Finite representation of real numbers

Fixed-point numbers

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Summary

1. Finite representation of real numbers in computers
2. Integers
3. Fixed-point
   - Scale factor
   - Dynamic range
   - How to determine the correct integer value (*m*)
4. Addition
   - Overflow
   - How to avoid overflow
5. Multiplication
   - Underflow
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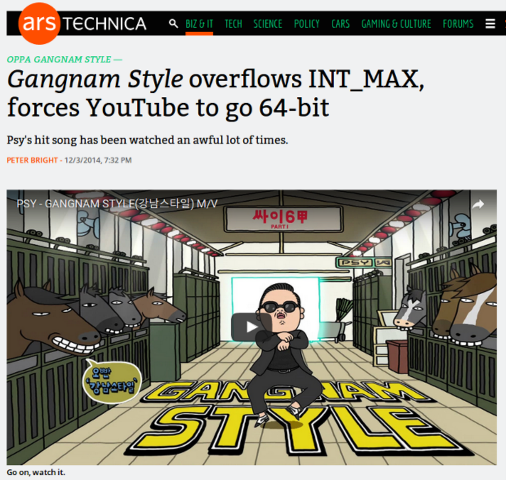
6. MAC operation
7. Shifts



OPPA GANGNAM STYLE —

Gangnam Style overflows INT_MAX, forces YouTube to go 64-bit

Psy's hit song has been watched an awful lot of times.

PETER BRIGHT - 12/3/2014, 7:32 PM

Go on, watch it.

https://arstechnica.com

Patriot Missile System problem

- On February 25th, 1991, a Patriot Missile system at Dhahran, Saudi Arabia failed to intercept a SCUD missile, killing 28 American soldiers.
- The radar of a Patriot missile system is designed to detect an incoming missile twice in order to avoid false alarms.
- Time is stored to an accuracy of 1/10th of a second in a 24-bit register.
- It results in 0.0001110111001100110011001100110 1... with an infinite number of bits.
- The error of representing 1/10th in 24-bit register is 0.000000095 decimal of seconds.
- After 100 hours of operation, cumulative error gives $0.000000095 \times 100 \times 60 \times 60 \times 10 = 0.34$ s.
- A SCUD travels at about 1,676 m/s. In 0.34 s, it travels more than half a kilometer.
- This error in the time calculation caused the Patriot system to expect an incoming missile at a wrong location for the second detection, causing it to consider the first detection as false alarm.

More information at https://blog.penjee.com/famous-number-computing-errors/

Unsigned integers

- An N-bit binary word can represent a total of 2^N separate values.
- Range: 0 to $2^N - 1$
- $n_{10} = 2^{N-1} b_{N-1} + 2^{N-2} b_{N-2} + \cdots + 2^1 b_1 + 2^0 b_0$

2's complement signed integers

- Range: -2^{N-1} to $2^{N-1} - 1$.
- $n_{10} = -b_{N-1} 2^{N-1} + \sum_{i=0}^{N-2} b_i 2^i$

Bit Pattern	Unsigned	2's Complement
0000 0000	0	0
0000 0001	1	1
0000 0010	2	2
•	•	•
•	•	•
0111 1110	126	126
0111 1111	127	127
1000 0000	128	-128
1000 0001	129	-127
•	•	•
•	•	•
1111 1110	254	-2
1111 1111	255	-1

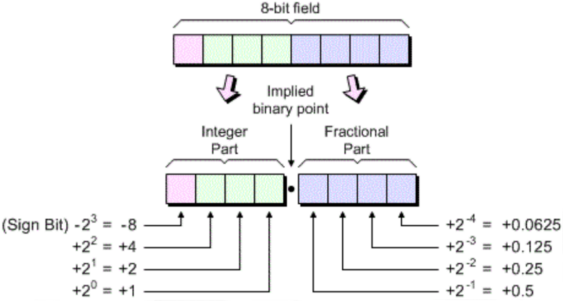
in C:

- 8 bits (`char, int8_t`): [-128, 127]]
- 16 bits (`short, int16_t`): [-32768, 32767]
- 32 bits (`int, long, int32_t`): [-2147483648, 2147483647]

Fixed-point representation

In fixed-point representation, a real number x is represented by an integer X with $N = m + n + 1$ bits, where:

- N is the wordlength.
- m represents the number of integer bits (to the left of the binary point).
- n represents the number of fractional bits (to the right of the binary point).
- The weights of bits to the right of the binary point are negative powers of 2: $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$ ... , etc.
- $n_{10} = -b_{m-1}2^{m-1} + \left(\sum_{i=0}^{m-1} b_i\, 2^i + \sum_{i=1}^{n} b_i\, 2^{-i}\right)$
- Precision: 2^{-n}.
- Range: -2^m to $2^m - 2^{-n}$.
- What happens if $n = 0$?



Qm.n notation

This naming convention does not take the MSB of the number (sign bit) into account.

For instance:

- Q0.15 (Q15)
  - 16 bits;
  - Range: -1 to 0.99996948;
  - Precision: 1/32768 (2^{-15}).
- Q3.12
  - 16 bits;
  - Range: -8 to 7.9998;
  - Precision: 1/4096 (2^{-12}).
- Q0.31 (Q31)
  - 32 bits;
  - Range: -1 to 0.999999999534339;
  - Precision: 4.6566129e-10 (2^{-31}).

Conversion to and from fixed point

Defining:

- Unit: $z = \; 1 << n \; = \; 1 \cdot 2^n$
- One half: $z = \; 1 << (n-1) \; = \; 1 \cdot 2^{(n-1)}$

Conversion from floating-point ("real") number to fixed-point number:

$$X := (int)(x \cdot (1 << n)) \tag{1}$$

$$X := (int)(x \cdot 2^n) \tag{2}$$

Conversion from fixed-point number to floating-point ("real") number:

$$x := (float)(X)/(1 << n) \tag{3}$$

$$x := (float)(X) \cdot 2^{-n} \tag{4}$$

Example 1: Represent x = 13.4 using Q4.3 format

$$X = round(13.4 \cdot 2^3) = 107 \; (01101011_2)$$

Example 2: Represent x = 0.052246 using Q4.11 format

$$X = round(0.052246 \cdot 2^{11}) = 107 \; (000000001101011_2)$$

- There is no difference at the CPU level (ALU) between both fixed-point and integer numbers.
- The difference is based on the concept of *scale*, which is almost completely in the head of the designer.
- Real values represented in Qm.n notation can be seen as a signed integer simply multiplied by 2^{-n}, the precision.
- In fact, the scale factor can be an arbitrary scale that may not be a power of two.
- **Example:** 16-bit 2's complement numbers between 8000H and 7FFFH can represent decimal values between –5 and +5, where the scale factor is 5/32768 (5 * 2^{-15}).
  1. Integer: -32768 to 32767 (8000H - 7FFFH).
  2. Fixed point Q15: $(-32768 * 2^{-15})$ to $(32767 * 2^{-15}) => -1$ to 0.99996948242.
  3. $(-1 * 5)$ to $(0.99996948242 * 5) => -5$ to 4.99984741211.

Scale factor, examples

Format	Scaling factor ( )	Range in Hex (fractional value)
(1.15)	$2^{15} = 32768$	0x7FFF (0.99) → 0x8000 (–1)
(2.14)	$2^{14} = 16384$	0x7FFF (1.99) → 0x8000 (–2)
(3.13)	$2^{13} = 8192$	0x7FFF (3.99) → 0x8000 (–4)
(4.12)	$2^{12} = 4096$	0x7FFF (7.99) → 0x8000 (–8)
(5.11)	$2^{11} = 2048$	0x7FFF (15.99) → 0x8000 (–16)
(6.10)	$2^{10} = 1024$	0x7FFF (31.99) → 0x8000 (–32)
(7.9)	$2^{9} = 512$	0x7FFF (63.99) → 0x8000 (–64)
(8.8)	$2^{8} = 256$	0x7FFF (127.99) → 0x8000 (–128)
(9.7)	$2^{7} = 128$	0x7FFF (511.99) → 0x8000 (–512)
(10.6)	$2^{6} = 64$	0x7FFF (1023.99) → 0x8000 (–1024)
(11.5)	$2^{5} = 32$	0x7FFF (2047.99) → 0x8000 (–2048)
(12.4)	$2^{4} = 16$	0x7FFF (4095.99) → 0x8000 (–4096)
(13.3)	$2^{3} = 8$	0x7FFF (4095.99) → 0x8000 (–4096)
(14.2)	$2^{2} = 4$	0x7FFF (8191.99) → 0x8000 (–8192)
(15.1)	$2^{1} = 2$	0x7FFF (16383.99) → 0x8000 (–16384)
(16.0)	$2^{0} = 1$(Integer)	0x7FFF (32767) → 0x8000h (–32768)

Dynamic range

Dynamic range is defined as,

$$DR_{db} = 20 \ log_{10} \left(\frac{\text{largest possible word value}}{\text{smallest possible word value}} \right) \quad \text{[dB]}$$

For N-bit signed integers,

$$DR_{dB} = 20 \ log_{10} \left[\frac{2^{(N-1)} - 1}{1} \right] \quad \text{[dB]}$$

$$DR_{dB} \approx 20 \ [(N-1) log_{10}(2)]$$

$$DR_{dB} \approx 20 \ log_{10}(2) \cdot (N-1)$$

$$DR_{dB} \approx 6.02 \cdot (N-1) \quad \text{[dB]}$$

Precision and Dynamic range examples

Format (N.M)		Largest positive value (0x7FFF)	Least negative value (0x8000)	Precision (0x0001)		DR(dB)
1	15	0,999969482421875	-1	3,05176E-05	2^-15	90,30873362
2	14	1,99993896484375	-2	6,10352E-05	2^-14	90,30873362
3	13	3,9998779296875	-4	0,00012207	2^-13	90,30873362
4	12	7,999755859375	-8	0,000244141	2^-12	90,30873362
5	11	15,99951171875	-16	0,000488281	2^-11	90,30873362
6	10	31,99902344	-32	0,000976563	2^-10	90,30873362
7	9	63,99804688	-64	0,001953125	2^-9	90,30873362
8	8	127,9960938	-128	0,00390625	2^-8	90,30873362
9	7	255,9921875	-256	0,0078125	2^-7	90,30873362
10	6	511,984375	-512	0,015625	2^-6	90,30873362
11	5	1023,96875	-1024	0,03125	2^-5	90,30873362
12	4	2047,9375	-2048	0,0625	2^-4	90,30873362
13	3	4095,875	-4096	0,125	2^-3	90,30873362
14	2	8191,75	-8192	0,25	2^-2	90,30873362
15	1	16383,5	-16384	0,5	2^-1	90,30873362
16	0	32767	-32768	1	2^-0	90,30873362

How to determine the correct integer value (m)

What is the correct value for m?

How much bits are needed to represent $-15 \leq x \leq 10$?

MATLAB

```
1 » INT_MIN = abs(-15); INT_MAX = 10;
2 » MAX = max( [ INT_MIN, INT_MAX ] );  % MAX = 15
3 » BITS = (log2 ( MAX ) + 2 );
4 » N = floor ( BITS );  % floor() rounds to -Inf
5 » N = 5.00
```

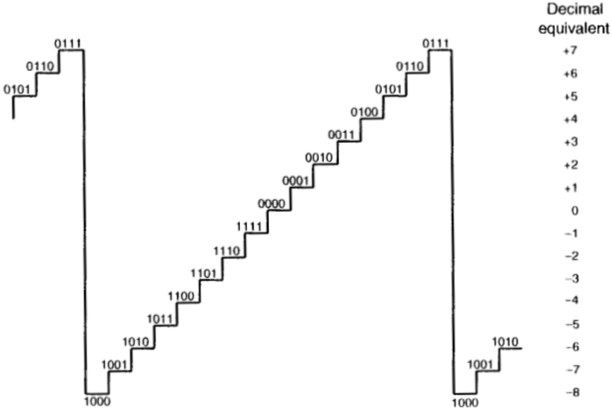
Addition in 2's complement

- Adding two N-bits numbers can produce a N+1 bits result.
- The result will have the same numbers of fractional bits.
- Only the integer part can grow.
- The last two bits of the carry row show if overflow occurs.

```
 11111 111    (carry)        0111     (carry)
  0000 1111   (15)            0111    (7)
+ 1111 1011   (-5)          + 0011    (3)
==================          =============
  0000 1010   (10)            1010    (-6)   invalid!
```

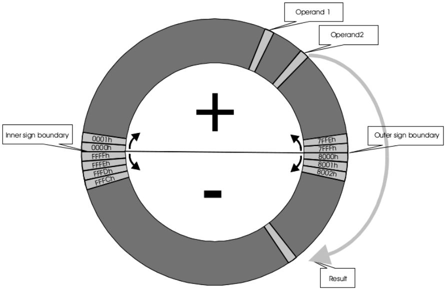
Overflow

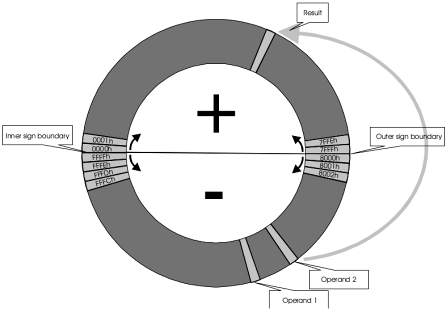
- An **overflow** occurs in a when a result is greater than $2^{N-1} - 1$ or lesser than -2^{N-1}.
- An overflow produces a **roll-over** (wrap).



Overflow II

- A roll-over usually has catastrophic consequences on a process.
- It only happens when two very large positive operands, or two very large negative operands, are added.
- It can never happen during the addition of a positive operand and a negative operand, whatever their magnitude.



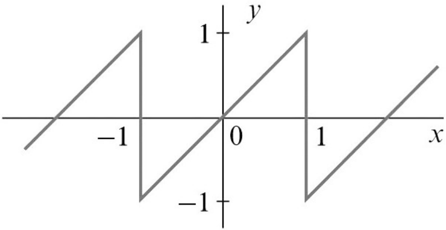


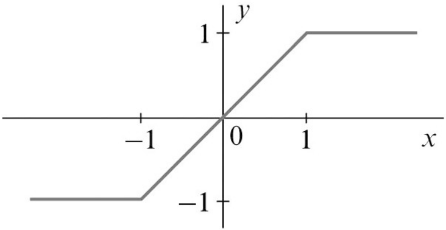
Longer word-length accumulator

- Saving the result in a N+1 word avoids overflows.
- The general rule is the sum of *s* individual *m*-bit can require as many as $m + log_2(s)$.
- **Example:** 256 8-bits words requires an accumulator whose word length is $8 + log_2(256) = 16$.
- DSP processors usually have 40-bit accumulators.
- How many sums are supported by a 40-bits accumulator for 16-bits numbers?

Saturation

- To avoid a rollover, overflow is detected and the result is saturated to the most positive or most negative value that can be represented.
- This procedure is called **saturation arithmetic**.
- DSP processors allows the results to be saturated automatically in hardware (In TI DSP C5505, SATD Bit at ST1_55 register).

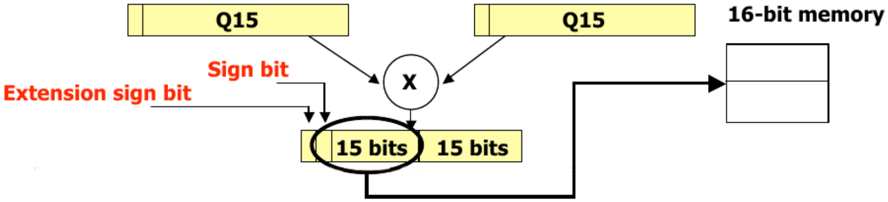




Be aware of non-linearity!

Multiplication in 2's complement

- The product of 2 N-bit numbers requires $2 \cdot N$ bits bits to contain all possible values.
- The 2 Most Significant Bits (MSB) are always equal (extension sign bit).
- Therefore, 2N-1 bits are enough to store the result.
- A Q15 multiplication produces Q1.30 result (extension sign bit).
- To transform the result into Q31 notation, it must be left-shifted by one bit.
- DSP processors have a special mode that allows its ALU to automatically perform the left shift when Q15xQ15.



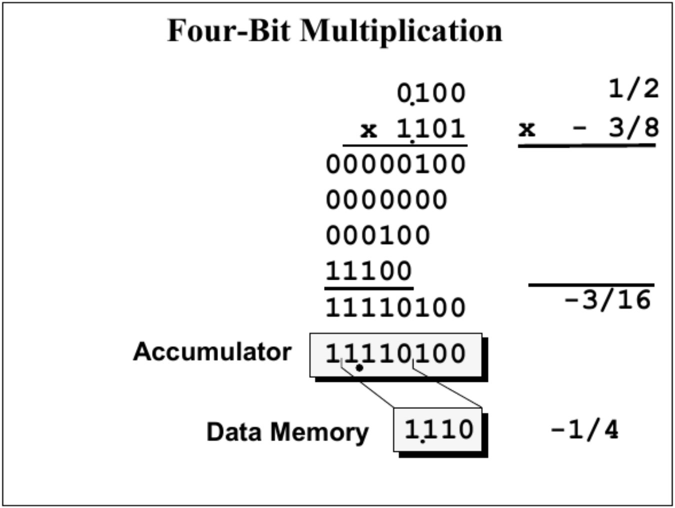
Four-Bit Integer Multiplication

```
    0100          4
  x 1101     x   -3
--------     ------
00000100
0000000
000100
11100
-----        ------
11110100        -12
```

Accumulator: 11110100 → −12

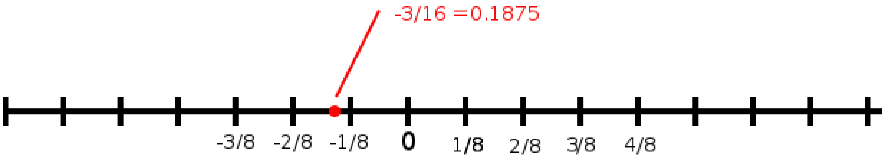
Data Memory: 11110100 → −12

Four-bit Q0.3 multiplication



Underflow

- After multiplication, 2N bits must be stored in a memory of N-bits word.
- An **underflow** occurs if the result is less than 2^{-n}.
- **Example:** Q0.3 precision is $2^{-3} = \frac{1}{8}$.



- What value should the multiplication result take? -1/8 or -2/8?
- In other words, what bits should be discarded from the multiplication result?

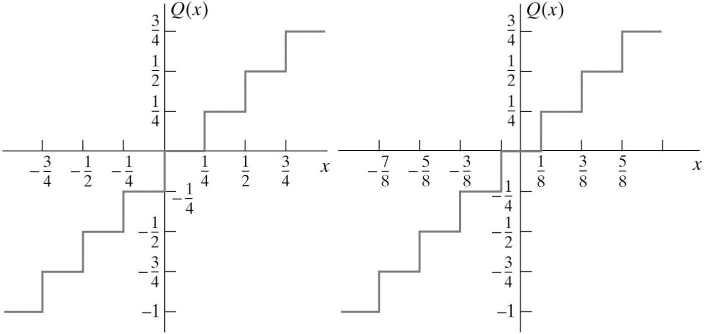
Truncation

- $y = Q(x)$.
- `floor()` function in MATLAB, also known as *round to minus infinity*.

Round-off

- $y = Q\left(x + 2^{-(n+1)}\right)$
- `round()` function in MATLAB, also known as *round to the nearest*.

What is $2^{-(n+1)}$?

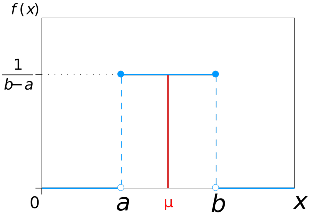


Error in rounding schemes

Error in rounding schemes is modeled as a uniform probability distribution.

$$\text{mean, } \mu = \frac{a+b}{2}$$

$$\text{variance, } \sigma^2 = \frac{(b-a)^2}{12}$$



- Truncation: $e = Q(x) - x, \quad -2^{-n} \leq e < 0,$ $\boxed{\mu = -\frac{2^{-n}}{2}}$ $\boxed{\sigma^2 = \frac{2^{-n}}{12}}$
- Round-off: $e = Q\left(x + 2^{-(n+1)}\right) - x, \quad -2^{-n}/2 < e \leq 2^{-n}/2,$ $\boxed{\mu = 0}$ $\boxed{\sigma^2 = \frac{2^{-n}}{12}}$

DSP processors manage truncation and round-off automatically.

MAC operation

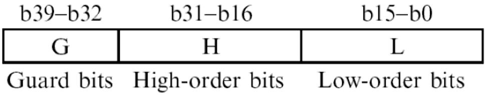
- MAC stands for **M**ultiply and **AC**cumulate.
- Since it represents the convolution operation, it is THE basic arithmetic operation in DSP.

In C:

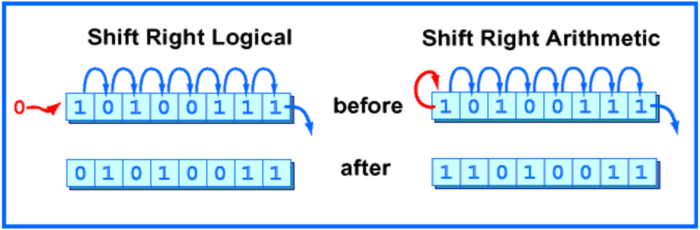
C code

```
c = 0;
for (i=0; i=N; i++)
{ c = c + (a * b); };
```

- A MAC operation summarizes the addition and multiplication problems, i.e., overflows and underflows.
- DSP processors have an accumulator with extra bits (guard bits) to avoid overflow during internal calculations (in TI DSP C5505, 40-bits accumulator).



- Multiplication by 2: all bits are shifted left by one position.
- Division by 2: all bits are shifted right by one position (**logical shift**).
- What happens with 2-complement numbers?
- The sign bit must be preserved! (**arithmetic shift**).
- Arithmetic shift $\neq$ logical shift.



In DSP processors:

- ALU can perform logical shifts of 32-bit operands in one cycle, from 16 bits to the right, to 15 bits to the left.
- Sign extension is performed during shifts to the right, if the Sign Extension Mode control bit (in C5505, SXM) is set.
- Result is saturated during shifts to the left if an overflow is detected, and Overflow bit (in C5505, OVM) is set.

Bibliography

1. Richard G. Lyons. *Understanding Digital Signal Processing, 3rd Ed*. Prentice Hill. 2010. Chapter 12.
2. Bruno Paillard. An Introduction To Digital Signal Processors, Chapter 5.