Specific identity types

$\times$-types.

$$A:\mathcal{U}, B:\mathcal{U}, p:A\times B, q:A\times B \vdash \dots$$

$$\left(\prod_{A,B:\mathcal{U}} \prod_{p,q:A\times B} p = q \simeq \dots\right)$$

Thm. For $p,q : A\times B$, we have

$$p =_{A\times B} q \simeq (\pi_1 p =_A \pi_1 q)\times(\pi_2 p =_B \pi_2 q).$$

Pf. $(\rightarrow)$. To produce a function

$$(p = q) \xrightarrow{f} (\pi_1 p = \pi_1 q)\times(\pi_2 p = \pi_2 q),$$

it suffices to define it on reflexivity, by path induction.

So set

$$f(r_p) :\doteq (r_{\pi_1 p}, r_{\pi_2 p}).$$

$(\leftarrow)$. To produce a function

$$\prod_{p,q:A\times B} (\pi_1 p = \pi_1 q)\times(\pi_2 p = \pi_2 q) \xrightarrow{g} (p = q)$$

it suffices to assume that p is of the form (a_1, b_1), q is of the form (a_2, b_2) and then define it on pairs $(e, f) : (a_1 = a_2)\times(b_1 = b_2)$ by the induction principle for $\times$-types. By path induction, we can assume that e

and f are of the form r_a and r_b respectively. Then we set

$$g(r_a, r_b) :\doteq r_{(a,b)}.$$

$(gf \sim id)$. Now we need to show that for all $e : p = q$,

$$gfe = e.$$

But again, it suffices to assume that e is $r_{(a,b)}$. Then since

$$\begin{aligned} gf\, r_{(a,b)} &\doteq g(r_a, r_b) \\ &\doteq r_{(a,b)} \end{aligned}$$

we are done.

$(fg \sim id)$ We need to show

$$fg(e,f) = (e,f).$$

Again, we assume e, f are of the form r_a, r_b respectively, so since

$$\begin{aligned} fg(r_a, r_b) &\doteq f\, r_{(a,b)} \\ &\doteq (r_a, r_b) \end{aligned}$$

we are done.

Thus, we have shown that the two types are quasi-equivalent. Therefore (by previous theorem) they are equivalent. $\square$

Σ-types

Thm. Given $p, q : \sum_{x:A} B(x)$, we have

$$(p = q) \simeq \sum_{e : \pi_1 p = \pi_1 q} tr_{\pi_1 p}\, \pi_2 p = \pi_2 q .$$

Pf. Exercise (see last week's exercises).

Example. Let hSpc be the type of types with a multiplication and identity:

$$hSpc := \sum_{T:U} \sum_{m : T \times T \to T} \sum_{e:T} \prod_{x:T} m(e,x) = x \times m(x,e) = x .$$

Then an identity between $G, H : hSpc$ is a tuple of an equality between the underlying spaces, an equality between the multiplications, etc.

$$(G =_{hSpc} H) \simeq \sum_{e : \pi_1 G \underset{U}{=} \pi_1 H} \sum_{f : tr_e \pi_2 G = \pi_2 H} \sum_{g : tr_e \pi_3 G = \pi_3 H} \cdots$$

In general.

Thm. Consider a type A and term $a : A$, consider also a dependent type $x : A \vdash B(x)$ and a term $b : B(a)$. The following are equivalent

1) $\sum_{x:A} B(x)$ is contractible.

2) We have equivalences $f_x : a = x \simeq B(x)$ for all x, where

$$f : \prod_{x:A} a = x \to B(x)$$

is defined by sending r_a to b.

Pf. By Thm 11.1.3 (in Exercises for today), each f_x is an equivalence if and only if the induced map

$$\sum_{x:A} a = x \xrightarrow{tot_f} \sum_{x:A} B(x)$$

is an equivalence. But $\sum_{x:A} B(x)$ is contractible (previous thm).

Since being contractible is equivalent to the canonical map $T \to$ unit being an equivalence (Ex 10.3 a in R), if tot_f is an equivalence, $\sum_{x:A} B(x)$ is contractible.

If $\sum_{x:A} B(x)$ is contractible, then its canonical map to the unit is an equivalence. Since equivalences satisfy 2-out-of-3 (Ex. 9.4c in R), we find that tot_f is an equivalence.

$\square$

The booleans.

Def. Define a function $E : \text{bool} \to \text{bool} \to \mathcal{U}$ by

$$E(\text{true},\text{true}) :\doteq \mathbb{1}$$
$$E(\text{false},\text{false}) :\doteq \mathbb{1}$$
$$E(\text{true},\text{false}) :\doteq \emptyset$$
$$E(\text{false},\text{true}) :\doteq \mathbb{1},$$

Def. Define the function

$$z : \prod_{x,y:\text{bool}} x = y \longrightarrow E(x,y)$$

by sending $r_{\text{true}} \longmapsto * : \mathbb{1}$, $r_{\text{false}} \longmapsto * : \mathbb{1}$.

Thm. For each $x,y : \text{bool}$, $=\text{-to-}E_{x,y}$ is an equivalence.

Pf. Using the previous theorem, we show that $\sum_{y:\text{bool}} E(x,y)$ is contractible for each $x:\text{bool}$.

We define the center of contraction to be $(x, z_{x,x}\, r_x)$.

We need a term of

$$\prod_{(y,e):\Sigma E} (x, 2_{xx} r_x) = (y,e).$$

Consider such y,e. By the characterization of $=$-types in Σ-types, we need to find a term in

$$\sum_{p:x=y} tr_p\, 2_{xx} r_x = e.$$

By induction on x,y there are four cases.

1. $\sum_{p:\mathrm{true}=\mathrm{true}} tr_p * =_{\mathbb{1}} e.$

   Since $\mathbb{1}$ is contractible, there is a path $c : tr_p * = e$ (by Exercises from last time).

   Then we choose (p,c).

2. $\sum_{p:\mathrm{false}=\mathrm{false}} tr_p * =_{\mathbb{1}} e.$

   Same as 1.

3. $\sum_{p:\mathrm{true}=\mathrm{false}} tr_p\, 2_{xx} r_x =_{\emptyset} e$

   In this case, we suppose $e : E(\mathrm{true}, \mathrm{false}) \doteq \emptyset$. By $\emptyset$-induction, we obtain a term.

4. $$\sum_{p:\text{false}=\text{true}} \mathrm{tr}_p * =_{\phi} e.$$

Same as 3. $\square$