The Z-transform

Dr. Ing. Rodrigo Gonzalez

rodrigo.gonzalez@ingenieria.uncu.edu.ar

Control y Sistemas

Facultad de Ingeniería,
Universidad Nacional de Cuyo

Summary

- The Z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- It is analogous to the Laplace transform for continuous systems.

The Laplace transform is defined as:

$$\mathcal{L}\{x(t)\} = F(s) = \int_0^{\infty} x(t)e^{-st}\,, \tag{1}$$

$$\mathcal{L}\{\dot{x}(t)\} = s\,F(s) \tag{2}$$

Eq. 2 enables us to find the transfer function of a linear continuous time system, given the differential equation description of that system.

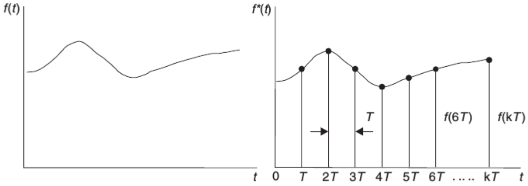
Bilateral z-transform

The bilateral or two-sided Z-transform of a discrete-time signal $x[n]$ is the formal power series $X(z)$ defined as

$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \,. \tag{3}$$

where $x[n]$ is the sampled version of $x(t)$, n is an integer and z is, in general, a complex number:

$$z = r\,e^{j\omega} = r \cdot (\cos\omega + j\sin\omega) \,. \tag{4}$$



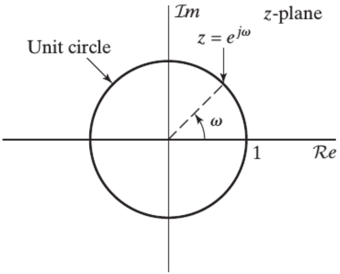
(a) Continuous Signal (b) Sampled Signal

Unilateral z-transform

Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$ the single-sided or unilateral Z-transform is defined as,

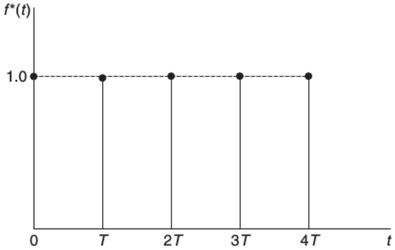
$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}. \tag{5}$$

In signal processing, this definition can be used to evaluate the Z-transform of the unit impulse response of a discrete-time causal system.



Example

Find the z-transform of the unit step function $u(t) = 1$.



$$\mathcal{Z}\{u[n]\} = U(z) = \sum_{n=0}^{\infty} u(kT) z^{-n}, \tag{6}$$

$$U(z) = z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-n} + \cdots \tag{7}$$

Equation 7 can be written in closed-form as,

$$\mathcal{Z}\{u[n]\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}. \tag{8}$$

The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$ROC = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\}. \tag{9}$$

Example (no ROC)

Let $x[n] = 0.5^n = \{\cdots, 0.5^{-3}, 0.5^{-2}, 0.5^{-1}, 1, 0.5, 0.5^2, \cdots\} \Rightarrow$

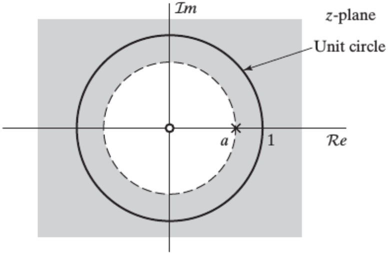
$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} \rightarrow \infty$$

Therefore, there are no values of z that satisfy this condition.

Causal ROC

Let $x[n] = 0.5^n u[n] = \{\cdots, 0, 0, 0, 1, 0.5, 0.5^2, 0.5^3, \cdots\}$.

$$\sum_{n=-\infty}^{\infty} (0.5^n u[n])\, z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n = \frac{1}{1 - 0.5z^{-1}} < \infty$$



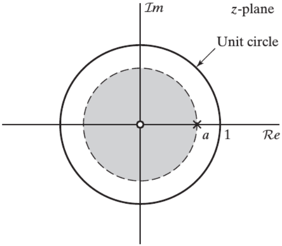
Using infinite geometric series, the equality only holds if $|0.5^{-1}\, z| < 1$, which can be rewritten in terms of z as $|z| > 0.5$. Thus, the ROC is $|z| > 0.5$. Generally, there will be one zero at $z = 0$, and one pole at $z = a$.

Anti causal ROC

Let $x[n] = -0.5^n u[-n-1] = \{\cdots, -0.5^{-3}, -0.5^{-2}, -0.5^{-1}, 0, 0, 0, \cdots\}$.

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} 0.5^n z^{-n} = -\sum_{m=1}^{\infty} \left(\frac{z}{0.5}\right)^m = -\frac{0.5^{-1}z}{1-0.5^{-1}z}$$

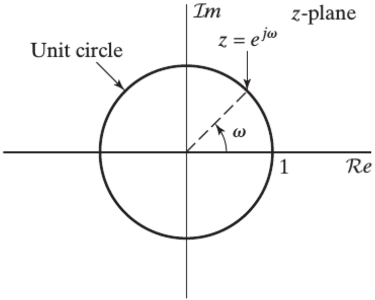
$$= -\frac{1}{0.5^{-1}z-1} = \frac{1}{1-0.5z^{-1}} < \infty$$



Using infinite geometric series, the equality only holds if $|0.5^{-1}\,z| < 1$, which can be rewritten in terms of z as $|z| < 0.5$. Thus, the ROC is $|z| < 0.5$. This is intentional to demonstrate that the transform result alone is insufficient.

Stability, Causality, and the ROC

- Z-transform X(z) of x[n] is unique when and only when specifying the ROC.
- The stability of a system can also be determined by knowing the ROC alone.
- If the ROC contains the unit circle (i.e., $|z| = 1$) then the system is stable.
- If a system is causal, then the ROC must contain infinity and the system function will be a right-sided sequence.
- If you need both, stability and causality, all the poles of the system function must be inside the unit circle.



Common z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

The z-transform of $x[n] = a_1 x_1[n] + a_2 x_2[n]$ is,

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} (a_1\, x_1[n] + a_2\, x_2[n]) z^{-n}\,, && (10)\\
&= a_1 \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n}\,, && (11)\\
&= a_1 X_1(z) + a_2 X_2(z)\,. && (12)
\end{aligned}
$$

Properties: Shift in time

The z-transform of $x[n-n_0]$ is,

$$X(x[n-n_0]) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}\,. \tag{13}$$

Let $m = n - n_0$,

$$\begin{aligned} X(x[n-n_0]) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}\,, && (14)\\ &= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m}\,, && (15)\\ &= z^{-n_0}X(z)\,. && (16)\end{aligned}$$

This leads directly to a property analogous to Eq. 2,

$$\mathcal{Z}\{x[n-1]\} = z^{-1}X(z)\,. \tag{17}$$

Properties: Convolution

The z-transform of

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\,, \tag{18}$$

is,

$$X(z) = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right]z^{-n}\,, \tag{19}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}\,, \tag{20}$$

Let $m = n - k$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{m=-\infty}^{\infty} x_2[m]z^{-(m+k)}\,, \tag{21}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{m=-\infty}^{\infty} x_2[m]z^{-m}\,, \tag{22}$$

$$= X_1(z)\, X_2(z)\,. \tag{23}$$

Constant-Coefficient Difference Equations

Consider a system described by the linear constant-coefficient difference equation,

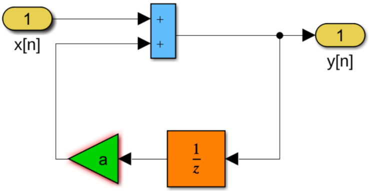
$$y[n] = x[n] + a\,y[n-1]\,, \tag{24}$$

$$y[n] - a\,y[n-1] = x[n]\,, \tag{25}$$

$$Y(z) - a\,Y(z)\,z^{-1} = X(z)\,, \tag{26}$$

$$Y(z)(1 - a\,z^{-1}) = X(z)\,, \tag{27}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - a\,z^{-1})} \tag{28}$$



In general,

$$\sum_{k=0}^{N} a_k\,y[n-k] = \sum_{k=0}^{M} b_k\,x[n-k]\,, \tag{29}$$

$$y[n] = \sum_{k=0}^{M} b_k\,x[n-k] - \sum_{k=1}^{N} a_k\,y[n-k]\,, \tag{30}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1\,z^{-1} + b_2\,z^{-2} + \ldots + b_M\,z^{-(M)}}{1 + a_1\,z^{-1} + a_2\,z^{-2} + \ldots + a_N\,z^{-(N)}}\,. \tag{31}$$

1 Wikipedia. Z-transform.

2 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed*. Prentice Hall. 2010. Chapter 3.

3 Paolo Prandoni and Martin Vetterli. *Signal processing for communications*. Taylor and Francis Group, LLC. 2008. Chapter 6.