MOODLE
- PROOF DINI'S THEOREM
- PROOF $f_{xy}=f_{yx}$
- PROOF CHAIN RULE

CHAIN RULE: $z=f(x,y)$ DIFF. AT (x_0,y_0), $x=x(u,v)$, $y=y(u,v)$ DIFF AT (u_0,v_0),
$(x_0,y_0)=(x(u_0,v_0),y(u_0,v_0))$.

IF $g(u,v)=f(x(u,v),y(u,v))$ THEN

$$\frac{\partial g}{\partial u}(u_0,v_0)=\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial u}(u_0,v_0)+\frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial u}(u_0,v_0)$$

$$\frac{\partial g}{\partial v}(u_0,v_0)=\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial v}(u_0,v_0)+\frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial v}(u_0,v_0)$$

PROOF.

f DIFF AT (x_0,y_0) $\Longleftrightarrow$ $f(x,y)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\underbrace{\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+E(x,y)\|(x,y)-(x_0,y_0)\|$, $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$

$$f(x(u,v),y(u,v))=f(x(u_0,v_0),y(u_0,v_0))+\frac{\partial f}{\partial x}(x(u_0,v_0),y(u_0,v_0))(x(u,v)-x(u_0,v_0))+\underline{\qquad}+E(x(u,v),y(u,v))\|(x(u,v),y(u,v))-(x(u_0,v_0),y(u_0,v_0))\|$$

$x=x(u,v)$ DIFF AT (u_0,v_0) $\Longleftrightarrow$ $x(u,v)=x(u_0,v_0)+\frac{\partial x}{\partial u}(u_0,v_0)(u-u_0)+\frac{\partial x}{\partial v}(u_0,v_0)(v-v_0)+E_1(u,v)\|(u,v)-(u_0,v_0)\|$ $E_1(u,v)\xrightarrow[(u,v)\to(u_0,v_0)]{}0$

IN PARTICULAR:

$$x(u_0+h,v_0)=x(u_0,v_0)+\frac{\partial x}{\partial u}(u_0,v_0)\cdot\underset{(u_0+h-u_0)}{h}+\dots\underset{0}{(v_0-v_0)}+E_1(u_0+h,v_0)\underset{\sqrt{h^2+0^2}=|h|}{\|(u_0+h,v_0)-(u_0,v_0)\|}$$

$$x(u_0+h,v_0)=x(u_0,v_0)+\frac{\partial x}{\partial u}(u_0,v_0)h+E_1(u_0+h,v_0)|h|$$

EXAMPLES: $f(x,y)=x+y^2-1$; $\frac{\partial f}{\partial x}(x,y)=1$; $\frac{\partial f}{\partial y}(x,y)=2y$

$x=x(u,v)=u^2+v^2$; $\frac{\partial x}{\partial u}(u,v)=2u$; $\frac{\partial x}{\partial v}(u,v)=2v$

$y(u,v)=2uv$; $\frac{\partial y}{\partial u}(u,v)=2v$; $\frac{\partial y}{\partial v}(u,v)=2u$

$$g(u,v)=f(x(u,v),y(u,v))=x(u,v)+y(u,v)^2-1$$
$$=u^2+v^2+4u^2v^2-1$$

$$\frac{\partial g}{\partial u}(u,v)=1\cdot 2u+2y(u,v)\cdot 2v$$
$$=2u+8uv^2=\frac{\partial g}{\partial u}(u,v)$$

$$\frac{\partial g}{\partial v}(u,v)=1\cdot 2v+2\cdot y(u,v)\cdot 2u$$
$$=2v+8u^2v=\frac{\partial g}{\partial v}(u,v)$$

2) $z=f(x,y)=x^2+y^2$, $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$ POLAR COORDINATES   rho

$\rho=\sqrt{x^2+y^2}$, $\rho\geq0$

$\theta=\arctan\left(\frac{y}{x}\right)$, $\theta\in[0,2\pi)$

$$g(\theta,\rho)=f(x(\rho,\theta),y(\rho,\theta))=\rho^2\cos^2\theta+\rho^2\sin^2\theta=\rho^2.$$

$$0=\frac{\partial g}{\partial\theta}(\theta,\rho)=2x(\rho,\theta)\cdot\frac{\partial x}{\partial\theta}+2y(\rho,\theta)\frac{\partial y}{\partial\theta}=2\rho\cos\theta\,\rho(-\sin\theta)+2\rho\sin(\theta)\,\rho\cos(\theta)=0$$

$$2\rho=\frac{\partial g}{\partial\rho}=2\rho\cos\theta\cdot\cos\theta+2\rho\sin\theta\cdot\sin(\theta)=2\rho$$

COROLLARY: $z=f(x,y)$ DIFF AT (x_0,y_0), $x=x(u)$, $y=y(u)$ DIFF AT u_0, $(x(u_0),y(u_0))=(x_0,y_0)$

IF $g(u)=f(x(u),y(u))$ THEN $g'(u)=\frac{\partial f}{\partial x}(x_0,y_0)\,x'(u)+\frac{\partial f}{\partial y}(x_0,y_0)\,y'(u)$

PROOF $x(u,v)=x(u)$

$$\frac{\partial x}{\partial u}(u,v)=\lim_{h\to0}\frac{x(u+h,v)-x(u,v)}{h}=\lim_{h\to0}\frac{x(u+h)-x(u)}{h}=x'(u)$$

TAYLOR POLYNOMIAL.

$y=h(x)$, LET US CONSIDER THE TAYLOR POLYNOMIAL AROUND $x=0$.

$$h(x)=h(0)+h'(0)(x-0)+\frac{h''(0)}{2}(x-0)^2+\dots+\frac{h^{(n)}(0)}{n!}(x-0)^n+\frac{h^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1},$$

C BETWEEN 0 AND X.

$z=f(x,y)$. CONSIDER THE TAYLOR POLYNOMIAL OF $f(x,y)$ AROUND (x_0,y_0)

$h(t)=f(x_0+t(x-x_0),y_0+t(y-y_0))$, $t\in[0,1]$

$h(0)=f(x_0,y_0)$

$h(1)=f(x,y)$.

WE CAN COMPUTE "EASILY" $f(x_0,y_0)$

WE WANT TO APPROXIMATE $f(x,y)$ BY USING $f(x_0,y_0)$.

$$f(x,y)=h(1)=h(0)+h'(0)+\frac{h''(0)}{2}+\dots+\frac{h^{(n)}(0)}{n!}+\frac{h^{(n+1)}(c)}{(n+1)!},\ c\in(0,1)$$
$$=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)$$

$\sqrt{4.02}\cdot\sqrt{9.032}=??$

$f(x,y)=\sqrt{x}\cdot\sqrt{y}$

$x=4.02$, $y=9.032$

$x_0=4$, $y_0=9$

$f(x_0,y_0)=\sqrt{4}\cdot\sqrt{9}=6$

$f(4.02,9.032)\sim$ TAYLOR POLYNOMIAL

$h(t)=f(x(t),y(t))$; $\begin{cases}x(t)=x_0+t(x-x_0)\\ y(t)=y_0+t(y-y_0)\end{cases}$, $x(0)=x_0$, $y(0)=y_0$, $x'(t)=x-x_0$, $y'(t)=y-y_0$

$$h'(t)=\frac{\partial f}{\partial x}(x(t),y(t))\,x'(t)+\frac{\partial f}{\partial y}(x(t),y(t))\cdot y'(t).$$

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$$f(x,y)=h(1)=\underline{h(0)}+\underline{h'(0)}+\frac{h''(0)}{2}+\dots+\frac{h^{(n)}(0)}{n!}+\frac{h^{(n+1)}(c)}{(n+1)!},\ c\in(0,1)$$
$$=\underline{f(x_0,y_0)}+\underline{\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+$$

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$h(t)=f(x(t),y(t))$; $\begin{cases}x(t)=x_0+t(x-x_0)\\ y(t)=y_0+t(y-y_0)\end{cases}$, $x(0)=x_0$, $y(0)=y_0$, $x'(t)=x-x_0$, $y'(t)=y-y_0$

$$h'(t)=\underbrace{\frac{\partial f}{\partial x}(x(t),y(t))}\,\underset{(x-x_0)}{x'(t)}+\underbrace{\frac{\partial f}{\partial y}(x(t),y(t))}\cdot\underset{(y-y_0)}{y'(t)}.$$

$$h''(t)=\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x(t),y(t))\cdot x'(t)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x(t),y(t))\cdot y'(t)\right)(x-x_0)+$$
$$\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x(t),y(t))\cdot x'(t)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x(t),y(t))\cdot y'(t)\right)(y-y_0)$$

WE CAN COMPUTE "EASILY" $f(x_0,y_0)$

WE WANT TO APPROXIMATE $f(x,y)$ BY USING $f(x_0,y_0)$.

$$f(x,y)=h(1)=\underline{h(0)}+\underline{h'(0)}+\frac{h''(0)}{\underline{2}}+\dots+\frac{h^{(n)}(0)}{n!}+\frac{h^{(n+1)}(c)}{(n+1)!},\ c\in(0,1)$$
$$=\underline{f(x_0,y_0)}+\underline{\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}+$$
$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x\partial x}(x_0,y_0)(x-x_0)^2+\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)(y-y_0)(x-x_0)+\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)(x-x_0)(y-y_0)+\right.$$

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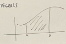
$$\left.+\underline{\frac{\partial^2 f}{\partial y\partial y}(x_0,y_0)(y-y_0)^2}\right)+\frac{1}{3!}\left(\frac{\partial^3 f}{\partial x\partial x\partial x}(x_0,y_0)(x-x_0)^3+\frac{\partial^3 f}{\partial x\partial x\partial y}(x_0,y_0)(x-x_0)^2(y-y_0)+\frac{\partial^3 f}{\partial x\partial y\partial x}(x_0,y_0)(x-x_0)^2(y-y_0)+\dots\right)$$

$+\dots$

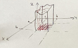
DOUBLE INTEGRALS / MULTIPLE INTEGRALS

$y=f(x)$ $\int_a^b f(x)\,dx=\text{AREA}\{(x,y(x)):a\leq x\leq b,\ 0\leq y(x)\leq f(x)\}$



$\iint_R f(x,y)\,dxdy$, $R=[a,b]\times[c,d]$, $f(x,y)\geq0$

$=\text{VOLUMEN}\{(x,y,z(x,y)):x\in[a,b],\ y\in[c,d],\ 0\leq z(x,y)\leq f(x,y)\}$



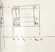
EXAMPLE:

$f(x,y)=2$, $R=[a,b]\times[c,d]$

$$\iint_R 2\,dxdy=2(b-a)(d-c)$$

$k(b-a)$

WE CONSIDER A PARTITION OF R.



$a=x_0<x_1<x_2<\dots<x_{n-1}<x_n=b$

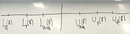
$c=y_0<y_1<y_2<\dots<y_{m-1}<y_m=d$

$R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$, $R=\bigcup_{\substack{1\leq i\leq m\\1\leq j\leq m}}R_{ij}$

$m_{ij}=\inf\{f(x,y),\ (x,y)\in R_{ij}\}$

$M_{ij}=\sup\{f(x,y),\ (x,y)\in R_{ij}\}$

$$L_p(f)=\sum_{ij}m_{ij}\,\text{AREA}(R_{ij})=\sum_{ij}m_{ij}(x_i-x_{i-1})(y_j-y_{j-1})\leq U_p(f)=\sum_{ij}M_{ij}(x_i-x_{i-1})(y_j-y_{j-1}).$$



DEF. WE SAY THAT A FUNCTION $z=f(x,y)$ IS INTEGRABLE ON $R=[a,b]\times[c,d]$ IFF

$\forall\varepsilon>0$, $\exists P$ A PARTITION OF R SUCH THAT $U_P(f)-L_P(f)<\varepsilon$.

IN THIS CASE. $\iint_R f(x,y)\,dxdy=\sup\{L_p(f),\text{ FOR ANY }P\}=\inf\{U_p(f),\text{ FOR ANY }P\}$.

$=L(f)$ $=U(f)$

EXAMPLE. $z=f(x,y)=x+y$, $R=[a,b]\times[c,d]$, $P=\{R_{ij},\ i,j\}$, $R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$

$m_{ij}=x_{i-1}+y_{j-1}$, $M_{ij}=x_i+y_j$, $m_{ij}\leq\frac{x_{i-1}+x_i}{2}+\frac{y_{j-1}+y_j}{2}\leq M_{ij}$.

$\begin{cases}x_{i-1}<\frac{x_{i-1}+x_i}{2}<x_i\\ y_{j-1}<\frac{y_{j-1}+y_j}{2}<y_j\end{cases}$

$$L_p(f)\leq\sum_{i,j}\left(\frac{x_{i-1}+x_i}{2}+\frac{y_{j-1}+y_j}{2}\right)(x_i-x_{i-1})(y_j-y_{j-1})\leq U_p(f)$$

$$\|$$

$$\frac{1}{2}\sum_{ij}(x_{i-1}+x_i)(x_i-x_{i-1})(y_j-y_{j-1})+\frac{1}{2}\sum(y_{j-1}+y_j)(y_j-y_{j-1})(x_i-x_{i-1})$$

$$=\frac{1}{2}\sum_i(x_i^2-x_{i-1}^2)\sum_j(y_j-y_{j-1})+\frac{1}{2}\sum_j(y_j^2-y_{j-1}^2)\sum_i(x_i-x_{i-1})=\frac{1}{2}(-x_0^2+x_n^2)(-y_0+y_m)+\frac{1}{2}(-y_0^2+y_m^2)(-x_0+x_n)$$

$$=\frac{1}{2}\left[(b^2-a^2)(d-c)+(d^2-c^2)(b-a)\right]$$