Series of functions

$\sum_{n=0}^{\infty} f_n(x)$ Partial sum: $S_n = \sum_{k=0}^{n} f_k(x)$

The series

- Pointwise converges to $f(x)$ in I if the sequence of partial sums S_n pointwise converges to f
- Uniformly converges to f if $\{S_n\}$ uniformly converges

Theorem Weierstrass M-test

If $\{f_n\}$ is a sequence of functions on I s.t. $\forall n\ \exists M_n$ so that $\forall x \in I$ we have

$|f_n(x)| \leq M_n$ and $\sum_{n} M_n < \infty$

$\Rightarrow \sum f_n(x)$ uniformly and absolutely conv. on I

Power series

$\sum_{n=0}^{\infty} a_n (x-x_0)^n$

Def: Domain of convergence $D \subseteq \mathbb{R}$ is the set of $x \in \mathbb{R}$ for which $\sum a_n(x-x_0)^n$ converges

Radius of convergence R: is the supremum over all r s.t. $\forall x$ s.t. $|x-x_0| < r$ the series converges

Theorem $\forall \sum_{n=0}^{\infty} a_n x^n$ there is $0 \leq R \leq \infty$ s.t. the series converges at x s.t. $|x| < R$ and $\forall x$ s.t. $|x| > R$ it diverges

(at $x = R$ we don't know)   diverges   converges   diverges

Theorem

Cauchy–Hadamard:

$L = \limsup_{n\to\infty} \left(|a_n|^{1/n}\right)$

$\Rightarrow R = \frac{1}{L}$ (If $L = 0$ then $R = \infty$)

Another way: $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \Rightarrow R = \frac{1}{L}$

Inside the domain of convergence we are allowed to differentiate term by term and integrate term by term of the power series

Exercise 1. Consider the series

$\sum_{n=0}^{\infty} x^n$

a) Show by definition that the series converges uniformly on $[-r, r]$ where $0 < r < 1$

b) Does it uniformly converge on $(-1, 1)$? Explain

a) Let $\varepsilon > 0$. We need to find $N(\varepsilon)$ s.t. $\forall n > N(\varepsilon) \Rightarrow \left|\sum_{k=0}^{n} x^k - f\right| < \varepsilon \quad \forall x \in [-r, r]$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x)$

$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

$\left|\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}\right| < \varepsilon$

$\Rightarrow \left|\frac{x^{n+1}}{1-x}\right| < \varepsilon$

$\left|\frac{x^{n+1}}{1-x}\right| \leq \frac{r^{n+1}}{1-r} < \varepsilon$

$g(x) = \frac{x^{n+1}}{1-x} \Rightarrow \left|\frac{x^{n+1}}{1-x}\right| \leq \frac{r^{n+1}}{1-r}$

Show by derivative test that $g(x)$ has a maximum at $x = r$ in the interval $[-r, r]$

$\frac{r^{n+1}}{1-r} < \varepsilon \Rightarrow r^{n+1} < \varepsilon(1-r) \Rightarrow (n+1)\ln r < \ln(\varepsilon(1-r))$

$\Rightarrow (n+1) > \frac{\ln(\varepsilon(1-r))}{\ln r}$

$n > -1 + \frac{\ln(\varepsilon(1-r))}{\ln r}$

$N(\varepsilon) = \max\left\{0, -1 + \frac{\ln(\varepsilon(1-r))}{\ln r}\right\}$

$\forall \varepsilon > 0,\ \exists N(\varepsilon)$ s.t. $\forall n > N(\varepsilon)$: $\left|\sum_{k=0}^{n} x^k - \frac{1}{1-x}\right| < \varepsilon \quad \forall x \in [-r, r]$

b) $(-1, 1)$?

The convergence is not uniform on the interval $(-1, 1)$ because we can't bound the distance from the limit function (when $x \to 1$, $\left|\frac{x^{n+1}}{1-x}\right| \to \infty$ and not to 0)

We can't find an N that is good $\forall x \in (-1, 1)$

Exercise 2

Determine if the following series converge uniformly and find their domain of convergence

1) $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ 2) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$

Weierstrass M-test

a) $f_n(x) = \frac{1}{x^2+n^2}$

$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely on $\mathbb{R}$

b) $f_n(x) = \frac{\sin(nx)}{n^2}$

Also here $\forall x$ $\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}$

and $\sum \frac{1}{n^2}$ converges so $\sum_{n=1}^{\infty} f_n(x)$ conv. unif. and absolutely on $\mathbb{R}$

Exercise 3

Find all intervals in $\mathbb{R}$ on which the series $\sum_{n=0}^{\infty} \left(\frac{x}{1+x}\right)^n$ uniformly converges.

$g(x) = \frac{x}{1+x}$

For $x < -1$: $\left|\frac{x}{1+x}\right| > 1$ so the series diverges

For $-1 < x < 0$: $\left|\frac{x}{1+x}\right| = -\frac{x}{1+x}$

For $-1 < x < 0$: $\left|\frac{x}{1+x}\right| = -\frac{x}{1+x} < 1 \iff -x < 1+x \iff -1 < 2x \iff x > -\frac{1}{2}$

If $x \geq 0$: $\left|\frac{x}{1+x}\right| < 1$

Conclusion: $\sum_{n=0}^{\infty} \left(\frac{x}{1+x}\right)^n$ pointwise converges on $\left(-\frac{1}{2}, \infty\right)$ and it diverges on every other interval

Def: V a vector space over $\mathbb{R}/\mathbb{C}$. $\langle -,- \rangle : V \times V \to \mathbb{K}$ is an inner product if

- linearity wrt 1st arg: $\langle x+x', y\rangle = \langle x,y\rangle + \langle x',y\rangle$, $\langle ax, y\rangle = a\langle x,y\rangle$
- (hermitian) symmetry: $\langle x,y\rangle = \overline{\langle y,x\rangle}$
- positiveness: if $x \in V\setminus 0$, $\langle x,x\rangle > 0$

semi-linearity wrt 2nd arg: $\langle x, y+y'\rangle = \langle x,y\rangle + \langle x,y'\rangle$, $\langle x, ay\rangle = \bar{a}\langle x,y\rangle$

Def: An inner product space (euclidean / unitary) is a vector space with inner product.

Ex:

$\mathbb{R}^n$: $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ } standard

$\mathbb{C}^n$: $\langle x,y\rangle = \sum_{i=1}^n x_i \overline{y_i}$

$\mathbb{R}(x)$: $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx$ ⇝ functional analysis, Hilbert space.

$V = M_n(\mathbb{R})$ (Frobenius) $\langle A,B\rangle = tr(A\overline{B}^t)$

$A = (a_{ij})_{i,j}$

$B = (b_{ij})_{i,j}$

$\overline{B}^t = (\overline{b_{ji}})_{i,j}$

$(A\overline{B}^t)_{ij} = \sum_{k=1}^n a_{ik}\overline{b_{jk}}$

$$tr(A\overline{B}^t) = \sum_{i=1}^n (A\overline{B}^t)_{ii} = \sum_{i=1}^n\left(\sum_{k=1}^n a_{ik}\overline{b_{ik}}\right) = \sum_{i,k=1}^n a_{ik}\overline{b_{ik}}$$

Lemma: V ip space

i) if $\langle x,y\rangle = 0$ for all $y \in V$, then $x = 0$

ii) if $\langle x,y\rangle = \langle x',y\rangle$ for all $y \in V$, then $x = x'$.

Lemma: Suppose that V is inner product space and that $W \subseteq V$ is a subspace. $\langle -,-\rangle_W := \langle -,-\rangle|_{W\times W} : W\times W \to \mathbb{R}$ is an inner product on W.

proof: if $x, x', y \in W$, then

$\langle x+x', y\rangle_W = \langle x+x', y\rangle = \langle x,y\rangle + \langle x',y\rangle = \langle x,y\rangle_W + \langle x',y\rangle_W$

$\langle ax,y\rangle_W = \langle ax,y\rangle = a\langle x,y\rangle = a\langle x,y\rangle_W$

Exercise: complete the proof. ∎

Example: $\mathbb{R}^3$, $\langle -,-\rangle$ std inner product. $W = \langle v_1 = e_1+e_2, v_2 = e_1+e_2+e_3\rangle \le V$ of dim 2

$\langle -,-\rangle_W = \langle -,-\rangle|_{W\times W} : W\times W \to \mathbb{R}$

$x, y \in W$

$x = av_1 + bv_2$

$y = cv_1 + dv_2$

$\boxed{\langle x,y\rangle_W} = \langle av_1+bv_2, cv_1+dv_2\rangle_W$
$= ac\langle v_1,v_1\rangle_W + ad\langle v_1,v_2\rangle_W + bc\langle v_2,v_1\rangle_W + bd\langle v_2,v_2\rangle_W$
$= \boxed{2ac + 2ad + 2bc + 3bd}$

$\langle v_1,v_1\rangle_W = 2$

$\langle v_1,v_2\rangle_W = 2$

$\langle v_2,v_1\rangle_W = 2$

$\langle v_2,v_2\rangle_W = 3$

$\langle v_2,v_2\rangle = \langle e_1+e_2+e_3, e_1+e_2+e_3\rangle = \langle e_1, e_1+e_2+e_3\rangle + \langle e_2, \dots\rangle + \langle e_3, \dots\rangle = 3$

Def: V ip space over $\mathbb{R}/\mathbb{C}$. $\|\cdot\| : V \to \mathbb{R}$ norm, $x \mapsto \|x\| := \langle x,x\rangle^{1/2}$ has the following properties:

i) $\|x\| = 0 \iff x = 0$

ii) $\|ax\| = |a|\,\|x\|$

iii) $\|x+y\| \le \|x\| + \|y\|$ (triangle inequality)

iv) $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$ (Cauchy–Bunyakovsky–Schwarz inequality)

Idea: $\|x\|$ is a measure of how long x is.

. if $V = \mathbb{R}^2$ with std inner product

$\|(x,y)\| = \langle (x,y),(x,y)\rangle^{1/2} = \sqrt{x^2+y^2}$

the function $\langle (x,y),(x',y')\rangle = 2xx' + 2yy'$ is also an inner product on $\mathbb{R}^2$

$\|(x,y)\|' = \sqrt{\langle (x,y),(x,y)\rangle'} = \sqrt{2x^2+2y^2}$

Prop: if V is an inner product space then the function $d : V\times V \to [0,\infty)$, $(v,w) \mapsto d(v,w) = \|v-w\|$ has the following properties

i) $d(x,y) = 0 \iff x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,z) \le d(x,y) + d(y,z)$

Idea: $d(x,y)$ is the distance between the vectors x, y

$\mathbb{R}^2$ std inner product

$x = (x_1,x_2)$, $y = (y_1,y_2)$

$$d(x,y) = \|x-y\| = \|(x_1-y_1, x_2-y_2)\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

Def: V ip space. Two vectors $x, y \in V$ are orthogonal if $\langle x,y\rangle = 0$. In that case we write $x \perp y$

Obs: $\langle x,y\rangle = 0 \iff \langle y,x\rangle = 0$
$x \perp y$, $y \perp x$
∴ $\perp$ is a symmetric relation

Ex: $M_n(\mathbb{R})$, Frobenius inner prod $\langle A,B\rangle = tr(AB^t)$

Lemma: . $0 \perp x$ for all $x \in V$

. if $x \in V$ is such that $x \perp y$ for all $y \in V$, then $x = 0$

proof: if $x \in V$, $\langle 0,x\rangle = 0$, so $x \perp 0$. Suppose $x \in V$ is such that $x \perp y$ $\forall y$ then $\langle x,y\rangle = 0$ $\forall y$, so $x = 0$ ∎

Prop: (Pythagoras') Suppose V ip space, $x,y \in V$.

i) if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

ii) More generally, $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x,y\rangle$

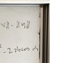
proof: $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$ ...

if $x \perp y$, then $\langle x,y\rangle = \langle y,x\rangle = 0$ so $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

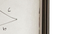
In general, $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x,y\rangle$ ∎

Def: if $x,y \in V\setminus 0$, then the angle between x and y is the unique number $\theta \in [0,\pi]$ s.t. $\cos\theta = \frac{\mathrm{Re}\langle x,y\rangle}{\|x\|\,\|y\|}$

$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$



$c^2 = a^2 + b^2 - 2ab\cos\alpha$



$c^2 = a^2 + b^2$

By CBS inequality for $x,y \ne 0$

$$\frac{|\langle x,y\rangle|}{\|x\|\,\|y\|} \le 1$$

so however $-1 \le \frac{\mathrm{Re}\langle x,y\rangle}{\|x\|\,\|y\|} \le 1$

So there is a unique $\theta \in [0,\pi]$ such that $\cos\theta = \frac{\mathrm{Re}\langle x,y\rangle}{\|x\|\,\|y\|}$

Geometry: if $x,y \in V\setminus 0$, then $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos(\theta)$

. V inner product space. $A = \{v_1,\dots,v_n\}$ subset of A. We say that

. A is orthogonal if $v_i \perp v_j$ for all $i \ne j$.

. A is orthonormal if it is orthogonal and $\|v_i\| = 1$ $\forall i$.

Ex: $\mathbb{R}^n$, $B = (e_1, e_2, \dots, e_n)$ is an orthonormal set: $\forall i \ne j$, $\langle e_i, e_j\rangle = 0$; $\forall i \in \{1,\dots,n\}$, $\langle e_i, e_i\rangle = 1$

Prop: V ip space, $A = \{v_1,\dots,v_n\}$ a subset of V.

i) if A is orthogonal and $0 \notin A$, then A is linearly independent.

proof: Suppose A is orthogonal, so that $\forall i \ne j$, $\langle v_i, v_j\rangle = 0$. Let $a_1,\dots,a_n \in \mathbb{K}$ be such that $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$. If $i \in \{1,\dots,n\}$ then

$0 = \langle 0, v_i\rangle$
$= \langle a_1v_1 + a_2v_2 + \dots + a_nv_n, v_i\rangle$
$= a_1\langle v_1,v_i\rangle + a_2\langle v_2,v_i\rangle + \dots + a_n\langle v_n,v_i\rangle$
$= a_i\underbrace{\langle v_i,v_i\rangle}_{\ne 0}$ because $\langle v_j,v_i\rangle = 0$ if $j \ne i$.

Since $0 \notin A \ni v_i$, $v_i \ne 0$ so $\langle v_i,v_i\rangle \ne 0$. ∴ $a_i = 0$.

∴ A is linearly independent.

Prop. ... a subset of V.

i) if A is orthogonal and $0 \notin A$, then A is linearly independent.

ii) if A is orthonormal, then A is linearly independent. If $x \in \langle A\rangle$, then
$$x = \langle x,v_1\rangle v_1 + \langle x,v_2\rangle v_2 + \dots + \langle x,v_n\rangle v_n$$

exist unique $a_1,\dots,a_n \in \mathbb{K}$ s.t. $x = a_1v_1 + a_2v_2 + \dots + a_nv_n$. Then if $i \in \{1,\dots,n\}$ then

$\boxed{\langle x,v_i\rangle} = \langle a_1v_1 + a_2v_2 + \dots + a_nv_n, v_i\rangle$
$= a_1\langle v_1,v_i\rangle + a_2\langle v_2,v_i\rangle + \dots + a_n\langle v_n,v_i\rangle$
$= a_i\underbrace{\langle v_i,v_i\rangle}_{=1 \text{ because } A \text{ is orthonormal}} = \boxed{a_i}$

iii) if A is orthonormal and $|A| = \dim V$, then A is a basis for V and $\forall x \in V$, $x = \langle x,v_1\rangle v_1 + \dots + \langle x,v_n\rangle v_n$

Ex: the standard basis in $\mathbb{R}^n$ or in $\mathbb{C}^n$ is orthonormal.

$B = (e_1,\dots,e_n)$

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

$\|e_i\| = \langle e_i, e_i\rangle^{1/2} = 1^{1/2} = 1$

$\langle e_i, e_j\rangle = \langle (0\dots 1\dots 0), (0\dots 1\dots 0)\rangle = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}$ ✓