

Beam estimation

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1 Direct estimation

To prevent the use of fitting methods to estimate the beam size, the main idea is to consider that (u, v) points were drawn from a 2D normal distribution. The beam shape is extracted from the empirical covariance of this distribution.

We have N points in the uv plane described by the vectors \mathbf{u} and \mathbf{v} (including the null frequency $(u, v) = (0, 0)$). We can define the modulation transfer function h of the interferometer as:

$$\hat{h}(u, v) = \begin{cases} 1 & \text{if } u \in \mathbf{u} \text{ and } v \in \mathbf{v}, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The dirty beam is h the Fourier transform of \hat{h} . One way to define the resolution of the interferometer is to compute the so called beam as half-width at half-maximum of a 2D Gaussian g fitted on the central peak of the dirty beam. The covariance matrix of the Gaussian \mathbf{C} are given by

$$\theta = \arg \min_{\mathbf{C}} \|g(\mathbf{C}) - h\|^2 \quad (2)$$

From Parseval-Plancherel theorem, this can be also written in Fourier domain as

$$\theta = \arg \min_{\mathbf{C}} \|\hat{g}(\mathbf{C}) - \hat{h}\|^2 \quad (3)$$

where the $\hat{g}(\mathbf{C})$ is also a Gaussian of covariance matrix $\mathbf{D} = \pi^{-2} \mathbf{C}^{-1}$.

This covariance matrix \mathbf{D} can be approximated from the distribution of the sampling point in the uv plane. As by construction $\langle \mathbf{u} \rangle = 0$ and $\langle \mathbf{v} \rangle = 0$ we have:

$$\mathbf{D} = \frac{2}{2N+1} \begin{pmatrix} \sum_n u_n^2 & \sum_n u_n v_n \\ \sum_n u_n v_n & \sum_n v_n^2 \end{pmatrix} \quad (4)$$

The $\frac{2}{2N+1}$ factor is to count twice all the (uv) points (due to the symmetry of the uv plane) excepted the null frequency. In addition to its covariance matrix \mathbf{C} , we can describe the beam with its principal angle θ and its half-widths at half maximum r_1 and

r_2 along both the major and minor axes respectively. These parameters can be extracted by the mean of the eigendecomposition:

$$\mathbf{D} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T \quad (5)$$

where \mathbf{Q} and $\mathbf{\Lambda}$ are the matrices of eigenvectors and eigenvalues respectively. The matrix $\mathbf{\Lambda}$ is a diagonal matrix that contains the eigenvalues (λ_1, λ_2) that are the variances along both axis. \mathbf{Q} is a rotation by the principal angle. As $\mathbf{C} = \pi^{-2} \mathbf{D}^{-1}$, its eigenvalues are $\mathbf{\Lambda}^{-1}$ and major and minor axis are inverted leading to a rotation of $\pi/2$ of the principal angle. As a result, these matrices can be expressed as a function of the beam parameters using as :

$$\mathbf{Q} = \begin{pmatrix} \cos(\pi/2 - \theta) & -\sin(\pi/2 - \theta) \\ \sin(\pi/2 - \theta) & \cos(\pi/2 - \theta) \end{pmatrix} \quad (6)$$

$$\mathbf{\Lambda} = \frac{\log(2)}{2\pi^2} \begin{pmatrix} r_1^{-2} & 0 \\ 0 & r_2^{-2} \end{pmatrix} \quad (7)$$

where $\frac{\sqrt{2\log(2)}}{2}$ is a factor to convert standard deviation to half-width at half-maximum. The beam ellipse parameters are then:

$$r_1 = \frac{\sqrt{2\log(2)}}{2\pi} \frac{1}{\sqrt{\lambda_1}} \quad (8)$$

$$r_2 = \frac{\sqrt{2\log(2)}}{2\pi} \frac{1}{\sqrt{\lambda_2}} \quad (9)$$

$$\theta = \arctan(Q[1, 1], Q[2, 1]) \quad (10)$$

In OImaging, we can plot the beam by applying the transformation matrix $\mathbf{T} = \frac{1}{\pi} \mathbf{Q} \mathbf{\Lambda}^{-1/2}$ to a circle of diameter 1.

2 Results

See notebook <https://jovian.com/ferreols/beamexample>