Tangency Factor Model: why we have $R^2 = 1$ In-Sample?

Remember that:

$$\mathbf{w}_p = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}$$

To understand why the cross-sectional regression of average asset returns on their betas yields an R^2 of 1 when the betas are obtained by regressing the assets on the tangency portfolio, we proceed as follows:

1. The Tangency Portfolio and Asset Expected Returns

In mean-variance optimization, the tangency portfolio is the portfolio on the efficient frontier that offers the highest Sharpe ratio. Its weights are proportional to the inverse of the covariance matrix Σ multiplied by the vector of expected returns $\boldsymbol{\mu}$:

$$\mathbf{w}_p = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}$$

where $\mathbf{1}$ is a vector of ones.

2. Covariance Between Assets and the Tangency Portfolio

The covariance between the return of asset i and the return of the tangency portfolio R_p is:

$$\mathrm{Cov}(R_i, R_p) = \mathrm{Cov}\left(R_i, \mathbf{w}_p^\top \mathbf{R}\right) = \mathbf{w}_p^\top \mathrm{Cov}(R_i, \mathbf{R}) = \mathbf{w}_p^\top \Sigma_{\cdot i} = (\Sigma \mathbf{w}_p)_i$$

Since $\Sigma \mathbf{w}_p = \boldsymbol{\mu}$ (a property of the tangency portfolio), we have:

$$\mathrm{Cov}(R_i, R_p) = \mu_i$$

3. Beta of Asset i with Respect to the Tangency Portfolio

The beta of asset i with respect to the tangency portfolio is defined as:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_p)}{\mathrm{Var}(R_p)} = \frac{\mu_i}{\sigma_p^2}$$

where $\sigma_p^2 = \mathrm{Var}(R_p)$.

4. Linear Relationship Between Expected Returns and Betas

Rewriting the equation for μ_i:

$$\mu_i = \beta_i \sigma_p^2$$

This establishes a perfect linear relationship between the expected returns μ_i and the betas β_i.

5. Cross-Sectional Regression

In the cross-sectional regression:

$$\mu_i = a + b\beta_i + \varepsilon_i$$

the perfect linear relationship $\mu_i = \beta_i \sigma_p^2$ implies:
- $a = 0$ - $b = \sigma_p^2$ - $\varepsilon_i = 0$

6. Resulting R^2 of 1

Since there is no residual error ($\varepsilon_i = 0$) in the regression model, all the variability in μ_i is explained by β_i. Therefore, the coefficient of determination is:

$$R^2 = 1$$

Conclusion

The perfect linear relationship between the expected returns and the betas (obtained from regressing asset returns on the tangency portfolio) ensures that the cross-sectional regression yields an R^2 of 1.

Summary of Formula Derivation

$$\begin{aligned}
\mathrm{Cov}(R_i, R_p) &= \mu_i \\
\beta_i &= \frac{\mu_i}{\sigma_p^2} \\
\mu_i &= \beta_i \sigma_p^2 \\
\mu_i &= 0 + \sigma_p^2 \beta_i + 0 \quad \text{(regression equation)}
\end{aligned}$$

Thus, $R^2 = 1$ because $\varepsilon_i = 0$.