

# Infinite impulse response filters

## IIR filtering

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## 4 FIR vs IIR

Table: Classification of discrete filters

	Finite impulse response (FIR)	Infinite impulse response (IIR)
Filtering in time domain	Moving average	Leaky Integrator
Filtering in frequency domain	Windowed Filters Equiripple Minimax	Bilinear z-transform

The MA filter equation,

$$y[n] = x[n] * h[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k], \quad (1)$$

$$y[n] = \frac{1}{M} \left[ \sum_{k=1}^{M-1} x[n-k] + x[n] \right]. \quad (2)$$

Since,

$$y[n-1] = \frac{1}{M-1} \left[ \sum_{k=1}^{M-1} x[n-k] \right]. \quad (3)$$

Then,

$$y[n] = \frac{1}{M} x[n] + \frac{M-1}{M} y[n-1]. \quad (4)$$

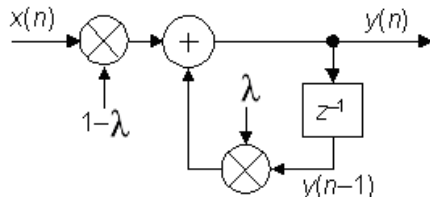
Defining  $\lambda = \frac{M-1}{M}$ ,

$$y[n] = \lambda y[n-1] + (1 - \lambda) x[n]. \quad (5)$$

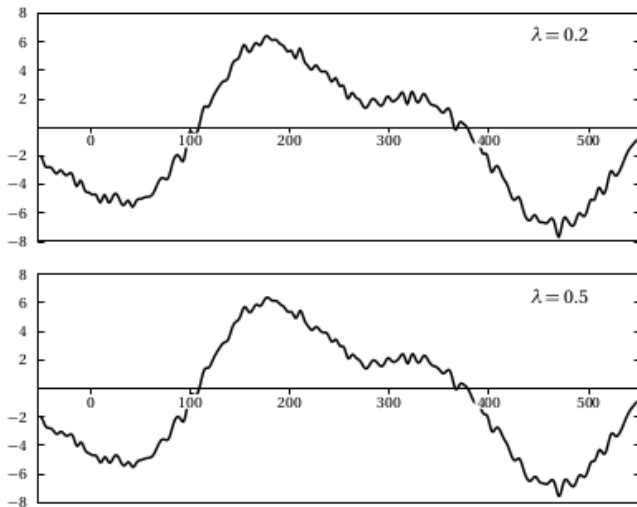
It can be seen that the leaky integrator filter is an IIR filter. Why?

$$y[n] = \lambda y[n - 1] + (1 - \lambda) x[n].$$

- No longer a convolution.
- Instead, a *constant coefficient difference equation*. Initial conditions must be set.
- The new system is LTI [2].
- System is stable for  $|\lambda| < 1$ .
- The value of  $\lambda$  (which is the pole of the system) determines the smoothing power of the filter.

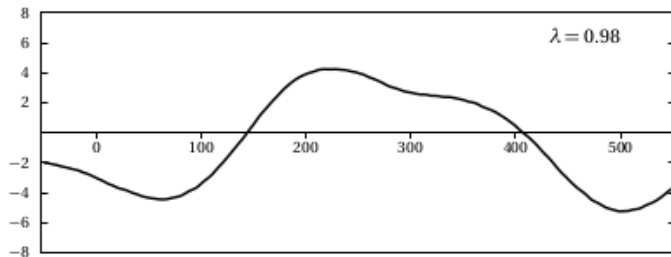
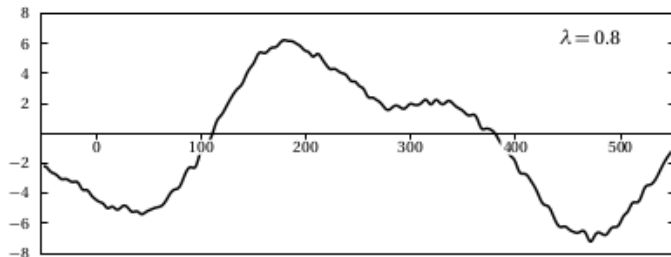


# Noise Reduction



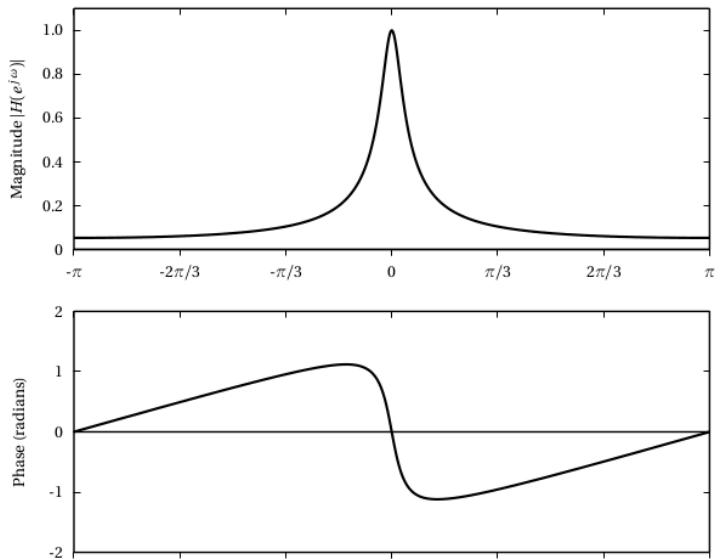
# Noise Reduction

- Note how the signal is delayed as  $\lambda$  grows.



# Frequency Response

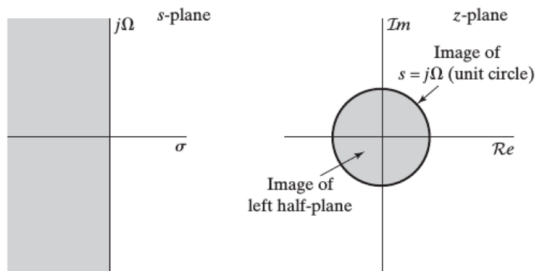
Magnitude and phase response of the leaky integrator for  $\lambda = 0.9$ .





# IIR filtering in frequency domain

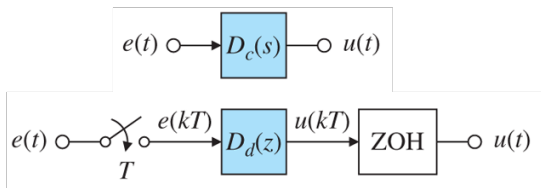
- The main idea is to transform an analog filter to the discrete domain.
- From  $s$  domain to  $z$  domain.
- This way, all the theory behind analog filter can be reused to implement a filter in a computer.



## Zero-Order Hold (ZOH) Method

- ZOH Method assumes that the input signal remains constant throughout the sample period.
- Given an input,  $e(k)$ , the system is essentially responding to a positive step at  $k$ , followed by a negative step one cycle delayed.
- In other words, one input sample produces a square pulse of height,  $e$ .

$$\frac{U(z)}{E(z)} = \mathcal{Z}\{D_c(s)\} - z^{-1} \mathcal{Z}\{D_c(s)\} = (1 - z^{-1}) \mathcal{Z}\{D_c(s)\}. \quad (6)$$



## Bilinear transform (Tustin's Method)

Suppose the following integrator,

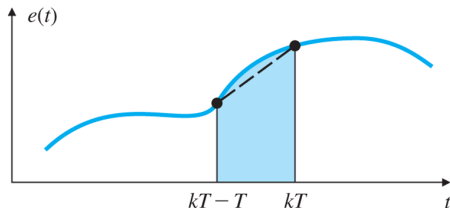
$$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}. \quad (7)$$

Therefore,

$$u(kT) = \int_0^{kT-T} e(t)dt + \int_{kT-T}^T e(t)dt, \quad (8)$$

$$u(kT) = u(kT - T) + \frac{T}{2} [e(kT - T) + e(kT)], \quad (9)$$

$$\Rightarrow \frac{U(z)}{E(z)} = \frac{T}{2} \left( \frac{1+z^{-1}}{1-z^{-1}} \right) = \frac{1}{\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}. \quad (10)$$



Comparing Eq. 7 and 10,

$$s = \frac{2}{T} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right). \quad (11)$$

The technique is an algebraic transformation between variables  $s$  and  $z$ .

Doing  $s = j\Omega$ , where  $\Omega$  is the analog frequency,  $-\infty, < \Omega < \infty$ ,

$$z = \frac{1 + (T/2)j\Omega}{1 - (T/2)j\Omega}. \quad (12)$$

The relationship between  $\Omega$  and  $\omega$ , the "digital" frequency,  $-\pi, < \omega < \pi$ , can be found by replacing  $z = e^{j\omega}$  in Eq. 11,

$$s = \frac{2}{T} \left( \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \right) = \sigma + j\Omega = \frac{2}{T} \left[ \frac{2e^{-j\omega/2}(j \sin \omega/2)}{2e^{-j\omega/2}(\cos \omega/2)} \right] = j \frac{2}{T_d} \tan(\omega/2). \quad (13)$$

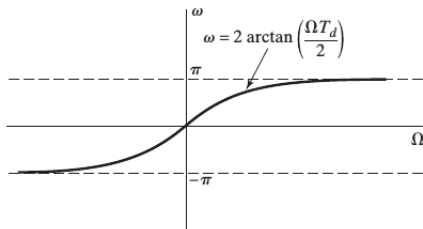
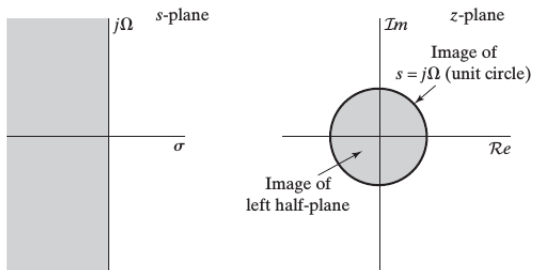
Real and imaginary parts on both sides of Eq. 13 are,

$$\sigma = 0, \quad (14)$$

$$\Omega = \frac{2}{T} \tan(\omega/2), \quad (15)$$

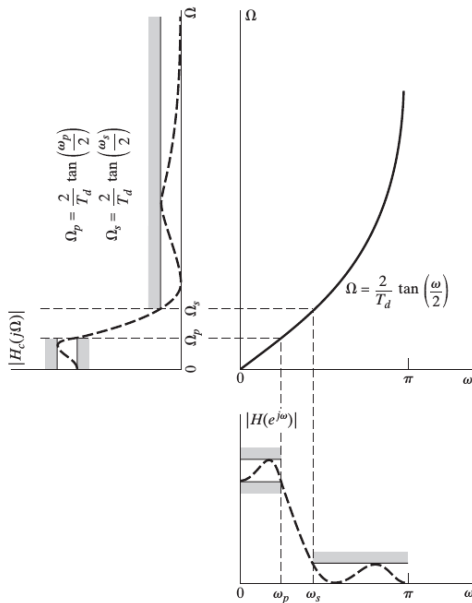
$$\omega = \arctan(\Omega T/2). \quad (16)$$

# Map from $s$ to $z$



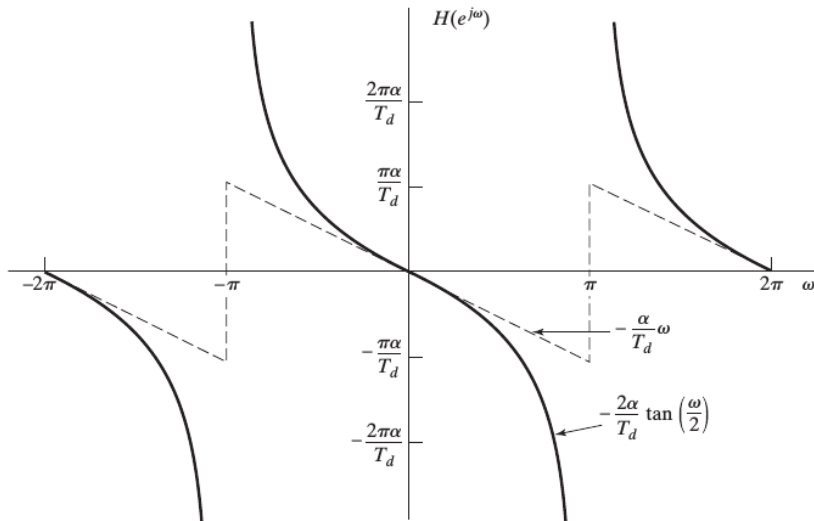
# Bilinear transform, Frequency Warping

- Non-linear compression of the frequency axis.
- The design of discrete-time filters using the bilinear transformation is useful only when this compression can be tolerated or compensated for.



## Bilinear transform, Phase response

Suppose a continuous-time filter with linear phase response. The nonlinear warping of the frequency axis introduced by the bilinear transformation will not preserve linearity in phase response.



## Example of IIR design using bilinear transform [3]

Design a digital filter equivalent of a 2<sup>nd</sup> order Butterworth low-pass filter with a cut-off frequency  $f_c = 100$  Hz and a sampling frequency  $f_s = 1000$  samples/sec. Derive the finite difference equation and draw the realisation structure of the filter. Given that the analogue prototype of the frequency-domain transfer function  $H(s)$  for a Butterworth filter is:

$$H(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1}$$

The normalised cut-off frequency of the digital filter is given by the following equation:

$$\Omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi 100}{1000} = 0.628$$

Now determine the equivalent analogue filter cut-off frequency  $\omega_{ac}$ , using the pre-warping function of Equation 5.9. The value of  $K$  is immaterial so let  $K = 1$ .

$$\omega_{ac} = K \cdot \tan\left(\frac{\Omega_c}{2}\right) = 1 \cdot \tan\left(\frac{0.628}{2}\right)$$

$$\omega_{ac} = 0.325 \text{ rads/sec}$$



## Example of IIR design using bilinear transform [3] II

Now denormalise the frequency-domain transfer function  $H(s)$  of the Butterworth filter, with the corresponding low-pass to low-pass frequency transformation of Equation 5.10. Hence the transfer function of the Butterworth filter becomes:

$$H(s) = \frac{1}{\left[\frac{s}{0.325}\right]^2 + \sqrt{2} \cdot \left[\frac{s}{0.325}\right] + 1}$$

Next, convert the analogue filter into an equivalent digital filter by applying the bilinear z-transform. This is achieved by making a substitution for  $s$  in the transfer function.

$$s = \frac{z-1}{z+1} \equiv \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{1}{\frac{1}{0.325^2} \cdot \left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + \frac{\sqrt{2}}{0.325} \cdot \left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 1}$$

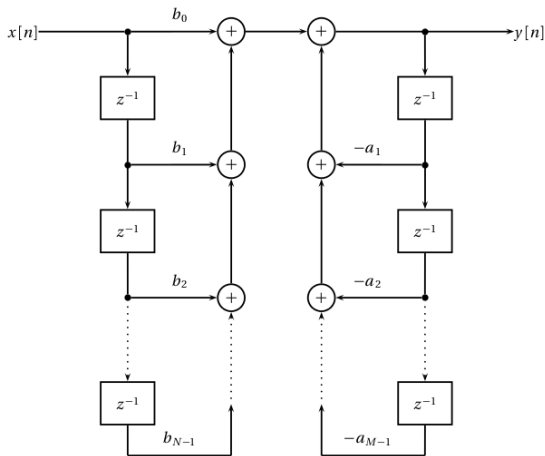
$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.067 + 0.135z^{-1} + 0.067z^{-2}}{1 - 1.1429z^{-1} + 0.4127z^{-2}}$$

The finite difference equation of the filter is found by inverting the transfer function.

$$y(n) = 1.1429y(n-1) - 0.4127y(n-2) + 0.067x(n) + 0.135x(n-1) + 0.067x(n-2)$$

# Direct form I IIR implementation

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{N-1} z^{N-1}}{1 + a_1 z^{-1} + \dots + a_{M-1} z^{M-1}}$$

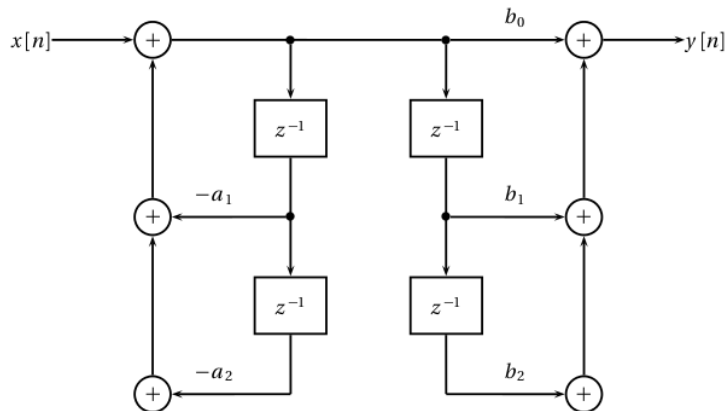


**Figure 7.24** Direct Form implementation of an IIR filter.

# Direct form I IIR implementation inverted

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

By the commutative properties of the z-transform, we can invert the order of computation to turn the Direct Form I structure into a new structure.

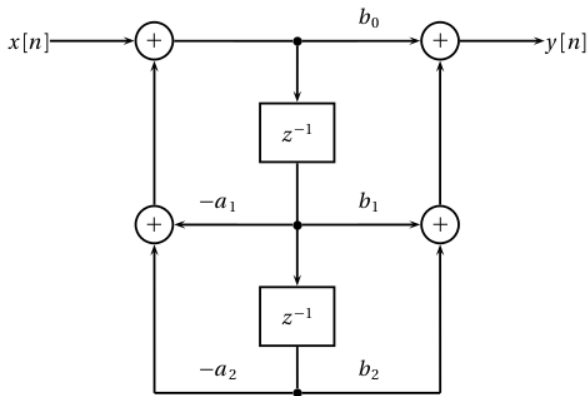


**Figure 7.25** Direct form I with inverted order.

## Direct form II IIR implementation

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

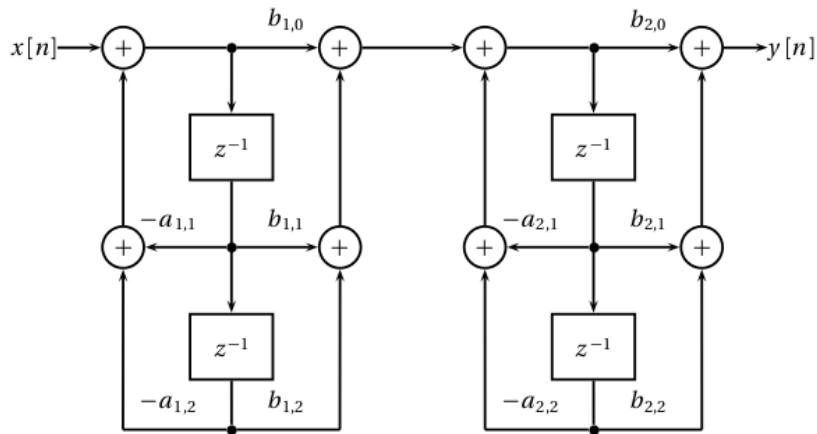
We can then combine the parallel delays together. This implementation is called Direct Form II; its obvious advantage is the reduced number of the required delay elements (hence of memory storage).



**Figure 7.26** Direct Form II implementation of a second-order section.

## IIR cascade implementation

The cascade structure of  $N$  second-order sections is much less sensitive to quantization than the previous Direct form II of order  $2 \cdot N$ .



**Figure 7.27** 4th order IIR: cascade implementation.

FIR, pros:

- Unconditional stability (no poles).
- Precise control of the phase response and, in particular, exact linear phase.
- Optimal algorithmic design procedures.
- Robustness with respect to finite numerical precision hardware.

FIR, cons:

- Longer input-output delay.
- Higher computational cost with respect to IIR solutions.

IIR, pros:

- Lower computational cost with respect to an FIR with similar behavior.
- Shorter input-output delay.
- Compact representation.

IIR, cons:

- Stability is not guaranteed.
- Phase response is difficult to control.
- Design is complex in the general case.
- Sensitive to numerical precision.

- 1 Alan V. Oppenheim and Ronald W. Schaffer. *Discrete-time signal processing*, 3rd Ed. Prentice Hall. 2010. Sections 7.2 and 7.3.
- 2 Paolo Prandoni and Martin Vetterli. *Signal processing for communications*. Taylor and Francis Group, LLC. 2008. Sections 5.3.2, 7.3, and 7.4.2.
- 3 Oliver Hinton. Digital Signal Processing Resources for EEE305 Course. Chapter 5. [www.staff.ncl.ac.uk/oliver.hinton/eee305/](http://www.staff.ncl.ac.uk/oliver.hinton/eee305/)
- 4 Gene F. Franklin, J. David Powell and Abbas Emami-Naeini. *Feedback Control of Dynamic Systems*. 7th Edition. 2014. Section 8.3.