Mathematical Modeling of Hydraulic Systems

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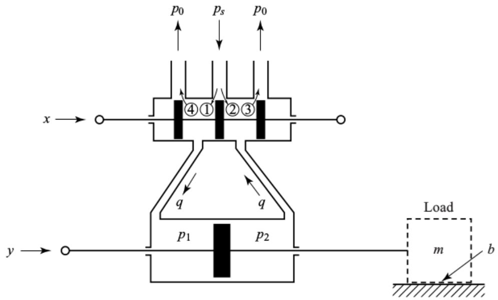
Summary

1. Hydraulic fluid acts as a lubricant, in addition to carrying away heat generated in the system to a convenient heat exchanger.
2. Comparatively small-sized hydraulic actuators can develop large forces or torques.
3. Hydraulic actuators have a higher speed of response with fast starts, stops, and speed reversals.
4. Hydraulic actuators can be operated under continuous, intermittent, reversing, and stalled conditions without damage.
5. Availability of both linear and rotary actuators gives flexibility in design.
6. Because of low leakages in hydraulic actuators, speed drop when loads are applied is small.

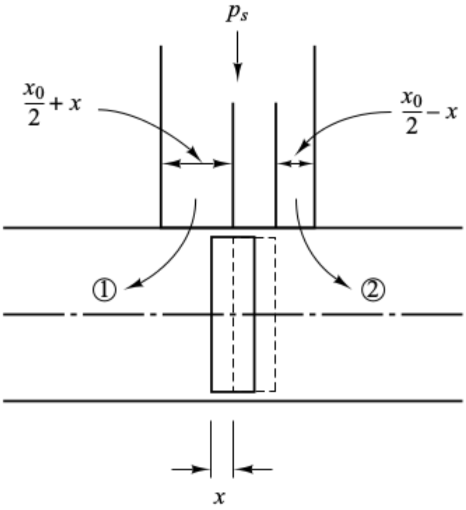
1. Hydraulic power is not readily available compared to electric power.
2. Cost of a hydraulic system may be higher than that of a comparable electrical system performing a similar function.
3. Fire and explosion hazards exist unless fire-resistant fluids are used.
4. Because it is difficult to maintain a hydraulic system that is free from leaks, the system tends to be messy.
5. Contaminated oil may cause failure in the proper functioning of a hydraulic system.
6. As a result of the nonlinear and other complex characteristics involved, the design of sophisticated hydraulic systems is quite involved.
7. Hydraulic circuits have generally poor damping characteristics. If a hydraulic circuit is not designed properly, some unstable phenomena may occur or disappear, depending on the operating condition.

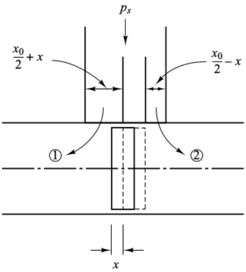
- Viscosity of hydraulic fluid can greatly affect damping and friction effects of the hydraulic circuits.
- Stability tests must be carried out at the highest possible operating temperature.
- Most hydraulic systems are nonlinear.
- A useful linearization technique for dealing with nonlinear systems must be used.



- A pilot-valve-controlled hydraulic power amplifier and actuator.
- The pilot valve is a balanced; pressure forces acting on it are all balanced.
- In practice, ports are often made wider than corresponding valves.
- Leakages improve both sensitivity and linearity.
- The valve is underlapped and symmetrical. It admits hydraulic fluid under high pressure into a power cylinder that contains a large piston.
- A large hydraulic force is established to move a load.

Diagram of the valve orifice area





Assuming the displacement x to be small, we obtain

$$A_1 = A_3 = k\left(\frac{x_0}{2} + x\right)$$

$$A_2 = A_4 = k\left(\frac{x_0}{2} - x\right)$$

where k is a constant.

Furthermore, we shall assume that the return pressure p_o in the return line is small and thus can be neglected. Then, referring to Figure 4–17(a), flow rates through valve orifices are

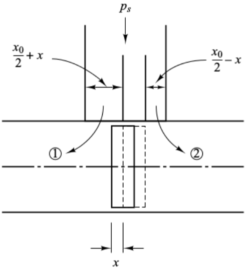
$$q_1 = c_1 A_1 \sqrt{\frac{2g}{\gamma}(p_s - p_1)} = C_1\sqrt{p_s - p_1}\left(\frac{x_0}{2} + x\right)$$

$$q_2 = c_2 A_2 \sqrt{\frac{2g}{\gamma}(p_s - p_2)} = C_2\sqrt{p_s - p_2}\left(\frac{x_0}{2} - x\right)$$

$$q_3 = c_1 A_3 \sqrt{\frac{2g}{\gamma}(p_2 - p_0)} = C_1\sqrt{p_2 - p_0}\left(\frac{x_0}{2} + x\right) = C_1\sqrt{p_2}\left(\frac{x_0}{2} + x\right)$$

$$q_4 = c_2 A_4 \sqrt{\frac{2g}{\gamma}(p_1 - p_0)} = C_2\sqrt{p_1 - p_0}\left(\frac{x_0}{2} - x\right) = C_2\sqrt{p_1}\left(\frac{x_0}{2} - x\right)$$

where $C_1 = c_1 k\sqrt{2g/\gamma}$ and $C_2 = c_2 k\sqrt{2g/\gamma}$, and γ is the specific weight and is given by $\gamma = \rho g$, where ρ is mass density and g is the acceleration of gravity. The flow rate q to the left-hand side of the power piston is



The flow rate q to the left-hand side of the power piston is

$$q = q_1 - q_4 = C_1\sqrt{p_s - p_1}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{p_1}\left(\frac{x_0}{2} - x\right) \qquad (4\text{–}25)$$

The flow rate from the right-hand side of the power piston to the drain is the same as this q and is given by

$$q = q_3 - q_2 = C_1\sqrt{p_2}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{p_s - p_2}\left(\frac{x_0}{2} - x\right)$$

In the present analysis we assume that the fluid is incompressible. Since the valve is symmetrical, we have $q_1 = q_3$ and $q_2 = q_4$. By equating q_1 and q_3, we obtain

$$p_s - p_1 = p_2$$

or

$$p_s = p_1 + p_2$$

If we define the pressure difference across the power piston as Δp or

$$\Delta p = p_1 - p_2$$

then

$$p_1 = \frac{p_s + \Delta p}{2}, \qquad p_2 = \frac{p_s - \Delta p}{2}$$

For the symmetrical valve shown in Figure 4–17(a), the pressure in each side of the power piston is $(1/2)p_s$ when no load is applied, or $\Delta p = 0$. As the spool valve is displaced, the pressure in one line increases as the pressure in the other line decreases by the same amount.

In terms of p_s and Δp, we can rewrite the flow rate q given by Equation (4–25) as

$$q = q_1 - q_4 = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + \Delta p}{2}}\left(\frac{x_0}{2} - x\right)$$

Noting that the supply pressure p_s is constant. the flow rate q can be written as a function of the valve displacement x and pressure difference Δp, or

$$q = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + \Delta p}{2}}\left(\frac{x_0}{2} - x\right) = f(x, \Delta p)$$

By applying the linearization technique presented in Section 3–10 to this case, the linearized equation about point $x = \bar{x}$, $\Delta p = \Delta\bar{p}$, $q = \bar{q}$ is

$$q - \bar{q} = a(x - \bar{x}) + b(\Delta p - \Delta\bar{p}) \qquad (4\text{–}26)$$

where

$$\bar{q} = f(\bar{x}, \Delta\bar{p})$$

$$a = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, \Delta p = \Delta\bar{p}} = C_1\sqrt{\frac{p_s - \Delta\bar{p}}{2}} + C_2\sqrt{\frac{p_s + \Delta\bar{p}}{2}}$$

$$b = \left.\frac{\partial f}{\partial \Delta p}\right|_{x=\bar{x},\, \Delta p = \Delta\bar{p}} = -\left[\frac{C_1}{2\sqrt{2}\sqrt{p_s - \Delta\bar{p}}}\left(\frac{x_0}{2} + \bar{x}\right) + \frac{C_2}{2\sqrt{2}\sqrt{p_s + \Delta\bar{p}}}\left(\frac{x_0}{2} - \bar{x}\right)\right] < 0$$

Coefficients a and b here are called *valve coefficients*. Equation (4–26) is a linearized mathematical model of the spool valve near an operating point $x = \bar{x}$, $\Delta p = \Delta \bar{p}$, $q = \bar{q}$. The values of valve coefficients a and b vary with the operating point. Note that $\partial f / \partial \Delta p$ is negative and so b is negative.

Since the normal operating point is the point where $\bar{x} = 0$, $\Delta \bar{p} = 0$, $\bar{q} = 0$, near the normal operating point Equation (4–26) becomes

$$q = K_1 x - K_2 \Delta p \qquad (4\text{–}27)$$

where

$$K_1 = (C_1 + C_2)\sqrt{\frac{p_s}{2}} > 0$$

$$K_2 = (C_1 + C_2)\frac{x_0}{4\sqrt{2}\sqrt{p_s}} > 0$$

Equation (4–27) is a linearized mathematical model of the spool valve near the origin ($\bar{x} = 0, \Delta\bar{p} = 0, \bar{q} = 0$.) Note that the region near the origin is most important in this kind of system, because the system operation usually occurs near this point.

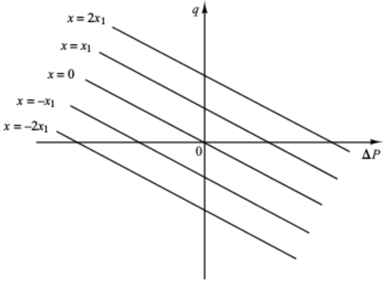


Figure 4–18
Characteristic curves of the linearized hydraulic servomotor.

Referring to Figure 4–17(a), we see that the rate of flow of oil q times dt is equal to the power-piston displacement dy times the piston area A times the density of oil ρ. Thus, we obtain

$$A\rho\, dy = q\, dt$$

Notice that for a given flow rate q the larger the piston area A is, the lower will be the velocity dy/dt. Hence, if the piston area A is made smaller, the other variables remaining constant, the velocity dy/dt will become higher. Also, an increased flow rate q will cause an increased velocity of the power piston and will make the response time shorter.

Equation (4–27) can now be written as

$$\Delta P = \frac{1}{K_2}\left(K_1 x - A\rho \frac{dy}{dt}\right)$$

The force developed by the power piston is equal to the pressure difference ΔP times the piston area A or

$$\begin{aligned} \text{Force developed by the power piston} &= A\, \Delta P \\ &= \frac{A}{K_2}\left(K_1 x - A\rho \frac{dy}{dt}\right) \end{aligned}$$

For a given maximum force, if the pressure difference is sufficiently high, the piston area, or the volume of oil in the cylinder, can be made small. Consequently, to minimize the weight of the controller, we must make the supply pressure sufficiently high.

Assume that the power piston moves a load consisting of a mass and viscous friction. Then the force developed by the power piston is applied to the load mass and friction, and we obtain

$$m\ddot{y} + b\dot{y} = \frac{A}{K_2}\left(K_1 x - A\rho\dot{y}\right)$$

or

$$m\ddot{y} + \left(b + \frac{A^2\rho}{K_2}\right)\dot{y} = \frac{AK_1}{K_2}x \tag{4–28}$$

where m is the mass of the load and b is the viscous-friction coefficient.

Assuming that the pilot-valve displacement x is the input and the power-piston displacement y is the output, we find that the transfer function for the hydraulic servomotor is, from Equation (4–28),

$$\frac{Y(s)}{X(s)} = \frac{1}{s\left[\left(\frac{mK_2}{AK_1}\right)s + \frac{bK_2}{AK_1} + \frac{A\rho}{K_1}\right]}$$

$$= \frac{K}{s(Ts + 1)} \tag{4–29}$$

where

$$K = \frac{1}{\frac{bK_2}{AK_1} + \frac{A\rho}{K_1}} \quad \text{and} \quad T = \frac{mK_2}{bK_2 + A^2\rho}$$

From Equation (4–29) we see that this transfer function is of the second order. If the ratio $mK_2/(bK_2 + A^2\rho)$ is negligibly small or the time constant T is negligible, the transfer function $Y(s)/X(s)$ can be simplified to give

$$\frac{Y(s)}{X(s)} = \frac{K}{s}$$

It is noted that a more detailed analysis shows that if oil leakage, compressibility (including the effects of dissolved air), expansion of pipelines, and the like are taken into consideration, the transfer function becomes

$$\frac{Y(s)}{X(s)} = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$

where T_1 and T_2 are time constants. As a matter of fact, these time constants depend on the volume of oil in the operating circuit. The smaller the volume, the smaller the time constants.

Operation of this hydraulic servomotor is as follows. If input x moves the pilot valve to the right, port II is uncovered, and so high-pressure oil enters the right-hand side of the power piston. Since port I is connected to the drain port, the oil in the left-hand side of the power piston is returned to the drain. The oil flowing into the power cylinder is at high pressure; the oil flowing out from the power cylinder into the drain is at low pressure. The resulting difference in pressure on both sides of the power piston will cause it to move to the left.

Note that the rate of flow of oil q (kg/sec) times dt (sec) is equal to the power-piston displacement dy (m) times the piston area A (m^2) times the density of oil ρ (kg/m^3). Therefore,

$$A\rho\, dy = q\, dt \tag{4–30}$$

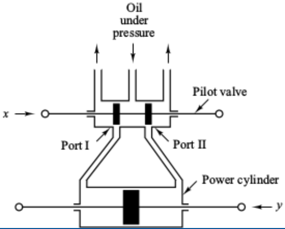
Because of the assumption that the oil flow rate q is proportional to the pilot-valve displacement x, we have

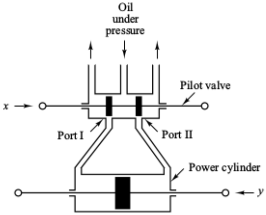
$$q = K_1 x \tag{4–31}$$

where K_1 is a positive constant. From Equations (4–30) and (4–31) we obtain

$$A\rho \frac{dy}{dt} = K_1 x$$

Figure 4–19 Hydraulic servomotor.





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$$q = K_1 x \qquad (4\text{–}31)$$

where K_1 is a positive constant. From Equations (4–30) and (4–31) we obtain

$$A\rho \frac{dy}{dt} = K_1 x$$

The Laplace transform of this last equation, assuming a zero initial condition, gives

$$A\rho s Y(s) = K_1 X(s)$$

or

$$\frac{Y(s)}{X(s)} = \frac{K_1}{A\rho s} = \frac{K}{s}$$

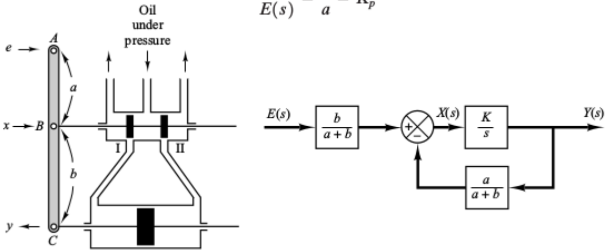
where $K = K_1/(A\rho)$. Thus the hydraulic servomotor shown in Figure 4–19 acts as an integral controller.

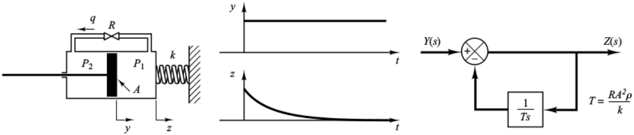
The controller here operates in the following way. If input e moves the pilot valve to the right, port II will be uncovered and high-pressure oil will flow through port II into the right-hand side of the power piston and force this piston to the left. The power piston, in moving to the left, will carry the feedback link ABC with it, thereby moving the pilot valve to the left. This action continues until the pilot piston again covers ports I and II. A block diagram of the system can be drawn as in Figure 4–20(b). The transfer function between $Y(s)$ and $E(s)$ is given by

$$\frac{Y(s)}{E(s)} = \frac{\dfrac{b}{a+b}\dfrac{K}{s}}{1 + \dfrac{K}{s}\dfrac{a}{a+b}}$$

Noting that under the normal operating conditions we have $|Ka/[s(a+b)]| \gg 1$, this last equation can be simplified to

$$\frac{Y(s)}{E(s)} = \frac{b}{a} = K_p$$





Let us derive the transfer function between the displacement z and displacement y. Define the pressures existing on the right and left sides of the piston as $P_1(\mathrm{lb_f/in.^2})$ and $P_2(\mathrm{lb_f/in.^2})$, respectively. Suppose that the inertia force involved is negligible. Then the force acting on the piston must balance the spring force. Thus

$$A(P_1 - P_2) = kz$$

where A = piston area, in.^{2}
k = spring constant, $\mathrm{lb_f/in.}$

The flow rate q is given by

$$q = \frac{P_1 - P_2}{R}$$

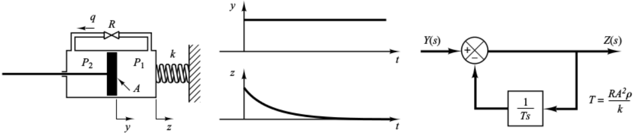
where q = flow rate through the restriction, lb/sec
R = resistance to flow at the restriction, $\mathrm{lb_f\text{-}sec/in.^2\text{-}lb}$

Since the flow through the restriction during dt seconds must equal the change in the mass of oil to the left of the piston during the same dt seconds, we obtain

$$q\,dt = A\rho(dy - dz)$$

where ρ = density, lb/in.^{3}. (We assume that the fluid is incompressible or ρ = constant.) This last equation can be rewritten as

$$\frac{dy}{dt} - \frac{dz}{dt} = \frac{q}{A\rho} = \frac{P_1 - P_2}{RA\rho} = \frac{kz}{RA^2\rho}$$



or

$$\frac{dy}{dt} = \frac{dz}{dt} + \frac{kz}{RA^2\rho}$$

Taking the Laplace transforms of both sides of this last equation, assuming zero initial conditions, we obtain

$$sY(s) = sZ(s) + \frac{k}{RA^2\rho} Z(s)$$

The transfer function of this system thus becomes

$$\frac{Z(s)}{Y(s)} = \frac{s}{s + \dfrac{k}{RA^2\rho}}$$

Let us define $RA^2\rho/k = T$. (Note that $RA^2\rho/k$ has the dimension of time.) Then

$$\frac{Z(s)}{Y(s)} = \frac{Ts}{Ts + 1} = \frac{1}{1 + \dfrac{1}{Ts}}$$

Hydraulic Proportional-Plus-Integral Control Action

The transfer function $Y(s)/E(s)$ is given by

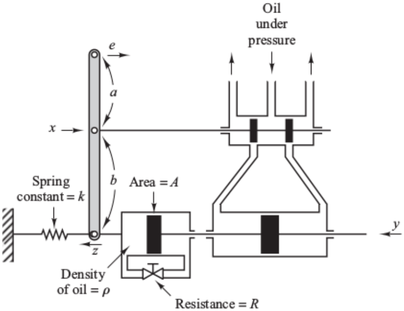
$$\frac{Y(s)}{E(s)} = \frac{\dfrac{b}{a+b}\dfrac{K}{s}}{1 + \dfrac{Ka}{a+b}\dfrac{T}{Ts+1}}$$

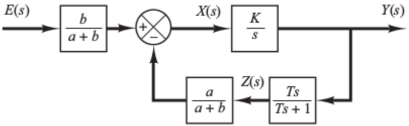
In such a controller, under normal operation $\left|KaT/[(a+b)(Ts+1)]\right| \gg 1$, with the result that

$$\frac{Y(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

where

$$K_p = \frac{b}{a}, \qquad T_i = T = \frac{RA^2\rho}{k}$$





$$k(y - z) = A(P_2 - P_1)$$

$$q = \frac{P_2 - P_1}{R}$$

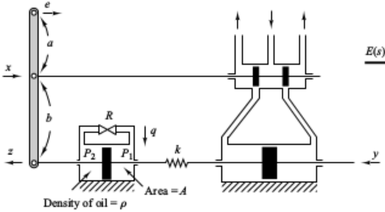
$$q\,dt = \rho A\,dz$$

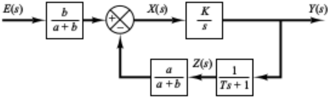
Hence

$$y = z + \frac{A}{k} qR = z + \frac{RA^2\rho}{k}\frac{dz}{dt}$$

or

$$\frac{Z(s)}{Y(s)} = \frac{1}{Ts + 1}$$





where

$$T = \frac{RA^2\rho}{k}$$

A block diagram for this system is shown in Figure 4–23(b). From the block diagram the transfer function $Y(s)/E(s)$ can be obtained as

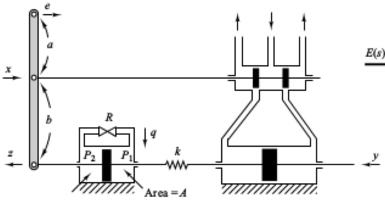
$$\frac{Y(s)}{E(s)} = \frac{\frac{b}{a+b}\frac{K}{s}}{1 + \frac{a}{a+b}\frac{K}{s}\frac{1}{Ts+1}}$$

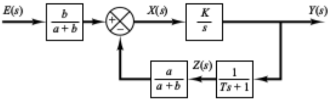
Under normal operation we have $|aK/[(a+b)s(Ts+1)]| \gg 1$. Hence

$$\frac{Y(s)}{E(s)} = K_p(1 + Ts)$$

where

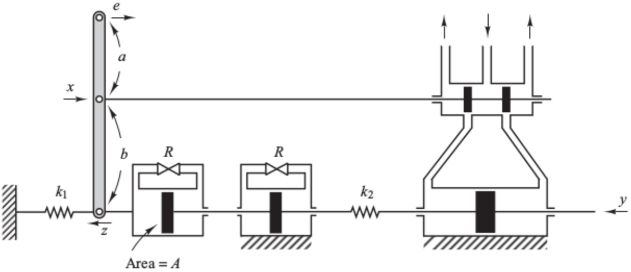
$$K_p = \frac{b}{a}, \qquad T = \frac{RA^2\rho}{k}$$





If the two dashpots are identical except the piston shafts, the transfer function $Z(s)/Y(s)$ can be obtained as follows:

$$\frac{Z(s)}{Y(s)} = \frac{T_1 s}{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1}$$



A block diagram for this system is shown in Figure 4–25. The transfer function $Y(s)/E(s)$ can be obtained as

$$\frac{Y(s)}{E(s)} = \frac{b}{a+b} \frac{\frac{K}{s}}{1 + \frac{a}{a+b}\frac{K}{s}\frac{T_1 s}{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1}}$$

Under normal circumstances we design the system such that

$$\left| \frac{a}{a+b}\frac{K}{s}\frac{T_1 s}{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1} \right| \gg 1$$

then

$$\frac{Y(s)}{E(s)} = \frac{b}{a}\frac{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1}{T_1 s}$$

$$= K_p + \frac{K_i}{s} + K_d s$$

where

$$K_p = \frac{b}{a}\frac{T_1 + 2T_2}{T_1}, \qquad K_i = \frac{b}{a}\frac{1}{T_1}, \qquad K_d = \frac{b}{a}T_2$$

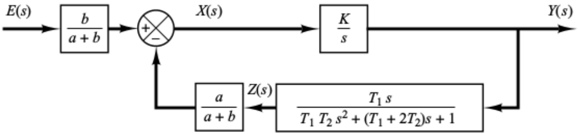
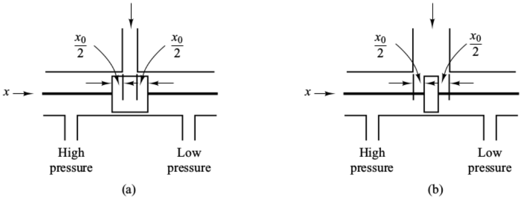
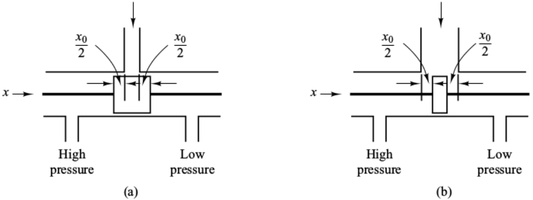


Figure 4–32
(a) Overlapped spool valve;
(b) underlapped spool valve.



A–4–6. Actual spool valves are either overlapped or underlapped because of manufacturing tolerances. Consider the overlapped and underlapped spool valves shown in Figures 4–32(a) and (b). Sketch curves relating the uncovered port area A versus displacement x.

Figure 4–32
(a) Overlapped spool valve;
(b) underlapped spool valve.

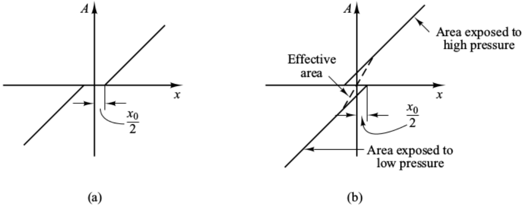


A–4–6. Actual spool valves are either overlapped or underlapped because of manufacturing tolerances. Consider the overlapped and underlapped spool valves shown in Figures 4–32(a) and (b). Sketch curves relating the uncovered port area A versus displacement x.

Solution. For the overlapped valve, a dead zone exists between $-\frac{1}{2}x_0$ and $\frac{1}{2}x_0$, or $-\frac{1}{2}x_0 < x < \frac{1}{2}x_0$. The curve for uncovered port area A versus displacement x is shown in Figure 4–33(a). Such an overlapped valve is unfit as a control valve.

For the underlapped valve, the curve for port area A versus displacement x is shown in Figure 4–33(b). The effective curve for the underlapped region has a higher slope, meaning a higher sensitivity. Valves used for controls are usually underlapped.

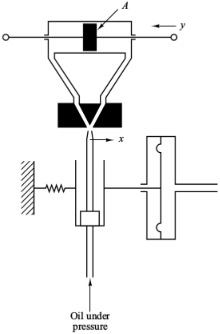
Figure 4–33
(a) Uncovered-port-area-A-versus displacement-x curve for the overlapped valve; (b) uncovered-port-area-A-versus-displacement-x curve for the underlapped valve.



A–4–7. Figure 4–34 shows a hydraulic jet-pipe controller. Hydraulic fluid is ejected from the jet pipe. If the jet pipe is shifted to the right from the neutral position, the power piston moves to the left, and vice versa. The jet-pipe valve is not used as much as the flapper valve because of large null flow, slower response, and rather unpredictable characteristics. Its main advantage lies in its insensitivity to dirty fluids.

Suppose that the power piston is connected to a light load so that the inertia force of the load element is negligible compared to the hydraulic force developed by the power piston. What type of control action does this controller produce?

Figure 4–34 Hydraulic jet-pipe controller.



A–4–8. Explain the operation of the speed control system shown in Figure 4–35.

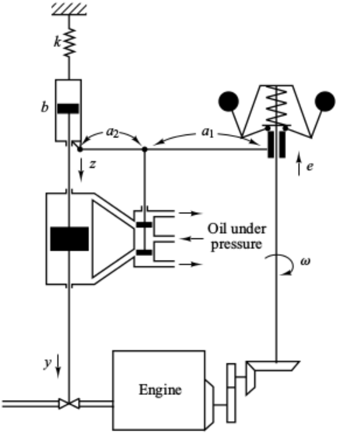
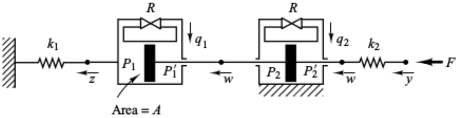


Figure 4–35 Speed control system.

A–4–9. Derive the transfer function $Z(s)/Y(s)$ of the hydraulic system shown in Figure 4–37. Assume that the two dashpots in the system are identical ones except the piston shafts.

Figure 4–37
Hydraulic system.



B–4–9. Figure 4–51 is a schematic diagram of an aircraft elevator control system. The input to the system is the deflection angle θ of the control lever, and the output is the elevator angle ϕ. Assume that angles θ and ϕ are relatively small. Show that for each angle θ of the control lever there is a corresponding (steady-state) elevator angle ϕ.

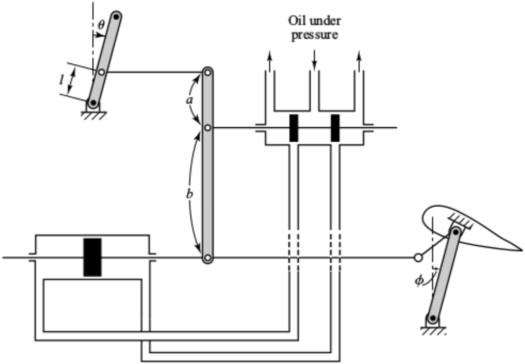


Figure 4–51 Aircraft elevator control system.

B–4–11. Consider the controller shown in Figure 4–53. The input is the air pressure p_i measured from some steady-state reference pressure $\bar{P}$ and the output is the displacement y of the power piston. Obtain the transfer function $Y(s)/P_i(s)$.

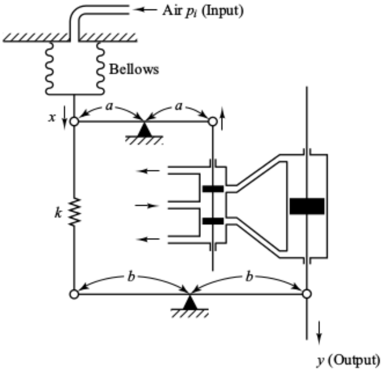


Figure 4–53
Controller.

Bibliography

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