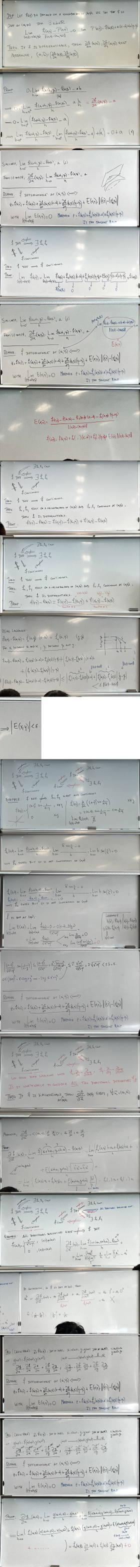

DEF LET $f(x,y)$ BE DEFINED ON A NEIGHBORHOOD OF (x_0,y_0). WE SAY THAT f IS DIFF AT (x_0,y_0) IFF $\exists\, a,b\in\mathbb{R}$:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-P(x,y)}{\|(x,y)-(x_0,y_0)\|}=0 \quad \text{WHERE } P(x,y)=f(x_0,y_0)+a(x-x_0)+b(y-y_0)$$

THEO. IF f IS DIFFERENTIABLE, THEN $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ EXIST.

MOREOVER, $(a,b)=\left(\frac{\partial f}{\partial x}(x_0,y_0),\frac{\partial f}{\partial y}(x_0,y_0)\right)$

PROOF. ....

$$0=\lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)-ah}{|h|}$$

$$\longrightarrow 0=\lim_{h\to0^+}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}-a\frac{h}{h}=\frac{\partial f}{\partial x}(x_0,y_0)-a$$

$$\longrightarrow 0=\lim_{h\to0^+}\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}-a\right)$$

$$\Longrightarrow \lim_{h\to0^+}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=\lim_{h\to0}\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}-a\right)+(a)=0+a \quad (1)$$

SIMILARLY, $\lim_{h\to0^-}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=a$ (2)

FROM (1) AND (2), $\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=a$

REMARK. f DIFFERENTIABLE AT (x_0,y_0) $\Longleftrightarrow$

$$z=f(x,y)=\underbrace{f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\,\|(x,y)-(x_0,y_0)\|$$

WITH $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$. MOREOVER $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ IS THE TANGENT PLANE

$$\begin{tikzcd} f \text{ DIFF} \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] & \exists f_x, f_y \arrow[l, Rightarrow, shift left, "/"] \arrow[dl, "/"] \\ f \text{ CONTINUOUS} \arrow[u, Rightarrow, shift left, "/"] & \end{tikzcd}$$

THEO. f DIFF $\longrightarrow$ f CONTINUOUS.

PROOF.

$$\lim_{(x,y)\to(x_0,y_0)}f(x,y)=\lim_{(x,y)\to(x_0,y_0)}\underbrace{f(x_0,y_0)}_{\downarrow\, f(x_0,y_0)}+\underbrace{f_x(x_0,y_0)(x-x_0)}_{\downarrow\, 0}+\underbrace{f_y(x_0,y_0)(y-y_0)}_{\downarrow\, 0}+\underbrace{E(x,y)}_{\downarrow\, 0}\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\downarrow\, 0}=f(x_0,y_0)$$

SIMILARLY, $\lim_{h\to0^-}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=a$ (2)

FROM (1) AND (2), $\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=a$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-a(x-x_0)-b(y-y_0)}{\|(x,y)-(x_0,y_0)\|}=0$$

$E(x,y)$

REMARK. f DIFFERENTIABLE AT (x_0,y_0) $\Longleftrightarrow$

$$z=f(x,y)=\underbrace{f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\,\|(x,y)-(x_0,y_0)\|$$

WITH $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$. MOREOVER $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ IS THE TANGENT PLANE

$$E(x,y)=\frac{f(x,y)-f(x_0,y_0)-f_x(x_0,y_0)(x-x_0)-f_y(x_0,y_0)(y-y_0)}{\|(x,y)-(x_0,y_0)\|}$$

$$f(x,y)=f(x_0,y_0)+f_x(\ ,\ )(x-x_0)+f_y(\ ,\ )(y-y_0)+E(x,y)\|(x,y)-(x_0,y_0)\|$$

$$\begin{tikzcd} & \exists f_x, f_y \text{ CONT} \arrow[dl, Rightarrow] & \\ f \text{ DIFF} \arrow[rr, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] & & \exists f_x, f_y \arrow[ll, Rightarrow, shift left, "/"] \arrow[dll, "/"] \\ f \text{ CONTINUOUS} \arrow[u, Rightarrow, shift left, "/"] & & \end{tikzcd}$$

THEO: f DIFF $\longrightarrow$ f CONTINUOUS.

THEO. f_x, f_y EXIST ON A NEIGHBORHOOD OF (x_0,y_0) AND f_x, f_y CONTINUOUS AT (x_0,y_0), THEN f IS DIFFERENTIABLE.

PROOF: $f(x,y)-f(x_0,y_0)=f(x,y)-f(x_0,y)+f(x_0,y)-f(x_0,y_0)$

$$\begin{tikzcd} & \exists f_x, f_y \text{ CONT} \arrow[dl, Rightarrow] & \\ f \text{ DIFF} \arrow[rr, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] & & \exists f_x, f_y \arrow[ll, Rightarrow, shift left, "/"] \arrow[dll, "/"] \\ f \text{ CONTINUOUS} \arrow[u, Rightarrow, shift left, "/"] & & \end{tikzcd}$$

THEO: f DIFF $\longrightarrow$ f CONTINUOUS.

THEO. f_x, f_y EXIST ON A NEIGHBORHOOD OF (x_0,y_0) AND f_x, f_y CONTINUOUS AT (x_0,y_0), THEN f IS DIFFERENTIABLE.

PROOF:

$$f(x,y)-f(x_0,y_0)=\underbrace{\overset{h(x)-h(x_0)}{f(x,y)-f(x_0,y)}}_{\text{FUNCTION IN } x}+\underbrace{\overset{k(y)-k(y_0)}{f(x_0,y)-f(x_0,y_0)}}_{\text{FUNCTION IN } y}$$

USING LAGRANGE:

$$f(x,y)-f(x_0,y_0)=f_x(x_1,y)(x-x_0)+f_y(x_0,y_1)(y-y_0)$$

FOR x_1 BETWEEN x_0 AND x, y_1 BETWEEN y_0 AND y.

$$f(x,y)-f(x_0,y_0)=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+\left(f_x(x_1,y)-f_x(x_0,y_0)\right)(x-x_0)+$$

$$+\left(f_y(x_0,y_1)-f_y(x_0,y_0)\right)(y-y_0)$$

$$\left|f(x,y)-f(x_0,y_0)-f_x(x_0,y_0)(x-x_0)-f_y(x_0,y_0)(y-y_0)\right|\le\overbrace{\left|f_x(x_1,y)-f_x(x_0,y_0)\right|}^{<\delta}\overset{\le\|(x,y)-(x_0,y_0)\|}{|x-x_0|}+\overbrace{\left|f_y(x_0,y_1)-f_y(x_0,y_0)\right|}^{<\delta}\overset{\le\|(x,y)-(x_0,y_0)\|}{|y-y_0|}$$

$$<\varepsilon\,\|(x,y)-(x_0,y_0)\|$$

$$\Longrightarrow |E(x,y)|<\varepsilon.$$

$$\begin{tikzcd} & \exists f_x, f_y \text{ CONT} \arrow[dl, Rightarrow, shift left] & \\ f \text{ DIFF} \arrow[rr, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] \arrow[ur, Rightarrow, shift left, "/"] & & \exists f_x, f_y \arrow[ll, Rightarrow, shift left, "/"] \arrow[dll, "/"] \\ f \text{ CONTINUOUS} \arrow[u, Rightarrow, shift left, "/"] & & \end{tikzcd}$$

$$\begin{tikzcd} & f \text{ DIFF} \arrow[dl, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] \arrow[r, Rightarrow, shift left, "/"] & \exists f_x, f_y \text{ CONT} \arrow[l, Rightarrow, shift left] \\ f \text{ CONT} \arrow[ur, Rightarrow, shift left, "/"] & & \exists f_x, f_y \arrow[ul, Rightarrow, shift left, "/"] \arrow[ll, Rightarrow, "/"] \end{tikzcd}$$

EXAMPLE: f DIFF $\not\Longrightarrow$ f_x, f_y EXIST AND CONTINUOUS

$$f(x,y)=\begin{cases}(x-y)^2\sin\frac{1}{x-y}, & x\neq y\\ 0 & x=y\end{cases}$$

$$f_x(x,y)=\frac{\partial}{\partial x}\left((x-y)^2\sin\frac{1}{x-y}\right)\qquad x\neq y$$

$$=2(x-y)\sin\frac{1}{x-y}-\cos\frac{1}{x-y}$$

$$\lim_{(x,y)\to(a,a)}f_x(x,y)\ \nexists$$

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{h^2\sin\frac1h-0}{h}=\lim_{h\to0}h\sin\left(\frac1h\right)=0.$$

$\longrightarrow$ f_x EXISTS BUT IT IS NOT CONTINUOUS ON (a,a).

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{h^2\sin\frac1h-0}{h}=\lim_{h\to0}h\sin\left(\frac1h\right)=0$$

$$f_y(0,0)=\dots\frac{f(0,k)-f(0,0)}{k}=\dots=0$$

$\longrightarrow$ f_x EXISTS BUT IT IS NOT CONTINUOUS ON (a,a).

f IS DIFF AT $(0,0)$?

$$\lim_{(x,y)\to(0,0)}E(x,y)=\lim_{(x,y)\to(0,0)}\frac{f(x,y)-0-0(x-0)-0(y-0)}{\sqrt{x^2+y^2}}=$$

$$=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}\begin{cases}x=y & \lim\frac{0}{\sqrt{2x^2}}=0\\ x\neq y & \lim\frac{(x-y)^2\sin\left(\frac{1}{x-y}\right)}{\sqrt{x^2+y^2}}\overset{?}{=}0\end{cases}$$

LAGRANGE

$h(x)=f(x,y)$ | $k(y)=f(x,y)$

$\exists\, h'(x)=f_x(x,y)$ | $k'(y)=f_y(x,y)$

$\Downarrow$

$h(x)$ CONT.

$$\left|\frac{(x-y)^2}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{x-y}\right)\right|\le\frac{(x-y)^2}{\sqrt{x^2+y^2}}=\frac{x^2-2xy+y^2}{\sqrt{x^2+y^2}}\le2\frac{x^2+y^2}{\sqrt{x^2+y^2}}=2\sqrt{x^2+y^2}<2\delta=\varepsilon.$$

$$0\le(x+y)^2=x^2+2xy+y^2\longrightarrow-2xy\le x^2+y^2$$

REMARK. f DIFFERENTIABLE AT (x_0,y_0) $\Longleftrightarrow$

$$z=f(x,y)=\underbrace{f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\,\|(x,y)-(x_0,y_0)\|$$

WITH $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$. MOREOVER $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ IS THE TANGENT PLANE

WE HAVE BEEN WORKING WITH $f_x=\frac{\partial}{\partial x}=\frac{\partial}{\partial(1,0)}$, $f_y=\frac{\partial}{\partial y}=\frac{\partial}{\partial(0,1)}$

IS IT WORTHWHILE TO CONSIDER <u>ALL</u> THE DIRECTIONAL DERIVATIVES $\frac{\partial f}{\partial\vec u}$?

THEO IF f IS DIFFERENTIABLE AT (x_0,y_0), THEN $\frac{\partial f}{\partial\vec u}(x_0,y_0)$ EXISTS, $\forall\,\vec u=(u_1,u_2)$.

MOREOVER, $\frac{\partial f}{\partial\vec u}=\left\langle(u_1,u_2),\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right\rangle=u_1\frac{\partial f}{\partial x}+u_2\frac{\partial f}{\partial y}$

PROOF:

$$\frac{\partial f}{\partial\vec u}(x_0,y_0)=\lim_{h\to0}\frac{f(\overset{x}{x_0+hu_1},\overset{y}{y_0+hu_2})-f(x_0,y_0)}{h}=\lim_{h\to0}\left(\frac{f_x(x_0,y_0)hu_1+f_y(x_0,y_0)hu_2+}{h}\right.$$

(x_0+hu_1,y_0+hu_2)

(x_0,y_0)

$$\left.+\frac{E(x_0+hu_1,y_0+hu_2)\sqrt{h^2u_1^2+h^2u_2^2}}{h}\right)=$$

$$=\lim_{h\to0}\left(f_x(x_0,y_0)u_1+f_y(x_0,y_0)u_2+\underbrace{E(x_0+hu_1,y_0+hu_2)}_{\downarrow}\underbrace{\frac{|h|}{h}}_{\text{IS BOUNDED}}\right)=f_x(\ ,\ )u_1+f_y(\ ,\ )u_2$$

f DIFF $\Longrightarrow$ ALL DIRECTIONAL DERIVATIVES EXIST.

ALL DIRECTIONAL DERIVATIVES EXIST $\not\Longrightarrow$ f DIFF.

EXAMPLE:

$$f(x,y)=\begin{cases}\frac{x^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial\vec u}(0,0)=\lim_{h\to0}\frac{f(0+hu_1,0+hu_2)-\overset{0}{f(0,0)}}{h}=$$

$$=\lim_{h\to0}\frac{(hu_1)^3}{h^2\underbrace{(u_1^2+u_2^2)}_{1}}\frac1h=\lim_{h\to0}\frac{h^3u_1^3}{h^3}=u_1^3$$

BY CONTRADICTION, IF f IS DIFF AT $(0,0)$, THEN:

$$u_1^3=\frac{\partial f}{\partial\vec u}(0,0)=u_1\underbrace{\frac{\partial f}{\partial(1,0)}(0,0)}_{f_x(0,0)}+u_2\underbrace{\frac{\partial f}{\partial(0,1)}(0,0)}_{f_y(0,0)}=u_1 1^3+u_2 0^3=u_1\quad\times$$

$$\vec u=\left(\frac{1}{\sqrt5},\frac{2}{\sqrt5}\right),\quad\left(\frac{1}{\sqrt5}\right)^2+\frac{4}{5}$$

THEO (CHAIN RULE) $z=f(x,y)$ DIFF AT (x_0,y_0), $x=x(u,v)$, $y=y(u,v)$ DIFF AT (u_0,v_0). $x(u_0,v_0)=x_0$, $y(u_0,v_0)=y_0$

LET $g(u,v)=f(x(u,v),y(u,v))$ $\qquad (u,v)\longrightarrow(x(u,v),y(u,v))\xrightarrow{f}\mathbb{R}$

$$\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u},\qquad\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

REMARK. f DIFFERENTIABLE AT (x_0,y_0) $\Longleftrightarrow$

$$z=f(x,y)=\underbrace{f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\,\|(x,y)-(x_0,y_0)\|$$

WITH $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$. MOREOVER $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ IS THE TANGENT PLANE

THEO (CHAIN RULE) $z=f(x,y)$ DIFF AT (x_0,y_0), $x=x(u,v)$, $y=y(u,v)$ DIFF AT (u_0,v_0). $x(u_0,v_0)=x_0$, $y(u_0,v_0)=y_0$

LET $g(u,v)=f(x(u,v),y(u,v))$ $\qquad (u,v)\longrightarrow(x(u,v),y(u,v))\xrightarrow{f}\mathbb{R}$

$$\frac{\partial g}{\partial u}^{(u_0,v_0)}=\frac{\partial f}{\partial x}^{(x_0,y_0)}\frac{\partial x}{\partial u}^{(u_0,v_0)}+\frac{\partial f}{\partial y}^{(x_0,y_0)}\frac{\partial y}{\partial u}^{(u_0,v_0)},\qquad\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

REMARK. f DIFFERENTIABLE AT (x_0,y_0) $\Longleftrightarrow$

$$z=f(x,y)=\underbrace{f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\,\|(x,y)-(x_0,y_0)\|$$

WITH $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$. MOREOVER $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ IS THE TANGENT PLANE

PROOF:

$$\frac{\partial g}{\partial u}(u_0,v_0)=\lim_{h\to0}\frac{g(u_0+h,v_0)-g(u_0,v_0)}{h}=\lim_{h\to0}\frac{f(x(u_0+h,v_0),y(u_0+h,v_0))-f(x(u_0,v_0),y(u_0,v_0))}{h}$$

$$=\lim_{h\to0}\left(f_x(x_0,y_0)\frac{x(u_0+h,v_0)-x(u_0,v_0)}{h}+f_y(x_0,y_0)\frac{y(u_0+h,v_0)-y(u_0,v_0)}{h}+E(x(u_0+h,v_0),y(u_0+h,v_0))\right.$$

$$E(x,y)\xrightarrow[(x,y)\to(x_0,y_0)]{}0$$

$$\left.+\ \cdots\cdots\right)=f_x(x_0,y_0)\frac{\partial x}{\partial u}(u_0,v_0)+f_y(x_0,y_0)\frac{\partial y}{\partial u}(u_0,v_0)+0$$