Def: V ips. We say a linear map $f: V \to V$ is orthogonal/unitary ($\mathbb{R}$ / $\mathbb{C}$) if $\langle f(x), f(y)\rangle = \langle x, y\rangle$ $\forall x, y \in V$.

This happens if and only if $\|f(x)\| = \|x\|$ $\forall x \in \mathbb{R}$

[these maps are the "isomorphisms" of inner product spaces]

Def: $f: V \to V$ is orthogonal/unitary iff $f^* f = \mathrm{Id}_V = f f^*$.

In particular, orthogonal/unitary maps are normal.

Prop: $f: V \to V$ is orth/unitary iff for every (some) O.N. ordered basis $\mathcal{B}$ the matrix $[f]_{\mathcal{B}}$ has orthonormal columns.

$f(v) \Rightarrow \mathcal{B}$ ONB $[f]_{\mathcal{B}}$ — check that the columns $\mathbb{R}^n/\mathbb{C}^n$ are orthonormal

Def: A matrix A in $M_n(\mathbb{R})$ or in $M_n(\mathbb{C})$ is orthogonal/unitary if $\langle Ax, Ay\rangle = \langle x, y\rangle$ $\forall x, y \in \mathbb{R}^n/\mathbb{C}^n$

Prop: i) If $A \in M_n(\mathbb{R})$ ($\mathbb{C}$), then TFAE:

a) A is orthogonal (unitary)

b) the function $f_A: x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^n$ ($\mathbb{C}^n$) is orthogonal (unitary)

c) $A A^t = I_n = A^t A$

d) the columns of A are an O.N.B. for $\mathbb{R}^n$ ($\mathbb{C}^n$)

e) the rows of A are an O.N.B. for $\mathbb{R}^n$ ($\mathbb{C}^n$)

* $O(n) = \{A \in M_n(\mathbb{R}) : A \text{ is orthogonal}\}$ the orthogonal group

* $U(n) = \{A \in M_n(\mathbb{C}) : A \text{ is unitary}\}$ the unitary group

Prop: If $A \in O(n)$ ($U(n)$), then
- A invertible, $A^{-1} = A^t \in O(n)$ ($U(n)$)
- $I_n \in O(n)$ ($U(n)$)
- $A, B \in O(n)$ ($U(n)$), $AB \in O(n)$ ($U(n)$)

Proof: If $A, B \in O(n)$, then
$(AB)(AB)^t = AB B^t A^t = A I_n A^t = A A^t = I_n$
$(AB)^t(AB) = \dots = I_n$
$\therefore AB \in O(n)$

See $A \in U(n)$
$A \overline{A^t} = I_n$
$\det(A)\overline{\det(A)} = \det(A) \det(\overline{A^t}) = \det I_n = 1$
$|\det(A)|^2$ $\quad |\det(A)| = 1$

* $SO(n) = \{A \in O(n) : \det(A) = 1\}$ — $I_n \in SO(n)$ — this is a group ($A, B \in SO(n) \Rightarrow AB \in SO(n)$) — this is the special orthogonal group
* $SU(n) = \{A \in U(n) : \det(A) = 1\}$ the special unitary group

$n = 1$?

$n = 2$: $SO(2)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$ iff $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ is an O.N.B. for $\mathbb{R}^2$

$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$0 = \left\langle \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle = b\cos\theta + d\sin\theta$

$\begin{pmatrix} b \\ d \end{pmatrix} = \pm \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

There exists a $\theta \in \mathbb{R}$ s.t.
$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ — $O(2)$

$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$ = rotation matrices

$n = 3$: $SO(3)$?

$A \in SO(3)$
- it is orthogonal
- $f: x \in \mathbb{R}^3 \mapsto Ax \in \mathbb{R}^3$ is an orthogonal map
- $\chi_f = \chi_A$ is a degree 3 elem of $\mathbb{R}[x]$, monic

there exists at least a real root: $\lambda \in \mathbb{R}$ s.t. $\chi_f(\lambda) = 0$

There exists an $x \in \mathbb{R}^3 \setminus 0$ s.t. $Ax = \lambda x$

$\langle Ax, Ax\rangle = \langle x, x\rangle > 0$
$= \langle \lambda x, \lambda x\rangle$ $\quad \lambda = \pm 1$
$= \lambda^2 \langle x, x\rangle$

Suppose χ_f has two real roots, then it has three.

$Ax = \lambda x$ $\quad \lambda = \pm 1$

$A\langle x\rangle^\perp \subseteq \langle x\rangle^\perp$

there is an O.N.B. $\{u, v\}$ of $\langle x\rangle^\perp$ s.t. $A|_{\langle x\rangle^\perp} \sim \begin{pmatrix} \cos\theta & \pm\sin\theta \\ \sin\theta & \mp\cos\theta \end{pmatrix}$

$\{x, u, v\}$ O.N.B. for $\mathbb{R}^3$

$P = (x|u|v)$ orthogonal matrix

$P^{-1}AP = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & \pm\sin\theta \\ 0 & \sin\theta & \mp\cos\theta \end{pmatrix}$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y\cos\theta + z\sin\theta \\ y\sin\theta - z\cos\theta \end{pmatrix}$

$\theta = 0$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \\ -z \end{pmatrix}$

$A \in M_n(\mathbb{K})$

$f_A: x \in \mathbb{K}^n \mapsto Ax \in \mathbb{K}^n$

$[f_A]_{std} = A$

$\mathcal{B} = (v_1, \dots, v_n)$ ordered basis for $\mathbb{K}^n$

$[f_A]_{\mathcal{B}} = P^{-1} A P$

$P = (v_1| \dots |v_n)$ is invertible

$A \in SO(3)$ $\quad f_A: x \in \mathbb{R}^3 \mapsto Ax \in \mathbb{R}^3$
- A is orth: $\langle Ax, Ay\rangle = \langle x, y\rangle$
- $\det A = 1$
- $\chi_f \in \mathbb{R}[x]$ has degree 3, so it has a real root λ $\Rightarrow x \in \mathbb{R}^3 \setminus 0$ s.t. $Ax = \lambda x$

$\lambda^2\langle x, x\rangle = \langle Ax, Ax\rangle = \langle x, x\rangle > 0$
$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

* $\langle x\rangle^\perp \subseteq \mathbb{R}^3$, $\dim \langle x\rangle^\perp = 2$
* $\langle x\rangle^\perp$ f_A-invariant

$y \in \langle x\rangle^\perp \Rightarrow 0 = \langle y, x\rangle = \langle Ay, Ax\rangle = \langle Ay, \lambda x\rangle = \lambda\langle Ay, x\rangle$
$\Rightarrow \langle Ay, x\rangle = 0$
$\Rightarrow f_A(y) \perp x$

* $f_A: \langle x\rangle^\perp \to \langle x\rangle^\perp$ is orthogonal

If (u, v) is an O.N.B. for $\langle x\rangle^\perp$

$[f_A|_{\langle x\rangle^\perp}]_{\mathcal{B}}$ = orthogonal matrix in $O(2)$
$= \begin{pmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{pmatrix}$

$\mathcal{B} = (x, u, v)$ is an O.N.B. for $\mathbb{R}^3$, $P = (x|u|v)$

$P^{-1}AP = [f_A]_{\mathcal{B}} = \begin{pmatrix} \pm 1 & \\ & \begin{matrix} \cos & \mp\sin \\ \sin & \pm\cos \end{matrix} \end{pmatrix}$

Since $\det PAP^{-1} = \det A = 1$,

$\begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} -1 & & \\ & \cos\theta & \sin\theta \\ & \sin\theta & -\cos\theta \end{pmatrix}$

Exercise: play the same game in $\mathbb{R}^4$

$A \in SO(4)$, χ_A deg 4

$P^{-1}AP = \begin{pmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & \cos\psi & -\sin\psi \\ & & \sin\psi & \cos\psi \end{pmatrix}$

Prop: i) If $A \in M_n(\mathbb{R})$ is symmetric, then

If $A \in M_n(\mathbb{R})$ is symmetric then $f: x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^n$, $[f]_{std} = A = A^t = [f]_{std}^t$ so f is self-adjoint.

So there is an O.N.O.B. $\mathcal{B} = (v_1, \dots, v_n)$ of eigenvects for f

$P = (v_1| \dots |v_n)$ is invertible matrix and orthogonal.

$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = [f]_{\mathcal{B}} = P^{-1} A P$

Prop:
i) If $A \in M_n(\mathbb{R})$ is symmetric, then there exists a orthogonal matrix P such that $P^{-1}AP$ is diagonal with real entries.
ii) If $A \in M_n(\mathbb{C})$ is hermitian ($A = \overline{A^t}$) then there is a unitary matrix P such $P^{-1}AP$ is diagonal.

Prop: Let $A \in M_n(\mathbb{R})$ (QR-decomposition). There exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

$A = (v_1|v_2|\dots|v_n)$

[I'm going to suppose that A is invertible]

then $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$

G-S $\rightsquigarrow \{w_1, \dots, w_n\}$ O.N.B. for $\mathbb{R}^n$

$(w_1|w_2|w_3)$ — Q is orthogonal; R upper triangular