Def. $f: V \circlearrowleft$, V ipus, $\dim V < \infty$

We say $g: V \circlearrowleft$ is adjoint to f if

$$\langle f(x), y\rangle = \langle x, g(y)\rangle \quad \forall x, y \in V.$$

In that case we write $f^* = g$.

Def. f is self-adjoint if $f = f^*$

f is normal if $f f^* = f^* f$.

(self-adjoint $\Rightarrow$ normal)

Thm. If f is self-adjoint then all the eigenvalues of f are real and there exists an orthonormal basis $\mathcal{B}$ for V s.t. $[f]_{\mathcal{B}}$ is diagonal.

Thm. If V is complex vector space and $f: V \circlearrowleft$ is normal, then there exists a orthonormal basis $\mathcal{B}$ for V s.t. $[f]_{\mathcal{B}}$ is diagonal.

Corollary: V complex vector space with an inner-product, $f: V \circlearrowleft$ linear. The following statements are equivalent:

i) f is normal $(f^* f = f f^*)$

ii) there is an orthonormal basis $\mathcal{B}$ for V s.t. $[f]_{\mathcal{B}}$ is diagonal.

Proof: The theorem tells us that i) $\Rightarrow$ ii)

ii) $\Rightarrow$ i) Suppose $\mathcal{B}$ is an O.N.B for V s.t. $[f]_{\mathcal{B}}$ is diagonal. $[f^*]_{\mathcal{B}} = \overline{[f]_{\mathcal{B}}}^t$

$$[f f^*]_{\mathcal{B}} = [f]_{\mathcal{B}} [f^*]_{\mathcal{B}} = \underbrace{[f]_{\mathcal{B}}}_{\text{diag}} \underbrace{\overline{[f]}^t_{\mathcal{B}}}_{\text{diag}}$$

(since all diagonal matrices commute)

$$= \overline{[f]}^t_{\mathcal{B}} [f]_{\mathcal{B}} = [f^*]_{\mathcal{B}}[f]_{\mathcal{B}} = [f^* f]_{\mathcal{B}}$$

$\therefore f f^* = f^* f$ $\therefore f$ is normal ∎

Def. V ipus, $f: V \to V$. If

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \quad \forall x, y \in V$$

then we say

- f is orthogonal if $K = \mathbb{R}$
- f is unitary if $K = \mathbb{C}$.

Obs. A unitary/orthogonal map preserves the inner-product $\langle x, y\rangle = \langle f(x), f(y)\rangle$

In particular, they preserve angles $x, y \in V \setminus 0$

$$\cos \text{angle}(x,y) = \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}$$

$$\cos \text{angle}(f(x), f(y)) = \frac{\langle f(x), f(y)\rangle}{\|f(x)\| \, \|f(y)\|} = \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|} = \cos \text{angle}(x,y)$$

$$\|f(x)\| = \langle f(x), f(x)\rangle^{1/2} = \langle x, x\rangle^{1/2} = \|x\|$$

Prop: V ipus, $f: V \circlearrowleft$

TFAE:

i) f is unitary/orthogonal

ii) $\forall x \in V$, $\|f(x)\| = \|x\|$.

Proof: i) $\Rightarrow$ ii) we already did!

ii) $\Rightarrow$ i)

Lemma: V ipus.

over $\mathbb{R}$: $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

over $\mathbb{C}$: $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2 + \frac{i}{4}\|x+iy\|^2 - \frac{i}{4}\|x-iy\|^2$

(related to the parallelogram law) ∎

Prop. V ipus, $f: V \circlearrowleft$

TFAE:

i) f is unitary/orthogonal

ii) $\forall x \in V$, $\|f(x)\| = \|x\|$.

Proof: i) $\Rightarrow$ ii) we already did!

ii) $\Rightarrow$ i) let's do the real case:

$$\langle f(x), f(y)\rangle = \frac{1}{4}\|f(x)+f(y)\|^2 - \frac{1}{4}\|f(x)-f(y)\|^2$$
$$= \frac{1}{4}\|f(x+y)\|^2 - \frac{1}{4}\|f(x-y)\|^2$$
$$= \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2 = \langle x, y\rangle$$

Exercise: Do the complex case.

$\langle f(x), f(y)\rangle = \cdots = \langle x, y\rangle$ ∎

Exercise: V ipus, $f: V \circlearrowleft$ linear. TFAE

i) f is ortho/unitary

ii) $d(f(x), f(y)) = d(x, y)$ $\forall x, y \in V$

(isometry)

Prop. V ipus, $f: V \circlearrowleft$ that has an adjoint

TFAE:

i) f is orthogonal/unitary

ii) $f^* f = \mathrm{id}_V = f f^*$, so that f is invertible and $f^{-1} = f^*$.

(In particular, ortho/unitary maps are normal)

i) $\Rightarrow$ ii) let's suppose that f is ortho/unitary, and let $x \in V$. If $y \in V$, then

$$\langle f^* f(x), y\rangle = \langle f^*(f(x)), y\rangle = \langle f(x), f^{**}(y)\rangle = \langle f(x), f(y)\rangle = \langle x, y\rangle$$

$\therefore f^* f(x) = x$

$\therefore f^* f = \mathrm{id}_V$

$$= \langle f(x), f(y)\rangle$$
$$= \langle x, y\rangle$$

$\therefore f^* f(x) = x$ $\qquad$ $\therefore f$ is invertible

$\therefore f^* f = \mathrm{id}_V$ $\qquad$ $f^* = f^* f f^{-1} = \mathrm{id} \cdot f^{-1}$ "f^{-1}"

$\det(f^*) \cdot \underbrace{\det(f)}_{\neq 0} = 1$ $\qquad$ $f f^* = f f^{-1} = \mathrm{id}_V$ ∎

Ex: $\theta \in \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$[f]_{std} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

f is a rotation of angle θ.

$$[f^*]_{std} = \overline{[f]}^t_{std} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotation of angle $-\theta$

$$= \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$\therefore f$ and f^* are inverse to each other

$\therefore f$ is orthogonal.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \underbrace{\cos^2\theta + \sin^2\theta}_{=1} & \sim \\ \underbrace{-\sin\theta\cos\theta + \cos\theta\sin\theta}_{=0} & \sim \end{pmatrix}$$

Prop. V ipus, $\dim V < \infty$, $f: V \circlearrowleft$

TFAE:

a) f is ortho/unitary.

b) there is some O.N.B $\mathcal{B}$ for V s.t. $f(\mathcal{B})$ is also orthonormal.

c) For all orthonormal bases $\mathcal{B}$ of V, $f(\mathcal{B})$ is an orthonormal basis

a) $\Rightarrow$ c) Suppose f is ortho/unitary. Let $\mathcal{B} = (v_1, \dots, v_n)$ be any orthonormal basis for V. For all $i, j \in \{1, \dots, n\}$

$$\langle f(v_i), f(v_j)\rangle = \langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then $f(\mathcal{B}) = (f(v_1), \dots, f(v_n))$ is orthonormal and a basis (since it is LI and has n elements)

c) $\Rightarrow$ b) We know that there exists some orthonormal basis $\mathcal{B}$ for V, and c) tells us that $f(\mathcal{B})$ is an O.N.B

b) $\Rightarrow$ a) Suppose that $\mathcal{B} = (v_1, \dots, v_n)$ is an orthonormal basis of V such that $f(\mathcal{B}) = (f(v_1), \dots, f(v_n))$ is also an orthonormal basis. Let x and y be two elements of V. There are scalars $a_1, \dots, a_n, b_1, \dots, b_n \in K$ such that $x = a_1 v_1 + \dots + a_n v_n$, $y = b_1 v_1 + \dots + b_n v_n$. We then have

$$\langle f(x), f(y)\rangle = \langle \sum_i a_i f(v_i), \sum_j b_j f(v_j)\rangle = \sum_{i,j} a_i \overline{b_j} \langle f(v_i), f(v_j)\rangle = \sum_i a_i \overline{b_i}$$

$$\langle x, y\rangle = \langle \sum_i a_i v_i, \sum_j b_j v_j\rangle = \sum_{i,j} a_i \overline{b_j}\langle v_i, v_j\rangle = \sum_i a_i \overline{b_i}$$

$\therefore \langle f(x), f(y)\rangle = \langle x, y\rangle$ $\quad \therefore f$ is ortho/unitary ∎

$$f(e_1) = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$f(e_2) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

(e_1, e_2) is an O.N.B for $\mathbb{R}^2$

$$(f(e_1), f(e_2)) = \left(\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right)$$

is also an O.N. basis for V

$\therefore f$ is orthogonal.

Ex: In general ($\mathbb{R}^n$ std i.p)

$f: \mathbb{R}^n \to \mathbb{R}^n$

with
$$[f]_{std} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = A \in M_n(\mathbb{R})$$

$\mathcal{B} = (e_1, \dots, e_n)$ is O.N.B for $\mathbb{R}^n$

f is orthogonal

iff $(f(e_1), f(e_2), \dots, f(e_n))$ is an O.N basis for $\mathbb{R}^n$

iff the columns of the matrix A are orthonormal!

Orthogonal and unitary matrices

- We say a matrix $A \in M_n(\mathbb{R})$ is orthogonal if $\langle Ax, Ay\rangle = \langle x, y\rangle$ $\forall x, y \in \mathbb{R}^n$
- We say $A \in M_n(\mathbb{C})$ is unitary if $\langle Ax, Ay\rangle = \langle x, y\rangle$ $\forall x, y \in \mathbb{C}^n$ (std inner prod in $\mathbb{C}^n$)

Prop. i) If $A \in M_n(\mathbb{R})$ ($\mathbb{C}$), then TFAE:

a) A is orthogonal (unitary)

b) the linear map $f_A: x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^n$ is orthogonal (unitary)

c) $A A^t = I = A^t A$

d) the columns of A form an orthonormal basis for $\mathbb{R}^n$ ($\mathbb{C}^n$)

Proof: i)

a) $\Rightarrow$ b) Suppose A is orthogonal. Then if $x, y \in \mathbb{R}^n$,

$$\langle f_A(x), f_A(y)\rangle = \langle Ax, Ay\rangle = \langle x, y\rangle$$

So f_A is orthogonal.

b) $\Rightarrow$ c) Suppose that $f_A: x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^n$ is orthogonal. We know that $[f_A]_{std} = A$, $[f_A^*]_{std} = \overline{[f_A]}^t_{std} = A^t$

$f_A f_A^* = \mathrm{id}_V = f_A^* f_A$ because f is orthogonal

$$\boxed{A A^t} = [f_A]_{std}[f_A^*]_{std} = [f_A f_A^*]_{std} = [\mathrm{id}]_{std} = I_n$$

Similarly,

$$A^t A = [f_A^*]_{std}[f_A]_{std} = [f_A^* f_A]_{std} = [\mathrm{id}]_{std} = I_n$$

c) $\Rightarrow$ d) Suppose that $A A^t = I_n = A^t A$.

Then $A^t A = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} \underbrace{a_{11}a_{11} + a_{21}a_{21} + \cdots + a_{n1}a_{n1}}_{\langle v_1, v_1\rangle} & \underbrace{a_{11}a_{12} + a_{21}a_{22} + \cdots + a_{n1}a_{n2}}_{\langle v_1, v_2\rangle} & \cdots \\ & & \end{pmatrix} = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \cdots & \langle v_2, v_n\rangle \\ \vdots & & & \vdots \\ \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}$$

$\langle v_i, v_i\rangle = 1$ $\forall i$

$\langle v_i, v_j\rangle = 0$ if $i \neq j$.

$(v_1, \dots, v_n)$ is O.N.

Proof: i)

d) $\Rightarrow$ b) Suppose the columns of A form an orthonormal basis for $\mathbb{R}^n$

$A = (v_1 | v_2 | \cdots | v_n)$

$$f_A: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto Ax$$

$f(e_i) = v_i$ $\forall i$

The image of $(e_1, \dots, e_n)$ by f_A is $(v_1, \dots, v_n)$, which is an O.N basis $\therefore f_A$ is orthogonal ∎

Obs: If the columns of $A \in M_n(\mathbb{R})$ form an orthonormal basis for $\mathbb{R}^n$, then the rows of A also form an orthonormal basis

Proof: A is orthogonal

$\Leftrightarrow A A^t = I = A^t A$

$\Leftrightarrow A^t (A^t)^t = I = (A^t)^t A^t$

$\Leftrightarrow A^t$ is orthogonal

$\Leftrightarrow$ the columns of A^t are an O.N basis

$\Leftrightarrow$ the rows of A are an O.N basis.

Ex:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ is orth} \qquad \left\{\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right\} \text{ is an O.N.B of } \mathbb{R}^2 \qquad \left\{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\right\} \text{ is an O.N.B for } \mathbb{R}^2_{rows}$$

A unitary/ortho map f is normal because $f^* f = f^{-1} f = \mathrm{id} = f f^{-1} = f f^*$

Prop. i) If $A \in M_n(\mathbb{R})$ is a symmetric matrix, there exists an orthogonal matrix C s.t. $C^t A C$ is diagonal.

ii) If $A \in M_n(\mathbb{C})$ is hermitian $(A = \overline{A}^t)$ then there exists a unitary matrix C s.t. $\overline{C}^t A C$ is diagonal

QR-decomposition