

Entanglement Entropy and Holography - Examples

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I. EXAMPLES OF EE

The general procedure is explained in the main paper.

A. EE of a disk in $d = 3$

A circular region \mathcal{A} with radius R inside a 2-dimensional layer $z = \delta \ll 1$ is represented in polar coordinates as

$$\mathcal{A} = \{(r, \theta, z, t) \mid t = 0, z = \delta, r \leq R\}, \quad (1)$$

and the corresponding surface of minimal area $\delta_{\mathcal{A}}$ will be represented by the z coordinate as a function of the polar coordinates of \mathcal{A} :

$$\gamma_{\mathcal{A}} = \{(r, \theta, z, t) \mid t = 0, z = f(r, \theta)\}. \quad (2)$$

There is no property on the AdS spacetime theory that will make the symmetry of $\partial\mathcal{A}$ on the coordinate θ not being transferred to $\gamma_{\mathcal{A}}$. Thus, $z = f(r)$.

In polar coordinates, the metric corresponding to a $(d+1)$ -dimensional AdS spacetime will be

$$ds_{\text{AdS}_4}^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{z^2} [-dt^2 + dz^2 + dr^2 + r^2 d\theta^2]. \quad (3)$$

Applying the restrictions on the coordinates for the definition of $\gamma_{\mathcal{A}}$, it is obtained the induced metric of the surface:

$$ds_{\gamma_{\mathcal{A}}}^2 = h_{\rho\sigma} dx^\rho dx^\sigma = \frac{L^2}{f(r)^2} [(1 + \dot{f}(r)^2) dr^2 + r^2 d\theta^2], \quad (4)$$

being $dz = \frac{\partial z}{\partial x^\rho} dx^\rho = \frac{\partial f(r)}{\partial r} dr = \dot{f}(r) dr$.

The determinant of the induced metric and its square root will be

$$h = \left(\frac{L}{f(r)}\right)^4 r^2 (1 + \dot{f}(r)^2), \quad \sqrt{h} = \left(\frac{L}{f(r)}\right)^2 r \sqrt{1 + \dot{f}(r)^2}. \quad (5)$$

The minimal value of the integral over the polar coordinates of the square root of the induced metric will correspond to the area of $\gamma_{\mathcal{A}}$. So, by the Ryu-Takayanagi formula, the entanglement entropy related to the region \mathcal{A} will be

$$\begin{aligned} S_{\mathcal{A}} &= \frac{1}{4G} \min \int_{\gamma_{\mathcal{A}}} \sqrt{h} dx^\rho = \\ &= \frac{1}{4G} \min \int_0^{2\pi} d\theta \int_0^R dr \left(\frac{L}{f(r)}\right)^2 r \sqrt{1 + \dot{f}(r)^2} \quad (6) \\ &= \frac{\pi L^2}{2G} \min \int_0^R dr \frac{r}{f(r)^2} \sqrt{1 + \dot{f}(r)^2}. \end{aligned}$$

The interior of this final integral looks like some type of Lagrangian $\mathcal{L}[r, f(r), \dot{f}(r)]$. Thus, the Euler-Lagrange

equation can be applied to find relations to find the extreme of this functional:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dr} \left[\frac{\partial \mathcal{L}}{\partial \dot{f}} \right] &= 0 \\ \rightarrow (1 + \dot{f}^2) (-2r - f\dot{f} - r f \ddot{f}) + r f \dot{f}^2 \ddot{f} &= 0 \end{aligned} \quad (7)$$

It is proven that $f(r) = \sqrt{R^2 - r^2}$ is solution of the previous solution and correspond to the function that minimise the functional of the entanglement entropy. Then:

$$\begin{aligned} S_{\mathcal{A}} &= \frac{\pi L^2}{2G} \int_0^R dr \frac{r}{(R^2 - r^2)^{3/2}} = \\ &= \frac{\pi L^2}{2G} \frac{R}{\sqrt{R^2 - r^2}} \Big|_{r=R} - \frac{\pi L^2}{2G} = \frac{\pi L^2}{2G} \frac{R}{\delta} - F, \end{aligned} \quad (8)$$

that is equivalent to the general expression for the entanglement entropy in a 3-dimensional QFT with

$$\begin{cases} c_1 = \frac{\pi L^2}{2G} \\ \delta = \lim_{r \rightarrow R} \sqrt{R^2 - r^2} \\ s_{\text{non-loc}} = -\frac{i\pi L^2}{2G} \end{cases}.$$