PROBLEM 1 *Numerical Coverage*

You are given a set of points $P = \{p_1, p_2, \ldots, p_n\}$ on the real line (you may assume these are given to you in sorted order).

1. Describe an algorithm that determines the smallest set of unit-length closed intervals that contains all of the given points. For example, the points $\{0.9, 1.2, 1.3, 2.1, 3.0\}$ can be covered by $[0.7, 1.7]$ and $[2.0, 3.0]$. State the runtime of your algorithm.

   Solution: Simply place an interval with it's left endpoint exactly on the next uncovered point. For example, if next uncovered point is x, add interval $[x, x+1]$ to the solution. Then repeat with next uncovered point.

2. Prove that the problem has *optimal substructure*.

   Solution: Suppose we have an optimal set of unit intervals $P = [p_1, p_1 + 1], [p_2, p_2 + 1], \ldots$ but somehow the intervals $P' = [p_2, P_2 + 1], \ldots$ are NOT optimal for covering the points at or above p_2. If P' is not optimal, then some other set of intervals can cover those points with fewer intervals, let's call this new optimal solution to the subproblem P''. We can take $P'' \cup [p_1, p_1 + 1]$ to get a solution that covers all the original points AND is more optimal than the originally assumed optimal solution, leading to a contradiction.

3. Prove that your algorithm's greedy choice returns the optimal result.

   Solution: Suppose the interval $[p_1, p_1 + 1]$ is NOT in the optimal solution. This means there is some OTHER interval $[x_1, x_1 + 1]$ that is optimal instead. That interval must cover the point p_1, so it must be the case that $x_1 < p_1$. Since p_1 is the first point in the range there are no points to cover before p_1, so we can remove interval starting at x_1 and exchange for the original interval starting at p_1 without changing the number of intervals in the solution and this is still gauranteed to cover the original points. Thus, $[p_1, p_1 + 1]$ is in some optimal solution.

PROBLEM 2 *Skiing*

Consider a situation in which you have n skiers with heights $p_1, p_2, \ldots, p_n$, and n pairs of skis with length $s_1, s_2, \ldots, s_n$. The problem is to assign each person a pair of skis so that the overall difference between the height of a skier and the length of their skis is minimized. More precisely, if $\alpha(i)$ returns the index of the skis assigned to person p_i, we want to minimize:

$$\frac{1}{n}\sum_{i=1}^{n}\left(\left|p_i - s_{\alpha(i)}\right|\right)$$

Consider two greedy algorithms that may solve this problem:

1. Find the skier and skis whose height difference is minimized. Assign those skis to that skier and repeat.
2. Pair the smallest skier with the smallest skis, second smallest skier to second shortest skis, etc. until all pairings are complete.

One of these greedy algorithms is correct. You must do three things:

1. Prove this problem has optimal substructure.

   Solution: Similar argument to previous problem: Suppose we have an optimal pairing of skis. Person 1 is paired with some set of skis (skis i). Suppose that the pairings for persons 2 through n are somehow NOT optimal. Then we can rearrange those skis between those people (2 through n) to get a lower average difference and add person 1 with skis i back to the solution. The overall solution cost just want down implying the original pairing was not actually optimal.

2. Claim which of the two algorithms is correct.

   Solution: The second algorithm is correct.

3. Show the greedy choice property of your chosen algorithm by using an exchange argument.

   Solution: **greedy choice property:** States that the smallest skier and the smallest skis are in some optimal solution. First assume they are not, this means that skier 1 (smallest skier) was paired with skis that were not the smallest. That means someone taller than skier 1 got the smallest skis. So for we have

   (a) p_1 paired with skis s_i where $i > 1$
   (b) p_j paired with skis s_1 where $j > 1$
   (c) $p_1 \le p_j$
   (d) $s_1 \le s_i$

   So the total cost of the pairing (where ℓ is the cost of the other skiers) is $\ell + |p_1 - s_i| + |p_j - s_1|$. There are three cases. For each, we will swap the ski assignment and show the overall cost of the "optimal solution" goes down or stays the same, leading to a contradiction because the smallest skier could have (or should have) been paired with the smallest skis:

   Case 1: Both persons smaller than both skis (or vice versa), $p_1 \le p_j \le s_1 \le s_i$. Note that it doesn't matter if people smaller than skis or skis smaller than skiers because we are dealing with 4 integers, so they are interchangeable. We begin with the optimal assignment (with terms rearranged to ensure absolute values are positive already) on the left and compare it with the exchanged assignment (which should be same or worse than optimal) on the right.

$$\ell + (s_i - p_1) + (s_1 - p_j) \le \ell + (s_1 - p_1) + (s_i - p_j)$$
$$-p_1 - p_j \le -p_1 - p_j)$$
$$0 \le 0$$

So solution is the same, and the exchange can be made.

Case 2: Alternating person,ski,person,ski heights, $p_1 \leq s_1 \leq p_j \leq s_i$. Again, doesn't matter if it is ski,person,ski,person instead.

$$\begin{aligned}
\ell + (s_i - p_1) + (p_j - s_1) &\leq \ell + (s_1 - p_1) + (s_i - p_j) \\
s_i - p_1 + p_j - s_1 &\leq s_1 - p_1 + s_i - p_j \\
p_j - s_1 &\leq s_1 + -p_j \\
2 * p_j &\leq 2 * s_1 \\
p_j &\leq s_1
\end{aligned}$$

Contradiction. This can only be the case if $p_j = s_1$, in which case the swap will keep the optimal solution in tact, if not, the solution will improve.

Case 3: persons completely "inside" skis or vice versa, $p_1 \leq s_1 \leq s_i \leq p_j$. Again, doesn't matter if it is ski,person,person,ski instead.

$$\begin{aligned}
\ell + (s_i - p_1) + (p_j - s_1) &\leq \ell + (s_1 - p_1) + (p_j - s_i) \\
s_i - p_1 + p_j - s_1 &\leq s_1 - p_1 + p_j - s_i \\
2 * s_i &\leq 2 * s_1 \\
s_i &\leq s_1
\end{aligned}$$

Contradiction. This can only be true if $s_1 = s_i$, in which case the solution will stay the same and the exchange works. Otherwise, the swap actually improves the optimal solution leading to a contradiction.