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Abstract

We study the optimal design of round-robin tournaments with three symmetric players. We characterize the subgame perfect equilibrium in these tournaments with either one or two prizes. Our results show that the players who wish to maximize their expected payoffs or their probabilities of winning have different preferences about the order of games under tournaments with one or two prizes. We analyze the optimal allocations of players for a designer who wishes to maximize the players' expected total effort in the tournaments with one and two prizes, and by comparing between them, it is demonstrated that in order to maximize the players' expected total effort the designer should allocate only one prize.

Keywords

Multi-stage contests, all-pay auctions, first-mover advantage, second-mover advantage, round-robin tournaments.

JEL Classification

D00, L00, D20, Z20, D44, O31.

1 Introduction

Recently, FIFA (the international governing body of football) came to the decision that starting from the 2026 World Cup, there will be 48 teams in the first round that will be divided into groups of three teams. In each group, the three teams will compete in a round-robin tournament and the best two teams from each group will qualify for the next round. This structure raises several questions: Does the order of the games (rounds) in each group affect the outcomes? If the designer wishes to maximize the teams' expected total effort, what is the optimal order of games for each group? Should the order of games be random or not, namely, should the allocation of teams in each round depend on the outcomes of the previous rounds? If the designer wishes to maximize the length of the tournament, namely, to minimize the probability that the tournament will be decided before the last round, what is then the optimal order of games for each group? Finally, how are the above issues approached if only one team would qualify for the next stage instead of two teams?

In this paper, we respond to these and other related issues by analyzing the optimal design of the round-robin tournament with three symmetric players. We study round-robin tournaments with three symmetric players, in which each player competes against all the other players in sequential pairwise games, where each game is modelled as an all-pay auction.[1] We consider two cases in which either one or two prizes are awarded. Each of these cases is divided into three subcases: in the first the players are randomly allocated throughout the rounds; in the second, the winner of the first round competes in the second round; and in the third, the winner of the first round competes in the third round.

We find that there is a significant difference between tournaments with one and two prizes. In tournaments with two prizes there is a second mover advantage, namely, a player who does not compete in the first round has the highest expected payoff as well as the highest probability of winning. In tournaments with a single prize, on the other hand, a player who does not compete in the first round has the lowest expected payoff as well the lowest probability of winning.

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[1] Applications of the all-pay contest have been made to rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, military and biological wars of attrition (see, for example, Hillman and Riley 1989, Baye, Kovenock and de Vries 1993, Amman and Leininger 1996, Krishna and Morgan 1997, Che and Gale 1998 and Siegel 2009).

The intuition behind these differences between the tournaments with one and two prizes is that in the tournament with two prizes, in a game between a player who already has one win and a player who has no wins, the latter has an advantage. The reason is that a player with one win has a high probability to be one of the winners of the tournament and therefore has a low incentive to compete in the ensuing games. For the player who has no wins this game is much more important, and therefore he wins with a high probability. On the other hand, in the tournament with one prize, in a game between a player who already has one win and a player who has no wins, the latter has a disadvantage. In that case, the player who already has a win leads, and unlike with two prizes, one win does not necessarily ensure winning the prize. Therefore, a player who already has a win has a higher value of winning and wins the second game with a higher probability than his opponent. The above argument demonstrates the advantage of not plying in the first game of the tournament with two prizes and the disadvantage of not playing in the first game of the tournament with only one prize.

We first assume that a contest designer wishes to maximize the expected total effort. Then, in the tournament with two prizes, the winner of the first round has to compete in the last round. On the other hand, in the tournament with one prize the winner of the first round has to compete in the second round. The intuition behind these results is that in the tournament with two prizes, it is preferable for the loser of the first round to compete in the second round since the winner of the first round is almost surely one of the winners of the tournament and therefore has a low incentive to compete in that round. On the other hand, in the tournament with a single prize, the winner of the first round is not for certain the winner of the tournament, and therefore it is preferable that he competes in the second round. Then, by comparing the optimal expected efforts with one and two prizes we find that the optimal design of the round-robin tournament with three symmetric players that maximizes the players' expected total effort is for the entire prize sum to be allocated as a single prize.

Another possible goal of the contest designer could be to maximize the length of the tournament, namely, to maximize the probability that the tournament will be decided in the last round. When only one prize is awarded, then by definition, if the winner of the first round competes in the third round the tournament has to be decided in the last round. Similarly, when two prizes are awarded, if the winner of the first round competes in the second round the tournament has to be decided in the last round as well. Therefore, we obtain that the optimal design of the round-robin tournament with three symmetric players that maximizes the probability that the tournament will be decided in the last round is the tournament with either one or two prizes. However, by comparing the optimal designs of the tournaments when the designer wishes to maximize the players' expected total effort and the length of the tournament, we observe that the contest designer cannot simultaneously maximize the expected total effort as well as the length of the tournament.

In the literature on contests and, in particular, on all-pay auctions, similarly to our present work, several studies have dealt with that question of what the optimal number of prizes is. Moldovanu and Sela (2001) showed that in all-pay auctions under incomplete information when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single first prize, but when cost functions are convex, several positive prizes may be optimal. Later (2006) these authors studied a two-stage all-pay auction with multiple prizes under incomplete information and showed that for a contest designer who maximizes the expected total effort, if the cost functions are linear in effort, it is optimal to allocate the entire prize sum to a single first prize. In symmetric all-pay auctions under complete information, Barut and Kovenock (1998) showed that a revenue maximizing prize structure allows any combination of k-1 prizes, where k is the number of players. That is, the contest designer is indifferent to whether he should allocate one or several prizes. However, Cohen and Sela (2008) studied all-pay auctions under complete information with heterogeneous prizes, and showed that the allocation of several prizes may be optimal for a contest designer who maximizes the total effort.

We also analyze how to allocate the players in the round-robin tournaments. Similarly, Rosen (1986) studied how to allocate players in an elimination tournament where the probability of winning a match is a stochastic function of the players' efforts. He considered an example with four players who can be either "strong" or "weak" and found numerically that a random seeding yields a higher total effort than the seeding where strong players meet weak players in the semifinals. Groh et al. (2012) studied an elimination all-pay auction with heterogeneous players whose ability is common knowledge. For tournaments with four players, they found optimal seedings for several criteria.

The literature on round-robin tournaments is quite scarce. Similarly to the recent work, Krumer, Megidish and Sela (2017) studied round-robin tournaments with three players and a single prize where the allocation of players is random. They found that a player who competes in the first and the third rounds has the highest expected payoff as well as the highest probability to win the tournament. Based on real-world data from wrestling Olympic tournaments, these findings were empirically confirmed by Krumer and Lechner (2017). The current paper extends these previous theoretical works by studying an order of games that depends on the outcomes of the previous rounds and also investigates how different numbers of prizes affect other criteria such as the expected total effort and the length of the tournament.

We also find that in round-robin tournaments, the way the players are allocated has a strong effect on their expected payoffs as well as on their winning probabilities. Therefore, our study also is applicable to the literature on the issue of fairness in economic environments, whose importance was extensively discussed by the renowned economist Adam Smith (Ashraf, Camerer and Loewenstein 2005). The role of fairness in tournament settings was also shown theoretically by Gill and Stone (2010).

The rest of the paper is organized as follows: Section 2 introduces round-robin tournaments with two prizes and in Section 3 we characterize their subgame perfect equilibrium for three different allocations of players. Section 4 introduces round-robin tournaments with one prize and in Section 5 we characterize their subgame perfect equilibrium for three different allocations of players. In Section 6 we compare between the tournaments with one and two prizes. Section 7 concludes.

2 Round-robin tournaments with three symmetric players and two identical prizes

We consider a round-robin all-pay tournament with three symmetric players $i \in \{1,2,3\}$. In each round $r, r \in \{1,2,3\}$ there is a different pair-wise game, such that each player competes in two different rounds (games). The two players with the highest number of wins receive an equal prize. In a case that each player wins in one round, then each of them wins a prize with the same probability of $\frac{2}{3}$. If one of the players loses in the first two rounds, the winners of the tournament are then decided and the players in the last round do not exert any effort (zero effort). Each round is modelled as an all-pay auction. In each round, both players exert efforts and the player with the higher effort wins the respective game. Without loss of generality, we assume that player i's value of winning is $V = \frac{1}{2}$ and his cost function is $C(x_i) = x_i$, where x_i is his effort.

We begin the analysis by explaining how the players' strategies are calculated in each game of the tournament. Suppose that players i and j compete in round $r, r \in \{1,2,3\}$. We denote by p_{ij} the probability that player i wins the game against player j and E_i and E_j are the expected payoffs of players i and j respectively. The mixed strategies of the players in each round are denoted by$F_{kr}(x), k \in \{i, j\}$. In addition, we assume that player *i*'s continuation value if he wins in round *r* is w_{ir} given the previous and possible future outcomes. Similarly, we assume that player *i*'s continuation value if he loses in round *r* is l_{ir}, given the previous and possible future outcomes. Without loss of generality, we assume that $w_{ir} - l_{ir} > w_{jr} - l_{jr}$.

Then, according to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), there is always a unique mixed-strategy equilibrium in which players *i* and *j* randomize on the interval $[0, w_{jr} - l_{jr}]$, according to their effort cumulative distribution functions, which are given by

$$E_i = w_{ir}F_{jr}(x) + l_{ir}\left(1 - F_{jr}(x)\right) - x = l_{jr} + w_{ir} - w_{jr}$$

$$E_j = w_{jr}F_{ir}(x) + l_{jr}\left(1 - F_{ir}(x)\right) - x = l_{jr}$$

Thus, player *i*'s equilibrium effort in round *r* is uniformly distributed; that is

$$F_{ir}(x) = \frac{x}{w_{jr} - l_{jr}}$$

while player *j*'s equilibrium effort in round *r* is distributed according to the cumulative distribution function

$$F_{jr}(x) = \frac{l_{jr} - l_{ir} + w_{ir} - w_{jr} + x}{w_{ir} - l_{ir}}$$

Player *i*'s probability of winning against player *j* is then

$$p_{ij} = 1 - \frac{w_{jr} - l_{jr}}{2(w_{ir} - l_{ir})} > \frac{1}{2}$$

and the expected total effort in the corresponding game is

$$TE_{ij} = \left(\frac{w_{jr} - l_{jr}}{2}\right)\left(1 + \frac{w_{jr} - l_{jr}}{w_{ir} - l_{ir}}\right)$$

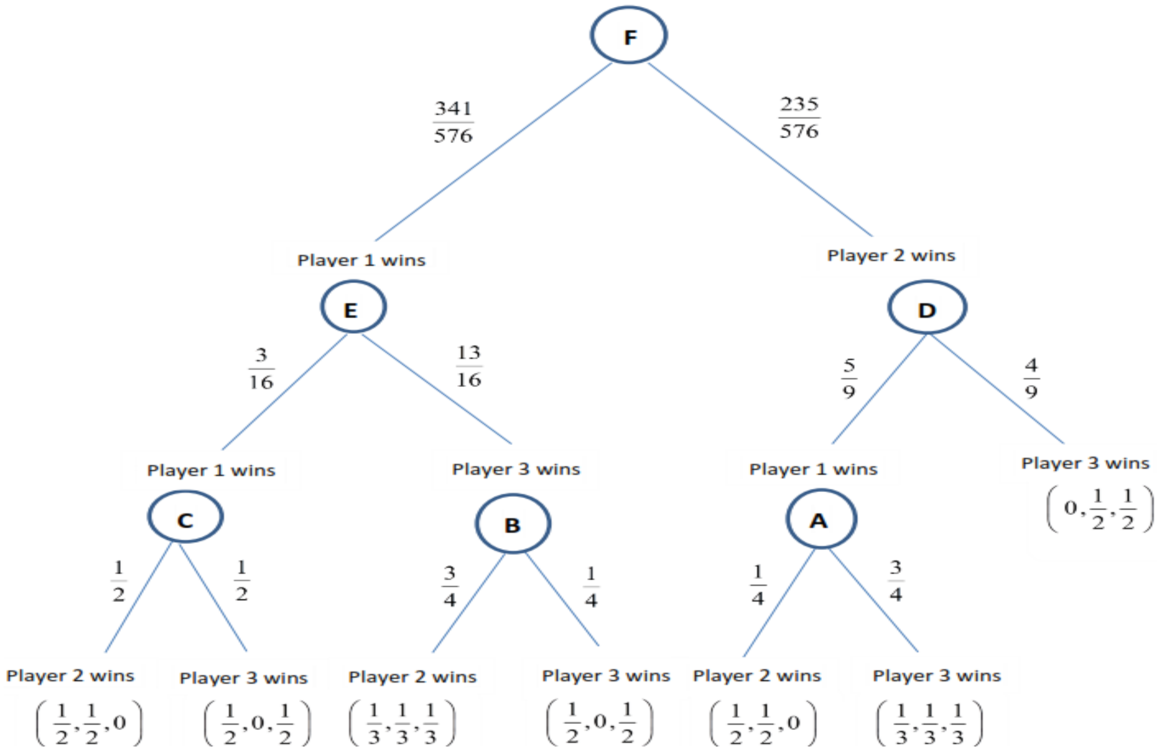
Below we describe three possible allocations of players in this tournament.

2.1 Case A: A random allocation of players

Figure 1 presents the symmetric round-robin tournament with two prizes as a game tree for case A where the order of games is random and is decided before the first game such that the players who compete in each round do not depend on the outcomes of the previous rounds. In the decision node *F*, players 1 and 2 compete in the first round, in the decision nodes *E* and *D*, players 1 and 3 compete in the second round, and in the decision nodes *A*, *B* and *C*, players

2 and 3 compete in the third round. For each decision node (*A*-*E*) there is a different path from the initial node *F*, namely, there is a different history of games in the previous rounds. The players' payoffs are indicated in the terminal nodes. The numbers on the sides of the branches in Figure 1 denote the winning probabilities of the players who compete in the appropriate decision nodes. These winning probabilities are explicitly analyzed in Appendix A.

Figure 1: Game tree for case A of the round-robin tournament with two prizes



2.2 Case B: The winner of the first round competes in the second round

Figure 2 presents the symmetric round-robin tournament with two prizes as a game tree for case B where the winner of the first round competes in the second round. In the decision node *G*, players 1 and 2 compete in the first round. If player 1 wins in the first round then in the decision node *F*, players 1 and 3 compete in the second round, and in the decision nodes *C* and *D*, players 2 and 3 compete in the third round. If player 2 wins in the first round, then in the decision node *E*, players 2 and 3 compete in the second round, and in the decision nodes *A* and *B* players 1 and 3 compete in the third round. The players' payoffs are indicated in the terminal nodes. The numbers on the sides of the branches in Figure 2 denote the winning probabilities of the players who compete in the appropriate decision nodes. These winning probabilities are analyzed in Appendix B.

Figure 2: Game tree for case B of the round-robin tournament with two prizes

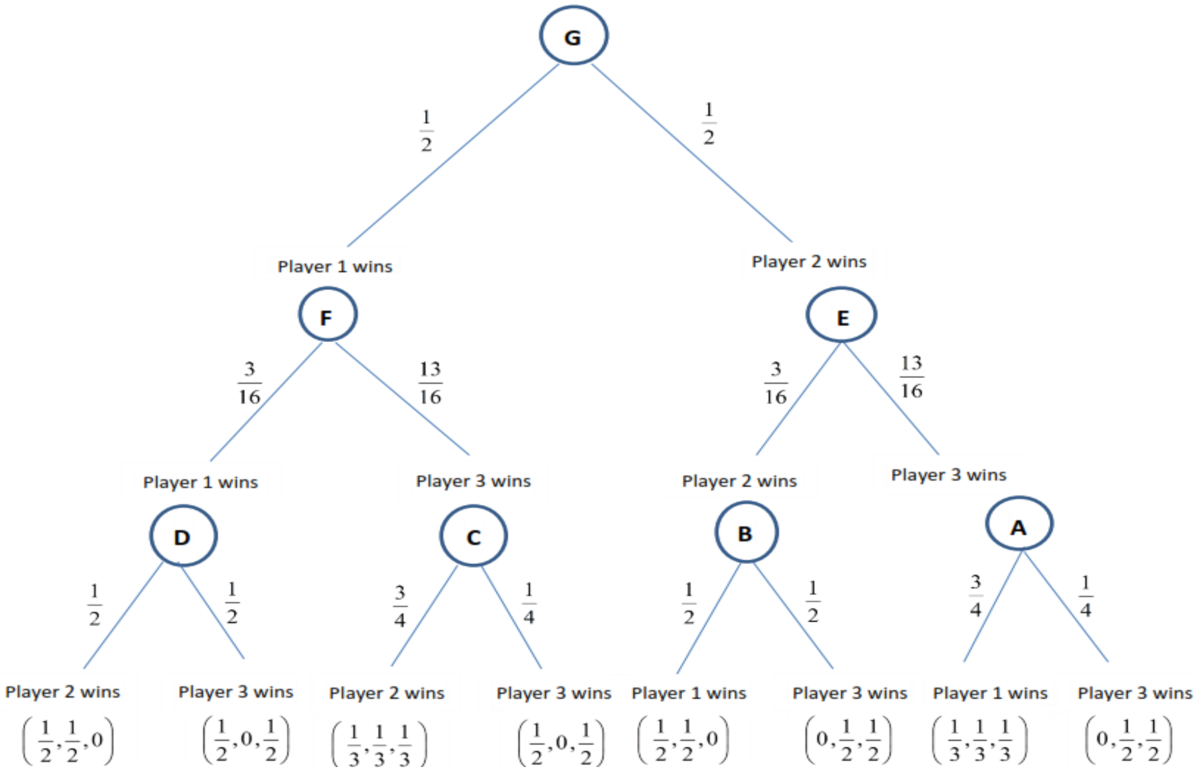
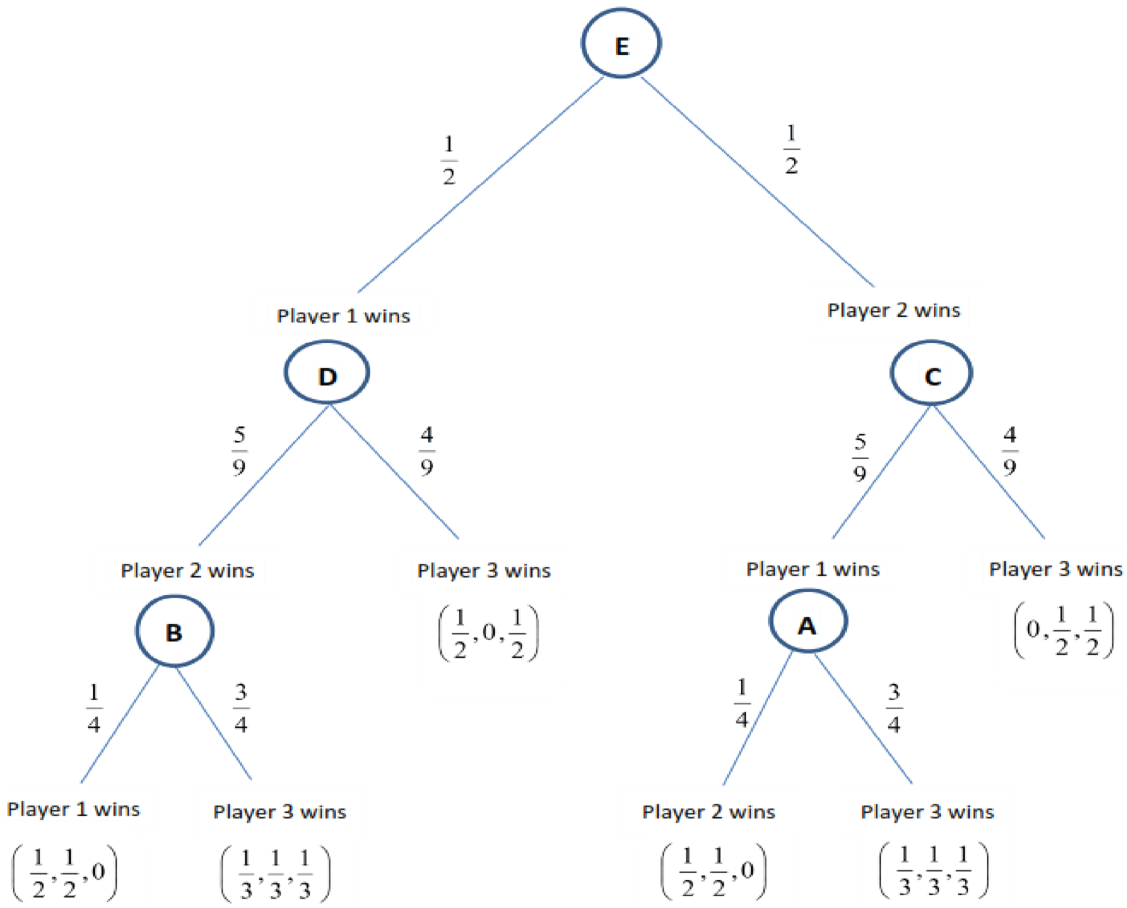


Figure 3: Game tree for case C of the round-robin tournament with two prizes



2.3 Case C: The winner of the first round competes in the third round

Figure 3 presents the symmetric round-robin tournament with two prizes as a game tree for case C, where the winner of the first round competes in the third round. In the decision node *E*, players 1 and 2 compete in the first round. If player 1 wins in the first round then in the decision node *D*, players 2 and 3 compete in the second round, and in the decision node *B*, players 1 and 3 compete in the third round. If player 2 wins in the first round, then in the decision node *C*, players 1 and 3 compete in the second round, and in the decision node *A*, players 2 and 3 compete in the third round. The players' payoffs are indicated in the terminal nodes. The numbers on the sides of the branches in Figure 3 denote the winning probabilities of the players who compete in the appropriate decision nodes. These winning probabilities are analyzed in Appendix C.

3 Equilibrium analysis of round-robin tournaments with three symmetric players and two identical prizes

3.1 The players' expected payoffs

Based on the analyses of the subgame perfect equilibrium in cases A, B and C that appear in appendices A, B and C, respectively, we obtain that the players' expected payoffs in the round-robin tournaments with two prizes are as follows:

Table 1: Comparison of players' expected payoffs in a tournament with two prizes

	Case A: A random allocation of players	**Case B:** The winner of the first round competes in the second round.	**Case C:** The winner of the first round competes in the third round
Player 1's expected payoff	0.103	0.135	0.042
Player 2's expected payoff	0.135	0.135	0.042
Player 3's expected payoff	0.19	0.208	0.166

We can conclude that

Proposition 1 *In the round-robin tournament with three symmetric players and two prizes, the player who competes in the last two rounds (player 3) has the highest expected payoff independent of whether the allocation of players is random or not.*

The intuition behind this result is that since there are two prizes, competing against a player who has already won once is an advantage. For example, if the winner of the first round competes in the second round, he already has a very high probability to be one of the winners. Therefore, the difference between his continuation values of a win and of a loss is relatively small. As such, a winner of the first round has no incentive to exert a high effort in the second round. However, for his opponent in the second round (player 3), this game is much more important and therefore he wins with a very high probability by exerting a relatively low effort. If, however, the loser of the first round competes in the second round, then the game is almost equally important for both players, since both players have zero wins at this point. Then, even if player 3 loses in the second round, he is still a favourite in the last round without exerting too much effort, since then he competes against a player who already won in the first round and therefore his incentive to exert a high effort is quite low. This implies that player 3, who does not compete in the first round, has the highest expected payoff.

3.2 The players' probabilities of winning

It is important to note that in absence of the effect of the order of rounds, each player would have the same probability of winning, which equals to $\frac{2}{3}$. However, based on the equilibrium analyses in Appendices A, B and C we obtain that the players' probabilities of winning in the round-robin tournaments with two prizes are as follows:

Table 2: Comparison of players' probabilities of winning in a tournament with two prizes

	Case A: A random allocation of players	**Case B:** The winner of the first round competes in the second round	**Case C:** The winner of the first round competes in the third round
Player 1's probability of winning	0.642	0.648	0.639
Player 2's probability of winning	0.647	0.648	0.639
Player 3's probability of winning	0.711	0.704	0.722

We can conclude that

Proposition 2 *In the round-robin tournament with three symmetric players and two prizes, the player who competes in the last two rounds (player 3) has the highest probability of winning independent of whether the allocation of players is random or not.*

The intuition behind this result is exactly the same as for the previous result when this player (player 3) has the highest expected payoff.

3.3 The players’ total effort

One of the possible goals of a contest designer is to maximize the players' expected total effort. Based on the equilibrium analyses in Appendices A, B and C, we obtain that if the allocation of players is random (case A) the players' expected total effort is 0.57, if the winner of the first round competes in the second round (case B), the expected total effort is 0.52, and if the winner of the first round competes in the third round (case C), the expected total effort is 0.75. Therefore, we can conclude that,

Proposition 3 *In the round-robin tournament with three symmetric players and two prizes, the expected total effort is maximized when the winner of the first round competes in the third round.*

The intuition behind Proposition 3 is as follows: The players' efforts in the first round are very similar in all the allocations of players, while the players' efforts in the third round are relatively small since this tournament has a high probability to be decided after two rounds. As such, the main difference among the players' total efforts in all the allocations of players occurs in the second round. Then, if player 3 competes in the second round against the loser of the first round, this round is almost equally important for both players since neither of them has a win and therefore they both exert a relatively high effort in the second round. On the other hand, if player 3 competes in the second round against the winner of the first round, both players exert relatively low efforts since for the winner of the first round this game is not so important as from his win in the first round he already has a high probability to win one of the two prizes. Hence, the expected total effort is maximized when the winner of the first round competes in the third one.

3.4 The length of the tournament

Another possible goal of the designer of round-robin tournaments is to minimize the probability that the tournament will be decided before the last round. In other words, the designer wants to maximize the length of the tournament. It is clear that the highest probability that the tournament will not be decided after two rounds occurs when the winner of the first round competes in the second round (case B) since then there is no chance that one of the players will have two losses after two rounds. If, however, the loser of the first round competes in the second round there is a relatively high probability that the loser of the first round will lose again and then the tournament will be decided before the third round. Therefore, the highest probability that the tournament will be decided before the last round occurs in case C. This intuition is confirmed by the analyses in Appendices A, B and C.

4 Round-robin tournaments with three symmetric players and one prize

This tournament differs from the one presented in Section 2 only by the number of prizes, namely, a player who wins two games wins the tournament, but if each player wins only once, each of them wins the tournament with the same probability of $\frac{1}{3}$. This means that if one of the players wins in the first two rounds, the winner of the tournament is then decided and the players in the last round do not exert any efforts (zero effort). Without loss of generality, we assume that player i's value of winning the tournament is $V = 1$. In this tournament as well as the one with two prizes, each round is modelled as an all-pay auction.

Similarly to the tournament with two prizes, we also consider here three possible allocations of players.

4.1 Case D: A random allocation of players

Figure 4 presents the symmetric round-robin tournament with one prize as a game tree for case D where the order of games is random such that the players who compete in each round do not depend on the outcomes in the previous rounds. Note that the structure of the game tree for this case is identical to Case A. The players' probabilities of winning as well as their expected payoffs have already been analyzed by Krumer et al. (2017) and are given in Appendix D.

4.2 Case E: The winner of the first round competes in the second round

Figure 5 presents the symmetric round-robin tournament with one prize as a game tree for case E where the winner of the first round competes in the second round. Note that the structure of the game tree for this case is identical to Case B but the players' expected payoffs and their probabilities of winning in both cases are different. The players' expected payoffs and their probabilities of winning in case E are analyzed in Appendix E.

Figure 4: Game tree for case D of the round-robin tournament with one prize

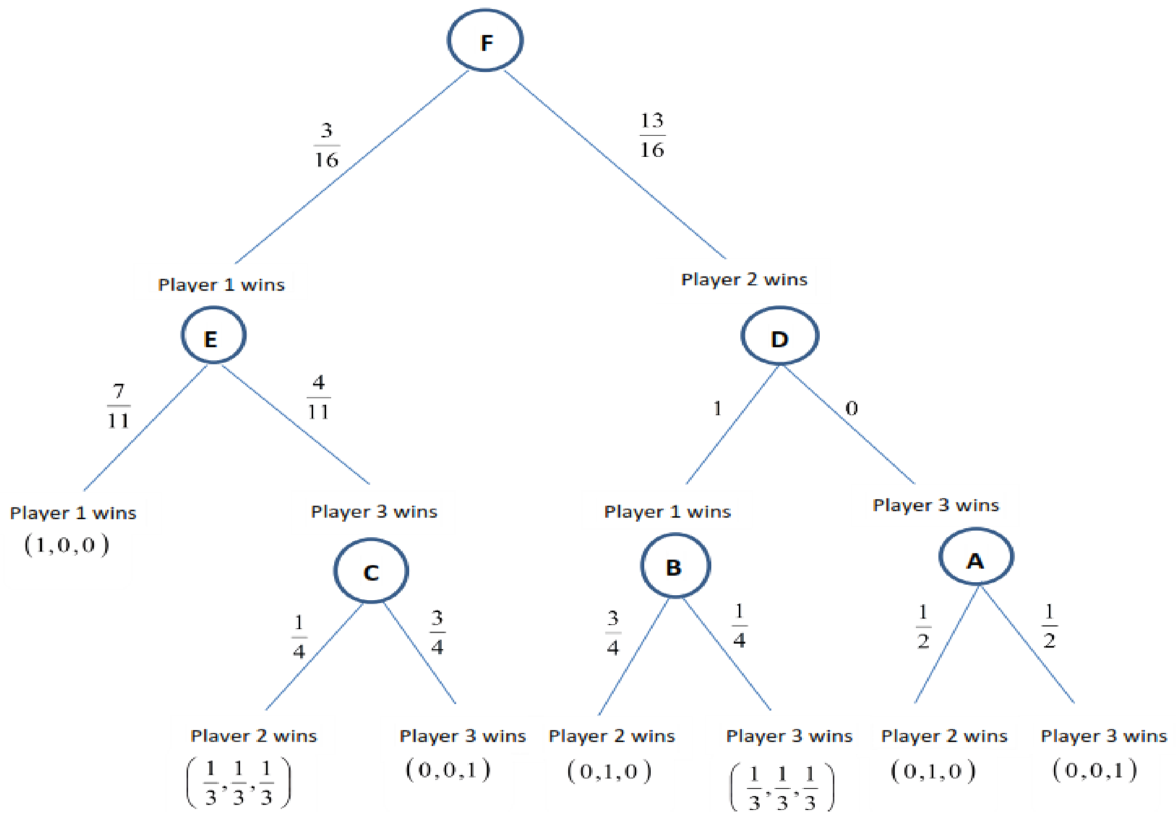
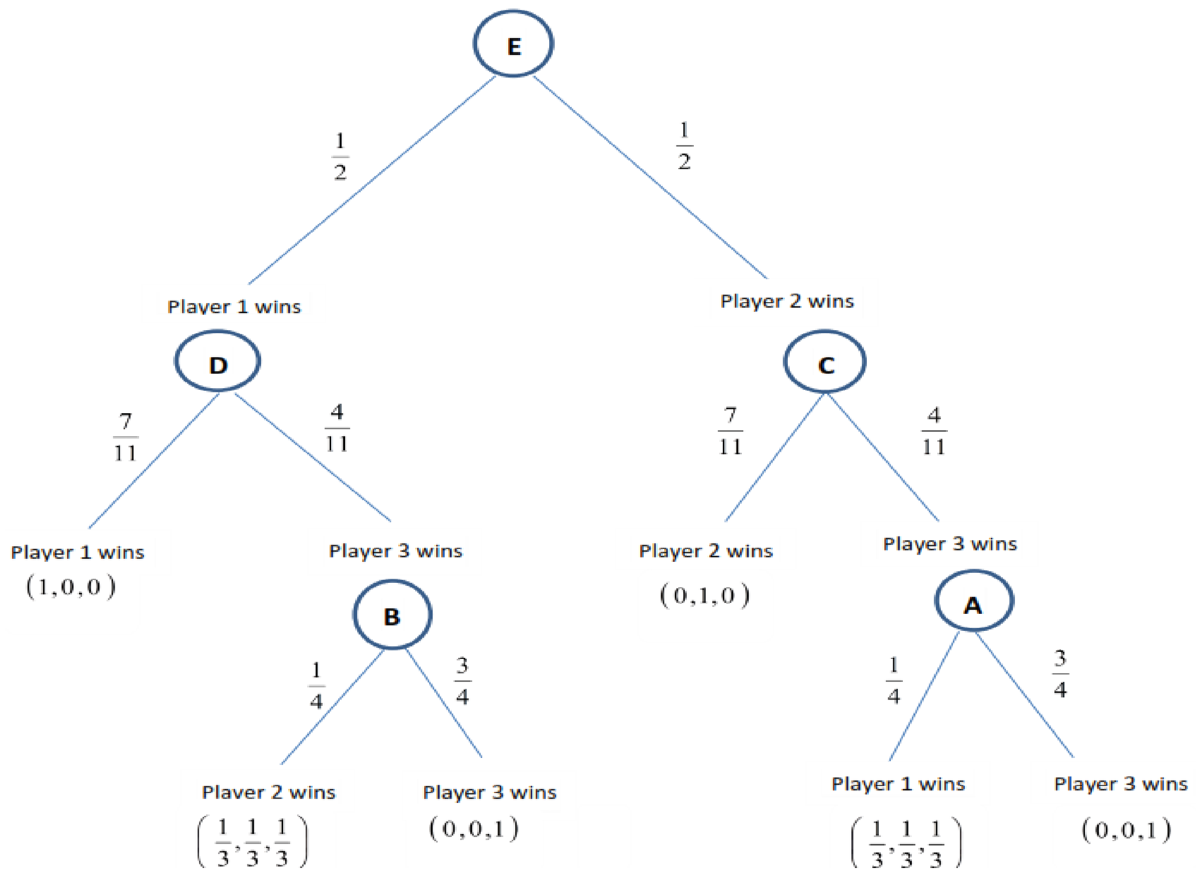


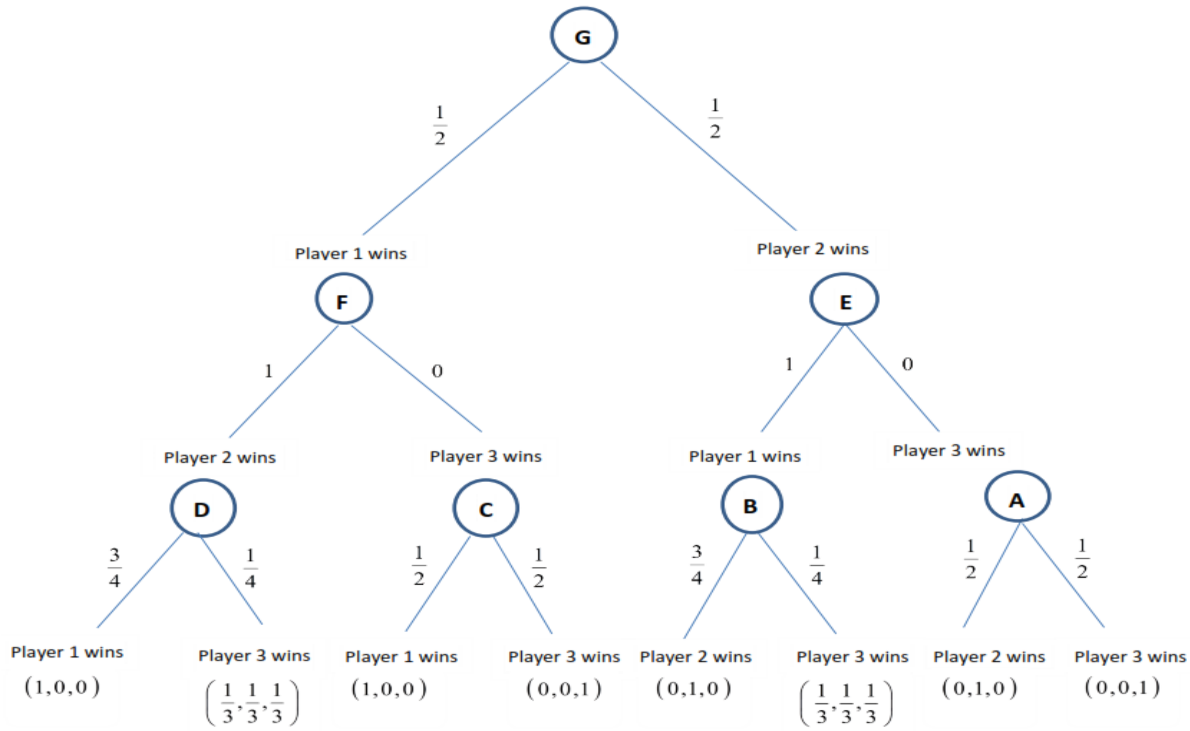
Figure 5: Game tree for case E of the round-robin tournament with one prize



4.3 Case F: The winner of the first round competes in the third round

Figure 6 presents the symmetric round-robin tournament with one prize as a game tree for case F where the winner of the first round competes in the third round. Note that the structure of the game tree for this case is identical to Case C but the players' expected payoffs and their probabilities of winning in both cases are different. The players' expected payoffs and their probabilities of winning in case F are analyzed in Appendix F.

Figure 6: Game tree for case F of the round-robin tournament with one prize



5 Equilibrium analysis of the round-robin tournament with three symmetric players and one prize

5.1 The players' expected payoffs

Based on the analyses of the subgame perfect equilibrium for cases D, E and F that appear in appendices D, E and F, respectively, we obtain that the players' expected payoffs in the round-robin tournaments with one prize are as follows:

Table 3: Comparison of players' expected payoffs in a tournament with one prize

	Case D: **A random allocation of players**	**Case E:** **The winner of the first round competes in the second round**	**Case F:** **The winner of the first round competes in the third round**
Player 1's expected payoff	0.083	0	0.083
Player 2's expected payoff	0.416	0	0.083
Player 3's expected payoff	0	0	0

We can conclude that

Proposition 4 *In the round-robin tournament with three symmetric players and one prize, the player who competes in the last two rounds (player 3) has the lowest expected payoff independent of whether the allocation of players is random or not.*

The intuition behind Proposition 4 is that since there is only one prize, if player 3 competes against the winner of the first round, the asymmetry in incentives acts against him. The reason is that the winner of the first round leads in the tournament, and unlike with two prizes, one win in the tournament does not necessarily ensure winning the tournament nor, in particular, a positive expected payoff. Therefore, the second round is much more important for the winner of the first one in the tournament with one prize than with two prizes. As a result, because of the asymmetry of the players in the second round the leader (player 1 or 2) of the tournament at that point of time has a higher probability to win as well as a higher expected payoff than his opponent in the second round (player 3). If, however, player 3 competes in the second round against a loser of the first round, then even if player 3 wins, in the third round he has to compete against a player who also has one win. Then, the expected payoff of player 3 in the last round will be zero.

5.2 The players' probabilities of winning

Based on the equilibrium analyses in Appendices D, E and F we obtain that the players' probabilities of winning in the round-robin tournaments with one prize are as follows:

Table 4: Comparison of players' probabilities of winning in a tournament with one prize

	Case D: **A random allocation of players**	**Case E:** **The winner of the first round competes in the second round**	**Case F:** **The winner of the first round competes in the third round**
Player 1's probability of winning	0.193	0.348	0.458
Player 2's probability of winning	0.683	0.348	0.458
Player 3's probability of winning	0.124	0.303	0.083

We can conclude that

Proposition 5 *In the round-robin tournament with three symmetric players and one prize, the player who competes in the last two rounds (player 3) has the lowest probability of winning independent of whether the allocation of players is random or not.*

The intuition behind this result is exactly the same as for the previous result which states that this player has the lowest expected payoff.

5.3 The players' total effort

Using the equilibrium analyses in Appendices D, E and F, we obtain that in the round-robin tournament with a random allocation of players (case D) the players' expected total effort is 0.5; in the round-robin tournament where the winner of the first round competes in the second round (case E), the players' expected total effort is 1; and in the round-robin tournament where the winner of the first round competes in the third round (case F), the players' expected total effort is 0.83. We can conclude that

Proposition 6 *In the round-robin tournament with three symmetric players and one prize, the expected total effort is maximized when the winner of the first round competes in the second round.*

The intuition behind Proposition 6 is as follows: if player 3 competes in the second round against the loser of the first round, his expected payoff in the last round would be zero either he wins or not, and therefore player 3 has no incentives to exert any effort already in the second round. However, if player 3 competes in the second round against the winner of the first round, despite asymmetry in the second round, player 3's expected payoff in the last round is positive if he wins, and therefore he has incentive to exert effort in the second round. Hence, the expected total effort is maximised if a winner of the first round competes in the second round.

5.4 Length of the tournament

In order to minimize the probability that the tournament will be decided after two rounds, the winner of the first round has to play in the third round (case F) since there is no chance that one of the players will have two wins after two rounds. If, however, the winner of the first round competes in the second round, there is a relatively high probability that the winner of the first round will win again, which means that the tournament will be decided before the third round. Therefore, the highest probability that the round-robin tournament will be decided after two rounds occurs in case E. This intuition is confirmed by the analyses in Appendices D, E and F.

6 The optimal round-robin tournament with three players

We now compare between the tournaments with one and two prizes. Table 5 summarises the results presented in Sections 3 and 5. We can see that the number of prizes completely affects the players' preferences about their allocations. If there is one prize, the players prefer to compete in the first and the last rounds. If, on the other hand, there are two prizes, the players prefer to compete in the second and the third rounds.

Table 5: Comparison between the tournaments

Number of prizes	**Case**	**Expected payoffs**			**Probabilities of winning**			**Total Effort**	**The probability that the tournament is decided after two rounds**
		$E(u_1)$	$E(u_2)$	$E(u_3)$	p_1	p_2	p_3		
Two prizes	A	0.103	0.135	0.19	0.642	0.647	0.711	0.57	0.181
	B	0.135	0.135	0.208	0.648	0.648	0.704	0.52	0
	C	0.042	0.042	0.166	0.639	0.639	0.722	0.75	0.44
One prize	D	0.083	0.416	0	0.193	0.683	0.124	0.5	0.119
	E	0	0	0	0.348	0.348	0.303	1	0.636
	F	0.083	0.083	0	0.458	0.458	0.083	0.83	0

The number of prizes also affects the optimal design of the round-robin tournament. Assume, first, that the contest designer wishes to maximize the players' expected total effort. If only one prize is awarded, the optimal design is when the winner of the first round competes in the second round and then the players' expected total effort is equal to 1. If, on the other hand, there are two identical prizes, each with a half value of the single prize, the optimal design is when the winner of the first round competes in the third round and then the expected total effort is equal to 0.75. Therefore, we obtain that

Theorem 1 *The optimal design of the round-robin tournament with three symmetric players that maximizes the players' expected total effort is when a single prize is awarded and the winner of the first round competes in the second round.*

Assume now that the contest designer wishes to maximize the length of the tournament, namely, the probability that the tournament will be decided in the last round. If only one prize is awarded, then if the winner of the first round competes in the third round, the tournament will necessarily be decided in the last round. If two prizes are awarded, then if the winner of the first round competes in the second round, the tournament will necessarily be decided in the last round. Therefore, we obtain that

Theorem 2 *The optimal design of the round-robin tournament with three symmetric players that maximizes the probability that the tournament will be decided in the last round could have either one or two prizes.*

By the above comparisons, we can see that the contest designer cannot simultaneously maximize the expected total effort and the length of the tournament.

7 Conclusion

In this paper we analyzed the subgame perfect equilibrium of round-robin tournaments with three symmetric players. We showed that the number of prizes has a crucial effect on the optimal allocation of players. More specifically, we found that in the tournament with two prizes there is a second mover advantage with regard to the expected payoff and the probability of winning independent of whether the allocation of players is random or depends on the outcomes. However, in the tournament with a single prize we found a second mover disadvantage independent of whether the allocation of players is random or depends on the outcomes.

We also saw that if a contest designer wishes to maximize the expected total effort, then he should allocate only one prize and then should allocate the players such that the winner of the first round will compete in the second round. If, on the other hand, he wishes to maximize the length of the tournament, he can ensure that the tournament will not be decided before the last round in both tournaments with either one or two prizes.

Our results have a practical implication to the recent change of the FIFA authorities about the structure of the World Cup, according to which from each group of three teams, the two best teams qualify for the next stage. Our results imply that the team that will compete in the last two rounds will theoretically have a higher probability to qualify for the next stage, which is definitely counter the fair play principles promoted by this organisation.

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Appendix A: A random allocation of players in the round-robin tournament with three players and two prizes

Round 3

As presented in Figure 1, which corresponds to this case, players 2 and 3 compete in the last round only if at least one of them lost in the previous rounds. Thus, we have the following three scenarios:

1. Assume first that player 2 won in the first round and player 1 won in the second (Figure 1, node A). Then, in the third round players 2 and 3 randomize on the interval $\left[0,\frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^A , $i = 2{,}3$, which are given by

$$\begin{aligned} E(u_2^A) &= \frac{1}{2}\cdot F_3^A(x)+\frac{1}{3}\cdot(1-F_3^A(x))-x=\frac{1}{3} \\ E(u_3^A) &= \frac{1}{3}\cdot F_2^A(x)-x=\frac{1}{6} \end{aligned} \tag{1}$$

Then, player 2's probability of winning against player 3 in the third round is $p_{23}^A=\frac{1}{4}$ and the expected total effort is $TE^A=\frac{1}{8}$.

2. Assume now that player 1 won in the first round and player 3 won in the second (Figure 1, node B). Then, in the third round players 2 and 3 randomize on the interval $\left[0, \frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^B , $i = 2,3$, which are given by

$$E(u_2^B) = \frac{1}{3} \cdot F_3^B(x) - x = \frac{1}{6}$$
$$E(u_3^B) = \frac{1}{2} \cdot F_2^B(x) + \frac{1}{3} \cdot (1 - F_2^B(x)) - x = \frac{1}{3} \qquad (2)$$

In that case, player 2's probability of winning against player 3 in the third round is $p_{23}^B = \frac{3}{4}$, and the expected total effort is $TE^B = \frac{1}{8}$.

3. Assume now that player 1 won both in the first and second rounds (Figure 1, node C). Then, in the third round players 2 and 3 randomize on the interval $\left[0, \frac{1}{2}\right]$ according to their effort cumulative distribution functions F_i^C , $i = 2,3$, which are given by

$$E(u_2^C) = \frac{1}{2} \cdot F_3^C(x) - x = 0$$
$$E(u_3^C) = \frac{1}{2} \cdot F_2^C(x) - x = 0 \qquad (3)$$

In that case, player 2's probability of winning against player 3 in the third round is $p_{23}^C = \frac{1}{2}$, and the expected total effort is $TE^C = \frac{1}{2}$.

Round 2

Based on results of the first round, we have two possible scenarios:

1. Assume first that player 2 won in the first round (Figure 1, node D). Then, in the second round players 1 and 3 randomize on the interval $\left[0, \frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^D , $i = 1,3$, which are given by

$$E(u_1^D) = \frac{3}{8} \cdot F_3^D(x) - x = \frac{1}{24}$$
$$E(u_3^D) = \frac{1}{2} \cdot F_1^D(x) + \frac{1}{6} \cdot (1 - F_1^D(x)) - x = \frac{1}{6} \qquad (4)$$

In that case, player 1's probability of winning against player 3 in the second round is $p_{13}^{D}=\frac{5}{9}$, and the expected total effort is $TE^{D}=\frac{17}{54}$.

2. Assume now that player 1 won in the first round (Figure 1, node E). Then, in the second round players 1 and 3 randomize on the interval $\left[0,\frac{1}{8}\right]$ according to their effort cumulative distribution functions F_i^E , $i=1{,}3$, which are given by

$$
(5)\qquad
\begin{aligned}
E(u_1^E) &= \frac{1}{2}\cdot F_3^E(x)+\frac{3}{8}\cdot(1-F_3^E(x))-x=\frac{3}{8}\\
E(u_3^E) &= \frac{1}{3}\cdot F_1^E(x)-x=\frac{5}{24}
\end{aligned}
$$

In that case, player 1's probability of winning against player 3 in the second round is $p_{13}^{E}=\frac{3}{16}$, and the expected total effort is $TE^{E}=\frac{11}{128}$.

Round 1

In the first round players 1 and 2 (Figure 1, node F) randomize on the interval $\left[0,\frac{235}{864}\right]$ according to their effort cumulative distribution functions $F_i^F, i=1{,}2$, which are given by

$$
(6)\qquad
\begin{aligned}
E(u_1^F) &= \frac{3}{8}\cdot F_2^F(x)+\frac{1}{24}\cdot(1-F_2^F(x))-x=\frac{89}{864}=0.103\\
E(u_2^F) &= \frac{11}{27}\cdot F_1^F(x)+\frac{13}{96}\cdot(1-F_1^F(x))-x=\frac{13}{96}=0.135
\end{aligned}
$$

Then player 1's probability of winning against player 2 in the first round is $p_{12}^{F}=\frac{341}{576}$, and the expected total effort is $TE^{F}=0.24696$.

The players' expected payoffs

Players 1 and 2's expected payoffs are given in (6). By (4), player 3's expected payoff in the tournament is $\frac{1}{6}$ only if player 2 wins in the first round, which happens with the probability of $1-\frac{341}{576}$ and by (5), player 3's expected payoff in the tournament is $\frac{5}{24}$ only if player 1 wins in the first round, which happens with the probability of $\frac{341}{576}$. Therefore, the expected payoff of player 3 is the highest among the players and equals 0.19. In sum, the players' expected payoffs are:

$$E(u_1)=\frac{89}{864}=0.103$$
$$E(u_2)=\frac{13}{96}=0.135$$
$$E(u_3)=\frac{2,645}{13,824}=0.19$$

The players' probabilities of winning

Player 1's probability to win a prize is

$$p_1=p_{12}^F\cdot p_{13}^E\cdot p_{23}^C+p_{12}^F\cdot p_{13}^E\cdot p_{32}^C+\frac{2}{3}\cdot(p_{12}^F\cdot p_{31}^E\cdot p_{23}^B)+p_{12}^F\cdot p_{31}^E\cdot p_{32}^B+p_{21}^F\cdot p_{13}^D\cdot p_{23}^A+\frac{2}{3}\cdot\left(p_{21}^F\cdot p_{13}^D\cdot p_{32}^A\right)=0.6417$$

Player 2's probability to win a prize is

$$p_2=p_{12}^F\cdot p_{13}^E\cdot p_{23}^C+\frac{2}{3}\cdot(p_{12}^F\cdot p_{31}^E\cdot p_{23}^B)+p_{21}^F\cdot p_{13}^D\cdot p_{23}^A+\frac{2}{3}\cdot\left(p_{21}^F\cdot p_{13}^D\cdot p_{32}^A\right)+p_{21}^F\cdot p_{31}^D=0.6473$$

And, player 3's probability to win a prize is

$$p_3=p_{12}^F\cdot p_{13}^E\cdot p_{32}^C+\frac{2}{3}\cdot(p_{12}^F\cdot p_{31}^E\cdot p_{23}^B)+p_{12}^F\cdot p_{31}^E\cdot p_{32}^B+\frac{2}{3}\cdot\left(p_{21}^F\cdot p_{13}^D\cdot p_{32}^A\right)+p_{21}^F\cdot p_{31}^D=0.7109$$

The players' expected total effort

The expected total effort in the tournament is

$$TE=TE^F+p_{12}^F\cdot TE^E+p_{21}^F\cdot TE^D+p_{12}^F\cdot p_{13}^E\cdot TE^C+p_{12}^F\cdot p_{31}^E\cdot TE^B+p_{21}^F\cdot p_{13}^D\cdot TE^A=0.57$$

The length of the tournament

The probability that the winners of the tournament will be determined before the last round is

$$p_{21}^F\cdot p_{31}^D=0.181$$

Appendix B: The winner of the first round competes in the second round of the round-robin tournament with three players and two prizes.

Round 3

As presented in Figure 2, which corresponds to this case, players 1 and 3 compete in the last round only if player 2 wins in the first round. Thus, we have the following two scenarios:

1. Assume first that player 2 won in the first round and then player 3 won in the second (Figure 2, node A). Then, players 1 and 3 randomize on the interval $\left[0,\frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^A , $i = 1,3$, which are given by

$$(7)\qquad \begin{aligned} E(u_1^A) &= \frac{1}{3}\cdot F_3^A(x) - x = \frac{1}{6} \\ E(u_3^A) &= \frac{1}{2}\cdot F_1^A(x) + \frac{1}{3}\cdot(1-F_1^A(x)) - x = \frac{1}{3} \end{aligned}$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^A = \frac{3}{4}$, and the expected total effort is $TE^A = \frac{1}{8}$.

2. Assume now that player 2 won both in the first and second rounds (Figure 2, node B). Then, players 1 and 3 randomize on the interval $\left[0,\frac{1}{2}\right]$ according to their effort cumulative distribution functions F_i^B , $i = 1,3$, which are given by

$$(8)\qquad \begin{aligned} E(u_1^B) &= \frac{1}{2}\cdot F_3^B(x) - x = 0 \\ E(u_3^B) &= \frac{1}{2}\cdot F_1^B(x) - x = 0 \end{aligned}$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^B = \frac{1}{2}$, and the expected total effort is $TE^B = \frac{1}{2}$.

Players 2 and 3 compete in the last round only if player 1 wins in the first round. Thus, we have the following two scenarios:

1. Assume that player 1 won in the first round and player 3 won in the second (Figure 2, node C). Then players 2 and 3 randomize on the interval $\left[0,\frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^C , $i = 2{,}3$, which are given by

$$E(u_2^C) = \frac{1}{3}\cdot F_3^C(x) - x = \frac{1}{6}$$
(9)
$$E(u_3^C) = \frac{1}{2}\cdot F_2^C(x) + \frac{1}{3}\cdot(1 - F_2^C(x)) - x = \frac{1}{3}$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^C = \frac{3}{4}$, and the expected total effort is $TE^C = \frac{1}{8}$.

2. Assume now that player 1 won in the first and second rounds (Figure 2, node D). Then, players 2 and 3 randomize on the interval $\left[0,\frac{1}{2}\right]$ according to their effort cumulative distribution functions F_i^D , $i = 2{,}3$, which are given by

$$E(u_2^D) = \frac{1}{2}\cdot F_3^D(x) - x = 0$$
(10)
$$E(u_3^D) = \frac{1}{2}\cdot F_2^D(x) - x = 0$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^D = \frac{1}{2}$, and the expected total effort is $TE^D = \frac{1}{2}$.

Round 2

Based on results of the first round, we have two possible scenarios:

1. Assume first that player 2 won in the first round (Figure 2, node E). Then, players 2 and 3 randomize on the interval $\left[0,\frac{1}{8}\right]$ according to their effort cumulative distribution functions F_i^E , $i = 2{,}3$, which are given by

$$E(u_2^E) = \frac{1}{2}\cdot F_3^E(x) + \frac{3}{8}\cdot(1 - F_3^E(x)) - x = \frac{3}{8}$$
(11)
$$E(u_3^E) = \frac{1}{3}\cdot F_2^E(x) - x = \frac{5}{24}$$

Then player 2's probability of winning against player 3 in the second round is $p_{23}^{E} = \frac{3}{16}$, and the expected total effort is $TE^{E} = \frac{11}{128}$.

2. Assume now that player 1 won in the first round (Figure 2, node F). Then, players 1 and 3 randomize on the interval $\left[0, \frac{1}{8}\right]$ according to their effort cumulative distribution functions F_i^F , $i = 1{,}3$, which are given by

$$
(12) \qquad \begin{aligned} E(u_1^F) &= \frac{1}{2} \cdot F_3^F(x) + \frac{3}{8} \cdot (1 - F_3^F(x)) - x = \frac{3}{8} \\ E(u_3^F) &= \frac{1}{3} \cdot F_1^F(x) - x = \frac{5}{24} \end{aligned}
$$

Then player 1's probability of winning against player 3 in the second round is $p_{13}^{F} = \frac{3}{16}$, and the expected total effort is $TE^{F} = \frac{11}{128}$.

Round 1

Players 1 and 2 (Figure 2, node G) randomize on the interval $\left[0, \frac{23}{96}\right]$ according to their effort cumulative distribution functions F_i^G , $i = 1{,}2$, which are given by

$$
(13) \qquad \begin{aligned} E(u_1^G) &= \frac{3}{8} \cdot F_2^G(x) + \frac{13}{96} \cdot (1 - F_2^G(x)) - x = \frac{13}{96} \\ E(u_2^G) &= \frac{3}{8} \cdot F_1^G(x) + \frac{13}{96} \cdot (1 - F_1^G(x)) - x = \frac{13}{96} \end{aligned}
$$

Then player 1's probability of winning against player 2 in the first round is $p_{12}^{G} = \frac{1}{2}$, and the expected total effort is $TE^{G} = \frac{23}{96}$.

The players' expected payoffs

Players 1 and 2's expected payoffs are given in (13). By (11), player 3's expected payoff is $\frac{5}{24}$ only if player 2 wins in the first round, which happens with the probability of $\frac{1}{2}$. Similarly, by (12), player 3's expected payoff is $\frac{5}{24}$ only if player 1 wins in the first round, which happens with the probability of $\frac{1}{2}$ as well. Therefore, the expected payoff of player 3 is the highest among the players and equals $\frac{5}{24}$. In sum, the players' expected payoffs are:

$$E(u_1)=\frac{13}{96}=0.135$$
$$E(u_2)=\frac{13}{96}=0.135$$
$$E(u_3)=\frac{5}{24}=0.208$$

The players’ probabilities of winning

Player 1’s probability to win a prize is

$$p_1=p_{12}^G\cdot p_{13}^F\cdot p_{23}^D+p_{12}^G\cdot p_{13}^F\cdot p_{32}^D+\frac{2}{3}\cdot(p_{12}^G\cdot p_{31}^F\cdot p_{23}^C)+p_{12}^G\cdot p_{31}^F\cdot p_{32}^C+p_{21}^G\cdot p_{23}^E\cdot p_{13}^B+\frac{2}{3}\cdot(p_{21}^G\cdot p_{32}^E\cdot p_{13}^A)=0.648$$

Player 2’s probability to win a prize is

$$p_2=p_{12}^G\cdot p_{13}^F\cdot p_{23}^D+\frac{2}{3}\cdot(p_{12}^G\cdot p_{31}^F\cdot p_{23}^C)+p_{21}^G\cdot p_{23}^E\cdot p_{13}^B+p_{21}^G\cdot p_{23}^E\cdot p_{31}^B+\frac{2}{3}\cdot(p_{21}^G\cdot p_{32}^E\cdot p_{13}^A)+p_{21}^G\cdot p_{32}^E\cdot p_{31}^A=0.648$$

And, player 3’s probability to win a prize is

$$p_3=p_{12}^G\cdot p_{13}^F\cdot p_{32}^D+\frac{2}{3}\cdot(p_{12}^G\cdot p_{31}^F\cdot p_{23}^C)+p_{12}^G\cdot p_{31}^F\cdot p_{32}^C+p_{21}^G\cdot p_{23}^E\cdot p_{31}^B+\frac{2}{3}\cdot(p_{21}^G\cdot p_{32}^E\cdot p_{13}^A)+p_{21}^G\cdot p_{32}^E\cdot p_{31}^A=0.704$$

The players' expected total effort

The expected total effort in the tournament is

$$\begin{aligned}TE&=TE^G+p_{12}^G\cdot TE^F+p_{21}^G\cdot TE^E+p_{12}^G\cdot p_{13}^F\cdot TE^D+p_{12}^G\cdot p_{31}^F\cdot TE^C+\\&+p_{21}^G\cdot p_{23}^E\cdot TE^B+p_{21}^G\cdot p_{32}^E\cdot TE^A=0.52\end{aligned}$$

The length of the tournament

The probability that the winners of the tournament will be determined before the last round is equal to zero, since there is no possibility that one of the players lost twice in the first two rounds.

Appendix C: The winner of the first round competes in the third round of the round-robin tournament with three players and two prizes.

Round 3

As presented in Figure 3, which corresponds to this case, players 2 and 3 compete in the last round only if player 2 wins in the first round and player 1 wins in the second (Figure 3, node A). Then, players 2 and 3 randomize on the interval $\left[0,\frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^A , $i = 2,3$, which are given by

$$E(u_2^A) = \frac{1}{2} \cdot F_3^A(x) + \frac{1}{3} \cdot (1 - F_3^A(x)) - x = \frac{1}{3} \quad (14)$$
$$E(u_3^A) = \frac{1}{3} \cdot F_2^A(x) - x = \frac{1}{6}$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^A = \frac{1}{4}$, and the expected total effort is $TE^A = \frac{1}{8}$.

Players 1 and 3 compete in the last round only if player 1 wins in the first round and player 2 wins in the second (Figure 3, node B). Then, players 1 and 3 randomize on the interval $\left[0,\frac{1}{6}\right]$ according to their effort cumulative distribution functions F_i^B , $i = 1,3$, which are given by

$$E(u_1^B) = \frac{1}{2} \cdot F_3^B(x) + \frac{1}{3} \cdot (1 - F_3^B(x)) - x = \frac{1}{3} \quad (15)$$
$$E(u_3^B) = \frac{1}{3} \cdot F_1^B(x) - x = \frac{1}{6}$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^B = \frac{1}{4}$, and the expected total effort is $TE^B = \frac{1}{8}$.

Round 2

Based on results of the first round, we have two possible scenarios:

1. Assume first that player 2 won in the first round (Figure 3, node C). Then, players 1 and 3 randomize on the interval $\left[0,\frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^C , $i = 1,3$, which are given by

$$\begin{aligned} E(u_1^C) &= \frac{3}{8}\cdot F_3^C(x) - x = \frac{1}{24} \\ E(u_3^C) &= \frac{1}{2}\cdot F_1^C(x) + \frac{1}{6}\cdot(1 - F_1^C(x)) - x = \frac{1}{6} \end{aligned} \tag{16}$$

Then player 1's probability of winning against player 3 in the second round is $p_{13}^C = \frac{5}{9}$, and the expected total effort is $TE^C = \frac{17}{54}$.

2. Assume now that player 1 won in the first round (Figure 3, node D). Then, players 2 and 3 randomize on the interval $\left[0,\frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^D , $i = 2,3$, which are given by

$$\begin{aligned} E(u_2^D) &= \frac{3}{8}\cdot F_3^D(x) - x = \frac{1}{24} \\ E(u_3^D) &= \frac{1}{2}\cdot F_2^D(x) + \frac{1}{6}\cdot(1 - F_2^D(x)) - x = \frac{1}{6} \end{aligned} \tag{17}$$

Then player 2's probability of winning against player 3 in the second round is $p_{23}^D = \frac{5}{9}$, and the expected total effort is $TE^D = \frac{17}{54}$.

Round 1

Players 1 and 2 (Figure 3, node E) randomize on the interval $\left[0,\frac{79}{216}\right]$ according to their effort cumulative distribution functions F_i^E , $i = 1,2$, which are given by

$$\begin{aligned} E(u_1^E) &= \frac{11}{27}\cdot F_2^E(x) + \frac{1}{24}\cdot(1 - F_2^E(x)) - x = \frac{1}{24} \\ E(u_2^E) &= \frac{11}{27}\cdot F_1^E(x) + \frac{1}{24}\cdot(1 - F_1^E(x)) - x = \frac{1}{24} \end{aligned} \tag{18}$$

Then player 1's probability of winning against player 2 in the first round is $p_{12}^E = \frac{1}{2}$, and the expected total effort is $TE^E = \frac{79}{216}$.

The players’ expected payoffs

Players 1 and 2's expected payoffs are given in (18). By (16), player 3’s expected payoff in the tournament is $\frac{1}{6}$ only if player 2 wins in the first round, which happens with the probability of $\frac{1}{2}$ and similarly, by (17), player 3’s expected payoff in the tournament is $\frac{1}{6}$, only if player 1 wins in the first round, which happens with the probability of $\frac{1}{2}$. Therefore, the expected payoff of player 3 is the highest among the players and is equal $\frac{1}{6}$.

In sum, the players' expected payoffs are:

$$E(u_1) = \frac{1}{24} = 0.042$$
$$E(u_2) = \frac{1}{24} = 0.042$$
$$E(u_3) = \frac{1}{6} = 0.166$$

The players’ probabilities of winning

Player 1’s probability to win a prize is

$$p_1 = p_{12}^E \cdot p_{23}^D \cdot p_{13}^B + \frac{2}{3} \cdot (p_{12}^E \cdot p_{23}^D \cdot p_{31}^B) + p_{12}^E \cdot p_{32}^D + p_{21}^E \cdot p_{13}^C \cdot p_{23}^A + \frac{2}{3} \cdot (p_{21}^E \cdot p_{13}^C \cdot p_{32}^A) = 0.639$$

Player 2’s probability to win a prize is

$$p_2 = p_{12}^E \cdot p_{23}^D \cdot p_{13}^B + \frac{2}{3} \cdot (p_{12}^E \cdot p_{23}^D \cdot p_{31}^B) + p_{21}^E \cdot p_{13}^C \cdot p_{23}^A + \frac{2}{3} \cdot (p_{21}^E \cdot p_{13}^C \cdot p_{32}^A) + p_{21}^E \cdot p_{31}^C = 0.639$$

And, player 3’s probability to win a prize is

$$p_3 = \frac{2}{3} \cdot (p_{12}^E \cdot p_{23}^D \cdot p_{31}^B) + p_{12}^E \cdot p_{32}^D + \frac{2}{3} \cdot (p_{21}^E \cdot p_{13}^C \cdot p_{32}^A) + p_{21}^E \cdot p_{31}^C = 0.722$$

The players’ expected total effort

The expected total effort in the tournament is

$$TE = TE^E + p_{12}^E \cdot TE^D + p_{21}^E \cdot TE^C + p_{12}^E \cdot p_{23}^D \cdot TE^B + p_{21}^E \cdot p_{13}^C \cdot TE^A = 0.75$$

The length of the tournament

The probability that the winners of the tournament will be determined before the last round is

$$p_{12}^{E}\cdot p_{32}^{D}+p_{21}^{E}\cdot p_{31}^{C}=0.44$$

Appendix D: A random allocation of players in the round-robin tournament with three players and one prize

This case which is presented in Figure 4, was already analyzed by Krumer, Megidish and Sela (2017). Therefore, we only summarize the main results.

The players' expected payoffs

$$E(u_1)=\frac{1}{12}=0.083$$
$$E(u_2)=\frac{5}{12}=0.416$$
$$E(u_3)=0$$

The players' probabilities of winning

$$p_1=p_{12}^{F}\cdot p_{13}^{E}+\frac{p_{12}^{F}\cdot p_{31}^{E}\cdot p_{23}^{C}}{3}+\frac{p_{21}^{F}\cdot p_{13}^{D}\cdot p_{32}^{B}}{3}=0.193$$
$$p_2=\frac{p_{12}^{F}\cdot p_{31}^{E}\cdot p_{23}^{C}}{3}+p_{21}^{F}\cdot p_{13}^{D}\cdot p_{23}^{B}+\frac{p_{21}^{F}\cdot p_{13}^{D}\cdot p_{32}^{B}}{3}+p_{21}^{F}\cdot p_{31}^{D}\cdot p_{23}^{A}=0.683$$
$$p_3=\frac{p_{12}^{F}\cdot p_{31}^{E}\cdot p_{23}^{C}}{3}+p_{12}^{F}\cdot p_{31}^{E}\cdot p_{32}^{C}+\frac{p_{21}^{F}\cdot p_{13}^{D}\cdot p_{32}^{B}}{3}+p_{21}^{F}\cdot p_{31}^{D}\cdot p_{32}^{A}=0.124$$

The players' expected total effort

The expected total effort in the tournament is

$$TE=TE^{F}+p_{12}^{F}\cdot TE^{E}+p_{21}^{F}\cdot TE^{D}+p_{12}^{F}\cdot p_{31}^{E}\cdot TE^{C}+p_{21}^{F}\cdot p_{13}^{D}\cdot TE^{B}+p_{21}^{F}\cdot p_{31}^{D}\cdot TE^{A}=0.5$$

The length of the tournament

The probability that the winner of the tournament will be determined before the last round is

$$p_{12}^{F}\cdot p_{13}^{E}=0.119$$

Appendix E: The winner of the first round competes in the second round of the round-robin tournament with three players and one prize.

Round 3

As presented in Figure 5, which corresponds to this case, players 1 and 3 compete in the last round only if player 2 wins in the first round and player 3 wins in the second (Figure 5, node A). Then, players 1 and 3 randomize on the interval $\left[0,\frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^A , $i = 1,3$, which are given by

$$(19) \qquad \begin{aligned} E(u_1^A) &= \frac{1}{3}\cdot F_3^A(x) - x = 0 \\ E(u_3^A) &= 1\cdot F_1^A(x) + \frac{1}{3}\cdot(1 - F_1^A(x)) - x = \frac{2}{3} \end{aligned}$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^A = \frac{1}{4}$, and the expected total effort is $TE^A = \frac{1}{4}$.

Players 2 and 3 compete in the last round only if player 1 wins in the first round and player 3 wins in the second (Figure 5, node B). Then, players 2 and 3 randomize on the interval $\left[0,\frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^B , $i = 2,3$, which are given by

$$(20) \qquad \begin{aligned} E(u_2^B) &= \frac{1}{3}\cdot F_3^B(x) - x = 0 \\ E(u_3^B) &= 1\cdot F_2^B(x) + \frac{1}{3}\cdot(1 - F_2^B(x)) - x = \frac{2}{3} \end{aligned}$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^B = \frac{1}{4}$, and the expected total effort is $TE^B = \frac{1}{4}$.

Round 2

Based on results of the first round, we have two possible scenarios:

1. Assume first that player 2 won in the first round (Figure 5, node C). Then, players 2 and 3 randomize on the interval $\left[0,\frac{2}{3}\right]$ according to their effort cumulative distribution functions F_i^C , $i = 2,3$, which are given by

(21)
$$E(u_2^C) = 1\cdot F_3^C(x) + \frac{1}{12}\cdot(1-F_3^C(x)) - x = \frac{1}{3}$$
$$E(u_3^C) = \frac{2}{3}\cdot F_2^C(x) - x = 0$$

Then player 2's probability of winning against player 3 in the second round is $p_{23}^C = \frac{7}{11}$, and the expected total effort is $TE^C = \frac{19}{33}$.

2. Assume now that player 1 won in the first round (Figure 5, node D). Then, players 1 and 3 randomize on the interval $\left[0,\frac{2}{3}\right]$ according to their effort cumulative distribution functions F_i^D , $i = 1,3$, which are given by

(22)
$$E(u_1^D) = 1\cdot F_3^D(x) + \frac{1}{12}\cdot(1-F_3^D(x)) - x = \frac{1}{3}$$
$$E(u_3^D) = \frac{2}{3}\cdot F_1^D(x) - x = 0$$

Then player 1's probability of winning against player 3 in the second round is $p_{13}^D = \frac{7}{11}$, and the expected total effort is $TE^D = \frac{19}{33}$.

Round 1

Players 1 and 2 (Figure 5, node E) randomize on the interval $\left[0,\frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^E , $i = 1,2$, which are given by

(23)
$$E(u_1^E) = \frac{1}{3}\cdot F_2^E(x) - x = 0$$
$$E(u_2^E) = \frac{1}{3}\cdot F_1^E(x) - x = 0$$

Then player 1's probability of winning against player 2 in the first round is $p_{12}^E = \frac{1}{2}$, and the expected total effort is $TE^E = \frac{1}{3}$.

The players' expected payoffs

Players 1 and 2's expected payoffs are given in (23). By (21) and (22), player 3's expected payoff is zero. In sum, the players' expected payoffs are:

$$E(u_1) = 0$$
$$E(u_2) = 0$$
$$E(u_3) = 0$$

The players' probabilities of winning

Player 1's probability to win the prize is

$$p_1 = p_{12}^E \cdot p_{13}^D + \frac{p_{12}^E \cdot p_{31}^D \cdot p_{23}^B}{3} + \frac{p_{21}^E \cdot p_{32}^C \cdot p_{13}^A}{3} = 0.348$$

Player 2's probability to win the prize is

$$p_2 = \frac{p_{12}^E \cdot p_{31}^D \cdot p_{23}^B}{3} + p_{21}^E \cdot p_{23}^C + \frac{p_{21}^E \cdot p_{32}^C \cdot p_{13}^A}{3} = 0.348$$

And, player 3's probability to win the prize is

$$p_3 = \frac{p_{12}^E \cdot p_{31}^D \cdot p_{23}^B}{3} + p_{12}^E \cdot p_{31}^D \cdot p_{32}^B + \frac{p_{21}^E \cdot p_{32}^C \cdot p_{13}^A}{3} + p_{21}^E \cdot p_{32}^C \cdot p_{31}^A = 0.303$$

The players' expected total effort

The expected total effort in the tournament is

$$TE = TE^E + p_{12}^E \cdot TE^D + p_{21}^E \cdot TE^C + p_{12}^E \cdot p_{31}^D \cdot TE^B + p_{21}^E \cdot p_{32}^C \cdot TE^A = 1$$

The length of the tournament

The probability that the winner of the tournament will be determined before the last round is

$$p_{12}^E \cdot p_{13}^D + p_{21}^E \cdot p_{23}^C = 0.636$$

Appendix F: The winner of the first round competes in the third round of the round-robin tournament with three players and one prize

Round 3

As presented in Figure 6, which corresponds to this case, players 2 and 3 compete in the last round only if player 2 wins in the first round. Thus, we have the following two scenarios:

1. If player 3 wins in the second round (Figure 6, node A), then, players 2 and 3 randomize on the interval $[0,1]$ according to their effort cumulative distribution functions F_i^A , $i = 2,3$, which are given by

$$
\begin{aligned}
E(u_2^A) &= 1 \cdot F_3^A(x) - x = 0 \\
E(u_3^A) &= 1 \cdot F_2^A(x) - x = 0
\end{aligned}
\tag{24}
$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^A = \dfrac{1}{2}$, and the expected total effort is $TE^A = 1$.

2. If player 1 wins in the second round (Figure 6, node B), then, players 2 and 3 randomize on the interval $\left[0, \frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^B , $i = 2,3$, which are given by

$$
\begin{aligned}
E(u_2^B) &= 1 \cdot F_3^B(x) + \frac{1}{3} \cdot (1 - F_3^B(x)) - x = \frac{2}{3} \\
E(u_3^B) &= \frac{1}{3} \cdot F_2^B(x) - x = 0
\end{aligned}
\tag{25}
$$

Then player 2's probability of winning against player 3 in the third round is $p_{23}^B = \dfrac{3}{4}$, and the expected total effort is $TE^B = \dfrac{1}{4}$.

Players 1 and 3 compete in the last round only if player 1 wins in the first round. Thus, we have the following two scenarios:

1. If player 3 wins in the second round (Figure 6, node C), then, players 1 and 3 randomize on the interval $[0,1]$ according to their effort cumulative distribution functions F_i^C , $i = 1,3$, which are given by

$$E(u_1^C) = 1 \cdot F_3^C(x) - x = 0 \qquad (26)$$
$$E(u_3^C) = 1 \cdot F_1^C(x) - x = 0$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^C = \frac{1}{2}$, and the expected total effort is $TE^C = 1$.

2. If player 2 wins in the second round (Figure 6, node D), then, players 1 and 3 randomize on the interval $\left[0, \frac{1}{3}\right]$ according to their effort cumulative distribution functions F_i^D , $i = 1,3$, which are given by

$$E(u_1^D) = 1 \cdot F_3^D(x) + \frac{1}{3} \cdot (1 - F_3^D(x)) - x = \frac{2}{3} \qquad (27)$$
$$E(u_3^D) = \frac{1}{3} \cdot F_1^D(x) - x = 0$$

Then player 1's probability of winning against player 3 in the third round is $p_{13}^D = \frac{3}{4}$, and the expected total effort is $TE^D = \frac{1}{4}$.

Round 2

Based on the results of the first round, we have two possible scenarios:

1. Assume first that player 2 won in the first round (Figure 6, node E). We can see that according to (24) and (25) player 3's expected payoff is zero, implying that he has no incentive to exert a positive effort and therefore we actually have no equilibrium. However, to overcome this problem, we can assume, similarly to Groh et al. (2012), that in every game each player obtains an additional prize for winning a single round of $m > 0$ with limit behaviour as $m \to 0$. This assumption does not affect the players' behavior, but ensures the existence of equilibrium. In that case, player 1 wins with certainty.

2. Assume now that player 1 won in the first round (Figure 6, node F). Then as in the first scenario, by (26) and (27), player 2 wins with certainty.

Round 1

Players 1 and 2 (Figure 6, node G) randomize on the interval $\left[0, \frac{7}{12}\right]$ according to their effort cumulative distribution functions F_i^G , $i = 1,2$, which are given by

(28)
$$E(u_1^G)=\frac{2}{3}\cdot F_2^G(x)+\frac{1}{12}\cdot(1-F_2^G(x))-x=\frac{1}{12}$$
$$E(u_2^G)=\frac{2}{3}\cdot F_1^G(x)+\frac{1}{12}\cdot(1-F_1^G(x))-x=\frac{1}{12}$$

Then player 1's probability of winning against player 2 in the first round is $p_{12}^G=\frac{1}{2}$, and the expected total effort is $TE^G=\frac{7}{12}$.

The players' expected payoffs

Players 1 and 2's expected payoffs are given in (28). Player 3 has no incentive to exert a positive effort and therefore, the expected payoff of player 3 is zero. In sum, the players' expected payoffs are:

$$E(u_1)=\frac{1}{12}=0.083$$
$$E(u_2)=\frac{1}{12}=0.083$$
$$E(u_3)=0$$

The players' probabilities of winning

Player 1's probability to win the prize is

$$p_1=p_{12}^G\cdot p_{23}^F\cdot p_{13}^D+\frac{p_{12}^G\cdot p_{23}^F\cdot p_{31}^D}{3}+p_{12}^G\cdot p_{32}^F\cdot p_{13}^C+\frac{p_{21}^G\cdot p_{13}^E\cdot p_{32}^B}{3}=0.458$$

Player 2's probability to win the prize is

$$p_2=\frac{p_{12}^G\cdot p_{23}^F\cdot p_{31}^D}{3}+p_{21}^G\cdot p_{13}^E\cdot p_{23}^B+\frac{p_{21}^G\cdot p_{13}^E\cdot p_{32}^B}{3}+p_{21}^G\cdot p_{31}^E\cdot p_{23}^A=0.458$$

And, player 3's probability to win the prize is

$$p_3=\frac{p_{12}^G\cdot p_{23}^F\cdot p_{31}^D}{3}+p_{12}^G\cdot p_{32}^F\cdot p_{31}^C+\frac{p_{21}^G\cdot p_{13}^E\cdot p_{32}^B}{3}+p_{21}^G\cdot p_{31}^E\cdot p_{32}^A=0.083$$

The players' expected total effort

The expected total effort in the tournament is

$$TE=TE^G+p_{12}^G\cdot TE^F+p_{21}^G\cdot TE^E+p_{12}^G\cdot p_{23}^F\cdot TE^D+p_{12}^G\cdot p_{32}^F\cdot TE^C+$$
$$+p_{21}^G\cdot p_{13}^E\cdot TE^B+p_{21}^G\cdot p_{31}^E\cdot TE^A=0.83$$

The length of the tournament

The probability that the winner of the tournament will be determined before the last round is equal to zero since there is no possibility that one of the players won twice in the first two rounds.