Derivation of Cross-Sectional $R^2 = 1$ for the Tangency Portfolio

The tangency portfolio maximizes the Sharpe ratio, and when we use it to form a single-factor model of expected returns, the R^2 of the cross-sectional regression of expected returns on betas is 100%. Here's the mathematical derivation.

1. Time-Series Regression to Estimate Betas

For each asset i, we perform a time-series regression of its excess returns on the excess returns of the tangency portfolio p:

$$\tilde{r}_{i,t} = \alpha_i + \beta_i \tilde{r}_{p,t} + \epsilon_{i,t} \tag{1}$$

where:

- $\tilde{r}i, t$ is the excess return of asset i at time t,
- $\tilde{r}p, t$ is the excess return of the tangency portfolio at time t,
- β_i is the estimated beta of asset i with respect to the tangency portfolio,
- α_i is the intercept term,
- $\epsilon_{i,t}$ is the regression residual.

2. Analytical Formula for the Tangency Portfolio

The tangency portfolio weights, which maximize the Sharpe ratio, are given by:

$$\mathbf{w}_p = \frac{\mathbf{\Sigma}^{-1}\tilde{\mathbf{r}}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\tilde{\mathbf{r}}} \tag{2}$$

where:

- $\mathbf{w}_p$ is the vector of weights in the tangency portfolio,
- $\mathbf{\Sigma}$ is the covariance matrix of asset returns,
- $\tilde{\mathbf{r}}$is the vector of expected excess returns,
- $\mathbf{1}$ is a vector of ones.

Let's denote the normalization constant by:

$$k = \frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\tilde{\mathbf{r}}} \tag{3}$$

Then, we can express the tangency portfolio weights as:

$$\mathbf{w}_p = k\mathbf{\Sigma}^{-1}\tilde{\mathbf{r}} \tag{4}$$

3. Expected Excess Returns and Covariances

The covariance between asset i and the tangency portfolio is:

$$\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p) = \mathbf{w}_p' \boldsymbol{\sigma}_i \tag{5}$$

where $\boldsymbol{\sigma}_i$ is the i -th column of $\boldsymbol{\Sigma}$.

The variance of the tangency portfolio is:

$$\mathrm{Var}(\tilde{r}_p) = \mathbf{w}_p' \boldsymbol{\Sigma} \mathbf{w}_p \tag{6}$$

4. Calculating Betas

The beta of asset i with respect to the tangency portfolio is:

$$\beta_i = \frac{\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p)}{\mathrm{Var}(\tilde{r}_p)} = \frac{\mathbf{w}_p' \boldsymbol{\sigma}_i}{\mathbf{w}_p' \boldsymbol{\Sigma} \mathbf{w}_p} \tag{7}$$

5. Relationship Between Expected Returns and Betas

Using the tangency portfolio weights, we have:

$$\tilde{\mathbf{r}} = \frac{1}{k} \boldsymbol{\Sigma} \mathbf{w}_p \tag{8}$$

For each asset i, we find:

$$E[\tilde{r}_i] = \frac{1}{k} (\boldsymbol{\Sigma} \mathbf{w}_p)_i = \frac{1}{k} \mathrm{Cov}(\tilde{r}_i, \tilde{r}_p) \tag{9}$$

Given that $\beta_i = \frac{\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p)}{\mathrm{Var}(\tilde{r}_p)}$, we obtain:

$$E[\tilde{r}_i] = \left(\frac{1}{k} \mathrm{Var}(\tilde{r}_p) \right) \beta_i \tag{10}$$

Calculating k using the expected return and variance of the tangency portfolio:

$$E[\tilde{r}_p] = \mathbf{w}_p' \tilde{\mathbf{r}} = \mathbf{w}_p' \left(\frac{1}{k} \boldsymbol{\Sigma} \mathbf{w}_p \right) = \frac{1}{k} \mathrm{Var}(\tilde{r}_p) \tag{11}$$

Thus,

$$k = \frac{\mathrm{Var}(\tilde{r}_p)}{E[\tilde{r}_p]} \tag{12}$$

Substituting back, we find:

$$E[\tilde{r}_i] = E[\tilde{r}_p] \beta_i \tag{13}$$

6. Cross-Sectional Regression and R^2 Calculation

The cross-sectional regression equation for expected returns is:

$$E[\tilde{r}_i] = \gamma_0 + \gamma_1 \beta_i + \varepsilon_i \tag{14}$$

where:

- $\gamma_0 = 0$,
- $\gamma_1 = E[\tilde{r}_p]$,
- $\varepsilon_i = 0$.

Since $\varepsilon_i = 0$ for all i, the sum of squared residuals (SSR) is zero:

$$\text{SSR} = \sum_i \varepsilon_i^2 = 0 \tag{15}$$

The R^2 of the regression is then:

$$R^2 = 1 - \frac{\text{SSR}}{\text{SST}} = 1 \tag{16}$$

Conclusion

Thus, by performing the time-series regression to estimate betas and using the analytical formula for the tangency portfolio, we have shown that expected excess returns are perfectly explained by betas in the cross-sectional regression, resulting in an R^2 of 1.