1.3 Some Special Types of Infinite Series

- ✓ GEOMETRIC SERIES
- ✓ HARMONIC SERIES
- ✓ p-SERIES
- ✓ TELESCOPING SERIES

Geometric Series

GEOMETRIC SERIES: $\sum_{n=1}^{+\infty} ar^{n-1}$

$$s_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

If $|r| < 1$: CONVERGENT with a sum of $\frac{a}{1-r}$

If $|r| \geq 1$: DIVERGENT

Harmonic Series

HARMONIC SERIES: $\sum_{n=1}^{+\infty} \frac{1}{n}$

is DIVERGENT

Compare $\sum_{n=1}^{+\infty} \frac{1}{n}$ *with* $\int_{1}^{+\infty} \frac{1}{x}\,dx$

p-Series

p-SERIES: $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ or $\sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)^p$

If $p > 1$: CONVERGENT

If $p \leq 1$: DIVERGENT

ILLUSTRATIONS Determine whether convergent or divergent. $\sum_{n=1}^{+\infty} ar^{n-1}$

1. $\sum_{n=1}^{+\infty} \frac{2}{3^{n-1}}$ $\qquad a = 2,\ r = \frac{1}{3}$

Since $|r| < 1$, the series converges with sum $\frac{2}{1 - \frac{1}{3}} = 3$

$\sum_{n=1}^{+\infty} ar^{n-1}$

2. $\sum_{n=2}^{+\infty} 3(2^{n-1}) = \sum_{n=1}^{+\infty} 3(2^{n}) = \sum_{n=1}^{+\infty} 3(2)(2^{n-1})$

$= \sum_{n=1}^{+\infty} 6(2^{n-1}) \qquad a=6,\ r=2$

Since $|r| \geq 1$ **, the series diverges.**

$\sum_{n=1}^{+\infty} ar^{n-1}$

3. $\sum_{n=0}^{+\infty} e^{-2n} = \sum_{n=1}^{+\infty} e^{-2(n-1)} = \sum_{n=1}^{+\infty} (e^{-2})^{n-1}$

$a=1,\ r=\frac{1}{e^2}$

Since $|r|<1$ **, the series Converges with sum** $\frac{1}{1-\frac{1}{e^2}} = \frac{e^2}{e^2-1}$

$\sum_{n=1}^{+\infty} ar^{n-1}$

4. $\sum_{n=1}^{+\infty} \left(\frac{4}{5}\right)^{n+2} = \sum_{n=1}^{+\infty} \left(\frac{4}{5}\right)^{3}\left(\frac{4}{5}\right)^{n-1} \qquad a=\frac{4^3}{5^3},\ r=\frac{4}{5}$

Since $|r|<1$ **, the series Converges with sum** $\frac{\frac{4^3}{5^3}}{1-\frac{4}{5}} = \frac{64}{25}$

5. $\sum_{n=1}^{+\infty} \frac{1}{n^e} \qquad p-series$

Since $p=e>1$ **, the series Converges**

6. $\sum_{n=1}^{+\infty} n^{-3/4} \qquad \sum_{n=1}^{+\infty} \frac{1}{n^{3/4}} \qquad p-series$

Since $p=\frac{3}{4}<1$ **, the series diverges**

Examples. Determine whether convergent or divergent.

1. $\sum_{n=1}^{+\infty} \frac{4}{3^{n-1}}$ **is convergent.**
2. $\sum_{n=1}^{+\infty} \frac{2^{n-1}}{5}$ **is divergent.**
3. $\sum_{n=1}^{+\infty} 5\left(-\frac{1}{4}\right)^{n-1}$ **is convergent.**

Examples. Determine whether convergent or divergent.

4. $\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ **is convergent.**
5. $\sum_{n=1}^{+\infty} n^{-1/2}$ **is divergent.**

Telescoping Series

TELESCOPING SERIES:

$$\sum_{n=1}^{+\infty} \frac{k}{f(n)\cdot f(n+1)}$$

$$= \sum_{n=1}^{+\infty} \frac{a}{f(n)} + \frac{b}{f(n+1)}$$

WARNING!

DECOMPOSITION OF PARTIAL FRACTIONS

ILLUSTRATION $\sum_{n=1}^{+\infty} \frac{k}{f(n)\cdot f(n+1)} = \sum_{n=1}^{+\infty} \frac{a}{f(n)} + \frac{b}{f(n+1)}$

1. $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ **is a telescoping series with** $f(n) = n$.

Using partial fractions…

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = (n+1)A + nB$$

$let\ n = 0 \qquad \Rightarrow 1 = A$

$let\ n = -1 \qquad \Rightarrow 1 = -B \ or \ B = -1$

$Thus,\ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad u_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\Rightarrow s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \left(1 - \cancel{\frac{1}{n+1}}^{\,0}\right) = 1$$

Thus, $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1$.

Also, $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ **is CONVERGENT.**

ILLUSTRATION $\sum_{n=1}^{+\infty} \frac{k}{f(n)\cdot f(n+1)} = \sum_{n=1}^{+\infty} \frac{a}{f(n)} + \frac{b}{f(n+1)}$

2. $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)}$ **is a telescoping series With** $f(n)=2n-1$ **.**

Using partial fractions…

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$1=(2n+1)A+(2n-1)B$$

$$let\ n=\tfrac{1}{2} \quad \Rightarrow 1=2A \quad \Rightarrow \quad A=\tfrac{1}{2}$$

$$let\ n=-\tfrac{1}{2} \quad \Rightarrow 1=-2B\ \ or\ \ B=-\tfrac{1}{2}$$

$$Thus,\ \frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{+\infty} \left(\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}\right)$$

$$\Rightarrow s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$= \left(\frac{1}{2} - \frac{1}{2(3)}\right) + \left(\frac{1}{2(3)} - \frac{1}{3(4)}\right) + \left(\frac{1}{3(4)} - \frac{1}{4(5)}\right) + \ldots + \left(\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}\right)$$

$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(\frac{1}{2} - \cancel{\frac{1}{2(2n+1)}}^{\,0}\right) = \frac{1}{2}$$

Thus, $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$ **.**

Also, $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)}$ **is CONVERGENT.**

Example. Determine whether convergent or divergent. If convergent, find the sum

$$\sum_{n=1}^{+\infty} \frac{1}{(3n+1)(3n-2)}$$

$$S_n = \frac{n}{3n+1} \qquad Sum = \frac{1}{3}$$