# An Introduction to Combinatorial Structures

Combinatorics

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**Aim of the paper:**
This work aims to introduce the theory of Dyck paths, highlight their connection to the Catalan numbers, and explore a few related objects such as Schröder and Motzkin paths, using a unifying solution method.

# 1 Dyck Paths

Dyck paths are lattice paths in the plane that start at $(0,0)$ and end at $(2n,0)$. A Dyck path consists of steps of the form $(x+1,y+1)$ (up) or $(x+1,y-1)$ (down), where $(x,y)$ denotes the starting point of the step. The fundamental condition is that the path never goes below the $x$–axis. We will refer to this requirement as the *deficit condition*. In Fig. 1 we have an example of a Dyck path, while Fig. 2 shows a path that violates the deficit condition. In both figures we took $n=6$.

Observe that the number of $(1,1)$ steps equals the number of $(1,-1)$ steps. In general, for a path of length $2n$ there are exactly $n$ up steps and $n$ down steps. The number of Dyck paths for a fixed $n$ is given by the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad \binom{2n}{n} = \frac{(2n)!}{n!\,n!}.$$

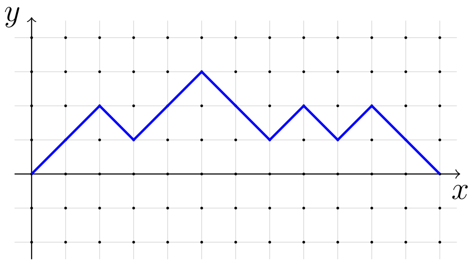


Fig. 1

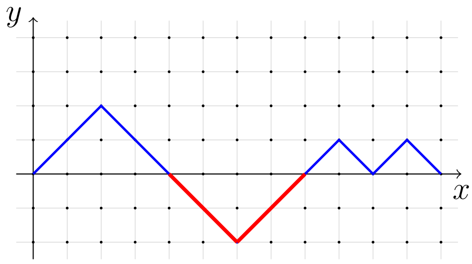


Fig. 2

Next we introduce a key concept for the problems that follow. The *deficit* of a path is the total number of steps taken strictly below the $x$–axis. For example, the path in Fig. 1 has deficit 0 (as do all Dyck paths), whereas the path in Fig. 2 has deficit 4 (the red segments). The deficit is always even. Our goal is to determine the number of paths with deficit 0, i.e., the Dyck paths.

## 1.1 Solution

We begin by counting all paths from $(0,0)$ to $(2n,0)$ without enforcing the deficit condition. To build a length-$2n$ path with $n$ up and $n$ down steps, we only need to choose which $n$ of the $2n$ positions are up steps. This yields $\binom{2n}{n}$ possibilities.

Let $D_k$ be the set of paths with deficit $k$ (with $k$ even). For any natural $n$ we have

$$\sum_{i=0}^{n} |D_{2i}| = \binom{2n}{n}.$$

We will prove that

$$|D_0| = |D_2| = \cdots = |D_{2n}|.$$

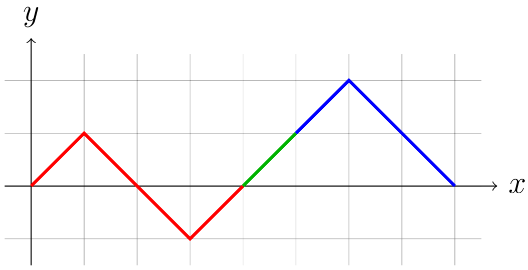


Fig. 3

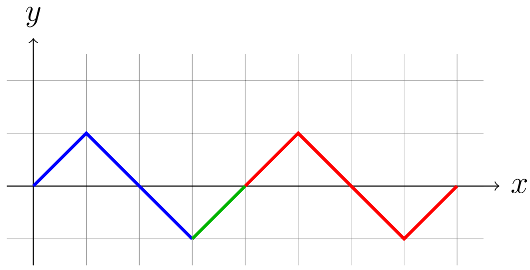


Fig. 4

To understand why these cardinalities are equal, consider a path with deficit 2.

It suffices to understand the behavior in Fig. 3 and Fig. 4 to generalize. In Fig. 3 the path (with deficit 2) is split into three subsequences (colored) that we recombine via translations: select the *last* up-step that leaves the $x$–axis (green). Split the remaining parts into a red and a blue block. Now move the blue block to the origin, attach the green step, and then append the red block. During this operation we do not change the total counts of up and down steps. From a sequence of deficit 2 we obtained one of deficit 4. Conversely, the process is invertible: a deficit-4 sequence can be uniquely mapped back to a deficit-2 sequence.

The algorithm does not rely essentially on the numbers 2 and 4. A path of deficit 6 can be transformed into one of deficit 8, and vice versa. In general there is a bijection between paths of deficit $2k$ and $2k+2$. Hence there exists a bijective map

$$f : D_{2k} \longrightarrow D_{2k+2},$$

and therefore

$$|D_{2k}| = |D_{2k+2}|.$$

Since $k$ was arbitrary, all sets $D_{2i}$ have the same size:

$$|D_0| = |D_2| = \cdots = |D_{2n}|.$$

Thus

$$\sum_{i=0}^{n} |D_{2i}| = (n+1)\,|D_0|,$$

so that

$$|D_0| = \frac{1}{n+1}\binom{2n}{n},$$

which is precisely the Catalan number.

### 1.2 Related viewpoints

A classical interpretation of Dyck paths is in terms of correctly balanced parentheses: map $(1,1)$ to "(" and $(1,-1)$ to ")". The condition of never going below the $x$–axis is equivalent to requiring that in every prefix the number of opening parentheses is at least the number of closing ones, and at the end the two counts are equal. Thus Dyck paths of order $n$ correspond bijectively to balanced parenthesis strings of length $2n$. For $n=3$ the five strings are

$$((())), \quad (()()), \quad (())(), \quad ()(()), \quad ()()().$$

## 2 Schröder Paths

A Schröder path of order $n$ is a lattice path from $(0,0)$ to $(2n,0)$ made of three elementary steps: $(1,1)$ (up), $(1,-1)$ (down), and $(2,0)$ (horizontal). The path must never go below the $x$–axis, just like in the Dyck case. The number of Schröder paths of order $n$ is denoted $S_n$ (the large Schröder number). These generalize Dyck paths by allowing horizontal steps. We now adapt the previous method.

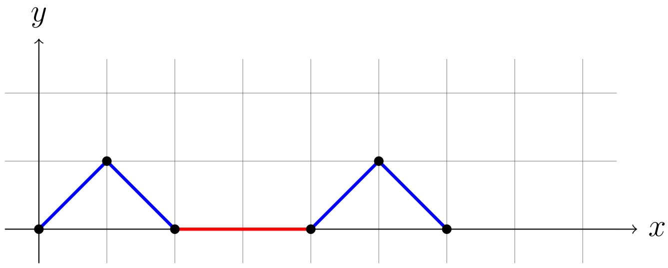


Fig. 5

### 2.1 Solution

Fix $k$ horizontal steps. Then there are $2(n-k)$ vertical steps; horizontals do not affect the deficit. Ignore the horizontals (keeping their relative positions) and apply the Dyck bijection on the $2(n-k)$ vertical steps. This yields, for fixed $k$,

$$|D_0^{(k)}| = |D_2^{(k)}| = \cdots = |D_{2(n-k)}^{(k)}|.$$

The total number of paths with exactly $k$ horizontals (without deficit restriction) is

$$\sum_{i=0}^{n-k} \left| D_{2i}^{(k)} \right| = \frac{(2n-k)!}{k!\,(n-k)!\,(n-k)!}.$$

Since there are $n-k+1$ equal classes, we get

$$\left|D_0^{(k)}\right| = \frac{1}{n-k+1} \cdot \frac{(2n-k)!}{k!\,(n-k)!\,(n-k)!}.$$

Summing over all $k$,

$$S_n = \sum_{k=0}^{n} \left|D_0^{(k)}\right| = \sum_{k=0}^{n} \frac{1}{n-k+1} \cdot \frac{(2n-k)!}{k!\,(n-k)!\,(n-k)!}.$$

## 2.2 Related problem

A well-known equivalent formulation: how many lattice paths from $(0,0)$ to $(n,n)$ using $(1,0)$, $(0,1)$, and $(1,1)$ never go *above* the diagonal $y=x$? The "stay under $y=x$" constraint plays the same role as "never go below the $x$–axis," and the diagonal steps $(1,1)$ correspond to the horizontal $(2,0)$ steps. Hence the count is the large Schröder number $S_n$.

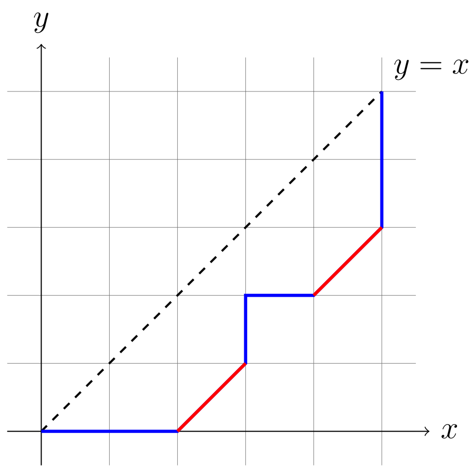


Fig. 6

# 3 Motzkin Paths

A Motzkin path of order $n$ starts at $(0,0)$ and ends at $(n,0)$ using steps $(1,1)$ (up), $(1,-1)$ (down), and $(1,0)$ (horizontal), and never goes below the $x$–axis.

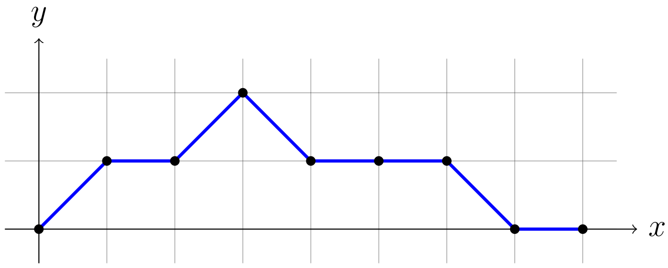


Fig. 7

## 3.1 Solution

Fix $k$ horizontal steps. Then the number of vertical steps is $n-k$, and to end at height 0 we must have exactly $j = \frac{n-k}{2}$ up steps and $j$ down steps (hence $n-k$ is even). Define the deficit as the number of vertical steps taken strictly below the $x$–axis; horizontals do not contribute.

Ignoring horizontals and looking only at the $2j$ vertical steps reduces to the Dyck case: the deficit can be $0, 2, \ldots, 2j$, so there are $(j+1)$ deficit classes. The same bijection on vertical steps shows that, for fixed $k$,

$$|D_0^{(k)}| = |D_2^{(k)}| = \cdots = |D_{2j}^{(k)}|.$$

The total number of sequences with $k$ horizontals and $j$ ups and $j$ downs is

$$\frac{n!}{k!\, j!\, j!},$$

which splits into $(j+1)$ equal classes. Hence

$$|D_0^{(k)}| = \frac{1}{j+1} \cdot \frac{n!}{k!\, j!\, j!}, \qquad j = \frac{n-k}{2}.$$

Summing over all admissible $k$ (with the same parity as $n$) yields

$$M_n = \sum_{\substack{0 \le k \le n \\ n-k \text{ even}}} \frac{1}{\frac{n-k}{2}+1} \cdot \frac{n!}{k!\left(\frac{n-k}{2}\right)!\left(\frac{n-k}{2}\right)!}.$$

Equivalently, with $j = \frac{n-k}{2}$,

$$M_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} C_j, \qquad \text{where} \quad C_j = \frac{1}{j+1}\binom{2j}{j}.$$

### 3.2 Related problems

Motzkin paths can also be interpreted in terms of noncrossing matchings. Consider $n$ points arranged on a line: some of them may be paired by connecting segments, while others may remain single. The restriction is that the segments must not intersect.

The correspondence is natural: an up step $(1,1)$ represents the opening of a pair, a down step $(1,-1)$ its closing, and a horizontal step $(1,0)$ a free (unmatched) point. The condition of never going below the $x$–axis reflects the fact that one cannot close a pair without having opened one first.

Thus, the number of noncrossing matchings with free points is exactly equal to the number of Motzkin paths of order $n$.

# Conclusions

We studied Dyck, Schröder, and Motzkin paths—central objects of combinatorics—and introduced the *deficit method* as a counting technique. This single idea unifies and clarifies several classical results around Catalan-like structures. Beyond theory, these paths arise in many contexts: balanced parenthesizations, binary trees, polygon dissections, and noncrossing matchings. Extending these ideas to weighted paths or additional constraints, as well as exploring links to probability and the analysis of algorithms, are natural directions for further work.

# References

[1] Wikipedia, *Dyck path*, `https://en.wikipedia.org/wiki/Dyck_path`.

[2] Wikipedia, *Schröder number*, `https://en.wikipedia.org/wiki/Schr%C3%B6der_number`.

[3] Wikipedia, *Motzkin number*, `https://en.wikipedia.org/wiki/Motzkin_number`.