UNIT 3

Differential Calculus of Functions of More Than One Variable

OBJECTIVES

By the end of the unit, you must be able to:

- ✓ determine domain and range of functions;
- ✓ sketch graphs and contour maps of functions;
- ✓ find the partial derivatives of a multivariable function;
- ✓ establish differentiability of a multivariable function;

OBJECTIVES

By the end of the unit, you must be able to:

- ✓ find directional derivatives;
- ✓ find equations of tangent planes and normal lines to surfaces; and
- ✓ find extreme values of a multivariable function

3.1

Functions of More Than One Variable

REVIEW

Function of a single variable:

$f : R \to R$ ***where*** $y = f(x)$

Also, $y = f(x)$ ***is a curve on the coordinate plane containing the points*** $(x, f(x))$.

The n-th dimensional space

R ***: the real number line***

$R^2 = \{(x,y) \mid x,y \in R\}$

: the plane

$R^3 = \{(x,y,z) \mid x,y,z \in R\}$

: the space

The n-dimensional space

R^n
$=\{(x_1, x_2, \ldots, x_n) \mid x_i \in R\}$

: the n-dimensional number space

$(x_1, x_2, \ldots, x_n)$ is a point in R^n.

GOAL

Functions of more than variable

$$f : R^n \to R$$

where $y = f(x_1, x_2, \ldots, x_n)$

Restrictions: $f : R^2 \to R$
$f : R^3 \to R$

Definition

A <u>function of n variables</u> is a set of ordered pairs (P, w), where $P \in R^n$, where <u>w is unique for each P</u>.

Details

$f : R^n \to R$

Notation: $w = f(P)$
where $P \in R^n$

Domain, D_f
: set of all admissible P

Range, R_f
: set of all resulting w

Graph of a function

Given: $w = f(x_1, x_2, \ldots, x_n)$

Graph of f
: set of all points
$(x_1, x_2, \ldots, x_n, w) \in R^{n+1}$
where $(x_1, x_2, \ldots, x_n) \in D_f$

Restrictions

Given: $z = f(x, y)$

Graph of f: surface in R^3

For functions of at least three variables,

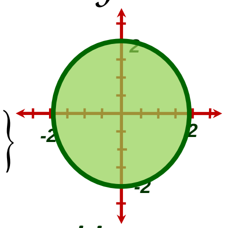
NO geometric graphs because of restriction to a 3D object.

Example #1

$$f(x,y)=\sqrt{4-x^2-y^2}$$

Domain, D_f:

$$\{(x,y)\mid x^2+y^2\le 4\}$$



Range, R_f: **Use the graph!**

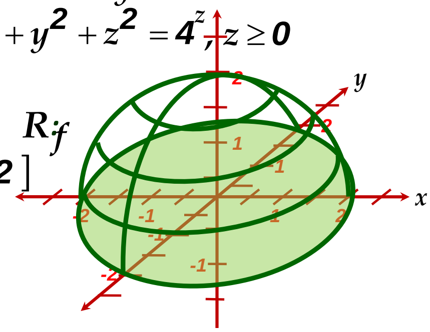
Continuation

$$z=\sqrt{4-x^2-y^2}$$

$$\Rightarrow x^2+y^2+z^2=4,\ z\ge 0$$

Range, R_f:

$$[0,2]$$



Example #2

$$f(x,y)=4-x^2-y^2$$

Domain, D_g: R^2

Range, R_g: **Use the graph!**

Graph: PARABOLOID

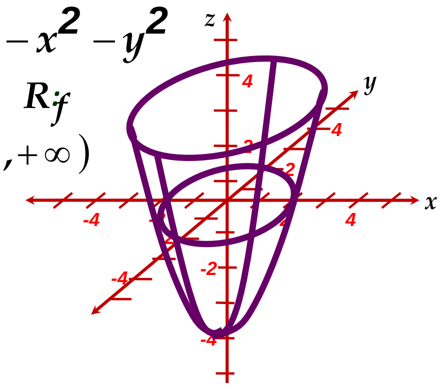
$$x^2+y^2+z-4=0$$

Continuation

$$z=4-x^2-y^2$$

Range, R_f:

$$[-4,+\infty)$$



Example #3

$$f(x,y,z)=\frac{1}{4-x^2-y^2-z^2}$$

Domain, D_f:

$$R^3-\{(x,y,z)\mid x^2+y^2+z^2=4\}$$

This is R^3 **except some sphere.**

Example #3

$$w=\frac{1}{4-x^2-y^2-z^2}$$

$$\Rightarrow x^2+y^2+z^2=4-\frac{1}{w}$$

Possible values of w**?** $\quad 4-\frac{1}{w}\ge 0$

Range, R_f: $\left(-\infty,0\right)\cup\left[\frac{1}{4},+\infty\right)$

Example #6

$$g(x,y,z) = \ln x + \sqrt{y} + \frac{1}{z}$$

Domain, D_f

$$\{(x,y,z) \mid x > 0, y \geq 0, z \neq 0\}$$

Range, R_f : R

Graph is in R^4

Levels

Consider $z = f(x,y)$

Level curve at $z = c$

$$\{(x,y) \mid f(x,y) = c\}$$

Consider $w = f(x,y,z)$

Level surface at $w = c$

$$\{(x,y,z) \mid f(x,y,z) = c\}$$

Contour map

A contour map of f is a set of level curves (or level surfaces) by considering different function values.

level curve ~ cross-section

contour map ~ topographic map

Example. Draw a contour map of

$$f(x,y) = \frac{1}{x^2 + y^2}$$

Consider level curves at

$$z = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$$

z CANNOT BE NEGATIVE!

Example (continued)

$$z = \frac{1}{x^2 + y^2}$$

At $z = k$

$$x^2 + y^2 = \frac{1}{k}$$

$$z = \frac{1}{4} : x^2 + y^2 = 4$$

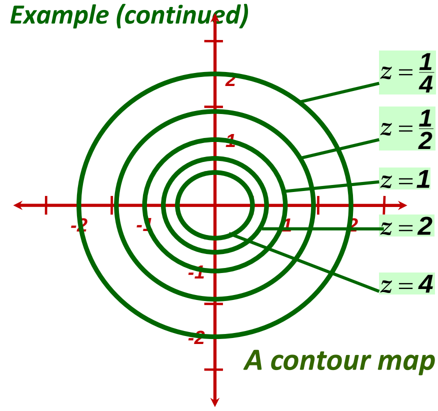
$$z = \frac{1}{2} : x^2 + y^2 = 2$$

$$z = 1 : x^2 + y^2 = 1$$

$$z = 2 : x^2 + y^2 = \frac{1}{2}$$

$$z = 4 : x^2 + y^2 = \frac{1}{4}$$

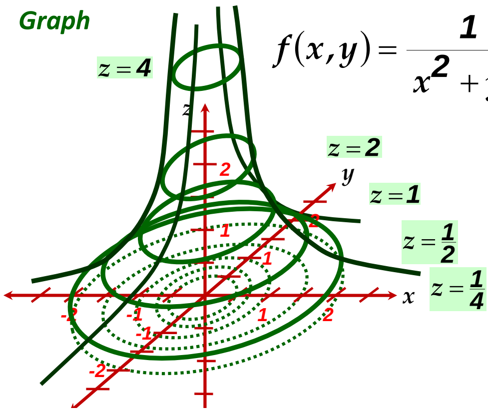
Example (continued)



A contour map of f

Graph

$$f(x,y)=\frac{1}{x^2+y^2}$$



Example. Draw a contour map of

$$g(x,y)=\frac{1}{x-y}$$

Consider level curves at

$$z=-3,-2,-1,1,2,3$$

Solution:

$$D_g:\{(x,y)\,|\,x-y\neq 0\}$$

$$R_g:\ R-\{0\}$$

Example (continued)

$$g(x,y)=\frac{1}{x-y}$$

$$z=-3:\ 3x-3y+1=0$$

$$z=-2:\ 2x-2y+1=0$$

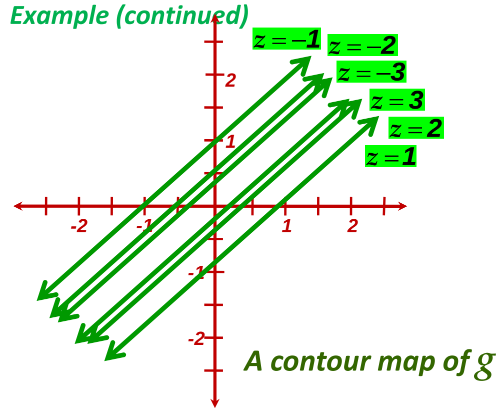
$$z=-1:\ x-y+1=0$$

$$z=1:\ x-y-1=0$$

$$z=2:\ 2x-2y-1=0$$

$$z=3:\ 3x-3y-1=0$$

Example (continued)



A contour map of g

Traces

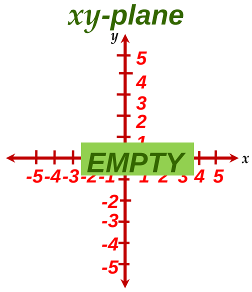
$$z=\frac{1}{x-y}$$

xy-plane: $\underset{z=0}{}\ 0=1$

yz-plane: $\underset{x=0}{}\ z=-\frac{1}{y}$

xz-plane: $\underset{y=0}{}\ z=\frac{1}{x}$

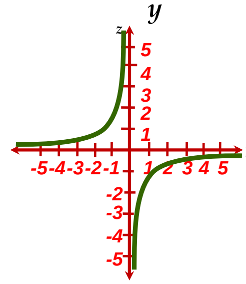
xy-plane



Traces

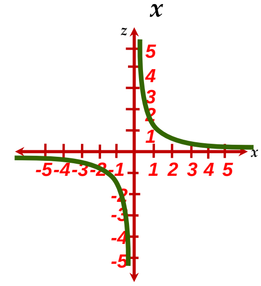
yz-plane

$$z=-\frac{1}{y}$$

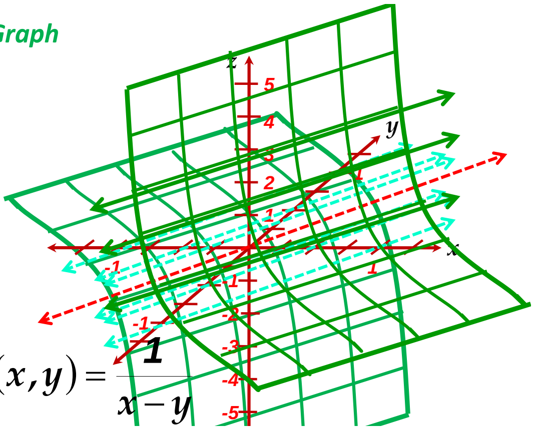


xz-plane

$$z=\frac{1}{x}$$



Graph



$$g(x,y)=\frac{1}{x-y}$$

Example. Draw a contour map of

$$f(x,y,z)=-x+y+2z$$

Consider level surfaces at

$$w=-2,-1,0,1,2$$

Reminder:

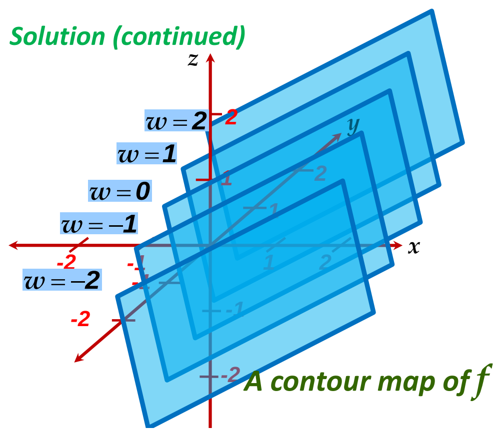
$ax+by+cz+d=0$ **is a plane.**

Solution:

$$f(x,y,z)=-x+y+2z$$

$$w=-2:\quad -x+y+2z+2=0$$
$$w=-1:\quad -x+y+2z+1=0$$
$$w=0:\quad -x+y+2z=0$$
$$w=1:\quad -x+y+2z-1=0$$
$$w=2:\quad -x+y+2z-2=0$$

Solution (continued)



Example. Draw a contour map of

$$g(x,y,z)=x^2-y+z$$

Consider level surfaces at

$$w=-2,-1,0,1,2$$

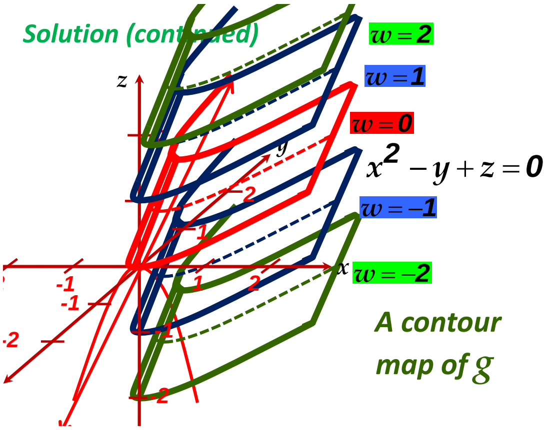
$$x^2+by+cz+d=0$$

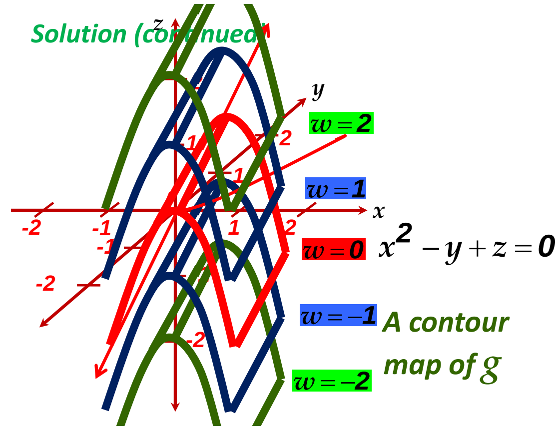
is a (slanting) parabolic cylinder.

Solution:

$$g(x,y,z)=x^2-y+z$$

$$w=-2:\quad x^2-y+z+2=0$$
$$w=-1:\quad x^2-y+z+1=0$$
$$w=0:\quad x^2-y+z=0$$
$$w=1:\quad x^2-y+z-1=0$$
$$w=2:\quad x^2-y+z-2=0$$


w = 2
w = 1
w = 0
$x^2 - y + z = 0$
w = −1
w = −2
A contour map of g


w = 2
w = 1
w = 0
$x^2 - y + z = 0$
w = −1
w = −2
A contour map of g

END