Indefinite integrals

Intro

$F(x) = \int f(x)\,dx$

$F(x)$ is the primitive function

antiderivative or indefinite integral of $f(x)$ in an inverval I

→ if $F(x)$ is differentiable in I

& $F'(x) = f(x)$

if F, G are primitive functions of $f(x)$

$\exists c$ st $F(x) = G(x) + c$

Immediate integrals:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

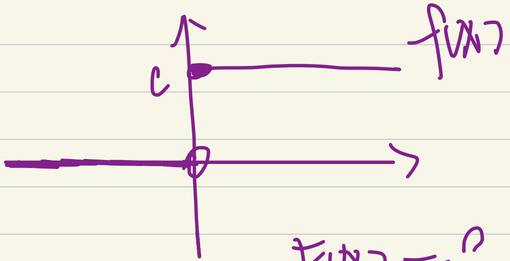
$\vdots$

$$\int \frac{1}{1+x^2} dx = \arctan x + c$$

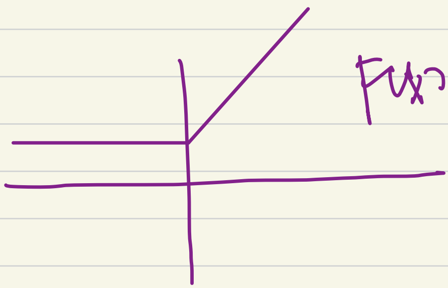
<u>Note</u>

Not every f has a primitive function.

Ex



F(x) = ?



$$F(x) = \begin{cases} \text{const} & x < 0 \\ x + \text{const} & x \geq 0 \end{cases}$$

F(x) is not differentiable at $x = 0$

→ f(x) not a primitive function of F(x)

All continuous functions have anti-derivatives

$$\int a f(x)\,dx = a\int f(x)\,dx$$

$\Rightarrow$

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Methods

① substitution

$$\int f(g(x))\,g'(x)$$

$$= \int f(u)\,du = F(u)+C$$

$$= F(g(x))+C$$

$u = g(x)$

$du = g'(x)\,dx$

② By parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

③ Integration of rational functions

$$\int \frac{u(x)}{v(x)}\,dx$$

$u(x)$, $v(x)$ are polynomials

Ex#1

$$\int x\cdot e^x\,dx = x\cdot e^x - \int e^x\,dx = xe^x - e^x + c$$

$u(x) = x \quad u'(x) = 1$

$v'(x) = e^x \quad v(x) = e^x$

Ex #2

$$\int e^{-|x|}\,dx$$

$$f(x) = e^{-|x|} = \begin{cases} e^{x} & x<0 \\ e^{-x} & x\geq 0 \end{cases}$$

$$F(x) = \begin{cases} e^{x}+C & x<0 \\ -e^{-x}+D & x\geq 0 \end{cases}$$

$F(x)$ has to be cont at $x=0$

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^-} F(x)$$

$$1+C = -1+D$$

$$\to D = 2+C$$

Ex #3

$$\int \tan dx = \int \frac{\sin x}{\cos x} dx \qquad u = \cos x$$

$$du = -\sin x \cdot dx$$

$$= \int \frac{-du}{u} = -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

Ex #4

$$F(x) = \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x \qquad u' = e^x$

$v' = \cos x \qquad v = \sin x$

$$-e^x \cdot \cos x + \int e^x \cdot \cos x \, dx$$

$u = e^x \qquad u' = e^x$

$v' = \sin x \qquad v = -\cos x$

$$e^x \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x \, dx$$

$$\rightarrow 2\int e^x \cos x \, dx = e^x \sin x + e^x \cdot \cos x$$

$$\rightarrow \int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cdot \cos x}{2} + C$$

Ex#5

$$I_n = \int x^n \sin x \, dx \qquad n \in \mathbb{N} \cup \{0\}$$

$u = x^n \qquad u' = n \cdot x^{n-1}$

$v' = \sin x \qquad v = -\cos x$

$$I_n = -x^n \cdot \cos x - \int n \cdot x^{n-1} \cdot (-\cos x) \, dx$$

$$I_n = -x^n \cdot \cos x + n \int x^{n-1} \cdot \cos x \, dx$$

$$u = x^{n-1} \quad u' = (n-1)x^{n-2}$$

$$v' = \cos x \quad v = \sin x$$

$$(x^{n-1}) \cdot \sin x - \int (n-1)x^{n-2} \cdot \sin x \, dx$$

$$\rightarrow I_n =$$

$$-x^n \cdot \cos x + n \cdot x^{n-1} \cdot \sin x - n(n-1) \underbrace{\int x^{n-2} \cdot \sin x \, dx}_{I_{n-2}}$$

$$I_n = -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}$$

For $n=0$

$$I_0 = -x^0 \cos x + C$$

For $n=1$

$$I_1 = -x\cdot\cos x + \sin x + C$$

Ex #6

$$\int \arctan x \, dx$$

$u = \arctan x \qquad u' = \dfrac{1}{1+x^2}$

$v' = 1 \qquad V = x$

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}$$

$$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

$u = 1+x^2$

$du = 2x\,dx$

$$-\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u|$$

Ex #7

$$\int \frac{1}{x(x^{10}+1)}\,dx \qquad \text{Find } F(x) \text{ for } x>0$$

$u = x^{10}$

$du = 10 \cdot x^{9} \cdot dx = 10 \cdot x^{10} \cdot \frac{dx}{x}$

$\rightarrow \frac{dx}{x} = \frac{du}{10u}$

$\rightarrow$

$$\frac{1}{10}\int \frac{du}{u(1+u)}$$

$$\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} = \frac{A(1+u)+Bu}{u(1+u)}$$

$$= \frac{u(A+B)+A}{u(1+u)}$$

$\rightarrow$ $A+B=0$, $A=1$ $\rightarrow$ $A=1$, $B=-1$

$$\rightarrow \frac{1}{10}\int \frac{du}{u(1+u)} = \frac{1}{10}\left[\int \frac{du}{u} - \int \frac{du}{u+1}\right]$$

$$= \frac{1}{10}\left[\ln|u| - \ln|1+u|\right] + C$$

$$= \frac{1}{10}\ln\left|\frac{u}{1+u}\right| + C$$

$$= \frac{1}{10}\ln\left|\frac{x^{10}}{x^{10}+1}\right| + C$$

Ex #8

$$\int \frac{dx}{\cos x}$$

$x \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$u = \sin x$

$du = \cos x \, dx$

$\cos x > 0$

$$= \int \frac{du}{\cos^2 x} \underset{\uparrow}{=} \int \frac{du}{1-u^2} = \int \frac{du}{(1+u)(1-u)}$$

$\sin^2 x + \cos^2 x = 1$

$$\frac{A}{1+u} + \frac{B}{1-u} = \frac{A(1-u) + B(1+u)}{(1+u)(1-u)}$$

$$= \frac{(B-A)u + A + B}{(1+u)(1-u)}$$

$B - A = 0$

$A + B = 1$ $\rightarrow$ $A = B = \frac{1}{2}$

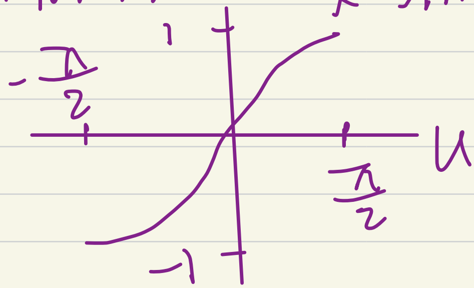
$$\int \frac{dx}{\cos x} = \frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du$$

$$= \frac{-1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c$$

Ex #9

(trigonometric substitutions)



$$\int \frac{dx}{\sqrt{1-x^2}} \qquad -1 < x < 1$$

$x = \sin u$

$dx = \cos u \, du$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = |\cos u| = \cos u$$

$$\int \frac{\cos u \, du}{\cos u} = u + c = \arcsin x + c$$

Ex#10

$$\int \frac{dx}{\sqrt{1+x^2}}$$

Define $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$

$\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

$$(\sinh t)' = \frac{e^t + e^{-t}}{2} = \cosh(t)$$

$$(\cosh t)' = \sinh t$$

Verify: $\cosh^2 t - \sinh^2 t = 1$

$x = \sinh t$

$dx = \cosh t \, dt$

$$\sqrt{1+x^2} = \sqrt{1+\sinh t} = \cosh t$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh t}{\cosh t}\, dt$$

$$= t + c$$

$$= \sinh^{-1}(x) + c$$

(Try $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$)

Another way

Euler's substitution

$t = x + \sqrt{1+x^2}$

...

Ex # 11

$$\int \frac{dx}{1+\sin x + \cos x}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\rightarrow 1+\sin x + \cos x$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2} + 1 - \sin^2\frac{x}{2} + \cos^2\frac{x}{2}$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2} + 2\cos^2\frac{x}{2}$$

$$1+\sin x + \cos x = 2\cos^2\frac{x}{2}\left(1+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$u = \frac{\sin x/2}{\cos x/2}$$

$$du = \frac{1}{2}\frac{dx}{\cos^2 x/2}$$

$$\rightarrow \int \frac{dx}{2\cos^2\frac{x}{2}\left(1+\frac{\sin x/2}{\cos\frac{x}{2}}\right)}$$

$$= \int \frac{du}{1+u} = \ln|1+u| + C$$

$$= \ln\left|1+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| + C$$

Rational functions

$\int \frac{P(x)}{g(x)}\,dx$ if $\deg(P(x)) \geq \deg(g(x))$

we can devide the polynomials

$$\rightarrow \int \left(r(x) + \frac{f(x)}{g(x)} \right) dx$$

$r(x)$ → Immediate

$\frac{f(x)}{g(x)}$ → $\deg(f(x)) < \deg(g(x))$

Ex

$$\int \frac{x^4}{1+x^2}\,dx$$

$\deg(x^4) > \deg(1+x^2)$

$\quad\;\; \shortparallel \qquad\qquad\quad \shortparallel$

$\quad\;\; 4 \qquad\qquad\quad\; 2$

$$\frac{x^4}{x^2+1} = x^2-1+\frac{1}{x^2+1}$$

$$\int \frac{x^4}{1+x^2}\,dx = \int \left(x^2-1+\frac{1}{x^2+1}\right)dx$$

$$= \frac{x^3}{3} - x + \arctan(x) + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x$$

$$\int \frac{-2x+4}{(x^2+1)(x^2-2x+1)}\,dx$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{(Ax+B)(x-1)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)}{(x^2+1)(x-1)(x-1)^2}$$

Final answer

$$\ln|x^2+1| + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C$$