Entanglement Entropy and Holography - Examples

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I. Examples of EE

The general procedure is explained in the main paper.

A. EE of a disk in $d = 3$

A circuilar region $\mathcal{A}$ with radius R inside a 2-dimensional layer $z = \delta \ll 1$ is represented in polar coordinates as

$$\mathcal{A} = \{(r, \theta, z, t) \mid t = 0, z = \delta, r \leq R\} \ , \tag{1}$$

and the corresponding surface of minimal area $\delta_{\mathcal{A}}$ will be represented by the z coordinate as a function of the polar coordinates of $\mathcal{A}$:

$$\gamma_{\mathcal{A}} = \{(r, \theta, z, t) \mid t = 0, z = f(r, \theta)\} \ . \tag{2}$$

There is no property on the AdS spacetime theory that will make the symetry of $\partial \mathcal{A}$ on the coordinate θ not being transfered to $\gamma_{\mathcal{A}}$. Thus, $z = f(r)$.

In polar coordinates, the metric corresponding to a $(d+1)$-dimensional AdS spacetime will be

$$ds^2_{\mathrm{AdS}_4} = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{z^2}[-dt^2 + dz^2 + dr^2 + r^2 d\theta^2] \ . \tag{3}$$

Aplying the restrictions on the coordinates for the definition of $\gamma_{\mathcal{A}}$, it is obtained the induced metric of the surface:

$$ds^2_{\gamma_{\mathcal{A}}} = h_{\rho\sigma} dx^\rho dx^\sigma = \frac{L^2}{f(r)^2} \left[\left(1 + \dot{f}(r)^2\right) dr^2 + r^2 d\theta^2\right] \ , \tag{4}$$

being $dz = \frac{\partial z}{\partial x^\rho} dx^\rho = \frac{\partial f(r)}{\partial r} dr = \dot{f}(r) dr$.

The determinant of the induced metric and its square rood will be

$$h = \left(\frac{L}{f(r)}\right)^4 r^2 (1 + \dot{f}(r)^2) \ , \ \sqrt{h} = \left(\frac{L}{f(r)}\right)^2 r \sqrt{1 + \dot{f}(r)^2} \ . \tag{5}$$

The minimal value of the integral over the polar coordinates of the square rood of the induced metric will correspond to the area of $\gamma_{\mathcal{A}}$. So, by the Ryu-Takayanagi formula, the entanglement entropy related to the region $\mathcal{A}$ will be

$$\begin{aligned} S_{\mathcal{A}} &= \tfrac{1}{4G} \min \textstyle\int_{\gamma_{\mathcal{A}}} \sqrt{h} dx^\rho = \\ &= \tfrac{1}{4G} \min \textstyle\int_0^{2\pi} d\theta \int_0^R dr \left(\frac{L}{f(r)}\right)^2 r \sqrt{1 + \dot{f}(r)^2} \\ &= \tfrac{\pi L^2}{2G} \min \textstyle\int_0^R dr \frac{r}{f(r)^2} \sqrt{1 + \dot{f}(r)^2} \ . \end{aligned} \tag{6}$$

The interior of this final integral looks like some type of Lagrangian $\mathcal{L}[r, f(r), \dot{f}(r)]$. Thus, the Euler-Lagrange equation can be aplied to find relations to find the extreme of this functional:

$$\begin{gathered} \frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dr}\left[\frac{\partial \mathcal{L}}{\partial \dot{f}}\right] = 0 \\ \to \left(1 + \dot{f}^2\right)\left(-2r - f\dot{f} - rf\ddot{f}\right) + rf\dot{f}^2\ddot{f} = 0 \end{gathered} \tag{7}$$

It is proven that $f(r) = \sqrt{R^2 - r^2}$ is solution of the previous solution and correspond to the function that minimise the functional of the entanglement entropy. Then:

$$\begin{gathered} S_{\mathcal{A}} = \tfrac{\pi L^2}{2G} \textstyle\int_0^R dr \frac{r}{(R^2 - r^2)^{3/2}} = \\ = \tfrac{\pi L^2}{2G} \left.\frac{R}{\sqrt{R^2 - r^r}}\right|_{r=R} - \tfrac{\pi L^2}{2G} = \tfrac{\pi L^2}{2G} \frac{R}{\delta} - F \ , \end{gathered} \tag{8}$$

that is equivalent to the general expression for the entanglement entropy in a 3-dimensional QFT with

$$\begin{cases} c_1 = \frac{\pi L^2}{2G} \\ \delta = \lim_{r \to R} \sqrt{R^2 - r^2} \\ s_{\text{non-loc}} = -\frac{i\pi L^2}{2G} \end{cases} .$$