

The z transform

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Control y Sistemas

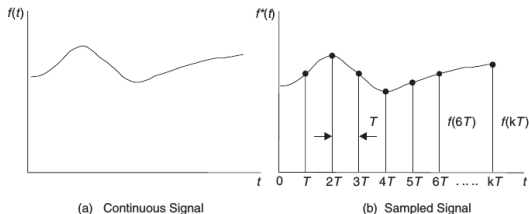
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- The z-transform is the principal analytical tool for single-input single-output discrete-time systems.
- It is analogous to the Laplace transform for continuous systems.



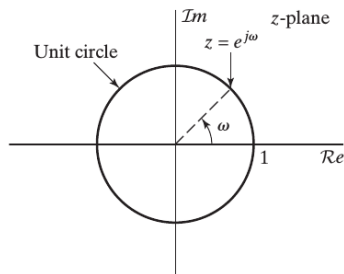
Taking z-transform of an ideal sampled signal gives,

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z), \quad (1)$$

$$x[n] \longleftrightarrow^Z X(z), \quad (2)$$

$$z = r e^{j\omega}, \quad (3)$$

$$X(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}, \quad (4)$$

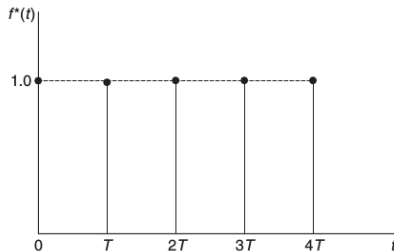


$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}.$$

- Equation 4 can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n} .
- For $r = 1$, Eq. 4 reduce to the Fourier transform of $x[n]$.
- Interpreting the Fourier transform as the z-transform on the unit circle in the z-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$.

Example

Find the z-transform of the unit step function $u(t) = 1$.



$$Z[u(t)] = \sum_{k=0}^{\infty} 1(kT)z^{-k}, \quad (5)$$

$$F(z) = z^0 + z^{-1} + z^{-2} + z^{-3} + \dots + z^{-k}. \quad (6)$$

Equation 6 can be written in closed-form as,

$$Z[1(t)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}}. \quad (7)$$

The convergence of the following power series for a given sequence depends only on $|z|$.

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z).$$

Since $|X(z)| < \infty$,

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty. \quad (8)$$

For example, the sequence $x[n] = u[n]$ (unit step) is not absolutely summable, and therefore, the Fourier transform power series does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with an ROC $|z| > 1$.

Common z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z > a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z > 0$

PROPERTY 1: The ROC will either be of the form $0 \leq r_R < |z|$, or $|z| < r_L \leq \infty$, or, in general the annulus, i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z -transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z -plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

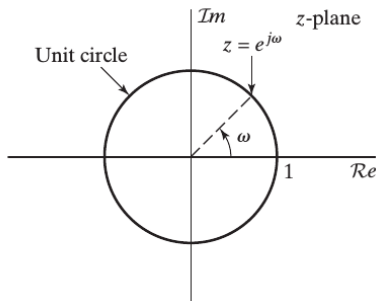
PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z -plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Stability, Causality, and the ROC

- The stability of a system can also be determined by knowing the ROC alone.
- If the ROC contains the unit circle (i.e., $|z| = 1$) then the system is stable.
- If you need a causal system then the ROC must contain infinity and the system function will be a right-sided sequence.
- If you need both, stability and causality, all the poles of the system function must be inside the unit circle.



The z-transform of $x[n] = a_1 x_1[n] + a_2 x_2[n]$ is,

$$X(z) = \sum_{n=-\infty}^{\infty} (a_1 x_1[n] + a_2 x_2[n]) z^{-n}, \quad (9)$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}, \quad (10)$$

$$= a_1 X_1(z) + a_2 X_2(z). \quad (11)$$

The z-transform of $x[n - n_0]$ is,

$$X(x[n - n_0]) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}. \quad (12)$$

Let $m = n - n_0$,

$$X(x[n - n_0]) = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}, \quad (13)$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m}, \quad (14)$$

$$= z^{-n_0} X(z). \quad (15)$$

The time-shifting property is often useful, in conjunction with other properties and procedures, for obtaining the inverse z-transform.

The z-transform of

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k], \quad (16)$$

is,

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] z^{-n}, \quad (17)$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}, \quad (18)$$

Let $m = n - k$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{m=-\infty}^{\infty} x_2[m]z^{-(m+k)}, \quad (19)$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{m=-\infty}^{\infty} x_2[m]z^{-m}, \quad (20)$$

$$= X_1(z) X_2(z). \quad (21)$$

Constant-Coefficient Difference Equations

Consider a system described by the linear constant-coefficient difference equation,

$$y[n] = x[n] + a y[n - 1], \quad (22)$$

$$y[n] - a y[n - 1] = x[n], \quad (23)$$

$$Y(z) - a Y(z) z^{-1} = X(z), \quad (24)$$

$$Y(z)(1 - a z^{-1}) = X(z), \quad (25)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - a z^{-1})} \quad (26)$$

In general,

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k], \quad (27)$$

$$y[n] = \sum_{k=0}^M b_k x[n - k] - \sum_{k=1}^N a_k y[n - k], \quad (28)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-(M)}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-(N)}}. \quad (29)$$

- 1 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed.* Prentice Hall. 2010. Chapter 3.
- 2 Paolo Prandoni and Martin Vetterli. *Signal processing for communications.* Taylor and Francis Group, LLC. 2008. Chapter 6.