is an admissible function.*

For $k \in \mathbb{R}$ fixed and f satisfiying the conditions asked in Proposition 3.1, we obtain a two-parameter family of solutions that depends solely on the base projection in the model manifold $(\mathcal{M}^n_k(F), g_k)$ and the maximum value of the function.

When $k > 0$, the solutions $U_{R,M,k}$ are determined up to reflection and it is sufficient to restrict to $R \in [0, \bar{r}_k]$ instead of $R \in [0, 2\bar{r}_k)$. Moreover, when $R = \bar{r}_k$, the solution is symmetric with respect to $r \to \bar{r}_k - r$.

Definition 3.2. *Let $f \in C(\mathbb{R})$ be an admissible function. Then, for any $k \in \mathbb{R}$, $M \in \mathcal{I}_f$ and $R \in \mathcal{R}_f$, set*
$$u_{R,M,k}(p) := U_{R,M,k}(r(p)), \qquad p \in \mathcal{M}^n_k(F),$$
where $U_{R,M,k}$ is the solution to (3.1)–(3.2)*, and define the domain*
$$\Omega_{R,M,k} := \begin{cases} \{\, p \in \mathcal{M}^n_k(F) : r(p) < r_+(0,M,k) \,\}, & R = 0, \\ \{\, p \in \mathcal{M}^n_k(F) : r_-(R,M,k) < r(p) < r_+(R,M,k) \,\}, & R \neq 0. \end{cases}$$
Then we say that the triple $(\Omega_{R,M,k}, u_{R,M,k}, f)$ is a model solution *in $\mathcal{M}^n_k(F)$. Moreover, if $R = 0$, we set*
$$\Gamma_{0,M,k} = u^{-1}_{0,M,k}(M) \text{ and } \Gamma^+_{0,M,k} = \{\, p \in \mathcal{M}^n_k(F) : r_k(p) = r_+(0,M,k) \,\},$$
and we also denote by $\Omega^+_{0,M,k} \subset \Omega_{0,M,k} \setminus \Gamma_{0,M,k}$ the subdomain of $\Omega_{0,M,k}$ without maximum points. Analogously, if either $R \in (0, +\infty)$ when $k \leq 0$ or $R \in (0, \bar{r}_k]$ when $k > 0$, we denote
$$\Gamma_{R,M,k} = u^{-1}_{R,M,k}(M) \text{ and } \Gamma^\pm_{R,M,k} = \{\, p \in \mathcal{M}^n_k(F) : r_k(p) = r_\pm(R,M,k) \,\},$$
and $\Omega^\pm_{R,M,k} \subset \Omega_{R,M,k} \setminus \Gamma_{R,M,k}$ the subdomains of $\Omega_{R,M,k}$ without maximum points such that $\mathrm{cl}(\Omega^\pm_{R,M,k}) \cap \partial\Omega_{R,M,k} = \Gamma^\pm_{R,M,k}$.

Remark 3.2. *The previous construction isolates all positive "radial" solutions dictated by* (3.1)*; it will serve as our reference class when contrasting with the nonradial families. These solutions were used in [12] as models to carry out comparison arguments for general solutions to the Dirichlet problem associated with* (2.5) *when $n = 2$.*

From now on, given a model manifold $(\mathcal{M}^n_k(F), g_k)$, an admissible function f and $M \in \mathcal{I}_f$, we denote by $\mathrm{Model}_{M,k,f}$ the family of model solutions $(\Omega^\pm_{R,M,k}, u_{R,M,k}, f)$, which depends only on $R \in \mathcal{R}_f$. Given any model solution$(\Omega^\pm_{R,M,k}, u_{R,M,k}, f)$, following [1], we will say that the parameter R is its *core radius*.

3.1 Model $\overline{\tau}$-function

For any element of $\mathrm{Model}_{M,k,f}$ with f satisfying the standard conditions and $M \in \mathcal{I}_f$ an important quantity that parametrizes model solutions can be defined:

$$\overline{\tau}^{\pm}_{M,k}(R) := \frac{U'_{R,M,k}(r_{\pm}(R,M,k))^2}{h_k(M)^2} \quad \forall\, R \in (0, 2\bar{r}_k), \tag{3.4}$$

this ration will be named *model $\overline{\tau}$-function* where the function $h_k(M)$ is defined as follows,

$$h_k(M) := U'_{0,M,k}(r_+(0,M,k)), \quad \forall M \in \mathcal{I}_f. \tag{3.5}$$

Note that, see appendix B.3

$$\lim_{R\to 0^+} \overline{\tau}^{-}_{M,k}(R) = +\infty.$$

Define

$$\tau^-_k(M) := \inf_{R\in[0,\bar{r}_k)} \overline{\tau}^{-}_{M,k}(R), \quad \tau^+_k(M) := \sup_{R\in[0,\bar{r}_k)} \overline{\tau}^{+}_{M,k}(R) \quad \text{and } \tau^0_k(M) := \inf_{R\in[0,\bar{r}_k)} \overline{\tau}^{+}_{M,k}(R).$$

[a] Revisa las referencias de lo serrin en los espacios forma KP y Se si está bien dicho

[a] When $k \le 0$, $\overline{\tau}^{-}_{M,k}$ and $\overline{\tau}^{+}_{M,k}$ are continuous functions. Moreover, it follows from the classical work of Serrin in [23] for the case of $k \equiv 0$ and the later work for the hyperbolic and sphere case in [16], that $\tau^-_k(M) \ge \tau^+_k(M)$ only happens in the case that the domain is ball in other cases we must have $\tau^-_k(M) \ge \tau^+_k(M)$ since . Observe also that, given the symmetry (3.3) of $U_{R,M,k}$ when $k > 0$, we have that $\tau^-_k(M) \le \tau^+_k(M)$ in this case. Furthermore, in general we have that $\tau^0_k(M) \le 1$.

Define the set

$$\mathrm{Adm}_{M,k,f} := \overline{\tau}^{-}_{M,k}((0,\bar{r}_k)) \cup \overline{\tau}^{+}_{M,k}((0,\bar{r}_k)), \tag{3.6}$$

which we call *admissible set*. Define also the set $\mathrm{Gap}_{M,k,f} := [\tau^+_k(M), \tau^-_k(M)] \ne \emptyset$, when $k \le 0$, which we call the *gap*. Note that we write $[a,a] = \{a\}$.

[b] No he quitado aquí lo de $\bar{M}$ porque la idea es que podamos tener el intervalo mas pequeño $(0,\bar{M})$

It is not clear, in general, whether the functions $\overline{\tau}^{\pm}_{M,k}$ are monotone. However, we can prove that they are indeed monotone provided that the function f satisfies an additional condition: [b]

Proposition 3.2. *Let $f \in \mathcal{C}^1(\mathbb{R})$ be a function satisfying the standard conditions. Suppose that there exists $\bar{M} > 0$ such that $(0,\bar{M}) \subset \mathcal{I}_f$ and*

$$f(x) \ge x f'(x), \quad \text{for all } x \in (0,\bar{M}). \tag{3.7}$$

Then, for any $M \in (0,\bar{M})$, there exists two values $\tau^-_k(M) \ge \tau^+_k(M)$ depending $\mathcal{C}^1$ on M such that

- $\overline{\tau}^{-}_{M,k} : (0,\bar{r}_k) \to (\tau^-_k(M), +\infty)$ *is decreasing,*

- $\overline{\tau}^{+}_{M,k} : [0, \bar{r}_k) \to [1, \tau^{+}_{k}(M))$ *is increasing.*

Furthermore, $\tau^{-}_{k}(M) = \tau^{+}_{k}(M)$ *when* $k > 0$*.*

The proof of this result can be found in Appendix B. Note that if f satisfies the conditions asked in the above proposition, we have that

$$\mathrm{Adm}_{M,k,f} := \begin{cases} [1, +\infty) & \text{if } k > 0, \\ [1, +\infty) \setminus \mathrm{Gap}_{M,k,f} & \text{if } k \leq 0. \end{cases}$$

4 Exotic solutions on $\mathbb{S}^n$ and on quotients

In contrast with space forms of nonpositive curvature, the sphere $\mathbb{S}^n$ admits rich isoparametric foliations. This allows non-rotational (*exotic*) solutions of (2.5) whose level sets are isoparametric hypersurfaces (see Appendix A). In this section we construct such solutions on $\mathbb{S}^n$ and show that, whenever the foliation is preserved by a group of isometries, the construction descends to domains in a smooth quotient $\mathbb{S}^n/G$.

4.1 Construction along an isoparametric foliation of $\mathbb{S}^n$

Let $\{\Gamma_r\}_{r\in[-1,1]}$ be an isoparametric family in $\mathbb{S}^n$, given by the restriction of a Cartan-Münzner polynomial $\rho \in \mathcal{C}^{\omega}(\mathbb{S}^n)$ via $\Gamma_r := \rho^{-1}(r)$. Let $\ell \in \{1, 2, 3, 4, 6\}$ be the number of distinct principal curvatures and let m_1, m_2 denote their multiplicities (alternating when ℓ is even, coincident when ℓ is odd). Along a normal geodesic to a focal submanifold, the Cartan–Münzner theory (see e.g. [9, §4.1]) yields

$$\rho(p) = \cos\big(\ell\, s(p)\big), \qquad s(p) := \mathrm{dist}_{\mathbb{S}^n}(p, \Gamma_1). \tag{4.1}$$

Let $f \in C(\mathbb{R})$ be a continuous real function with $\mathcal{I}_f \neq \emptyset$. Seeking $v = V \circ s$, a direct computation shows that v solves the PDE in (2.5) if and only if V solves

$$V''(s) + \left((n-1)\cot(\ell s) - \frac{c}{\ell \sin(\ell s)}\right) V'(s) + f\big(V(s)\big) = 0, \tag{4.2}$$

where $c = \frac{\ell^2}{2}(m_2 - m_1)$. We prescribe the Cauchy data

$$V(S) = M > 0, \qquad V'(S) = 0, \qquad S \in [0, \pi/\ell]. \tag{4.3}$$

Standard arguments, presented in Appendix B, on ODE and focal geometry allow us to show: [c]

Lemma 4.1. *Let* $f \in C(\mathbb{R})$ *be a function that is locally Lipschitz in the intervals where is positive, denoted by* $\mathcal{P}_f$*. Then for any* $S \in [0, \pi/\ell]$ *and* $M \in \mathcal{P}_f$ *there exists a unique solution* $V_{S,M}$ *to* (4.2)–(4.3) *with the following properties:*

[c] Aquí creo que se puede pedir que f sea localmente lipschitz en los intervalos que es positiva entonces tenemos que u va a estar en $C^(2,1)_{loc}$

i. If $S = 0$ (resp. $S = \pi/l$), there exists $0 < s_+(0, M) < \pi/\ell$ (resp. $0 < s_-(\pi/l, M) < \pi/\ell$) such that $V_{0,M} > 0$ on $[0, s_+(0, M))$, $V_{0,M}(0) = M$, $V_{0,M}(s_+(0, M)) = 0$, and $V'_{0,M} < 0$ on $(0, s_+(0, M)]$ (resp. $V_{\pi/l,M} > 0$ on $(s_-(\pi/l, M), \pi/l]$, $V_{\pi/l,M}(\pi/l) = M$, $V_{\pi/l,M}(s_-(\pi/l, M)) = 0$, and $V'_{\pi/l,M} > 0$ on $[s_-(\pi/l, M), \pi/l)$).

ii. If $S \in (0, \pi/\ell)$, there exist $0 < s_-(S, M) < S < s_+(S, M) < \pi/\ell$ such that $V_{S,M} > 0$ on (s_-, s_+), $V_{S,M}(S) = M$, $V_{S,M}(s_-(S, M)) = V_{S,M}(s_+(S, M)) = 0$, $V'_{S,M} > 0$ on $(s_-(S, M), S)$, and $V'_{S,M} < 0$ on $(S, s_+(S, M))$.

Then, writing $R = \cos(\ell S) \in [-1, 1]$ for $S \in [0, \pi/\ell]$, the function $v_{R,M} := V_{S,M} \circ s$ defined on

$$\Omega_{R,M} := \begin{cases} \{\, p \in \mathbb{S}^n : \ s(p) < s_+(0, M) \,\} & \text{if } R = 1, \\ \{\, p \in \mathbb{S}^n : \ s(p) > s_-(\pi/l, M) \,\} & \text{if } R = -1, \\ \{\, p \in \mathbb{S}^n : \ s_-(S, M) < s(p) < s_+(S, M) \,\} & \text{if } R \in (-1, 1), \end{cases}$$

solves (2.5) in $\Omega_{R,M}$ with $(v_{R,M})_{\max} = M$ and $\mathrm{Max}(v_{R,M}) = \Gamma_R$.

Theorem 4.1 (Exotic solutions on $\mathbb{S}^n$)**.** *Let $f \in C(\mathbb{R})$ be a Lipschitz function positive in $\mathbb{R}^*_+$. For every isoparametric family $\{\Gamma_r\} \subset \mathbb{S}^n$, every $R \in [-1, 1]$ and every $M > 0$, there exists an f-extremal domain $(\Omega_{R,M}, v_{R,M}, f)$ for (2.5) whose level sets are the leaves of $\{\Gamma_r\}$. In particular, when the leaves are not umbilic (e.g. Cartan's families with $\ell = 3$), these are non-rotational (exotic) solutions.*

Remark 4.1. *For $f(u) = \lambda u$ or $f \equiv 1$, the existence of extremal domains associated to isoparametric foliations (including cases where the top level set is a focal submanifold) is classical; see Savo [22] and Shklover [24] . Our construction yields the same phenomenon for general positive f.*

4.2 Descent to quotients

Let $G \leq \mathrm{Iso}(\mathbb{S}^n)$ be a closed subgroup and $\{\Gamma_r\}_{r\in[-1,1]}$ be an isoparametric family in $\mathbb{S}^n$, $\Gamma_r := \rho^{-1}(r)$. We say that G *preserves* the foliation if $\rho \circ \xi = \rho$ for all $\xi \in G$. Fix an interval of regular leaves $[a, b] \subset (-1, 1)$ and assume the action of G on each Γ_r, $r \in [a, b]$, is free. Then the projection $\pi : \mathbb{S}^n \to \mathbb{S}^n/G$ restricts to smooth embedded hypersurfaces $\pi(\Gamma_r) \subset \mathbb{S}^n/G$, and

$$\Omega^{\Gamma}_{[a,b]} := \bigcup_{r\in[a,b]} \Gamma_r/G \ \subset \ \mathbb{S}^n/G$$

is a smooth domain with smooth boundary $\partial\Omega^{\Gamma}_{[a,b]} = \pi(\Gamma_a) \cup \pi(\Gamma_b)$.

Theorem 4.2 (Exotic solutions on smooth quotients)**.** *Assume G preserves $\{\Gamma_r\}$ and acts freely on the leaves Γ_r for $r \in [a,b]$. Let $(\Omega_{R,M}, v_{R,M}, f)$ be as above with* $\mathrm{Max}(v_{R,M}) = \Gamma_R$ *and $R \in [a,b]$. Then $u := v_{R,M}$ is G-invariant and descends to a function u_G on $\Omega^\Gamma_{[a,b]}$ solving* (2.5) *in the quotient, with the same Dirichlet data and a constant Neumann datum on each connected boundary component.*

Proof. Since u depends only on s and $\rho = \cos(\ell s)$ is G-invariant, u is G-invariant. The free action yields a smooth manifold quotient where the projected leaves are embedded and have constant mean curvature inherited from the isoparametric family, hence the Neumann datum is constant along each boundary component. □

Remark 4.2 (Lens spaces and projective spaces)**.** *(1) If $n = 2m+1$ and the foliation is invariant under the Hopf $\mathbb{S}^1$-action on $\mathbb{S}^{2m+1}$, then it descends to an isoparametric foliation on $\mathbb{CP}^m$ (and analogously $\mathbb{S}^{4m+3} \to \mathbb{HP}^m$); see [9, Prop. 8.1]. Hence, for any interval $[a,b]$ of regular leaves in $\mathbb{S}^{2m+1}$, the corresponding exotic solutions descend to $\mathbb{CP}^m$, and further to any lens space $L(p,q) = \mathbb{S}^{2m+1}/\mathbb{Z}_p$ since the $\mathbb{Z}_p$-action is free.*

(2) For the antipodal quotient $\mathbb{RP}^n = \mathbb{S}^n/\{\pm 1\}$, Cartan–Münzner families with even ℓ produce $\{\pm 1\}$-invariant ρ and thus descend; odd ℓ do not.

Remark 4.3 (Immersed case and singular quotients)**.** *If the G-action has fixed points on some Γ_r, then $\mathbb{S}^n/G$ is an orbifold near the image of the fixed set and $\pi(\Gamma_r)$ is an immersed hypersurface with lower-dimensional singular strata. The function u still descends and solves* (2.5) *on the regular part; the Neumann datum is constant on each connected piece of $\partial\Omega^\Gamma_{[a,b]}$ away from singular strata. For the purposes of a smooth boundary problem, we restrict to free actions on the chosen interval of leaves.*

5 Pseudo-radial Functions

In this section, $(\mathcal{M}, g)$ denotes a n-dimensional riemannian manifold satisfying the curvature bound Ric $\geq (n-1)kg$, for some $k \in \mathbb{R}$. The goal here is to extend the comparison algorithm developed in [1] to the context of general Riemannian manifolds. To do so, given a $\mathcal{C}^2$-domain $\Omega \subset \mathcal{M}$ and a real function f, we consider the general Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

5.1 The comparison triples

Consider a solution (Ω, u, f) to (5.1), where f is a positive function on $\mathbb{R}_+$, and let $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ be an open connected component. We aim to define a canonical association between the triple $(\mathcal{U}, u, f)$ and a component $(\Omega^{\pm}_{R,M,k}, u_{R,M,k}, f) \in \mathrm{Model}_{M,k,f}$ of a model solution in a model manifold. The following definition extends [12, Definition 5.1].

Definition 5.1. *Let* (Ω, u, f) *be a solution to* (5.1). *Let* $\Gamma \in \pi_0(\partial\Omega)$ *be a connected component of the boundary. Then we define*

$$\overline{\tau}(\Gamma) := \max\left\{ \frac{|\nabla u|^2(p)}{h_k(u_{\max})^2} \;:\; p \in \Gamma \right\}. \tag{5.2}$$

where h_k *is defined in* (3.5). *If* $\mathcal{U}$ *is a connected component of* $\Omega \setminus \mathrm{Max}(u)$, $\partial\Omega \cap \mathrm{cl}(\mathcal{U}) \neq \emptyset$, *we extend the previous definition as*

$$\overline{\tau}(\mathcal{U}) := \max\left\{ \overline{\tau}(\Gamma) \;:\; \Gamma \in \pi_0(\partial\Omega \cap \mathrm{cl}(\mathcal{U})) \right\}. \tag{5.3}$$

Otherwise, we set $\overline{\tau}(\mathcal{U}) = 0$.

Note that the quantity (5.3) doesn't depends on the model manifold $(\mathcal{M}^n_k(F), g_k)$ where the model solution is considered, and in fact we have that

$$\overline{\tau}(\Omega^{\pm}_{R,M,k}) = \overline{\tau}^{\pm}_{M,k}(R), \quad \forall R \in \mathcal{R}_f,$$

where $\overline{\tau}^{\pm}_{M,k}$ are the model $\overline{\tau}$-functions defined in Section 3. These observations lead us to define a correspondence between a general solution to (5.1) and our model solutions in this case using the $\overline{\tau}$-map. The following definitions generalizes [12, Definition 5.4].

Definition 5.2. *Let* (Ω, u, f) *be a solution to* (5.1), *let* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *and set* $M = u_{\max}$. *Suppose that* $M \in \mathcal{I}_f$, *take* $\Gamma \in \pi_0(\mathrm{cl}(\mathcal{U}) \cap \partial\Omega)$ *such that* $\overline{\tau}(\Gamma) = \overline{\tau}(\mathcal{U})$, *and write* g_0 *for the Riemannian metric induced by* g *over* Γ.

Let $R \in \mathcal{R}_f$ *be such that* $\overline{\tau}(\mathcal{U}) < \overline{\tau}(\Omega^{i}_{R,M,k})$ *for some* $i \in \{+,-\}$. *Then, we say that* $(\Omega^{i}_{R,M,k}, u_{R,M,k}, f)$ *in* $\mathcal{M}^n_k(\Gamma)$ *is a* comparison triple *associated to* $(\mathcal{U}, u, f)$, *where we consider the fiber in* $\mathcal{M}^n_k(\Gamma)$ *equipped with the Riemannain metric* g_0. *We will also say that* $u_{R,M,k}$ *is a* comparison function *associated to* u *inside* $\mathcal{U}$.

If instead of strict inequality we have that $\overline{\tau}(\mathcal{U}) = \overline{\tau}(\Omega^{i}_{R,M,k})$, *we say that* $(\Omega^{i}_{R,M,k}, u_{R,M,k}, f)$ *is an* model triple associated *to* $(\mathcal{U}, u, f)$. *We will also say that* $u_{R,M,k}$ *is a* model function *associated to* u *inside* $\mathcal{U}$.

Remark 5.1. *It is worth noting that in the above definition, the associated model solution is not necessarily unique. Indeed, if the functions* $\overline{\tau}^{-}_{M,k}$ *and* $\overline{\tau}^{+}_{M,k}$ *are not monotone, there might exist multiple model solutions corresponding to the same triple* $(\mathcal{U}, u, f)$. *This, however, does not affect the validity of the comparison arguments developed in [1], since sharp estimates for* $(\mathcal{U}, u, f)$ *can still be obtained even when more than one model solution is available for comparison.*

Remark 5.2. *It should be noted that, even when $u_{\max} \in \mathcal{I}_f$, the existence of an associated model triple for $(\mathcal{U}, u, f)$ cannot, in general, be guaranteed. Indeed, such existence would require that $\overline{\tau}(\mathcal{U}) \in \mathrm{Adm}_{M,k,f}$, a condition that is not necessarily satisfied in general. However, we can ensure the existence of associated model solutions in some cases, as shown by the following result.*

Lemma 5.1. *Let (Ω, u, f) be a solution to* (5.1) *and $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. Suppose that f satisfies the standard conditions and that $u_{\max} \in \mathcal{I}_f$.*

1. *If the $(n-1)$-dimensional part to $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ contains a regular point and u is smooth in a neighborhood of this point, then it must be $\overline{\tau}(\mathcal{U}) > \tau_k^0(M)$.*

2. *If $(\mathcal{M}, g) = \mathbb{M}^n(k)$ with $k \leq 0$, then we have that $\overline{\tau}(\mathcal{U}) \notin \mathrm{Gap}_{M,k,f}$.*

In particular, if $(\mathcal{M}, g) = \mathbb{M}^n(k)$, $f \in \mathcal{C}^\infty(\mathbb{R})$ satisfies (5.4) *within $(0, \bar{M}) \subset \mathcal{I}_f$ with $M \in (0, \bar{M})$ and $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ contains a smooth hypersurface, then there exists a unique associated model triple to $(\mathcal{U}, u, f)$ with core radius greater than zero.*

We will prove this result in Section 5.3.

In the remainder of paper, we derive estimates for the gradient of a solution to (5.1), as well as for the mean curvature and volume of its level sets. These estimates will be obtained by comparing with the corresponding quantities associated to the model solutions introduced in Section 3.

5.2 Gradient estimates

In this section we will consider solutions to (5.1) with $f \in \mathcal{C}^1(\mathbb{R})$ satisfying the standard conditions and

$$f'(x) \geq nk \quad \forall x \in \mathcal{I}_f, \tag{5.4}$$

where we recall that $(\mathcal{M}, g)$ satisfies the curvature condition $\mathrm{Ric} \geq (n-1)kg$. Note that (5.4) reduces to [12, (4.7)] in the particular case of $n = 2$ and $k = 1$.

Remark 5.3. *If the function f is such that $f(0) > 0$ when $k < 0$ and satisfies* (5.4)*, then we have that f satisfies the standard conditions. In particular, there exists $\mathcal{I}_f \subset \mathbb{R}_+^*$ such that for $M \in \mathcal{I}_f$ we have a well-defined family of model solutions with $\mathcal{R}_f = [0, 2\bar{r}_k)$.*

Let (Ω, u, f) be a solution to (5.1) and take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. Fix a comparison triple $(\bar{\mathcal{U}}, \bar{u}, f)$. For the case in which $(\bar{\mathcal{U}}, \bar{u}, f)$ is an associated model solution to $(\mathcal{U}, u, f)$, we will need the following definition:

Definition 5.3. *We say that the triple $(\mathcal{U}, u, f)$ is* equivalent *to its associated model solution $(\bar{\mathcal{U}}, \bar{u}, f)$, and we write $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$, if $\mathcal{U}$ is isometric to $\pi_B(\bar{\mathcal{U}}) \times_{s_k} \Gamma$ and the triples $(\mathcal{U}, u, f)$ and $(\bar{\mathcal{U}}, \bar{u}, f)$ are congruent.*

Remark 5.4. *From now on, we write $M = u_{\max}$. Furthermore, we write $\bar{U}$ for the solution to the O.D.E.* (3.1) *defining $\bar{u}$, $\bar{R}$ for its core radius and we denote also $r_\pm(\bar{R}, M, k) = \bar{r}_\pm$. We will also write $(\bar{\Omega}, \bar{u}, f)$ for the model solution containing the triple $(\bar{\mathcal{U}}, \bar{u}, f)$.*

We follow [12, Subsection 5.2]. Define the function $G : [0, M] \times [\bar{r}_-, \bar{r}_+] \to \mathbb{R}$ as

$$G(u, r) = u - \bar{U}(r).$$

Then we have that $\frac{\partial G}{\partial r}(u, r) = 0$ if, and only if $\bar{U}' = 0$, that is if, and only if $r = \bar{R}$, so the Implicit Function Theorem implies the existence of two $\mathcal{C}^3$-functions

$$\chi_- : [0, M] \to [\bar{r}_-, \bar{R}] \quad \text{and} \quad \chi_+ : [0, M] \to [\bar{R}, \bar{r}_+] \tag{5.5}$$

such that

$$G(u, \chi_\pm(u)) = 0 \quad \text{for all} \quad u \in [0, M].$$

Definition 5.4. *Let (Ω, u, f) be a solution to* (5.1) *with $u_{\max} =: M \in \mathcal{I}_f$, and let $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. Let $(\bar{\mathcal{U}}, \bar{u}, f)$ be a comparison triple associated to $(\mathcal{U}, u, f)$. Then we define the pseudo-radial function Ψ relating the triples $(\bar{\mathcal{U}}, \bar{u}, f)$ and $(\mathcal{U}, u, f)$ as*

$$\Psi := \begin{cases} \begin{array}{l} \mathcal{U} \to [\bar{r}_-, \bar{R}] \\ \ p \mapsto \Psi(p) := \chi_-(u(p)) \end{array} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M), \\ \\ \begin{array}{l} \mathcal{U} \to [\bar{R}, \bar{r}_+] \\ \ p \mapsto \Psi(p) := \chi_+(u(p)) \end{array} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M). \end{cases}$$

Define the function

$$\bar{W} = |\nabla \bar{u}|^2 \circ \Psi = \bar{U}'(\Psi)^2 \tag{5.6}$$

and set

$$W = |\nabla u|^2 \ \text{in}\, \mathcal{U}. \tag{5.7}$$

We are now ready to state the main result of this section, which is the fact that $W \leq \bar{W}$ in $\mathcal{U}$. This result is significant as it yields sharp estimates on the geometry of the level sets of a solution to (5.1). The consequences of Theorem 5.1 are presented in the following section. Many of the results stated there are generalizations of those in [2], and therefore, many proofs are only sketched. In some instances, however, we include the full computations in the appendix for completeness.

Theorem 5.1. *Let (Ω, u, f) be a solution to* (5.1) *with $f \in \mathcal{C}^1$ satisfying the standard conditions and* (5.4)*. Suppose that $u_{\max} \in \mathcal{I}_f$. Take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$, and consider $(\bar{\mathcal{U}}, \bar{u}, f)$ to be a comparison triple associated to $(\mathcal{U}, u, f)$. Then, it holds*

$$W(p) \leq \bar{W}(p) \ \textit{for all}\ p \in \mathcal{U}.$$

Moreover, if the equality holds at one single point of $\mathcal{U}$, then $\mathrm{Ric}(\nabla u, \nabla u) = (n-1)k\,|\nabla u|^2$ inside $\mathcal{U}$ and $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.

The proof of this result follows from the maximum principle after the following lemma, whose proof is given in Appendix C.

Lemma 5.2. *For $(\mathcal{U}, u, f)$ and $(\bar{\mathcal{U}}, \bar{u}, f)$ satisfying the conditions of Theorem 5.1, define the function*

$$F_\beta = \left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}} (W - \bar{W}) \quad \text{in}\, \mathcal{U}.$$

Then we have that F_β satisfies the following inequality in $\mathcal{U}$:

$$\Delta F_\beta - 2\frac{n-1}{n}\lambda(\Psi)\left\langle \nabla F_\beta, \nabla u\right\rangle - 2\frac{n-1}{n}\frac{|\nabla u|^2}{\bar{U}'(\Psi)^2}\mu(\Psi)F_\beta \geq 0, \tag{5.8}$$

where we have that

$$\lambda(r) = \frac{-\bar{U}''(r) + \cot_k(r)\bar{U}'(r)}{\bar{U}'(r)^2}, \quad \forall r \in [\bar{r}_-, \bar{R}) \cup (\bar{R}, \bar{r}_+], \tag{5.9}$$

and $\mu(r)$ is non-negative in $(\bar{r}_-, \bar{R}) \cup (\bar{R}, \bar{r}_+)$.

Proof of Theorem 5.1. As a consequence of the above lemma and the strong maximum principle, it is straightforward to check that $W \leq \bar{W}$ and $W(p) = \bar{W}(p)$ in one single point if, and only if $W \equiv \bar{W}$ on $\mathcal{U}$. We shall discuss here the rigidity statement.

By looking at the proof of Lemma 5.2 in Appendix C, we get that the equality $W \equiv \bar{W}$ implies that $\mathrm{Ric}(\nabla u, \nabla u) = (n-1)k\,|\nabla u|^2$ and the hessian of u satisfies the identity

$$\nabla^2 u = -\lambda(\Psi) du \otimes du + \frac{1}{n}\left(\lambda(\Psi)|\nabla u|^2 - f(u)\right) g_0,$$

where λ is defined in (C.4) and g_0 denotes the restriction of the metric g to the hypersurface Γ.

Then, we get that the function $S_k \circ \Psi$ satisfies that $\nabla^2(S_k \circ \Psi) = s_k' \cdot g$, so Brinkmann's Theorem (see for example [20, Theorem 4.3.3]) implies that the metric g has the warped product structure

$$g = d\Psi^2 + s_k(\Psi) g_0, \quad \forall p \in \mathcal{U},$$

where g_0 is the metric g restricted to some level set of u inside $\mathcal{U}$. Thus, the result follows. □

Remark 5.5. *Note that if $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$, then the level sets of u in $\mathcal{U}$ must be compact, embedded hypersurfaces with constant and equal principal curvatures. This, in itself, imposes a restriction on the geometry of the set $\mathcal{U}$. For example, if $(\mathcal{M}, g)$ is the space form $\mathbb{M}^n(k)$, then by the classification of isoparametric hypersurfaces with $g = 1$, it must be that $\Gamma = \mathbb{S}^{n-1}$ and $g_0 = g_{\mathbb{S}^{n-1}}$, so that $\mathcal{U}$ is an annular domain and u is radially symmetric.*

Moreover, in general, if $\overline{\tau}(\mathcal{U}) = 1$, extending Ψ to $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ we have that $\nabla^2 S_k(\Psi) = s_k'(\Psi) g$ with $S_k \circ \Psi = 1$ and $\nabla(S_k \circ \Psi) = 0$ on this set, so it follows again from [20, Theorem 4.3.3] that it must be $g_0 = g_{\mathbb{S}^{n-1}}$ and $\Gamma = \mathbb{S}^n$. This, in turn, implies that $\mathrm{Max}(u) = \{o\}$, $(\mathcal{M}, g) = \mathbb{M}^n(k)$, Ω is a ball and u is radially symmetric with respect to o.

5.3 Consequences of the gradient estimates

In this section we gather here several consequences of Theorem 5.1. First, we present a generalization—adapted to our setting—of the results established in [2]. Since the proofs follow very closely those in [2] (see also [12]), we omit the detailed arguments, providing brief comments when appropriate. We then establish an isoperimetric-type inequality, derived from the co-area formula, which represents a novel contribution of this work. Finally, we discuss the extension of the estimate for the location of hot spots obtained in [1, Section 5].

5.3.1 Curvature and volume estimates

Here, we compare the curvature of the level sets of a general solution u within a connected component $\mathcal{U} \subset \Omega \setminus \mathrm{Max}(u)$, with those of a corresponding comparison function $\bar{u}$. The results obtained here constitute generalizations of those presented in [11, Subsection 4.2] to the higher-dimensional setting.

In contrast to the case when the dimension is $n = 2$, curvature estimates for the level sets of u within a component $\mathcal{U}$ become less effective in higher dimensions, particularly when $n > 3$. This limitation arises from the fact that we only have control over the Laplacian of u, which in turn restricts our ability to estimate only the mean curvature of the level sets. Nevertheless, by generalizing the results of [12, Proposition 5.1] and [12, Proposition 5.2], we are able to derive sharp estimates for the mean curvature of the zero level set and the regular part of the top stratum of u within $\mathrm{cl}(\mathcal{U})$.

Proposition 5.1. *Let* (Ω, u, f) *be a solution to* (5.1) *with* $f \in \mathcal{C}^1$ *satisfying the standard conditions and* (5.4). *Suppose that* $M = u_{\max} \in \mathcal{I}_f$. *Take* $\mathcal{U} \in \pi_0 (\Omega \setminus \mathrm{Max}(u))$ *a connected component and* $(\bar{\mathcal{U}}, \bar{u}, f)$ *the associated model triple to* $(\mathcal{U}, u, f)$. *Let* $p \in \partial\Omega$ *be a point such that*

$$|\nabla u|^2 (p) = \max_{\partial\Omega \cap \mathrm{cl}(\mathcal{U})} |\nabla u|^2 . \tag{5.10}$$

Then, if $H(p)$ *denotes the mean curvature of* $\partial\Omega$ *at* p *with respect the inner orientation to* $\mathcal{U}$*, it holds*

$$H(p) \leq \cot_k(\bar{r}_+) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \quad \text{and} \quad H(p) \leq -\cot_k(\bar{r}_-) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M),$$

where $\bar{r}_-$ *and* $\bar{r}_+$ *are defined as the zeros of the solution to* (3.1)-(3.2) *defining* $(\bar{\Omega}, \bar{u}, f)$.

Proof. Let $p \in \partial\Omega \cap \mathrm{cl}(\mathcal{U})$ be a point satisfying (5.10) and define $\gamma : [0, \bar{r}) \to \mathcal{M}^n$ to be the unit speed geodesic with initial data $\gamma(0) = p$ and $\gamma'(0) = \nabla u / |\nabla u| (p)$. Then, for t close to zero, one has the Taylor expansions

$$W(\gamma(t)) = W(p) + 2 \left((n-1)\sqrt{W}(p) H(p) - f(0) \right) \sqrt{W}(p) t + O(t^2)$$

and

$$\bar{W}(\gamma(t)) = \bar{W}(p) + 2\left(\mp(n-1)\cot_k(\bar{R})\sqrt{\bar{W}}(p) - f(0)\right)\sqrt{\bar{W}}(p)t + O(t^2).$$

proved in Lemma D.1, where we take $\bar{r}_+$ when $\overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M)$ and $\bar{r}_-$ otherwise. Since (5.10) implies that $W(p) = \bar{W}(p)$, the result follows from Theorem 5.1 by comparing the first order terms of the previous expansions. $\square$

Remark 5.6. *Observe that the above result only applies when $(\bar{\mathcal{U}}, \bar{u}, f)$ is an associated model triple to $(\mathcal{U}, u, f)$, because this is the only case in which a point $p \in \partial\Omega$ satisfying* (5.10) *exists.*

Now we derive an estimate that works for smooth solutions to (5.1). Although the regularity can certainly be relaxed, we only deal with smooth functions for the sake of simplicity.

Take $p \in \mathrm{Max}(u)$ to be a regular point of the top stratum of u. Then we have the following result.

Proposition 5.2. *Let (Ω, u, f) be a solution to* (5.1) *with $f \in \mathcal{C}^\infty$ satisfying the standard conditions and* (5.4). *Suppose that $M = u_{\max} \in \mathcal{I}_f$. Take $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ a connected component and $(\bar{\mathcal{U}}, \bar{u}, f)$ a comparison triple with core radius $\bar{R} > 0$. Let $\Gamma \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$ be a closed set and $p \in \Gamma$ such that Γ is a smooth hypersurface around p. Then, if $H(p)$ denotes the mean curvature of Γ at p with respect the inner orientation to $\mathcal{U}$, it holds*

$$H(p) \leq -\cot_k(\bar{R}) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \quad \text{and} \quad H(p) \leq -\cot(\bar{R}) \quad \text{if } \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M).$$

Proof. First, we note that since $u = 0$ along $\partial\Omega$ and $u > 0$ inside Ω, it follows that $\Gamma \subset \Omega$. Thus, since $p \in \Gamma$ is a regular point, it exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Gamma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Gamma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_-. \end{cases}$$

Then, by Lemma D.2, we have the following expansions around $p \in \Gamma$:

$$W = f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4),$$

$$\bar{W} = f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right)s\right) + O(s^4),$$

where $H(p)$ is defined with respect to the normal pointing to $\mathcal{V}_+$ and in the expansion for $\bar{W}$ we take the positive sign when $\overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M)$ and the negative one otherwise. Then the result follows as in the previous proposition, taking into account that $W \leq \bar{W}$ inside $\mathcal{V}_+$ by Theorem 5.1. $\square$

As a consequence of the above result and using the maximum principle, we can now give the proof of Lemma 5.1.

Proof of Lemma 5.1. We begin by proving the first item of the lemma, assuming that the $(n-1)$-dimensional part of $\mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$ contains a regular point. We argue by contradiction. Let $p \in \mathrm{Max}(u)$ be such a point, and suppose that $\overline{\tau}(\mathcal{U}) \leq 1$. Then, by Proposition 5.2, the mean curvature of $\mathrm{Max}(u)$ at p satisfies

$$|H(p)| \geq \cot_k(R), \quad \forall R \in (0, \bar{r}_k),$$

which implies that $H(p)$ is unbounded—a contradiction, since p is a regular point of $\mathrm{Max}(u)$.

For the second item, we again proceed by contradiction. Since f satisfies the standard conditions and $M = u_{\max} \in \mathcal{I}_f$, the maximum principle ensures that $\mathrm{cl}(\mathcal{U}) \cap \partial\Omega \neq \emptyset$. Thus, choose a connected component $\Gamma \subset \mathrm{cl}(\mathcal{U}) \cap \partial\Omega$. Because Γ is part of the level set of a $\mathcal{C}^2$ function and Ω is bounded, Γ is a properly embedded, compact $\mathcal{C}^2$-hypersurface in $\mathbb{M}^n(k)$. By the Jordan–Brouwer Separation Theorem, we have $\mathbb{M}^n(k) \setminus \Gamma = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1$ is open and bounded. It follows that either $\mathcal{U} \subset \mathcal{V}_1$ or $\mathcal{U} \subset \mathcal{V}_2$.

Suppose first that $\mathcal{U} \subset \mathcal{V}_1$. For any point $p \in \mathcal{U}$, let $u_{R,M,k}$ denote the model solution centered at p, depending on r_p, the distance to p. We use the notation $\Omega^{\pm}_{R,M,k}$ introduced in Section 3 for the regions where $u_{R,M,k}$ is positive and does not attain its maximum value. Choose $R_0 > 0$ sufficiently large so that $\Omega^{+}_{R_0,M,k} \cap \mathcal{V}_1 = \emptyset$. Then, for $R < R_0$, define

$$w = u_{R,M,k} - u \quad \text{in} \quad \Omega^{+}_{R_0,M,k} \cap \mathcal{U}.$$

By decreasing R slightly from R_0, we reach a configuration where $w \leq 0$ in $\Omega^{+}_{R,M,k} \cap \mathcal{U}$, and there exists a point q either in $\Omega^{+}_{R,M,k} \cap \mathcal{U}$ or in $\Gamma^{+}_{R,M,k} \cap \partial\mathcal{U}$ such that $w(q) = 0$. However, since $\overline{\tau}(\Omega^{+}_{R,M,k}) < \overline{\tau}(\mathcal{U})$, the strong maximum principle implies that $u_{R,M,k} = u$ in a neighborhood of q, which is a contradiction.

If instead $\mathcal{U} \subset \mathcal{V}_2$, we choose $p \in \mathcal{V}_1$ and $R_0 > 0$ such that $\Omega^{-}_{R_0,M,k} \subset \mathcal{V}_1$. Define

$$w = u - u_{R,M,k} \quad \text{in} \quad \Omega^{-}_{R,M,k} \cap \mathcal{U},$$

and, by taking $R > R_0$ sufficiently large, we again obtain a contradiction with the strong maximum principle. $\square$

Now we shall present some estimates for the area of the level sets of a given smooth solution (Ω, u, f) to (5.1), and also for the volume of a given domain $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$. This result is a generalization of the estimate established in [12, Proposition 5.4], extending it to the n-dimensional setting and to more general level sets.

Proposition 5.3. *Let* (Ω, u, f) *be a solution to* (5.1)*, where* $f \in \mathcal{C}^\infty(\mathbb{R})$ *is a function satisfying the standard conditions and* (5.4)*. Suppose that* $u_{\max} \in \mathcal{I}_f$*. Let* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *be a connected component and* $(\bar{\mathcal{U}}, \bar{u}, f)$ *a comparison triple with core radius* $\bar{R} > 0$*.*

Let $t \in [0,M)$ be a regular value of u and set $\Gamma_t := u^{-1}(t)$. Assume that the $(n-1)$-dimensional part of $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by $\Gamma_M \subset \mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$, consists in a (possibly disconnected) smooth hypersurface. Then,*

$$\mathcal{H}^{n-1}(\Gamma_M) \leq \begin{cases} \left(\dfrac{s_k(\bar{R})}{s_k(\chi_+(t))}\right)^{n-1} \mathcal{H}^{n-1}(\Gamma_t) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \\ \left(\dfrac{s_k(\bar{R})}{s_k(\chi_-(t))}\right)^{n-1} \mathcal{H}^{n-1}(\Gamma_t) & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M), \end{cases} \tag{5.11}$$

where $\chi_\pm$ is given in (5.5)*. Furthermore, equality holds in* (5.11) *if, and only if $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.*

Proof. The proof of this result follows closely that of [12, Proposition 5.4], where it was proved only for the case $t = 0$ in dimension 2. However, we present here a detailed proof to show how the arguments made for the zero level sets Γ_0 in [12, Proposition 5.4] work when a general level set Γ_t is considered.

First, define the vector field

$$\mathcal{X}(p) := \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)} \nabla u \text{ for all } p \in \Omega \setminus \mathrm{Max}(u). \tag{5.12}$$

Then, using that u is a solution to (5.1), $\bar{U}$ solves (3.1) and $\nabla\Psi = \nabla u / \bar{U}'(\Psi)$, one can check that its divergence is given by

$$\mathrm{div}(\mathcal{X}) := \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \left(f(u)(W - \bar{W}) - \cot(\Psi) W \right).$$

Now, since $u \in \mathcal{C}^2(\Omega)$, Sard's Theorem implies that the set of critical values of u consists of isolated points. Thus, almost all $t \in [0, M)$ is a regular value of u. Given $\varepsilon > 0$, define the set

$$\mathcal{U}_{t,\varepsilon} = \{ p \in \mathcal{U} \, : \, t < u(p) < M - \varepsilon \},$$

where, as always, we denote $M = u_{\max}$. By the above observation, we can consider $\varepsilon > 0$ as small as we want and such that $M - \varepsilon$ is a regular value of u. For fixed $\varepsilon > 0$, the set $\partial\mathcal{U}_{t,\varepsilon}$ consists in a family of complete smooth hypersurfaces, and by applying the usual version of Divergence Theorem in $\mathcal{U}_{t,\varepsilon}$ to the vector field $\mathcal{X}$, given by (5.12), we obtain

$$\begin{aligned} \int_{\mathcal{U}_{t,\varepsilon}} \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \left(f(u)(W - \bar{W}) - \cot_k(\Psi) W \right) d\mu = \\ \int_{\Gamma_t} \frac{-|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)} \, d\sigma + \int_{\Gamma_{M-\varepsilon}} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)} \, d\sigma, \end{aligned} \tag{5.13}$$

where ν is the inner unit normal to $\partial\mathcal{U}_{t,\varepsilon}$ and $d\mu$ and $d\sigma$ denote the n-dimensional and $(n-1)$-dimensional volume elements induced by the metric of $\mathbb{S}^n$, respectively. Note that $\nu = \nabla u/|\nabla u|$ along Γ_t and $\nu = -\nabla u/|\nabla u|$ along $\Gamma_{M-\epsilon}$.

We first analyze the boundary integrals in the previous identity. Along each (regular) level set Γ_t, the pseudo-radial function is constant, that is, $\Psi = \chi_\pm(t)$, where $\chi_\pm$ is given in (5.5). Hence, using that $W \leq \bar{W}$ by Theorem 5.1, we conclude

$$\int_{\Gamma_t} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)}\, d\sigma \begin{cases} \geq -\dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{s_k^{n-1}(\chi_+(t))} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq \dfrac{\mathcal{H}^{n-1}(\Gamma_t)}{s_k^{n-1}(\chi_-(t))} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{5.14}$$

On the other hand, observe that taking a sequence $\varepsilon_n \to 0$ such that $M - \varepsilon_n$ is a regular value of u for all n, and using that $\lim_{x\in\mathcal{U}, x\to \mathrm{Max}(u)} \frac{W}{\bar{W}} = 1$ (this follows by the expansions given in Lemma D.2), we obtain

$$\lim_{n\to+\infty} \int_{\Gamma_{M-\varepsilon_n}} \frac{|\nabla u|}{s_k(\Psi)^n \bar{U}'(\Psi)}\, d\sigma = \begin{cases} -\dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k^{n-1}(\bar{R})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \dfrac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k^{n-1}(\bar{R})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M), \end{cases} \tag{5.15}$$

Finally, observe that $\bar{U}'(\Psi)^3 < 0$ if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and $\bar{U}'(\Psi)^3 > 0$ if $\overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M)$, so we conclude that

$$\int_{\mathcal{U}} \frac{1}{s_k(\Psi)^n \bar{U}'(\Psi)^3} \Big(f(u)(W - \bar{W}) - \cot_k(\Psi) W^2 \Big)\, d\mu \begin{cases} \geq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M), \\ \leq 0 & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_0(M). \end{cases} \tag{5.16}$$

Making $\varepsilon_n \to 0$ in (5.13), we get (5.11) after substituting (5.14), (5.15) and (5.16).

For the rigidity statement, note that equality in (5.11) implies that the right hand side of (5.13) is equal to zero. But this implies that $W \equiv \bar{W}$ on $\mathcal{U}$, and then $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$ from the rigidity statement of Theorem 5.1. □

Remark 5.7. *A crucial aspect in the proof of this result is the requirement that the level sets near Γ_M be regular, which necessitates the use of Sard's Theorem. To apply this theorem, it is sufficient that u be of class $\mathcal{C}^n$. Consequently, by regularity theory, Proposition 5.2 remains valid under the weaker assumptions $f \in \mathcal{C}^{n-2}(\mathbb{R})$ when $n > 2$, and $f \in \mathcal{C}^1(\mathbb{R})$ when $n = 2$ (since f must satisfy condition* (5.4)*). For the same reason, Proposition 5.4 also holds under this relaxed regularity for f. In both cases, however, we have chosen to assume f is smooth for the sake of clarity.*

5.3.2 An isoperimetric inequality

In this section, we present an isoperimetric inequality for domains $\mathcal{U}$ bounded by an hypersurface of maximum points of a solution to (5.1).

We recall that, for a model solution $(\Omega_{R,M,k}, u_{R,M,k}, f_k)$ with $R \in \mathcal{R}_f$, we write $\Omega_{R,M,k} \setminus \Gamma_{R,M,k} = \Omega^+_{R,M,k} \cup \Omega^-_{R,M,k}$, and we write $\mathcal{H}^l$ for the l-dimensional Hausdorff measure associated to the metric of $\mathcal{M}$. Then, as consequence of the estimate given in Proposition 5.3 and the coarea formula we obtain the following result.

Proposition 5.4. *Let* (Ω, u, f) *be a solution to* (5.1)*. Suppose that* $f \in \mathcal{C}^\infty(\mathbb{R})$ *is a smooth function satisfying the standard conditions and* (5.4)*. Suppose that* $M = u_{\max} \in \mathcal{I}_f$*, and let* $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))$ *be a connected component and* $(\bar{\mathcal{U}}, \bar{u}, f)$ *a comparison triple with core radius* $\bar{R} > 0$*. Assume that the* $(n-1)$*-dimensional part of* $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u)$*, denoted by* Γ_M*, consists in a (possibly disconnected) smooth hypersurface. Then we have that*

$$\frac{\mathcal{H}^n(\mathcal{U})}{\mathcal{H}^{n-1}(\Gamma_M)} \geq \begin{cases} \frac{\int_{\bar{R}}^{\bar{r}_+} s_k(r)^{n-1}\, dr}{s_k(\bar{R})^{n-1}} & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M) \\ \\ \frac{\int_{\bar{r}_-}^{\bar{R}} s_k(r)^{n-1}\, dr}{s_k(\bar{R})^{n-1}} & \textit{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M). \end{cases} \tag{5.17}$$

Furthermore, equality holds in (5.17) *if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$*.*

Proof. We will only consider the case in which $\overline{\tau}(\mathcal{U}) \leq \tau_k^+(M)$, and thus $\bar{\mathcal{U}} = \Omega^+_{\bar{R},M,k}$. The other case follows with exactly the same argument.

Observe that $u \in \mathcal{C}^\infty(\mathcal{U})$, thus Sard's Theorem implies that the set of critical values of u is finite, and we can write $\{t_1, \ldots, t_k\} \subset (0, M)$ for this set. Note that, for any $t \in (0, M) \setminus \{t_1, \ldots, t_k\}$, $\Gamma_t := u^{-1}(t)$ is a (possibly disconnected) smooth hypersurface. Then, since any level set of u has zero n-dimensional Hausdorff measure, the coarea formula (see [15]) implies that

$$\mathcal{H}^n(\mathcal{U}) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|}\, d\sigma_t \right) dt,$$

where we set $t_0 = 0$ and $t_{k+1} = M$, and we write $d\sigma_t$ for the $(n-1)$-dimensional area element along each level set Γ_t. Then, recalling that $\bar{W} = (\bar{U}' \circ \Psi)^2$, where $\bar{U}$ is the solution to (3.1) defining the comparison function $\bar{u}$ and Ψ is given in Definition 5.4, by Theorem 5.1 we get that

$$\mathcal{H}^n(\mathcal{U}) \geq \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}}\, d\sigma_t \right) dt = \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}}\, d\sigma_t \right) dt, \tag{5.18}$$

where the last identity follows from the fact that $\bar{W}$ is always positive when restricted to Γ_t with $t < M$.

Now, observe that $\bar{W}$ is constant along any Γ_t and can be expressed in terms of the functions defined in (5.5) as

$$\bar{W} = \bar{U}(\chi_\pm(t))^2, \quad \forall t \in (0, M),$$

where we take χ_+ when $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and χ_- in the other case. Hence, we conclude that

$$\begin{aligned} \int_0^M \left(\int_{\Gamma_t} \frac{1}{\sqrt{\bar{W}}}\, d\sigma_t \right) dt &= \int_0^M \left(\frac{\mathcal{H}^{n-1}(\Gamma_t)}{|\bar{U}'(\chi_+(t))|}\, d\sigma_t \right) dt \\ &\geq \frac{\mathcal{H}^{n-1}(\Gamma_M)}{s_k(\bar{R})^{n-1}} \int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|}\, dt, \end{aligned} \tag{5.19}$$

where we have used the estimate given in Proposition 5.3 for the last inequality.

Finally, noting that $\chi_+ : [0, M] \to [\bar{R}, \bar{r}_+]$ is monotonically decreasing (where $\bar{r}_+ = r_+(\bar{R}, M)$), we can perform the change of variable $t = \chi_+^{-1}(r)$ in the last integral of above to conclude that

$$\int_0^M \frac{s_k(\chi_+(t))^{n-1}}{|\bar{U}'(\chi_+(t))|}\, dt = \int_{\bar{R}}^{\bar{r}_+} s_k(r)^{n-1}\, dr.$$

Now (5.17) follows from (5.18) and (5.19).

For the rigidity statement, we note that equality in (5.17) implies equality in (5.18), and then we can argue as in the proof of Proposition 5.3. $\square$

Remark 5.8. *Note that in the case in which $(\mathcal{M}, g) = \mathbb{M}^n(k)$ we have that the n-dimensional volume of a geodesic ball of radius R, denoted by B_R, and the $(n-1)$-dimensional area of its boundary are given by*

$$\mathcal{H}^n(B_R) = \omega_{n-1} \cdot \int_0^R s_k(r)^{n-1}\, dr \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B_R) = \omega_{n-1} \cdot s_k(R)^{n-1}$$

respectively, where we denote by ω_{n-1} the euclidean volume of $\mathbb{S}^{n-1}$.

It follows that in this case (5.17) *takes the form*

$$\frac{\mathcal{H}^n(\mathcal{U})}{\mathcal{H}^{n-1}(\Gamma_M)} \geq \begin{cases} \frac{\mathcal{H}^n(\Omega^+_{\bar{R},M,k})}{\mathcal{H}^{n-1}(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) < \tau_k^+(M) \\ \\ \frac{\mathcal{H}^n(\Omega^-_{\bar{R},M,k})}{\mathcal{H}^{n-1}(\Gamma_{\bar{R},M,k})} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \geq \tau_k^-(M). \end{cases}$$

5.3.3 Location of the hot-spots

In this section, we provide estimates for the distance to the hot spots of a solution (Ω, u, f) to (5.1) in terms of the expected core radius of the solution. These results generalizes those contained in [1, Section 5].

The study of the location of critical points of differentiable functions is a classical topic in geometric analysis, tracing back to the works of Gauss. More recently, Magnanini and Poggesi have obtained estimates for the distance to extremum points of solutions to certain differential equations posed in the Euclidean space.

Among other interesting results, the authors obtain in [17, Theorem 1.1] an estimate for the distance to the set of maximum points (the *hot spots*) of a solution to the Dirichlet problem associated with Serrin's equation (that is, (5.1) with $f(u) \equiv 1$) in Euclidean space. The proof of this result relies on a gradient estimate obtained via a suitable P-function (see [17, Lemma 2.2]). Subsequently, in [1, Section 5], the authors derive sharp estimates for the distance to the set of maximum points in terms of the expected core radius of a solution. The results in [1, Section 5] also rely on a gradient estimate for solutions to Serrin's problem, which is a particular case of Theorem 5.1 (see [1, Theorem 3.5]). These parallels naturally motivate the formulation of a generalization of [1, Theorem 5.1] in our current setting.

Proposition 5.5. *Let* (Ω, u, f) *be a solution to* (5.1) *with* $f \in \mathcal{C}^1(\mathbb{R})$ *satisfying the standard conditions and* (5.4). *Suppose further that* $u_{\max} \in \mathcal{I}_f$. *Take* $\mathcal{U} \in \pi_0\left(\Omega \setminus \mathrm{Max}(u)\right)$, *and consider* $(\bar{\mathcal{U}}, \bar{u}, f)$ *to be a comparison triple associated to* $(\mathcal{U}, u, f)$. *Then, for any point* $p \in \mathrm{Max}(u) \cap \mathrm{cl}(\mathcal{U})$, *the following holds*

$$\mathrm{dist}(p, \partial\Omega \cap \mathrm{cl}(\mathcal{U})) \geq \begin{cases} \bar{r}_+ - \bar{R} & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) \leq \tau_k^+(M) \\ \\ \bar{R} - \bar{r}_- & \text{if} \quad \overline{\tau}(\bar{\mathcal{U}}) > \tau_k^-(M), \end{cases} \tag{5.20}$$

where $\bar{r}_-$ *and* $\bar{r}_+$ *are defined as the zeros of the solution to* (3.1)-(3.2) *defining* $(\bar{\Omega}, \bar{u}, f)$.

Furthermore, equality in (5.20) *holds if, and only if* $(\mathcal{U}, u, f) \equiv (\bar{\mathcal{U}}, \bar{u}, f)$.

The proof of the above result follows exactly as in [1, Theorem 5.1], so we omit it here. The generalization of [1, Theorem E] deserves a bit more of discussion.

First, we note that condition (5.4) implies that we can bound from above the distance from any point $p \in \Omega$ to the boundary of the domain as done in [17, Lemma 2.1]. This is the content of the following result:

Lemma 5.3. *Let* (Ω, u, f) *be a solution to* (5.1) *with* $f \in \mathcal{C}^1$ *satisfying* (5.4) *and* $f(0) = 1$. *Suppose that* Ω *is contained in a convex domain of* $\mathcal{M}$, *and define the function* $d(x) = \mathrm{dist}(p, \partial\Omega)$ *for all* $p \in \mathrm{cl}(\Omega)$. *Then we have that*

$$u(p) \geq \frac{2}{n} \cdot \frac{s_k\left(\frac{d(p)}{2}\right)^2}{s_k'(d(p))}, \quad \forall p \in \mathrm{cl}(\Omega). \tag{5.21}$$

Moreover, if the identity happens at some point $p \in \Omega$, *then* u *attains its unique maximum at* p, Ω *is a geodesic ball centered at* p, u *only depends on the distance to* p *and* $f(x) = nkx + 1$.

Proof. The idea is to compare with a solution to Serrin's equation, following the approach used in the proof of [17, Lemma 2.1].

For any $p \in \Omega$, let $r_p(q) = \mathrm{dist}(p, q)$ denote the distance from p to q for all $q \in \mathrm{cl}(\Omega)$. Since the curvature bound $\mathrm{Ric} \geq (n-1)k\, g$ holds, we have

$$\Delta r_p(q) \leq (n-1)\cot_k\big(r_p(q)\big), \quad \forall\, q \in \mathrm{cl}(\Omega).$$

Let

$$W^R(r) = \frac{2\left[s_k\left(\frac{R}{2}\right)^2 - s_k\left(\frac{r}{2}\right)^2\right]}{n\, s_k'(r)}, \quad \forall r \in [0, R),$$

be the radial function defining the solution to Serrin's equation on a geodesic ball of radius R in $\mathbb{M}^n(k)$.

Since $f(x) \geq nkx + 1$, it follows that $w^R := W^R \circ r_x$ is a subsolution to (5.1) on the geodesic ball of radius R centered at x, denoted $B_R(x)$. By the maximum principle, we conclude that

$$u \geq w^{d(p)} \quad \text{in} \quad B_{d(p)}(p),$$

which yields

$$u(p) \geq w^{d(p)}(p) = \frac{2}{n} \cdot \frac{s_k\left(\frac{d(p)}{2}\right)^2}{s_k'(d(p))}.$$

Now, the rigidity statement follows from the strong maximum principle. □

Given a solution (Ω, u, f) to (5.1), following [1, Section 5] we define the $\overline{\tau}$-function associated to the whole domain Ω as

$$\overline{\tau}(\Omega) := \max\left\{\overline{\tau}(\mathcal{U}) \,:\, \mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(u))\right\}.$$

Let $\Gamma \in \pi_0(\partial\Omega)$ be such that $\overline{\tau}(\Gamma) = \overline{\tau}(\Omega)$. Then we extend Definition 5.2 by saying that a model solution $(\bar{\Omega}, \bar{u}, f)$ in $\mathcal{M}_k^n(\Gamma)$ is a comparison triple associated to (Ω, u, f) if $u_{\max} = \bar{u}_{\max}$ and $\overline{\tau}(\Omega) \leq \overline{\tau}(\bar{\Omega})$, and we say that it is an associated model triple if $\overline{\tau}(\Omega) = \overline{\tau}(\bar{\Omega})$. When we have fixed a comparison triple $(\bar{\Omega}, \bar{u}, f)$, we denote its core radius by $\bar{R}$, and we will also write $\bar{r}_\pm = r_\pm(\bar{R}, u_{\max}, k)$.

We also define the inner radius of Ω as

$$r_\Omega = \max\left\{\mathrm{dist}(p, \partial\Omega) \,:\, p \in \Omega\right\}.$$

Now we are in a position to prove a generalization of [1, Theorem E] to our setting.

Theorem 5.2. *Let* (Ω, u, f) *be a solution to* (5.1) *with* $f \in \mathcal{C}^1(\mathbb{R})$ *satisfying* (5.4) *and with* $f(0) = 1$. *Suppose that* $M = u_{\max} \in \mathcal{I}_f$, *and fix* $(\bar{\Omega}, \bar{u}, f)$ *a comparison triple associated to* (Ω, u, f).

Suppose that Ω is contained in a convex domain of $\mathcal{M}$ and that $\partial\Omega$ is mean convex. If $k > 0$, suppose further that $\bar{r}_+ \leq \bar{\pi}/2\sqrt{k}$. Then we have that

$$\frac{\mathrm{dist}(p, \partial\Omega)}{\tan_k(r_\Omega)/n} \geq \frac{\bar{r}_+ - \bar{R}}{\sqrt{\frac{2}{n}M + kM^2}}, \quad \forall p \in \Omega. \tag{5.22}$$

Moreover, if the equality holds at one single point of Ω, then $(\mathcal{M}, g)$ is the space form $\mathbb{M}^n(k)$, $f(x) = nkx + 1$ and (Ω, u, f) is a ball solution to Serrin's problem.

Proof. Write $M = u_{\max}$ for simplicity. First, we note that the mean convexity of $\partial\Omega$, together with the estimates provided in Proposition 5.1 and the fact that $\bar{r}_+ \leq \pi/2\sqrt{k}$ if $k > 0$ implies that it must be $\overline{\tau}(\Omega) \leq \tau_k^+(M)$.

Let $o \in \Omega$ be such that $\mathrm{dist}(o, \partial\Omega) = r_\Omega$. Then, Lemma 5.3 implies that

$$\frac{\tan_k(r_\Omega)^2}{n^2} \leq \frac{2}{n}u(o) + ku(o)^2 \leq \frac{2}{n}M + kM^2,$$

so we get (5.22) from this inequality together with Proposition 5.5.

To prove the rigidity statement, note that equality in (5.22) implies equality in (5.21), so get that $f(x) = nkx + 1$, Ω is a geodesic ball centered at o, u is a function depending only on the distance to o and $\mathrm{Max}(u) = \{o\}$. Now, since $\Omega \setminus \mathrm{Max}(u)$ is connected and we must also have equality in (5.20), we get that g admits the warped product decomposition

$$g = d\Psi^2 + s_k(\Psi)^2 g_0$$

in Ω, where here Ψ is defined in Definition 5.4 and g_0 is the metric restricted to Γ, the connected component of $\partial\Omega$ with $\overline{\tau}(\Gamma) = \overline{\tau}(\Omega)$. Now, since it must be $\overline{\tau}(\Omega) = 1$, the rigidity statement follows from Remark 5.5 □

A Appendix: Isoparametric Hypersurfaces of $\mathbb{S}^n$

Isoparametric hypersurfaces have been extensively studied in the literature in space forms and are defined as those whose principal curvatures are all constant (cf. [8, Chapter 3]). In $\mathbb{H}^n$ or $\mathbb{R}^n$, isoparametric hypersurfaces are relatively simple: they are totally umbilic submanifolds, spherical cylinders, or equidistant hypersurfaces to totally geodesic submanifolds of codimension greater than one. The latter case only arises in hyperbolic n-space. In $\mathbb{S}^n$, these hypersurfaces are of particular interest because they provide natural foliations of the sphere by level sets of a polynomial.

A fundamental result in the theory of isoparametric hypersurfaces in $\mathbb{S}^n$ states that they can be described as the level sets of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$, known as a *Cartan-Münzner polynomial*, which satisfies the following differential equations:

$$\bar{\Delta}P(x) = c\,|x|^{l-2}, \quad \text{and} \quad |\bar{\nabla}P(x)|^2 = l^2|x|^{2l-2}, \tag{A.1}$$

where $l, m_1, m_2 \in \mathbb{N}$ and $c = l^2(m_2 - m_1)/2$, and $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplacian and gradient in the ambient Euclidean space $\mathbb{R}^{n+1} \supset \mathbb{S}^n$. When restricted to the sphere, the function $\rho := P_{|\mathbb{S}^n}$ satisfies:

Theorem A.1 ([8], Theorem 3.32). *Let $\Gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a connected isoparametric hypersurface with l distinct principal curvatures κ_i, with respective multiplicities m_i, $i \in \{1, \dots, l\}$. Then, Γ is an open subset of a level set of the restriction to $\mathbb{S}^n$ of a homogeneous polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree l, and $\rho = P_{|\mathbb{S}^n}$ satisfies the differential equations:*

$$\Delta\rho(p) = -l(l+n-1)\rho(p) + c \quad \text{and} \quad |\nabla\rho|^2(p) = l^2(1-\rho^2(p)), \tag{A.2}$$

for each $p \in \mathbb{S}^n$, where $c = l^2(m_2 - m_1)/2$ and $2(n-1) = l(m_1 + m_2)$. Furthermore, we have $m_i = m_{i+2}$ (indices modulo l) for each $i \in \{1, \dots, l\}$. Here, Δ and ∇ denote the Laplacian and gradient on $\mathbb{S}^n$.

It follows from (A.2) that $\rho(\mathbb{S}^n) = [-1, 1]$, which means that $\{\Gamma_r := \rho^{-1}(r)\}_{r\in[-1,1]}$ forms a foliation of $\mathbb{S}^n$, and for each $r \in (-1, 1)$, the level set Γ_r is a connected isoparametric hypersurface in $\mathbb{S}^n$. The level sets Γ_1 and Γ_{-1} are minimal submanifolds of $\mathbb{S}^n$ of codimension $m_1 + 1$ and $m_2 + 1$, respectively, and they serve as the focal submanifolds of each isoparametric hypersurface in the family $\{\Gamma_r\}_{r\in(-1,1)}$.

A.1 Examples of Isoparametric Hypersurfaces

Isoparametric hypersurfaces in $\mathbb{S}^n$ are characterized by the number of distinct principal curvatures l and their multiplicities $0 < m_1, m_2 \leq n-1$. After a series of papers by Münzner, it was shown that the only possible values for the number of distinct principal curvatures of an isoparametric hypersurface are $l \in \{1, 2, 3, 4, 6\}$ (see [8, Theorem 3.49]).

Below, we describe some families of isoparametric hypersurfaces corresponding to each possible value of l.

Umbilic Hypersurfaces ($l = 1$): The simplest example of an isoparametric hypersurface is a *geodesic sphere*, defined as the set of points at a fixed distance r from a given point $p \in \mathbb{S}^n$. These hypersurfaces are totally umbilic, meaning that all their principal curvatures are equal. Up to an isometry, they can be described as level sets of the height function:

$$\rho(x) = x_{n+1}, \quad x = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n.$$

Each geodesic sphere is given by

$$\Gamma_r = \{x \in \mathbb{S}^n \mid x_{n+1} = r\}, \quad r \in (-1, 1).$$

This family foliates $\mathbb{S}^n$ into parallel hypersurfaces, except for the two limiting cases $r = \pm 1$, which correspond to the focal submanifolds—single points (the poles). It is clear that $\Gamma_r \cong \mathbb{S}^{n-1}$, $r \in (-1, 1)$, and this foliation describes cylindrical coordinates in $\mathbb{S}^n$.

Generalized Clifford Tori ($l = 2$): The *generalized Clifford tori* are a natural extension of the classical Clifford torus in $\mathbb{S}^3$. Let $a, b \in \mathbb{N}$ such that $a \leq b$ and $a + b = n + 1$. For $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$, define the Cartan-Münzner polynomial

$$P^{a,b}(x) = (x_1^2 + \cdots + x_a^2) - (x_{a+1}^2 + \cdots + x_{n+1}^2),$$

and write $\rho^{a,b} = P^{a,b}|_{\mathbb{S}^n}$. A generalized Clifford torus is defined as a level set

$$\mathbb{T}_r^{a,b} = \left\{ p \in \mathbb{S}^n \,\middle|\, \rho^{a,b}(p) = r \right\},$$

for some $r \in (-1, 1)$. Each $\mathbb{T}_r^{a,b}$ is homeomorphic to $\mathbb{S}^{a-1} \times \mathbb{S}^{b-1}$, and the multiplicities of the principal curvatures are $m_1 = a - 1$ and $m_2 = b - 1$. As shown in [8, Section 3.8.2], each $\mathbb{T}_r^{a,b}$ is a homogeneous hypersurface generated by the subgroup of isometries $SO(a) \times SO(b)$, and its two focal submanifolds are totally geodesic spheres that are polar to each other.

It is evident that the generalized Clifford torus $\mathbb{T}_r^{a,b}$, being a level set of $P^{a,b}$ on $\mathbb{S}^n$, is antipodally symmetric.

Cartan's Isoparametric Hypersurfaces ($l = 3$): É. Cartan classified isoparametric hypersurfaces Γ in $\mathbb{S}^n$ with three distinct principal curvatures ($l = 3$).

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where $\mathbb{C}, \mathbb{H}, \mathbb{O}$ denote the complex numbers, quaternions, and Cayley numbers (or octonions), respectively, and set $m = \dim(\mathbb{F})$. There is a canonical representation of $\mathbb{F}$ as $\mathbb{R}^m$ equipped with a suitable inner product. A point in $\mathbb{R}^{2+3m}$ can then be expressed as (x, y, X, Y, Z), where $x, y \in \mathbb{R}$ and $X, Y, Z \in \mathbb{F}$. Here, $\overline{X}$ denotes the conjugate of $X \in \mathbb{F}$.

An isoparametric hypersurface $\Gamma \subset \mathbb{S}^{1+3m}$ with three distinct principal curvatures is given as a level set of the Cartan-Münzner polynomial

$$\begin{aligned} P(x, y, X, Y, Z) =& x^3 - 3xy^2 + \frac{3}{2}x\left(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}\right) + \frac{3\sqrt{3}}{2}y\left(X\overline{X} - Y\overline{Y}\right) + \\ &+ \frac{3\sqrt{3}}{2}\left(XYZ + \overline{ZYX}\right). \end{aligned}$$

All principal curvatures of Γ have the same multiplicity $m \in \{1, 2, 4, 8\}$. The focal submanifolds of Γ are a pair of antipodal embeddings of $\mathbb{FP}^2$, where $\mathbb{FP}^2$ denotes the projective plane over the division algebra $\mathbb{F}$.

In a seminal work, Cartan showed that Γ is homogeneous, it arises as the orbit of a closed subgroup of $SO(2 + 3m)$ acting on the sphere. The structure of this group depends on the parameter $m = \dim(\mathbb{F})$, as explained in [8, Subsection 3.8.3].

More generally, following Cartan's foundational results, subsequent works established that any homogeneous isoparametric hypersurface in a Euclidean sphere arises as a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two. These symmetric spaces have been fully classified. In particular, [9, Table 1] provides a complete list of all homogeneous isoparametric hypersurfaces in spheres, indicating:

- the number of distinct principal curvatures l,
- the corresponding multiplicities (m_1, m_2),
- and the symmetric space whose isotropy representation generates the associated homogeneous isoparametric hypersurface.

For example, in the case $m = 1$, we have that Γ is a principal orbit of the isotropy representation of $SU(3)/SO(3)$ in $\mathbb{S}^4$.

Exotic Isoparametric Hypersurfaces ($l = 4$): Unlike the previous cases, the family of isoparametric hypersurfaces with four distinct principal curvatures in Euclidean spheres includes both homogeneous and non-homogeneous examples.

Let $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ denote the pair of multiplicities of the principal curvatures of an isoparametric hypersurface. When $l = 4$, all such hypersurfaces are classified by their multiplicities, and—except for two homogeneous examples with multiplicities $(2, 2)$ and $(4, 5)$—all belong to an infinite family constructed by Ferus, Karcher, and Münzner in [13]. This construction is based on representations of *Clifford algebras*, and a detailed description can be found in [8, Section 3.9]. We briefly outline how to construct the Cartan-Münzner polynomials associated with this family.

Let $m_1 = m$ and fix $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let $\mathrm{Sym}_k(\mathbb{R})$ denote the space of $k \times k$ symmetric matrices with real entries. Then, an $(m+1)$-tuple $(P_0, \dots, P_m)$ of elements in $\mathrm{Sym}_{2n+2}(\mathbb{R})$ is called a *Clifford system* if the matrices P_i satisfy:

$$P_i^2 = \mathrm{Id}_{2n+2}, \quad P_i P_j = -P_j P_i, \quad i \neq j, \quad 0 \leq i, j \leq m,$$

where Id_{2n+2} denotes the identity matrix of dimension $(2n+2)$. Then the function $P : \mathbb{R}^{2n+2} \to \mathbb{R}$ defined by

$$P(x) = \langle x, x \rangle^2 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2, \quad \forall x \in \mathbb{R}^{2n+2}, \tag{A.3}$$

is a Cartan-Münzner polynomial defining an isoparametric hypersurface with four distinct principal curvatures and multiplicities $(m, n-m)$. A level set of this polynomial is called a *Clifford isoparametric hypersurface*.

Endowing $\mathrm{Sym}_k(\mathbb{R})$ with the standard inner product for symmetric matrices, given by

$$\langle A, B \rangle = k^{-1} \mathrm{Trace}(A \cdot B), \quad \forall A, B \in \mathrm{Sym}_k(\mathbb{R}),$$

we define the Clifford sphere associated with the Clifford system $(P_0, \dots, P_m)$ as

$$\mathcal{C}(P_0, \dots, P_m) = \left\{ A \in \mathrm{span}\, \{P_0, \dots, P_m\} \ : \ |A|^2 = 1 \right\} \subset \mathrm{Sym}_{2n+2}(\mathbb{R}).$$

Two Clifford systems generate the same isoparametric family if and only if they produce the same *Clifford sphere*. It is known that infinitely many Clifford systems with distinct Clifford spheres can be constructed, producing an infinite family of non-congruent Clifford isoparametric hypersurfaces. It follows from the classification of homogeneous examples that most hypersurfaces in this family are inhomogeneous; this can also be verified independently, see [8, Theorem 3.77].

The topology of Clifford hypersurfaces is studied in [25]. If $\rho := P_{\mathbb{S}^{2n+1}}$ for some P as in (A.3), then the focal submanifolds of the family $\{\Gamma_r := \rho^{-1}(r)\}_{r\in(-1,1)}$ are given by $\Gamma_{-1} = \rho^{-1}(-1)$ and $\Gamma_1 = \rho^{-1}(1)$. It follows that Γ_{-1} is diffeomorphic to an $\mathbb{S}^{n+1}$-bundle over $\mathbb{S}^m$, while Γ_1 is a Clifford–Stiefel manifold (see [8] for a detailed description of these submanifolds). Furthermore, $\Gamma_{\frac{n-2m+1}{n+1}} = \rho^{-1}((n-2m+1)/(n+1))$ is always a minimal hypersurface diffeomorphic to $\Gamma_1 \times \mathbb{S}^m$.

As a particular example, prior to the work of Ferus et al., É. Cartan discovered an isoparametric family of Clifford type, later generalized by Nomizu [19]. Consider $\mathbb{C}^{n+1}$ as the complex vector space of dimension $n+1$, and write $z = (x, y) \in \mathbb{R}^{n+1} \oplus i\mathbb{R}^{n+1}$. Then, using the Euclidean inner product in $\mathbb{R}^{n+1}$,

$$P((x, y)) = 4\,|x|^2\,|y|^2 - 4\,\langle x, y\rangle^2\,, \quad \forall (x, y) = z \in \mathbb{C}^{n+1},$$

is a Cartan-Münzner polynomial that defines an isoparametric family with four distinct principal curvatures in $\mathbb{S}^{2n+1}$. Each hypersurface in this family is an orbit of the isometric action of $SO(2) \times SO(n+1)$ on $\mathbb{S}^{2n+1}$. In particular, all hypersurfaces in this family are homogeneous.

Exotic Isoparametric Hypersurfaces ($l = 6$) There are only two known isoparametric families with six distinct principal curvatures. Both families are homogeneous and consist of hypersurfaces with principal curvatures of equal multiplicity $m_1 = m_2 = m \in \{1, 2\}$.

In the case $m = 1$, Miyaoka proved in [18] that if $\Gamma \subset \mathbb{S}^7$ is an isoparametric hypersurface with $l = 6$ distinct principal curvatures, then Γ arises as the inverse image, under the Hopf fibration $h : \mathbb{S}^7 \to \mathbb{S}^4$, of a Cartan isoparametric hypersurface with three distinct principal curvatures. It is also shown that the two focal submanifolds of Γ are non-congruent minimal embeddings of $\mathbb{RP}^2 \times \mathbb{S}^3$.

When $m = 2$, the isoparametric family arises from the adjoint orbits of the exceptional compact Lie group G_2 (the automorphism group of the octonions $\mathbb{O}$). The group G_2 acts on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^{14}$ by isometries with respect to the bi-invariant metric. Miyaoka showed that these hypersurfaces are fiber bundles over $\mathbb{S}^6$, with fibers given by Cartan's isoparametric hypersurfaces with three distinct principal curvatures of multiplicity two. Thus, isoparametric hypersurfaces with $(l, m) = (6, 2)$ are closely related to those with $(l, m) = (3, 2)$. It follows that an isoparametric hypersurface $\Gamma \subset \mathbb{S}^{13}$ with six distinct principal curvatures of multiplicity two is diffeomorphic to the homogeneous space G_2/T^2, where $T^2 \subset G_2$ is a subgroup isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ as a Lie group. The focal submanifolds of Γ are both diffeomorphic to $G_2/U(2)$, where $U(2)$ denotes the unitary group of dimension two.

B Appendix: ODE toolkit for f-Extremal Domains in model manifolds

Throughout this appendix we collect and prove the ODE statements used in Sections 3 and 4. We first treat the general (isoparametric) reduction on $\mathbb{S}^n$ and then specialize to the equation related to f-extremal domains in model manifolds. Since most of the arguments are standard and already considered in [12], we only sketch the proofs, referring to our previous paper when necessary.

B.1 Solutions related to the isoparametric case

Fix a continuous nonlinearity $f \in C(\mathbb{R})$, positive in $\mathbb{R}^*_+$. We study the ODE

$$V''(s) + \Big((n-1)\cot(\ell s) - \frac{c}{\ell\sin(\ell s)}\Big)\, V'(s) + f\big(V(s)\big) = 0 \quad \text{on } (0, \pi/\ell), \tag{B.1}$$

where $c = \ell^2(m_2 - m_1)/2$ and m_1, m_2 are the principal-curvature multiplicities of the family.

We analyze the Cauchy problem with interior or endpoint critical data

$$V(S) = M > 0, \qquad V'(S) = 0, \qquad S \in [0, \pi/\ell]. \tag{B.2}$$

First we deal with the non-singular case, i.e., when $S \in (0, \pi/l)$.

Proof of Item (ii) from Lemma 4.1. Existence and uniqueness of a solution $V_{S,M}$ to (B.1)-(B.2) defined within an open interval containing S follows from classical theory of differential equations, so we only have to prove the monotonicity properties of the solution.

Writing the equation as $(\mu V')' + \mu f(V) = 0$ with the smooth integrating factor

$$\mu(s) = \exp\left(\int_S^s \Big((n-1)\cot(\ell x) - \frac{c}{\ell\sin(\ell x)}\Big) dx\right),$$

we obtain

$$V'_{S,M}(s) = -\,\mu(s)^{-1}\int_S^s \mu(x)\, f\big(V(x)\big)\, dx < 0 \quad \text{for } s \in (e_1, e_2),$$

being $(e_1, e_2) \subset (0, \pi/l)$ the maximal interval of definition of $V_{S,M}$. Thus, it follows that $V'_{S,M} > 0$ if $s < S$ and $V'_{S,M} < 0$ if $s > S$. Now the existence of $s_-(S, M)$ and $s_+(S, M)$ follows using the same argument of the proof of Item 2 in [12, Theorem 4.1]. □

Now we deal with the existence in the singular case, i.e., when $S \in \{0, \pi/l\}$. In this case, we can proceed with a standard fixed point argument, as done in [10, 12].

Proof of Item (i) from Lemma 4.1. We only deal with the case $S = 0$. The other case is completely analogous. As in [10, Subsection 3.2.1], for a non-negative integer k and $\varepsilon \in (0, \pi)$, we denote by $\mathcal{C}_e^k(\varepsilon)$ the space of even $\mathcal{C}^k$-functions on $[-\varepsilon, \varepsilon]$, equipped with the norm

$$\|V\|_{k,\varepsilon} = \sum_{i=1}^{k} \sup_{s\in[-\varepsilon,\varepsilon]} \left|V^{(i)}(s)\right|, \quad \forall V \in \mathcal{C}_e^k(\varepsilon).$$

We also define $\mathcal{C}_e(\varepsilon) := \mathcal{C}_e^0(\varepsilon)$. For any $g \in \mathcal{C}_e(\varepsilon)$, define the function given by

$$A(g)(s) = -\int_0^s \mu(y)^{-1}\Big(\int_0^y \mu(x)\, f\big(W(x)\big)\, dx\Big)dy, \quad \forall s \in (-\pi, \pi) \setminus \{0\} \tag{B.3}$$

and $A(g)(0) = 0$. It is straightforward to check that $A(g)$ is $\mathcal{C}^2$ in $(-\pi/l, \pi/l)$.

Now define the operator $\mathcal{A} : \mathcal{C}_e^1(\varepsilon) \times \mathbb{R}_+ \to \mathcal{C}_e^1(\varepsilon)$ given by

$$\mathcal{T}(V) = A(f(M + V)).$$

Then, taking into account that

$$\cot(ls) = \frac{1}{ls} + O(s) \quad \text{and} \quad \sin(ls) = ls + O(s^3)$$

near zero, we check that there exists constants $C_1, C_2 > 0$ such that

$$C_1 s^{lm_1} \le \mu(s) \le C_2 s^{lm_1}$$

for s close to zero. In particular, we get that

$$\|\mathcal{T}(V)\|_{0,\varepsilon} \le C_3 \cdot \varepsilon^2 \cdot \sup_{[M-\|V\|_{0,\varepsilon}, M+\|V\|_{0,\varepsilon}]} |f|,$$

for some $C_3 > 0$. Thus, if ε is small enough, we get that $\mathcal{T}(B) \subsetneq B$, where $B \subset \mathcal{C}_e(\varepsilon)$ is the closed ball of radius 1 centered at 0. Consequently, by Schauder's Fixed Point Theorem, $\mathcal{T}$ has a fixed point $V_{0,M} \in B$, which is a weak solution to (B.1)-(B.2). But $V_{0,M}$ is $\mathcal{C}^2$ on $(-\pi/l, \pi/l)$ by construction, so it is a classical solution to the equation.

Now the existence of $s_+(0, M)$ and the properties of the solution follows as in the other case. □

B.2 Solutions related to the model manifolds

Consider $f \in \mathcal{C}(\mathbb{R})$ satisfying the standard conditions. We study the equation

$$U''(r) + (n-1)\cot_k(r)\, U'(r) + f\big(U(r)\big) = 0, \qquad r \in (0, 2\bar{r}_k), \tag{B.4}$$

together with the initial conditions

$$U(R) = M > 0, \qquad U'(R) = 0, \qquad R \in [0, 2\bar{r}_k), \tag{B.5}$$

with $\bar{r}_k = +\infty$ if $k \leq 0$ and $\bar{r}_k = \pi/\sqrt{k}$ if $k > 0$.

Now we have to check the monotonicity properties of $U_{R,M,k}$. In particular, we prove that f is an admissible function if it satisfies the standard assumptions.

Proof of Proposition 3.1. When $k > 0$, it is a particular case of that of Lemma 4.1, so we assume that $k \leq 0$. Since the argument is the same in both cases, we only show the proof when $k < 0$.

Given any $R \in [0, +\infty)$ and $M \in \mathcal{I}_f$, the existence of a solution $U_{R,M,k}$ to (B.4)-(B.5) is proved by employing standard arguments (as done in the previous section), so we omit the proof here. Let (e_1, e_2) be the maximal interval of definition of $U_{R,M,k}$. Then, from the implicit representation

$$U_{R,M,k}(r) = M - \int_R^r \frac{1}{s_k(y)^{n-1}} \left(\int_R^y s_k(x)^{n-1} f(U_{R,M,k}(x))\, dx \right) dy,$$

we get that, whenever $U_{R,M,k}(r)$ is positive, $U'_{R,M,k} > 0$ if $r < R$ and $U'_{R,M,k} < 0$ if $r > R$.

The existence of $r_-(R, M, k)$ and of $r_+(R, M, k)$ when $e_2 < +\infty$ is immediate by the boundary behaviour of solutions to differential equations with Lipschitz coefficients. Thus, we only prove the existence of $r_+(R, M, k) > R$ when $e_2 = +\infty$. The case if e_1 is the as e_2

Since f satisfies the standard conditions, suppose first that $f(x) \geq \lambda x$ with $\lambda > \lambda_1(\mathbb{H}^n(k))$. Given a fixed point $p \in \mathbb{H}^n(k)$, define the function $r_p = \mathrm{dist}_{\mathbb{H}^n(k)}(p, \cdot)$. Then, define the function $\tilde{u} = U_{R,M,k} \circ r_p$ inside the set $\mathcal{U} := \{r_p(q) > R\}$. Note that $\tilde{u}$ would be a piece of a model solution in $\mathbb{H}^n(k)$ if there exists $r_-(R, M, k) \leq R < r_+(R, M, k)$ as in Definition 3.1. Now, given that $\lambda > \lambda_1(\mathbb{H}^n(k))$, there exists a function v and a geodesic ball $B \subset \mathbb{H}^n(k)$ centered at p such that v is positive and solves $\Delta v + \lambda v = 0$ inside B, and gets the value zero on ∂B. Arguing by contradiction, suppose that $\tilde{u}$ is positive in all $\mathcal{U}$. Then, there exists $\alpha_0 > 0$ such $\alpha \cdot v < \tilde{u}$ inside $\mathcal{U}$. If α is increases then there exists an $\alpha_1 > \alpha_0$ such that the graphs of $\alpha_1 \cdot v$ and $\tilde{u}$ touches tangentially for the first time. But then, since $f(x) \geq \lambda x$, we conclude that it must be $\tilde{u} \equiv \alpha_1 v$ by the maximum principle, which provides the desired contradiction.

If $f \geq g(x) = nkx + C$ with $C > 0$, we can use the same argument by comparing with a solution to the equation $\Delta v + g(v) = 0$ inside some geodesic ball of $\mathbb{H}^n(k)$, provided that $M \in \mathcal{I}_g = (0, -1/nkC)$ as it is proved in [3, Lemma 5.2]. $\square$

Now suppose further that $f \in \mathcal{C}^1(\mathbb{R})$ and there exists $\bar{M} > 0$ such that

$$f(x) \geq f'(x)x \quad \forall x \in (0, \bar{M}) \subset \mathcal{I}_f$$

In this case, we can prove the monotonicity properties of the functions $\overline{\tau}^{\pm}_{M,k}$ defined in (3.4) stated in Proposition 3.2. The key observation is that the function $Z_{R,k} := \partial_R(U_{R,M,k})$ satisfies the differential equation

$$Z''(r) + (n-1)\cot_k(r)\, Z'(r) + f'\big(U(r)\big)Z(r) = 0, \quad \forall r \in [\bar{r}_-, \bar{r}_+],$$

where we write $\bar{r}_{\pm} = r_{\pm}(R, M, k)$. Hence, we can follow the proof of Item 4 in [12, Theorem 4.1].

Proof of Proposition 3.2. To get the monotonicity properties of the functions $\overline{\tau}^{\pm}_{M,k}$, it is enough to study the sign of the functions

$$\varphi^{\pm}_{M,k}(R) := \partial_R(U_{R,M,k}(\bar{r}_{\pm})^2) = -2\left(U''_{R,M,k}(\bar{r}_{\pm})Z_{R,k}(\bar{r}_{\pm}) - Z'_{R,k}(\bar{r}_{\pm})U'_{R,M,k}(\bar{r}_{\pm})\right).$$

By proceeding as in the proofs of Claims A and B in [12, Theorem 4.1], using the Killing vector field

$$\tilde{Y} = \cos\theta_{n-1}\prod_{j=1}^{n-2}\sin\theta_j\partial_r + \cot_k(r)\left(\cos\theta_{n-1}\sum_{i=1}^{n-2}\frac{\prod_{j=1}^{n-2}\sin\theta_j}{\prod_{j=1}^{i-1}\sin\theta_j}\partial_{\theta_i} - \frac{\sin\theta_{n-1}}{\prod_{j=1}^{n-2}\sin\theta_j}\partial_{\theta_{n-1}}\right)$$

in $\mathbb{M}^n(k)$, we get that $\varphi^{-}_{M,k}$ and $\varphi^{+}_{M,k}$ are negative and positive respectively in $[0, \bar{r}_k)$. In particular, $\overline{\tau}^{-}_{M,k}$ and $\overline{\tau}^{+}_{M,k}$ are strictly decreasing and increasing functions respectively. Note that $\overline{\tau}^{+}_{M,k}(0) = 1$ by definition, and it is clear that

$$\lim_{R\to 0}\overline{\tau}^{-}_{M,k}(R) = +\infty.$$

On the other hand, we have that

$$\lim_{R\to\infty}\overline{\tau}^{\pm}_{M,k}(R) = \tau^{\pm}_{k}(M) > 0,$$

and the Implicit Function Theorem implies that these quantities depends $\mathcal{C}^1$ in M

In the case $k > 0$, the symmetry of the functions $U_{R,M,k}$ with respect to $\bar{r}_k$ implies that $\overline{\tau}^{-}_{M,k}(\bar{r}_k) = \overline{\tau}^{+}_{M,k}(\bar{r}_k)$. ☐

B.3 Behaviour of the τ-function

Lemma B.1. *We want to study the equation* (3.1) *in the case of* $f(x) = f_k(x) = knx + 1$. *Let* $V_{R,M,k}(r)$ *be a solution for the previously stated problem such that* $V_{R,M,k}(R) = M$ *and* $V'_{R,M,k}(R) = 0$. *Therefore,*

$$\lim_{R\to 0^+}\bar{\tau}_{M.k}(R) = +\infty,$$

where $\bar{\tau}_{M.k}(R)$ *is as in definition 3.4.*

Proof. For the case of $k = 0$ the behaviour of $bar\tau_{M.k}(R)$ is proved in [1, 2]. Thus, we are going to focus in the case $k = \pm 1$. Then, the function $V_{R,M,k}$ can be express as follows,

$$V_{R,M,k}(r) = M + c_k(r)\left(A(R,M,k) + B(R,M,k)\left(\frac{-k}{s_k(r)} + G_k(r)\right)\right)$$

for some $A(R, M, k)$ and $B(R, M, k)$ and

$$G_k(r) := \begin{cases} \int_0^{\cosh(r)} \frac{1-(1-t^2)^{\frac{n}{2}}}{t^2(1-t^2)^{\frac{n}{2}}} \text{if} & k < 0, \\ \int_0^{\cos(r)} \frac{1-(t^2-1)^{\frac{n}{2}}}{t^2(t^2-1)^{\frac{n}{2}}} \text{ if} & k > 0. \end{cases}$$

Since, the function $G_k(r)$ near the zero has the following expansion, $\frac{1}{r^{n-1}}$, we obtain the desired result. □

Proposition B.1. *Let $f(0) = 0$ and $f'(x) \geq kn$. Then, there exists $0 < \tau_0(M) < \tau_k^+(M)$ and $\tau_k^-(M) > 0$ such that,*

$$\bar{\tau}^+_{M,k} : [0, \bar{r}_k) \mapsto [\tau_0(M), \tau_k^+(M)) \quad \textit{and} \quad \bar{\tau}^-_{M,k} : [0, \bar{r}_k) \mapsto [\tau_k^-(M), +\infty),$$

with $\tau_k^+(M) < \tau_k^-(M)$ if $k \leq 0$.

Proof. First we are going to show that the $\lim_{R \mapsto 0^+} U'_{R,M,k}(r_(R, M, k)^2 = +\infty$. Let's assume that there exits $\tau_1 = \sup_{R \in (0,\bar{r}_k)} \bar{\tau}^-_{M,k}(R) < \infty$, this hypothesis contradict the Lemma **??**. Thus, there exists $R_0 > 0$ such that for all $R \in (0, R_0)$ it satisfies that $\tau_k(R) > \tau_1$. We consider a solution to the equation (3.1) in the case of $f(x) = f_k(x) = knx + 1$, denoted by $V_{R,M,k}(r)$. Then there exists $r_- \in (R_1, R_0)$ such that $V_{R,M,k}(r_-) = 0$. Hence, there exists two associated model solutions in $\mathbb{M}^n(k)$, $(\Omega^-_{R,M,k}, u, f)$ and $(\tilde{\Omega}_{R,M,k}, v, f_k)$.

For $R \leq R_0$ we can define,

$$\omega_R = V_{R,M,k} - u_{R,M,k} \quad \text{in } \mathcal{O}_R := \Omega^-_{R,M,k} \cap \tilde{\Omega}_{R,M,k}.$$

Then $\omega_R < 0$ in $\mathcal{O}_R$ if $|R - R_0|$ is small enough. Let $R_2 < R$ such that $\omega_R \leq 0$ thus there exists a point $x \in \bar{\mathcal{O}}_{R_2}$ with $\omega_R(x) = 0$. But we have that,

$$\Delta\omega_R = -f_k(v) + f(u) \geq -f_k(v) + f_k(u).$$

Hence we obtain that $\Delta\omega_R + \omega_R \frac{f_k(v)-f_k(u)}{v-u} \geq 0$. Then, since $\tau_k(R) > \bar{\tau}^-_{R,M,k}$, the interior (or boundary) maximum principle implies that $\omega_R \equiv 0$ in $\mathcal{O}_R$, which is a contradiction. We conclude that

$$\lim_{R \to 0^+} \bar{\tau}^-_{M,k}(R) = +\infty.$$

Then, we are going to show $\tau_k^+(M) < \tau_k^-(M)$. If there exists a $R_0 > 0$ such that $\bar{\tau}^+_{M,k}(R_0) = \bar{\tau}^-_{M,k}(R_0)$ this implies that $\Omega_{R_0,M,k}$ is a geodesic ball by the Serrin's theorem. Thus,

$\tau^{\bar{+}}_{M,k}(R_0) \neq \tau^{\bar{-}}_{M,k}(R_0)$. Moreover, since we have that $\lim_{R\to 0^+} \bar{\tau}_{M.k}(R) = +\infty$ we can conclude with $\tau_k^+(M) < \tau_k^-(M)$.

Finally, defining,

$$\tau_0(M) = \bar{\tau}^+_{M.k}(0), \quad \tau_k^+(M) = \sup_R \bar{\tau}^+_{M,k}(R) \quad \text{ and } \tau_k^-(M) = \sup_R \bar{\tau}^-_{M,k}(R)$$

□

Remark B.1.

C Appendix: Gradient estimates

In this appendix we present the proof of Lemma 5.2. We follow [12, Section 5.2].

Remark C.1. *Following the notation of [11], we will denote $\Psi_\pm = \Psi$ and $\chi_\pm = \chi$ and we will do the computations considering both possibilities at the same time. We will also denote the derivatives with respect to u with a dot, and the derivatives with respect to Ψ with a $'$.*

The proof follows as in [1, Theorem 3.5]. The idea is to find an elliptic inequality involving the functions W and $\bar{W}$ defined in Section 5.2. Here we follow [12, Section 5.5].

We recall that

$$W = |\nabla u|^2 \quad \text{and} \quad \bar{W} = |\nabla \bar{u}|^2 \circ \Psi \quad \text{in}\, \mathcal{U},$$

where $\Psi : \mathcal{U} \to (\bar{r}_-, R) \cup (R, \bar{r}_+)$ is the pseudo-radial function defined in Definition 5.4. We have that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (5.4) and $\bar{U}$ satisfies the differential equation

$$\bar{U}''(r) + (n-1)\cot_k(r)\bar{U}'(r) + f(\bar{U}) = 0. \tag{C.1}$$

From the definition of Ψ, we have then that $u = \bar{U} \circ \Psi$. Write $F_\beta = \beta \cdot (W - \bar{W})$, where

$$\beta(p) = \left(\frac{s_k(\Psi(p))}{\bar{U}'(\Psi(p))}\right)^{2\frac{n-1}{n}}, \quad \forall p \in \mathcal{U}.$$

Then, the Laplacian of F_β defined in terms of $\bar{U}(\Psi)$ satisfies,

$$\begin{aligned} \Delta F_\beta =& \Delta\left(\left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}}\right)(W - \bar{W}) + \left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}} \Delta(W - \bar{W}) \\ &+ 2g\left(\nabla\left(\left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}}\right), \nabla(W - \bar{W})\right). \end{aligned} \tag{C.2}$$

Consequently, each term in the right-hand side of (C.2) can be analyzed separately in order to control the sign of ΔF_β.

Analysing the function $\beta(\Psi) = \left(\frac{s_k(\Psi)}{\bar{U}'(\Psi)}\right)^{2\frac{n-1}{n}}$ a direct computation yields $\frac{\beta'(\Psi)}{\beta(\Psi)} = \frac{2(n-1)}{n}\lambda(\Psi)\bar{U}'(\Psi)$. Consequently, since $u = \bar{U} \circ \Psi$, we obtain

$$\nabla(\beta(\Psi)) = \frac{\beta'(\Psi)}{\bar{U}'(\Psi)}\nabla u. \tag{C.3}$$

In order to improve the notation we introduce the following auxiliary function,

$$\lambda(r) := \frac{-\bar{U}''(r) + \cot_k(r)\bar{U}'(r)}{\bar{U}'(r)^2}, \quad \forall r \in [\bar{r}_-, \bar{R}) \cup (\bar{R}, \bar{r}_+]. \tag{C.4}$$

Using relation (C.3), the last term in (C.2) can be expressed entirely in terms of u and F_β. Indeed, we have

$$g(\nabla\beta(\Psi), \nabla(W - \bar{W})) = \frac{2(n-1)}{n}\lambda(\Psi)g(\nabla u, \nabla F_\beta) - F_\beta\left(\frac{2(n-1)}{n}\right)^2\lambda(\Psi)^2|\nabla u|^2 \tag{C.5}$$

We next compute the Laplacian of $\beta(\Psi)$. From the relation (C.3) we have that,

$$\Delta\beta(\Psi) = \operatorname{div}(\nabla\beta(\Psi)) = \operatorname{div}\left(\frac{\beta'(\Psi)}{\bar{U}'(\Psi)}\nabla u\right).$$

Then, extending the previous computation using (C.4) and the relation with $\frac{\lambda'(\psi)}{\bar{U}'(\psi)}$ we obtain that

$$\begin{aligned}\Delta\beta(\Psi) =& \frac{2(n-1)}{n}\beta(\Psi)\Bigg(\Bigg(\frac{f'(\bar{U}) - nk}{\left(\bar{U}'(\Psi)\right)^2} + \frac{n+2}{n}\frac{\lambda(\Psi)g(\bar{U})}{\left(\bar{U}'(\Psi)\right)^2} \\ &+ \frac{3n-4}{n}\lambda(\Psi)^2\Bigg)|\nabla u|^2 - \lambda(\Psi)g(\bar{U})\Bigg)\end{aligned} \tag{C.6}$$

Concluding that,

$$\begin{aligned}(W - \bar{W})\Delta\beta(\Psi) =& \frac{2(n-1)}{n}F_\beta\Bigg(\Bigg(\frac{f'(\bar{U}) - nk}{\left(\bar{U}'(\Psi)\right)^2} + \frac{n+2}{n}\frac{\lambda(\Psi)f(\bar{U})}{\left(U'(\Psi)\right)^2} \\ &+ \frac{3n-4}{n}\lambda(\Psi)^2\Bigg)|\nabla u|^2 - \lambda(\Psi)g(\bar{U})\Bigg)\end{aligned} \tag{C.7}$$

We develop the term second form (C.2) individually. On one hand, the Bochner formula for the Laplacian of W together with the bounds of the Ricci curvature tensor,

$$\frac{1}{2}\Delta W \geq |\nabla^2 u|^2 + (k(n-1) - f'(u))\,|\nabla u|^2 \tag{C.8}$$

On the other hand, the Laplacian for $\bar{W}$ can be computed using the chain rule,

$$\Delta \bar{W} = \frac{\partial \bar{W}}{\partial u}\Delta(u) + \frac{\partial^2 \bar{W}}{\partial u^2}|\nabla u|^2.$$

From the definition of $u = \bar{U} \circ \Psi$ we have that $\frac{\partial \Psi}{\partial u} = \frac{1}{\bar{U}'(\Psi)}$. We now expand each derivative term appearing in the expression of the laplacian of $\bar{W}$,

- The first derivative of $\bar{W}$ with respect to u is given by

$$\frac{\partial \bar{W}}{\partial u} = \frac{\partial(\bar{U}' \circ \Psi(u))^2}{\partial u} = -2(g(\bar{U}) + (n-1)\cot_k(\Psi)\bar{U}'(\Psi)).$$

- The second derivative of $\bar{W}$ with respect to u is given by $\frac{\partial^2 W}{\partial u^2} = 2\frac{\partial \bar{U}''(\Psi(u))}{\partial u}$ then joining with $\frac{\partial \cot_k(\Psi)}{\partial u} = -\frac{1}{s_k^2(\Psi)}$ for every $k \in \mathbb{R}$ we can conclude that,

$$\begin{aligned}
\frac{\partial^2 W}{\partial u^2} =& -2\ \left(\frac{\partial g(\bar{U})}{\partial u} + (n-1)\frac{\partial \cot_k(\Psi)\bar{U}'(\Psi)}{\partial u}\right) \\
=& -2\left(f'(\bar{U})\bar{U}'(\Psi) + (n-1)\left(\frac{\partial \cot_k(\Psi)}{\partial u}\bar{U}'(\Psi) + \cot_k(\Psi)\frac{\partial \bar{U}'(\Psi)}{\partial u}\right)\right)\frac{\partial \Psi}{\partial u} \\
=& -2\left(f'(\bar{u}) + (n-1)\left(-\frac{1}{s_k(\Psi)} + \cot_k(\Psi)\frac{U''(\Psi)}{\bar{U}'(\Psi)}\right)\right).
\end{aligned}$$

As consequence, we obtain that the gradient of $\bar{W}$ is,

$$\nabla \bar{W} = -2\left((n-1)\cot_k(\Psi)\bar{U}'(\Psi) + g(\bar{U})\right)\nabla u \tag{C.9}$$

It can be concluded that,

$$\begin{aligned}
\Delta \bar{W} =& 2\left(g(\bar{U}) + (n-1)\cot_k(\Psi)\bar{U}'(\Psi)\right) f(u) \\
&- 2\left(f'(\bar{u}) + (n-1)\left(-\frac{1}{s_k(\Psi)} + \cot_k(\Psi)\frac{U''(\Psi)}{\bar{U}'(\Psi)}\right)\right)|\nabla u|^2
\end{aligned} \tag{C.10}$$

Therefore, $\Delta \bar{W}$ can be written in terms of $\bar{W}$ and $\lambda(\Psi)$,

$$\begin{aligned}
\Delta \bar{W} = 2\Bigg(&\frac{1}{n}\left((n-1)\lambda(\Psi) + f(\bar{U})\right) f(u) \\
&+ (n-1)\left(k + \cot_k(\Psi)\lambda(\Psi)\bar{U}'(\Psi) - \frac{f'(\bar{U})}{n-1}\right)|\nabla u|^2\Bigg)
\end{aligned} \tag{C.11}$$

Thus, joining the equations (C.8) and (C.11) to obtain a lower bound for the Laplacian of $W-\bar{W}$,

$$\begin{aligned}\Delta(W-\bar{W}) \geq & |\nabla^2 u|^2 + (k(n-1) - f'(u))\,|\nabla u|^2 \\ & -2\Bigg(\frac{1}{n}\left((n-1)\lambda(\Psi) + f(\bar{U})\right) f(u) \\ & +(n-1)\left(k + \cot_k(\Psi)\lambda(\Psi)\bar{U}'(\Psi) - \frac{f'(\bar{U})}{n-1}\right)|\nabla u|^2\Bigg)\end{aligned} \tag{C.12}$$

Now, we want to give a bound for the norm of the Hessian of u. For a general radial function $v = V\circ r$ then the Hessian form is,

$$\nabla^2 v = (V''(r) - V'\cot_k(r))dr\otimes dr + V'\cot_k(r) g_{S^{n-1}}$$

Then, applying the following form to $\bar{u}$,

$$\nabla^2\bar{u} = -\lambda(r)d\bar{u}\otimes d\bar{u} + \frac{1}{n}\left(\lambda(r)|\nabla u|^2 - g(\bar{u})\right) g_{S^{n-1}}, \tag{C.13}$$

Thus, the norm of the Hessian of u satisfies that,

$$\begin{aligned}|\nabla^2 u + \lambda(r)du\otimes du + & \frac{1}{n}\left(-\lambda(r)|\nabla u|^2 + g(\bar{u})\right) g_{S^{n-1}}|^2 = \\ & |\nabla^2 u|^2 + 2\lambda(\Psi) g(\nabla^2 u, du\otimes du) + \frac{n-1}{n}\lambda(\Psi)^2|\nabla u|^4 \\ & - \frac{g(\bar{U})^2}{n} + \frac{2}{n}\lambda(\Psi)g(\bar{U})|\nabla u|^2.\end{aligned} \tag{C.14}$$

Therefore, using the equation (C.9) the previous equation can be simplified as,

$$g(\nabla^2 u, du\otimes du) = \frac{1}{2}g(\nabla(W-\bar{W}),\nabla u) - 2\lambda(u)\left[\frac{n-1}{n}\lambda(\Psi)\bar{W} + \frac{g(\bar{U})}{n}\right]|\nabla u|^2. \tag{C.15}$$

Then, we have that,

$$\begin{aligned}|\nabla^2 u|^2 + & \lambda(\Psi)g(\nabla(W-\bar{W}),\nabla u) - \lambda(u)\left[\frac{2(n-1)}{n}\lambda(\Psi)^2\bar{W} + \frac{g(\bar{U})}{n}\right]|\nabla u|^2 \\ & + \frac{n-1}{n}\lambda(\Psi)^2|\nabla u|^4 - \frac{g(\bar{U})^2}{n} + \frac{2}{n}\lambda(\Psi)g(\bar{U})|\nabla u|^2 \geq 0.\end{aligned}$$

Since,

$$\frac{1}{n^2}\lambda(\Psi)^2\bar{W} - \cot_k(\Psi)\lambda(\Psi)\bar{U}'(\Psi) = \frac{1}{n}\lambda(\Psi)\bar{U}'(\Psi)g(\bar{U}).$$

Then

$$\begin{aligned}\Delta(W-\bar{W}) \geq & -2\lambda(\Psi)g(\nabla(W-\bar{W}),\nabla u)+2\frac{n-1}{n}\lambda(\Psi)\left[g(\bar{U})-\lambda(\Psi)|\nabla u|^2\right](W-\bar{W})\\ & +\frac{2}{n}g(u)(g(u)-f(u))+2\left(g'(\bar{U})-f'(u)\right)|\nabla u|^2\\ \geq & -2\lambda(\Psi)g(\nabla(W-\bar{W}),\nabla u)+2\frac{n-1}{n}\lambda(\Psi)\left[f(u)+\lambda(\Psi)|\nabla u|^2\right](W-\bar{W})\end{aligned}$$

Then we want to write the previous equation in terms of F_β,

$$\beta(\Psi)\Delta(W-\bar{W}) \geq -2\lambda(\Psi)g(\nabla u,\nabla F_\beta)+2\frac{n-1}{n}\lambda(\Psi)F_\beta\left(f(u)+\lambda(\Psi)|\nabla u|^2\right) \tag{C.16}$$

Joining the previous equation together with (C.5) and (C.7)

$$\begin{aligned}\Delta F_\beta \geq & \frac{2(n-2)}{n}\lambda(\Psi)g(\nabla u,\nabla F_\beta)\\ & +2\frac{n-1}{n}\frac{|\nabla u|^2}{\left(\bar{U}'(\Psi)\right)^2}F_\beta\left(f'(U)-nk+\frac{n+2}{n}\lambda(\Psi)f(U)\right).\end{aligned} \tag{C.17}$$

Then, if the function

$$\mu(r) := f'(\bar{U}(r))-nk+\frac{n+2}{n}\lambda(r)f(\bar{U}(r)) \tag{C.18}$$

defined above were non-negative in $\mathcal{U}$, the inequality (5.8) would be elliptic. Now we show that in fact $\mu \geq 0$ when f satisfies hypothesis (5.4).

Lemma C.1. *Let μ be the function defined by* (C.18)*, and suppose that f satisfies condition* (5.4)*. Then $\mu \geq 0$ in* $(\bar{r}_-, R)\cup(R,\bar{r}_+)$.

Proof. First, note that up to take a dilation of the solution and of the metric, we can restrict to the cases in which $k=-1,0$, or 1.

Observe that when $k=-1,0$ and $r\in(\bar{r}_-,\bar{R})$ or $k=1$ and $r\in(\bar{r}_-,\bar{R})$ or $r\in[\bar{r}/2,\bar{r}_+)$ when $\bar{r}_+>\bar{r}/2$ it is straightforward to check that $\mu(r)>0$ using equation (3.1). For the other cases, consider the change of variable

$$r(t)=\begin{cases}\operatorname{arccosh}(t) & \text{if} \quad k=-1\\ \sqrt{2t} & \text{if} \quad k=0,\\ \arccos(t) & \text{if} \quad k=1,\end{cases}$$

and define the function $\bar{V}(t):=\bar{U}(r(t))$. Then it is a straightforward computation to check that

$$\mu(t)=\frac{1}{r'(t)^2}\left(\frac{n+2}{n}\bar{V}''(t)^2-\bar{V}'''(t)\bar{V}'(t)\right), \quad \forall t\in(\bar{R},\bar{r}_+).$$

The idea now is to argue as in Item 6 in [12, Theorem 4.1]. Since the three cases $k=-1,0,1$ are very similar, we only write the proof when $k=0$.

Define $\bar{T}=\bar{R}^2/2$ and $\bar{t}_+=\bar{r}_+^2/2$. Then, using that $\bar{U}$ solves (3.1), we get that $\bar{V}$ solves

$$2t\bar{V}''(t)+n\bar{V}'(t)+f(\bar{V})=0,$$

and thus $\bar{V}'$ solves

$$2t\bar{V}'''(t)+(n+2)\bar{V}''(t)+f'(\bar{V})\bar{V}'(t)=0.$$

Integrating this equation between $\bar{T}$ and t, we obtain that

$$\bar{V}''(t)=-\frac{1}{(2t)^{\frac{n+2}{2}}}\left((2\bar{T})^{\frac{n}{2}}f(M)+\int_{\bar{T}}^{t}(2s)^{\frac{n}{2}}f'(\bar{V}(s))\bar{V}'(s)\,ds\right),$$

where we write $M=u_{\max}$. Note that $f'\geq 0$ and $\bar{V}'(t)<0$ if $t\in(\bar{T},+\infty)$, so the integral in the above formula is negative when t is in this interval. Now take $\bar{T}<a<b$. Then we have that

$$\begin{aligned}\bar{V}''(a)-\bar{V}''(b)=&\left(\frac{1}{(2b)^{\frac{n+2}{2}}}-\frac{1}{(2a)^{\frac{n+2}{2}}}\right)\left((2\bar{T})^{\frac{n}{2}}f(M)-\int_{\bar{T}}^{a}(2s)^{\frac{n}{2}}f'(\bar{V}(s))\bar{V}'(s)\,ds\right)\\&+\frac{1}{(2b)^{\frac{n+2}{2}}}\int_{a}^{b}(2s)^{\frac{n}{2}}f'(\bar{V}(s))\bar{V}'(s)\,ds\leq 0.\end{aligned}$$

This implies that $\bar{V}(t)'''\geq 0$ in $(\bar{T},\bar{t}_+)$, so we conclude that $\mu(t)\geq 0$ in this interval. This concludes the proof of the lemma. □

D Appendix: Taylor expansions on level sets

In this appendix, we prove two useful expansions for the functions $W=|\nabla u|^2$ and $\bar{W}=|\nabla u_{\bar{R},M}|\circ\Psi$ near their zero and top level sets. These expansions are employed in Subsection 5.3.1 to derive curvature estimates. We briefly recall the setting.

Let $(\mathcal{M},g)$ a n-dimensional Riemannian manifold with bounded Ricci curvature $\mathrm{Ric}\geq(n-1)kg$. Let Ω be a compact C^2 domain and $\mathcal{U}\in\pi_0(\Omega\setminus\mathrm{Max}(u))$ and take $p\in\mathrm{cl}(\mathcal{U})\cap\partial\Omega$. Consider $(\bar{\mathcal{U}},\bar{u},f)$ to be a comparison triple associated to $(\mathcal{U},u,f)$. Write $M=u_{\max}$ and $\bar{R}$ for the core radius of the triple $(\bar{\mathcal{U}},\bar{u},f)$.

Define $\gamma:[0,\varepsilon)\to\mathcal{M}$ as the unit speed geodesic with initial data $\gamma(0)=p$ and $\gamma'(0)=\nabla u/|\nabla u|(p)$. We have the following expansions around p.

Lemma D.1. *For any $p\in\mathrm{cl}(\mathcal{U})\cap\partial\Omega$ there exists $\varepsilon>0$ such that for any $t\in[0,\varepsilon)$, the following expansions holds:*

$$\begin{aligned}W(\gamma(t))&=W(p)+2\left((n-1)\sqrt{W}(p)H(p)-f(0)\right)\sqrt{W}(p)t+O(t^2),\\\bar{W}(\gamma(t))&=\bar{W}(p)+2\left(\mp(n-1)\cot_k(\Psi)\sqrt{\bar{W}}(p)-f(0)\right)\sqrt{\bar{W}}(p)t+O(t^2).\end{aligned}\tag{D.1}$$

Proof. First, observe that $W \circ \gamma$ is a real valued function which is $\mathcal{C}^2$ in a neighborhood of $t = 0$. Thus, it exists $\varepsilon > 0$ such that its Taylor expansion is given by

$$W(\gamma(t)) = W(p) + \langle \nabla W, \nabla u / |\nabla u| \rangle (p)t + O(t^2),$$

for all $t \in (0, \varepsilon)$. Now, by [8, Theorem 3.3] it follows that

$$H = \frac{1}{(n-1)|\nabla u|^3} \left(\frac{1}{2} \left\langle \nabla |\nabla u|^2, \nabla u \right\rangle - |\nabla u|^2 \Delta u \right),$$

and then, since $\Delta u(p) = -f(0)$ from (5.1), we conclude that

$$W(\gamma(t)) = W(p) + 2\left((n-1)\sqrt{W}(p)H(p) - f(0) \right) \sqrt{W}(p)t + O(t^2),$$

which is the first of the expansions in (D.1).

Now, in order to compute the Taylor expansion of the $\mathcal{C}^2$ function $\bar{W} \circ \gamma$, we remind that

$$\nabla \bar{W} = 2\left((n-1)\cot_k(\Psi)\bar{U}'(\Psi) - f(u) \right) \bar{U}'(\Psi)$$

where $\bar{U}$ is such that $\bar{u} = \bar{U} \circ \Psi$. Then, we conclude that

$$\begin{aligned} \bar{W}(\gamma(t)) &= \bar{W}(p) + \left\langle \nabla \bar{W}, \nabla u / |\nabla u| \right\rangle (p) + O(t^2) \\ &= \bar{W}(p) + 2\left((n-1)\cot_k(\Psi)\sqrt{\bar{W}}(p) - f(0) \right) \sqrt{\bar{W}}(p)t + O(t^2). \end{aligned}$$

□

Now take a smooth hypersurface $\Gamma \subset \mathrm{Max}(u)$, and consider $p \in \Gamma$. Note that it exists $\mathcal{V} \subset \Omega$ a small neighborhood of p such that $\mathcal{V} \cap \Gamma$ is an embedded, two-sided smooth hypersurface that divides $\mathcal{V} \setminus \Gamma$ into two connected components $\mathcal{V}_+, \mathcal{V}_-$, where we suppose that $\mathcal{V}_+ \subset \mathcal{U}$. Define the signed distance function

$$s(x) := \begin{cases} +\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_+, \\ -\mathrm{dist}(x, \Gamma) & \text{if } x \in \mathcal{V}_-. \end{cases} \tag{D.2}$$

Observe that s is smooth in $\mathcal{V}_+$ and $\mathcal{V}_-$ by the regularity results of [14].

Now suppose that the function f appearing in (5.1) is smooth. Our goal is to extend the expansions from [12, Lemma 5.2] to the n-dimensional setting. While the expansions in that lemma are derived under the assumption that the solution to (5.1) is analytic, it is important to note that the arguments remain valid assuming only $\mathcal{C}^4$ regularity. In this appendix, however, we work with smooth functions for the sake of simplicity.

Lemma D.2. *Suppose that the function f appearing in* (5.1) *is smooth. Then for any $x \in \mathcal{V}_+$, the following expansions holds:*

$$
\begin{aligned}
W &= f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4),\\
\bar{W} &= f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right) s\right) + O(s^4),
\end{aligned}
\tag{D.3}
$$

where $H(p)$ is computed with respect to the inner normal to $\mathcal{U}$.

Proof. First, write $z = \Psi - \bar{R}$, where Ψ is the pseudo-radial function defined in Definition 5.4. Using the equation (B.1), we compute the Taylor expansion of $M - U \circ \Psi$ as a function of the variable Ψ:

$$
u - M = \frac{f(M)}{2} z^2 - \frac{(n-1)f(M)\cot_k(R)}{6} z^3 + O(z^4). \tag{D.4}
$$

Next, recalling that $\bar{W} = \bar{U}'(\Psi)^2$, where $\bar{U}$ is the solution to (B.1) defining the comparison triple associated to $(\mathcal{U}, u, f)$, we get that

$$
\bar{W}''(\bar{R}) = 2f(M)^2 \quad \text{and} \quad \bar{W}'''(R) = -6(n-1)\cot_k(\bar{R}) f(M)^2.
$$

Thus, we compute the following Taylor expansion in a neighborhood of $z = 0$,

$$
\frac{\bar{W}}{M-u} = 2f(M) + \frac{4(n-1)\cot_k(\bar{R}) f(M)}{3} z + O(z^2). \tag{D.5}
$$

Finally, using (D.4) we get that

$$
z = \pm\sqrt{\frac{2}{f(M)}}\sqrt{M-u} + O(M-u),
$$

where we take the positive sign if $\overline{\tau}(\mathcal{U}) \geq \tau_0(M)$ and the negative sign if $\overline{\tau}(\mathcal{U}) < \tau_0(M)$, so substituting this into (D.5) we conclude that

$$
\bar{W} = 2f(M)(M-u) - \frac{4(n-1)\sqrt{2f^3(M)}}{3}\cot_k(\bar{R})(M-u)^{3/2} + O((M-u)^2). \tag{D.6}
$$

Next, observe that since f is of class $\mathcal{C}^\infty$ standard regularity results implies that u is smooth, and we have a Taylor expansion of u at $p \in \Gamma$ given by [6, Theorem 3.1] (which works also in the smooth setting, as remarked in [6, Remark 3.2]):

$$
u = M - \frac{f(M)}{2} s^2 - \frac{(n-1)f(M)}{6} H(p) s^3 + O(s^4),
$$

where $s \in \mathcal{C}^\infty(\mathcal{V}_+)$ is defined in (D.2). Then, we get that

$$
\nabla u = -s f(M)\left(1 + \frac{(n-1)H(p)}{2} s + O(s^3)\right)\nabla s, \tag{D.7}
$$

so taking into account that $|\nabla s| = 1$, we conclude that

$$W = f(M)^2 s^2 \left(1 + (n-1)H(p)s\right) + O(s^4).$$

In addition, substituting (D.7) into (D.6), we have that

$$\bar{W} = f(M)^2 s^2 \left(1 + (n-1)\left(\frac{H(p)}{3} \pm \frac{2\cot_k(\bar{R})}{3}\right) s\right) + O(s^4),$$

where we recall that the positive sign is taken if $\overline{\tau}(\bar{\mathcal{U}}) \geq \tau_0(M)$ and the negative is taken otherwise. This finishes the proof of the lemma. $\square$

E Top stratum as C^1–limit of regular levels

Let $(\mathcal{M}^n, g)$ be a smooth Riemannian manifold and $\Omega \subset M$ a relatively compact domain with C^2-boundary. Consider u solving (2.5) with $f \in C^{n-2}(\mathbb{R})$ and satisfying the standing structural hypotheses of Section 3 when needed. By classical elliptic regularity, $u \in C^n(\Omega)$ (and $u \in C^{n-1}(\overline{\Omega})$ when $\partial\Omega$ is C^{n-1}). Hence Sard's theorem applies to u and for a.e. $c \in \mathcal{I}_f$ the level Γ_c is a smooth embedded compact hypersurface, possibly disconnected; moreover, at every regular level $\nabla u \neq 0$ and Σ_c admits a C^n local graph representation by the Implicit Function Theorem on manifolds.

E.1 Upgrading the top stratum from C^1 to C^{n-1} via Morse–Bott

We write $M := \max_\Omega u$, $\Gamma_M \subset \mathrm{Max}(u)$, and assume throughout that Γ_M is a $(n-1)$-dimensional connected component which is an embedded C^1 hypersurface (top stratum). We also assume that $f \in C^{n-2}$ so that $u \in C^n(\Omega)$ by Schauder theory, and that u is *Morse–Bott* along Γ_M namely

$$\ker \nabla^2 u|_{\Gamma_M} = T\Gamma_M, \qquad \nabla^2 u \text{ is nondegenerate on the normal bundle of } \Gamma_M.$$

In our setting, the Moser-Bott condition is warantied since u is a solution to (2.5) and hence $\Delta u = -f(M) < 0$ along Γ_M. This means that the lowest eigenvalue of the hessian $\nabla^2 u$ is always negative and, therefore, $\nabla^2 u$ is nondegenerate along the normal direction of Γ_M, which is well-defined since we are assuming Γ_M is C^1.

Let $U \Subset \Omega$ be a geodesic neighborhood of p such that $\Gamma_M \cap U$ is a C^1-hypersurface and U decomposes as $U = U^- \cup (\Gamma_M \cap U) \cup U^+$, where U^+ is the component of $U \cap \{u < M\}$ such that the signed distance s to Γ_M is positive on U^+.

Proposition E.1 (Morse–Bott upgrade and C^1 convergence of regular levels)**.** *Let $p \in \Gamma_M$ and assume Γ_M is an embedded C^1 hypersurface near p, and that u is Morse–Bott along Γ_M. Then:*

1. *(C^{n-1} **upgrade of the top stratum**) There exist a neighborhood $U \Subset \Omega$ of p, a C^{n-1} diffeomorphism*
$$\Phi : (\Gamma_M \cap U) \times (-\varepsilon, \varepsilon) \longrightarrow U,$$
with $\Phi(\,\cdot\,, 0) = \mathrm{id}$ on $\Gamma_M \cap U$, and a function $\lambda \in C^{n-1}(\Gamma_M \cap U)$ with $\lambda > 0$ such that
$$(u \circ \Phi)(y, s) \;=\; M - \frac{\lambda(y)}{2}\, s^2, \qquad (y, s) \in (\Gamma_M \cap U) \times (-\varepsilon, \varepsilon). \tag{E.1}$$
In particular, $\Gamma_M \cap U = \{s = 0\}$ is a C^{n-1} hypersurface (hence the initial C^1 regularity bootstraps to C^{n-1}).

2. *(**Structure and C^1 limit of regular levels**) By Sard's theorem, there exists a sequence of regular values $c_j \uparrow M$. For j large, the level set*
$$\Sigma_{c_j} := u^{-1}(c_j) \cap U$$
is an embedded C^{n-1} hypersurface that can be written as a normal graph *over $\Gamma_M \cap U$:*
$$\Sigma_{c_j} \cap U \;=\; \big\{(y, s) : s = \sigma_{c_j}(y)\big\}, \qquad \sigma_{c_j}(y) \;=\; \sqrt{\frac{2\,(M - c_j)}{\lambda(y)}} \;\in C^{n-1}(\Gamma_M \cap U), \tag{E.2}$$
where the positive branch is chosen towards the side where $u < M$. Moreover, as $j \to \infty$,
$$\sigma_{c_j} \;\xrightarrow{C^1_{\mathrm{loc}}(\Gamma_M \cap U)}\; 0,$$
so that $\Sigma_{c_j} \cap U \to \Gamma_M \cap U$ in the C^1 (graphical) sense. In particular,
$$\nu_{\Sigma_{c_j}} \to \nu_{\Gamma_M} \quad \textit{and} \quad \mathrm{II}_{\Sigma_{c_j}} \to \mathrm{II}_{\Gamma_M} \quad \textit{in } C^0_{\mathrm{loc}}(U).$$

Proof. Step 1 (Morse–Bott normal form). Fix $p \in \Gamma_M$ as in the statement and let s be the signed distance to Γ_M in U, positive towards the component $U^+ \subset U$. Since $u \in C^n$ and u is Morse–Bott along Γ_M, the Morse–Bott lemma (in the C^p category), shrinking U is necessary, provides a C^{n-1} change of coordinates straightening u to the quadratic normal form (E.1); see [4, Thm 2]. The proof yields that Φ is a C^{n-1} diffeomorphism and $\lambda \in C^{n-1}$, with $\lambda > 0$ along Γ_M (the positivity corresponds to the fact that M is a local maximum in the normal direction). Therefore $\Gamma_M \cap U = \{s = 0\}$ is C^{n-1}.

Step 2 (Explicit form of nearby regular levels). Fix $c \in (M - \varepsilon', M)$ and consider the equation $u \circ \Phi(y, s) = c$. By (E.1), this is equivalent to
$$M - \frac{\lambda(y)}{2}\, s^2 = c, \qquad \text{i.e.} \qquad s = \sigma_c(y) := \sqrt{\frac{2\,(M - c)}{\lambda(y)}}.$$

Since $\lambda \in C^{n-1}$ and is strictly positive on $\Gamma_M \cap U$, the map $y \mapsto \sigma_c(y)$ belongs to $C^{n-1}(\Gamma_M \cap U)$ for each such c. Sard's theorem for u (which holds as $u \in C^n$ with $n = \dim \mathcal{M}$) gives a sequence $c_j \uparrow M$ with c_j regular; for j large the above representation defines an embedded C^{n-1} hypersurface Σ_{c_j} in U.

Step 3 (C^1 convergence). From (E.2) we have

$$\|\sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \;\le\; \sqrt{\frac{2\,(M-c_j)}{\inf_{\Gamma_M \cap U} \lambda}}, \qquad \nabla^\top \sigma_{c_j}(y) \;=\; -\frac{1}{2}\,\sigma_{c_j}(y)\,\nabla^\top\big(\log \lambda(y)\big),$$

hence

$$\|\nabla^\top \sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \;\le\; C\,\|\sigma_{c_j}\|_{C^0(\Gamma_M \cap U)} \;\xrightarrow[j\to\infty]{}\; 0.$$

Therefore $\sigma_{c_j} \to 0$ in $C^1_{\mathrm{loc}}(\Gamma_M \cap U)$, and the graphical convergence follows. Since (in normal graph coordinates) ν_{Σ_c} and II_{Σ_c} depend smoothly on $\nabla^\top \sigma_c$ and $\nabla^2_y \sigma_c$, the C^1 convergence implies C^0 convergence of normals and second fundamental forms on compact subsets of U.

Moreover, the Taylor expansion proved in Appendix D yields, as $s \downarrow 0$,

$$u = M - \tfrac{f(M)}{2}\,s^2 - \tfrac{(n-1)f(M)}{6}\,H(p)\,s^3 + O(s^4), \qquad |\nabla u|^2 = f(M)^2 s^2\,\Big(1 + (n-1)H(p)\,s + O(s^2)\Big),$$

with $H(p)$ the mean curvature of Γ_M at p (inner normal to U^+); see Lemma D2. Consequently, $|\nabla u| \simeq f(M)\,s$ and, for c close to M, the level $\Sigma_c \cap U$ can be written as a normal graph $s = \sigma_c(y)$ over $y \in \Gamma_M \cap U$ with $\sigma_c \to 0$ in C^1 uniformly on compact subsets (invert the expansion $u(y,s) = c$ by the IFT). □

Remark E.1. Analytic case: *If, in addition, u is real-analytic and Morse–Bott along Γ_M, the Morse–Bott normal form is analytic and Γ_M is a real-analytic hypersurface; the functions σ_c in* (E.2) *are then analytic in y.*

Lower regularity: *If the top stratum Γ_M fails to be a C^1 hypersurface, one still has Hausdorff convergence of Σ_c to Γ_M; however, C^1 convergence may fail across singular points.*

Remark E.2 (Para nosotros)**.**

- ***Morse–Bott lemma (regularity).*** *If $u \in C^p$, $p \ge 2$, then there exist local coordinates of class C^{p-1} in which u assumes the quadratic normal form along a critical submanifold; see Banyaga–Hurtubise [4, Theorem 2]. In our setting $u \in C^n$, hence the chart and the coefficient λ are C^{n-1}.*

  `BanyagaHurtubise04`: *A. Banyaga and D. E. Hurtubise, "A proof of the Morse–Bott lemma,"* Expositiones Mathematicae *22 (2004), 365–373.*

- ***Sard and regular level sets.*** *Since $u \in C^n$ on an n-dimensional manifold, Sard's theorem applies; regular values $c < M$ yield C^{n-1} embedded level hypersurfaces by the Implicit Function Theorem (IFT). Standard references: Lee,* Introduction to Smooth Manifolds *(regular level set theorem and IFT); Krantz–Parks,* The Implicit Function Theorem.

- ***Elliptic bootstrap to*** C^n***.*** *With* $f \in C^{n-2}$ *and smooth coefficients for* Δ_g*, classical Schauder theory implies* $u \in C^n$ *(interior, and up to the boundary under standard compatibility); see Evans or Gilbarg–Trudinger.*

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