The z transform

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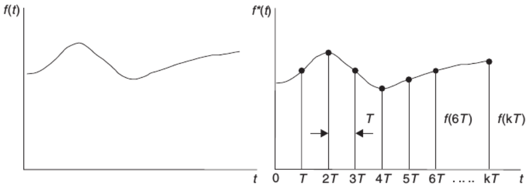
Resumen

Introduction

- The z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- It is analogous to the Laplace transform for continuous systems
- Conceptually, the symbol z can be associated with discrete time shifting in a difference equation in the same way that s can be associated with differentiation in a differential equation.

Definition



(a) Continuous Signal (b) Sampled Signal

Taking Laplace transforms of an ideal sampled signal gives,

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t - kT)\,, \tag{1}$$

$$F(s) = \mathscr{L}\{f^*(t)\} = \sum_{k=0}^{\infty} f^*(t)e^{-kTs}\,, \tag{2}$$

$$F(s) = \sum_{k=0}^{\infty} f^*(t)(e^{Ts})^{-k}\,, \tag{3}$$

Definition II

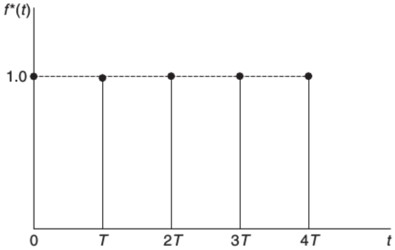
Define z as,

$$z = e^{Ts}\,, \tag{4}$$

$$F(z) = \sum_{k=0}^{\infty} f^*(t) z^{-k} = Z[f^*(t)]\,. \tag{5}$$

Example

Find the z-transform of the unit step function $f(t) = 1$.



$$Z[1(t)] = \sum_{k=0}^{\infty} 1(kT)z^{-k} , \tag{6}$$

$$F(z) = z^0 + z^{-1} + z^{-2} + z^{-3} + ... + +z^{-k} . \tag{7}$$

Equation 7 can be written in closed-form as,

$$Z[1(t)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}} . \tag{8}$$

Properties: Linearity

The z-transform of $x(n) = a_1 x_1(n) + a_2 x_2(n)$ is,

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} (a_1 x_1(n) + a_2 x_2(n)) z^{-n}\,, && (9) \\ &= a_1 \sum_{n=0}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=0}^{\infty} x_2(n) z^{-n}\,, && (10) \\ &= a_1 X_1(z) + a_2 X_2(z)\,. && (11) \end{aligned}$$

Properties: Shift in time

The z-transform of $x(n - n_0)$ is,

$$X(x(n - n_0)) = \sum_{n=n_0}^{\infty} x(n - n_0) z^{-n} \,. \tag{12}$$

Let $m = n - n_0$,

$$X(x(n - n_0)) = \sum_{m=0}^{\infty} x(m) z^{-(m+n_0)} \,, \tag{13}$$

$$= z^{-n_0} \sum_{m=0}^{\infty} x(m) z^{-m} \,, \tag{14}$$

$$= z^{-n_0} X(z) \,. \tag{15}$$

Properties: Convolution

The z-transform of

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\,, \tag{16}$$

is,

$$X(z) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} x_1(k)x_2(n-k) \right] z^{-n}\,, \tag{17}$$

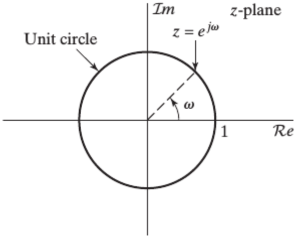
$$= \sum_{k=0}^{\infty} x_1(k) \sum_{n=0}^{\infty} x_2(n-k)z^{-n}\,, \tag{18}$$

Let $n-k=m$

$$X(z) = \sum_{k=0}^{\infty} x_1(k) \sum_{n=0}^{\infty} x_2(m)z^{-(m+k)}\,, \tag{19}$$

$$= \sum_{k=0}^{\infty} x_1(k)z^{-k} \sum_{n=0}^{\infty} x_2(m)z^{-m}\,, \tag{20}$$

$$= X_1(z)\,X_2(z)\,. \tag{21}$$



More generally, z can be expressed as a complex variable in polar form as,

$$z = r\ e^{j\omega}\ . \tag{22}$$

Then,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(n)(re^{j\omega})^{-n}\,, \tag{23}$$

$$X(re^{-j\omega n}) = \sum_{n=0}^{\infty} x(n) r^{-n} e^{-j\omega n}\,. \tag{24}$$

For $r = 1$ Eq. 24 reduces to the Fourier transform.

Region of Convergence II

For any given sequence, the set of values of z for which the z-transform power series converges is called the region of convergence (ROC), of the z-transform.

This criterion leads to the condition,

$$|X(re^{-j\omega n})| \leq \sum_{n=0}^{\infty} |x(n)r^{-n}| < \infty \,. \tag{25}$$

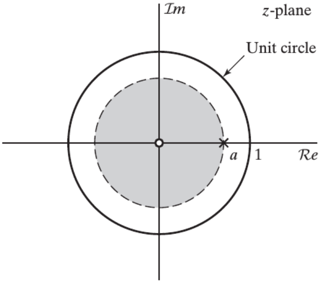
For example, what is the ROC of the step signal from Eq. 8?,

$$Z[1(t)] = \frac{z}{z-1} = z^{0} + z^{-1} + z^{-2} + \ldots + z^{-k} \,. \tag{26}$$

$$|z| > 1 \,. \tag{27}$$

What is the ROC of this step signal?,

$$Z[a\,1(t)] = \frac{z}{z-a}\,. \tag{28}$$



Common Laplace and z-transforms

	$f(t)$ or $f(kT)$	$F(s)$	$F(z)$
1	$\delta(t)$	1	1
2	$\delta(t-kT)$	e^{-kTs}	z^{-k}
3	$1(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
4	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
5	e^{-at}	$\frac{1}{(s+a)}$	$\frac{z}{z-e^{-aT}}$
6	$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
7	$\frac{1}{a}(at-1+e^{-at})$	$\frac{a}{s^2(s+a)}$	$\frac{z\{(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})\}}{a(z-1)^2(z-e^{-aT})}$
8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
9	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
10	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
11	$e^{-at}\cos\omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

Difference equation method

Consider a system of the form

$$\frac{X_o}{X_i}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots}$$

Thus

$$(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots)X_o(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots)X_i(z)$$

or

$$X_o(z) = (-a_1 z^{-1} - a_2 z^{-2} - \cdots)X_o(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots)X_i(z)$$

Equation (7.21) can be expressed as a difference equation of the form

$$\begin{aligned} x_o(kT) = &-a_1 x_o(k-1)T - a_2 x_o(k-2)T - \cdots \\ &+ b_0 x_i(kT) + b_1 x_i(k-1)T + b_2 x_i(k-2)T + \cdots \end{aligned}$$

Difference equation method II

$$\frac{X_o}{X_i}(s) = \frac{1}{1+s}$$
$$= \frac{z}{z - e^{-T}} = \frac{z}{z - 0.368}$$

Equation (7.23) can be written as

$$\frac{X_o}{X_i}(z) = \frac{1}{1 - 0.368z^{-1}}$$

Equation (7.24) is in the same form as equation (7.19). Hence

$$(1 - 0.368z^{-1})X_o(z) = X_i(z)$$

or

$$X_o(z) = 0.368z^{-1}X_o(z) + X_i(z)$$

Equation (7.25) can be expressed as a difference equation

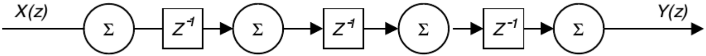
$$x_o(kT) = 0.368x_o(k-1)T + x_i(kT)$$

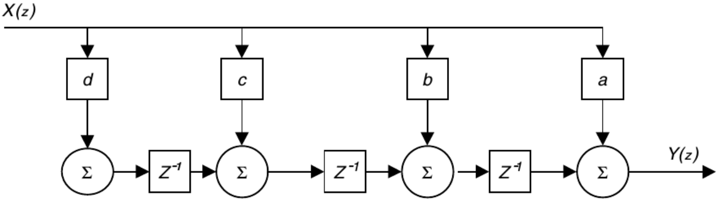
Assume that $x_o(-1) = 0$ and $x_i(kT) = 1$, then from equation (7.26)

$$x_o(0) = 0 + 1 = 1, \quad k = 0$$
$$x_o(1) = (0.368 \times 1) + 1 = 1.368, \quad k = 1$$
$$x_o(2) = (0.368 \times 1.368) + 1 = 1.503, \quad k = 2 \quad \text{etc.}$$

Consider the general third-order transfer function

$$\frac{Y(z)}{X(z)} = \frac{az^3 + bz^2 + cz + d}{z^3 + ez^2 + fz + g}$$





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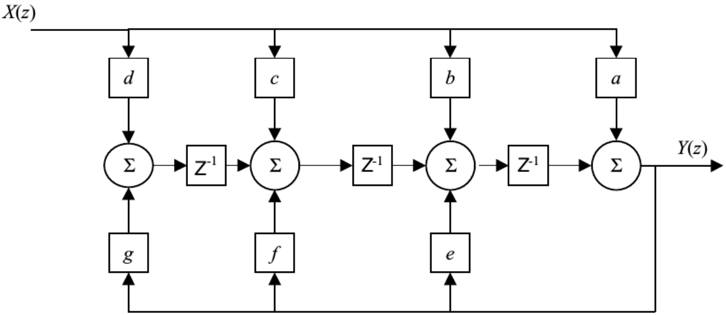

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