THEO (DIRICHLET)

$\lim a_n \overset{(1)}{=} 0$, $\underbrace{\sum_1^\infty |a_{n+1} - a_n| \text{ conv}}_{(3)}$, $\{B_m = b_1 + b_2 + \dots + b_m\}$ bounded $\overset{(2)}{}$ $\Longrightarrow \sum a_n b_n$ conv.

THEO (ABEL)

$\sum_1^\infty b_n$ conv (1), $\{a_n\}$ $\underbrace{\text{bounded and monotone}}_{(2)}$ $\Longrightarrow \sum a_n b_n$ conv.

PROOF: $\sum_1^m a_k b_k = \underbrace{a_m B_m}_{\text{LIMIT EXISTS}} - \underbrace{\sum_1^{m-1} B_k (a_{k+1} - a_k)}_{\text{ABSOLUTELY CONV} \longrightarrow \text{CONV}}$

$(1)+(2)$ $\quad 0 \le \sum |B_k(a_{k+1} - a_k)| \underset{(2)}{\le} M \cdot \sum |a_{k+1} - a_k|$ $\quad (3)$ + COMPARISON TEST

$(1)+(2)$ $\quad 0 \le \sum |B_k(a_{k+1} - a_k)| \underset{(2)}{\le} M \cdot \sum |a_{k+1} - a_k| \underset{(2)}{=}$

$= M \left| \sum_1^{m-1} (a_{k+1} - a_k) \right| = M |(a_m - a_1)| \le M (|a_m| + |a_1|) \underset{(2)}{\le} 2MK$

$\Longrightarrow \left\{ \sum_1^{m-1} |B_k(a_{k+1} - a_k)| \right\}$ BOUNDED + INCREASING $\Longrightarrow$ CONV.

THEO (SPARSITY)

$S_m = a_1 + a_2 + \dots + a_m \le S_{m+1} = a_1 + a_2 + \dots + a_m + a_{m+1} = S_m + \overset{\overset{0}{\wedge}}{a_{m+1}} \Longrightarrow$ INCREASING

$T_m = 2a_2 + 4a_4 + \dots + 2^m a_{2^m} \le T_{m+1} = T_m + \underbrace{2^{m+1} a_{2^{m+1}}}_{\vee\!|\, 0}$

$\sum a_i$ conv $\Longleftrightarrow \lim S_m$ EXISTS

$= M \left| \sum_1^{m-1} (a_{k+1} - a_k) \right| = M |(a_m - a_1)| \le M (|a_m| + |a_1|) \underset{(2)}{\le} 2MK$

$\Longrightarrow \left\{ \sum_1^{m-1} |B_k(a_{k+1} - a_k)| \right\}$ BOUNDED + INCREASING $\Longrightarrow$ CONV.

THEO (SPARSITY)

$S_m = a_1 + a_2 + \dots + a_m \le S_{m+1} = a_1 + a_2 + \dots + a_m + a_{m+1} = S_m + \overset{\overset{0}{\wedge}}{a_{m+1}} \overset{(*)}{\Longrightarrow}$ INCREASING $(*)$

$T_m = 2a_2 + 4a_4 + \dots + 2^m a_{2^m} \le T_{m+1} = T_m + \underbrace{2^{m+1} a_{2^{m+1}}}_{\vee\!|\, 0}$

$\sum a_i$ conv $\Longleftrightarrow \lim S_m$ EXISTS $\underset{(*)}{\Longleftrightarrow} \{S_m\}$ BOUNDED

$\sum 2^i a_{2^i}$ conv $\Longleftrightarrow \lim T_m$ EXISTS $\underset{(*)}{\Longleftrightarrow} \{T_m\}$ BOUNDED

$\{S_m\}$ BOUNDED $\Longleftrightarrow \{T_m\}$ BOUNDED

$\Longrightarrow 0 \le |S_m| = S_m \le M$

$0 \le T_m \le 2 S_{2^m} \le 2M$

$\Longleftarrow 0 \le T_m = |T_m| \le K \ \forall m$

$0 \le S_m \le S_{2^m - 1} \le a_1 + T_{m-1} \le a_1 + K$

$m \le 2^m - 1$

SEQUENCES $\{a_n\}$, $\lim a_n = ?$

OF ~~REAL NUMBERS~~ FUNCTIONS

$\sum_1^\infty a_n = \lim_{m \to \infty} \Big(\overbrace{\sum_1^m a_n}^{S_m} \Big) = \lim_{m\to\infty} S_m$

SERIES OF $\boxed{\text{REAL NUMERS}}$ FUNCTIONS $\quad \sum_1^\infty a_n$

SEQUENCE OF PARTIAL SUMS $\{S_m\}$

SEQUENCES OF FUNCTIONS $\{f_n(x), n \in \mathbb{N}\}$

SERIES OF FUNCTIONS $\sum_1^\infty f_n(x)$ CONNECTED WITH THE SEQUENCE $\left\{ S_m(x) = \sum_1^m f_n(x) \right\}$

DEF. LET $\{f_n(x), n \in \mathbb{N}\}$ BE A SEQUENCE OF FUNCTIONS, $f_n : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$

1) WE SAY THAT THE SEQUENCE CONVERGES POINTWISE TO A FUNCTION $f(x)$ IFF FOR ANY $a \in D$ THE SEQUENCE OF REAL NUMBERS $\{f_n(a)\}$ CONVERGES TO $f(a)$

$\dots \{f_n(a)\} \dots$

1) WE SAY THAT THE SERIES $\sum_1^\infty f_n(x)$ CONVERGES POINTWISE TO A FUNCTION $f(x)$ IFF FOR ANY $a \in D$ THE SEQUENCE OF PARTIAL SUMS $\left\{ S_m(a) = \sum_1^m f_n(a) \right\}$ CONVERGES TO $S(a)$.

WE WILL USE THE NOTATION

$f_n(x) \underset{n \to \infty}{\longrightarrow} f(x)$

THAT IS, FOR ANY $a \in D$:

$\forall \varepsilon > 0, \exists N = N(\varepsilon, a) \in \mathbb{N}$

$n > N \Longrightarrow |f_n(a) - f(a)| < \varepsilon$

2) WE SAY THAT THE SEQUENCE CONVERGES UNIFORMLY TO $f(x)$ IFF

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}$:

$n > N \Longrightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x$

NOTATION: $f_n \longrightarrow f$

$\sum_1^\infty f_n(x) = S(x)$, $S_m(x) = \sum_1^m f_n(x) \underset{m \to \infty}{\longrightarrow} S(x)$ POINTWISE

THAT IS, FOR ANY $a \in D$:

$\forall \varepsilon > 0, \exists N = N(\varepsilon, a) \in \mathbb{N}$

$m > N \Longrightarrow |S_m(a) - S(a)| < \varepsilon$

2) WE SAY THAT THE SERIES $\sum_1^\infty f_n(x)$ CONVERGES TO $S(x)$ IFF

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N}$:

$m > N \Longrightarrow |S_m(x) - S(x)| < \varepsilon, \ \forall x$

$\sum_1^\infty f_n = S$, $S_m = \sum_1^m f_n \longrightarrow S$ UNIFORMLY

$f_n(x) \longrightarrow f(x)$ MEANS POINTWISE CONVERGENCE

$f_n \longrightarrow f$ MEANS UNIFORMLY CONVERGENCE

EXAMPLES: 1) $\left\{ f_n(x) = \frac{\sin(nx)}{n}, n \in \mathbb{N} \right\}$ SEQUENCE OF FUNCTIONS.

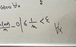
POINTWISE CONVERGENCE: $\lim_{n \to \infty} \frac{\sin(n \cdot a)}{n} = 0$ SINCE $\underset{\downarrow\, 0}{\frac{-1}{n}} \le \frac{\sin(na)}{n} \le \underset{\downarrow\, 0}{\frac{1}{n}}$

$\Longrightarrow f_n(a) \longrightarrow f(a) = 0 \longrightarrow f(x) = 0 \ \forall x$

UNIFORM CONVERGENCE: ✓

$\forall \varepsilon > 0, \exists N > \varepsilon^{-1} : n > N \Longrightarrow \left| \frac{\sin(nx)}{n} - 0 \right| \le \frac{1}{n} < \varepsilon \quad \forall x$

2) $f_n(x) = \begin{cases} n, & x \in (0, \frac{1}{n}) \\ 0, & x \in [0,1] \setminus (0, \frac{1}{n}) \end{cases}$ $\quad f_n(x) \overset{??}{\longrightarrow} 0$



POINTWISE CONVERGENCE:

$\lim_{n \to \infty} f_n(a)$: $= 0$ IF $a = 0$; $= 0$ IF $a = 1$; IF $0 < a < 1$, CHOOSE N: $\frac{1}{N} < a \longrightarrow f_n(a) = 0 \ \forall n > N$ $\left(n > N \to \frac{1}{n} < \frac{1}{N} < a,\ a \notin (0, \frac{1}{n}) \right)$

$\Longrightarrow \lim f_n(a) = 0 \ \forall a$

$\forall \varepsilon > 0, \exists N = N(a) > a^{-1} : n > N \Longrightarrow |f_n(a) - 0| = 0 < \varepsilon.$ $\quad 0 < \frac{1}{2n} < \frac{1}{n}$

UNIFORM CONVERGENCE: NO

$\forall \varepsilon > 0, \exists N = N(\varepsilon) : n > N \overset{??}{\Longrightarrow} |f_n(x) - 0| < \varepsilon, \forall x$ $\quad$ NO: $\left| f_n\left(\frac{1}{2n}\right) - 0 \right| = n$

$0 \quad 1/2n \quad 1/n$

REMARK: IF $f_n \longrightarrow f$ UNIFORMLY $\Longrightarrow f_n(x) \longrightarrow f(x)$ POINTWISE.

THEOREM 1 (CAUCHY CRITERIUM FOR UNIFORM CONVERGENCE FOR SEQUENCES)

$f_n \longrightarrow f$ UNIFORMLY $\Longleftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} : n, m > N \Longrightarrow |f_n(x) - f_m(x)| < \varepsilon \quad \forall x.$

THEOREM: (CAUCHY CRITERIUM FOR UNIFORM CONVERGENCE FOR SERIES OF FUNCTIONS).

$\sum_1^\infty f_n \longrightarrow S$ UNIFORMLY $\Longleftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} : n > m > N \longrightarrow |f_{m+1}(x) + \dots + f_n(x)| < \varepsilon$

DEF $\Updownarrow$ $\{S_m\}$ converge uniformly to $S \underset{\text{THEO 1}}{\Longleftrightarrow} \forall \varepsilon > 0 : \exists N = N(\varepsilon) \in \mathbb{N} : n > m > N \Longrightarrow |S_n(x) - S_m(x)| < \varepsilon \quad \forall x$

BEHAVIOUR OF LIMITS OF SEQUENCES/SERIES OF FUNCTIONS WITH RESPECT TO CONTINUITY, DIFFERENTIATION AND INTEGRATION.

$f_n \to f$

LEMMA: IF $f_n \longrightarrow f$ UNIFORMLY, $\lim_{x \to c} f_n(x) = L_n$ AND $\lim_{n \to \infty} L_n = L$

$\Longrightarrow \lim_{x \to c} f(x) = L.$

$L = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \left(\lim_{x \to c} f_n(x) \right)$

$= \lim_{x \to c} f(x) = \lim_{x \to c} \left(\lim_{n \to \infty} f_n(x) \right)$

$S_m \longrightarrow S \quad \left(\sum_1^m f_n \longrightarrow S \right)$

LEMMA: IF $\sum_1^\infty f_n$ CONVERGES UNIFORMLY TO S

AND $\lim_{x \to c} S_m(x) = A_m$ AND $\lim_{m \to \infty} A_m = A$

THEN $\lim_{x \to c} S(x) = A$.

$A \overset{?}{=} \lim_{x \to c} \left(\sum_1^\infty f_n(x) \right) \overset{?}{=} \sum_1^\infty \lim_{x \to c} f_n(x)$

$\overset{?}{=} \lim_{m \to \infty} \left(\sum_1^m \lim_{x \to c} f_n(x) \right) \overset{?}{=} \lim_{m \to \infty} \lim_{x \to c} S_m(x)$

$\overset{?}{=} \lim_{m \to \infty} A_m.$

LEMMA 3 IF $\sum_1^\infty f_n$ CONV. UNIFORMLY TO S

AND $\boxed{\lim_{x \to c} f_n(x) = L_n}$ AND

$\boxed{\sum L_n \text{ CONVERGES TO } A}$ THEN

$\lim_{x \to c} S(x) = A$

$\lim_{x \to c} S(x) = \lim_{x \to c} \sum_1^\infty f_n(x)$

$= \sum_1^\infty \lim_{x \to c} f_n(x)$

PROOF (LEMMA 3)

IF $\lim_{x \to c} f_n(x) = L_n \Longrightarrow$

$\lim_{x \to c} S_m(x) = \lim_{x \to c} \left(f_1(x) + f_2(x) + \dots + f_m(x) \right) = L_1 + L_2 + \dots + L_m = A_m$

$= \sum_1^m L_n$

$A = \sum_1^\infty L_n = \lim_{m \to \infty} \left(\sum_1^m L_n \right) = \lim_{m \to \infty} A_m$

THE PROOF FOLLOWS BY LEMMA 2.

[illegible]

b) $\lim_{n\to\infty} \frac{x}{n}\ln\frac{x}{n}$ denote $t=\frac{x}{n}$

$=\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+} -t = 0$

$f_n(x) = \frac{x}{n}\ln\frac{x}{n} \xrightarrow[n\to\infty]{} f(x)=0$ pointwise

$M_n = \sup_{x\in(0,1]}\left\{\left|\frac{x}{n}\ln\frac{x}{n} - 0\right|\right\}$ (If $M_n \xrightarrow[n\to\infty]{} 0$ the convergence is uniform)

$\left|\frac{x}{n}\ln\frac{x}{n} - 0\right| = \left|\frac{x}{n}\ln\frac{x}{n}\right| = -\frac{x}{n}\ln\frac{x}{n}$ $x\in(0,1]$

$\lim_{x\to 0} -\frac{x}{n}\ln\frac{x}{n} = 0$

$h_n(x) = \begin{cases} 0 & x=0 \\ -\frac{x}{n}\ln\frac{x}{n} & 0<x\le 1\end{cases}$

M_n is the maximum of $h_n(x)$

$h_n'(x) = -\frac{1}{n}\ln\frac{x}{n} - \frac{x}{n}\cdot\frac{\frac{1}{n}}{\frac{x}{n}}$

$= -\frac{1}{n}\left[1+\ln\frac{x}{n}\right] = 0$

$\ln\frac{x}{n} = -1 \Rightarrow x = \frac{n}{e}$

$x=\frac{n}{e}$ for $n\ge 3$ $x>1$

$h_n'(x) > 0$ for $n\ge 3$ $x\in(0,1]$

$\Rightarrow h_n(x)$ is monotonically increasing.

$M_n = \max_{x\in(0,1]}\{h_n(x)\} = h_n(1) = -\frac{1}{n}\ln\frac{1}{n}$ denote $t=\frac{1}{n}$

$\lim_{n\to\infty} M_n = \lim_{n\to\infty} -\frac{1}{n}\ln\frac{1}{n} = \lim_{t\to 0^+} -t\ln t = 0$ $\Rightarrow f_n(x) \xrightarrow[n\to\infty]{\text{uniformly}} f(x)$

c) $f_n(x) = x^n e^{-nx}$ $x\in(0,\infty)$

$\lim_{n\to\infty}\left(\frac{x}{e^x}\right)^n = 0$

$f_n(x) \xrightarrow[n\to\infty]{\text{pointwise}} f(x)=0$

$e^x > x$ $\forall x\in(0,\infty)$

$g(x) = e^x - x$ $g(0)=1$

$g'(x) = e^x - 1 > 0$ $\forall x\in(0,\infty)$

g is monotonically increasing

$e^x > x$

claim $f_n(x) \xrightarrow{\text{uniformly}} f(x)$ on $(0,\infty)$

proof (by definition)

(we need to show $\forall\varepsilon>0$, $\exists N(\varepsilon)$ s.t. $\forall x\in(0,\infty)$ $\forall n>N(\varepsilon)$

$\left|\left(\frac{x}{e^x}\right)^n - 0\right| < \varepsilon$

$g(x) = \frac{x}{e^x} = xe^{-x}$ (let's find the max of g)

$g'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) = 0 \Rightarrow x=1$

$g'(x)<0$ $x>1$ $\Rightarrow x=1$ is a max of $\frac{x}{e^x}$

$g'(x)>0$ $x<1$

$\Rightarrow \frac{x}{e^x} \le \frac{1}{e}$ $\forall x\in(0,\infty)$

$\left|\left(\frac{1}{e}\right)^n\right| < \varepsilon \Rightarrow e^{-n} < \varepsilon \Rightarrow n > -\ln\varepsilon$

take $N_0 = \begin{cases} -\ln\varepsilon & 0<\varepsilon<1 \\ 1 & \varepsilon\ge 1\end{cases}$

$\Rightarrow \forall\varepsilon>0$ $\exists N=N_0$ s.t. $\forall n>N_0$ $\forall x\in(0,\infty)$

$\Rightarrow \left|\left(\frac{x}{e^x}\right)^n - 0\right| \le \left(\frac{1}{e}\right)^n < \varepsilon$

$\Rightarrow$ the convergence is uniform.

__Exercise 2__ Given $f_n = \cos^n(x)$ $0\le x\le\frac{\pi}{2}$

a) Find $\lim_{n\to\infty} f_n(x)$

b) Does $\{f_n\}$ converge uniformly?

c) calculate $\lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx$

a) $\lim_{n\to\infty}(\cos x)^n = \begin{cases} 1 & x=0 \\ 0 & 0<x\le\frac{\pi}{2}\end{cases} = f(x)$

b) No. Each $f_n(x)$ is continuous but the limit function is not continuous

$\Rightarrow$ the convergence is not uniform

c) $\lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx$

(note that since the convergence is not uniform we can't say $\lim\int_0^{\pi/2} f_n\,dx = \int_0^{\pi/2} f\,dx$)

on $\left[\frac{\varepsilon}{2}, \frac{\pi}{2}\right]$ $f_n = \cos^n x$ converges uniformly to $f(x)=0$

By Dini's theorem (because f_n converges monotonically)

$\int_0^{\pi/2}\cos^n x\,dx = \underbrace{\int_0^{\varepsilon/2}\cos^n x\,dx}_{\le\int_0^{\varepsilon/2}1\,dx = \frac{\varepsilon}{2}} + \underbrace{\int_{\varepsilon/2}^{\pi/2}\cos^n x\,dx}$

$\left|\lim\int_{\varepsilon/2}^{\pi/2}\cos^n x\,dx\right| < \frac{\varepsilon}{2}$

$\lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ $\forall\varepsilon$

$\Rightarrow \lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx = 0$

$\int_0^{\pi/2}\cos^n x\,dx = \underbrace{\int_0^{\varepsilon/2}\cos^n x\,dx}_{\le\int_0^{\varepsilon/2}1\,dx = \frac{\varepsilon}{2}} + \underbrace{\int_{\varepsilon/2}^{\pi/2}\cos^n x\,dx}$

$\left|\lim\int_{\varepsilon/2}^{\pi/2}\cos^n x\,dx\right| < \frac{\varepsilon}{2}$

$\lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ $\forall\varepsilon$

$\Rightarrow \lim_{n\to\infty}\int_0^{\pi/2}\cos^n x\,dx = 0$

$f_n \xrightarrow{\text{uniformly}} 0$ on $\left[\frac{\varepsilon}{2}, \frac{\pi}{2}\right]$

__Exercise 3__ Suppose that g is differentiable on all $\mathbb{R}$ and $|g'(x)| \le r < 1$ $\forall x\in\mathbb{R}$. Let f_1 be a bounded function on $\mathbb{R}$, and define $f_n(x) = g(f_{n-1}(x))$ $n\ge 2$.

Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

__Solution__ we will use Cauchy's criteria

$|f_{n+1}(x) - f_n(x)| = |g(f_n(x)) - g(f_{n-1}(x))|$

mean value theorem, on every interval $[a,b]$, $\exists c\in(a,b)$ s.t. $g'(c) = \frac{g(b)-g(a)}{b-a} \Rightarrow |g(b)-g(a)| = \underbrace{|g'(c)|}_{\le r}|b-a|$

$|f_{n+1}(x) - f_n(x)| = |g(f_n(x)) - g(f_{n-1}(x))| \le r\,|f_n(x) - f_{n-1}(x)| =$

$= r\,|g(f_{n-1}(x)) - g(f_{n-2}(x))| \le r^2\,|f_{n-1}(x) - f_{n-2}(x)| \dots$

$\dots \le r^n\,|f_2(x) - f_1(x)|$

assume $n>m$

$|f_n - f_m| = |f_n - f_{n-1} + f_{n-1} - f_{n-2} + f_{n-2} - \dots + f_{m+1} - f_m| \le$

$\le \underbrace{|f_n - f_{n-1}|}_{\le r^{n-2}|f_2-f_1|} + \underbrace{|f_{n-1} - f_{n-2}|}_{\le r^{n-3}|f_2-f_1|} + \dots + \underbrace{|f_{m+1} - f_m|}_{\le r^{m-1}|f_2-f_1|} \le$

$\le |f_2-f_1|\left(r^{n-2} + r^{n-3} + \dots + r^{m-1}\right) = |f_2-f_1|\,r^{m-1}\left(1+r+r^2+\dots+r^{n-m-1}\right)$

$(1+r+r^2+\dots+r^{n-m-1}) \le \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ $r<1$

$\Rightarrow |f_n - f_m| \le \frac{|f_2-f_1|\,r^{m-1}}{1-r} < \varepsilon$

$\Rightarrow r^{m-1} < \frac{\varepsilon(1-r)}{|f_2-f_1|}$

$\Rightarrow (m-1)\ln r < \ln\left(\frac{\varepsilon(1-r)}{|f_2-f_1|}\right) \Rightarrow m > \frac{\ln\left(\frac{\varepsilon(1-r)}{|f_2-f_1|}\right)}{\ln r} + 1$

$\Rightarrow \forall\varepsilon>0$ $\exists N_0$ s.t. $\forall n>m>N_0$

$|f_n - f_m| < \varepsilon$ $\Rightarrow \{f_n(x)\}$ converges uniformly by Cauchy's criteria

We still need to show that $|f_2 - f_1|$ is bounded

$f_2(x) = \underbrace{g(f_1(x)) - g(0)}_{g'(c)(f_1(x)-0)} + g(0)$ $c\in(0, f_1(x))$

We know that f_1 is bounded. g is differentiable $\Rightarrow g$ is continuous everywhere $\Rightarrow g$ is bounded in every closed interval

$|f_2(x)| \le |r\,f_1(x) + g(0)| \le r\,\underbrace{|f_1(x)|}_{\text{bounded}} + \underbrace{|g(0)|}_{\text{bounded}}$

$\Rightarrow |f_2(x) - \underbrace{f_1(x)}_{\text{bounded}}|$ is bounded

What happens if $f_1(x) = f_2(x)$