V inner product space

$V^* = \hom(V, \mathbb{K})$ = the "dual space" of V

$= \{\text{functions } V \to \mathbb{K} \text{ linear}\}$

Lemma: For every $y \in V$ the function $\varphi_y : x \in V \longmapsto \langle x, y \rangle \in \mathbb{K}$ is linear, so gives a map

$\Phi : y \in V \longmapsto \varphi_y \in V^*$

that is semilinear and injective.

$f : V \to W$ is semilinear if

$$\begin{cases} f(v+v') = f(v) + f(v') \\ f(\lambda v) = \bar{\lambda} f(v) \end{cases}$$

(* if $\mathbb{K} = \mathbb{R}$, then semilinear = linear)

And if V is finite-dimensional, then Φ is surjective: Riesz's theorem tells us that if $\varphi \in V^*$, then there exists a unique $x_\varphi \in V$ such that

$$\varphi(z) = \langle z, x_\varphi \rangle \quad \forall z \in V$$

and this means that

$$\Phi(x_\varphi) = \varphi$$

If $\mathcal{B} = (x_1, \dots, x_n)$ is an orthonormal basis for V, then

$$x_\varphi = \sum_{i=1}^{n} \overline{f(x_i)} \cdot x_i$$

Ex: $V = \mathbb{R}^3 / \mathbb{C}^3$ with std inner product.

Fix $a, b, c \in \mathbb{R}$, $\varphi : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \longmapsto ax + by + cz$

$\mathcal{B} = (e_1, e_2, e_3)$ the std basis for $\mathbb{R}^3$ is orthonormal so we can use it compute the vector that represents φ

$x_\varphi = \overline{f(e_1)} e_1 + \overline{f(e_2)} e_2 + \overline{f(e_3)} e_3$
$= \bar{a} e_1 + \bar{b} e_2 + \bar{c} e_3$

then we have

$\varphi(v) = \langle v, x_\varphi \rangle \quad \forall v \in \mathbb{R}^3$

If $v = (x, y, z)$,

$ax + by + cz = \langle (x,y,z), (\bar{a}, \bar{b}, \bar{c}) \rangle$

this is really obvious

Ex: $V = \mathbb{R}^2$

$\langle (x,y), (x', y') \rangle = 2xx' + 2yy' + xy' + x'y$

Fix $a, b \in \mathbb{R}$, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \longmapsto ax + by$

If we want to represent φ using the standard inner product, then the representing vector is just $(a,b) \in \mathbb{R}^3$: $\varphi(v) = \langle v, (a,b) \rangle$ just as in the previous example.

We know (e_1, e_2) is a basis for $\mathbb{R}^2$

$w_1 = \frac{e_1}{\|e_1\|} = \left(\frac{1}{\sqrt{2}}, 0\right)$ $\quad \|e_1\|^2 = \langle e_1, e_1 \rangle = 2$ $\quad \langle e_2, w_1 \rangle = \langle (0,1), (\frac{1}{\sqrt{2}}, 0) \rangle = \frac{1}{\sqrt{2}}$

$\tilde{w}_2 = e_2 - \langle e_2, w_1 \rangle w_1 = \left(-\frac{1}{2}, 1\right)$

$w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \frac{\left(-\frac{1}{2}, 1\right)}{\sqrt{3/2}}$ $\quad \|\tilde{w}_2\|^2 = \langle \tilde{w}_2, \tilde{w}_2 \rangle = 2\left(-\frac{1}{2}\right)^2 + 2 \cdot 1^2 - \frac{1}{2} - \frac{1}{2} = \frac{1}{2} + 2 - 1 = \frac{3}{2}$

If $\mathcal{B} = (x_1, \dots, x_n)$ is an orthonormal basis for V, then

$$x_\varphi = \sum_{i=1}^{n} \overline{\varphi(x_i)} \cdot x_i$$

$x_\varphi = \overline{f(w_1)} w_1 + \overline{f(w_2)} w_2$
$= \frac{a}{\sqrt{2}} w_1 + \sqrt{\frac{2}{3}}\left(-\frac{a}{2} + b\right) w_2$

$\varphi(v) = \langle v, x_\varphi \rangle \quad \forall v \in \mathbb{R}^2$

Ex: $V = \mathbb{R}[x]_{\leq 2}$

$\langle p, q \rangle = \int_0^1 p \cdot q$

$\varphi : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}$, $p \longmapsto p'(1)$

$\varphi(p) = \langle p, v_\varphi \rangle \quad \forall p \in \mathbb{R}[x]_{\leq 2}$
$\quad \| \qquad\qquad \|$
$p'(1) \qquad \int_0^1 p\, v_\varphi$

$w_1 = \frac{1}{\|1\|} = 1$ $\quad \|1\|^2 = \langle 1, 1 \rangle = \int_0^1 1 = 1$

$\tilde{w}_2 = x - \langle x, 1 \rangle 1$ $\quad \langle x, 1 \rangle = \int_0^1 x = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$
$= x - 1/2$

$w_2 = \frac{x - 1/2}{\|x - 1/2\|}$
$= (x - 1/2) 2\sqrt{3}$

$\|x - 1/2\|^2 = \langle x - \frac{1}{2}, x - \frac{1}{2} \rangle$
$= \int_0^1 \left(x - \frac{1}{2}\right)^2$
$= \frac{\left(x - \frac{1}{2}\right)^3}{3}\Big|_0^1$
$= \frac{(1/2)^3}{3} - \frac{(-1/2)^3}{3}$
$= \frac{2(1/2)^3}{3} = \frac{1}{3 \cdot 2^2}$

$\tilde{w}_3 = x^2 - \langle x^2, 1 \rangle 1 - \langle x^2, x - 1/2 \rangle (x - 1/2)$
$= x^2 - \frac{1}{3} - \frac{1}{12}(x - 1/2)$
$= x^2 - \frac{1}{12} x + \left(\frac{1}{12 \cdot 2} - \frac{1}{3}\right)$

$w_3 = \frac{\tilde{w}_3}{\|\tilde{w}_3\|}$ $\quad \|\tilde{w}_3\|^2 = \dots$

(w_1, w_2, w_3)

$v_\varphi = \overline{\varphi(w_1)} w_1 + \varphi(w_2) w_2 + \varphi(w_3) w_3$
$= 0 w_1 + 2\sqrt{3} w_2 + \varphi(w_3) w_3$
$\in \mathbb{R}[x]_{\leq 2}$

$\varphi(p) = \langle p, v_\varphi \rangle$
$p'(1) = \int_0^1 p\, v_\varphi$ ∎

Def: If $f : V \to V$ is a linear map, then a linear $g : V \to V$ is adjoint to f if

$$\langle f(x), y \rangle = \langle x, g(y) \rangle \quad \forall x, y \in V.$$

Lemma: If $f : V \to V$ has an adjoint, then it has exactly one. (Not every linear map has an adjoint!)

If f has an adjoint we write it $f^* : V \to V$

Prop: V i.p.s

i) If $f, g : V \circlearrowleft$ have adjoints then for all $a, b \in \mathbb{K}$ the map $af + bg : V \circlearrowleft$ also has an adjoint, and

$$(af + bg)^* = \bar{a} f^* + \bar{b} g^*$$

ii) If $f : V \to V$ has an adjoint, then $f^* : V \to V$ also has an adjoint and

$$(f^*)^* = f.$$

Proof (i): let $h = \bar{a} f^* + \bar{b} g^* : V \circlearrowleft$

We will show that h is the adjoint of $af + bg : V \circlearrowleft$, that is, that for all $v, w \in V$,

$\langle (af + bg)(v), w \rangle = \langle v, h(w) \rangle$

We compute:

$\langle (af + bg)(v), w \rangle$
$= \langle a f(v) + b g(v), w \rangle$
$= a \langle f(v), w \rangle + b \langle g(v), w \rangle$
$= a \langle v, f^*(w) \rangle + b \langle v, g^*(w) \rangle$
$= \langle v, \bar{a} f^*(w) \rangle + \langle v, \bar{b} g^*(w) \rangle$
$= \langle v, \bar{a} f^*(w) + \bar{b} g^*(w) \rangle$
$= \langle v, (\bar{a} f^* + \bar{b} g^*)(w) \rangle$

(ii) Suppose that $f : V \to V$ has an adjoint $f^* : V \to V$.

We want to show that f^* has an adjoint and that $(f^*)^* = f$, and for that we have to prove that

$\langle f^*(v), w \rangle = \langle v, f(w) \rangle$

for all $v, w \in V$. This is easy:

$\langle f^*(v), w \rangle = \overline{\langle w, f^*(v) \rangle}$
$= \overline{\langle f(w), v \rangle}$
$= \overline{\overline{\langle v, f(w) \rangle}} = \langle v, f(w) \rangle$ ∎

$\langle f(v), w \rangle = \langle v, f^*(w) \rangle \quad \forall v, w \in V$

Def: $g : V \to V$ is adjoint to $f : V \to V$ if

$\langle f(v), w \rangle = \langle v, g(w) \rangle \quad \forall v, w \in V$

- $(af + bg)^* = \bar{a} f^* + \bar{b} g^* \quad \forall a, b \in \mathbb{K}$
- $(f^*)^* = f$
- $(f \circ g)^* = g^* \circ f^*$.

Theorem: If $\dim V < \infty$, then every linear map $f : V \to V$ has an adjoint.

Lemma: If $f, g : V \to V$ have adjoints, then $f \circ g : V \to V$ has an adjoint and

$$(f \circ g)^* = g^* \circ f^*$$

Proof: We have to show that

$\langle (f \circ g)(v), w \rangle \overset{?}{=} \langle v, (g^* \circ f^*)(w) \rangle$
$\quad \| \qquad\qquad\qquad\qquad \|$
$\langle f(g(v)), w \rangle$
$\quad \|$
$\langle g(v), f^*(w) \rangle = \langle v, g^*(f^*(w)) \rangle$ ∎

Proof: Suppose that V is fin. dim. and let $f : V \circlearrowleft$ be a linear map.

If $y \in V$, then the function

$\varphi_y : x \in V \longmapsto \langle f(x), y \rangle \in \mathbb{K}$

is linear:

$\varphi_y(ax + bx') = \langle f(ax + bx'), y \rangle$
$= \langle a f(x) + b f(x'), y \rangle$
$= a \langle f(x), y \rangle + b \langle f(x'), y \rangle$
$= a \varphi_y(x) + b \varphi_y(x')$

So $\varphi_y \in V^*$. Riesz's representation theorem tell us that there is a unique vector $v_y \in V$ s.t.

$\varphi_y(x) = \langle x, v_y \rangle \quad \forall x \in V.$

Let's define the function

$g : V \to V$, $y \longmapsto v_y$

We will show that g is linear and adjoint to f

Suppose that $x, y \in V$

$\langle f(x), y \rangle = \varphi_y(x) = \langle x, v_y \rangle = \langle x, g(y) \rangle$

let $y, y' \in V$, $a, b \in \mathbb{K}$

$g(ay + by') = v_{ay + by'}$

is the unique vector such that

$\varphi_{ay+by'}(x) = \langle x, v_{ay+by'} \rangle \quad \forall x$

We know that

$\langle f(x), ay + by' \rangle$
$\quad \|$
$\bar{a} \langle f(x), y \rangle + \bar{b} \langle f(x), y' \rangle = \bar{a} \varphi_y(x) + \bar{b} \varphi_{y'}(x)$

$= \bar{a} \langle x, v_y \rangle + \bar{b} \langle x, v_{y'} \rangle$
$= \langle x, a v_y + b v_{y'} \rangle$

$\therefore \varphi_{ay+by'}(x) = \langle x, a v_y + b v_{y'} \rangle \quad \forall x$

By definition of $v_{ay+by'}$

$\varphi_{ay+by'}(x) = \langle x, v_{ay+by'} \rangle \quad \forall x$

$\therefore a v_y + b v_{y'} = v_{ay+by'} = g(ay + by')$
$\quad \|$
$a g(y) + b g(y') \qquad \therefore g$ is linear ∎

Theorem: Suppose that $n = \dim V < \infty$ and let $\mathcal{B} = (x_1, \dots, x_n)$ be an orthonormal ordered basis for V.

If $f : V \to V$ is a linear map and $[f]_{\mathcal{B}} = (a_{ij}) \in M_n(\mathbb{K})$ is the matrix of f w.r.t. $\mathcal{B}$, then the matrix of the linear map $f^* : V \to V$ is

$$[f^*]_{\mathcal{B}} = \overline{[f]_{\mathcal{B}}}^{t}$$

Proof: If $v \in V$, then

$v = \sum_{i=1}^{n} \langle v, x_i \rangle x_i$

In particular,

$f^*(x_k) = \sum_{i=1}^{n} \langle f^*(x_k), x_i \rangle x_i$
$= \sum_{i=1}^{n} \overline{\langle x_i, f^*(x_k) \rangle} x_i$
$= \sum_{i=1}^{n} \overline{\langle f(x_i), x_k \rangle} x_i$
$= \sum_{i=1}^{n} \overline{a_{ki}} x_i$

$f(x_\ell) = \sum_{i=1}^{n} \underbrace{\langle f(x_\ell), x_i \rangle}_{} x_i = a_{i\ell}$

$\therefore \langle f(x_\ell), x_i \rangle = a_{i\ell}$

$$[f^*]_{\mathcal{B}} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & \bar{a}_{22} & & \bar{a}_{n2} \\ \bar{a}_{13} & \bar{a}_{23} & \cdots & \vdots \\ \vdots & \vdots & & \\ \bar{a}_{1n} & \bar{a}_{2n} & & \bar{a}_{nn} \end{pmatrix} = \overline{(a_{ij})}^{t}$$ ∎

Ex: $\mathbb{R}^3$ with std inner product

and $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x + z \\ y - z \\ x + y + z \end{pmatrix}$$

$\mathcal{B} = (e_1, e_2, e_3)$ std basis is O.N.

$$[f]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$[f^*]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$f^*\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ y + z \\ x - y + z \end{pmatrix}$$ ∎