Adjoints V ips

If $f: V \to V$, we say g is adjoint to f if $\langle f(x), y\rangle = \langle x, g(y)\rangle \ \forall x, y$

If f has an adjoint, it has only one and we write f^*.

$\langle f(x), y\rangle = \langle x, f^*(y)\rangle \ \forall x, y$

$(\lambda f + \mu g)^* = \bar{\lambda} f^* + \bar{\mu} g^*$

$(f^*)^* = f$, $(g \circ f)^* = f^* \circ g^*$

If $\dim V < \infty$, then every linear map $f: V \to V$ has an adjoint.

If $B = (x_1, \dots, x_n)$ is an ONB for V, then

$[f^*]_B = \overline{[f]_B}^t$

Ex: $V = \mathbb{R}^2$, std inner product

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x \end{pmatrix}$

$B = (e_1, e_2)$ is an ONB for $\mathbb{R}^2$

$[f]_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$[f^*]_B = \overline{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^t = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = [f]_B$

$f = f^*$

$g: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+2y \\ x \end{pmatrix}$

$[g]_B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$[g^*]_B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \neq [g]_B$

$g^* \neq g$, $g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \end{pmatrix}$

Suppose $y \in \ker(f^*)$

$\langle x, f^*(y)\rangle = \langle x, 0\rangle = 0$

$\langle f(x), y\rangle$

If $y \in \ker(f^*)$, then $\langle f(x), y\rangle = 0 \ \forall x \in V$

If $z \in \operatorname{Im}(f)$, there is an $x \in V$ s.t. $z = f(x)$, $\langle z, y\rangle = \langle f(x), y\rangle = 0$

$\therefore \ker(f^*) \subseteq \operatorname{Im}(f)^\perp$

Suppose $y \in \operatorname{Im}(f)^\perp$

then $\forall z \in \operatorname{Im}(f)$, $\langle z, y\rangle = 0$

Let $x \in V$. then $f(x) \in \operatorname{Im}(f)$

$\langle x, f^*(y)\rangle = \langle f(x), y\rangle = 0$

$\therefore \forall x \in V$, $\langle x, f^*(y)\rangle = 0$

$\therefore f^*(y) = 0$ $\therefore y \in \ker(f^*)$

Prop: V ips, $f: V \to V$ linear map, $\dim V < \infty$

i) $\ker(f^*) = \operatorname{Im}(f)^\perp$

ii) $\operatorname{Im}(f^*) = \ker(f)^\perp$

. Suppose $y \in \operatorname{Im}(f^*)$

then there is an $x \in V$ s.t. $y = f^*(x)$

If $z \in \ker(f)$, then $f(z) = 0$ so

$0 = \langle f(z), x\rangle = \langle z, f^*(x)\rangle = \langle z, y\rangle$

$\therefore \forall z \in \ker(f)$, $\langle z, y\rangle = 0$

$\therefore y \perp \ker(f)$ $\therefore y \in \ker(f)^\perp$

Prop: V ips, $f: V \to V$ linear map, $\dim V < \infty$

i) $\ker(f^*) = \operatorname{Im}(f)^\perp$ ✓

ii) $\operatorname{Im}(f^*) = \ker(f)^\perp$ ✓

iii) $\operatorname{Im}(f^*)^\perp = \ker(f)$ ✓

proof. ☺

(iii) let $x \in \operatorname{Im}(f^*)^\perp$.

then $\forall y \in V$, $\langle x, f^*(y)\rangle = 0$

$\langle f(x), y\rangle$

$\therefore f(x) = 0$

$\therefore x \in \ker(f)$

. Finally, from ii)

$\ker(f) = \ker(f)^{\perp\perp} \subseteq \operatorname{Im}(f^*)^\perp$

$\boxed{\ker(f)^\perp \subseteq \operatorname{Im}(f^*)^{\perp\perp} = \operatorname{Im}(f^*)}$

$S \subseteq T$ subspaces of V

$\Downarrow$

$T^\perp \subseteq S^\perp$

Self-adjoint maps

V ips

Def: A linear map $f: V \to V$ is self-adjoint if it is its own adjoint, $f^* = f$.

In that case we have $\langle f(x), y\rangle = \langle x, f(y)\rangle \ \forall x, y$

Prop: V finite-dim ips, B ONB for V

A linear function $f: V \to V$ is self-adjoint if and only if $[f]_B = \overline{[f]_B}^t$

In particular

. if $K = \mathbb{R}$, then f is self-adjoint if and only if $[f]_B$ is symmetric

. if $K = \mathbb{C}$, then f is self-adjoint if and only if the matrix $[f]_B$ is hermitian

Def. $A \in M_n(\mathbb{C})$ is hermitian if $A = \bar{A}^t$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x \end{pmatrix}$ is self-adjoint because $[f]_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is symmetric

Example: ($\dim V < \infty$)

Prop: A projection $p: V \to V$ is self-adjoint if and only if it is orthogonal

proof: $\Rightarrow$) Suppose that p is self-adjoint

let $x \in \ker(p)$, $y \in \operatorname{Im}(p)$

$\langle x, y\rangle = \langle x, p(y)\rangle = \langle x, p^*(y)\rangle$

$= \langle p(x), y\rangle$

$= \langle 0, y\rangle$

$= 0$

$\therefore \ker(p) \perp \operatorname{Im}(p)$

$\Leftarrow$) Suppose that p is orthogonal, so that $\operatorname{Im}(p) \perp \ker(p)$

let $x, y \in V = \operatorname{Im}(p) \oplus \ker(p)$

$x = \underbrace{x'}_{\ker} + \underbrace{x''}_{\operatorname{Im}}$, $y = \underbrace{y'}_{\ker} + \underbrace{y''}_{\operatorname{Im}}$

$\langle p(x), y\rangle = \langle p(x') + p(x''), y' + y''\rangle$

$= \langle p(x'), y'\rangle + \langle p(x''), y''\rangle$

$= \langle x'', y''\rangle$

$\langle x, p(y)\rangle = \langle x' + x'', p(y') + p(y'')\rangle = \langle x', p(y'')\rangle$

$+ \langle x'', p(y'')\rangle = \langle x'', y''\rangle$ ∎

Prop: V ips, $\dim V < \infty$, $f: V \to V$ self-adjoint

i) Every eigenvalue of f is real. [All the roots χ_f are real in $\mathbb{C}$]

ii) if x, y are eigenvectors of with different eigenvalues then $x \perp y$

proof: i) let $\lambda \in K$ be an eigenvalue of f, so that there is a non-zero $x \in V$ s.t. $f(x) = \lambda x$

$\boxed{\lambda \underbrace{\langle x, x\rangle}_{\neq 0}} = \langle \lambda x, x\rangle$

$= \langle f(x), x\rangle$

$= \langle x, f^*(x)\rangle$

$= \langle x, f(x)\rangle$

$= \langle x, \lambda x\rangle$

$= \boxed{\bar{\lambda} \underbrace{\langle x, x\rangle}_{\neq 0}}$

$\therefore \lambda = \bar{\lambda}$, $\lambda \in \mathbb{R}$

ii) Suppose $\lambda, \mu \in \mathbb{R}$, $x, y \in V \setminus 0$

such that $\lambda \neq \mu$, $f(x) = \lambda x$, $f(y) = \mu y$

$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle$

$= \langle f(x), y\rangle$

$= \langle x, f(y)\rangle$

$= \langle x, \mu y\rangle$

$= \mu\langle x, y\rangle$

$\therefore \underbrace{(\lambda - \mu)}_{\neq 0}\langle x, y\rangle = 0$ $\therefore \langle x, y\rangle = 0$, $x \perp y$ ∎

Prop: V ips, $\dim V < \infty$, $f: V \to V$ self-adjoint.

i) Every eigenvalue of f is real. [All the roots χ_f are real in $\mathbb{C}$]

ii) if x, y are eigenvectors of with different eigenvalues then $x \perp y$

Prop. If $f: V \to V$ is self-adjoint, then f has at least one eigenvalue. ($0 < \dim V < \infty$)

proof. if $K = \mathbb{C}$, then $\chi_f \in \mathbb{C}[x]$ has some root in $\mathbb{C}$, because $\mathbb{C}$ is algebraically closed, and that root is an eigenvalue of f

We have to consider the case in which $K = \mathbb{R}$. Show that $\chi_f \in \mathbb{R}[x]$ has some root in $\mathbb{R}$

let $B = (v_1, \dots, v_n)$ be an ONB for V, and let $A = [f]_B \in M_n(\mathbb{R})$

Since f is self-adjoint, $A = A^t$

let $g: v \in \mathbb{C}^n \longmapsto Av \in \mathbb{C}^n$

this is a $\mathbb{C}$-linear map

let's put on $\mathbb{C}^n$ the standard inner product

$B' = (e_1, e_2, \dots, e_n)$ the standard basis for $\mathbb{C}^n$

B' is ON for $\mathbb{C}^n$

$[g]_{B'} = A$

therefore

$\overline{[g]_{B'}}^t = \bar{A}^t = \bar{A} = A = [g]_{B'}$

$[g]_{B'}$ is hermitian, so $g = g^*$

$\therefore g: \mathbb{C}^n \to \mathbb{C}^n$ is self-adjoint.

Since g is an endomorphism of a $\mathbb{C}$-vector space it has some eigenvalue $\lambda \in \mathbb{C}$

Since $g = g^*$, $\lambda \in \mathbb{R}$

$\chi_g = \chi_{[g]_{B'}} = \chi_A = \chi_{[f]_B} = \chi_f$

$0 = \chi_g(\lambda) = \chi_f(\lambda)$ λ is a real root of χ_f

and therefore it is an eigenvalue for f ∎

$\chi^2 +$

Prop. If $f: V \to V$ is self-adjoint, then f has at least one eigenvalue. ($0 < \dim V < \infty$)

thm: let V be a fin. dim ips and let $f: V \to V$ be a self-adjoint linear map. there exists an orthonormal basis B of V whose elements are eigenvectors of f and therefore f is diagonalizable

proof: We will do induction with respect to $n = \dim V$

. Suppose first that $n = 1$.

If $v \in V \setminus 0$, then $w = \frac{v}{\|v\|}$ and $B = (w)$ is an ONB for V.

$[f]_B = (\lambda) \in M_1(K)$ is diagonal ✓

. Suppose now that $n = \dim V > 1$.

Since f is self-adjoint, there is some eigenvalue $\lambda \in K$

so there is some non-zero vector $v \in V \setminus 0$ s.t. $f(v) = \lambda v$

let $w_1 = \frac{v}{\|v\|}$, a unit vector

$f(w_1) = \lambda w_1$

let $W := \langle w_1\rangle^\perp$, a subspace of V

We know that $V = \langle w_1\rangle \oplus \langle w_1\rangle^\perp$

let's check that W is f-invariant:

$v \in W \Rightarrow f(v) \in W$

let $v \in W = \langle w_1\rangle^\perp$

$\langle f(v), w_1\rangle = \langle v, f(w_1)\rangle$

$= \langle v, \lambda w_1\rangle$

$= \bar{\lambda}\langle v, w_1\rangle$

$= 0$

$f(v) \in \langle w_1\rangle^\perp = W$

We can consider $f_W: W \to W$, $x \mapsto f(x)$

$W \subseteq V$ so it an inner-product space

let's show that f_W is self-adjoint

let $x, y \in W$

$\langle f_W(x), y\rangle = \langle f(x), y\rangle = \langle x, f(y)\rangle = \langle x, f_W(y)\rangle$

$\dim W = n - 1$.

By inductive hypothesis, there exists an ordered ONB for W of eigenvectors of f_W

let $\underbrace{(w_2, \dots, w_n)}_{n-1}$ be that $\overset{ON}{\text{basis}}$

. w_i eigenvect of f_W

. $w_i \perp w_j$ if $i \neq j$

$\|w_i\| = 1 \ \forall i$

let $B = (w_1, \underbrace{w_2, \dots, w_n})$

$\langle w_1\rangle \oplus \langle w_1\rangle^\perp = V$

is a basis of V

it is ON ∎

Ex: if $A \in M_n(\mathbb{R})$ is symmetric, then A is diagonalizable with real eigenvalues in an orthonormal basis

proof: $f: x \in \mathbb{R}^n \longmapsto Ax \in \mathbb{R}^n$

$[f]_{B_0} = A$ is symm, $f = f^*$, f is diagonalizable ∎

An application

Suppose G is a graph

The adjacency matrix of G:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in M_7(\mathbb{R})$$

$v = (v_1, v_2, \dots, v_7)$ $\quad Av = \lambda_7 v$ $\quad v \in \mathbb{R}^7 \setminus 0$

$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7$

Thm: V ips, $\dim V < \infty$, $\to V$ self-adjoint

Prop. If $f: V \to V$ is self-adjoint, then f has at least one eigenvalue. ($0 < \dim V < \infty$)

$Av = \lambda_{max} v$

$v \in \mathbb{R}^n \setminus 0$

$v = (v_1, v_2, \dots, v_n)$