PSE APE Masters
Quantitative Macroeconomics II

Problem Set #1

Due Date: 6 March 2026

You should do this problem set in groups of 2. Please hand in your answers by email to moritz.scheidenberger@psemail.eu.

Consider the following income fluctuation problem for $t = 0, 1, ...,$.

$$\begin{aligned} max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma} \\ s.t.\ C_t + k_{t+1} = Rk_t + z_t w \\ k_t > -\underline{b} \\ n_t = 1 \end{aligned} \tag{1}$$

where C_t denotes consumption and labor supply is exogenous normalised to 1. So labor income $z_t w N_t$ equals the product of the constant hourly wage rate w and the individual productivity z_t. Note hours worked are normalised to 1. k_t are holdings of a riskless asset, R is a constant gross interest rate, and $\underline{b}$ is a borrowing limit (lower bound on asset holdings).

$\tilde{z} = log(z_t)$ follows an exogenous AR(1) process

$$\tilde{z}_{t+1} = \rho \tilde{z}_t + \varepsilon_{t+1}$$

with $\varepsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$

Consider the following parameter values:

β	ρ	σ	σ_ϵ^2	$\underline{b}$
0.98	0.95	2	0.05	0

Question 1

1. Use Rouwenhorst's method, with Matlab programmes provided, to transform the AR(1) stochastic process for $\tilde{z}_t = log(z_t)$ to an N state Markov chain, defined by a support $Z_1 < Z_2 < ... < Z_5$ and a $N \times N$ Matrix of transition probabilities P. Choose $N = 5$. Normalise $Z_1, ..., Z_5$ such that average productivity equals 1.

2. For $w = 1$, $R = 1/\beta * 0.995$, solve the above problem in two ways:

A. Value function iteration (VFI) (you can use your code from QM 1)

1. Solve the consumer's problem using the following algorithm

   (a) Choose a criterion ε for convergence of the value function (choose e.g. 10^{-6}).

   (b) Choose a grid $\mathbb{K} = \{k_1 < k_2 < ...k_N\}$, $N = 20$ with $k_1 = 0$ and $k_N = 10 * w$. Choose a grid whose points are equally spaced in terms of $log(k)$ (thus giving more points at the bottom of the k grid).

   (c) Choose an initial value function $v_0(k_i, z_j)$, an $5 \times N$ matrix.

   (d) Calculate for all $i = 1, .., N$ $j = 1, ..., 5$

   $$v_{s+1}(k, z) = \max_{k'}\{\frac{c(z, k, k')^{1-\sigma} - 1}{1 - \sigma} + \beta E[v_s(k', z')|z\} \tag{2}$$

   for $s = 0, 1, 2, ...$ Use golden-ratio search to find the maximising k' for each combination of k and z.

   (e) If $\max_{i,j} |v_s[i, j] - v_{s-1}[i, j]| \geq \varepsilon$ go back to 3. Otherwise stop.

2. Plot the policy function $k'(k, z)$.

3. Simulate a panel of income, consumption and asset holdings of 2000 individuals over 1400 periods. Calculate the mean asset holdings discarding the first 400 periods. Store the panel of income shocks $\{z_{i,t}\}$.

4. Evaluate the Euler equation errors. In particular, for every value k_i, z_j, calculate the percentage difference between the consumption according to your policy function $c(k_i)$, and that which makes the Euler equation hold with equality given tomorrow's consumption and marginal productivity

   $$c(k_i, z_j)^{imp} = \left(\beta RE[c(k'(k_i), z')^{-\sigma}|z_j]\right)^{-1/\sigma} \tag{3}$$

   Report the maximum across i and j.

B. Using Carroll (2006)'s endogenous-gridpoints method (EGM)

1. Use Carroll (2006)'s method to solve for the optimal policy. In particular, define cash-on-hand $x(k, z) = zw + Rk$ for all k, z. Choose a finite grid K for capital <u>in the next period</u> k_{t+1}.

   0. Guess a policy $k'_0(x, z)$.

1. For all $k' \in K$, $z \in Z$, given $k'_i(x, z)$, calculate

$$C_{i+1}(k', z) = u'_{inv}\{\beta R E[u'(x(k', z') - k'_i(k', z'))]\} \tag{4}$$

for inverse marg utility u'_{inv}.

2. Calculate 'endogenous grid point' $x(k', z) = C_{i+1}(k', z) + k'$
3. Update $k'_{i+1}(x, z) = k'(x(k', z), z)$
4. If k'_{i+1} is close to k'_i, stop. O/w go back to 1.

2. Compare the run-time with Value-function iteration, and compare the policy functions.

Question 2

Now consider an economy with a continuum of individuals of measure 1. Capital and labor are used in a standard Cobb-Douglas production function operated by competitive firms

$$y_t = K_t^{\alpha} N_t^{1-\alpha} \tag{5}$$

where $K_t = \int k_t$ is the average capital stock across individuals, and $N_t = \int n_t$ the average labor supply, whose marginal products in equilibrium equal R and w, respectively.

1. For a given interest rate R, derive the demand for capital and the wage from the firm's first-order conditions.
2. Use your solution from Question 1.B (or if this was too difficult 1.A, and a bisection algorithm on R, to find the equilibrium prices and quantities. Do not change the panel of exogenous income shocks. Use the cross-sectional distribution from the previous guess on R as the starting point for the next simulation.
3. Plot a histogram, or a kernel density, of the wealth distribution and calculate the Gini coefficient, 90/10 and 99/1 percentile ratios.

Question 3

Consider a version of the economy where the return on individual asset holdings is heterogeneous, in addition to individual earnings. So the return of individual i from holding a unit of the only asset betwen today and next period is

$$R_t^i = 1 + \bar{r} + \tilde{r}_t^i \tag{6}$$

In particular, the excess return $\tilde{r}_t$ follows a first-order markov process

$$\tilde{r}_{t+1} = \rho^r \tilde{r}_t + \epsilon_{t+1}$$

with the i.i.d. innovation $\epsilon_{t+1} \sim N(0, \sigma_r^2)$ independent of $\varepsilon_{i,s+1}$ $\forall i, s$.

1. Consider $\bar{r} = \frac{0.985}{\beta} - 1$, two cases $\rho^r = 0$ and $\rho^r = 0.9$, and $\sigma_r^2 = (1 - \rho^2)0.002$. Discretise the return process using Rouwenhorst's method with 5 states. Construct a 25x25 transition matrix for the exogenous states with associated values of z and r. Solve the consumer's problem for a given $\bar{r}$, using EGM. If this is too difficult, use VFI. If this is too difficult, abstract from earnings heterogeneity. Plot the policy functions for the highest and lowest return state.

2. Simulate the economy as in Question 2. Calculate the excess demand for capital and derive the equilibrium $\bar{r}$.

3. Plot a histogram, or a kernel density, of the wealth distribution and calculate the Gini coefficient, 90/10 and 99/1 percentile ratios, for both values of ρ^r. Does the right-tail have an approximate Pareto shape? (For this, estimate the CDF of the stationary distribution, and plot the logarithm of the estimated survival function against the logarithm of k.)