$z = f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} = f_x(x_0,y_0)$$

PARTIAL DERIVATIVES

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} = f_y(x_0,y_0)$$

EXAMPLE: $f(x,y) = x^3y^2 + x^2y$

$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = y^2 \cdot 3x^2 + y\, 2x$

$f_y(x,y) = x^3 \cdot 2y + x^2$

SINCE $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ ARE TWO VARIABLE FUNCTIONS, WE CAN DEFINE PARTIAL DERIVATIVES OF HIGHER ORDER:

$$(f_x)_x = \frac{\partial^2}{\partial x \partial x} f \ ; \quad (f_x)_y = \frac{\partial^2}{\partial y \partial x} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right),$$

$$(f_y)_y = \frac{\partial^2 f}{\partial y \partial y}$$

EXAMPLE: $f(x,y) = x^3y^2 + x^2y$

$f_x(x,y) = 3x^2y^2 + 2xy$ — $f_{xx}(x,y) = 6xy^2 + 2y$, $f_{xy}(x,y) = 6x^2y + 2x$

$f_y(x,y) = 2x^3y + x^2$ — $f_{yx}(x,y) = 6x^2y + 2x$, $f_{yy}(x,y) = 2x^3$

$f_{xy}(x,y) = f_{yx}(x,y)$

THEO. ASSUME f, f_x, f_y, f_{xy} EXIST IN $N(x_0,y_0)$, f_{xy} IS CONTINUOUS AT (x_0,y_0). THEN f_{yx} EXISTS AT (x_0,y_0) AND $f_{yx}(x_0,y_0) = f_{xy}(x_0,y_0)$, WHERE $N(x_0,y_0)$ IS SOME NEIGHBORHOOD OF (x_0,y_0).

PROOF: WE KNOW THAT $\exists \delta > 0$:

$(x,y) \in N_\delta(x_0,y_0) \subseteq N(x_0,y_0) \longrightarrow |f_{xy}(x,y) - f_{xy}(x_0,y_0)| < \varepsilon$

THEN $\exists \delta' > 0$: $S_{\delta'}(x_0,y_0) = \{(x,y) : |x-x_0| < \delta', |y-y_0| < \delta'\} \subseteq N_\delta(x_0,y_0)$

THEN: $(x,y) \in S_{\delta'}(x_0,y_0) \Longrightarrow |f_{xy}(x,y) - f_{xy}(x_0,y_0)| < \varepsilon$.

LET: $F(h,t) = \underbrace{f(x_0+h, y_0+t) - f(x_0+h, y_0)}_{} + f(x_0,y_0) - f(x_0, y_0+t), \quad \forall |h| < \delta', |t| < \delta'$

$= g(x_0+h) - g(x_0) \underset{\text{LAGRANGE}}{=} g'(x_1)(x_0+h-x_0), \quad x_1 \in (x_0, x_0+h)$ or (x_0+h, x_0)

WHERE $g(x) = f(x, y_0+t) - f(x, y_0)$

THEN: $F(h,t) = g'(x_1) \cdot h = \underbrace{\left(f_x(x_1,y_0+t) - f_x(x_1,y_0)\right)}_{\text{APPLY LAGRANGE TO}} \cdot h = f_{xy}(x_1,y_1) \cdot t \cdot h$

APPLY LAGRANGE TO $S(y) = f_x(x_1,y)$: $S(y_0+t) - S(y_0) = S'(y_1)(y_0+t-y_0)$, $y_1 \in (y_0, y_0+t)$ or (y_0+t, y_0)

$g : (a,b) \to \mathbb{R}$

$g(b) - g(a) = g'(z) \cdot (b-a)$

FOR SOME $z \in (a,b)$

$\Longrightarrow \left|\frac{F(h,t)}{ht} - f_{xy}(x_0,y_0)\right| = |f_{xy}(x_1,y_1) - f_{xy}(x_0,y_0)| < \varepsilon$ SINCE $|x_1-x_0| < |h| < \delta'$, $|y_1-y_0| < |t| < \delta'$

$f_{xy}(x_0,y_0) - \varepsilon < \frac{F(h,t)}{ht} < f_{xy}(x_0,y_0) + \varepsilon$

$$\lim_{t\to 0} \frac{F(h,t)}{ht} = \lim_{t\to 0}\left(\frac{f(x_0+h, y_0+t) - f(x_0+h,y_0)}{t} \cdot \frac{1}{h} - \frac{f(x_0,y_0+t) - f(x_0,y_0)}{t} \cdot \frac{1}{h}\right)$$

$$= \frac{f_y(x_0+h,y_0) - f_y(x_0,y_0)}{h} \Longrightarrow f_{xy}(x_0,y_0) - \varepsilon \leq \frac{f_y(x_0+h,y_0) - f_y(x_0,y_0)}{h} \leq f_{xy}(x_0,y_0) + \varepsilon$$

$$\lim_{h\to 0} \frac{f_y(x_0+h,y_0) - f_y(x_0,y_0)}{h} = f_{yx}(x_0,y_0)$$

AND $f_{xy}(x_0,y_0) - \varepsilon \leq f_{yx}(x_0,y_0) \leq f_{xy}(x_0,y_0) + \varepsilon \Longleftrightarrow |f_{yx}(x_0,y_0) - f_{xy}(x_0,y_0)| \leq \varepsilon, \ \forall \varepsilon$

COROLLARY: IF f AND ALL ITS PARTIAL DERIVATIVES (OF ORDER $\leq r$) EXIST AT $N(x_0,y_0)$ AND ARE CONT. AT (x_0,y_0)

THEN $\frac{\partial^r f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_r}} = \frac{\partial^r f}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_r}}$ IF $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$, $x_{i_k}, x_{j_k} \in \{x, y\}$.

f

$f_x \quad f_y$

$f_{xx} \quad f_{xy} \quad f_{yx} \quad f_{yy}$

$f_{xxx} \quad f_{xxy}$

PROOF: INDUCTION

REMARK: $f_{xy}(x_0,y_0) - \varepsilon < \frac{F(h,t)}{ht} < f_{xy}(x_0,y_0) + \varepsilon$,

1) WE COMPUTE $\lim_{t\to 0} \lim_{h\to 0} \frac{F(h,t)}{ht} = f_{xy}(x_0,y_0) \Longrightarrow$ BY ASSUMPTION THIS LIMIT EXIST

2) WE COMPUTE $\lim_{h\to 0} \lim_{t\to 0} \frac{F(h,t)}{ht} = f_{yx}(x_0,y_0)$

EXAMPLE. $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ $\quad \lim_{(x,y)\to(0,0)} f(x,y) \nexists$ (USING $y = x$: $\frac{2x^2}{x^2+x^2} = 1$, $y = -x$: $\frac{-2x^2}{x^2+x^2} = -1$)

f IS NOT CONTINUOUS AT $(0,0)$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{2xy}{x^2+y^2}\right) = \frac{2y(x^2+y^2) - 2xy \cdot 2x}{(x^2+y^2)^2}, \quad \forall (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\frac{2xy}{x^2+y^2}\right) = \frac{2x(x^2+y^2) - 2xy \cdot 2y}{(x^2+y^2)^2}, \quad \forall (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0, \quad f(t,0) = \frac{2t0}{t^2+0^2} = 0, \quad t \neq 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0, 0+t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

f IS DIFFERENTIABLE AT $x_0 \Longleftrightarrow f(x) \sim$ TANGENT LINE

INFI 1: IF $f'(x_0)$ EXISTS $\longrightarrow$ f IS CONTINUOUS AT x_0.

INFI 2: IF $f_x(0,0)$, $f_y(0,0)$ EXIST BUT f IS NOT CONTINUOUS

INFI 2: IF f IS DIFFERENTIABLE AT $(x_0,y_0) \Longrightarrow f$ IS CONTINUOUS AT (x_0,y_0)

$f(x,y) \sim$ TANGENT PLANE

DEF. LET $z = f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. WE SAY THAT f IS DIFFERENTIABLE AT (x_0,y_0) IFF

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \left[\overbrace{f(x_0,y_0) + a(x-x_0) + b(y-y_0)}^{P(x,y)}\right]}{\|(x,y)-(x_0,y_0)\|} = 0, \quad \text{FOR SOME } a, b \in \mathbb{R}$$

MOREOVER, $z = P(x,y) = f(x_0,y_0) + a(x-x_0) + b(y-y_0)$ IS THE EQUATION OF A PLANE GOING THROUGH $(x_0, y_0, f(x_0,y_0))$, AND IT IS CALLED THE TANGENT PLANE

INFI 1

$f'(x_0)$ EXISTS $\Longleftrightarrow$

$$\lim_{x\to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x-x_0)]}{x-x_0} = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right) = 0$$

f DIFFERENTIABLE $\Longrightarrow$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ EXIST

$\not\Longleftarrow$

THEO. IF f IS DIFFERENTIABLE AT (x_0,y_0) THEN $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ EXIST

MOREOVER, THE TANGENT PLANE IS:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

DEF. LET $z = f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. WE SAY THAT f IS DIFFERENTIABLE AT (x_0,y_0) IFF

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \left[\overbrace{f(x_0,y_0) + a(x-x_0) + b(y-y_0)}^{P(x,y)}\right]}{\|(x,y)-(x_0,y_0)\|} = 0, \quad \text{FOR SOME } a, b \in \mathbb{R}$$

MOREOVER, $z = P(x,y) = f(x_0,y_0) + a(x-x_0) + b(y-y_0)$ IS THE EQUATION OF A PLANE GOING THROUGH $(x_0, y_0, f(x_0,y_0))$, AND IT IS CALLED THE TANGENT PLANE

INFI 1

$f'(x_0)$ EXISTS $\Longleftrightarrow$

$$\lim_{x\to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x-x_0)]}{x-x_0} = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right) = 0$$

f DIFFERENTIABLE $\Longrightarrow$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ EXIST

$\not\Longleftarrow$

THEO. IF f IS DIFFERENTIABLE AT (x_0,y_0) THEN $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ EXIST

MOREOVER, THE TANGENT PLANE IS:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

PROOF: SINCE f IS DIFFERENTIABLE, WE HAVE $\exists a, b \in \mathbb{R}$:

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - a(x-x_0) - b(y-y_0)}{\|(x,y)-(x_0,y_0)\|} \underset{\substack{x-x_0=h \\ y-y_0=t}}{=} \lim_{(h,t)\to(0,0)} \frac{f(x_0+h, y_0+t) - f(x_0,y_0) - a\,h - b\,t}{\sqrt{h^2+t^2}}$$

IF WE APPROXIMATE TO $(0,0)$ IN THE DIRECTION

$\begin{cases} x = x_0 + h \\ y = y_0 \end{cases}$ $\quad h \to 0$, $t = 0$.

$$0 = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0) - a\,h}{\sqrt{h^2}} = \lim_{h\to 0}\left(\frac{f(x_0+h,y_0) - f(x_0,y_0)}{|h|} - a \cdot \frac{h}{|h|}\right)$$

$$\lim_{h\to 0^+}\left(\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} - a\,\frac{h}{h}\right) = \frac{\partial f}{\partial x}(x_0,y_0) - a = 0$$

$$\lim_{h\to 0^-}\left(\frac{f(x_0+h,y_0) - f(x_0,y_0)}{-h} + a\,\frac{h}{h}\right) = -\frac{\partial f}{\partial x}(x_0,y_0) + a = 0$$

$\Longrightarrow a = \frac{\partial f}{\partial x}(x_0,y_0)$.

EXAMPLE (CONTINUED) $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = (0,0)$

f IS NOT CONT. AT $(0,0)$.

IS f DIFFERENTIABLE AT $(0,0)$? NO

$$0 \overset{?}{=} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left[f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)\right]}{\sqrt{(x-0)^2+(y-0)^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{2xy}{x^2+y^2}}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{(x^2+y^2)^3}} \nexists \qquad y = x: \ \frac{2x^2}{\sqrt{(x^2+x^2)^3}} = \frac{2x^2}{\sqrt{8}\, x^2 |x|} = \frac{1}{\sqrt{2}\,|x|} \nexists$$