# Equivalences and contractibility

## Equivalences

We now have a notion of sameness for terms ($=$ type)
but what about types?

We can also use $=$-types for types:

$$A =_{U} B$$

but how can we construct an equality?

Ex. $? : \mathrm{bool} =_{U} \mathrm{unit} + \mathrm{unit}$

We introduce a notion of equivalence so that

Ex. $\mathrm{bool} \simeq \mathrm{unit} + \mathrm{unit}$

Would expect this to consist of

$$f : \mathrm{bool} \to \mathrm{unit} + \mathrm{unit} \qquad g \circ f = \mathrm{id}_{\mathrm{bool}}$$

$$\mathrm{bool} \leftarrow \mathrm{unit} + \mathrm{unit} : g \qquad f \circ g = \mathrm{id}_{\mathrm{unit}+\mathrm{unit}}$$

Two problems.

## Problem 1:

How to prove $g \circ f = id_{bool}$? Same as before.

<u>Definition</u>. Consider $A : U$, $x : A \vdash B(x)$ type, $x : A \vdash f(x), g(x) : B(x)$.
Define
$$f \sim g := \prod_{x:A} fx = gx \,;$$
the type of <u>homotopies</u> from $f$ to $g$.

Now we can show

<u>Lem</u>. $g \circ f \sim id_{bool}$.

<u>Pf</u>. We need to show $g \circ f(x) = id_{bool}(x)$ for all $x : bool$.
But when $x$ is true, both sides are definitionally true.
When $x$ is false, both sides are false. $\square$

## <u>Problem 2</u>.

If we define (for $f : A \to B$)
$$\text{isqEquiv } f := \sum_{g : B \to A} g \circ f \sim id_A \times f \circ g \sim id_B$$

this type is way more complicated than we expect.

**Ex.** $\mathrm{isqEquiv}\ \mathrm{id}_{S^1} :\equiv \sum_{g : S^1 \to S^1} g \circ \mathrm{id}_{S^1} \sim \mathrm{id}_{S^1} \times \mathrm{id}_{S^1} \circ g \sim \mathrm{id}_{S^1}$

$$\simeq \sum_{g : S^1 \to S^1} g \sim \mathrm{id}_{S^1} \times g \sim \mathrm{id}_{S^1}$$

consider $\mathrm{id}_{S^1}$ for $g$. Then we have a component

$$\underbrace{\mathrm{id}_{S^1} \sim \mathrm{id}_{S^1}} \times \mathrm{id}_{S^1} \sim \mathrm{id}_{S^1}$$

$$\prod_{x : S^1} x = x \simeq \mathbb{Z}$$

(Actually $\mathrm{isqEquiv}\ \mathrm{id}_{S^1} \simeq \mathbb{Z}$.)

We want isEquiv to be simple, i.e. a proposition.

**Def.** Consider $P : \mathcal{U}$. $P$ is a (homotopy) proposition if

$$\mathrm{isProp}(P) :\equiv \prod_{p,q : P} p = q.$$

So 'classically' $P$ is either empty or a singleton.

**Def.** Let $f : A \to B$. We define

$$\mathrm{isEquiv}\ f :\equiv \sum_{g : B \to A} g \circ f \sim \mathrm{id}_A \times \sum_{h : A \to B} f \circ g \sim \mathrm{id}_B$$

If, isEquiv $f$ is inhabited we say <u>$f$ is an equivalence</u>. We define

$$A \simeq B := \sum_{f : A \to B} \mathrm{isEquiv}(f),$$

<u>the type of equivalences</u> from $A$ to $B$.

<u>Thm</u>. There are functions

$$\text{equiv-to-qequiv} : \mathrm{isEquiv}\, f \rightleftarrows \mathrm{isqEquiv}\, f : \text{qequiv-to-equiv}$$

<u>Pf</u>. Let

$$\text{qequiv-to-equiv}\,(g, H, K) := (g, H, g, K).$$

Consider $(g, H, h, K) : \mathrm{isEquiv} f$.

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[rr, bend left, "K \Uparrow"] & A \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}"', "H \Downarrow"] & B \arrow[r, "g"] & A
\end{tikzcd}$$

Then $g \circ f \circ h : B \to A$, and

$$H \cdot (\mathrm{id}_g \circ K \circ \mathrm{id}_f) : g \circ f \circ h \circ f \sim \mathrm{id}$$

$$K \cdot (\mathrm{id} \circ H \circ \mathrm{id}) : f \circ g \circ f \circ h \sim \mathrm{id}.$$

Thus, set

$$\text{equiv-to-qequiv}(g, H, h, K) := \left(g \circ f \circ h,\; H \cdot (\mathrm{id}_g \circ K \circ \mathrm{id}_f),\; K \cdot (\mathrm{id} \circ H \circ \mathrm{id})\right). \qquad \square$$

Thus, they are logically equivalent.

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## <u>Contractibility</u>

The definition of proposition is a way of expressing that a type has at most one element.

Now, we define the notion of having exactly one element.

**Def.** Consider $A : U$. We define

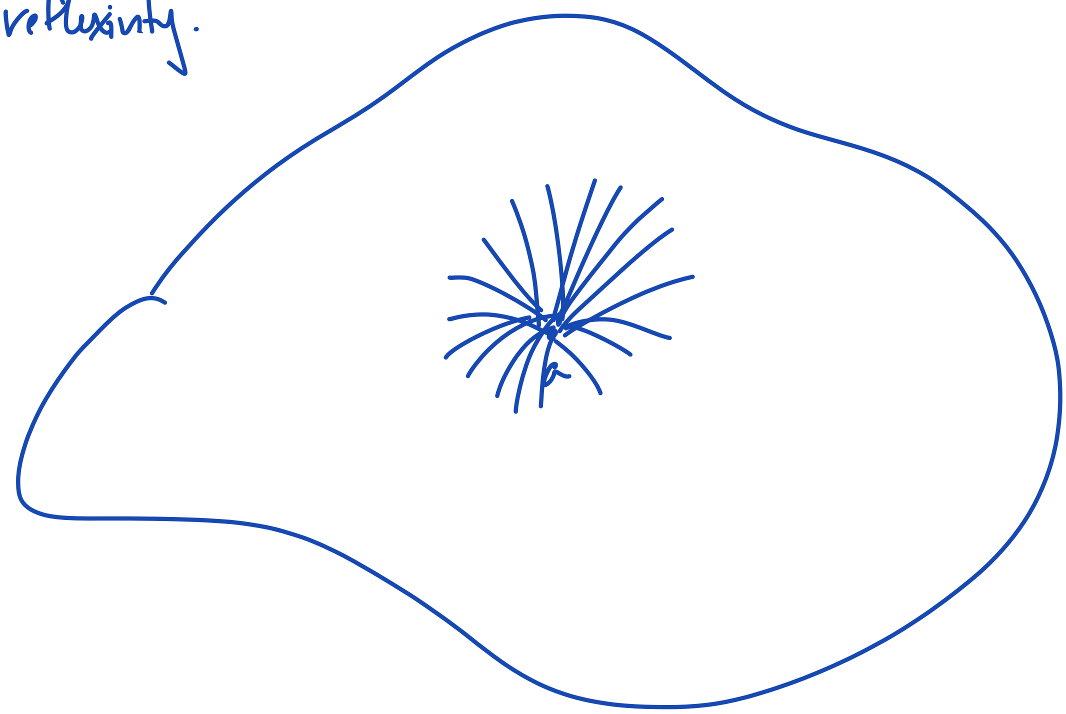
$$\mathrm{isContr}\, A := \sum_{a:A} \prod_{x:A} a = x.$$

When $\mathrm{isContr}\, A$ is inhabited, we say that $A$ is contractible. We call $c : \mathrm{isContr}\, A$ a <u>contraction</u> of $A$, and $\pi_1 c : A$ the <u>center of contraction</u>.

**Thm.** For any type $A$ and any $a:A$, $\sum_{x:A} a=x$ is contractible.

(Stands for $\prod_{A:U} \prod_{a:A} \mathrm{isContr} \sum_{x:A} a=x$)

**Pf.** Let $(a, r_a)$ be the center of contraction.
Now consider $(x,p) : \sum_{x:A} a=x$. By path induction, we take $p$ to be $r_a$. We need an equality $(a,r_a) = (a,r_a)$, but we take reflexivity. $\square$



# Back to equivalences

In classical math, we can define a bijection as a function whose fibers are all singletons.

preimages of singletons

**Def.** Consider $f : A \to B$ and $b : B$. The (homotopy) fiber of $f$ at $b$ is defined to be

$$\mathrm{fib}_f(b) :\equiv \sum_{a:A} fa = b.$$

**Def.** A function $f : A \to B$ is contractible (in reality an equivalence) if all of its fibers are contractible: that is,

$$\mathrm{isContr}(f) :\equiv \prod_{b:B} \mathrm{isContr}(\mathrm{fib}_f(b)).$$

**Thm.** $\mathrm{isContr}(f) \simeq \mathrm{isEquiv}(f)$.

**Lem.** Consider a contractible function $f : A \to B$. Then there exists a function $g : B \to A$.

**Pf.** Let $\alpha : \mathrm{isContr}(f) \doteq \prod_{b:B} \sum_{c : \sum_{a:A} fa = b} \prod_{x : \sum_{a:A} fa = b} c = x$

be the contraction.

Then $\chi b : \sum_{c : \sum_{a:A} fa = b} \prod_{x : \sum_{a:A} fa = b} c = x$, so $\pi_1 \chi b : \sum_{a:A} fa = b$.

Thus, we take $gb :\equiv \pi_1 \pi_1 \chi b$. Since $\pi_2 \pi_1 \chi b : fgb = b$, we have $fg \sim id_B$.

Now consider $a : A$ with the goal of showing $gfa = a$, thus $gf \sim id_A$. We have $gfa \doteq \pi_1 \pi_1 \chi fa$.

Now $\pi_2 \chi fa : \prod_{x : \sum_{a:A} fa = fa} \pi_1 \chi fa = x$, so

$$\pi_2 \chi fa (a, r_{fa}) : \pi_1 \chi fa = (a, r_{fa})$$

Under $ap_{\pi_1}$, we find an equality

$$\pi_1 \pi_2 \chi fa (a, r_{fa}) : \pi_1 \pi_1 \chi fa = a.$$

$\square$