# CHAPTER 2

# TECHNIQUES OF INTEGRATION



***Objectives:***

At the end of the chapter, you should be able to

1. demonstrate understanding of ***integration by parts***,
2. determine an appropriate technique to evaluate an integral and
3. evaluate improper integrals.



## CHAPTER OUTLINE





## WARM UP!

**Evaluate the following integrals:**

1. $\int cos(2x)dx = \frac{1}{2}sin(2x)+C.$
2. $\int sin(3x)dx = \frac{-1}{3}cos(3x)+C.$
3. $\int sec^2(4x)dx = \frac{1}{4}tan(4x)+C.$
4. $\int csc^2(5x)dx = \frac{-1}{5}cot(5x)+C.$



5. $\int sec(6x)tan(6x)dx = \frac{1}{6}sec(6x)+C.$
6. $\int csc(7x)cot(7x)dx = \frac{-1}{7}csc(7x)+C.$
7. $\int e^{2x}dx = \frac{1}{2}e^{2x}+C.$
8. $\int 3^{4x}dx = \frac{3^{4x}}{4\,ln3}+C.$
9. $\int \frac{dx}{1+x} = ln|1+x|+C.$
10. $\int \frac{dx}{1+2x} = \frac{1}{2}ln|1+x|+C.$



11. $\int \frac{dx}{\sqrt{4-x^2}} = Arc\,sin\left(\frac{x}{2}\right)+C.$
12. $\int \frac{dx}{4+x^2} = \frac{1}{2}Arc\,tan\left(\frac{x}{2}\right)+C.$
13. $\int \frac{dx}{4+9x^2} = \frac{1}{6}Arc\,tan\left(\frac{3x}{2}\right)+C.$
14. $\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}Arc\,sec\left(\frac{x}{3}\right)+C.$
15. $\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}Arc\,sec\left(\frac{2x}{3}\right)+C.$

There are basically two methods of integration:

1. integration by parts
2. substitution/transformation



## 2.1 Integration by parts

From MATH 36, we learned that

$$d(uv) = udv + vdu$$

Equivalently,

$$udv = d(uv) - vdu.$$

Integrating both sides,

$$\int udv = \int d(uv) - \int vdu,$$

$$\boxed{\int udv = uv - \int vdu.}$$



**Example 2.1.1** Evaluate $\int \ln x dx.$

*solution:*

$$\int \ln x dx = \int udv \; = uv - \int vdu$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}dx, & v = x \end{array}}$$

$$= (\ln x)x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$



*Ans.* $\int \ln x dx = x \ln x - x + C.$

*Checking:*

$$\frac{d(x \ln x - x)}{dx} = \cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x - 1$$

$$= 1 + \ln x - 1$$

$$= \ln x.$$



**Example 2.1.2** Evaluate $\int x \cos x dx.$

*solution:*

$$\int x \cos x dx = \int udv = uv - \int vdu$$

There are two possibilities:

$$\boxed{\begin{array}{ll} u = x & dv = \cos x dx \\ du = dx & v = \sin x \end{array}} \qquad \boxed{\begin{array}{ll} u = \cos x & dv = x dx \\ du = -\sin x dx & v = \frac{x^2}{2} \end{array}}$$

The second possibility will result to a more complicated integral so we consider the first one.



**Example 2.1.2** Evaluate $\int x \cos x dx.$

*solution:*

$$\int x \cos x dx = \int udv \; = uv - \int vdu$$

$$\boxed{\begin{array}{ll} u = x, & dv = \cos x dx \\ du = dx, & v = \sin x \end{array}}$$

$$= x \cdot \sin x - \int \sin x dx$$

$$= x \cdot \sin x - -\cos x + C$$

$$= x \sin x + \cos x + C.$$

*Ans.* $\int x\cos x dx = x\sin x + \cos x + C.$

*Checking:*

$$\frac{d(x\sin x + \cos x)}{dx} = \frac{d(x\sin x)}{dx} + \frac{d(\cos x)}{dx}$$
$$= x\cdot\cos x + \cancel{\sin x} - \cancel{\sin x}$$
$$= x\cos x.$$



**Example 2.1.3** Evaluate $\int x^2 \cos x dx.$

*solution:*

$$\int x^2\cos x dx = \int u dv \quad = uv - \int v du$$

$$\boxed{\begin{array}{ll} u = x^2, & dv = \cos x dx \\ du = 2x dx, & v = \sin x \end{array}}$$

$$= x^2\cdot\sin x - \int \sin x\cdot 2x dx$$
$$= x^2\sin x - 2\underline{\int x\sin x dx}$$



$$\int x\sin x dx \quad = \int u dv \quad = uv - \int v du$$

$$\boxed{\begin{array}{ll} u = x, & dv = \sin x dx \\ du = dx, & v = -\cos x \end{array}}$$

$$= x(-\cos x) - \int -\cos x dx$$
$$= -x\cos x + \int \cos x dx$$
$$= -x\cos x + \sin x + k.$$

$$\int x^2\cos x dx \quad = x^2\sin x - 2\int x\sin x dx$$
$$= x^2\sin x - 2(-x\cos x + \sin x + k)$$
$$= x^2\sin x + 2x\cos x - 2\sin x + C.$$



*Ans.* $\int x^2\cos x dx = x^2\sin x + 2x\cos x - 2\sin x + C.$

*Checking:*

$$\frac{d(x^2\sin x + 2x\cos x - 2\sin x + C)}{dx}$$
$$= x^2\cdot\cos x + \sin x\cdot 2x + 2x\cdot -\sin x + \cos x\cdot 2 - 2\cos x$$
$$= x^2\cos x + \cancel{2x\sin x} - \cancel{2x\sin x} + \cancel{2\cos x} - \cancel{2\cos x}$$
$$= x^2\cos x.$$



**Example 2.1.4** Evaluate $\int \sec^3 x dx.$

*solution:*

$$\int \sec^3 x dx = \int \sec x\cdot\sec^2 x dx$$

$$\boxed{\begin{array}{ll} u = \sec x, & dv = \sec^2 x dx \\ du = \sec x\tan x dx, & v = \tan x \end{array}}$$

$$\int \sec^3 x dx \quad = \int u dv \quad = uv - \int v du$$
$$= \sec x\tan x - \int \tan x\cdot\sec x\tan x dx$$



$$\int \sec^3 x dx = \sec x\tan x - \int \sec x\tan^2 x dx$$
$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)dx$$
$$= \sec x\tan x - \int \sec^3 x dx + \int \sec x dx$$
$$= \sec x\tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|$$
$$= \sec x\tan x + \ln|\sec x + \tan x| - \int \sec^3 x dx$$
$$2\int \sec^3 x dx = \sec x\tan x + \ln|\sec x + \tan x|$$
$$\int \sec^3 x dx = \frac{\sec x\tan x + \ln|\sec x + \tan x|}{2} + C.$$

$$\int sec^3 xdx = \frac{sec\,x\tan x + \ln|sec\,x + \tan x|}{2} + C.$$

***Checking:***

$$\frac{1}{2}\frac{d\left(sec\,x\tan x + \ln|sec\,x + \tan x|\right)}{dx}$$

$$= \frac{1}{2}\Big[\left(sec\,x \cdot sec^2 x + \tan x \cdot sec\,x\tan x\right) + \frac{1}{sec\,x + \tan x}\cdot\left(sec\,x\tan x + sec^2 x\right)\Big]$$

$$= \frac{1}{2}\left[\left(sec\,x \cdot sec^2 x + \tan x \cdot sec\,x\tan x\right) + \frac{sec\,x\cancel{(sec\,x + \tan x)}}{\cancel{sec\,x + \tan x}}\right]$$

$$= \frac{1}{2}\left[sec^3 x + sec\,x\tan^2 x + sec\,x\right]$$

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$$= \frac{1}{2}\left[sec^3 x + sec\,x\tan^2 x + sec\,x\right]$$

$$= \frac{1}{2}\left[sec^3 x + sec\,x\left(sec^2 x - 1\right) + sec\,x\right]$$

$$= \frac{1}{2}\left[sec^3 x + sec^3 x - \cancel{sec\,x} + \cancel{sec\,x}\right]$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2}sec^3 x\right)$$

$= sec^3 x.$ **YES!**

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**In using integration parts, there are three possibilities:**

**1. After applying the formula,**

$$\int udv = uv - \int vdu,$$

$\int vdu$ **can be easily evaluated as in Examples 2.1.1 and 2.1.2.**

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**2. Successive applications of integration by parts are necessary as in Example 2.1.3.**

**3. After applying the formula**

$$\int udv = uv - \int vdu,$$

**a multiple of** $\int udv$ **appears on the right-hand side as in Example 2.1.4.**

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## Additional examples of integrals under case 1

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**Example 2.1.5** Evaluate $\int Arc\sin xdx$ .

***solution:***

$$\int Arc\sin xdx = \int udv = uv - \int vdu$$

$$\boxed{\begin{array}{ll} u = Arc\sin x, & dv = dx \\ du = \dfrac{dx}{\sqrt{1-x^2}}, & v = x \end{array}}$$

$$\int Arc\sin xdx = \left(Arc\sin x\right)x - \int x\cdot\frac{dx}{\sqrt{1-x^2}}$$

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$$\int Arc\sin x\,dx = (Arc\sin x)x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$\boxed{\begin{array}{l} s = 1-x^2 \\ ds = -2x\,dx \\ \dfrac{-ds}{2} = x\,dx \end{array}}$$

$$= xArc\sin x + \frac{1}{2}\int \frac{ds}{s^{1/2}}$$

$$= xArc\sin x + \frac{1}{2}\int s^{-1/2}ds$$

$$= xArc\sin x + \frac{1}{\cancel{2}}\cdot\frac{s^{1/2}}{1/\cancel{2}} + C$$

$$= xArc\sin x + (1-x^2)^{1/2} + C$$

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Ans. $\int Arc\sin x\,dx = xArc\sin x + (1-x^2)^{1/2} + C$

**Checking:**

$$\frac{d\left(xArc\sin x + (1-x^2)^{1/2}\right)}{dx}$$

$$= x\cdot\frac{1}{\sqrt{1-x^2}} + Arc\sin x + \frac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot -\cancel{2}x$$

$$= \cancel{\frac{x}{\sqrt{1-x^2}}} + Arc\sin x - \cancel{\frac{x}{\sqrt{1-x^2}}} = Arc\sin x.$$

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**Example 2.1.6** Evaluate $\int Arc\tan(2x)dx$ .

***solution:***

$$\int Arc\tan(2x)dx = \int u\,dv = uv - \int v\,du$$

$$\boxed{\begin{array}{ll} u = Arc\tan(2x), & dv = dx \\ du = \dfrac{2dx}{1+4x^2}, & v = x \end{array}}$$

$$\int Arc\tan(2x)dx = (Arc\tan(2x))x - \int x\cdot\frac{2dx}{1+4x^2}$$

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$$\int Arc\tan(2x)dx = (Arc\tan(2x))x - \int x\cdot\frac{2dx}{1+4x^2}$$

$$\boxed{\begin{array}{l} s = 1+4x^2 \\ ds = 8x\,dx \\ \dfrac{ds}{8} = x\,dx \end{array}}$$

$$= xArc\tan(2x) - 2\int\frac{x\,dx}{1+4x^2}$$

$$= xArc\tan(2x) - 2\cdot\frac{1}{8}\int\frac{ds}{s}$$

$$= xArc\tan(2x) - \frac{1}{4}ln|s| + C$$

$$= xArc\tan(2x) - \frac{1}{4}ln|1+4x^2| + C$$

$$= xArc\tan(2x) - \frac{1}{4}ln(1+4x^2) + C$$

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$$\int Arc\tan(2x)dx = xArc\tan(2x) - \frac{1}{4}ln(1+4x^2) + C$$

**Checking:**

$$\frac{d\left(xArc\tan(2x) - \frac{1}{4}ln(1+4x^2)\right)}{dx}$$

$$= x\cdot\frac{2}{1+4x^2} + Arc\tan(2x) - \frac{1}{4}\cdot\frac{8x}{1+4x^2}$$

$$= \cancel{\frac{2x}{1+4x^2}} + Arc\tan(2x) - \cancel{\frac{2x}{1+4x^2}}$$

$$= Arc\tan(2x)$$

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## Additional examples

## of integrals under case 2

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**Example 2.1.7.** Evaluate $\int x^4 \cos(2x)dx$.

***solution:***

$$\int x^4 \cos(2x)dx = \int u\,dv$$

$$\boxed{\begin{array}{ll} u = x^4, & dv = \cos(2x)dx \\ du = 4x^3dx, & v = \frac{1}{2}\sin(2x) \end{array}}$$

$$= uv - \int v\,du$$

$$= x^4 \cdot \frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\cdot 4x^3dx$$

$$= \frac{x^4 \sin(2x)}{2} - 2\int \underline{x^3 \sin(2x)dx}$$



$$\int x^3 \sin(2x)dx = \int u\,dv$$

$$\boxed{\begin{array}{ll} u = x^3, & dv = \sin(2x)dx \\ du = 3x^2dx, & v = \frac{-1}{2}\cos(2x) \end{array}}$$

$$= uv - \int v\,du$$

$$= x^3 \cdot \frac{-1}{2}\cos(2x) - \int \frac{-1}{2}\cos(2x)\cdot 3x^2dx$$

$$= \underline{\frac{-x^3 \cos(2x)}{2} + \frac{3}{2}\int x^2 \cos(2x)dx}$$

$$\int x^4 \cos(2x)dx = \frac{x^4 \sin(2x)}{2} - 2\int \underline{x^3 \sin(2x)dx}$$

$$= \frac{x^4 \sin(2x)}{2} - 2\left(\frac{-x^3 \cos(2x)}{2} + \frac{3}{2}\int x^2 \cos(2x)dx\right)$$



$$= \frac{x^4 \sin(2x)}{2} - 2\left(\frac{-x^3 \cos(2x)}{2} + \frac{3}{2}\int x^2 \cos(2x)dx\right)$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - 3\int \underline{x^2 \cos(2x)dx}$$

$$\int x^2 \cos(2x)dx$$

$$\boxed{\begin{array}{ll} u = x^2, & dv = \cos(2x)dx \\ du = 2xdx, & v = \frac{1}{2}\sin(2x) \end{array}}$$

$$= \int u\,dv$$

$$= uv - \int v\,du$$

$$= x^2 \cdot \frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\cdot 2xdx$$

$$= \underline{\frac{x^2 \sin(2x)}{2} - \int x \sin(2x)dx}$$



$$\int x^2 \cos(2x)dx = \frac{x^2 \sin(2x)}{2} - \int x \sin(2x)dx$$

$$\int x^4 \cos(2x)dx$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - 3\int x^2 \cos(2x)dx$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - 3\left(\frac{x^2 \sin(2x)}{2} - \int x \sin(2x)dx\right)$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - \frac{3x^2 \sin(2x)}{2} + 3\int \underline{x \sin(2x)dx}$$



$$\int x \sin(2x)dx$$

$$\boxed{\begin{array}{ll} u = x, & dv = \sin(2x)dx \\ du = dx, & v = \frac{-1}{2}\cos(2x) \end{array}}$$

$$= \int u\,dv = uv - \int v\,du$$

$$= x \cdot \frac{-1}{2}\cos(2x) - \int \frac{-1}{2}\cos(2x)\cdot dx$$

$$= \frac{-x\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)dx$$

$$= \frac{-x\cos(2x)}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\sin(2x)$$

$$= \frac{-x\cos(2x)}{2} + \frac{1}{4}\cos(2x) + k$$



$$\int x^4 \cos(2x)dx$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - \frac{3x^2 \sin(2x)}{2} + 3\int x \sin(2x)dx$$

$$= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - \frac{3x^2 \sin(2x)}{2}$$

$$+ 3\left(\frac{-x\cos(2x)}{2} + \frac{\cos(2x)}{4} + k\right)$$

$$\boxed{= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - \frac{3x^2 \sin(2x)}{2} - \frac{3\cos(2x)}{2} + \frac{3\sin(2x)}{4} + C}$$

## Another solution …



$$\int x^4 \cos(2x)dx = \int udv$$

$$= x^4 \cdot \tfrac{1}{2}\sin(2x) - 4x^3 \cdot \tfrac{-1}{4}\cos(2x) + 12x^2 \cdot \tfrac{-1}{8}\sin(2x) - 24x \cdot \tfrac{1}{16}\cos(2x) + 24 \cdot \tfrac{1}{32}\sin(2x) + C$$

$$\boxed{= \frac{x^4 \sin(2x)}{2} + x^3\cos(2x) - \frac{3x^2 \sin(2x)}{2} - \frac{3x\cos(2x)}{2} + \frac{3\sin(2x)}{4} + C}$$

| $u$ | | $dv$ |
|---|---|---|
| $x^4$ | + | $\cos(2x)dx$ |
| $4x^3$ | − | $\tfrac{1}{2}\sin(2x)$ |
| $12x^2$ | + | $\tfrac{-1}{4}\cos(2x)$ |
| $24x$ | − | $\tfrac{-1}{8}\sin(2x)$ |
| $24$ | + | $\tfrac{1}{16}\cos(2x)$ |
| $0$ | | $\tfrac{1}{32}\sin(2x)$ |



**Example 2.1.8.** Evaluate $\int x^5 e^{3x}dx.$

$$\int x^5 e^{3x}dx = \int udv$$

$$= x^5 \cdot \tfrac{1}{3}e^{3x} - 5x^4 \cdot \tfrac{1}{9}e^{3x} + 20x^3 \cdot \tfrac{1}{27}e^{3x} - 60x^2 \cdot \tfrac{1}{81}e^{3x} + 120x \cdot \tfrac{1}{243}e^{3x} - 120 \cdot \tfrac{1}{729}e^{3x} + C$$

$$\boxed{= \frac{x^5 e^{3x}}{3} - \frac{5x^4 e^{3x}}{9} + \frac{20x^3 e^{3x}}{27} - \frac{20x^2 e^{3x}}{27} + \frac{40xe^{3x}}{81} - \frac{40e^{3x}}{243} + C.}$$

| $u$ | | $dv$ |
|---|---|---|
| $x^5$ | + | $e^{3x}dx$ |
| $5x^4$ | − | $\tfrac{1}{3}e^{3x}$ |
| $20x^3$ | + | $\tfrac{1}{9}e^{3x}$ |
| $60x^2$ | − | $\tfrac{1}{27}e^{3x}$ |
| $120x$ | + | $\tfrac{1}{81}e^{3x}$ |
| $120$ | − | $\tfrac{1}{243}e^{3x}$ |
| $0$ | | $\tfrac{1}{729}e^{3x}$ |



## Additional examples of integrals under case 3



**Example 2.1.9** Evaluate $\int \csc^3 xdx.$

*solution:*

$$\int \csc^3 xdx = \int \csc x \cdot \csc^2 xdx$$

$$\boxed{\begin{array}{l} u = \csc x, \quad dv = \csc^2 xdx \\ du = -\csc x \cot xdx, \; v = -\cot x \end{array}}$$

$$\int \csc^3 xdx = \int udv = uv - \int vdu$$

$$= \csc x \cdot -\cot x - \int -\cot x \cdot -\csc x \cot xdx$$



$$\int \csc^3 xdx = -\csc x \cot x - \int \csc x \cot^2 xdx$$

$$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)dx$$

$$= -\csc x \cot x - \int (\csc^3 x - \csc x)dx$$

$$= -\csc x \cot x - \int \csc^3 xdx + \int \csc xdx$$

$$\int \csc^3 xdx + \int \csc^3 xdx = -\csc x \cot x + \int \csc xdx$$

$$2\int \csc^3 xdx = -\csc x \cot x + \ln|\csc x - \cot x|$$

$$\int \csc^3 xdx = \frac{-\csc x \cot x + \ln|\csc x - \cot x|}{2} + C$$

**Example 2.1.10** Evaluate $\int \sin(2x)\cos(3x)dx.$

***solution:***

$$\int \sin(2x)\cos(3x)dx = \int udv$$

$$\boxed{\begin{array}{ll} u = \sin(2x), & dv = \cos(3x)dx \\ du = 2\cos(2x)dx & v = \frac{1}{3}\sin(3x) \end{array}}$$

$$\int \sin(2x)\cos(3x)dx = uv - \int vdu$$

$$= \sin(2x)\cdot\frac{1}{3}\sin(3x) - \int \frac{1}{3}\sin(3x)\cdot 2\cos(2x)dx$$



$$\int \sin(2x)\cos(3x)dx$$

$$= \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\int \cos(2x)\sin(3x)dx$$

$$\int \cos(2x)\sin(3x)dx = \int udv$$

$$\boxed{\begin{array}{ll} u = \cos(2x), & dv = \sin(3x)dx \\ du = -2\sin(2x)dx & v = -\frac{1}{3}\cos(3x) \end{array}}$$

$$\int \cos(2x)\sin(3x)dx = uv - \int vdu$$

$$= \cos(2x)\cdot -\frac{1}{3}\cos(3x) - \int -\frac{1}{3}\cos(3x)\cdot -2\sin(2x)dx$$



$$\int \cos(2x)\sin(3x)dx$$

$$= -\cdot\frac{1}{3}\cos(2x)\cos(3x) - \frac{2}{3}\int \sin(2x)\cos(3x)dx$$

$$\int \sin(2x)\cos(3x)dx$$

$$= \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\int \cos(2x)\sin(3x)dx$$

$$\int \sin(2x)\cos(3x)dx = \frac{1}{3}\sin(2x)\sin(3x)$$

$$-\frac{2}{3}\left(-\cdot\frac{1}{3}\cos(2x)\cos(3x) - \frac{2}{3}\int \sin(2x)\cos(3x)dx\right)$$



$$\int \sin(2x)\cos(3x)dx = \frac{1}{3}\sin(2x)\sin(3x)$$

$$+\frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)dx$$

$$\left(1 - \frac{4}{9}\right)\int \sin(2x)\cos(3x)dx$$

$$= \frac{1}{3}\sin(2x)\sin(2x) + \frac{2}{9}\cos(2x)\cos(3x)$$

$$\frac{5}{9}\int \sin(2x)\cos(3x)dx = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)$$



$$\int \sin(2x)\cos(3x)dx = \frac{9}{5}\left(\frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\right)$$

$$\int \sin(2x)\cos(3x)dx = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C$$



## 2.2 Powers of trigonometric functions

A. $\int \sin^m u \cos^n u\, du$

**Case 1.** ***m*** **or** ***n*** **is odd.**

**If *m* is odd, write** $\sin^m u$ **as**

$$\sin^{m-1} u \cdot \sin u$$

**and express the remaining even powers of** $\sin u$ **in terms of** $\cos u$**.**

A. $\int \sin^m u \cos^n u\, du$

If $n$ is odd, write $\cos^n u$ as

$\cos^{n-1} u \cdot \cos u$

and express the remaining even powers of $\cos u$ in terms of $\sin u$.

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**Example 2.2.1** Evaluate $\int \sin^3(2x)\cos^2(2x)dx$.

*solution:*

$$\boxed{\begin{aligned} u &= \cos(2x) \\ du &= -2\sin(2x)dx \\ \frac{-du}{2} &= \sin(2x)dx \end{aligned}}$$

$$\begin{aligned}
&\int \sin^3(2x)\cos^2(2x)dx \\
&= \int \sin^2(2x)\cos^2(2x)\cdot \sin(2x)dx \\
&= \int \left[1-\cos^2(2x)\right]\cos^2(2x)\sin(2x)dx \\
&= \int \left[\cos^2(2x)-\cos^4(2x)\right]\sin(2x)dx \\
&= \int \left(u^2-u^4\right)\cdot\frac{-du}{2}
\end{aligned}$$

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$$\begin{aligned}
\int \sin^3(2x)\cos^2(2x)dx &= \frac{-1}{2}\int \left(u^2-u^4\right)du \\
&= \frac{-1}{2}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C \\
&= \frac{-u^3}{6}+\frac{u^5}{10}+C \\
&\boxed{= \frac{-\cos^3(2x)}{6}+\frac{\cos^5(2x)}{10}+C.}
\end{aligned}$$

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**Example 2.2.2** Evaluate $\int \sin^2(2x)\cos^5(2x)dx$.

*solution:*

$$\begin{aligned}
&\int \sin^2(2x)\cos^5(2x)dx \\
&= \int \sin^2(2x)\cos^4(2x)\cos(2x)dx \\
&= \int \sin^2(2x)\left[\cos^2(2x)\right]^2\cos(2x)dx \\
&= \int \sin^2(2x)\left[1-\sin^2(2x)\right]^2\cos(2x)dx \\
&= \int \sin^2(2x)\left[1-2\sin^2(2x)+\sin^4(2x)\right]\cos(2x)dx
\end{aligned}$$

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$$\begin{aligned}
&= \int \left[\sin^2(2x)-2\sin^4(2x)+\sin^6(2x)\right]\cos(2x)dx \\
&= \int \left(u^2-2u^4+u^6\right)\frac{du}{2} \\
&= \frac{1}{2}\int \left(u^2-2u^4+u^6\right)du \\
&= \frac{1}{2}\left(\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right)+C \\
&= \frac{u^3}{6}-\frac{u^5}{5}+\frac{u^7}{14}+C \\
&\boxed{= \frac{\sin^3(2x)}{6}-\frac{\sin^5(2x)}{5}+\frac{\sin^7(2x)}{14}+C.}
\end{aligned}$$

$$\boxed{\begin{aligned} u &= \sin(2x) \\ du &= 2\cos(2x)dx \\ \frac{du}{2} &= \cos(2x)dx \end{aligned}}$$

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**Case 2. $m$ and $n$ are even.**

Use the following half-angle formulas:

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

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**Example 2.2.3** Evaluate $\int sin^2(2x)cos^2(2x)dx$ .

***solution:***

$$\int sin^2(2x)cos^2(2x)dx = \int\left(\frac{1-cos(4x)}{2}\right)\left(\frac{1+cos(4x)}{2}\right)dx$$

$$= \frac{1}{4}\int\left(1-cos^2(4x)\right)dx \quad = \frac{1}{4}\int\left(1-\left(\frac{1+cos(8x)}{2}\right)\right)dx$$

$$= \frac{1}{4}\int\left(1-\frac{1}{2}-\frac{1}{2}\cdot cos(8x)\right)dx \quad = \frac{1}{4}\int\left(\frac{1}{2}-\frac{1}{2}\cdot cos(8x)\right)dx$$

$$= \frac{1}{4}\left(\frac{1}{2}\cdot x-\frac{1}{2}\cdot\frac{1}{8}sin(8x)\right)+C \quad \boxed{= \frac{x}{8}-\frac{sin(8x)}{64}+C}$$



**Example 2.2.4** Evaluate $\int sin^6 xdx$ .

***solution:***

$$\int sin^6 xdx = \int\left(sin^2 x\right)^3 dx$$

$$= \int\left(\frac{1-cos(2x)}{2}\right)^3 dx \quad = \int\frac{(1-cos(2x))^3}{2^3}dx$$

$$= \frac{1}{8}\int\left(1-3cos(2x)+3cos^2(2x)-cos^3(2x)\right)dx$$

$$= \frac{1}{8}\int\left(1-3cos(2x)+3\cdot\frac{1+cos(4x)}{2}-cos^3(2x)\right)dx$$

$$= \frac{1}{8}\int\left(1-3cos(2x)+\frac{3}{2}+\frac{3}{2}cos(4x)-cos^3(2x)\right)dx$$



$$\int sin^6 xdx$$

$$= \frac{1}{8}\int\left(\frac{5}{2}-3cos(2x)+\frac{3}{2}cos(4x)-cos^3(2x)\right)dx$$

$$= \frac{1}{8}\left(\frac{5}{2}x-\frac{3}{2}sin(2x)+\frac{3}{8}sin(4x)\right)-\frac{1}{8}\int cos^3(2x)dx$$

$$= \frac{5}{16}x-\frac{3}{16}sin(2x)+\frac{3}{64}sin(4x)-\frac{1}{8}\int cos^3(2x)dx$$



$$\int cos^3(2x)dx = \int cos^2(2x)\cdot \underline{cos(2x)}dx$$

$$= \int\left(1-sin^2(2x)\right)\underline{cos(2x)}dx$$

$$= \int cos(2x)dx - \int sin^2(2x)\cdot cos(2x)dx$$

$$= \frac{1}{2}sin(2x)-\frac{1}{2}\int u^2du$$

$$= \frac{1}{2}sin(2x)-\frac{1}{2}\cdot\frac{u^3}{3}+k$$

$$= \frac{1}{2}sin(2x)-\frac{sin^3(2x)}{6}+k$$



$$\int sin^6 xdx$$

$$= \frac{5}{16}x-\frac{3}{16}sin(2x)+\frac{3}{64}sin(4x)-\frac{1}{8}\int cos^3(2x)dx$$

$$= \frac{5}{16}x-\frac{3}{16}sin(2x)+\frac{3}{64}sin(4x) -\frac{1}{8}\left(\frac{1}{2}sin(2x)-\frac{sin^3(2x)}{6}+k\right)$$

$$= \frac{5}{16}x-\frac{3}{16}sin(2x)+\frac{3}{64}sin(4x)-\frac{sin(2x)}{16}+\frac{sin^3(2x)}{48}+C$$

$$= \frac{5x}{16}-\frac{sin(2x)}{4}+\frac{3}{64}sin(4x)+\frac{sin^3(2x)}{48}+C$$



## B. $\int sec^m u\, tan^n u\, du$

**Case 1. *m* is even.**

**If *m* is even, write $sec^m u$ as**

$$sec^{m-2} u\cdot sec^2 u$$

**and express the remaining even powers of *sec u* in terms of *tan u*.**

**Example 2.2.5** Evaluate $\int sec^4(2x)tan(2x)dx$.

***solution:***

$$u = tan(2x)$$
$$du = 2\,sec^2(2x)dx$$
$$\frac{du}{2} = sec^2(2x)dx$$

$$\int sec^4(2x)tan(2x)dx$$
$$= \int sec^2(2x)tan(2x)\underline{sec^2(2x)dx}$$
$$= \int \left(1+tan^2(2x)\right)tan(2x)\underline{sec^2(2x)dx}$$
$$= \int \left(tan(2x)+tan^3(2x)\right)\underline{sec^2(2x)dx}$$
$$= \int \left(u+u^3\right)\cdot\frac{du}{2}$$

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$$\int sec^4(2x)tan(2x)dx = \frac{1}{2}\int\left(u+u^3\right)du$$
$$= \frac{1}{2}\left(\frac{u^2}{2}+\frac{u^4}{4}\right)+C$$
$$= \frac{u^2}{4}+\frac{u^4}{8}+C$$
$$= \frac{tan^2(2x)}{4}+\frac{tan^4(2x)}{8}+C.$$

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**Example 2.2.6** Evaluate $\int sec^6(2x)dx$ .

***solution:***

$$u = tan(2x)$$
$$du = 2\,sec^2(2x)dx$$
$$\frac{du}{2} = sec^2(2x)dx$$

$$\int sec^6(2x)dx$$
$$= \int sec^4(2x)\underline{sec^2(2x)dx}$$
$$= \int \left(sec^2(2x)\right)^2\underline{sec^2(2x)dx}$$
$$= \int \left(1+tan^2(2x)\right)^2\underline{sec^2(2x)dx}$$
$$= \int \left(1+2\,tan^2(2x)+tan^4(2x)\right)\underline{sec^2(2x)dx}$$
$$= \int \left(1+2u^2+u^4\right)\cdot\frac{du}{2}$$

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$$\int sec^6(2x)dx = \frac{1}{2}\int\left(1+2u^2+u^4\right)du$$
$$= \frac{1}{2}\left(u+\frac{2u^3}{3}+\frac{u^5}{5}\right)+C$$
$$= \frac{u}{2}+\frac{u^3}{3}+\frac{u^5}{10}+C$$
$$= \frac{tan(2x)}{2}+\frac{tan^3(2x)}{3}+\frac{tan^5(2x)}{10}+C$$

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B. $\int sec^m u\,tan^n u\,du$

**Case 2. *n* is odd.**

**If *n* is odd, factor out $sec\,u\,tan\,u$ and express the remaining even powers of $tan\,u$ in terms of $sec\,u$.**

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**Example 2.2.7** Evaluate $\int sec^3 x\,tan^3 x\,dx$.

***solution:***

$$\int sec^3 x\,tan^3 x\,dx$$
$$= \int sec^2 x\,tan^2 x\cdot \underline{sec\,x\,tan\,x\,dx}$$
$$= \int sec^2 x\left(sec^2 x-1\right)\underline{sec\,x\,tan\,x\,dx}$$
$$= \int \left(sec^4 x-sec^2 x\right)\underline{sec\,x\,tan\,x\,dx}$$
$$= \int \left(u^4-u^2\right)du$$
$$= \frac{u^5}{5}-\frac{u^3}{3}+C = \frac{sec^5 x}{5}-\frac{sec^3 x}{3}+C$$

$$u = sec\,x$$
$$du = sec\,x\,tan\,x\,dx$$

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**Example 2.2.8** Evaluate $\int tan^5(2x)dx$ .

***solution:***

$$\int tan^5(2x)dx$$
$$= \int sec^{-1}(2x)tan^4(2x)\cdot \underline{sec(2x)tan(2x)}dx$$
$$= \int sec^{-1}(2x)[tan^2(2x)]^2 \cdot \underline{sec(2x)tan(2x)}dx$$
$$= \int sec^{-1}(2x)[sec^2(2x)-1]^2 \cdot \underline{sec(2x)tan(2x)}dx$$
$$= \int sec^{-1}(2x)[sec^4(2x)-2sec^2(2x)+1]\underline{sec(2x)tan(2x)}dx$$

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$$\int tan^5(2x)dx$$
$$= \int sec^{-1}(2x)[sec^4(2x)-2sec^2(2x)+1]\underline{sec(2x)tan(2x)}dx$$
$$= \int [sec^3(2x)-2sec(2x)+sec^{-1}(2x)]\underline{sec(2x)tan(2x)}dx$$
$$= \frac{1}{2}\int\left(u^3-2u+\frac{1}{u}\right)du$$
$$= \frac{1}{2}\left(\frac{u^4}{4}-u^2+ln|u|\right)+C$$
$$= \frac{1}{2}\left(\frac{sec^4(2x)}{4}-sec^2(2x)+ln|sec(2x)|\right)+C$$

$$\boxed{\begin{aligned} u &= sec(2x) \\ du &= 2sec(2x)tan(2x)dx \\ \frac{du}{2} &= sec(2x)tan(2x)dx \end{aligned}}$$

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B. $\int sec^m u\, tan^n\, udu$

**Case 3. *m* is odd and *n* is even.**

**If *m* is odd and *n* is even, express the even power of *tan u* in terms of *sec u* and use integration by parts.**

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**Example 2.2.9** Evaluate $\int sec\, x\, tan^2\, xdx$ .

***solution:***

$$\int sec\, x\, tan^2\, xdx$$
$$= \int sec\, x(sec^2 x-1)dx$$
$$= \int (sec^3 x - sec\, x)dx$$
$$= \frac{sec\, x\, tan\, x + ln|sec\, x+tan\, x|}{2} - ln|sec\, x+tan\, x| + C$$
$$= \frac{sec\, x\, tan\, x - ln|sec\, x+tan\, x|}{2} + C$$

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C. $\int csc^m u\, cot^n\, udu$

**Case 1. *m* is even.**

**If *m* is even, write $csc^m\ u$ as**

$$csc^{m-2}\ u\cdot csc^2\ u$$

**and express the remaining even powers of *csc u* in terms of *cot u*.**

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**Example 2.2.10** Evaluate $\int csc^4(2x)cot(2x)dx$.

***solution:***

$$\boxed{\begin{aligned} u &= cot(2x) \\ du &= -2csc^2(2x)dx \\ \frac{-du}{2} &= csc^2(2x)dx \end{aligned}}$$

$$\int csc^4(2x)cot(2x)dx$$
$$= \int csc^2(2x)cot(2x)csc^2(2x)dx$$
$$= \int (1+cot^2(2x))cot(2x)\underline{csc^2(2x)}dx$$
$$= \int (cot(2x)+cot^3(2x))\underline{csc^2(2x)}dx$$
$$= \int (u+u^3)\cdot\frac{-du}{2} = \frac{-1}{2}\int (u+u^3)du = \frac{-1}{2}\left(\frac{u^2}{2}+\frac{u^4}{4}\right)+C$$
$$= \frac{-u^2}{4}-\frac{u^4}{8}+C = \frac{-cot^2(2x)}{4}-\frac{cot^4(2x)}{8}+C.$$

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C. $\int csc^m u \, cot^n u \, du$

**Case 2. $n$ is odd.**

**If $n$ is odd, factor out $csc\,u\,cot\,u$ and express the remaining even powers of $cot\,u$ in terms of $csc\,u$.**



**Example 2.2.11** Evaluate $\int csc^3 x \, cot^3 x dx$ .

***solution:***

$$\int csc^3 x \, cot^3 x dx$$
$$= \int csc^2 x \, cot^2 x \cdot \underline{csc\,x\,cot\,xdx}$$
$$= \int csc^2 x(csc^2 x - 1)\underline{csc\,x\,cot\,xdx}$$
$$= \int (csc^4 x - csc^2 x)\underline{csc\,x\,cot\,xdx}$$
$$= \int (u^4 - u^2)\cdot -du = -\int (u^4 - u^2)du$$
$$= \frac{-u^5}{5} + \frac{u^3}{3} + C = \frac{-csc^5 x}{5} + \frac{csc^3 x}{3} + C$$

$$\boxed{\begin{array}{c} u = csc\,x \\ du = -csc\,x\,cot\,xdx \\ -du = csc\,x\,cot\,xdx \end{array}}$$



C. $\int csc^m u \, cot^n u \, du$

**Case 3. $m$ is odd and $n$ is even.**

**If $n$ is even, express the even power of $cot\,u$ in terms of $csc\,u$.**



**Example 2.2.12 Evaluate** $\int csc\,x\,cot^2 xdx$ .

***solution:***

$$\int csc\,x\,cot^2 xdx = \int csc\,x(csc^2 x - 1)dx = \int (\underline{csc^3 x} - csc\,x)dx$$

**From Example 2.1.9,**

$$\int csc^3 xdx = \frac{-csc\,x\,cot\,x + ln|csc\,x - cot\,x|}{2} + C$$

$$\int csc\,x\,cot^2 xdx$$
$$= \frac{-csc\,x\,cot\,x + ln|csc\,x - cot\,x|}{2} - ln|csc\,x - cot\,x| + C$$
$$= \frac{-csc\,x\,cot\,x - ln|csc\,x - cot\,x|}{2} + C$$



**Summary:**

A. $\int sin^m u \, cos^n u \, du$

- **Case 1. $m$ or $n$ is an odd + integer.**
- **Case 2. $m$ and $n$ are even + integers.**

B. $\int sec^m u \, tan^n u \, du$

- **Case 1. $m$ is an even + integer.**
- **Case 2. $n$ is an odd + integer.**
- **Case 3. $m$ is odd and $n$ is an even + integer.**

C. $\int csc^m u \, cot^n u \, du$

- **Case 1. $m$ is an even + integer.**
- **Case 2. $n$ is an odd + integer.**
- **Case 3. $m$ is odd and $n$ is an even + integer.**



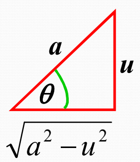
**2.3 Trigonometric substitution**

**In this section, we study how to evaluate integrals involving any of the following:**

$$\sqrt{a^2 - u^2}, \sqrt{a^2 + u^2}, \sqrt{u^2 - a^2},$$

**where $a$ is a constant and $u$ is a differentiable function of $x$.**

**Case 1.** *The integral involves* $\sqrt{a^2-u^2}$.



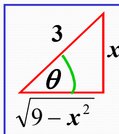

$$\sin\theta=\frac{u}{a}$$
$$u=a\sin\theta$$
$$du=a\cos\theta\,d\theta$$
$$\cos\theta=\frac{\sqrt{a^2-u^2}}{a}$$
$$\sqrt{a^2-u^2}=a\cos\theta$$



**Example 2.3.1** Evaluate $\int\sqrt{9-x^2}\,dx$ .

*solution:*

$$\int\sqrt{9-x^2}\,dx$$
$$=\int 3\cos\theta\cdot 3\cos\theta\,d\theta$$
$$=9\int\cos^2\theta\,d\theta$$
$$=9\int\frac{1+\cos 2\theta}{2}\,d\theta$$
$$=\frac{9}{2}\int(1+\cos 2\theta)\,d\theta$$



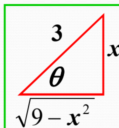

$$\sin\theta=\frac{x}{3}$$
$$x=3\sin\theta$$
$$dx=3\cos\theta\,d\theta$$
$$\cos\theta=\frac{\sqrt{9-x^2}}{3}$$
$$\sqrt{9-x^2}=3\cos\theta$$



$$\int\sqrt{9-x^2}\,dx=\frac{9}{2}\int(1+\cos 2\theta)\,d\theta$$
$$=\frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$
$$=\frac{9\theta}{2}+\frac{9\sin 2\theta}{4}+C$$
$$=\frac{9\,Arc\sin\left(\frac{x}{3}\right)}{2}+\frac{9\sin\left(2\,Arc\sin\left(\frac{x}{3}\right)\right)}{4}+C$$
$$\boxed{=\frac{9\,Arc\sin\left(\frac{x}{3}\right)}{2}+\frac{x\sqrt{9-x^2}}{2}+C}$$



$$\int\sqrt{9-x^2}\,dx$$
$$=\frac{9\theta}{2}+\frac{9\sin 2\theta}{4}+C$$
$$=\frac{9\theta}{2}+\frac{9}{4}\cdot 2\sin\theta\cos\theta+C$$
$$=\frac{9\theta}{2}+\frac{9}{2}\sin\theta\cos\theta+C$$
$$=\frac{9\,Arc\sin\left(\frac{x}{3}\right)}{2}+\frac{\cancel{9}}{2}\cdot\frac{x}{\cancel{3}}\cdot\frac{\sqrt{9-x^2}}{\cancel{3}}+C$$
$$=\frac{9\,Arc\sin\left(\frac{x}{3}\right)}{2}+\frac{x\sqrt{9-x^2}}{2}+C$$




$$\sin\theta=\frac{x}{3}$$
$$x=3\sin\theta$$
$$dx=3\cos\theta\,d\theta$$
$$\cos\theta=\frac{\sqrt{9-x^2}}{3}$$
$$\sqrt{9-x^2}=3\cos\theta$$

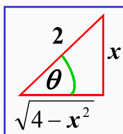


**Example 2.3.2** Evaluate

$$\int\frac{dx}{\left(\sqrt{4-x^2}\right)^3}.$$

*solution:*

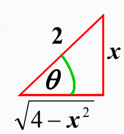
$$\int\frac{dx}{\left(\sqrt{4-x^2}\right)^3}=\int\frac{2\cos\theta\,d\theta}{(2\cos\theta)^3}$$
$$=\int\frac{2\cos\theta\,d\theta}{8\cos^3\theta}$$
$$=\frac{1}{4}\int\frac{d\theta}{\cos^2\theta}$$



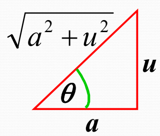

$$\sin\theta=\frac{x}{2}$$
$$x=2\sin\theta$$
$$dx=2\cos\theta\,d\theta$$
$$\cos\theta=\frac{\sqrt{4-x^2}}{2}$$
$$\sqrt{4-x^2}=2\cos\theta$$



$$\int\frac{dx}{\left(\sqrt{4-x^2}\right)^3}=\frac{1}{4}\int\frac{d\theta}{\cos^2\theta}$$
$$=\frac{1}{4}\int\sec^2\theta\,d\theta$$
$$=\frac{1}{4}\tan\theta+C$$
$$=\frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}}+C$$
$$=\frac{x}{4\sqrt{4-x^2}}+C$$



**Case 2.** *The integral involves* $\sqrt{a^2+u^2}$.




$$\tan\theta=\frac{u}{a}$$

$$u=a\tan\theta$$

$$du=a\sec^2\theta\,d\theta$$

$$\sec\theta=\frac{\sqrt{a^2+u^2}}{a}$$

$$\sqrt{a^2+u^2}=a\sec\theta$$

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**Example 2.3.3** Evaluate

$$\int\frac{dx}{\sqrt{9+x^2}}.$$

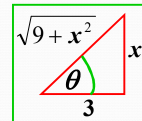
*solution:*

$$\int\frac{dx}{\sqrt{9+x^2}}=\int\frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$=\int\sec\theta\,d\theta$$

$$=\ln\left|\sec\theta+\tan\theta\right|+C$$

$$\boxed{=\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|+C}$$




$$\tan\theta=\frac{x}{3}$$

$$x=3\tan\theta$$

$$dx=3\sec^2\theta\,d\theta$$

$$\sec\theta=\frac{\sqrt{9+x^2}}{3}$$

$$\sqrt{9+x^2}=3\sec\theta$$

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**Example 2.3.4** Evaluate

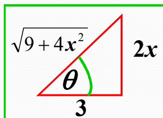
$$\int\sqrt{9+4x^2}\,dx.$$

*solution:*

$$\int\sqrt{9+4x^2}\,dx$$

$$=\int 3\sec\theta\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\frac{9}{2}\int\sec^3\theta\,d\theta$$




$$\tan\theta=\frac{2x}{3}$$

$$x=\frac{3}{2}\tan\theta$$

$$dx=\frac{3}{2}\sec^2\theta\,d\theta$$

$$\sec\theta=\frac{\sqrt{9+4x^2}}{3}$$

$$\sqrt{9+4x^2}=3\sec\theta$$

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$$=\frac{9}{2}\left(\frac{\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|}{2}\right)+C$$

$$=\frac{9}{4}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)+C$$

$$\tan\theta=\frac{2x}{3}\qquad\sec\theta=\frac{\sqrt{9+4x^2}}{3}$$

$$=\frac{9}{4}\left(\frac{\sqrt{9+4x^2}}{3}\cdot\frac{2x}{3}+\ln\left|\frac{\sqrt{9+4x^2}}{3}+\frac{2x}{3}\right|\right)+C$$

$$=\frac{x\sqrt{9+4x^2}}{2}+\frac{9}{4}\ln\left|\frac{\sqrt{9+4x^2}}{3}+\frac{2x}{3}\right|+C$$

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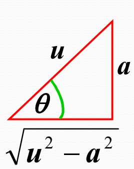
**Case 3.** *The integral involves* $\sqrt{u^2-a^2}$.




$$\csc\theta=\frac{u}{a}$$

$$u=a\csc\theta$$

$$du=-a\csc\theta\cot\theta\,d\theta$$

$$\cot\theta=\frac{\sqrt{u^2-a^2}}{a}$$

$$\sqrt{u^2-a^2}=a\cot\theta$$

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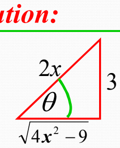
**Example 2.3.5** Evaluate $\displaystyle\int\frac{dx}{\left(\sqrt{4x^2-9}\right)^3}$

*solution:*




$$\csc\theta=\frac{2x}{3}$$

$$x=\tfrac{3}{2}\csc\theta$$

$$dx=\tfrac{-3}{2}\csc\theta\cot\theta\,d\theta$$

$$\cot\theta=\frac{\sqrt{4x^2-9}}{3}$$

$$\sqrt{4x^2-9}=3\cot\theta$$

$$\int\frac{dx}{\left(\sqrt{4x^2-9}\right)^3}$$

$$=\int\frac{\frac{-3}{2}\csc\theta\cot\theta\,d\theta}{(3\cot\theta)^3}$$

$$=\frac{-1}{18}\int\frac{\csc\theta\,d\theta}{\cot^2\theta}$$

$$=\frac{-1}{18}\int\frac{\frac{1}{\sin\theta}}{\frac{\cos^2\theta}{\sin^2\theta}}\,d\theta$$

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$$= \frac{-1}{18}\int \frac{\sin\theta}{\cos^2\theta}d\theta$$

$$= \frac{-1}{18}\int \frac{-du}{u^2} = \frac{1}{18}\int u^{-2}du \;= \frac{1}{18}\cdot\frac{-1}{u} + C$$

$$= \frac{-1}{18\cos\theta} + C \;= \frac{-\sec\theta}{18} + C$$

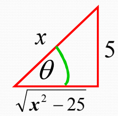
$$= \frac{-1}{18}\cdot\frac{2x}{\sqrt{4x^2-9}} + C$$

$$\boxed{= \frac{-x}{9\sqrt{4x^2-9}} + C}$$



**Example 2.3.6** Evaluate $\int \sqrt{x^2 - 25}\,dx$.

***solution:***




$\csc\theta = \frac{x}{5}$

$x = 5\csc\theta$

$dx = -5\csc\theta\cot\theta\,d\theta$

$\cot\theta = \frac{\sqrt{x^2-25}}{5}$

$\sqrt{x^2-25} = 5\cot\theta$

$$\int\sqrt{x^2-25}\,dx$$

$$= \int 5\cot\theta\cdot -5\csc\theta\cot\theta\,d\theta$$

$$= -25\int \csc\theta\cot^2\theta\,d\theta \quad \text{(Example 2.2.12)}$$

$$= -25\left(\frac{-\csc\theta\cot\theta - \ln\left|\csc\theta - \cot\theta\right|}{2}\right) + C$$

$$= \frac{25}{2}\left(\csc\theta\cot\theta + \ln\left|\csc\theta - \cot\theta\right|\right) + C$$

$$= \frac{25}{2}\left(\frac{x}{5}\cdot\frac{\sqrt{x^2-25}}{5} + \ln\left|\frac{x}{5} - \frac{\sqrt{x^2-25}}{5}\right|\right) + C$$



## 2.4 Partial Fractions

**Suppose we wish to evaluate**

$$\int \frac{P(x)}{Q(x)}dx$$

**where $P$ and $Q$ are polynomials and the degree of $P$ is less than the degree of $Q$.**



**Case 1. The factors of $Q(x)$ are linear and distinct.**

**Suppose**

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\ldots(a_nx + b_n).$$

**Then there exist constants $A_1$, $A_2$, …, $A_n$ such that**

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \ldots + \frac{A_n}{(a_nx + b_n)}.$$



**Example 2.4.1** Write $\frac{2x+1}{x^2-3x-4}$ as a sum of partial fractions.

***solution:***

$$x^2 - 3x - 4 = (x-4)(x+1)$$

$$\frac{2x+1}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$\Rightarrow \frac{2x+1}{x^2-3x-4} = \frac{A(x+1) + B(x-4)}{(x-4)(x+1)}$$

$$\Rightarrow 2x + 1 = A(x+1) + B(x-4)$$



$$2x + 1 = A(x+1) + B(x-4)$$

0 0

**When $x = 4$,**

$$2\cdot 4 + 1 = A(4+1)$$

$$9 = 5A$$

$$\boxed{A = \frac{9}{5}}$$

**When $x = -1$,**

$$2\cdot -1 + 1 = B(-1-4)$$

$$-1 = -5B$$

$$\boxed{B = \frac{1}{5}}$$

$$\frac{2x+1}{x^2-3x-4} = \frac{\frac{9}{5}}{x-4} + \frac{\frac{1}{5}}{x+1}$$

$$\frac{2x+1}{x^2-3x-4} = \frac{9}{5(x-4)} + \frac{1}{5(x+1)}$$

**Example 2.4.2** Write $\dfrac{x^2+3}{(2x-1)(x-2)(x+2)}$ as a sum of partial fractions.

*solution:*

$$\frac{x^2+3}{(2x-1)(x-2)(x+2)}=\frac{A}{2x-1}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$=\frac{A(x-2)(x+2)+B(2x-1)(x+2)+C(2x-1)(x-2)}{(2x-1)(x-2)(x+2)}$$

$$x^2+3=A(x-2)(x+2)+B(2x-1)(x+2)+C(2x-1)(x-2)$$



$$x^2+3=A(x-\overset{0}{\cancel{2}})(x+2)+B(2\overset{0}{\cancel{x}}-1)(x+2)+C(2\overset{0}{\cancel{x}}-1)(x\overset{0}{\cancel{-2}})$$

**When $x = 1/2$,**

$$\left(\tfrac{1}{2}\right)^2+3=A\left(\tfrac{1}{2}-2\right)\left(\tfrac{1}{2}+2\right)$$

$$\tfrac{1}{4}+3=A\cdot\tfrac{-3}{2}\cdot\tfrac{5}{2}$$

$$\frac{13}{\cancel{4}}=\frac{-15}{\cancel{4}}A$$

$$\boxed{A=\frac{-13}{15}}$$

**When $x = 2$,**

$$2^2+3=B(2\cdot2-1)(2+2)$$

$$7=B\cdot3\cdot4$$

$$7=12B$$

$$\boxed{B=\frac{7}{12}}$$



$$x^2+3=A(x-2)(x\overset{0}{\cancel{+2}})+B(2x-1)(x\overset{0}{\cancel{+2}})+C(2x-1)(x-2)$$

**When $x = -2$,**

$$(-2)^2+3=C(2\cdot-2-1)(-2-2)$$

$$7=C\cdot-5\cdot-4$$

$$7=20C$$

$$\boxed{C=\frac{7}{20}}$$



$$\frac{x^2+3}{(2x-1)(x-2)(x+2)}=\frac{A}{2x-1}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$A=\frac{-13}{15}\qquad B=\frac{7}{12}\qquad C=\frac{7}{20}$$

$$\frac{x^2+3}{(2x-1)(x-2)(x+2)}=\frac{\frac{-13}{15}}{2x-1}+\frac{\frac{7}{12}}{x-2}+\frac{\frac{7}{20}}{x+2}$$

$$\frac{x^2+3}{(2x-1)(x-2)(x+2)}=\frac{-13}{15(2x-1)}+\frac{7}{12(x-2)}+\frac{7}{20(x+2)}$$



Recall …

$$\int\frac{du}{u}=ln|u|+C$$

Thus,

$$\int\frac{dx}{ax+b}=\frac{1}{a}ln|ax+b|+C$$

$$u=ax+b$$
$$du=adx$$
$$\frac{1}{a}du=dx$$

For example,

$$\int\frac{dx}{3x+4}=\frac{1}{3}ln|3x+4|+C$$

$$\int\frac{dx}{5-2x}=\frac{-1}{2}ln|5-2x|+C$$



**Evaluate the following integrals mentally:**

$$\int\frac{dx}{2x+1}=\frac{1}{2}ln|2x+1|+C$$

$$\int\frac{dx}{1-2x}=\frac{-1}{2}ln|1-2x|+C$$

$$\int\frac{dx}{4x+5}=\frac{1}{4}ln|4x+5|+C$$

$$\int\frac{dx}{5-4x}=\frac{-1}{4}ln|5-4x|+C$$

**Example 2.4.3** Evaluate $\int \frac{2x+1}{x^2-3x-4}dx$

***solution:***

**From Example 2.4.1,**

$$\frac{2x+1}{x^2-3x-4} = \frac{9}{5(x-4)} + \frac{1}{5(x+1)}$$

$$\int \frac{2x+1}{x^2-3x-4}dx = \int \left[\frac{9}{5(x-4)} + \frac{1}{5(x+1)}\right]dx$$

$$= \frac{9\,ln|(x-4)|}{5} + \frac{ln|(x+1)|}{5} + C$$

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**Example 2.4.4** Evaluate

$$\int \frac{(x^2+3)dx}{(2x-1)(x-2)(x+2)}$$

***solution:***

**From Example 2.4.2,**

$$\frac{x^2+3}{(2x-1)(x-2)(x+2)} = \frac{-13}{15(2x-1)} + \frac{7}{12(x-2)} + \frac{7}{20(x+2)}$$

$$\int \frac{(x^2+3)dx}{(2x-1)(x-2)(x+2)} = \int \left[\frac{-13}{15(2x-1)} + \frac{7}{12(x-2)} + \frac{7}{20(x+2)}\right]dx$$

$$= \frac{-13\,ln|2x-1|}{30} + \frac{7\,ln|x-2|}{12} + \frac{7\,ln|x+2|}{20} + C$$

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**Case 2.** **The factors of $Q(x)$ are linear but some are repeated.**

**Suppose $a_i x + b_i$ is a factor of $Q(x)$ of multiplicity $k$.**

**Then corresponding to the factor $a_i x + b_i$,**

**there corresponds $k$ fractions of the form**

$$\frac{A_1}{(a_i x + b_i)}, \frac{A_2}{(a_i x + b_i)^2}, \ldots, \frac{A_n}{(a_i x + b_i)^k}.$$

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**Example 2.4.5** **Write** $\frac{2x+1}{(2x-1)x^2}$ **as a sum of partial fractions.**

***solution:***

$$\frac{2x+1}{(2x-1)x^2} = \frac{A}{2x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$$\frac{2x+1}{(2x-1)x^2} = \frac{Ax^2 + Bx(2x-1) + C(2x-1)}{(2x-1)x^2}$$

$$2x+1 = Ax^2 + Bx(2x-1) + C(2x-1)$$

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$$2x+1 = A\cancel{x}^{0\,2} + B\cancel{x}^{0}(2\cancel{x}^{0}-1) + C(2\cancel{x}^{0}-1)$$

**When $x = 1/2$,**

$$2\cdot\tfrac{1}{2}+1 = A\left(\tfrac{1}{2}\right)^2$$

$$2 = \tfrac{1}{4}A$$

$$\boxed{A = 8}$$

**When $x = 0$,**

$$2\cdot 0+1 = C(2\cdot 0-1)$$

$$1 = -C$$

$$\boxed{C = -1}$$

**When $x = 1$,**

$$2\cdot 1+1 = 8\cdot 1^2 + B\cdot 1(2\cdot 1-1) - 1(2\cdot 1-1)$$

$$3 = 8 + B - 1$$

$$\boxed{B = -4}$$

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$$\frac{2x+1}{(2x-1)x^2} = \frac{A}{2x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$$A = 8 \qquad B = -4 \qquad C = -1$$

$$\frac{2x+1}{(2x-1)x^2} = \frac{8}{2x-1} - \frac{4}{x} - \frac{1}{x^2}$$

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**Example 2.4.6** Write $\dfrac{x^2}{(2x+1)^3}$ as a sum of partial fractions.

*solution:*

$$\frac{x^2}{(2x+1)^3}=\frac{A}{2x+1}+\frac{B}{(2x+1)^2}+\frac{C}{(2x+1)^3}$$

$$\frac{x^2}{(2x+1)^3}=\frac{A(2x+1)^2+B(2x+1)+C}{(2x+1)^3}$$

$$x^2=A(2x+1)^2+B(2x+1)+C$$

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$$x^2=A(\cancel{2x+1}^{\,0})^2+B(\cancel{2x+1}^{\,0})+C$$

When $x=-1/2$,

$$\left(\tfrac{-1}{2}\right)^2=C$$

$$\boxed{C=\frac{1}{4}}$$

When $x=1/2$,

$$\left(\tfrac{1}{2}\right)^2=A\left(2\cdot\tfrac{1}{2}+1\right)^2+B\left(2\cdot\tfrac{1}{2}+1\right)+\frac{1}{4}$$

$$\frac{1}{4}=4A+2B+\frac{1}{4}$$

$$4A+2B=0\Leftrightarrow \boxed{2A+B=0}$$

When $x=0$,

$$0^2=A(2\cdot0+1)^2+B(2\cdot0+1)+\frac{1}{4}$$

$$\boxed{A+B=-\frac{1}{4}}$$

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$$A+B=-\frac{1}{4}$$

$$2A+B=0$$

$$A=\frac{1}{4}\quad\Rightarrow\frac{1}{4}+B=-\frac{1}{4}\quad\Rightarrow B=-\frac{1}{4}-\frac{1}{4}=\frac{-2}{4}=\frac{-1}{2}$$

$$\frac{x^2}{(2x+1)^3}=\frac{A}{2x+1}+\frac{B}{(2x+1)^2}+\frac{C}{(2x+1)^3}$$

$$\frac{x^2}{(2x+1)^3}=\frac{\frac{1}{4}}{2x+1}+\frac{\frac{-1}{2}}{(2x+1)^2}+\frac{\frac{1}{4}}{(2x+1)^3}$$

$$\frac{x^2}{(2x+1)^3}=\frac{1}{4(2x+1)}-\frac{1}{2(2x+1)^2}-\frac{1}{4(2x+1)^3}$$

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**Example 2.4.7** Evaluate $\displaystyle\int\frac{2x+1}{(2x-1)x^2}\,dx$ .

*solution:*

$$\frac{2x+1}{(2x-1)x^2}=\frac{8}{2x-1}-\frac{4}{x}-\frac{1}{x^2}$$

$$\int\frac{2x+1}{(2x-1)x^2}\,dx=\int\left[\frac{8}{2x-1}-\frac{4}{x}-\frac{1}{x^2}\right]dx$$

$$=8\cdot\frac{1}{2}\,ln|2x-1|-4\,ln|x|+\frac{1}{x}+C.$$

$$=4\,ln|2x-1|-4\,ln|x|+\frac{1}{x}+C.$$

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$$\int\frac{1}{x^2}\,dx=\int x^{-2}\,dx=\frac{x^{-1}}{-1}+C=\frac{-1}{x}+C$$

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**Example 2.4.8** Evaluate $\displaystyle\int\frac{x^2}{(2x+1)^3}\,dx$ .

*solution:*

$$\frac{x^2}{(2x+1)^3}=\frac{1}{4(2x+1)}-\frac{1}{2(2x+1)^2}-\frac{1}{4(2x+1)^3}$$

$$\int\frac{x^2dx}{(2x+1)^3}=\int\left[\frac{1}{4(2x+1)}-\frac{1}{2(2x+1)^2}-\frac{1}{4(2x+1)^3}\right]dx$$

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$$\int \frac{x^2dx}{(2x+1)^3} = \int\left[\frac{1}{4(2x+1)} - \frac{1}{2(2x+1)^2} - \frac{1}{4(2x+1)^3}\right]dx$$

$$\boxed{\begin{aligned} u &= 2x+1 \\ du &= 2dx \\ \tfrac{1}{2}du &= dx \end{aligned}}$$

$$= \int\left[\frac{1}{4u} - \frac{1}{2}u^{-2} - \frac{1}{4}u^{-3}\right]\cdot\frac{du}{2}$$

$$= \frac{1}{2}\left(\frac{1}{4}ln|u| - \frac{1}{2}\cdot\frac{u^{-1}}{-1} - \frac{1}{4}\cdot\frac{u^{-2}}{-2}\right) + C$$

$$= \frac{1}{8}ln|2x+1| + \frac{1}{4(2x+1)} + \frac{1}{16(2x+1)^2} + C$$

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**Case 3.** **The factors of $Q(x)$ are linear and quadratic and the quadratic factors are distinct.**

**Suppose $ax^2+bx+c$ is a distinct factor of $Q(x)$.**

**Then corresponding to this quadratic factor**

$$ax^2+bx+c$$

**there corresponds a fraction of the form**

$$\frac{Ax+B}{ax^2+bx+c}$$

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**Example 2.4.9** **Write** $\dfrac{x^2+3}{(2x-1)(x^2+1)}$ **as a sum of partial fractions.**

***solution:***

$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(2x-1)}{(2x-1)(x^2+1)}$$

$$x^2+3 = A(x^2+1)+(Bx+C)(2x-1)$$

$$x^2+3 = A(x^2+1)+Bx(2x-1)+C(2x-1)$$

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$$x^2+3 = A(x^2+1)+B\cancelto{0}{x}(2x-1)+C(2x\cancelto{0}{-1})$$

**When $x = 1/2$,**

$$\left(\tfrac{1}{2}\right)^2+3 = A\left(\left(\tfrac{1}{2}\right)^2+1\right)$$

$$\tfrac{1}{4}+3 = A\left(\tfrac{1}{4}+1\right)$$

$$\tfrac{13}{4} = A\left(\tfrac{5}{4}\right)$$

$$\boxed{A = \frac{13}{5}}$$

**When $x = 0$,**

$$3 = \tfrac{13}{5}(0^2+1)+C(2\cdot 0-1)$$

$$3 = \tfrac{13}{5}-C$$

$$C = \tfrac{13}{5}-3$$

$$\boxed{C = \frac{-2}{5}}$$

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$$x^2+3 = A(x^2+1)+Bx(2x-1)+C(2x-1)$$

**When $x = 1$,**

$$1^2+3 = \tfrac{13}{5}(1^2+1)+B\cdot 1(2\cdot 1-1)-\tfrac{2}{5}(2\cdot 1-1)$$

$$4 = \tfrac{13}{5}(2)+B-\tfrac{2}{5}$$

$$4 = \tfrac{26}{5}+B-\tfrac{2}{5}$$

$$B = 4-\tfrac{24}{5}$$

$$\boxed{B = \frac{-4}{5}}$$

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$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$$

$$\boxed{\begin{aligned} A &= \frac{13}{5} \\ B &= \frac{-4}{5} \\ C &= \frac{-2}{5} \end{aligned}}$$

$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{\frac{13}{5}}{2x-1} + \frac{\frac{-4}{5}\cdot x+\frac{-2}{5}}{x^2+1}$$

$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{13}{5(2x-1)} - \frac{4x}{5(x^2+1)} - \frac{2}{5(x^2+1)}$$

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**Example 2.4.10** **Evaluate** $\displaystyle\int \frac{(x^2+3)dx}{(2x-1)(x^2+1)}$ .

*solution:*

$$\frac{x^2+3}{(2x-1)(x^2+1)} = \frac{13}{5(2x-1)} - \frac{4x}{5(x^2+1)} - \frac{2}{5(x^2+1)}$$

$$\int \frac{(x^2+3)dx}{(2x-1)(x^2+1)} = \int \left[\frac{13}{5(2x-1)} - \frac{4x}{5(x^2+1)} - \frac{2}{5(x^2+1)}\right] dx$$

$$= \frac{13}{5}\cdot\frac{1}{2} ln|2x-1| - \frac{4}{5}\cdot\frac{1}{2} ln|x^2+1| - \frac{2}{5}\cdot Arc\tan x + C$$

$$= \frac{13\, ln|2x-1|}{10} - \frac{2\, ln(x^2+1)}{5} - \frac{2\, Arc\tan x}{5} + C$$

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## 2.5 Algebraic substitution

**In this section, we study how to evaluate integrals involving radicals but to which the method of trigonometric substitution does not apply.**

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**Example 2.5.1** Evaluate $\displaystyle\int \frac{dx}{2+\sqrt{x}}$ .

*solution:*

Let $y = 2+\sqrt{x}$. Then

$$y-2 = \sqrt{x} \Rightarrow (y-2)^2 = x.$$

$$\Rightarrow 2(y-2)dy = dx$$

$$\int \frac{dx}{2+\sqrt{x}} = \int \frac{2(y-2)dy}{y} = 2\int \left(1-\frac{2}{y}\right)dy$$

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$$\int \frac{dx}{2+\sqrt{x}} = 2\int \left(1-\frac{2}{y}\right)dy$$

$\boxed{y = 2+\sqrt{x}}$

$$= 2(y - 2\, ln|y| + C)$$

$$= 2y - 4\, ln|y| + C$$

$$= 2(2+\sqrt{x}) - 4\, ln|2+\sqrt{x}| + C$$

$$= 4 + 2\sqrt{x} - 4\, ln|2+\sqrt{x}| + C$$

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## Another solution

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**Example 2.5.1** Evaluate $\displaystyle\int \frac{dx}{2+\sqrt{x}}$ .

*solution:*

Let $y = \sqrt{x}$. Then

$$y^2 = x \Rightarrow 2ydy = dx$$

$$\int \frac{dx}{2+\sqrt{x}} = \int \frac{2ydy}{2+y}$$

$$= 2\int \frac{ydy}{2+y}$$

1
$y+2 \,\big)\, y$
$y+2$
$-2$

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$$\int \frac{dx}{2+\sqrt{x}} = 2\int \left(1-\frac{2}{2+y}\right)dy$$

$$= 2\left(y - 2\,ln\left|2+y\right|\right) + C$$

$$= 2y - 4\,ln\left|2+y\right| + C$$

$$= 2\sqrt{x} - 4\,ln\left|2+\sqrt{x}\right| + C$$

$$ans.\ \left(4+2\sqrt{x}-4\,ln\left|2+\sqrt{x}\right|+C\right)$$

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**Example 2.5.2** Evaluate $\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$.

*solution:*

Let $y = \sqrt[6]{x}$. Then

$$y^6 = x \Rightarrow 6y^5 dy = dx$$

$$\Rightarrow \sqrt[3]{x} = x^{1/3} = \left(y^6\right)^{1/3} = y^2,$$

$$\sqrt{x} = x^{1/2} = \left(y^6\right)^{1/2} = y^3$$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} = \int \frac{6y^5 dy}{y^2+y^3} = 6\int \frac{y^5 dy}{y^2(1+y)}$$

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$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$$

$$= 6\int \frac{y^3 dy}{(1+y)}$$

$$= 6\int \left(y^2 - y + 1 - \frac{1}{(y+1)}\right)dy$$

$$\begin{array}{r|l} & y^2 - y + 1 \\ \hline y+1 & y^3 \\ & y^3 + y^2 \\ \hline & -y^2 \\ & -y^2 - y \\ \hline & y \\ & y+1 \\ \hline & -1 \end{array}$$

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$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} = 6\int \left(y^2 - y + 1 - \frac{1}{(y+1)}\right)dy$$

$$= 6\left(\frac{y^3}{3} - \frac{y^2}{2} + y - ln\left|y+1\right|\right) + C$$

$$= 2y^3 - 3y^2 + 6y - 6\,ln\left|y+1\right| + C$$

$$= 2\left(x^{1/6}\right)^3 - 3\left(x^{1/6}\right)^2 + 6x^{1/6} - 6\,ln\left|x^{1/6}+1\right| + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\,ln\left|\sqrt[6]{x}+1\right| + C$$

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**Example 2.5.3** Evaluate $\int \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx$

*solution:*

Let $y = \sqrt{1+\sqrt{1+\sqrt{x}}}$. Then

$$y^2 = 1 + \sqrt{1+\sqrt{x}}$$

$$y^2 - 1 = \sqrt{1+\sqrt{x}}$$

$$\left(y^2-1\right)^2 = 1+\sqrt{x}$$

$$\left(y^2-1\right)^2 - 1 = \sqrt{x}$$

$$y^4 - 2y^2 + \cancel{1} - \cancel{1} = \sqrt{x}$$

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$$y^4 - 2y^2 = \sqrt{x} \Rightarrow \left(y^4-2y^2\right)^2 = x$$

$$\Rightarrow 2\left(y^4-2y^2\right)\left(4y^3-4y\right)dy = dx$$

$$\int \sqrt{1+\sqrt{1+\sqrt{x}}}\,dx = \int y \cdot 2\left(y^4-2y^2\right)\left(4y^3-4y\right)dy$$

$$= 2\int y\left(4y^7 - 4y^5 - 8y^5 + 8y^3\right)dy$$

$$= 2\int y\left(4y^7 - 12y^5 + 8y^3\right)dy$$

$$= 2\int \left(4y^8 - 12y^6 + 8y^4\right)dy$$

$$= 2\left(\frac{4y^9}{9} - \frac{12y^7}{7} + \frac{8y^5}{5}\right) + C$$

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$$= 2\left(\frac{4y^9}{9} - \frac{12y^7}{7} + \frac{8y^5}{5}\right) + C$$

$$= \frac{8y^9}{9} - \frac{24y^7}{7} + \frac{16y^5}{5} + C$$

$$= \frac{8\left(\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^9}{9} - \frac{24\left(\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^7}{7} + \frac{16\left(\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^5}{5} + C.$$

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## 2.6 Rational Functions of *sinx* and *cosx*

Let $z = tan\left(\frac{x}{2}\right)$. Then

$$dz = sec^2\left(\tfrac{x}{2}\right)\cdot\frac{1}{2}dx$$

$$2dz = sec^2\left(\tfrac{x}{2}\right)dx$$

$$2dz = \left[1 + tan^2\left(\tfrac{x}{2}\right)\right]dx$$

$$2dz = \left(1 + z^2\right)dx$$

$$dx = \frac{2dz}{1+z^2}$$

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$$sin\,x = sin\left(2\cdot\tfrac{x}{2}\right) \qquad \boxed{sin(2\theta) = 2\,sin(\theta)cos(\theta)}$$

$$= 2\,sin\left(\tfrac{x}{2}\right)cos\left(\tfrac{x}{2}\right)$$

$$= 2\frac{sin\left(\frac{x}{2}\right)}{cos\left(\frac{x}{2}\right)}cos^2\left(\tfrac{x}{2}\right) = 2\frac{sin\left(\frac{x}{2}\right)}{cos\left(\frac{x}{2}\right)}\cdot\frac{1}{sec^2\left(\frac{x}{2}\right)}$$

$$= 2\,tan\left(\tfrac{x}{2}\right)\cdot\frac{1}{sec^2\left(\frac{x}{2}\right)}$$

$$= 2z\cdot\frac{1}{1+z^2}$$

$$\boxed{sin\,x = \frac{2z}{1+z^2}}$$

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$$cos\,x = cos\left(2\cdot\tfrac{x}{2}\right) \qquad \boxed{cos(2\theta) = 2\,cos^2(\theta) - 1}$$

$$= 2\,cos^2\left(\tfrac{x}{2}\right) - 1$$

$$= 2\cdot\frac{1}{sec^2\left(\frac{x}{2}\right)} - 1$$

$$= 2\cdot\frac{1}{1+z^2} - 1 \quad = \frac{2}{1+z^2} - 1$$

$$= \frac{2-1-z^2}{1+z^2}$$

$$\boxed{cos\,x = \frac{1-z^2}{1+z^2}}$$

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**Summary**:

$$z = tan\left(\frac{x}{2}\right)$$

$$dx = \frac{2dz}{1+z^2}$$

$$sin\,x = \frac{2z}{1+z^2}$$

$$cos\,x = \frac{1-z^2}{1+z^2}$$

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**Example 2.6.1** Evaluate $\int \frac{dx}{1 - sin\,x}$ .

***solution:***

$$\int \frac{dx}{1 - sin\,x} = \int \frac{\frac{2dz}{1+z^2}}{1 - \frac{2z}{1+z^2}}$$

$$= \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{1+z^2-2z}{\cancel{1+z^2}}}$$

$$= \int \frac{2dz}{1+z^2-2z}$$

$$\boxed{\begin{aligned} z &= tan\left(\frac{x}{2}\right) \\ dx &= \frac{2dz}{1+z^2} \\ sin\,x &= \frac{2z}{1+z^2} \end{aligned}}$$

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$$= \int \frac{2dz}{(1-z)^2}$$

$$= -2\int \frac{du}{u^2} \quad \text{where} \quad u = 1 - z, \quad du = -dz$$

$$= -2\int u^{-2}\,du$$

$$= -2\frac{u^{-1}}{-1} + C$$

$$= \frac{2}{u} + C = \frac{2}{1-z} + C \boxed{= \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C}$$

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**Example 2.6.2** **Evaluate** $\int \frac{dx}{4\sin x - 3\cos x}$.

*solution:*

$$\int \frac{dx}{4\sin x - 3\cos x}$$

$$= \int \frac{\frac{2dz}{1+z^2}}{4\cdot\frac{2z}{1+z^2} - 3\cdot\frac{1-z^2}{1+z^2}}$$

$$\boxed{\begin{aligned} z &= \tan\left(\frac{x}{2}\right) \\ dx &= \frac{2dz}{1+z^2} \\ \sin x &= \frac{2z}{1+z^2} \\ \cos x &= \frac{1-z^2}{1+z^2} \end{aligned}}$$

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$$\int \frac{dx}{4\sin x - 3\cos x} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{8z - 3 + 3z^2}{\cancel{1+z^2}}}$$

$$= \int \frac{2dz}{8z - 3 + 3z^2}$$

$$= 2\int \frac{dz}{3z^2 + 8z - 3}$$

$$= 2\int \frac{dz}{(3z-1)(z+3)}$$

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$$\frac{1}{(3z-1)(z+3)} = \frac{A}{(3z-1)} + \frac{B}{(z+3)}$$

$$1 = A(z+3) + B(3z-1)$$

$$A = \frac{3}{10} \qquad B = \frac{-1}{10}$$

$$\frac{1}{(3z-1)(z+3)} = \frac{\frac{3}{10}}{(3z-1)} + \frac{\frac{-1}{10}}{(z+3)}$$

$$\frac{1}{(3z-1)(z+3)} = \frac{3}{10(3z-1)} - \frac{1}{10(z+3)}$$

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$$\int \frac{dx}{4\sin x - 3\cos x} = 2\int \frac{dz}{(3z-1)(z+3)}$$

$$= 2\int \left[\frac{3}{10(3z-1)} - \frac{1}{10(z+3)}\right] dz$$

$$= 2\left(\frac{3}{10}\cdot\frac{1}{3}\ln|3z-1| - \frac{1}{10}\ln|z+3|\right) + C$$

$$= \frac{1}{5}\ln\left|3\tan\left(\tfrac{x}{2}\right) - 1\right| - \frac{1}{5}\ln\left|\tan\left(\tfrac{x}{2}\right) + 3\right| + C$$

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## Check your understanding.

**Determine the technique of integration or method of substitution that will enable you to evaluate each of the following integrals:**

$\int \frac{dx}{3 + \sqrt[4]{x}}$ **Algebraic substitution**

$\int \frac{x^2\,dx}{\sqrt{9x^2 - 4}}$ **Trigonometric substitution**

$\int \frac{(2x-3)dx}{(9x^2-4)(3x+1)}$ **Partial fractions**

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$$\int \frac{dx}{\sin x + 2\cos x}$$

**Rational function of *sinx and cosx***

$$\int \sin^4(2x)\cos^3(2x)dx$$

**Powers of trigo. functions**

$$\int x^3 \cos(2x)dx$$

**Integration by parts**



## 2.7 Improper Integrals

**Definition 2.7.1** An integral is ***improper*** if either limits of integration are infinite or if the integrand is discontinuous at a number or at numbers within the interval of integration.



### TYPE UB

**An integral $\int_a^b f(x)dx$ is improper of type UB if**

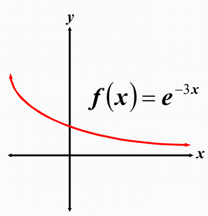
**1. $f$ is continuous for all $x \geq a$ and**

**2. $b = +\infty$**

**Illustration 2.7.1**

$$\int_0^{+\infty} e^{-3x}dx$$

**is improper of type UB.**






### TYPE UA

**An integral $\int_a^b f(x)dx$ is improper of type UA if**

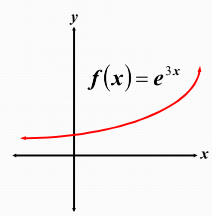
**1. $f$ is continuous for all $x \leq b$ and**

**2. $a = -\infty$**

**Illustration 2.7.2**

$$\int_{-\infty}^{1} e^{3x}dx$$

**is improper of type UA.**






### TYPE UAB

**An integral $\int_a^b f(x)dx$ is improper of type UAB if**

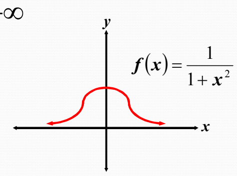
**1. $f$ is continuous for all $x$ and**

**2. $a = -\infty$ and $b = +\infty$**

**Illustration 2.7.3**

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx$$

**is improper of type UAB.**






### TYPE DB

**An integral $\int_a^b f(x)dx$ is improper of type DB if**

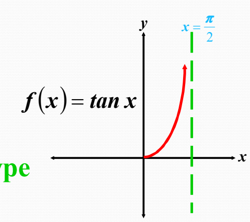
**1. $f$ is continuous for all $a \leq x < b$ and**

**2. $\lim_{x \to b^-} f(x) = \pm\infty$**

**Illustration 2.7.4**

$$\int_0^{\frac{\pi}{2}} \tan x dx$$

**is improper of type DB.**



## TYPE DA

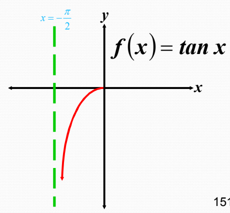
An integral $\int_a^b f(x)dx$ is improper of type DA if

1. $f$ is continuous for all $a < x \le b$ and
2. $\lim\limits_{x \to a^+} f(x) = \pm\infty$

**Illustration 2.7.5**

$$\int_{-\frac{\pi}{2}}^{0} \tan x dx$$

is improper of type DA.






## TYPE DC

An integral $\int_a^b f(x)dx$ is improper of type DC if there is a number $c$ such that $a < c < b$ and

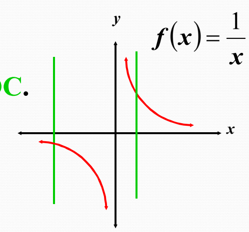
1. $f$ is continuous for all $a \le x < c$ and for all $c < x \le b$
2. $\lim\limits_{x \to c^-} f(x) = \pm\infty$ or $\lim\limits_{x \to c^+} f(x) = \pm\infty$.



**Illustration 2.7.6**

$$\int_{-2}^{1} \frac{1}{x} dx$$

is improper of type DC.




**How do we evaluate improper integrals?**



**Definition 2.7.2**

| Type | Form | Prep. | Definition |
|---|---|---|---|
| UB | $\int_a^{+\infty} f(x)dx$ | $\int_a^t f(x)dx$ | $\lim\limits_{t\to+\infty} \int_a^t f(x)dx$ |
| UA | $\int_{-\infty}^{b} f(x)dx$ | $\int_t^b f(x)dx$ | $\lim\limits_{t\to-\infty} \int_t^b f(x)dx$ |
| UAB | $\int_{-\infty}^{+\infty} f(x)dx$ | $\int_0^{+\infty} f(x)dx + \int_{-\infty}^{0} f(x)dx$ | $\lim\limits_{t\to+\infty} \int_0^t f(x)dx + \lim\limits_{t\to-\infty} \int_t^0 f(x)dx$ |
| DB | $\int_a^b f(x)dx$ | $\int_a^t f(x)dx$ | $\lim\limits_{t\to b^-} \int_a^t f(x)dx$ |
| DA | $\int_a^b f(x)dx$ | $\int_t^b f(x)dx$ | $\lim\limits_{t\to a^+} \int_t^b f(x)dx$ |
| DC | $\int_a^b f(x)dx$ | $\int_a^c f(x)dx + \int_c^b f(x)dx$ | $\lim\limits_{t\to c^-} \int_a^t f(x)dx + \lim\limits_{t\to c^+} \int_t^b f(x)dx$ |



**Example 2.7.1** Evaluate $\int_0^{+\infty} e^{-x} dx$.

*Solution:*

Let $f(x) = e^{-x}$. Since $f$ is continuous everywhere, the given integral is improper of type UB.

$$\int_0^{+\infty} e^{-x}dx = \lim_{t\to+\infty} \int_0^t e^{-x}dx = \lim_{t\to+\infty} -e^{-x}\Big]_0^t$$

$$= -\lim_{t\to+\infty} \frac{1}{e^x}\Big]_0^t = -\lim_{t\to+\infty}\left(\cancelto{0}{\frac{1}{e^t}} - \frac{1}{e^0}\right)$$

$$= -(0-1) = 1.$$



**Example 2.7.2** Evaluate $\int_1^{+\infty} \frac{dx}{x}$.

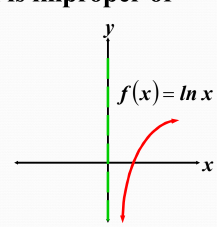
*Solution:*

Let $f(x) = \frac{1}{x}$. Since $f$ is continuous for all $x \ge 1$, the given integral is improper of type UB.

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{t\to+\infty} \int_1^t \frac{dx}{x}$$

$$= \lim_{t\to+\infty} \ln|x|\Big]_1^t = \lim_{t\to+\infty} \left(\ln|t| - \cancelto{0}{\ln 1}\right)$$

$$= \lim_{t\to+\infty} \ln|t| = +\infty$$



**Example 2.7.3** Evaluate $\int_{-\infty}^{0} x^2 e^x dx$.

*Solution:*

Let $f(x) = x^2e^x$. Since $f$ is continuous everywhere, the given integral is improper of type UA.

$$\int_{-\infty}^{0} x^2e^x dx = \lim_{t\to-\infty}\int_{t}^{0} x^2e^x dx$$

$$= \lim_{t\to-\infty}\left(x^2e^x - 2xe^x + 2e^x\right)\Big]_t^0$$

$$= \lim_{t\to-\infty}\left[2 - \left(\cancel{t^2e^t}^{\,0} - \cancel{2te^t}^{\,0} + \cancel{2e^t}^{\,0}\right)\right]$$

$$= 2.$$

| $u$ | $dv$ |
|---|---|
| $x^2$ (+) | $e^x$ |
| $2x$ (−) | $e^x$ |
| $2$ (+) | $e^x$ |
| $0$ | $e^x$ |



**Example 2.7.4** Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

*Solution:*

Let $f(x) = \frac{1}{1+x^2}$. Since $f$ is continuous everywhere, the given integral is improper of type UAB.

Type UA

Type UB

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{+\infty}\frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty}\int_{t}^{0}\frac{dx}{1+x^2} + \lim_{t\to+\infty}\int_{0}^{t}\frac{dx}{1+x^2}$$



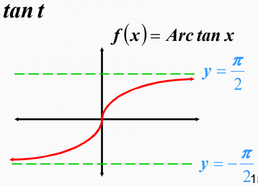
$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \lim_{t\to-\infty}\int_{t}^{0}\frac{dx}{1+x^2} + \lim_{t\to+\infty}\int_{0}^{t}\frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty} Arc\tan x\Big]_t^0 + \lim_{t\to+\infty} Arc\tan x\Big]_0^t$$

$$= \lim_{t\to-\infty}\left(\cancel{Arc\tan 0}^{\,0} - Arc\tan t\right) + \lim_{t\to+\infty}\left(Arc\tan t - \cancel{Arc\tan 0}^{\,0}\right)$$

$$= -\lim_{t\to-\infty} Arc\tan t + \lim_{t\to+\infty} Arc\tan t$$

$$= -\,-\frac{\pi}{2} + \frac{\pi}{2} = \pi$$






**Example 2.7.5** Evaluate $\int_{1}^{5} \frac{dx}{\sqrt{5-x}}$.

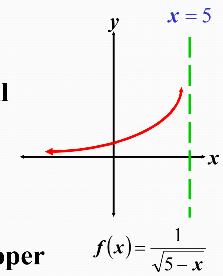
*Solution:*

Let $f(x) = \frac{1}{\sqrt{5-x}}$.

Since $f$ is continuous for all $x < 5$ and

$$\lim_{x\to5^-}\frac{1}{\sqrt{5-x}} = +\infty$$

the given integral is improper of type DB.






$$\int_{1}^{5}\frac{dx}{\sqrt{5-x}} = \lim_{t\to5^-}\int_{1}^{t}\frac{dx}{\sqrt{5-x}}$$

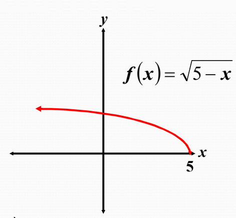
$$= \lim_{t\to5^-} -2\sqrt{5-x}\,\Big]_1^t$$

$$= -2\lim_{t\to5^-}\sqrt{5-x}\,\Big]_1^t$$

$$= -2\lim_{t\to5^-}\left(\sqrt{5-t} - \sqrt{5-1}\right)$$

$$= -2\lim_{t\to5^-}\left(\cancel{\sqrt{5-t}}^{\,0} - 2\right)$$

$$= -2(-2) = 4.$$






**Example 2.7.6** Evaluate $\int_{-\frac{\pi}{2}}^{0} \tan x dx$.

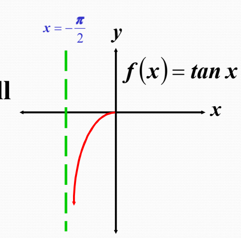
*Solution:*

Let $f(x) = \tan x$.

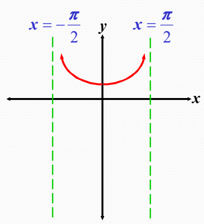
Since $f$ is continuous for all $\frac{-\pi}{2} < x \le 0$ and

$$\lim_{x\to\frac{-\pi}{2}^-} \tan x = -\infty$$

the given integral is improper of type DA.



$$\int_{-\frac{\pi}{2}}^{0} \tan x dx$$
$$= \lim_{t \to -\frac{\pi}{2}^+} \int_t^0 \tan x dx$$
$$= \lim_{t \to -\frac{\pi}{2}^+} \ln|\sec x|\Big]_t^0$$
$$= \lim_{t \to -\frac{\pi}{2}^+} \left(\ln|\sec 0| - \ln|\sec t|\right)$$
$$= \lim_{t \to -\frac{\pi}{2}^+} \left(\cancelto{0}{\ln 1} - \ln|\sec t|\right)$$
$$= -\lim_{t \to -\frac{\pi}{2}^+} \ln|\sec t| \;= -\infty.$$




$f(x) = \sec x$

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**Example 2.7.7** Evaluate $\int_{-1}^{1} \frac{dx}{x^2}$ .

***Solution:***

Let $f(x) = \frac{1}{x^2}$ .

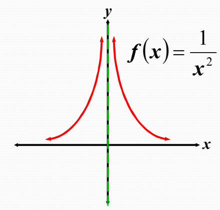
Since $f$ is continuous for all

$-1 \le x < 0$ and $-0 < x \le 1$

and

$\lim_{x \to 0} \frac{1}{x^2} = +\infty$

the given integral is improper of type UC.

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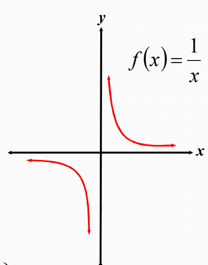
$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$
$$= \lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^2} + \lim_{t \to 0^+} \int_{t}^{1} \frac{dx}{x^2}$$
$$= \lim_{t \to 0^-} \frac{-1}{x}\Big]_{-1}^{t} + \lim_{t \to 0^+} \frac{-1}{x}\Big]_{t}^{1}$$
$$= -\lim_{t \to 0^-} \frac{1}{x}\Big]_{-1}^{t} - \lim_{t \to 0^+} \frac{1}{x}\Big]_{t}^{1}$$
$$= -\lim_{t \to 0^-}\left(\cancelto{-\infty}{\frac{1}{t}} + 1\right) - \lim_{t \to 0^+}\left(1 - \cancelto{+\infty}{\frac{1}{t}}\right)$$
$$= - - \infty \; - - \infty = +\infty + \infty = +\infty.$$

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An improper integral is said to be ***convergent*** if it has a real value. Otherwise, it is said to be ***divergent***.

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## SUMMARY

**In this chapter, we studied**

1. **integration by parts**
2. **methods of substitution**
   a. **For powers of trigo. functions**
   b. **Trigonometric substitution**
   c. **Partial fractions**
   d. **Algebraic substitution**
   e. **For rational functions of *sinx* and *cosx***
3. **how to evaluate improper integrals.**

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# THE END

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