Leibniz Rule and Double integrals

① $f(x,y): [a,b]\times[c,d] \longrightarrow \mathbb{R}$ is cont

$\Longrightarrow F(y) = \int_a^b f(x,y)\,dx$ is cont on $[c,d]$

② Leibniz: Assume $\frac{\partial f}{\partial y}(x,y)$ is continuous on $[a,b]\times[c,d]$

$\Longrightarrow F'(y) = \frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

③ Generalization of Leibniz

$F(y) = \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx$ $\quad a \le \varphi(y),\ \psi(y) \le b$

If $\frac{\partial f}{\partial y}$ is cont and $\psi(y), \varphi(y)$ are diff

$F'(y) = \int_{\varphi(y)}^{\psi(y)} \frac{\partial f}{\partial y}(x,y)\,dx + f(\psi(y),y)\,\psi'(y) - f(\varphi(y),y)\,\varphi'(y)$

④ Fubini

If f is cont

$\Longrightarrow \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$

If the domain is not a rectangle, we still can change the order of integration carefully

Exercise

$F(x) = \int_1^2 \sin(xe^y)\,dy$ calculate $F'(x)$

$F'(x) = \frac{d}{dx}\int_1^2 \sin(xe^y)\,dy = \int_1^2 \frac{\partial}{\partial x}(\sin(xe^y))\,dy = \int_1^2 e^y\cos(xe^y)\,dy$

$t = xe^y$

$dt = xe^y\,dy$

$\Longrightarrow F'(x) = \int_{xe}^{xe^2} \frac{dt}{x}\cos(t) = \frac{\sin t}{x}\Big|_{xe}^{xe^2} = \frac{1}{x}\left[\sin(xe^2) - \sin(xe)\right]$

Example

Evaluate $\int_0^1 \frac{x}{(1+2x)^2}\,dx$

Sol Define $F(y) = \int_0^1 \frac{x}{(1+yx)^2}\,dx$ (We want $F(2)$)

$\frac{\partial}{\partial y}\left(-\frac{1}{1+yx}\right)$

$F(y) = \int_0^1 \frac{\partial}{\partial y}\left(-\frac{1}{1+xy}\right)dx = -\frac{d}{dy}\int_0^1 \frac{1}{1+xy}\,dx$

$= -\frac{d}{dy}\left(\frac{1}{y}\ln(1+xy)\Big|_{x=0}^{x=1}\right) = -\frac{d}{dy}\left(\frac{1}{y}\ln(1+y) - \frac{1}{y}\ln 1\right)$

$\Longrightarrow F(y) = -\left[-\frac{1}{y^2}\ln(1+y) + \frac{1}{y(1+y)}\right]$

$\Longrightarrow F(2) = \int_0^1 \frac{x\,dx}{(1+2x)^2} = \frac{\ln(3)}{4} - \frac{1}{6}$

You can verify that the result is correct

$\frac{x}{(1+2x)^2} = \frac{1}{2}\frac{(2x+1-1)}{(1+2x)^2} = \frac{1}{2}\left[\frac{1}{1+2x} - \frac{1}{(1+2x)^2}\right]$

Solve the integral to verify $\frac{\ln 3}{4} - 1/6$

Exercise Evaluate $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

Hint Consider the function $F(y) = \int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx$

Sol

$F'(y) = \int_0^y \frac{\partial}{\partial y}\left(\frac{\ln(1+xy)}{1+x^2}\right)dx + \frac{\ln(1+y^2)}{1+y^2}\cdot y' - \frac{\ln(1)}{1}\cdot 0'$

$F'(y) = \int_0^y \frac{x}{(1+xy)(1+x^2)}\,dx + \frac{\ln(1+y^2)}{1+y^2}$

$\frac{x}{(1+xy)(1+x^2)} = \frac{A}{1+xy} + \frac{Bx+C}{1+x^2} = \frac{A + Ax^2 + Bx + Bx^2y + Cxy + C}{(1+xy)(1+x^2)}$

$A + C = 0$

$B + Cy = 1$

$A + By = 0$

$\to By = C$

$\to B + By^2 = 1 \Rightarrow B = \frac{1}{1+y^2}$

$C = \frac{y}{1+y^2}$

$A = \frac{-y}{1+y^2}$

$F'(y) = \frac{-y}{1+y^2}\int_0^y \frac{1}{1+xy}\,dx + \frac{1}{1+y^2}\int_0^y \frac{x}{1+x^2}\,dx + \frac{y}{1+y^2}\int_0^y \frac{1}{1+x^2}\,dx + \frac{\ln(1+y^2)}{1+y^2}$

$= \frac{-y}{1+y^2}\frac{1}{y}\ln(1+xy)\Big|_0^y + \frac{1}{1+y^2}\frac{1}{2}\ln(1+x^2)\Big|_0^y + \frac{y}{1+y^2}\arctan(x)\Big|_0^y + \frac{\ln(1+y^2)}{1+y^2}$

$= -\cancel{\frac{\ln(1+y^2)}{1+y^2}} + \frac{1}{2}\frac{\ln(1+y^2)}{1+y^2} + \frac{y}{1+y^2}\arctan(y) + \cancel{\frac{\ln(1+y^2)}{1+y^2}}$

$\Longrightarrow F'(y) = \frac{1}{2}\frac{\ln(1+y^2)}{1+y^2} + \frac{y\arctan y}{1+y^2}$

$F'(y) = \frac{1}{2}\left(\ln(1+y^2)\arctan y\right)'$

$\Longrightarrow F(y) = \frac{1}{2}\ln(1+y^2)\arctan y + \text{const}$

Exercise

$f(x,y) = \begin{cases} \frac{y^2-x^2}{(x^2+y^2)^2} & (x,y)\ne(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Is this correct? $\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy = \int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx$

Is f continuous at $(0,0)$?

$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{(x^2+y^2)^2} \underset{\substack{\text{polar}\\\text{coord.}}}{=} \lim_{r\to 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^4} \not\exists$

f is not continuous

It is not correct. You can check that the result is $\pi/4$ or $-\pi/4$

Simple domain $D = \{(x,y) \,/\, a \le x \le b,\ y_1(x) \le y \le y_2(x)\}$

$\iint_D f(x,y)\,dx\,dy = \int_a^b\left(\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\right)dx$

or $D = \{(x,y) \,/\, c \le y \le d,\ x_1(y) \le x \le x_2(y)\}$

$\iint_D f(x,y)\,dx\,dy = \int_c^d\left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right)dy$

Exercise Evaluate $I = \iint_D e^{x/y}\,dx\,dy$

D is the domain bounded between $x=0$, $y=1$ and $x=y^2$

we can choose

$0 \le x \le 1$

$\sqrt{x} \le y \le 1$

or $0 \le y \le 1$

$0 \le x \le y^2$

$\int_0^1\left(\int_0^{y^2} e^{x/y}\,dx\right)dy = \int_0^1 y e^{x/y}\Big|_{x=0}^{x=y^2}dy = \int_0^1 y(e^{y^2/y} - 1)\,dy$

$= \int_0^1 (ye^y - y)\,dy = \left(ye^y - e^y - \frac{y^2}{2}\right)\Big|_0^1 =$

$u = y \quad v' = e^y$

$u' = 1 \quad v = e^y$

$= e - e - \frac{1}{2} - 0 + 1 + 0 = \boxed{\frac{1}{2}}$

Exercise

Calculate $\iint_D (x^3+y^3)\,dx\,dy$ where D is the domain bounded by $y = x$ $\quad y = \frac{x}{2}$ $\quad x = 4$

$0 \le x \le 4$

$\frac{x}{2} \le y \le x$

(side note:

$\iint_D f(x,y)\,dx\,dy$ = Volume of under the graph of $f(x,y)$ above D)

$I = \int_0^4\left(\int_{x/2}^{x} (x^3+y^3)\,dy\right)dx = \int_0^4\left(x^3y + \frac{y^4}{4}\right)\Big|_{y=x/2}^{y=x}dx$

$= \int_0^4\left(x^4 + \frac{x^4}{4} - \frac{x^4}{2} - \frac{x^4}{64}\right)dx = \int_0^4 \frac{47}{64}x^4\,dx = \frac{47}{64\times 5}x^5\Big|_0^4$

$= 752/5$

Exercise

Calculate $\iint_D (x^3+y^3)\,dx\,dy$ where D is the domain bounded by $y = x$ $\quad y = \frac{x}{2}$ $\quad x = 4$

$0 \le x \le 4$

$\frac{x}{2} \le y \le x$

??

$0 \le y \le 4$

$y \le x \le 2y$

X NO

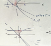
$I = \int_0^2\left(\int_y^{2y} (x^3+y^3)\,dx\right)dy + \int_2^4\left(\int_y^4 (x^3+y^3)\,dx\right)dy$

$= 752/5$

Exercise Let D be the region bounded between $y=1$, $x=y^2$ and $x+2y+1=0$

Find the area of D

$A(D) = \iint_D 1\,dx\,dy$



$-1 \le y \le 1$

$-2y-1 \le x \le y^2$

$A(D) = \int_{-1}^1\left(\int_{-2y-1}^{y^2} 1\,dx\right)dy$

$= \int_{-1}^1 (y^2+2y+1)\,dy =$

$\left(\frac{y^3}{3} + y^2 + y\right)\Big|_{-1}^1 = 8/3$