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Applications of the Radiation from Fast Electron Beams

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The radiation from fast electron beams passing through a succession of electric or magnetic fields of alternating polarity is examined. The radiation of maximum frequency is emitted in the forward direction. If the deflecting fields are not too large, a semiqualitative argument shows that the maximum frequency is the lowest possible harmonic. The frequencies emitted are determined by studying the Doppler effect, and the angular distribution of radiated energy as well as the total radiation are calculated in a simple straightforward manner with reference to well-known formulas of special relativity. The question of the coherence of the radiation is discussed. The spectral distribution of radiated energy is then calculated more exactly. It is concluded that several applications of the radiation appear possible. A scheme for obtaining millimeter-waves of considerable power is outlined. The upper limit of the power in a band extending down to a wavelength of 1 millimeter is calculated to be of the order of several kilowatts for a beam of one ampere and an energy of 1.5 megavolt. The use of the radiation for speed monitoring of beams with energies up to 1000 megavolt is discussed.

I. INTRODUCTION

An electron, traveling through a stationary electric or magnetic field distribution, radiates. The frequency spectrum of the emitted radiation depends on the speed of the electron. We shall see that the entire spectrum of electromagnetic radiation starting from microwaves and extending to hard x-rays may be easily obtained from electrons with speeds ranging from a megavolt, say, to 1000 Mev.

Several applications suggest themselves. We shall discuss them under the following headings:

(1) production of energy in rather inaccessible spectral bands, *viz.*, millimeter to infrared radiation;*

(2) speed monitoring for electron beams produced by linear or other accelerating devices;

(3) speed measurement for fast individual electrons or other particles (mesons, protons).

We shall also briefly consider the radiation emitted by an electron moving through an electromagnetic wave traveling in the opposite direction. This occurs, e.g., in a linear accelerator in which a wave is reflected from the output end and travels back to the starting section.

The stationary field distribution we have in mind is a succession of electric or magnetic fields of alternating polarity, regularly spaced, as in Fig. 1. The radiation in transverse fields turns out to be much larger than that emitted by electrons crossing an array of longitudinal fields. Hence, we shall be mainly interested in the case of transverse fields, as shown in Fig. 1. The spatial field distribution may be fourier analyzed. Let the fundamental wavelength be l_0. It will be shown that the frequency of the emitted radiation varies with the angle of observation. The fundamental component of the radiation emitted under the angle θ (see Fig. 1) has a frequency

$$\omega=\Omega_\theta=2\pi\beta c/l_0(1-\beta\cos\theta). \tag{1}$$

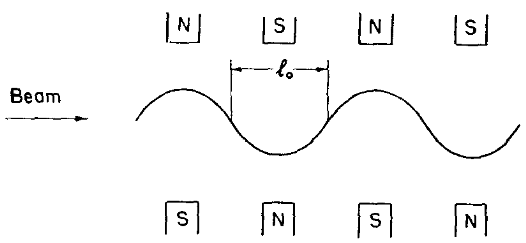

Fig. 1. Schematic arrangement of undulator magnets.

* *Notes added in proof:* According to reports of the Electronics Research Laboratory at M.I.T., P. D. Coleman is working on such a scheme. Prior reference to the problem treated in this paper has also been found in a paper by V. L. Ginsburg, Radiation of microwaves and their absorption in air, Bull. Acad. Sci. U.R.S.S. Ser. Phys. 9 (1947), No. 2, 165 (in Russian).

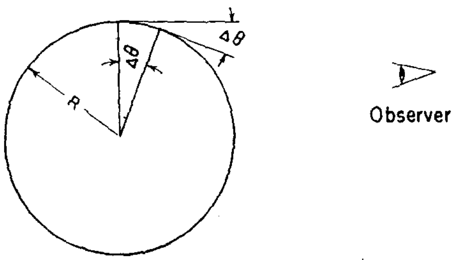

FIG. 2. Illustrates the radiation cone emanating from an electron in a circular orbit.

In the extreme relativistic case, when the electron velocity is near the velocity of light and the accelerations and decelerations suffered by the electrons in the periodic field are considerable, most of the energy is in a high harmonic of this fundamental frequency, which is determined by the maximum instantaneous acceleration or deceleration.[1] However, even for extremely relativistic electron speeds, the energy is predominantly in the fundamental provided that the deflecting fields are small, i.e., of the order of a megavolt/cm.

To show this we run through the following qualitative argument. Assume that the electrons in the periodic field describe arcs of circles, reminiscent of a sine wave. The radiation is mainly forward and contained in a narrow cone of opening

$$\Delta\theta = m_0c^2/E, \tag{2}$$

where E is the beam energy. As the electrons move along, the cone swings to and fro and may intermittently pass out of the view of an observer situated on the axis. This observer may then receive short pulses of radiation only, which explains why the energy is radiated into a higher harmonic.

Consider Fig. 2. As the electron moves along a circular arc of radius R, the radiation cone sweeps past the observer and is out of his view when the electron has moved through $\Delta\theta$ radians. Thus, the time during which radiation from the electron is received is given by

$$\tau = (1/c)R\Delta\theta(1-\beta) \sim \tfrac{1}{2}(1/c)(1-\beta^2)R\Delta\theta, \tag{3}$$

the radius of the circle is given by $m\beta^2c^2/R = e\mathcal{E}$, where $\mathcal{E}$ is the deflecting field. Hence,

$$\tau = (1-\beta^2)^{\frac{1}{2}} m\beta^2c^2/2ce\mathcal{E}. \tag{4}$$

The wavelength of the radiation is then approximately given by

$$\lambda_1 = \tfrac{1}{2}(1-\beta^2)^{\frac{1}{2}}\beta^2mc^2/e\mathcal{E} = \tfrac{1}{2}(1-\beta^2)\beta^2m_0c^2/e\mathcal{E}. \tag{5}$$

The fundamental frequency (1) corresponds to a wavelength

$$\lambda_2 = l_0(1-\beta)/\beta, \tag{6}$$

in the forward direction.

If the wavelength λ_1 is small compared with λ_2, the radiation is indeed mostly contained in a high harmonic. If, however,

$$\lambda_1 \geqq \lambda_2, \tag{7}$$

the radiation is mostly of fundamental frequency. This means we can expect to find most of the energy in the fundamental if

$$e\mathcal{E}l_0/m_0c^2 < 1. \tag{8}$$

In the following section we shall determine the radiated energy and frequencies under the assumption that Eq. (8) holds. This will first be done in a simple manner which clearly shows the physical principles involved. Later a more complete analysis of the frequency spectrum is carried out.

II. FREQUENCY OBSERVED IN THE LABORATORY SYSTEM

1. The Doppler Effect

To determine the observed frequencies we must apply the theory of the relativistic Doppler effect which we shall now review briefly. Let the moving system E have primed coordinates and let it emit a spherical wave of amplitude

$$S' = (A'/r')\exp[2\pi i\nu'(t'-r'/c)+i\delta']. \tag{9}$$

Let the observer belong to a system L (laboratory system) with unprimed coordinates z, y, r, t, and let the directions z and z' be parallel to the relative (constant) velocity of the two systems. We call this velocity v and define $\beta = v/c$.

Let the observer be located at the origin 0 of system L and let the direction $\langle 00'\rangle$ define an angle θ (in L) with the direction of movement, the positive z-direction. Let the corresponding angle between $\langle 00'\rangle$ and z be θ' in E. Then

$$r' = z'\cos\theta' + y'\overset{\sin\theta'}{\cancel{\cos\theta'}}$$

and

$$S' = (A'/r')\exp\{2\pi i\nu'[t'-(z'\cos\theta' + y'\sin\theta')/c]+i\delta\}'. \tag{10}$$

The observer in L interprets this as a spherical wave

$$S = (A/r)\exp\{2\pi i\nu[t-(z\cos\theta + y\sin\theta)/c]+i\delta\}. \tag{11}$$

The primed and unprimed coordinates are, of course, related by the Lorentz transformations

$$z' = (z-vt)/(1-\beta^2)^{\frac{1}{2}}, \quad t' = (t-vz/c^2)/(1-\beta^2)^{\frac{1}{2}}, \quad y = y', \quad x = x'. \tag{12}$$

Thus we can eliminate the primed coordinates in Eq. (10).

Comparing Eqs. (10) and (11) we must get relations which hold good for any time and place; thus, the coefficients of z, y, t must be equal. In this way one obtains

$$\nu = \nu'(1+\beta\cos\theta')/(1-\beta^2)^{\frac{1}{2}}, \tag{13}$$

$$\cos\theta = (\cos\theta'+\beta)/(1+\beta\cos\theta'), \tag{14a}$$

$$\nu = \nu'(1-\beta^2)^{\frac{1}{2}}/(1-\beta\cos\theta). \tag{14b}$$

[1] Compare L. I. Schiff, Rev. Sci. Instr. 17, 6 (1946); J. Schwinger, Phys. Rev. 75, 1912 (1949).

2. Frequency Seen by the Moving Electrons

(a) *By Riding through a Succession of Magnetic or Electric Fields of Alternating Polarity (See Fig. 1)*

The fundamental fourier component of the field has a wavelength l_0 in L, the length over which the periodic field extends is l and the number of full fundamental waves $n=l/l_0$.

An observer at rest in E (the system attached to the electron) takes a time $t'=l'/v$ to cover the distance l' as he rides past it. He receives $n=l/l_0=l'/l_0'$ fundamental waves; hence, he sees a frequency (number of waves/unit time)

$$\nu'=l'v/l_0'l'=v/l_0',$$

but

$$l_0'=(1-\beta^2)^{\frac{1}{2}}l_0.$$

Hence, he sees a frequency

$$\nu'=v/l_0(1-\beta^2)^{\frac{1}{2}}. \quad (15)$$

(b) *Situation in a Section of a Linear Accelerator Tube Electron Moving with Velocity v Relative to the L-System, Encountering a Traveling Wave with Velocity v Relative to the L-System*

Let there be ν^* waves emitted by the end of the accelerator tube (reflected waves) in unit L-time.

Measure coordinates and time in the system W attached to the wave and denote them by double primes.

During the time dt'' the wave has moved a distance

$$dz''=-\beta c dt''=-\beta c dt/(1-\beta^2)^{\frac{1}{2}}.$$

During the time dt'' the W-system has received $\nu^* dt$ full waves emitted by the accelerator end, the length of a full wave appears to be (for a W-observer)

$$\beta c/\nu^*(1-\beta^2)^{\frac{1}{2}}$$

The relative velocity between electron and wave, by Einstein's velocity addition theorem is

$$u/c=\{1-[(1-\beta^2)/(1+\beta^2)]^2\}^{\frac{1}{2}}. \quad (16)$$

Thus in time dt' the electron will cover a distance

$$dz_1''=-u dt_1''=u dt'/(1-u^2/c^2)^{\frac{1}{2}}$$

along the wave and so during time dt' the electron encounters

$$\nu' dt'=u dt'(1-\beta^2)^{\frac{1}{2}}\nu^*/(1-u^2/c^2)^{\frac{1}{2}}\beta c$$

waves.

We find that this latter expression equals

$$\nu' dt'=2\nu^* dt'/(1-\beta^2)^{\frac{1}{2}}.$$

Thus, the frequency as seen by the electron is

$$\nu'=2\nu^*/(1-\beta^2)^{\frac{1}{2}}. \quad (17)$$

3. Frequency Received in L-System

In case (a): the frequency emitted by the electron is ν' in its own system. This is much higher than microwave frequency in the cases which interest us. Thus, we can treat it as a light wave propagated with velocity c and calculate the frequency ν received by the observer in L by means of the Doppler formula (13).

$$\nu=v(1+\beta\cos\theta')/l_0(1-\beta^2)=v/l_0(1-\beta\cos\theta). \quad (18)$$

In case (b): We apply the same Doppler formula (13) and obtain

$$\nu=2\nu^*(1+\beta\cos\theta')/(1-\beta^2)=2\nu^*/(1-\beta\cos\theta). \quad (19)$$

III. ENERGY RADIATED BY AN ELECTRON

Total energy calculated in the E-system and later referred to the L-system.—We want to calculate the energy radiated by an electron riding through a succession of electric or magnetic fields of alternating polarity. We shall call the arrangement an undulator.

First we shall calculate the energy radiated as it appears to an observer moving with the electron (E-system).

We apply the well-known formula for the Poynting vector

$$\mathbf{S}'=e^2\dot{v}'^2\sin^2\psi'/4\pi c^3 r'^2, \quad (20)$$

which measures the energy radiated per unit time through unit area of a spherical surface element at distance r' from the source whose normal makes an angle ψ' with the acceleration vector.

To obtain the total energy radiated we have to multiply Eq. (20) with the time l'/v during which the electron radiates and integrate Eq. (20) over the sphere. We obtain

$$W'=4\pi\tfrac{2}{3}(e^2\dot{v}^2/4\pi c^4)l'/\beta=\tfrac{2}{3}(e^2\dot{v}'^2/c^4)l'/\beta. \quad (21)$$

Note that we have used primed quantities because we have calculated the energy in the E-system.

The undulator may consist of a succession of magnetic fields of strength H with alternating polarity, let us call it the magnetic undulator. Alternatively, we may consider an analogous electric undulator—in this case let the maximum field strength be E. (More exactly, E and H are the amplitudes of the fundamental component of the spatial field distribution).

The fields which the electron sees are given by the relativistic field transformation formulas:

$$H_z'=H_z,\quad H_y'=(H_y+\beta E_x)/(1-\beta^2)^{\frac{1}{2}},$$
$$H_x'=(H_x-\beta E_y)/(1-\beta^2)^{\frac{1}{2}}. \quad (22a)$$

$$E_z'=E_z,\quad E_y'=(E_y-\beta H_x)/(1-\beta^2)^{\frac{1}{2}},$$
$$E_x'=(E_x+\beta H_y)/(1-\beta^2)^{\frac{1}{2}}. \quad (22b)$$

Looking at these expressions we see that the longitudinal field components are unchanged, while the transverse components increase with β.

Accordingly, the electron acceleration in E is much higher for transverse fields.

Therefore, we first consider an undulator with purely transverse field:

(a) electric,
(b) magnetic.

TABLE I.

Beam energy in Mev	Beam current	Power produced	Minimum fundamental wavelength
1000	1 μa	2.8 mW	25A
100	100 μa	2.8 mW	2500A
10	10 ma	2.8 mW	2.5 10^{-3} cm

Consider case (a). The transverse field $E_y=E$. Hence,

$$E'=E/(1-\beta^2)^{\frac{1}{2}}, \tag{23}$$

the acceleration

$$\dot{v}=(e/m_0E)/(1-\beta^2)^{\frac{1}{2}} \quad (m_0 \text{ rest mass}). \tag{24}$$

The length l appears shortened to the observer in E

$$l'=l(1-\beta^2)^{\frac{1}{2}}. \tag{25}$$

With these expressions for l' and v the energy radiated in E becomes

$$\tfrac{2}{3}e^2E^2l/m_0c^2(1-\beta^2)^{\frac{1}{2}}. \tag{26}$$

Thus, energy must now be transformed into the L-system by means of transformation formulas

$$W=(W'-\beta p_z')/(1-\beta^2)^{\frac{1}{2}},\ p_z=(p_z'-\beta W')/(1-\beta^2)^{\frac{1}{2}} \tag{27}$$

for the energy momentum tensor. The momentum in the E-system is zero. Thus,

$$W=(2/3\beta)(e^2/m_0c^2)^2E^2l/(1-\beta^2). \tag{28}$$

It is interesting to note that this is just the field energy swept out by the electron with the classical radius e^2/mc^2.

Now consider case (b). Let there be a magnetic field $H=H_x$. The field H_x' does not produce a force on the electron at rest in the E-system. But there is an electric force $E_y'=-\beta H_x/(1-\beta^2)^{\frac{1}{2}}$ which produces an acceleration

$$\dot{v}=-e\beta H/m_0(1-\beta^2)^{\frac{1}{2}},$$

and the energy W becomes

$$W=\tfrac{2}{3}(e^2/m_0c^2)^2\beta H^2l/(1-\beta^2). \tag{29}$$

IV. RADIATION BY THE ENTIRE BEAM: QUESTION OF COHERENCE

We show briefly, that for large β-radiation from N electrons is N times the radiation from the single electron, and not N^2 times this quantity, as one might hope to get from a coherently radiating beam. The field produced by N electrons is, of course, a superposition of the individual fields

$$E=E_1+E_2+E_3+E_4+\cdots,$$

and the energy is proportional to the square of this

$$(E_1=E_2+E_3+\cdots)^2;$$

but the contributions of individual electrons are random in phase, and therefore the cross terms of the squared bracket drop out, and we are left with the sum of N squares

$$E_1^2+E_2^2+E_3^2+\cdots,$$

which is N times the energy radiated by the individual electron. This must be so, unless it be possible to bunch the electrons so that a whole bunch arrives at successive elements of the grating simultaneously. However, this is hard to realize. In the E-system the distance l_0 appears shortened to $l_0(1-\beta^2)^{\frac{1}{2}}$ and the bunches would have to be short compared with this to be effective.

It turns out that there is a small region of the spectrum which is interesting, while the bunching is yet possible: the millimeter wave range.

V. NUMERICAL VALUES

Let the fundamental wavelength of the undulator field be $l=1$ cm. Let us further consider a magnetic undulator with $H=15{,}000$ gauss. Then the total energy radiated incoherently amounts to the values shown in Table I.

For any harmonic the radiation of maximum frequency (minimum wavelength) is emitted in the forward direction ($\theta=0$).

Another way of stating the result is to express the radiation per electrons in terms of quanta of maximum fundamental frequency. We can then say that the total radiated (emitted into the entire spectrum radiated) amount to 3 such quanta.

The field strength E of an electric undulator can probably not be made much greater than 1 Mev/cm, and under these conditions the energy output is about 1/25 of that of the magnetic undulator considered above.

VI. LONGITUDINAL FIELD ARRAYS

The longitudinal fields are the same in the E and L systems, and therefore the radiated energy is smaller by a factor $1-\beta^2$. This, no doubt, is the reason why the linear accelerator compares so favorably with other devices relying on circular electron orbits involving transverse fields. For the electric case we obtain an energy

$$W=\tfrac{2}{3}(e^2/m_0c^2)^2E^2l/\beta. \tag{30}$$

VII. THE ANGULAR DISTRIBUTION OF RADIATED ENERGY

The radiation of an electron in arbitrary motion is given[2] by

$$\frac{\mathbf{H}}{e}=\frac{(\mathbf{r}\times\dot{\mathbf{v}})(\mathbf{r}\cdot\mathbf{v}-cr)-(\mathbf{r}\times\mathbf{v})(\mathbf{r}\cdot\dot{\mathbf{v}})}{(cr-\mathbf{r}\cdot\mathbf{v})^3}, \tag{31a}$$

$$\frac{\mathbf{E}}{e}=\frac{(\mathbf{r}c-r\mathbf{v})(\mathbf{r}\cdot\dot{\mathbf{v}})-\dot{\mathbf{v}}\mathbf{r}(\mathbf{r}\cdot\mathbf{v}-cr)}{(cr-\mathbf{r}\cdot\mathbf{v})^3}. \tag{31b}$$

[2] See P. Frank and R. v. Mises, *Differential- und Integralgleichungen der Mechanik und Physik*, Vol. II, p. 788, formula 19. (Mary S. Rosenberg, New York 25, N. Y.)

Since

$$H=\mathbf{r}\times\mathbf{E}/r \quad\text{and}\quad \mathbf{r}\cdot\mathbf{E}=0,$$

it is seen immediately that E and H are both perpendicular to r and to each other. We assume v to be parallel to z and $\dot{v}$ parallel to y (this is the transverse field case). With definitions,

$$\begin{aligned} \mathbf{r}\times\mathbf{y}/y&=r\cos\psi, & \mathbf{r}\cdot\mathbf{z}/z&=r\cos\theta,\\ \mathbf{r}\times\mathbf{y}/y&=r\sin\psi, & \mathbf{r}\cdot\mathbf{z}/z&=r\sin\theta, \end{aligned} \tag{32}$$

we find for the Poynting vector (which has the dimension of power)

$$\mathbf{S}=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})=\frac{\mathbf{r}e^2|\dot{v}|^2}{4\pi rc^3r^2}\left[\frac{1}{(1-\beta\cos\theta)^4}-\frac{(1-\beta^2)\cos\psi}{(1-\cos\theta)^6}\right]. \tag{33}$$

For the case of an electron moving in the direction of the acceleration vector (longitudinal undulator fields) it is found that

$$S=e^2\dot{v}^2\sin^2\theta/4\pi c^3r^2(1-\beta\cos\theta)^6. \tag{34}$$

To check the previous calculations carried out in the E-system we have to integrate Eq. (34) over the surface of the sphere after multiplication with the time during which the radiation is received in order to find the total energy. This time depends on the angle of observation. Consider formula (5a) for the Doppler effect of frequency. The frequency received varies with the angle θ, between the observer and the z-axis. However, the electron encounters n full waves of the grating and the observer cannot receive more or less oscillations of fundamental frequency from the radiating electron. Thus, the time during which radiation is received by the observer varies with θ. It is given by

$$T_\theta=l(1-\beta\cos\theta)/\beta c. \tag{35}$$

This is easily seen from Fig. 3 where the arrow points to an observer at large distance. It is seen that the radiation coming from β is delayed by a time interval $-(l/c\beta)\cos\theta$.

For the integration over the angles it is preferable to introduce an azimuth χ in the y-x plane (Fig. 4). From the cosine theorem of solid geometry

$$\cos\psi=\cos\chi\sin\theta. \tag{36}$$

The total energy radiated by the transverse undulator is thus given by

$$\frac{le^2\dot{v}^2}{4c^4\beta}\left[\int_{-1}^{+1}\frac{2du}{(1-\beta u)^3}-(1-\beta^2)\int_{-1}^{+1}\frac{(1-u^2)du}{(1-\beta u)^5}\right], \tag{37}$$

which indeed yields

$$\tfrac{2}{3}e^2\dot{v}^2l/c^4\beta(1-\beta^2)^2. \tag{38}$$

The corresponding total energy radiated in a longitudinal field array turns out to be

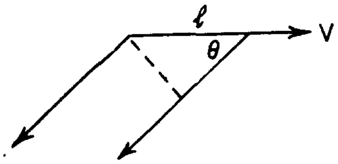

FIG. 3. Illustrates retardation of radiation from a moving electron.

$$\tfrac{2}{3}e^2\dot{v}^2l/c^4\beta(1-\beta^2)^3. \tag{39}$$

Of course, in the transverse case (38) we have to use the transverse mass

$$m_0/(1-\beta^2)^{\frac{1}{2}}, \tag{40}$$

while in the longitudinal case (39) we have to use the longitudinal mass

$$m_0/(1-\beta^2)^{\frac{3}{2}} \tag{41}$$

when computing the acceleration and thus fall back on formulas (28) and (29), respectively, and on Eq. (30).

VIII. CALCULATION OF THE FREQUENCY SPECTRUM RADIATED BY THE UNDULATOR

1. A Fundamental Radiation Formula

In order to calculate the frequency spectrum in more detail, two different procedures are available. We choose to work with a fourier expansion of the current density rather than with fourier expansions of the fields. In this case a convenient starting point is an expression for the average energy $P_{\mathbf{k}\omega}(\mathbf{r}')d\omega$ emitted into a frequency range between ω and $\omega+d\omega$ by a current with density distribution $J(\mathbf{r}')$. Let the observer be situated at a point with radius vector $\mathbf{r}$ drawn from the origin of the coordinate system. Let the distance $|\mathbf{r}'-\mathbf{r}|$ be large compared with the extension of the radiating current system, and let $\mathbf{k}$ be a vector of absolute value $|k|=\omega/c$ which points from the current element at $\mathbf{r}'$ to the observer at $\mathbf{r}$. Then the average energy is given by

$$P_{\mathbf{k}\omega}(\mathbf{r})d\omega=(k^2/2\pi r^2c)\left|\int J_{\perp\mathbf{k}\omega}(\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'}d\tau'\right|^2 d\omega, \tag{42}$$

where $J_{\perp\mathbf{k}}(\mathbf{r}')$ is the fourier amplitude of the current density vector projected on to the plane perpendicular to $\mathbf{k}$. For a derivation of this formula, see L. I. Schiff, *Quantum Mechanics*.[3]

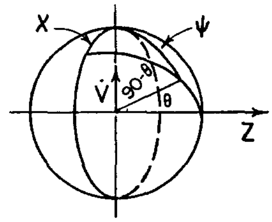

FIG. 4. Diagram showing relation between the spherical angles.

[3] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 264.

2. Electron Orbits in an Undulator

To calculate the radiation emitted by fast electrons crossing the undulator we first determine the motion of the electrons in a field given by the fundamental components of a fourier expansion of the undulator field. Let this be an electric field,

$$E_x=\mathcal{E}\sin(2\pi/l_0)z=\mathcal{E}\sin(2\pi\beta c/l_0)t=\mathcal{E}\sin\omega_0 t,\qquad \omega_0=2\pi\beta c/l_0,\quad (43)$$

in the transverse direction x. The exact equations for the relativistic motion,

$$d(mv_x)/dt=e\mathcal{E}\sin(2\pi/l_0)z,\quad v_x=dz/dt;$$
$$d(mv_z)/dt=0,\quad m=m_0/[1-(v_x^2+v_z^2)/c^2]^{\frac{1}{2}},\quad v_z=dx/dt,$$

are difficult to solve. We shall, in fact, assume that the transverse velocity is small compared with the longitudinal one, and also that the change in longitudinal velocity is negligible. We then find that the transverse velocity v_x is given by

$$v_x=-e\mathcal{E}(1-\beta^2)^{\frac{1}{2}}\cos(\omega_0 t)/m_0\omega_0,\quad (44)$$

and the transverse deflection x by

$$x=e\mathcal{E}(1-\beta^2)^{\frac{1}{2}}\sin(\omega_0 t)/m_0\omega_0^2=-A\sin\omega_0 t.\quad (45)$$

A better approximation is afforded by the assumption that the absolute value of the velocity is constant along the path. With this constant velocity $v=\beta c$ we have again, putting at first $z=vt$,

$$v_x=\overset{-\mathcal{E}}{e}(1-\beta^2)^{\frac{1}{2}}\cos(\omega_0 t)/m_0\omega_0$$

and for the longitudinal velocity we obtain

$$v_z=v(1-v_x^2/v^2)^{\frac{1}{2}}=\beta c(1-\tfrac{1}{2}v_x^2/v_2)$$
$$=v[1-(1-\beta^2)^{\frac{1}{2}}e^2\mathcal{E}^2\cos^2(\omega_0 t)/2\beta^2c^2\omega_0^2m_0^2].\quad (46)$$

Integrating this with respect to t we find

$$z=(v-a^2)t-a^2\sin(2\omega_0 t)/2\omega_0,$$
$$a^2=\tfrac{1}{4}(1-\beta^2)e^2\mathcal{E}^2/\beta^2c^2m_0^2\omega_0^2\ll\beta^2.\quad (47)$$

We now assume t in the form

$$t=(z/v)-\alpha f(z),\quad (48)$$

where α is a small quantity and $f(z)$ is an unknown function which we now set out to determine. Substituting Eq. (48) into Eq. (47) we find

$$(za/v)f(v-a^2)\alpha f(z)$$
$$+(a^2/2\omega_0)\sin 2\omega_0[(z/v)-\alpha f(z)]=0.\quad (49)$$

From this we find by expanding the sine function in terms of the small quantity α,

$$-\alpha f(z)=\frac{1}{v-a^2}\left\{\frac{za^2}{v}+\frac{a^2}{2\omega_0}\sin\frac{2\omega_0 z}{v}\right\}\left\{1+\frac{a^2}{v}\cos 2\omega_0\frac{z}{v}\right\},\quad (50)$$

the leading term of this expression is of order $a^2/2\omega_0 v$. We observe that we should now recalculate v_x using Eq. (57). It is easily seen that this would not change the terms of order $a^2/2\omega_0 v$ in Eq. (50).

We shall proceed to calculate the radiation neglecting terms of this order. It will then be possible to see what the corrections amount to. However, we know already that longitudinal oscillations of the electrons give rise to much weaker radiation than transverse ones.

3. Calculation of the Radiation

The components of current density J_z and J_x due to a single electron may be given in the form of products of δ-functions,

$$J_x=ev_x\delta(y)\delta(x+A\sin\omega_0 z/\beta c)\delta(z-c\beta t),\quad (51)$$

$$J_z=e\beta c\delta(y)\delta(x+A\sin\omega_0 z/\beta c)\delta(z-\beta ct).\quad (52)$$

The fourier components of current density of angular frequency ω, i.e., the amplitudes $J_{x\omega}$, $J_{z\omega}$, of the fourier integral expansions

$$J_z(\mathbf{r},t)=\int_0^\infty(J_{z\omega}(\mathbf{r})e^{i\omega t}+\text{c.c.})d\omega,\quad (53)$$

$$J_x(\mathbf{r},t)=\int_0^\infty(J_{x\omega}(\mathbf{r})e^{i\omega t}+\text{c.c.})d\omega,\quad (54)$$

of Eqs. (51), (52) are given by

$$J_{x\omega}=(ev_x/2\pi\beta c)\delta(y)\delta(x+A\sin\omega_0 z/\beta c)e^{i\omega z/\beta c},\quad (55)$$

$$J_{z\omega}=e[\delta(y)/2\pi]\delta(x+A\sin\omega_0 z/\beta c)e^{i\omega z/\beta c}.\quad (56)$$

This can be seen by substituting Eqs. (55), (56) back into Eqs. (53), (54) and using well-known rules concerning the operation with δ-functions.

Let the direction of observation $\mathbf{k}$ have polar angles ϕ, θ, with respect to the z-axis. $J_{\perp\mathbf{k}\omega}$ is given by

$$J_{\perp\mathbf{k}\omega}=J_{x\omega}(1-\cos^2\phi\sin^2\theta)^{\frac{1}{2}}\mathbf{u}_1+J_{z\omega}\sin\theta\mathbf{u}_2,\quad (57)$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors in the directions of the components $\perp\mathbf{k}$ of $J_{x\omega}$ and $J_{z\omega}$.

We substitute Eq. (57) into Eq. (42) and find

$$P_{\mathbf{k}\omega}=\frac{k^2e^2}{8\pi^3r^2c}\left|\int\int\delta\left(x'+A\sin\frac{\omega_0 z'}{\beta c}\right)\exp[-ik(x'\sin\theta\cos\phi+z'\cos\theta)+i\omega z'/\beta c]\times[\mathbf{u}_1\sin\theta+\mathbf{u}_2(v_x/\beta c)(1-\cos^2\phi\sin^2\theta)^{\frac{1}{2}}]\times dx'dz'\right|^2,\quad (58)$$

$u_1 + u_2$ have been interchanged

where the integration over dy', which is trivial, has been carried out. We have further

$$P_{\mathbf{k}\omega}=\frac{k^2e^2}{8\pi^3r^2c}\left|\int\left(\exp\left\{-ik\left[-A\sin\frac{\omega_0 z'}{\beta c}\sin\theta\cos\phi-z'\left(\frac{1}{\beta}-\cos\theta\right)\right]\right\}\right)[\mathbf{u}_1\sin\theta+\mathbf{u}_2\times(v_x/\beta c)(1-\cos^2\phi\sin^2\theta)^{\frac{1}{2}}]dz'\right|^2.\quad (59)$$

The integral is extended from $-l/2$ to $l/2$ (l is the total length of the undulator).

Introducing the following variables:

$$\xi=2\pi z'/l_0=\omega_0 z'/\beta c \quad (\omega_0=2\pi\beta c/l_0), \tag{60a}$$

$$\Omega_\theta=\frac{\omega_0}{1-\beta\cos\theta} \quad \left(\Omega_0=\frac{\omega_0}{1-\beta}\sim\frac{\omega_0 2\beta}{1-\beta^2}\sim\frac{2\omega_0}{1-\beta^2}\right), \tag{60b}$$

$$Z=\frac{\omega}{\Omega_\theta}\,\frac{e\mathcal{E}(1-\beta^2)^{\frac{1}{2}}l_0\sin\theta\cos\phi}{2\pi\beta m_0c^2(1-\beta\cos\theta)}, \tag{60c}$$

the expression (59) takes the form

$$P_{\mathbf{k}\omega}=C\left|\int_{-\pi/2}^{\pi/2}\left(\exp\left\{-iZ\sin\xi+i\frac{\omega}{\Omega_\theta}\xi\right\}\right)\right.$$
$$\left.\times\left[\mathbf{u}_1\sin\theta+\mathbf{u}_2\sin X\frac{l_0 e\mathcal{E}(1-\beta^2)^{\frac{1}{2}}}{\beta^2 m_0c^2 2\pi}\cos\xi\right]d\xi\right|^2, \tag{61}$$

where we have written $\sin X$ for $(1-\cos^2\phi\sin^2\theta)^{\frac{1}{2}}$ and C for $(e^2\beta^2/8\pi^3r^2c)(\omega/\omega_0)^2$. Under the integrand we expand

$$e^{-iZ\sin\xi}=\sum_{n=-p}^{\infty}J_n(Z)e^{-in\xi}, \tag{62}$$

and obtain the result

$$C\left|\mathbf{u}_1\sin\theta\sum_{n=-\infty}^{\infty}J_n(Z)\int\exp\left[i\left(\frac{\omega}{\Omega_\theta}-n\right)\xi\right]d\xi\right.$$
$$-\mathbf{u}_2\sin X\frac{l_0e\mathcal{E}(1-\beta^2)^{\frac{1}{2}}}{\beta^2m_0c^22\pi}\sum J_n(Z)$$
$$\left.\times\int\exp\left[i\left(\frac{\omega}{\Omega_\theta}-n\right)\xi\right]\cos\xi d\xi\right|^2. \tag{63}$$

After an elementary trigonometric manipulation this becomes

$$C\left|\sum_{n=-\infty}^{\infty}J_n(Z)\left[\mathbf{u}_1\sin\theta\int\exp\left[i\left(\frac{\omega}{\Omega_\theta}-n\right)\xi\right]d\xi\right.\right.$$
$$-\frac{\mathbf{u}_2}{2}\sin X\frac{e\mathcal{E}l_0(1-\beta^2)^{\frac{1}{2}}}{\beta^2m_0c^22\pi}\left(\int\exp\left\{i\left[\frac{\omega}{\Omega_\theta}-(n-1)\right]\xi\right\}d\xi\right.$$
$$\left.\left.\left.+\int\exp\left\{i\left[\frac{\omega}{\Omega_\theta}-(n+1)\right]\xi\right\}d\xi\right)\right]\right|^2. \tag{64}$$

To discuss Eq. (64) we remark that the integrals

$$S_q=\int_{-l\pi/l_0}^{l\pi/l_0}\exp\left[i\left(\frac{\omega}{\Omega_\theta}-q\right)\xi\right]d\xi$$
$$=\frac{2\sin[(\omega/\Omega_\theta)-q]l\pi/l_0}{(\omega/\Omega_\theta)-q} \tag{65}$$

$q=n,\quad n-1,\quad n+1$

are well known in diffraction theory. They show a large maximum in a narrow frequency range

$$|\delta\omega|<(l_0/2l)\Omega_\theta$$

and, of course, subsidiary maxima. It is seen that ω is restricted to small frequency bands centered at the "harmonics" of the "fundamental frequency" Ω_θ.

The energy radiated per unit area into the pth harmonic of Ω_θ is given by

$$\frac{1}{(1-\beta\cos\theta)^2}\left(\frac{\omega}{\Omega_\theta}\right)^2\frac{e^2\beta^2S_p{}^2}{4\pi^2r^2c}\left\{\sin^2\theta[J_p(Z)]^2\right.$$
$$\left.+\tfrac{1}{4}\sin^2X[J_{p+1}(Z)+J_{p-1}(Z)]^2\cdot\frac{e^2\mathcal{E}^2l_0{}^2(1-\beta^2)}{4\pi^2\beta^2(m_0c^2)^2}\right\}. \tag{66}$$

The result obtained by putting $p=0$ deserves special mention. It is very small in our case. However, it is this term which would give the contribution of the Cerenkov effect to the total energy radiated if the undulator were filled with a medium of refractive index $\epsilon^{\frac{1}{2}}$. In this case, the velocity of light, c, where it represents the propagation of velocity of the radiation, has to be replaced by $c/\epsilon^{\frac{1}{2}}$ and the effect of this is that Z in Eq. (66) has to be replaced by $\epsilon^{\frac{1}{2}}Z$ and that the whole expression (66) is multiplied by $\epsilon^{\frac{1}{2}}$, while Ω_θ is now given by

$$\Omega_\theta=\omega_0/(1-\beta\epsilon^{\frac{1}{2}}\cos\theta). \tag{67}$$

The integrals S_0 give a contribution when $\omega/\Omega_\theta=0$, i.e., when

$$\beta(\epsilon)^{\frac{1}{2}}\cos\theta=1. \tag{68}$$

The angle θ as defined by Eq. (68) is real only when

$$\beta>1/\epsilon^{\frac{1}{2}}. \tag{69}$$

Thus, no contribution is obtained in free space. It is not intended to study the Cerenkov effect in this paper.

For the discussion of Eq. (66) first note that in case of the electric undulator we have always

$$e\mathcal{E}l_0\ll\pi mc^2. \tag{70}$$

We can replace $J_p(Z)$ by $Z^p/p!2^p$, and we note that the energy radiated into higher harmonics is small compared with that radiated into the fundamental frequency Ω_θ.

For the energy radiated into the fundamental ($p=1$) frequency band under angles ranging from 0 to θ we obtain

$$\frac{e^2\beta^2}{4\pi^2c}\int_0^\theta d\theta\int d\phi\left(\frac{\omega}{\Omega_\theta}\right)^2\left[\int S_p{}^2d\omega\right]\left\{\frac{\sin^4\theta\cos^2\phi}{(1-\beta\cos\theta)^4}\right.$$
$$\left.+\frac{1-\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^2}\right\}\frac{e^2\mathcal{E}^2(1-\beta^2)l_0{}^2\sin\theta}{4\cdot4\pi^2\beta^2(m_0c^2)^2}. \tag{71}$$

The integration over $d\omega$ may be carried out first. It involves

$$\int S_p^2 d\omega = \int_{\omega/\Omega_\theta=1}^{\infty} \frac{4\sin^2[(\omega/\Omega_\theta)-1]l\pi/l_0}{[(\omega/\Omega_\theta)-1]^2} d\omega = 4\frac{l\pi^2}{l_0}\Omega_\theta. \quad (72)$$

We are mainly interested in the result of the integration

$$(e^2/mc^2)\beta l\mathcal{E}^2(1-\beta^2)/8\pi.$$

$$\left\{\int_0^{2\pi} d\phi \int_0^{\theta} d\theta \left[\frac{\sin^5\theta\cos^2\phi}{(1-\beta\cos\theta)^4} + \frac{\sin\theta - \sin^3\cos\phi}{(1-\beta\cos\theta)^2}\right]\left(\frac{\omega}{\Omega_\theta}\right)^2\right\} \quad (73)$$

for small angles θ.

The frequency spread depends on the angle θ under which the radiation is received. Let us say we aim at a frequency spread of 10 percent. Then

$$d\Omega_\theta/\Omega_\theta = \beta\sin\theta d\theta/(1-\beta\cos\theta) \sim \beta\theta^2/(1-\beta) = 1/10,$$
$$\theta^2 \sim (1/10)(1-\beta^2)/2\beta. \quad (74)$$

We can put $\cos\theta = 1$ in the integrand, and we find that the first integral can be neglected compared with the second. Integrating over ϕ we find that the energy radiated into a cone of opening θ is approximately 1/20 of the total energy as given by Eq. (27).

4. Correction for the Constant Velocity Orbit Approximation

It is now fairly easily seen how the calculation could be modified to include terms of order a^2l_0/v^2 arising from the constant velocity orbit approximation. The integrand of formula Eq. (73) is multiplied with

$$\exp\{i\omega[(a^2z'/2v^2)+(l_0a^2/4\pi v^2)\sin 2\omega_0 z'/v]\}.$$

For $\exp[i\omega(l_0a^2/4\pi v)\sin 2\omega_0 z'/v]$ we write

$$\sum_{r=\infty}^{\infty} J_r\left(\frac{l_0a^2}{4\pi v^2}\right)e^{2ir\xi}. \quad (75)$$

However, as the argument of the bessel functions is small, the series reduce to the terms $r=0$, which is unity, in our case. The term $\exp(i\omega a^2z'/2v^2)$ gives rise to a small change of the frequency Ω_θ which is now given by

$$\Omega_\theta' = \omega_0/[1+(a^2/2v^2)-\beta\cos\theta]. \quad (76)$$

IX. CONCLUSIONS

1. Production of Energy in the Infrared and Millimeter Band

The efficiency of energy conversion depends entirely on whether the energy is radiated coherently or not, and this, in turn, depends on whether the electrons can be so bunched that, seen from the L-system, they look no longer than half the wavelength of the radiation received in the L-system.

The undulator considered above delivers the equivalent of 3 quanta of maximum fundamental frequency per electron. In the millimeter band this corresponds to a total radiation of $2\cdot 10^{-15}$ ergs/electron. Let the beam contain 1 coulomb/sec, then we have $\frac{3}{5}\cdot 10^{19}$ electrons per second. The power radiated coherently by electrons which are suitably bunched depends on the number of electrons in the bunch and the nch repetition rate. Let us consider millimeter waves.

If we assume that is is possible to bunch all the electrons contained in one millimeter length of the beam into half that length, and to repeat a half-millimeter bunch every millimeter along the beam. The number of electrons/bunch is then

$$N = \tfrac{1}{5}\cdot 10^8$$

and the power is $N^2\cdot 2\cdot 10^{-15}$ number of bunche/sec.

Power radiated coherently thus becomes

$$(\tfrac{1}{5}\cdot 10^8)^2 2\cdot 10^{-15} 3\cdot 10^{11} = 2.4\cdot 10^{11}\ \text{erg/sec} = 2.4\cdot 10^4\ \text{watts},$$

while the power radiated incoherently would have been $1.2\cdot 10^{-3}$ watts. However, if it were possible to bunch all the electrons contained in one-cm length of beam into $\frac{1}{2}$-millimeter bunches, we should get $2.4\cdot 10^5$ watts radiated coherently.

This is the total energy radiated. The energy contained in a band of 10 percent frequency spread collected under an angle θ given by Eq. (74) is 1/20 of this, i.e., $0.6\cdot 10^{-4}$ watt incoherent or $1.2\cdot 10^4$ watts coherent radiation, on the second assumption and $1.2\cdot 10^3$ watts on the first.

In the first case the power radiated coherently is $2\cdot 10^7$ times the power radiated noncoherently from a uniform beam, while in the second case it is $2\cdot 10^8$ times that quantity.

However, the total beam energy is only $1\cdot 15\cdot 10^6$ watts, so the energy radiated in the second case is 21 percent of the beam energy. We have obviously reached the limit of validity of our theoretical assumptions. The radiation reaction per electron is no longer negligible with respect to the external magnetic forces acting on the beam and should be considered in the dynamics of the electron.

We note that we may start with an undulator period $1_0 = 16$ mm, use a beam of 1.5 Mev and obtain $1_0' = 4$ mm in the E-system. Then, for coherent radiation the bunches in the E-system must not be longer than 2 mm, while they should not exceed $\frac{1}{2}$ mm in the L-system. This seems just feasible.

The bunching system should be such that the speed variation within the bunch is small.

The maximum frequency emitted is given by

$$\nu = (l_0/v)1/(1-\beta).$$

Hence, the frequency spread due to velocity spread is given by

$$d\nu/\nu = -d\beta/(1-\beta).$$

It is seen that speed variations in a bunch lead to an amplified frequency spread of the emitted radiation.

2. Speed Monitoring for the Whole Beam

Speed monitoring is practicable if the emitted radiation can be detected and its frequency determined with sufficient accuracy. We are now thinking about noncoherent radiation, but it is easily seen that the energy involved is sufficient for the purpose. Even at millimeter wavelength the power is of the order of milliwatts, and much more if for higher beam energies. There are 3 quanta emitted per electron crossing an undulator of length of one meter with $l_0 = 1$ cm. A beam of one microamp, mean current contains $\frac{3}{5}\cdot 10^{13}$ electrons/sec, or $\frac{3}{5}\cdot 10^{7}$ per microsecond, and there will be about $2\cdot 10^{7}$ quanta emitted per microsecond, or 10^{6} in the 10 percent bandwidth considered above.

It seems convenient to use such an apparatus which can determine $1-\beta^2$ directly and a train of waves 100 long affords ample precision for wavelength measurement.

3. Speed Determination for Individual Particles

Here the figure of 3 quanta per electron seems rather small. It seems necessary to go to a greater length of undulator. An undulator 10 meters long would produce 30 quanta which could be detected by a scintillation counter.

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Electrical Breakdown over Insulators in High Vacuum

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In the investigated range of 5×10^{-3} to 10^{-7} mm Hg, the breakdown voltage over insulators in high vacuum is independent of pressure. Currents of 10^{-11} to 10^{-8} ampere were observed in the region below breakdown voltage by detecting x-ray quanta with a Geiger-Mueller counter. Pinhole camera x-ray pictures revealed that practically all radiations originate from an area on the anode a distance from the insulator, with a weaker radiation coming from a ring immediately adjacent to the insulator. Current-voltage relationships as usually observed in these experiments indicate a roughness factor and an emitting area on the cathode similar to previous findings in vacuum gaps. Current bursts were observed which did not develop into complete breakdown. Oscilloscopic observations revealed that sometimes at breakdown over insulators the voltage on the test sample drops to 2.5 kv; in other cases it falls to less than 100 volts. The low voltage arc-like discharge extinguishes at a current of about one ampere for copper electrodes in contact with Pyrex glass. When a resistance in series with the test sample is increased to keep the maximum current below one ampere, no stable discharge is observed. As in a vacuum gap, the breakdown voltage over an insulator is increased by successive breakdowns. Part of this "conditioning" is permanent. The nonpermanent part is dependent on the state of the test sample prior to conditioning. The anode does not appear to influence conditioning. When the resistance in series with the test sample allows a discharge current above one ampere to flow, a fast conditioning usually occurs which results in a high permanent breakdown voltage.

INTRODUCTION

THE main limitation of high vacuum insulation is a spontaneous discharge, often called breakdown. The characteristics of high vacuum breakdown are high currents, preceded by very low pre-breakdown current; low voltage drops in the vacuum gap during breakdown; and in practically all cases the independence of breakdown voltage on pressure, provided that the pressure is below that at which a glow discharge may develop. Since the residual gas cannot supply a sufficient number of charged particles to carry the high breakdown current, gas or vapor must be released by the breakdown process.

If electrodes in a high vacuum are separated by an insulator, breakdown over the insulator, without its puncture, takes place at lower voltage than in a vacuum gap with the same separation of electrodes.

Preliminary work by Dr. M. J. Kofoid[1] showed that the breakdown voltage over a hollow glass cylinder which rests on a flat electrode is increased by more than 100 percent when one end of the insulator fits into a groove in the cathode. No similar effect is found for the opposite polarity. Breakdowns over a hollow glass cylinder fused to the cathode occur at higher voltages than when this glass piece rests on a well-polished electrode. Kofoid's tests with such metal-to-glass seals at the cathode indicate that the breakdown voltage changes little with the length of the insulator.

The writer made further investigations on this

[1] Unpublished work done at these Laboratories.