Simulation of a normal distribution with an unbiased estimator

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To estimate the variance σ^2 of a normal distribution, a commonly used unbiased estimator is the corrected variance estimator, also called the unbiased variance estimator adjusted for the degrees of freedom (Bessel's correction). This estimator is based on the sum of the squares of the deviations from the mean, adjusted for the degrees of freedom. The corrected variance estimator s^2 for a random sample of n observations from a normal distribution is given by the following formula :

$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

1. s^2 is the corrected variance estimator
2. n is the number of observations in the sample
3. X_i is each observation in the sample
4. $\overline{X}$ is the average of the observations

The proof that s^2 is an unbiased estimator of the variance σ^2 of a normal distribution relies on the properties of moments and expectation. Here is a simplified proof :
Let $X_1, X_2, ..., X_n$ be a random sample of size nn coming from a normal distribution with variance σ^2.

1. Calculation of the corrected variance estimator s^2 :

$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

2. Calculation of the expectation of s^2 :

$$\mathbf{E}(s^2) = \mathbf{E}\left(\frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2\right)$$

3. Simplifying the expression using properties of expectation :

$$\mathbf{E}(s^2) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \mathbf{E}\left(\left(X_i - \overline{X}\right)^2\right)$$

4. Using the relation $\mathbf{Var}(X) = \mathbf{E}(X2) - [\mathbf{E}(X)]^2$:

$$\mathbf{E}(s^2) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left[\mathbf{Var}(X_i) + \left[\mathbf{E}(\overline{X})\right]^2\right]$$

5. For a normal distribution:

$$\mathbf{E}(s^2) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left[\sigma^2 + \mu\right]$$

6. Simplification of the expression :

$$\mathbf{E}(s^2) = \sigma^2$$

This shows that the corrected variance estimator s^2 is unbiased for the variance σ^2 of the normal distribution from which the sample comes.
This demonstration uses properties of moments and expectation to show that, on average, the estimator σ^2 is equal to the true variance σ^2, which justifies its use as an unbiased estimator.