The z transform

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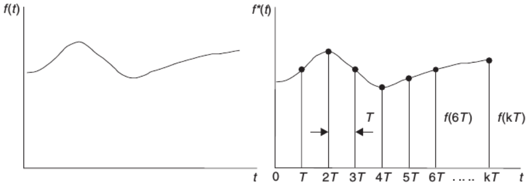
Summary

- The z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- It is analogous to the Laplace transform for continuous systems.

Bilateral z-transform



(a) Continuous Signal (b) Sampled Signal

Taking z-transform of an ideal sampled signal gives,

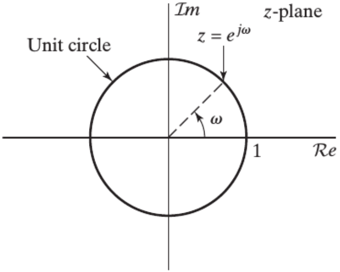
$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)\,, \tag{1}$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z)\,, \tag{2}$$

$$z = r\,e^{j\omega}\,, \tag{3}$$

$$X(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}\,, \tag{4}$$

Complex z-plane

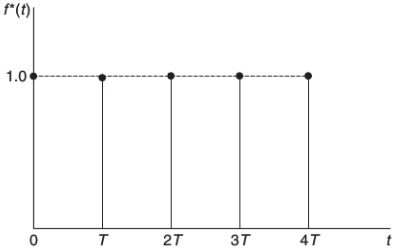


$$X(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}\,.$$

- Equation 4 can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}.
- For $r = 1$, Eq. 4 reduce to the Fourier transform of $x[n]$.
- Interpreting the Fourier transform as the z-transform on the unit circle in the z-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$.

Example

Find the z-transform of the unit step function $u(t) = 1$.



$$Z[u(t)] = \sum_{k=0}^{\infty} 1(kT)z^{-k}\,, \tag{5}$$

$$F(z) = z^{0} + z^{-1} + z^{-2} + z^{-3} + \ldots + z^{-k}\,. \tag{6}$$

Equation 6 can be written in closed-form as,

$$Z[1(t)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}}\,. \tag{7}$$

The convergence of the following power series for a given sequence depends only on $|z|$.

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)\,.$$

Since $|X(z)| < \infty$,

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty\,. \tag{8}$$

For example, the sequence $x[n] = u[n]$ (unit step) is not absolutely summable, and therefore, the Fourier transform power series does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with an ROC $|z| > 1$.

Common z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

ROC propieties

PROPERTY 1: The ROC will either be of the form $0 \leq r_R < |z|$, or $|z| < r_L \leq \infty$, or, in general the annulus, i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

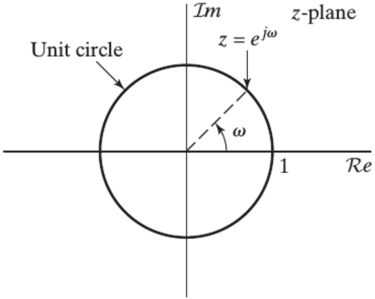
PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Stability, Causality, and the ROC

- The stability of a system can also be determined by knowing the ROC alone.
- If the ROC contains the unit circle (i.e., $|z| = 1$) then the system is stable.
- If you need a causal system then the ROC must contain infinity and the system function will be a right-sided sequence.
- If you need both, stability and causality, all the poles of the system function must be inside the unit circle.



Properties: Linearity

The z-transform of $x[n] = a_1 x_1[n] + a_2 x_2[n]$ is,

$$X(z) = \sum_{n=-\infty}^{\infty} (a_1\, x_1[n] + a_2\, x_2[n]) z^{-n}\,, \tag{9}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n}\,, \tag{10}$$

$$= a_1 X_1(z) + a_2 X_2(z)\,. \tag{11}$$

Properties: Shift in time

The z-transform of $x[n-n_0]$ is,

$$X(x[n-n_0]) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}\,. \tag{12}$$

Let $m = n - n_0$,

$$\begin{aligned} X(x[n-n_0]) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}\,, && (13)\\ &= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m}\,, && (14)\\ &= z^{-n_0}X(z)\,. && (15)\end{aligned}$$

The time-shifting property is often useful, in conjunction with other properties and procedures, for obtaining the inverse z-transform.

Properties: Convolution

The z-transform of

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\,, \tag{16}$$

is,

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] z^{-n}\,, \tag{17}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}\,, \tag{18}$$

Let $m = n - k$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{m=-\infty}^{\infty} x_2[m]z^{-(m+k)}\,, \tag{19}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{m=-\infty}^{\infty} x_2[m]z^{-m}\,, \tag{20}$$

$$= X_1(z)\, X_2(z)\,. \tag{21}$$

Constant-Coefficient Difference Equations

Consider a system described by the linear constant-coefficient difference equation,

$$y[n] = x[n] + a\,y[n-1]\,, \tag{22}$$

$$y[n] - a\,y[n-1] = x[n]\,, \tag{23}$$

$$Y(z) - a\,Y(z)\,z^{-1} = X(z)\,, \tag{24}$$

$$Y(z)(1 - a\,z^{-1}) = X(z)\,, \tag{25}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - a\,z^{-1})} \tag{26}$$

In general,

$$\sum_{k=0}^{N} a_k\,y[n-k] = \sum_{k=0}^{M} b_k\,x[n-k]\,, \tag{27}$$

$$y[n] = \sum_{k=0}^{M} b_k\,x[n-k] - \sum_{k=1}^{N} a_k\,y[n-k]\,, \tag{28}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1\,z^{-1} + b_2\,z^{-2} + ... + b_M\,z^{-(M)}}{1 + a_1\,z^{-1} + a_2\,z^{-2} + ... + a_N\,z^{-(N)}}\,. \tag{29}$$

Bibliography

1 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed*. Prentice Hall. 2010. Chapter 3.

2 Paolo Prandoni and Martin Vetterli. *Signal processing for communications*. Taylor and Francis Group, LLC. 2008. Chapter 6.