— ICTP Summer School on Superstring Theory and Related Topics —

TUTORIAL on Entanglement in QFT

Pablo Bueno[1]

§Introduction. At this point you have probably noticed that performing explicit calculations of entanglement entropy (EE) for (discretized) quantum field theories (QFTs) is not an easy task in general. Even in the "simple" cases of $(1+1)$-dimensional free theories, some degree of nontrivial computational technology is required.

Part of the success of the Ryu-Takayanagi prescription [1, 2] for evaluating EE for holographic CFTs dual to Einstein gravity in the bulk comes from its computational simplicity.

In this tutorial I will show you a (toy?) model for which computations are even simpler in general dimensions. Then, we will explicitly evaluate EE for various regions in $d=3$ and $d=4$ in this model. Along the way, we will comment on the different kinds of EE universal terms and their nature as well as on various aspects of the dependence of EE on the shape of the entangling region.

§Color code for EE terms. non-universal and local; universal and local; universal and non-local; non-universal and local+non-local.

§General structure of EE for CFTs. Given a smooth entangling region V on a time slice of a (discretized) d-dimensional conformal field theory (CFT), the entanglement entropy (EE) takes the generic form —see *e.g.*, [3, 4],

$$S_{\rm EE}^{(d)} = b_{d-2}\frac{H^{d-2}}{\delta^{d-2}} + b_{d-4}\frac{H^{d-4}}{\delta^{d-4}} + \cdots + \begin{cases} b_1 \frac{H}{\delta} + (-1)^{\frac{d-1}{2}} s^{\rm univ}\,, & (\text{odd } d)\,, \\ b_2 \frac{H^2}{\delta^2} + (-1)^{\frac{d-2}{2}} s^{\rm univ} \log\left(\frac{H}{\delta}\right) + b_0\,, & (\text{even } d)\,. \end{cases} \tag{0.1}$$

In this expression, H is some characteristic length of V, δ is a UV regulator[2].

In any state, the leading term is always the "area-law" piece, which diverges as $\sim 1/\delta^{d-2}$ except for $d=2$ theories. For those, in the simplest case of a region V corresponding to an interval of length H, one finds

$$S_{\rm EE}^{(2)} = \frac{c}{3}\log\left(\frac{H}{\delta}\right) + b_0\,, \tag{0.2}$$

where c is the Virasoro central charge of the theory [7, 8].

[1]pablo.bueno@cab.cnea.gov.ar

[2]Recall that, strictly speaking, the EE of subregions is an ill-defined quantity in the continuum. From a lattice point of view, as we go to that regime, more and more entanglement accumulates between modes at both sides of ∂V, leading to an infinite result in the limit. From the QFT perspective, the reason has to do with the fact that algebras of observables associated to subregions are von Neumann algebras of type-III, which do not admit a finite notion of trace —see *e.g.*, [5, 6].

The coefficients $b_{d-2}, \dots, b_1$ are "non-universal" in the sense that they are not well-defined in the continuum: different regularization procedures give different answers for them (*e.g.,* if I redefine the cutoff as $\delta \to 2\delta$, then $b_{d-2} \to \tilde{b}_{d-2} \equiv b_{d-2}/2^{d-2}$). In addition to being non-universal, all these terms have a "local nature", in the sense that they are controlled by integrals (of various intrinsic and extrinsic curvatures) over the entangling surface ∂V.

On the other hand, s^{univ} are "universal": they are well-defined in the continuum and capture meaningful information about the CFT. In even dimensions, the universal term is logarithmic and s^{univ} is given by a linear combination of local integrals over the entangling surface ∂V weighted by theory-dependent coefficients which can be shown to coincide with the trace-anomaly charges [9–12]. These are dimensionless numbers characteristic of a given CFT, which appear weighting various terms in the expectation value of the trace of the stress-tensor of the theory when this is put on a curved background. The general expression takes the form [13–15]

$$\langle T^\mu_\mu \rangle = -2(-)^{d/2} A \mathcal{X}_d + \sum_n B_n I_n \,, \tag{0.3}$$

where $\mathcal{X}_d$ is the Euler characteristic of the corresponding background, and the I_n are different independent (order-$d/2$ in curvature) contractions of the Weyl tensor. In the four-dimensional case, there is only one of the latter, and eq. (0.3) becomes $\langle T^\mu_\mu \rangle = -\frac{a}{16\pi^2}\mathcal{X}_4 + \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$. For comparison, in the $d=6$ case there are three "B-type" charges, B_1, B_2, B_3, besides the "A-type" one.

Then, in a $d=4$ CFT, the EE for a smooth region is given by [9, 10]

$$S^{(4)}_{\text{EE}} = b_2 \frac{H^2}{\delta^2} - \left[\frac{a}{2\pi} \int_{\partial V} d^2y \sqrt{h} \mathcal{R} + \frac{c}{2\pi} \int_{\partial V} d^2y \sqrt{h} \left(\text{tr} k^2 - \frac{1}{2} k^2 \right) \right] \log(H/\delta) + b_0 \,, \tag{0.4}$$

where $\mathcal{R}$ is the Ricci scalar of ∂V and the other integral involves various contractions of extrinsic curvatures of ∂V. Observe that the dependence on the geometry of ∂V and the dependence on the theory for which we are computing EE are strongly disentangled from each other. The latter appears only through the constants a and c. This will be very different for odd-dimensional theories. The charges a and c can be isolated, for instance, by considering entangling surfaces corresponding to spheres and cylinders, respectively,

$$S^{(4)}_{\text{EE}}|_{\text{sphere}} \supset -4a \log(R/\delta) \,, \quad S^{(4)}_{\text{EE}}|_{\text{cylinder}} \supset -\frac{c}{2} \frac{L}{R} \log(R/\delta) \,, \tag{0.5}$$

where R is the radius of the sphere or the cylinder, respectively, and L the length of the former.

The coefficients appearing in the trace-anomaly and EE expressions can be constrained imposing certain physical requirements for the theories under consideration. For instance, given any initial state, imposing the energy flux measured at infinity integrated over time to be positive leads to the bounds [16]

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18} \,. \tag{0.6}$$

These have been proven to hold for any unitary CFT in $d = 4$ [17].

In odd dimensions no logarithmic term is present for smooth entangling surfaces, and the universal contribution is a constant term which no longer corresponds to an integral over ∂V. The simplest case corresponds to three-dimensional CFTs, for which[3]

$$S_{\rm EE}^{(3)} = b_1 \frac{H}{\delta} - F\,. \tag{0.7}$$

For $\partial V = \mathbb{S}^1$, F actually equals the free energy of the corresponding theory on $\mathbb{S}^3$ [20, 21], which reveals its non-local nature. A similar relation holds for $d = 5, 7, \ldots$, namely $s^{\rm univ}$ for $\partial V = \mathbb{S}^{d-2}$ equals the free energy on $\mathbb{S}^d$. In F, $s^{\rm univ}$, the dependence on the geometric details of V and the dependence on the details of the CFT for which we are computing EE are no longer disentangled from each other.

No one has ever payed much attention to the constant coefficient b_0 appearing for even-dimensional theories. Just like its cousins $b_{d-2}, \ldots, b_1$, this is a non-universal piece, in the sense that you can pollute it by changing the regulator. However, all this pollution has a local origin and b_0 also contains a universal non-local part which does not depend on the regulator details...

§Singular regions. When geometric singularities are present in ∂V, the structure of divergences in eq. (0.1) gets modified. The prototypical case is that of an entangling region bounded by a corner of opening angle Ω in $d = 3$ CFTs. In that case, a new logarithmic universal contribution appears

$$S_{\rm EE}^{(3)}|_{\rm corner} = b_1 \frac{H}{\delta} - a^{(3)}(\Omega) \log\left(\frac{H}{\delta}\right) + \tilde{b}_0\,, \tag{0.8}$$

where $a^{(3)}(\Omega)$ is a cutoff-independent function of the opening angle. This has been extensively studied in the literature —see *e.g.,* [22] for an updated list of references.

The dependence of $a^{(3)}(\Omega)$ on the opening angle changes from one CFT to another —*e.g.,* compare the relatively simple holographic result [23] with the highly complicated resulting expressions for free fields [24–26]. It satisfies some general properties. On the one hand, $a^{(3)}(\Omega) = a^{(3)}(2\pi - \Omega)$, which follows from $S_{\rm EE}(V) = S_{\rm EE}(\bar{V})$, a consequence of the purity of the ground state. Besides, using strong subadditivity and Lorentz invariance one can show that [25]

$$a^{(3)}(\Omega) \geq 0\,, \quad \partial_\Omega a^{(3)}(\Omega) \leq 0\,, \quad \partial_\Omega^2 a^{(3)}(\Omega) \geq -\frac{\partial_\Omega a^{(3)}(\Omega)}{\sin\Omega}\,, \quad \text{for} \quad \Omega \in [0, \pi]\,. \tag{0.9}$$

[3] Constant terms such as F are less robust than their even-dimensional logarithmic counterparts. This is because we cannot resolve the relevant IR scales of the entangling region with more precision than the UV cutoff. If we shift the relevant characteristic scale as $R \to R + a\delta$, with $a = \mathcal{O}(1)$, we will pollute the putative universal contribution as $F \to F(1 - b_1 a)$. This pollution —which does not occur for logarithmic contributions— can be remedied using mutual information as a geometric regulator [18, 19].

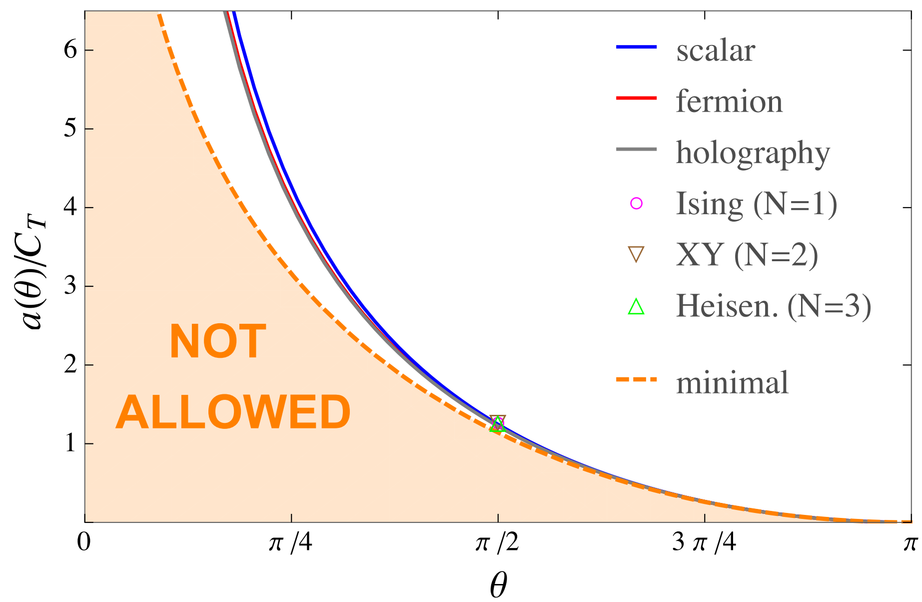

Figure 1. Corner function $a^{(3)}(\Omega)$ normalized by the two-point function charge C_T for various theories: free scalar, free fermion, Einstein gravity and various $O(N)$ models [27–29], as well as general lower bound function $\mathfrak{a}_{\rm min}(\Omega)$. Normalization by C_T makes all curves lie very close to one another throughout the whole range [30].

In the very-sharp and almost-smooth limits, the function behaves, respectively, as [24–26]

$$a^{(3)}(\Omega \simeq 0) = \frac{k}{\Omega} + \mathcal{O}(\Omega)\,, \qquad a^{(3)}(\Omega \simeq \pi) = \sigma \cdot (\Omega - \pi)^2 + \sum_{p=2} \sigma^{(p-1)} \cdot (\Omega - \pi)^{2p}\,. \tag{0.10}$$

In the first expression, k is a constant which coincides with the universal coefficient corresponding to a slab region for general theories —see *e.g.,* [31, 32]. Note that in the second formula only even powers appear in the expansion. The leading coefficient, σ, turns out to be related to the stress-energy tensor two-point function coefficient C_T through[4] [30, 34]

$$\sigma = \frac{\pi^2}{24} C_T\,, \tag{0.11}$$

for general CFTs. Using eq. (0.11) and the third relation in eq. (0.9), one can find a lower bound on $a^{(3)}(\Omega)$ valid for general CFTs [35]. This takes the form

$$a^{(3)}(\Omega) \geq \mathfrak{a}_{\rm min}(\Omega)\,, \quad \text{where} \quad \mathfrak{a}_{\rm min}(\Omega) \equiv \frac{\pi^2 C_T}{3} \log\left[1/\sin(\Omega/2)\right]\,, \tag{0.12}$$

[4]For any CFT in d dimensions, the stress-tensor correlator behaves as $\langle T_{ab}(x) T_{cd}(0)\rangle = C_T I_{ab,cd}(x)/|x|^{2d}$ where $I_{ab,cd}(x)$ is a fixed tensorial structure, and the only theory-dependent quantity is the charge C_T [33]. Note also that in $d = 4$, C_T is proportional to the trace-anomaly charge c.

where C_T is to be understood as the one corresponding to the theory we are comparing with. The bound turns out to be pretty tight for all theories considered so far, even for considerably small values of the opening angle [35].

The nature of $a^{(3)}(\Omega)$ is different from the one of the analogous coefficient corresponding to a conical entangling surface in four-dimensions. In that case, a similar logarithmic enhancement of the universal term does occur, and the EE reads[5]

$$S_{\rm EE}^{(4)}|_{\rm cone} = b_2 \frac{H^2}{\delta^2} - a^{(4)}(\Omega) \log^2 (H/\delta) + \tilde{b}_0 \log (H/\delta) + b_0 \, . \tag{0.13}$$

For the cone, the universal function $a^{(4)}(\Omega)$ is much more constrained than $a^{(3)}(\Omega)$. In fact, the explicit angular dependence is the same for all four-dimensional CFTs, namely [31, 36]

$$a^{(4)}(\Omega) = \frac{c}{4} \cdot \frac{\cos^2 \Omega}{\sin \Omega} \, . \tag{0.14}$$

The only dependence on the theory under consideration appears through the charge c. The contrast with $a^{(3)}(\Omega)$ can be understood from the fact that both $a^{(3)}(\Omega)$ and $a^{(4)}(\Omega)$ can be (sort of) thought of as emerging from the respective contributions $s^{\rm univ}$ in eq. (0.1) due to a logarithmically divergent build-up of Fourier modes which arises on a smooth surface when a singularity is included [37, 38]. While in $d = 3$, $s^{\rm univ}$ is a constant and non-local term, in $d = 4$ it is a geometric integral over the entangling surface.

This difference between the odd- and even-dimensional cases persists for higher d. For instance, for general even-dimensional theories, it can be shown that a (hyper)conical entangling surface gives rise to a universal term of the form [22]

$$S_{\rm EE}|_{\rm (hyper)cone} \supset (-1)^{\frac{d-2}{2}} a^{(d)}(\Omega) \log^2 \left(\frac{H}{\delta} \right) \, , \quad a^{(d)}(\Omega) = \frac{\cos^2 \Omega}{\sin \Omega} \sum_{j=0}^{\frac{d-4}{2}} \left[\gamma_j^{(d)} \cos(2j\Omega) \right] , \tag{0.15}$$

where again the functional dependence on the opening angle is completely fixed for any CFT up to $(d/2 - 1)$ coefficients $\gamma_j^{(d)}$ related to the corresponding trace-anomaly charges. On the other hand, for odd-d one finds something like

$$S_{\rm EE}|_{\rm (hyper)cone} \supset (-1)^{\frac{d-1}{2}} a^{(d)}(\Omega) \log \left(\frac{H}{\delta} \right) \, , \tag{0.16}$$

where in the function $a^{(d)}(\Omega)$ the dependence on the theory and the opening angle are no longer disentangled from each other —see *e.g.,* eqs. 3.25 and 3.26 in [31] for implicit formulas for $a^{(5)}(\Omega)$ in the case of holographic Einstein gravity. Still, the connection with C_T in the almost-smooth limit persists, and versions of eq. (0.11) for general d (both odd and even) exist [34, 37, 39].

[5]Note that here, $\tilde{b}_0$ does not really contain a non-local contribution, but still it can contain well-defined information (local) contaminated by cut-off dependent pollution coming from the double-log term. In that sense, it should not be colored in green, but I preferred not to introduce a fifth color...

One can consider other types of singular regions, with their own peculiarities and features. These include wedges, cones with non-circular sections, curved corners and cones, polyhedral corners, etc. —see [22, 31] and references therein.

§The "extensive mutual information" model. For some time during the Neolithic age of QFT entanglement entropy, it was believed that the analytic result obtained for the $(1+1)$-dimensional Dirac fermion in the multi-interval case in [40, 41] was valid for general CFTs in that number of dimensions. While it was later understood that this was not the case, the result exhibited an interesting property, namely, its "tripartite information" was vanishing

$$I_3(A;B,C) \equiv I(A,B) + I(A,C) - I(A,B\cup C) = 0\,. \tag{0.17}$$

Roughly speaking, this means that there is no overlap between the information shared by A and B and the one shared by A and C. Mutual information is "extensive" in this model, as opposed to the $I_3(A;B,C) > 0$ ("subextensive") and $I_3(A;B,C) < 0$ ("superextensive") cases, both possible on general grounds.

Interestingly, imposing eq. (0.17) along with some physically reasonable requirements such as causality and Poincaré invariance, strongly restricts the form of the EE and the mutual information in general dimensions. In particular, the entanglement entropy of a region A in this "Extensive Mutual Information Model" (EMI) reads [40, 42, 43]:

$$S_{\rm EE}^{\rm EMI} = \kappa_{(d)} \int_{\partial A} \mathrm{d}^{d-2}\sigma_1 \int_{\partial A} \mathrm{d}^{d-2}\sigma_2\, n^i(x_1) n^j(x_2)\, \frac{\delta_{ij}}{|x_1 - x_2|^{2(d-2)}}\,, \tag{0.18}$$

where $n^i(x_1)$ is the unit normal vector to the boundary of A, ∂A, at the point x_1 and $\kappa_{(d)}$ is a positive parameter[6].

It is an open problem to find out whether eq. (0.18) is the EE of an actual CFT in $d \geq 3$. Independently of the answer, the model respects all general principles of EE (such as strong subadditivity) and is a useful tool for understanding various features.

§EE of a disk in $d = 3$. Let us start with the simple case of a disk region of radius R. We work in polar coordinates and put the center of the disk in the origin of coordinates. In principle, we have to perform two integrals, but due to the symmetry of the problem, we can fix $x_2 = (R, 0)$, and then $\vec{n}(x_2) = (1, 0)$, which makes one of them trivial. On the other hand, we have $x_1 = (R\cos\theta_1, R\sin\theta_1)$, $\vec{n}(x_1) = (\cos\theta_1, \sin\theta_1)$. Also, $\mathrm{d}\sigma_1 = R\mathrm{d}\theta_1$, $\mathrm{d}\sigma_2 = R\mathrm{d}\theta_2$. From this, one finds $|x_1 - x_2|^2 = 2R^2(1-\cos\theta_1) = 4R^2\sin^2(\theta/2)$. Then, we have

$$S_{\rm EE}^{\rm EMI} = \kappa_{(3)} \int_0^{2\pi} R\mathrm{d}\theta_2 \int R\mathrm{d}\theta_1 \frac{\cos\theta_1}{4R^2\sin^2(\theta_1/2)}\,. \tag{0.19}$$

[6] Recall that $\mathrm{d}^{d-2}\sigma_1 \equiv \mathrm{d}^{d-2}x_1\sqrt{h(x_1)}$ where h is the determinant of the induced metric on ∂A.

The second integral diverges when $|x_1 - x_2|^2 \to 0$, so we need to regulate it. One possibility is to allow only for angles larger than δ/R. In that case, one finds

$$S_{\rm EE}^{\rm EMI} = \frac{\pi\kappa_{(3)}}{2}\cdot 2\int_{\delta/R}^{\pi}\frac{\cos\theta_1 {\rm d}\theta_1}{\sin^2(\theta_1/2)} = 4\pi\kappa_{(3)}\frac{R}{\delta} - 2\pi^2\kappa_{(3)}\,. \tag{0.20}$$

We find an "area-law" term with coefficient $b_1 = 4\pi\kappa_{(3)}$ and a universal piece $F = 2\pi^2\kappa_{(3)}$. This should match the free energy on $\mathbb{S}^3$ if the EMI corresponded to an actual model.

In principle we could have chosen a different procedure to regulate our integral. For instance, we could have replaced $|x_1 - x_2|^2 \to |x_1 - x_2|^2 + \delta^2$ instead. If we do that, we find

$$S_{\rm EE}^{\rm EMI} = 2\pi R^2\kappa_{(3)}\int_0^{2\pi}\frac{\cos\theta_1 {\rm d}\theta_1}{[4R^2\sin^2(\theta/2) + \delta^2]} = 2\pi^2\kappa_{(3)}\frac{R}{\delta} - 2\pi^2\kappa_{(3)}\,. \tag{0.21}$$

As we can see, the universal piece is unchanged but, unsurprisingly, b_1 differs now from the result obtained using the other regulator.

§EE across a sphere in $d = 4$. Let us now move to four dimensions. Consider first a spherical entangling surface. Similarly to the disk case, all points are equivalent on the sphere surface, so we can fix $x_2 = (0, 0, R)$ and $\vec{n}(x_2) = (0, 0, 1)$. On the other hand, $x_1 = R(\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$ and $\vec{n}(x_1) = (\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$. From this, we find $\vec{n}(x_1)\cdot\vec{n}(x_2) = \cos\theta_1$, and $|x_1 - x_2|^4 = 16R^4\sin^2(\theta_1/2)$. Also, ${\rm d}^2\sigma_1 = R^2\sin\theta_1{\rm d}\theta_1{\rm d}\phi_1$, and so on. Putting the pieces together, we can perform three of the four integrals to get

$$S_{\rm EE}^{\rm EMI} = \kappa_{(4)}\cdot 4\pi R^2\cdot 2\pi R^2\int {\rm d}\theta_1\frac{\sin\theta_1\cos\theta_1}{16R^4\sin^4(\theta_1/2)}\,. \tag{0.22}$$

Just like for the disk, we can regulate this integral by allowing only for angles larger than δ/R. This leads to

$$S_{\rm EE}^{\rm EMI} = \frac{\pi^2\kappa_{(4)}}{2}\int_{\delta/R}^{\pi}{\rm d}\theta_1\frac{\sin\theta_1\cos\theta_1}{\sin^4(\theta/2)} \tag{0.23}$$

$$= 4\pi^2\kappa_{(4)}\frac{R^2}{\delta^2} - 4\pi^2\kappa_{(4)}\log(R/\delta) - \pi^2\kappa_{(4)}\left[\frac{2}{3} + 4\log 2\right]. \tag{0.24}$$

All expected terms show up. We have the area-law piece, a universal logarithmic contribution, and a constant one. Comparing with eq. (0.4), we can guess the value of the would-be trace-anomaly coefficient $a_{\rm EMI}$. One finds $a_{\rm EMI} = \pi^2\kappa_{(4)}$.

Let us repeat the experiment with a different regulator. We try again with $|x_1 - x_2|^4 \to |x_1 - x_2|^4 + \delta^4$. In that case, we have

$$S_{\rm EE}^{\rm EMI} = 8\pi^2R^4\kappa_{(4)}\int_0^{\pi}{\rm d}\theta_1\frac{\sin\theta_1\cos\theta_1}{[16R^4\sin^4(\theta/2) + \delta^4]} \tag{0.25}$$

$$= 2\pi^3\kappa_{(4)}\frac{R^2}{\delta^2} - 4\pi^2\kappa_{(4)}\log(R/\delta) - \pi^2\kappa_{(4)}[1 + 4\log 2]\,. \tag{0.26}$$

We see that the non-universal piece b_2 changes, whereas the universal one is not affected by the change of regulator. Regarding the often forgotten b_0, we see that one of its constants gets altered, whereas the other (the $-4\pi^2 \log 2\kappa_{(4)}$ piece) is not. This would be the kind of behavior expected for a non-universal constant containing both local and non-local contributions, where the latter are not affected by cutoff changes. In this case, however, this is not really the case for $-4\pi^2 \log 2\kappa_{(4)}$, which can be contaminated by shifting δ.

§EE across a cylinder in $d = 4$. Consider now an entangling region consisting of an infinite cylinder of radius R. In cylindrical coordinates we can write $x_1 = (R\cos\phi_1, R\sin\phi_1, z_1)$, $\vec{n}(x_1) = (\cos\phi_1, \sin\phi_1, 0)$, and $x_2 = (R, 0, z_2)$, $\vec{n}(x_2) = (1, 0, 0)$, where we already took advantage of the circular symmetry of the surface. Now, $\mathrm{d}^2\sigma_1 = R\mathrm{d}\phi_1\mathrm{d}z_1$ and the same for 2. We have now $|x_1 - x_2|^4 = [4R^2\sin^2(\phi_1/2) + (z_1 - z_2)^2]^2$. So we find

$$S_{\text{EE}}^{\text{EMI}} = \kappa_{(4)} \cdot 2\pi \int R\mathrm{d}z_2 \int R\mathrm{d}z_1 \int \mathrm{d}\phi_1 \frac{\cos\phi_1}{[4R^2\sin^2(\phi_1/2) + (z_1 - z_2)^2]^2}\,, \tag{0.27}$$

where we performed the trivial integral over ϕ_2. We could have safely set $z_1 = 0$ from the beginning, regulating its integral with an IR cutoff L, so we get a contribution $\int_{-L/2}^{L/2} \mathrm{d}z_1 = L$. In the resulting expression, we can perform the integral over z_2, which requires no regulation, which yields

$$S_{\text{EE}}^{\text{EMI}} = 2\pi R^2 \cdot L \cdot \kappa_{(4)} \cdot \int \mathrm{d}\phi_1 \frac{\pi\cos\phi_1}{16R^3\sin^3(\phi_1/2)}\,. \tag{0.28}$$

Regulating the angular integral as usual, we are finally left with

$$S_{\text{EE}}^{\text{EMI}} = 2\pi R^2 \cdot L \cdot \kappa_{(4)} \cdot 2\int_{\delta/R}^{\pi} \mathrm{d}\phi_1 \frac{\pi\cos\phi_1}{16R^3\sin^3(\phi_1/2)} \tag{0.29}$$

$$= \pi^2\kappa_{(4)}\frac{RL}{\delta^2} - \frac{3\pi^2\kappa_{(4)}}{4}\frac{L}{R}\log(R/\delta) - \frac{\pi^2\kappa_{(4)}}{2}\left[\frac{1}{12} + 3\log 2\right]\frac{L}{R}\,. \tag{0.30}$$

We find a similar structure to the spherical case, with all the expected terms. Now we can identify the other would-be trace-anomaly coefficient. This gives $c_{\text{EMI}} = 3\pi^2\kappa_{(4)}/2$. Hence, we find for the EMI model $a_{\text{EMI}}/c_{\text{EMI}} = 2/3$. It is an interesting fact of nature that the quotient between both coefficients comfortably lies within the range of allowed values compatible with unitarity (eq. (0.6)).

If we use the $|x_1 - x_2|^4 \to |x_1 - x_2|^4 + \delta^4$ regularization instead, we get

$$S_{\text{EE}}^{\text{EMI}} = \pi^3\kappa_{(4)}\frac{RL}{\delta^2} - \frac{3\pi^2\kappa_{(4)}}{4}\frac{L}{R}\log(R/\delta) - \frac{\pi^2\kappa_{(4)}}{2}\left[-\frac{1}{2} + \frac{9}{2}\log 2\right]\frac{L}{R}\,. \tag{0.31}$$

Same story, the non-universal pieces change, whereas the universal one is not affected. Note that in this case b_0 "changes completely" with respect to the other regularization.

§EE of a corner in $d = 3$. Let us now turn to entangling surfaces with singularities. Consider first a corner region with opening angle Ω. We use Cartesian coordinates this time.

The entangling surface is defined by the lines $Y(X) = 0$ and $Y(X) = X \cdot \tan\Omega$ with $X \geq 0$. In this case there are two contributions, one from considering 1 and 2 on the same line, and another one from considering 1 on the $Y(X) = 0$ line and 2 on the $Y(X) = X \cdot \tan\Omega$ one. Each of these appears twice in the EE expression. We write: $S_{\rm EE}^{\rm EMI} = 2(s_{\rm I} + s_{\rm II})$. Let us look at the first contribution. For that, we have $\vec{n}(X_1) = \vec{n}(X_2) = (0,1)$ and $\mathrm{d}\sigma_1 = \mathrm{d}X_1$, $\mathrm{d}\sigma_2 = \mathrm{d}X_2$,

$$s_{\rm I} = \kappa_{(3)} \int \mathrm{d}X_1 \int \mathrm{d}X_2 \frac{1}{(X_1 - X_2)^2} \,. \tag{0.32}$$

In the case of the second contribution, we have $\vec{n}(X_1) = (0,-1)$, $\vec{n}(X_2) = (-\sin\Omega, \cos\Omega)$, $\mathrm{d}\sigma_1 = \mathrm{d}X_1$, $\mathrm{d}\sigma_2 = \mathrm{d}X_2 / \cos\Omega$. Putting the pieces together,

$$s_{\rm II} = -\kappa_{(3)} \int \mathrm{d}X_1 \int \mathrm{d}X_2 \frac{1}{(X_1 - X_2)^2 + \tan^2\Omega X_2^2} \,. \tag{0.33}$$

We can regulate the integrals as

$$\int \mathrm{d}X_1 \to \int_\delta^H \mathrm{d}X_1 \,, \quad \int \mathrm{d}X_2 \to \left[\int_0^{x_1 - \delta} \mathrm{d}X_2 + \int_{x_1+\delta}^\infty \mathrm{d}X_2 \right] \,. \tag{0.34}$$

The results read (see [22] for some help with the integrals)

$$s_{\rm I} = \frac{2\kappa_{(3)} H}{\delta} - \kappa_{(3)} \log(H/\delta) + \mathcal{O}(\delta^0) \,, \tag{0.35}$$

$$s_{\rm II} = -\kappa_{(3)} (\pi - \Omega) \cot\Omega \log(H/\delta) + \mathcal{O}(\delta^0) \,. \tag{0.36}$$

The final result reads then

$$S_{\rm EE}^{\rm EMI} = \frac{4\kappa_{(3)} H}{\delta} - a_{\rm EMI}^{(3)}(\Omega) \log(H/\delta) + \mathcal{O}(\delta^0) \,, \tag{0.37}$$

where

$$a_{\rm EMI}^{(3)}(\Omega) = 2\kappa_{(3)} [1 + (\pi - \Omega)\cot\Omega] \,. \tag{0.38}$$

We can verify that this function satisfies all general properties explained above for a decent EE corner function. In particular, in the very-sharp and almost-smooth limits, we find eq. (0.10) indeed holds with $k = 2\pi\kappa_{(3)}$ and $\sigma = 2\kappa_{(3)}/3$. Using eq. (0.11), we can obtain the value of the would-be stress-tensor two-point function charge C_T for the EMI model. This reads $C_T^{\rm EMI} = 16\kappa_{(3)}/\pi^2$.

§EE across a cone in $d = 4$. Let us now see how the EMI reproduces the general CFT result for the EE across a conical region. We can parametrize the cone surface in cylindrical coordinates by $z = \rho/\tan\Omega$, $t = 0$. The induced metric on the cone surface is given by $\mathrm{d}s_h^2 = \mathrm{d}\rho^2 / \sin^2\Omega + \rho^2 \mathrm{d}\phi$, so $\mathrm{d}^2\sigma_1 = [\rho_1 / \sin\Omega] \mathrm{d}\rho_1 \mathrm{d}\phi_1$ and analogously for 2. Given the symmetry of the problem, we can just set $\phi_2 = 0$ everywhere and multiply the remainder integrals by an overall 2π. The unit normal vector to the cone surface is given $\vec{n} = \vec{u}_\rho \cos\Omega - \vec{u}_z \sin\Omega$, where $\vec{u}_\rho = \cos\phi \vec{u}_x + \sin\phi \vec{u}_y$. Using this, it is straightforward to find $\vec{n}(x_1) \cdot \vec{n}(x_2) = \cos^2\Omega \cos\phi_1 +$

$\sin^2\Omega$. Similarly, we find $|x_1-x_2|^4=\left[\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi_1+(\rho_1-\rho_2)^2/\tan^2\Omega\right]^2$. Then, after some trivial manipulations, we are left with the integrals

$$S_{\rm EE}^{\rm EMI}=\frac{2\pi\kappa_{(4)}}{\sin^2\Omega}\int\rho_1{\rm d}\rho_1\int\rho_2{\rm d}\rho_2\int_0^{2\pi}{\rm d}\phi_1\frac{[\cos^2\Omega\cos\phi_1+\sin^2\Omega]}{[a-b\cos\phi_1]^2}\,, \tag{0.39}$$

where $a\equiv\rho_1^2+{\rho_2}^2+(\rho_1-\rho_2)^2/\tan^2\Omega$, $b\equiv 2\rho_1\rho_2$. We can regulate the radial integrals in various ways. For instance, as follows:

$$\int{\rm d}\rho_1\to\int_\delta^H{\rm d}\rho_1\,,\quad\int{\rm d}\rho_2\to\left[\int_0^{\rho_1-\delta}{\rm d}\rho_2+\int_{\rho_1+\delta}^\infty{\rm d}\rho_2\right]. \tag{0.40}$$

Performing the angular integrals, we are left with

$$S_{\rm EE}^{\rm EMI}=\frac{4\pi^2\kappa_{(4)}}{\sin^2\Omega}\left[\cos^2\Omega\, s_{\rm I}+\sin^2\Omega\, s_{\rm II}\right]\,, \tag{0.41}$$

where

$$s_{\rm I}=\int\int\frac{b\,\rho_1\rho_2}{(a^2-b^2)^{3/2}}{\rm d}\rho_1{\rm d}\rho_2\,,\quad s_{\rm II}=\int\int\frac{a\,\rho_1\rho_2}{(a^2-b^2)^{3/2}}{\rm d}\rho_1{\rm d}\rho_2\,. \tag{0.42}$$

Doing the radial ones we are left with

$$s_{\rm I}=\frac{1}{64}\sin\Omega\left[\cos(2\Omega)-7\right]\log^2\left(H/\delta\right)\,,\quad s_{\rm II}=\frac{1}{32}\cos^2\Omega\sin\Omega\log^2\left(H/\delta\right)\,, \tag{0.43}$$

up to nonuniversal contributions which I omit this time. Putting both pieces together, we are left with [22]

$$S_{\rm EE}^{\rm EMI}\supset-\frac{3\pi^2\kappa_{(4)}}{8}\cdot\frac{\cos^2\Omega}{\sin\Omega}\log^2\left(H/\delta\right)\,. \tag{0.44}$$

We observe that the angular dependence is the expected one. Also, comparing with eq. (0.14), we get $c_{\rm EMI}=3\pi^2\kappa_{(4)}/2$, which matches the cylinder result.

§Bonus Track: EE across a deformed sphere. In [38], Mezei studied the EE of a slightly deformed spherical entangling surface in a d-dimensional CFT. Parametrizing the deformed sphere as

$$r(\Omega_{d-2})=1+\varepsilon\sum_{\ell,m_1,\dots,m_{d-3}}a_{\ell,m_1,\dots,m_{d-3}}Y_{\ell,m_1,\dots,m_{d-3}}(\Omega_{d-2})\,, \tag{0.45}$$

where the $Y_{\ell,m_1,\dots,m_{d-3}}(\Omega_{d-2})$ are real (hyper)spherical harmonics, he found that, for general higher-curvature holographic CFTs, the universal contribution $s^{\rm univ}$ takes the form

$$s^{\rm univ}=s_0^{\rm univ}+\varepsilon^2 s_2^{\rm univ}+\mathcal{O}(\varepsilon^3)\,, \tag{0.46}$$

where $s_0^{\rm univ}$ is the result for the round sphere in each case (*e.g.*, $s_0^{\rm univ}=F$ in $d=3$ and $s_0^{\rm univ}=4a$ in $d=4$) and

$$s_2^{\rm univ}=C_T\,\frac{\pi^{\frac{d+2}{2}}(d-1)}{2^{d-2}\Gamma(d+2)\Gamma(d/2)}\sum_{\ell,m_1,\dots,m_{d-3}}a^2_{\ell,m_1,\dots,m_{d-3}}\frac{\Gamma(d+\ell-1)}{\Gamma(\ell-1)}\times\begin{cases}\pi/2 & (d\text{ odd})\\ 1 & (d\text{ even})\,.\end{cases} \tag{0.47}$$

Hence, the sphere is a local extremum of the EE, and the leading correction in the deformation is controlled by the stress-tensor two-point function charge C_T. This result was later proven to hold for general CFTs in [34] (along with relation (0.11) for the almost-smooth limit of the corner function).

As far as I know, nobody has verified that eq. (0.47) indeed holds for the EMI...

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