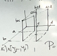
$$\iint_R f(x,y)\,dx\,dy = \text{VOLUME}\left(\{(x,y,z(x,y)) : (x,y)\in R,\ 0\le z(x,y)\le f(x,y)\}\right)$$

$z = f(x,y) = x+y$



$a = x_0 < x_1 < \dots < x_n = b$

$c = y_0 < y_1 < \dots < y_m = d$, $R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$, $P = \bigcup R_{ij}$

$m_{ij} = \text{INF}\{f(x,y),\ (x,y)\in R_{ij}\} = x_{i-1} + y_{j-1}$; $M_{ij} = x_i + y_j$

$$\sum c_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

$$L_P(f) = \sum m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \le \frac{1}{2}(b^2-a^2)(d-c) + \frac{1}{2}(d^2-c^2)(b-a) \le U_P(f) = \sum M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

LET'S SEE THAT f IS INTEGRABLE:

$m_{ij} \le c_{ij} = \frac{x_i + x_{i-1}}{2} + \frac{y_j + y_{j-1}}{2} \le M_{ij}$

$\sum \frac{(x_i + x_{i-1})}{2}(x_i - x_{i-1})(y_j - y_{j-1}) + \dots$

$\frac{1}{2}\sum_{ij}(x_i^2 - x_{i-1}^2)(y_j - y_{j-1}) + \dots$

$= \frac{1}{2}\sum_i (x_i^2 - x_{i-1}^2)\cdot\sum_j (y_j - y_{j-1})$

$$U_P(f) - L_P(f) = \sum (M_{ij} - m_{ij})(x_i - x_{i-1})(y_j - y_{j-1})$$
$$= \sum\left[(x_i - x_{i-1}) + (y_j - y_{j-1})\right](x_i - x_{i-1})(y_j - y_{j-1})$$

ASSUME: $x_i - x_{i-1} = \frac{b-a}{m}$, $y_j - y_{j-1} = \frac{d-c}{m}$

$$U_P(f) - L_P(f) = \left(\frac{b-a}{m} + \frac{d-c}{m}\right)\sum_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$
$$= \left(\frac{b-a}{m} + \frac{d-c}{m}\right)\sum_i (x_i - x_{i-1})\cdot\sum_j (y_j - y_{j-1}) = \left(\frac{b-a}{m} + \frac{d-c}{m}\right)(b-a)(d-c) < \varepsilon$$

FOR m, m BIG ENOUGH.

$\Longrightarrow f$ IS INTEGRABLE AND

$$\iint_R (x+y)\,dx\,dy = \sup\{L_P(f), P\} = \text{INF}\{U_P(f), P\} = \frac{1}{2}(b^2-a^2)(d-c) + \frac{1}{2}(d^2-c^2)(b-a)$$

ITERATED INTEGRALS

EXAMPLE: $f(x,y) = x+y$, FIX y_0 AND CONSIDER $h(x) = f(x,y_0) = x+y_0$

$$\int_a^b h(x)\,dx = \int_a^b (x+y_0)\,dx = \frac{x^2}{2} + y_0 x\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2} + y_0(b-a)$$

$$g(y) = \int_a^b f(x,y)\,dx = \frac{b^2-a^2}{2} + y(b-a)$$

$\frac{1}{2}(b^2-a^2)(d-c) + \frac{1}{2}(d^2-c^2)(b-a)$

$$\int_c^d g(y)\,dy = \int_c^d\left(\frac{b^2-a^2}{2} + y(b-a)\right)dy = \frac{b^2-a^2}{2}y + (b-a)\frac{y^2}{2}\Big|_c^d =$$
$$= \frac{1}{2}(b^2-a^2)(d-c) + \frac{1}{2}(b-a)(d^2-c^2)$$

$$I_1 = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy \quad ; \quad I_2 = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

THEO: IF $z = f(x,y)$ IS CONTINUOUS ON $R = [a,b]\times[c,d]$, THEN BOTH ITERATED INTEGRALS

$$I_1 = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy \quad \text{AND} \quad I_2 = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx \quad \text{EXIST.}$$

PROOF. THE FUNCTION $h(x) = f(x,y)$ IS CONTINUOUS ON $[a,b]$ SINCE

$f(x,y)$ IS CONT. ON ANY $(x_0,y_0)\in R$,

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \longrightarrow y = y_0$; $\lim_{x\to x_0} f(x,y_0) = f(x_0,y_0)$, $\forall y$

$\lim_{x\to x_0} h(x) \quad h(x_0)$

$\longrightarrow h(x)$ IS INTEGRABLE ON $[a,b] \Longrightarrow F(y) = \int_a^b f(x,y)\,dx$ EXISTS.

NOW WE HAVE TO PROVE THAT $F(y)$ IS INTEGRABLE ON $[c,d]$

WE WILL SEE THAT $F(y)$ IS CONTINUOUS AT $[c,d] \longrightarrow$ IT IS INTEGRABLE.

$f(x,y)$ IS CONT ON R, R IS CLOSED AND BOUNDED $\Longrightarrow f(x,y)$ IS UNIFORMLY CONT

$\forall\varepsilon>0, \exists\delta>0 \quad \|(x,y)-(x',y')\| < \delta \Longrightarrow |f(x,y) - f(x',y')| < \varepsilon$

IF $|y-y'| < \delta \Longrightarrow \|(x,y)-(x,y')\| = \sqrt{(x-x)^2 + (y-y')^2} = |y-y'| < \delta \Longrightarrow |f(x,y) - f(x,y')| < \varepsilon$

$$\longrightarrow |F(y) - F(y')| = \left|\int_a^b f(x,y)\,dx - \int_a^b f(x,y')\,dx\right| = \left|\int_a^b (f(x,y) - f(x,y'))\,dx\right| \le$$
$$\le \int_a^b |f(x,y) - f(x,y')|\,dx < \int_a^b \varepsilon\,dx = \varepsilon(b-a) = \varepsilon'$$

$\Longrightarrow F(y)$ IS (UNIFORMLY) CONT. ON $[c,d]$, THEN IT IS INTEGRABLE $\longrightarrow I_1$ EXISTS

THEO (LEIBNITZ'S THEOREM ABOUT THE DERIVATIVE)

LET $z = f(x,y)$ BE SUCH THAT $\frac{\partial f}{\partial y}$ IN $(a,b)\times(c,d)$ AND IT IS CONT. ON $R = [a,b]\times[c,d]$

THEN $F(y) = \int_a^b f(x,y)\,dx$ IS DIFFERENTIABLE ON (c,d)

AND $F'(y) = \left(\int_a^b f(x,y)\,dx\right)' = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

PROOF: $\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$ EXISTS SINCE $\frac{\partial f}{\partial y}(x,y)$ IS CONT. ON ANY $(x_0,y_0)\in R$.

THEN $h(x) = \frac{\partial f}{\partial y}(x,y)$ IS CONT. ON x, $\forall x\in[a,b]$.

$\longrightarrow h(x)$ IS INTEGRABLE ON $[a,b]$ AND $\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

EXISTS. LET'S SEE THAT $F'(y) = \lim_{h\to 0}\frac{F(y+h) - F(y)}{h} = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

$$\frac{F(y+h) - F(y)}{h} = \int_a^b \frac{f(x,y+h) - f(x,y)}{h}\,dx \underset{\text{LAGRANGE}}{=\!=} \int_a^b \frac{\frac{\partial f}{\partial y}(x,y+s)\,\cancel{(y+h-y)}}{\cancel{h}}\,dx$$

WITH $y+s$ BETWEEN y AND $y+h \longrightarrow |s| < |h|$

SINCE $\frac{\partial f}{\partial y}(x,y)$ IS CONT., BY THE PREVIOUS THEOREM, $\int_a^b \frac{\partial f}{\partial y}(x,y+s)\,dx$ IS CONTIN.

$$\lim_{h\to 0}\frac{F(y+h) - F(y)}{h} = \lim_{h\to 0}\int_a^b \frac{\partial f}{\partial y}(x,y+s)\,dx = \lim_{s\to 0}\int_a^b \frac{\partial f}{\partial y}(x,y+s)\,dx$$
$$= \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

$\Longrightarrow s\to 0$

COROLLARY: IF $z = f(x,y)$ IS CONTINUOUS ON R THEN BOTH ITERATED INTEGRALS EXIST AND THEY COINCIDE.

PROOF: WE HAVE ALREADY PROVEN THAT $I_1 = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$ AND

$I_2 = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$ EXIST.

WE DEFINE:

$g(t) = \int_c^d\left(\int_a^t f(x,y)\,dx\right)dy \qquad g(a) = 0$

$h(t) = \int_a^t\left(\int_c^d f(x,y)\,dy\right)dx \qquad h(a) = 0$

$$g'(t) = \left(\int_c^d\left(\int_a^t f(x,y)\,dx\right)dy\right)' = \int_c^d \frac{\partial}{\partial t}\left(\int_a^t f(x,y)\,dx\right)dy \quad \text{BY LEIBNITZ SINCE}$$

$\frac{\partial}{\partial t}\left(\int_a^t f(x,y)\,dx\right) \underset{\text{FTC}}{=} f(t,y)$ WHICH IS CONTINUOUS $\Longrightarrow g'(t) = \int_c^d f(t,y)\,dy$

$$h'(t) = \left(\int_a^t\left(\int_c^d f(x,y)\,dy\right)dx\right)' \underset{\text{FTC}}{=} \int_c^d f(t,y)\,dy$$

$=$

$\Longrightarrow (g-h)'(t) = 0\ \forall t \Longrightarrow (g-h)(t) = \text{CONSTANT } K$

BUT $K = (g-h)(a) = g(a) - h(a) = 0 - 0 \Longrightarrow K = 0$ AND $g(t) = h(t)$.

THEO. ASSUME $z = f(x,y)$ IS INTEGRABLE ON $R = [a,b]\times[c,d]$ AND

$F(y) = \int_a^b f(x,y)\,dx$ EXISTS FOR EACH $y\in[c,d]$

THEN F IS INTEGRABLE ON $[c,d]$ AND $\iint_R f(x,y)\,dx\,dy = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

$\Longrightarrow (g-h)'(t) = 0\ \forall t \Longrightarrow (g-h)(t) = \text{CONSTANT } K$

BUT $K = (g-h)(a) = g(a) - h(a) = 0 - 0 \Longrightarrow K = 0$ AND $g(t) = h(t)$.

THEO. ASSUME $z = f(x,y)$ IS INTEGRABLE ON $R = [a,b]\times[c,d]$ AND

$F(y) = \int_a^b f(x,y)\,dx$ EXISTS FOR EACH $y\in[c,d]$

THEN F IS INTEGRABLE ON $[c,d]$ AND $\iint_R f(x,y)\,dx\,dy = \int_c^d\underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{F(y)}dy = \int_c^d F(y)\,dy$

PROOF. WE HAVE TO PROVE THAT $\forall\varepsilon>0$, $\left|\iint_R f\,dx\,dy - \int_c^d F(y)\,dy\right| < \varepsilon$

LET $P = \{a = x_0 < x_1 < \dots < x_n = b\}$, $Q = \{c = y_0 < y_1 < \dots < y_m = d\}$ BE TWO PARTITIONS, AND CHOOSE $c_j\in[y_{j-1}, y_j]$, FOR ANY j.

$m_{ij} = \text{INF}\{f(x,y), (x,y)\in R_{ij}\}$

$M_{ij} = \text{SUP}\{f(x,y), (x,y)\in R_{ij}\}$

$m_{ij} \le f(x,y) \le M_{ij} \quad \forall (x,y)\in R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$.

IN PARTICULAR:

$m_{ij} \le f(x, c_j) \le M_{ij} \quad \forall x\in[x_{i-1}, x_i]$, $\forall i$

$m_{ij}(x_i - x_{i-1}) = \int_{x_{i-1}}^{x_i} m_{ij}\,dx \le \int_{x_{i-1}}^{x_i} f(x,c_j)\,dx \le \int_{x_{i-1}}^{x_i} M_{ij}\,dx = M_{ij}(x_i - x_{i-1})$ $\forall i$

$$\Longrightarrow \sum_i m_{ij}(x_i - x_{i-1}) \le \underbrace{\int_a^b f(x,c_j)\,dx}_{F(c_j)} \le \sum_i M_{ij}(x_i - x_{i-1})$$

$$\longrightarrow \underbrace{\sum_j\sum_i m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})}_{L_P(f)} \le \underbrace{\sum_j F(c_j)(y_j - y_{j-1})} \le \underbrace{\sum_j\sum_i M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})}_{U_P(f)}$$

f INTEGRABLE $\longrightarrow U_P(f) - L_P(f) < \varepsilon$, $\sup\{L_P(f), P\} = \text{INF}\{U_P(f), P\} = \iint_R f(x,y)\,dx$