Mathematical Modeling of Pneumatic Systems

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Symmary

Resistance of Pressure Systems

The gas flow resistance R may be defined as follows:

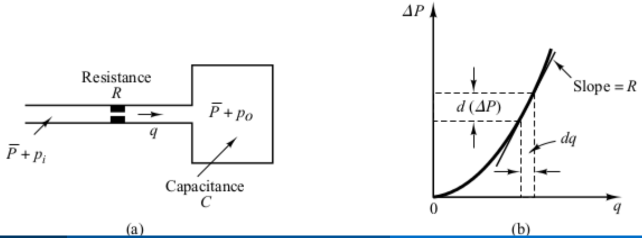
$$R = \frac{\text{change in gas pressure difference, lb}_f/\text{ft}^2}{\text{change in gas flow rate, lb/sec}}$$

or

$$R = \frac{d(\Delta P)}{dq} \qquad (4\text{–}8)$$

- where $d(\Delta P)$ is a small change in the gas pressure difference, and dq is a small change in the gas flow rate.
- Computation of the value of the gas flow resistance R may be quite time consuming.
- Experimentally, it can be determined from a plot of the pressure difference versus flow rate.

Figure 4–4
(a) Schematic diagram of a pressure system; (b) pressure-difference-versus-flow-rate curve.



The capacitance of the pressure vessel may be defined by

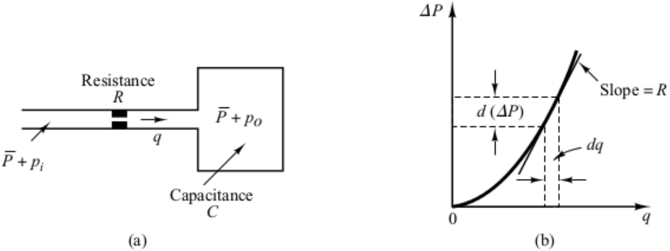
$$C = \frac{\text{change in gas stored, lb}}{\text{change in gas pressure, lb}_f/\text{ft}^2}$$

or

$$C = \frac{dm}{dp} = V\frac{d\rho}{dp} \qquad (4\text{–}9)$$

$$C = \frac{V}{nR_{\text{gas}}T} \qquad (4\text{–}12)$$

Figure 4–4
(a) Schematic diagram of a pressure system; (b) pressure-difference-versus-flow-rate curve.



Let us define

$\bar{P}$ = gas pressure in the vessel at steady state (before changes in pressure have occurred), lb_f/ft^2

p_i = small change in inflow gas pressure, lb_f/ft^2

p_o = small change in gas pressure in the vessel, lb_f/ft^2

V = volume of the vessel, ft^3

m = mass of gas in the vessel, lb

q = gas flow rate, lb/sec

ρ = density of gas, lb/ft^3

For small values of p_i and p_o, the resistance R given by Equation (4–8) becomes constant and may be written as

$$R = \frac{p_i - p_o}{q}$$

The capacitance C is given by Equation (4–9), or

$$C = \frac{dm}{dp}$$

Since the pressure change dp_o times the capacitance C is equal to the gas added to the vessel during dt seconds, we obtain

$$C\,dp_o = q\,dt$$

or

$$C\frac{dp_o}{dt} = \frac{p_i - p_o}{R}$$

which can be written as

$$RC\frac{dp_o}{dt} + p_o = p_i$$

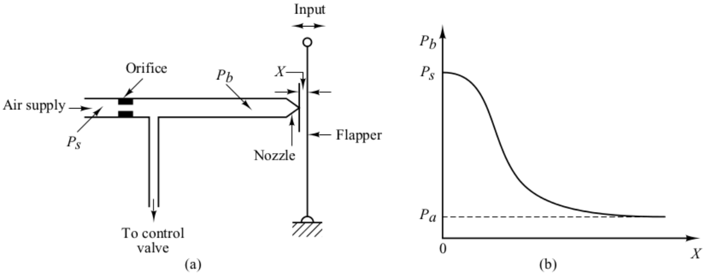
If p_i and p_o are considered the input and output, respectively, then the transfer function of the system is

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs + 1}$$

Pneumatic Nozzle–Flapper Amplifiers

- The nozzle–flapper amplifier converts displacement into a pressure signal.
- The power source for this amplifier is a supply of air at constant pressure.
- The nozzle–flapper amplifier converts small changes in the position of the flapper into large changes in the back pressure in the nozzle.
- Thus a large power output can be controlled by the very little power that is needed to position the flapper.
- To ensure proper functioning of the amplifier, the nozzle diameter must be larger than the orifice diameter (0.4 mmm vs 0.25 mm).
- The lowest possible pressure will be the ambient pressure P_a.

Figure 4–5 (a) Schematic diagram of a pneumatic nozzle–flapper amplifier; (b) characteristic curve relating nozzle back pressure and nozzle–flapper distance.



- As the nozzle back pressure P_b increases, the diaphragm valve moves downward.The opening to
- The atmosphere decreases and the opening to the pneumatic valve increases, thereby increasing the control pressure P_c.
- P_c can vary from 0 psi (pound-force per square inch) to full supply pressure, usually 20 psi.
- Since P_c changes almost instantaneously with changes in P_b, the time constant of the pneumatic relay is negligible.

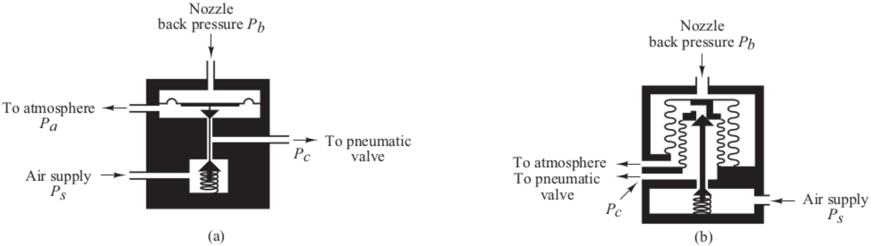
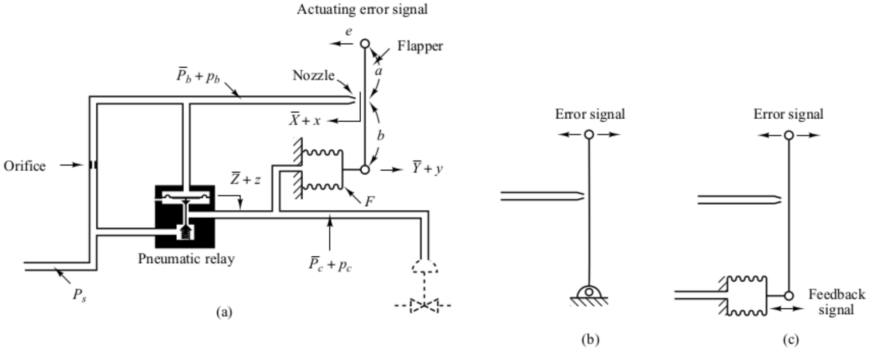


Figure 4–6
(a) Schematic diagram of a bleed-type relay; (b) schematic diagram of a nonbleed-type relay.

- The nozzle-flapper amplifier constitutes the first-stage amplifier, and the relay-type amplifier constitutes the second-stage amplifier.
- Feedback of the pneumatic output reduces the amount of actual movement of the flapper.
- Error signal and bellows F now move the flapper.
- If these two movements were equal, no control action would result.
- The effect of the feedback bellows is thus to reduce the sensitivity of the controller.



Assuming that the relationship between the variation in the nozzle back pressure and the variation in the nozzle–flapper distance is linear, we have

$$p_b = K_1 x \tag{4–13}$$

where K_1 is a positive constant. For the diaphragm valve,

$$p_b = K_2 z \tag{4–14}$$

where K_2 is a positive constant. The position of the diaphragm valve determines the control pressure. If the diaphragm valve is such that the relationship between p_c and z is linear, then

$$p_c = K_3 z \tag{4–15}$$

where K_3 is a positive constant. From Equations (4–13), (4–14), and (4–15), we obtain

$$p_c = \frac{K_3}{K_2} p_b = \frac{K_1 K_3}{K_2} x = Kx \tag{4–16}$$

where $K = K_1 K_3 / K_2$ is a positive constant. For the flapper, since there are two small movements (e and y) in opposite directions, we can consider such movements separately and add up the results of two movements into one displacement x. See Figure 4–8(d). Thus, for the flapper movement, we have

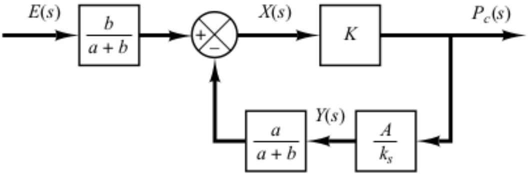
$$x = \frac{b}{a+b} e - \frac{a}{a+b} y \tag{4–17}$$

The bellows acts like a spring, and the following equation holds true:

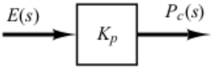
$$A p_c = k_s y \tag{4–18}$$

Pneumatic Proportional Controllers (Force-Distance Type), cont'd II

$$\frac{P_c(s)}{E(s)} = \frac{\dfrac{b}{a+b}K}{1 + K\dfrac{a}{a+b}\dfrac{A}{k_s}} = K_p \qquad (4\text{–}19)$$

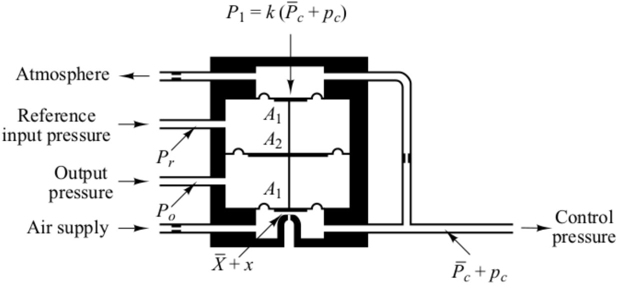


(e)



(f)

- The main advantage of the force-balance type controller is that it eliminates many mechanical linkages and pivot joints, thereby reducing the effects of friction.



$$p_e = P_r - P_o \tag{4–20}$$

If $p_e = 0$, there is an equilibrium state with the nozzle–flapper distance equal to $\bar{X}$ and the control pressure equal to $\bar{P}_c$. At this equilibrium state, $P_1 = \bar{P}_c k$ (where $k < 1$) and

$$\bar{X} = \alpha(\bar{P}_c A_1 - \bar{P}_c k A_1) \tag{4–21}$$

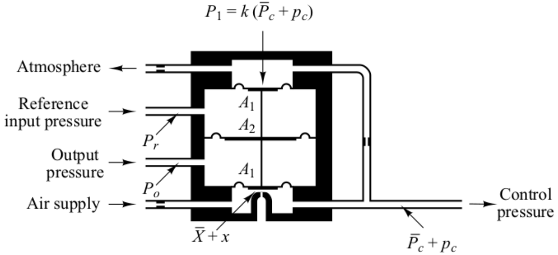
where α is a constant.

Let us assume that $p_e \neq 0$ and define small variations in the nozzle–flapper distance and control pressure as x and p_c, respectively. Then we obtain the following equation:

$$\bar{X} + x = \alpha\left[(\bar{P}_c + p_c)A_1 - (\bar{P}_c + p_c)kA_1 - p_e(A_2 - A_1)\right] \tag{4–22}$$

From Equations (4–21) and (4–22), we obtain

$$x = \alpha\left[p_c(1 - k)A_1 - p_e(A_2 - A_1)\right] \tag{4–23}$$



Pneumatic Proportional Controllers (Force-Balance Type), cont'd II

At this point, we must examine the quantity x. In the design of pneumatic controllers, the nozzle–flapper distance is made quite small. In view of the fact that x/α is very much smaller than $p_c(1-k)A_1$ or $p_e(A_2 - A_1)$—that is, for $p_e \neq 0$

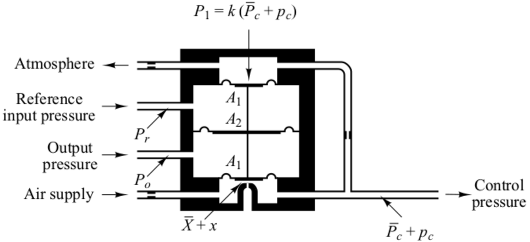
$$\frac{x}{\alpha} \ll p_c(1-k)A_1$$
$$\frac{x}{\alpha} \ll p_e(A_2 - A_1)$$

we may neglect the term x in our analysis. Equation (4–23) can then be rewritten to reflect this assumption as follows:

$$p_c(1-k)A_1 = p_e(A_2 - A_1)$$

and the transfer function between p_c and p_e becomes

$$\frac{P_c(s)}{P_e(s)} = \frac{A_2 - A_1}{A_1}\,\frac{1}{1-k} = K_p$$



Pneumatic Actuating Valves

$$Ap_c = m\ddot{x} + b\dot{x} + kx$$

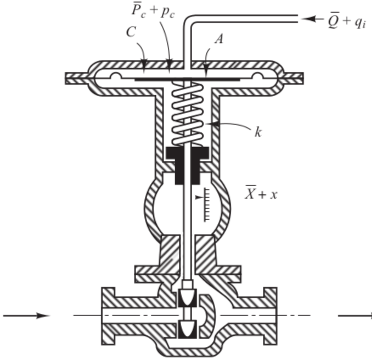
where m = mass of the valve and valve stem
b = viscous-friction coefficient
k = spring constant

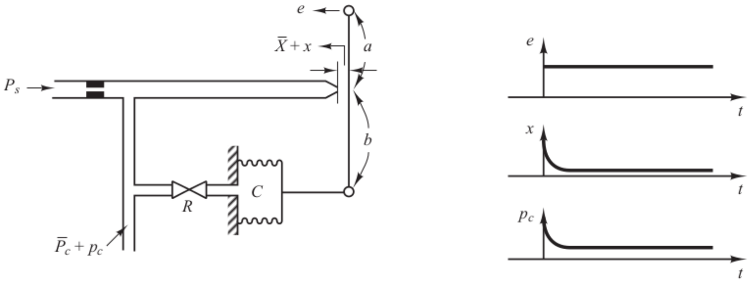
If the force due to the mass and viscous friction are negligibly small, then this last equation can be simplified to

$$Ap_c = kx$$

The transfer function between x and p_c thus becomes

$$\frac{X(s)}{P_c(s)} = \frac{A}{k} = K_c$$




e
$\bar{X}+x$
a
P_s
b
C
R
$\bar{P}_c + p_c$
e
t
x
t
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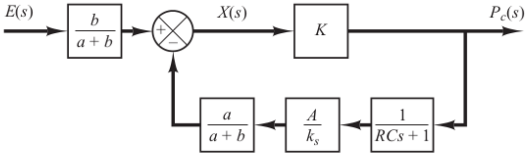
t

(a) (b)


$E(s)$
$\frac{b}{a+b}$
+
−
$X(s)$
K
$P_c(s)$
$\frac{a}{a+b}$
$\frac{A}{k_s}$
$\frac{1}{RCs+1}$

Proportional-Plus-Derivative Control Action (PD)

$$\frac{P_c(s)}{E(s)} = \frac{\dfrac{b}{a+b}K}{1 + \dfrac{Ka}{a+b}\dfrac{A}{k_s}\dfrac{1}{RCs+1}}$$

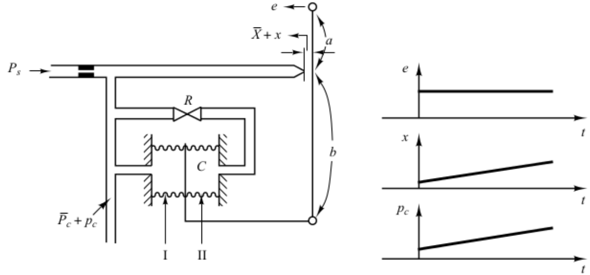
In such a controller the loop gain $\left|KaA/\left[(a+b)k_s(RCs+1)\right]\right|$ is made much greater than unity. Thus the transfer function $P_c(s)/E(s)$ can be simplified to give

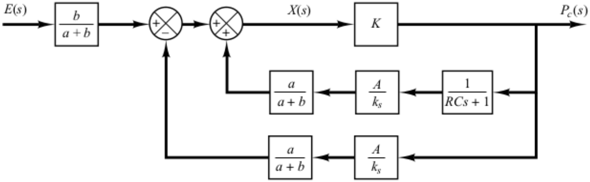
$$\frac{P_c(s)}{E(s)} = K_p(1 + T_d s)$$

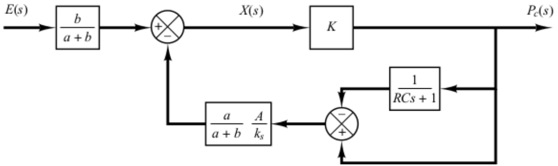
where

$$K_p = \frac{bk_s}{aA}, \qquad T_d = RC$$

Proportional-Plus-Integral Control Action







$$\frac{P_c(s)}{E(s)} = \frac{\dfrac{b}{a+b}K}{1 + \dfrac{Ka}{a+b}\dfrac{A}{k_s}\left(1 - \dfrac{1}{RCs+1}\right)}$$

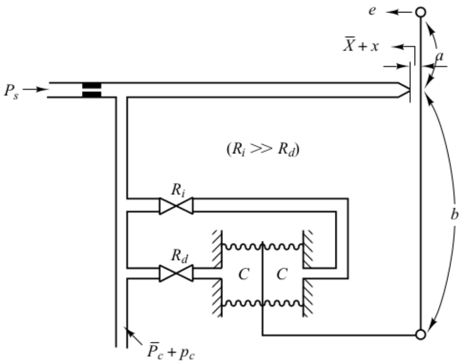
where K is a constant, A is the area of the bellows, and k_s is the equivalent spring constant of the combined bellows. If $\left|KaARCs/\left[(a+b)k_s(RCs+1)\right]\right| \gg 1$, which is usually the case, the transfer function can be simplified to

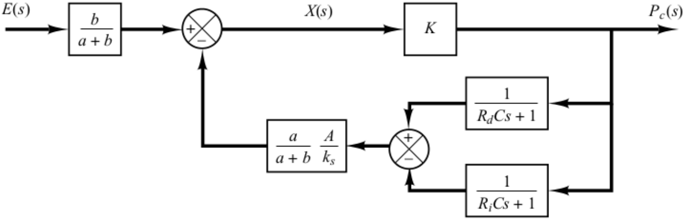
$$\frac{P_c(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

where

$$K_p = \frac{bk_s}{aA}, \qquad T_i = RC$$

Proportional-Plus-Integral-Plus-Derivative Control Action





The transfer function of this controller is

$$\frac{P_c(s)}{E(s)} = \frac{\dfrac{bK}{a+b}}{1 + \dfrac{Ka}{a+b}\dfrac{A}{k_s}\dfrac{(R_iC - R_dC)s}{(R_dCs+1)(R_iCs+1)}}$$

By defining

$$T_i = R_iC, \qquad T_d = R_dC$$

and noting that under normal operation $\left|KaA(T_i - T_d)s/[(a+b)k_s(T_ds+1)(T_is+1)]\right| \gg 1$ and $T_i \gg T_d$, we obtain

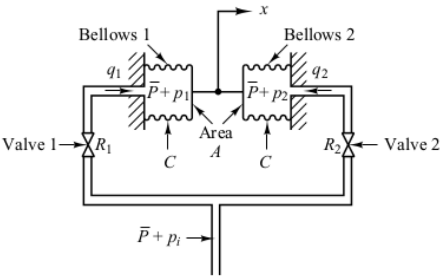
$$\begin{aligned}\frac{P_c(s)}{E(s)} &\doteqdot \frac{bk_s}{aA}\frac{(T_ds+1)(T_is+1)}{(T_i - T_d)s} \\ &\doteqdot \frac{bk_s}{aA}\frac{T_dT_is^2 + T_is + 1}{T_is} \\ &= K_p\left(1 + \frac{1}{T_is} + T_ds\right) \qquad (4\text{–}24)\end{aligned}$$

where

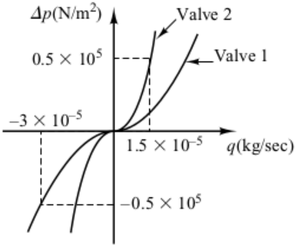
$$K_p = \frac{bk_s}{aA}$$

Exercice 1

A–4–4. In the pneumatic pressure system of Figure 4–29(a), assume that, for $t < 0$, the system is at steady state and that the pressure of the entire system is $\bar{P}$. Also, assume that the two bellows are identical. At $t = 0$, the input pressure is changed from $\bar{P}$ to $\bar{P} + p_i$. Then the pressures in bellows 1 and 2 will change from $\bar{P}$ to $\bar{P} + p_1$ and from $\bar{P}$ to $\bar{P} + p_2$, respectively. The capacity (volume) of each bellows is 5×10^{-4} m^3, and the operating-pressure difference Δp (difference between p_i and p_1 or difference between p_i and p_2) is between -0.5×10^5 N/m^2 and 0.5×10^5 N/m^2. The corresponding mass flow rates (kg/sec) through the valves are shown in Figure 4–29(b). Assume that the bellows expand or contract linearly with the air pressures applied to them, that the equivalent spring constant of the bellows system is $k = 1 \times 10^5$ N/m, and that each bellows has area $A = 15 \times 10^{-4}$ m^2.



(a)

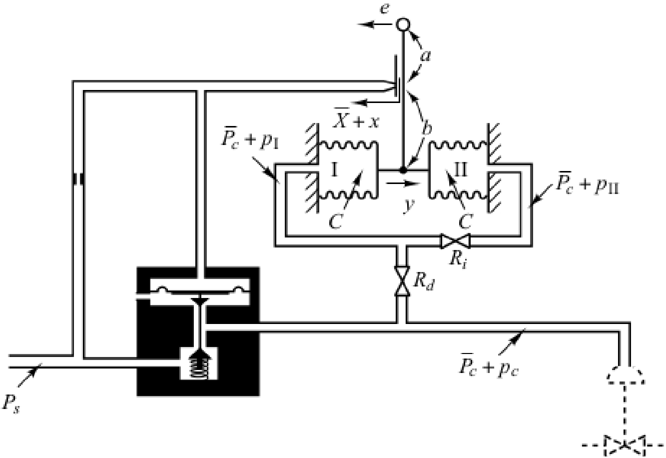


(b)

Exercice 2

A–4–5. Draw a block diagram of the pneumatic controller shown in Figure 4–30. Then derive the transfer function of this controller. Assume that $R_d \ll R_i$. Assume also that the two bellows are identical.

If the resistance R_d is removed (replaced by the line-sized tubing), what control action do we get? If the resistance R_i is removed (replaced by the line-sized tubing), what control action do we get?



Exercice 3

B–4–4. Figure 4–45 shows a pneumatic controller. The pneumatic relay has the characteristic that $p_c = K p_b$, where $K > 0$. What kind of control action does this controller produce? Derive the transfer function $P_c(s)/E(s)$.

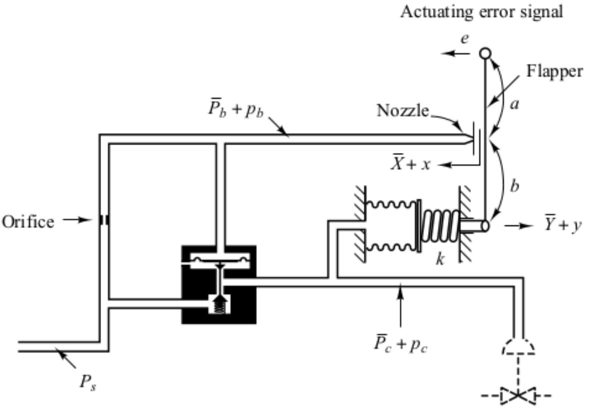


Figure 4–45
Pneumatic controller.

Exercice 4

B–4–5. Consider the pneumatic controller shown in Figure 4–46. Assuming that the pneumatic relay has the characteristics that $p_c = K p_b$ (where $K > 0$), determine the control action of this controller. The input to the controller is e and the output is p_c.

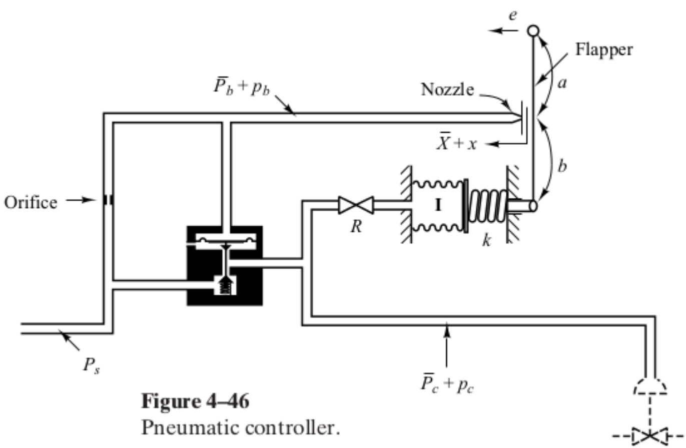


Figure 4–46
Pneumatic controller.

Exercice 5

B–4–6. Figure 4–47 shows a pneumatic controller. The signal e is the input and the change in the control pressure p_c is the output. Obtain the transfer function $P_c(s)/E(s)$. Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

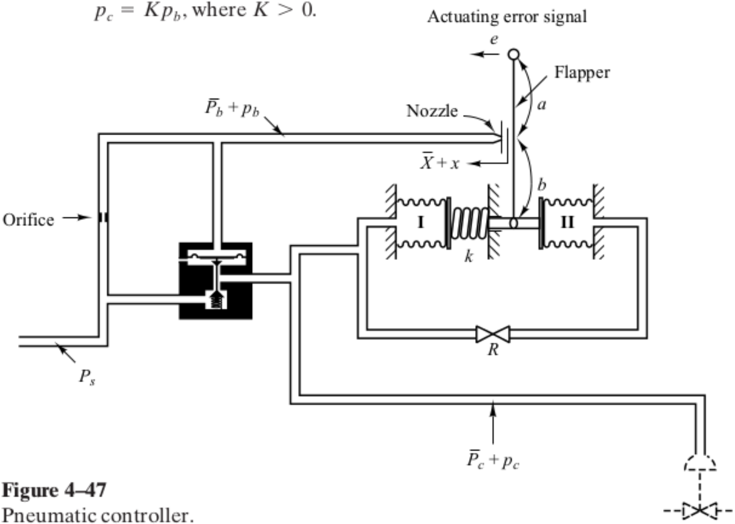


Figure 4–47
Pneumatic controller.

B–4–7. Consider the pneumatic controller shown in Figure 4–48. What control action does this controller produce? Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

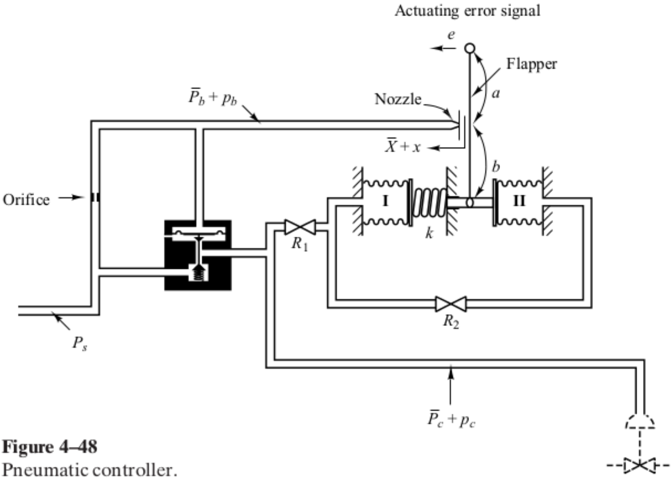


Figure 4–48
Pneumatic controller.

Bibliography

- Ogata, Katsuhiko. Modern Control Engineering. Fifth Edition. Prentice Hall. 2009. Chapter 4.