Last time:
- Structure of dependent type theory
- Π - types
- $\to$ - types
- $\wedge / \times$ - types

} §1-2 of Rijke

This time:
- Inductive types } §3-4 of Rijke

Next time
- The identity type $\to$ homotopy } §5 of Rijke

General idea

- Type formers: The rules that define Π, $\to$, $\wedge/\times$, inductive types are all type formers.
  When we study type theory, we can choose which type formers to include.
  When people talk about HoTT, they often mean a type theory with particular type formers (the ones introduced in this course) + an axiom.

- Comparison with ZF-based mathematics:
  - Products, functions, etc have to be encoded in ZF.
  - Thus, everyday mathematics is far away from foundational rules.
  - In type theory, we postulate that products, functions, etc exist.
  - In type theory, everyday mathematics is closer to foundations.

- Inductive types are <u>freely generated</u> by <u>canonical terms</u>.

- In Agda:

```
data Bool : Type where
  true false : Bool
```

Bool is generated by the two canonical terms true, false.

To define a (dependent) function out of Bool, it suffices to define it on its canonical elements true and false.

The

data _ where
_

syntax tells Agda that we are defining an inductive Type.

In pen-and-paper HoTT, we specify the behavior of inductive types by hand.

<u>The booleans</u>: bool

bool-form: $\dfrac{}{\vdash \text{bool type}}$

bool-intro:

$$\frac{}{\vdash \mathrm{true} : \mathrm{bool}} \qquad \frac{}{\vdash \mathrm{false} : \mathrm{bool}}$$

bool-elim:

$$\frac{\Gamma, x : \mathrm{bool} \vdash D \ \mathrm{type} \qquad \Gamma \vdash a : D[\mathrm{true}/x] \qquad \Gamma \vdash b : D[\mathrm{false}/x]}{\Gamma, x : \mathrm{bool} \vdash \mathrm{ind\text{-}bool}_{a,b} : D}$$

bool-comp:

$$\frac{\Gamma, x : \mathrm{bool} \vdash D \ \mathrm{type} \qquad \Gamma \vdash a : D[\mathrm{true}/x] \qquad \Gamma \vdash b : D[\mathrm{false}/x]}{\begin{array}{c}\Gamma \vdash \mathrm{ind\text{-}bool}_{a,b}[a/x] \doteq a : D[\mathrm{true}/x] \\ \Gamma \vdash \mathrm{ind\text{-}bool}_{a,b}[b/x] \doteq b : D[\mathrm{false}/x]\end{array}}$$

<u>Ex</u>. not : bool → bool

(Remember →-intro:

$$\frac{x : P \vdash q : Q}{\lambda x . q : P \to Q}$$
)

weakened bool-form

$$\frac{\dfrac{\dfrac{}{x : \mathrm{bool} \vdash \mathrm{bool}\ \mathrm{type}} \qquad \dfrac{}{\vdash \mathrm{false} : \mathrm{bool} \doteq \mathrm{bool}[\mathrm{true}/x]} \qquad \dfrac{}{\vdash \mathrm{true} : \mathrm{bool} \doteq \mathrm{bool}[\mathrm{false}/x]}}{x : \mathrm{bool} \vdash \mathrm{ind\text{-}bool}_{\mathrm{false},\mathrm{true}} : \mathrm{bool}}}{\vdash \lambda x . \mathrm{ind\text{-}bool}_{\mathrm{false},\mathrm{true}} : \mathrm{bool} \to \mathrm{bool}}$$

Coproducts +

+ - form:

$$\frac{\Gamma \vdash P \text{ type} \qquad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P + Q \text{ type}}$$

+ - intro:

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \mathrm{inl}(p) : P + Q} \qquad \frac{\Gamma \vdash q : Q}{\Gamma \vdash \mathrm{inr}(q) : P + Q}$$

+ - elim:

$$\frac{\begin{array}{c}\Gamma, x : P + Q \vdash D \text{ type} \\ \Gamma, p : P \vdash a : D[\mathrm{inl}(p)/x] \\ \Gamma, q : Q \vdash b : D[\mathrm{inr}(q)/x]\end{array}}{\Gamma, x : P + Q \vdash \text{ind-}+_{a,b} : D}$$

+ - elim:

$$\frac{\begin{array}{c}\Gamma, x : P + Q \vdash D \text{ type} \\ \Gamma, p : P \vdash a : D[\mathrm{inl}(p)/x] \\ \Gamma, q : Q \vdash b : D[\mathrm{inr}(q)/x]\end{array}}{\begin{array}{c}\Gamma, p : P \vdash \text{ind-}+_{a,b}[\mathrm{inl}(p)/x] \doteq a : D[\mathrm{inl}(p)/x] \\ \Gamma, q : Q \vdash \text{ind-}+_{a,b}[\mathrm{inl}(q)/x] \doteq b : D[\mathrm{inl}(q)/x]\end{array}}$$

Logical interpretation:

- We can prove (produce a term of) $P+Q$ if we can prove P or we can prove Q.
- To prove something from $P+Q$ we do a proof by cases.
- So $+$ behaves like disjunction (or).

Ex. For any types A, B, C, there is a function
$$A\times B + A\times C \to A\times(B+C).$$

$$\dfrac{\dfrac{\dfrac{}{x_1 : A\times B \vdash (pr_1 x_1, \mathrm{inl}\, pr_2 x_1) : A\times(B+C)} \qquad \dfrac{}{x_2 : A\times C \vdash (pr_1 x_2, \mathrm{inr}\, pr_2 x_2) : A\times(B+C)}}{x : A\times B + A\times C \vdash ? : A\times(B+C)}}{\vdash ? : A\times B + A\times C \to A\times(B+C).}$$

Dependent pair types (aka dependent sum types, Sigma types) Σ

Σ-form:
$$\dfrac{\Gamma, x:P \vdash Q}{\Gamma \vdash \sum_{x:P} Q}$$

Σ-intro:
$$\frac{\Gamma \vdash p : P \qquad \Gamma \vdash q : Q[p/x]}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q}$$

Σ-elim:
$$\frac{\Gamma, y : \sum_{x:P} Q \vdash D \text{ type} \qquad \Gamma, x:P, z:Q \vdash a : D[\mathrm{pair}(x,z)/y]}{\Gamma, y : \sum_{x:P} Q \vdash \mathrm{ind}_\Sigma(a,y) : D}$$

Σ-comp:
$$\frac{\Gamma, y : \sum_{x:P} Q \vdash D \text{ type} \qquad \Gamma, x:P, z:Q \vdash a : D[\mathrm{pair}(x,z)/y]}{\Gamma, x:P, z:Q \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,z)) \doteq a : D[\mathrm{pair}(x,z)/y]}$$

<u>Logical interpretation</u>: To prove $\sum_{x:P} Q$ (thinking of P as a set and Q as a predicate on P), we need to produce a term $p:P$ and prove $Q[p/x]$. Thus, it behaves like $\exists_{x:P} Q(x)$.

<u>Set interpretation</u>: The canonical terms are $\mathrm{pair}(p,q)$. It behaves like $\coprod_{x:P} Q(x)$.

<u>Ex</u>. Let $\mathrm{Vect}(n)$ denote the type of vectors of length $n:\mathbb{N}$. Then $\sum_{n:\mathbb{N}} \mathrm{Vect}(n)$ is the type of all vectors.

<u>Ex</u>. For any $x:P \vdash Q$, there is a projection function $\pi : \sum_{x:P} Q \to P$.
$$\cfrac{\cfrac{x:P, z:Q \vdash x : P}{y : \sum_{x:P} Q \vdash \mathrm{ind}_\Sigma(x,y) : P}}{\lambda x.\mathrm{ind}_\Sigma(x,y) : \sum_{x:P} Q \longrightarrow P}$$

Ex. There is a projection function

$$\begin{array}{ccccc} \sum_{n:\mathbb{N}} \mathrm{Vect}(n) & & \text{cf.} & \coprod_{n:\mathbb{N}} \mathrm{Vect}(n) & \text{in sets} \\ \downarrow & & & \downarrow & \\ \mathbb{N} & & & \mathbb{N} & \end{array}$$

Ex. Consider a dependent type $x:\mathrm{bool} \vdash D(x)$ type.
There is a function $\left[\sum_{x:\mathrm{bool}} D(x)\right] \to D(\mathrm{true}) + D(\mathrm{false})$.

$$\dfrac{\dfrac{}{\vdash \lambda z.\mathrm{inl}\,z : D(\mathrm{true}) \to D(\mathrm{true}) + D(\mathrm{false})} \qquad \dfrac{}{\vdash \lambda z.\mathrm{inr}\,z : D(\mathrm{false}) \to D(\mathrm{true}) + D(\mathrm{false})}}{x:\mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x) : D(x) \to D(\mathrm{true}) + D(\mathrm{false})}$$

$$\dfrac{x:\mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x) : D(x) \to D(\mathrm{true}) + D(\mathrm{false})}{x:\mathrm{bool},\ y:D(x) \vdash \mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x)\, y : D(\mathrm{true}) + D(\mathrm{false})}$$

$$\dfrac{x:\mathrm{bool},\ y:D(x) \vdash \mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x)\, y : D(\mathrm{true}) + D(\mathrm{false})}{z : \sum_{x:\mathrm{bool}} D(x) \vdash \mathrm{ind}_{\Sigma}(\mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x)\, y, z) : D(\mathrm{true}) + D(\mathrm{false})}$$

$$\dfrac{z : \sum_{x:\mathrm{bool}} D(x) \vdash \mathrm{ind}_{\Sigma}(\mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x)\, y, z) : D(\mathrm{true}) + D(\mathrm{false})}{\vdash \lambda z.\, \mathrm{ind}_{\Sigma}(\mathrm{ind}_{\mathrm{bool}}(\lambda z.\mathrm{inl}\,z, \lambda z.\mathrm{inr}\,z, x)\, y, z) : \left[\sum_{x:\mathrm{bool}} D(x)\right] \to D(\mathrm{true}) + D(\mathrm{false}).}$$

__The natural numbers__ $\mathbb{N}$

$\mathbb{N}$-form:
$$\dfrac{}{\vdash \mathbb{N}\ \text{type}}$$

$\mathbb{N}$-intro:
$$\dfrac{}{\vdash 0 : \mathbb{N}} \qquad \dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \ldots}$$

$\vdash 0 : \mathbb{N}$ $\qquad\qquad$ $\Gamma \vdash sn : \mathbb{N}$

$\mathbb{N}$-elim:

$$\frac{\Gamma, x:\mathbb{N} \vdash D \text{ type} \qquad \Gamma \vdash a : D[0/x] \qquad \Gamma, x:\mathbb{N}, y:D \vdash b : D[sx/x]}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D}$$

$\mathbb{N}$-comp:

$$\frac{\Gamma, x:\mathbb{N} \vdash D \text{ type} \qquad \Gamma \vdash a : D[0/x] \qquad \Gamma, x:\mathbb{N}, y:D \vdash b : D[sx/x]}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D[0/x] \\ \Gamma, x:\mathbb{N}, y:D \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq b : D[sx/x]\end{array}}$$

Ex. $sss0$ is (in usual parlance) 3.

Ex. $\iota : \mathrm{bool} \to \mathbb{N}$

$$\frac{\dfrac{\vdash s0 : \mathbb{N} \qquad \vdash 0 : \mathbb{N}}{x:\mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(s0,0,x) : \mathbb{N}}}{\vdash \lambda x . \mathrm{ind}_{\mathrm{bool}}(s0,0,x) : \mathrm{bool} \to \mathbb{N}}$$

Ex. $\mathrm{add} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$x:\mathbb{N} \vdash 0 : \mathbb{N}$

$x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash sz : \mathbb{N}$

$$x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash sz:\mathbb{N}$$

$$x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N}$$

$$x:\mathbb{N} \vdash \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N} \to \mathbb{N}$$

$$\lambda x. \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, sz, y): \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$

<u>Ex</u>. mult: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$x:\mathbb{N} \vdash 0:\mathbb{N}$$

$$x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash \mathrm{add}(z, y)$$

$$x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,y)):\mathbb{N}$$

$$x:\mathbb{N} \vdash \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,y)):\mathbb{N} \to \mathbb{N}$$

$$\lambda x. \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,y)): \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$