Category theory

Def. Grpd $:= \sum_{T:U}$ is of h-level $3\ T$.

We have a univalence principle

Thm. $(G =_{Grpd} H) \simeq (G \simeq H)$.

But what about categories?

How do we define categories in UF?

- A category is normally defined as a bunch of sets.
- We could do this, but
  - we stay in the set level (everything in mathematics is a bunch of sets)
  - the SIP for structures on sets tells us that

    $$(\mathcal{C} =_{Cat} \mathcal{D}) \simeq (\mathcal{C} \cong \mathcal{D})$$

    and isomorphism is not the right kind of sameness for categories.

Instead, we take a groupoid and put extra structure on it.

- Every category has a 'core groupoid'
  - the objects and all invertible morphisms.

Def. A category $\mathscr{C}$ consists of

- a ~~set~~ type $ob\mathscr{C}$
- a set $hom(X,Y)$ for every pair $X,Y \in ob\mathscr{C}$

  $X,Y : ob\mathscr{C} \vdash hom(X,Y) : Set$
- an ~~element~~ term $1_X \in hom(X,X)$ for every $X \in ob\mathscr{C}$

  $X : ob\mathscr{C} \vdash 1_X : hom(X,X)$
- a function $\circ : hom(X,Y) \times hom(Y,Z) \to hom(X,Z)$ for every $X,Y,Z \in ob\mathscr{C}$.

  $X,Y,Z : ob\mathscr{C} \vdash \circ : hom(X,Y) \to hom(Y,Z) \to hom(X,Z)$
- such that

univalence
- the morphism $idtoiso : (X = Y) \to iso(X,Y)$ is an equivalence

  $\left(iso(X,Y) := \sum_{f : hom(X,Y)} \sum_{g : hom(X,Y)} g \circ f = 1_X \times f \circ g = 1_Y \right).$

. . .

i.e., the type $\mathcal{C}at := \sum_{ob\mathscr{C} : Grpd} \; \sum_{hom : ob\mathscr{C} \to ob\mathscr{C} \to Set} \; \sum_{1 : \prod_{X : ob\mathscr{C}} hom\, X} \cdots$

NB. This is often called a univalent category and the requirement that idtoiso is an equivalence is called (internal) univalence.

Cf. complete Segal spaces

<u>Thm (univalence for categories)</u> $(\mathcal{C} =_{\mathrm{Cat}} \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$.

<u>Cor</u> $\mathcal{C}at$ is a 2-groupoid (h-level 4).

This is a great acheivement for UF.
→ evil vs non evil, practice of CT
→ now part of the theory

The type $\mathcal{C}at$ consists of
- terms → categories
- equalities → equivalence of categories

What about functors, natural transformations?

<u>Def</u>. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of
- a function $\mathrm{ob}F : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$
- functions $\hom_{\mathcal{C}}(X,Y) \to \hom_{\mathcal{D}}(FX, FY)$ for all $X, Y \in \mathcal{C}$
- such that ...

The type of functors $[\mathcal{C}, \mathcal{D}]$ is not a set
- terms → functors
- equalities → natural transformations

Def. A natural transformation $\eta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ consists of

- $\eta_X : \hom(FX, GX)$ for all $X : \mathrm{ob}\,\mathcal{C}$
- $$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\overset{\varepsilon_f}{=}"] & FY \arrow[d, "\eta_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$ for all $X, Y : \mathrm{ob}\,\mathcal{C}$, $f : \hom(X, Y)$

The type of natural transformations $F \Rightarrow G$ is a <u>set</u>.

We can form a <u>bicategory</u> of categories.

→ Have univalence for bicategories.

→ Have univalence for any higher (but finite) algebraic str.

Back to lower dimensions:

- can put category structure on Grp (this is univalent)
or rings ...

<u>NB</u>. We can also define categories in a naive way (sets of objects, morphisms): call these <u>set</u> categories. They form a univalent category where.

NB. There is no 1-category of univalent categories.

More precisely.

Def.
An equivalence of categories $\mathcal{C} \simeq \mathcal{D}$ is a functor
$F : \mathcal{C} \to \mathcal{D}$
which is fully faithful and essentially surjective.

NB. Since our notion of exists is constructive, we don't need AC to find an inverse.

Thm. Given univalent categories $\mathcal{C}, \mathcal{D}$, we find
$$\text{id-to-equiv} : \mathcal{C} = \mathcal{D} \to \mathcal{C} \simeq \mathcal{D}$$
given by refl $\mapsto$ id. Then id-to-equiv is an equivalence.

Rezk completion

Given a nonunivalent category, we want a way to make it univalent.

Eg. The Kleisli category is not necessarily univalent.

Given such a $\mathcal{C}$, we seek a universal univalent category $\hat{\mathcal{C}}$.

That is, a category $\hat{\mathcal{C}}$ with an equivalence $\mathcal{C} \to \hat{\mathcal{C}}$.

Thm. There is such a $\hat{\mathcal{C}}$ for every $\mathcal{C}$.

<u>Pf sketch</u>. We take $\hat{\mathcal{C}}$ to be the essential image of the embedding

$$y : \mathcal{C} \to [\mathcal{C}^{op}, Set].$$

$[\mathcal{C}^{op}, Set]$ is a univalent category (since Set is), and so is any full subcategory of a univalent category.

Since y is fully faithful, $y : \mathcal{C} \to \hat{\mathcal{C}}$ is an equivalence.

<u>Higher inductive types</u>

In a nonunivalent category, we have a type of objects and an equivalence relation presented by the isos that we would like to quotient by.

So we take a higher inductive type for the new type of objects given by the following constructors:

$$\frac{X : \mathrm{ob}\,\mathcal{C}}{i\,X : \mathrm{ob}\,\hat{\mathcal{C}}} \qquad \frac{e : X \cong Y}{je : iX = iY} \qquad \frac{x : A}{p : j1_x = r_{ix}} \qquad \frac{e : X \cong Y \quad f : Y \cong Z}{\chi : (je \cdot jf) = j(e \circ f)} \qquad \frac{\alpha\,\beta : p \underset{x=y}{=} q}{\alpha = \beta}$$

This gives us a groupoid-level truncation of $\mathrm{ob}\,\hat{\mathscr{C}}$ by the isomorphisms.
Tedious but straightforward to check $\hat{\mathscr{C}}$ is a category and $\mathscr{C} \to \hat{\mathscr{C}}$ is an equivalence.

<u>Point</u>: HoTT (homotopy in general) is a robust language for talking about 'proof relevant sameness'.

This is important in category theory and elsewhere...