# CHAPTER 3

# APPLICATIONS OF THE DEFINITE INTEGRAL



## OUTLINE





## CHAPTER OBJECTIVES

At the end of the chapter, you should be able to use the definite integral to find/compute for

1. the area of a plane region using vertical or horizontal strips,
2. the volume of a solid of revolution using the cylindrical or shell method,
3. the center of mass of a rod, plane region and of a solid of revolution,
4. the length of an arc of a plane curve,
5. the area of a surface of revolution and
6. work and force due to liquid pressure.



## 3.1 Area of a plane region

Let $f$ be a function of $x$ which is **continuous** and **non-negative** on the closed interval $[a, b]$.

Subdivide the closed interval $[a, b]$ into $n$ sub-intervals by choosing $(n - 1)$ intermediate numbers

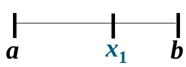
$$x_1, x_2, \ldots, x_{n-1}$$

where

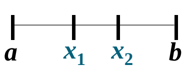
$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$



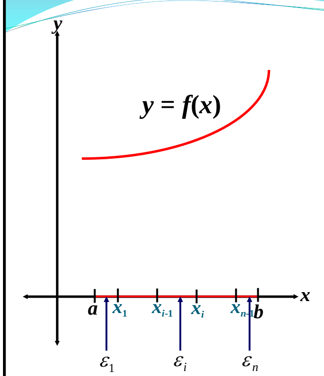
For example, if we want to subdivide $[a,b]$ into 2 sub-intervals, we only choose one number $x_1$ between $a$ and $b$.




If we want to subdivide $[a,b]$ into 3 sub-intervals, we choose two numbers $x_1$ and $x_2$ between $a$ and $b$.






Denote the $i$th sub-interval by $I_i$ so that

$$I_1 = [x_0, x_1]$$
$$I_2 = [x_1, x_2]$$
$$I_3 = [x_2, x_3]$$
$$I_i = [x_{i-1}, x_i]$$
$$I_n = [x_{n-1}, x_n]$$

For each $i = 1, 2, \ldots, n$, choose a number $\varepsilon_i \in I_i$.

Denote the length of the $i$th sub-interval by

$$\Delta_i x$$

so that

$$\Delta_1 x = x_1 - x_0$$

$$\Delta_2 x = x_2 - x_1$$

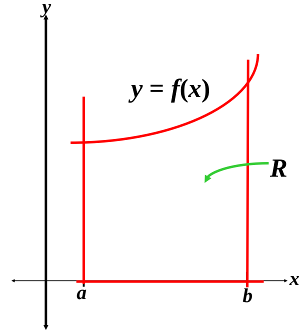
$$\Delta_3 x = x_3 - x_2$$

$$\Delta_i x = x_i - x_{i-1}$$

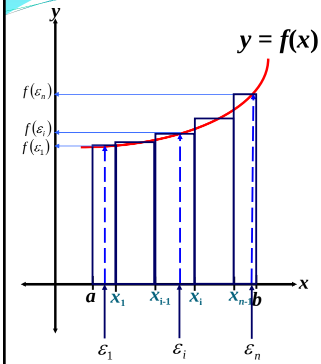
$$\Delta_n x = x_n - x_{n-1}$$



Our objective is to find a formula for the area of the region $R$ enclosed by the graph of $f$, the $x$-axis and the lines given by $x = a$ and $x = b$.






On the $i$th sub-interval, construct a rectangle of length

$$f(\varepsilon_i)$$

and whose width is

$$\Delta_i x.$$



The area of the $i$th rectangular region is

$$f(\varepsilon_i)\cdot\Delta_i x.$$

Thus, the sum of areas of all $n$ rectangular regions is

$$f(\varepsilon_1)\Delta_1 x + f(\varepsilon_2)\Delta_2 x + \ldots + f(\varepsilon_i)\Delta_i x + \ldots + f(\varepsilon_n)\Delta_n x$$

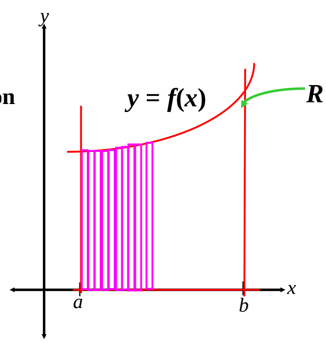
or

$$\sum_{i=1}^{n} f(\varepsilon_i)\Delta_i x.$$



The area of the region $R$ is given by

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\varepsilon_i)\Delta_i x.$$






By definition,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\varepsilon_i)\Delta_i x = \int_a^b f(x)dx.$$

Thus, the area of the region $R$ is given by

$$A(R) = \int_a^b f(x)dx.$$

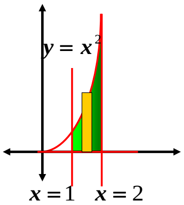
width of a rectangular element

length of a rectangular element

**Example 3.1.1** Find the area of the region enclosed by the graphs of

$$y = x^2,\ x = 1,\ x = 2 \text{ and } y = 0.$$

***Solution:***




$$A(R) = \int_1^2 x^2 dx = \left.\frac{x^3}{3}\right]_1^2$$

$$= \frac{2^3}{3} - \frac{1}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

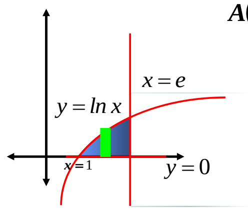
**Thus, the area of the region is 7/3 sq. units.**



**Example 3.1.2** Find the area of the region enclosed by the graphs of

$$y = \ln x,\ x = e \text{ and } y = 0.$$

***Solution:***




$$A(R) = \int_1^e \ln x\, dx$$

$$= x \ln x - x\Big]_1^e$$

$$= (e \cancelto{1}{\ln e} - e) - (1 \cdot \cancelto{0}{\ln 1} - 1)$$

$$= (e - e) + 1 = 1$$

**Thus, the area of the region *R* is 1 sq. unit.**



Let *g* be a function of *y* which is **continuous** and **non-negative** on the closed interval **[*c*, *d*]**.

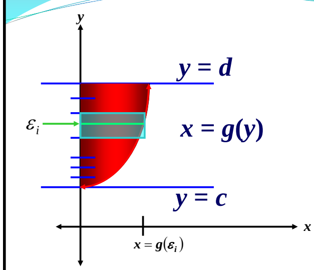
Subdivide the closed interval **[*c*, *d*]** into ***n*** sub-intervals by choosing **(*n* - 1)** intermediate numbers

$$y_1, y_2, \ldots, y_{n-1}$$

where

$$c = y_0 < y_1 < y_2 < \ldots < y_{n-1} < y_n = d.$$






For each $i = 1, 2, \ldots, n$, choose a number

$$\varepsilon_i \in I_i.$$

The area of the *i*th rectangular region is

$$g(\varepsilon_i) \cdot \Delta_i y.$$

The area of the region *R* is given by

$$\lim_{n \to \infty} \sum_{i=1}^{n} g(\varepsilon_i) \Delta_i y.$$

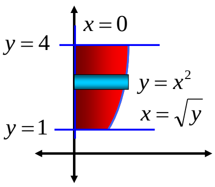
$$A(R) = \int_c^d g(y) dy.$$



**Example 3.1.3** Find the area of the region in the first quadrant enclosed by the graphs of

$$y = x^2,\ y = 1,\ y = 4 \text{ and } x = 0.$$

***Solution:***




$$A(R) = \int_1^4 \sqrt{y}\, dy$$

$$= \left.\frac{y^{3/2}}{3/2}\right]_1^4 = \frac{2}{3} y^{3/2}\Big]_1^4$$

$$= \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{3}(8 - 1) = \frac{14}{3}$$

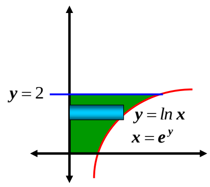
**Thus, the area of the region is 14/3 sq. units.**



**Example 3.1.4** Find the area of the region in the first quadrant enclosed by the graphs of

$$y = \ln x \text{ and } y = 2.$$

***Solution:***




$$A(R) = \int_0^2 e^y\, dy$$

$$= e^y\Big]_0^2$$

$$= e^2 - e^0$$

$$= e^2 - 1$$

**Thus, the area of the region is ($e^2$ - 1) sq. units.**

**Theorem 3.1.1** Let $f$ and $g$ be functions of $x$ which are continuous on $[a, b]$ and

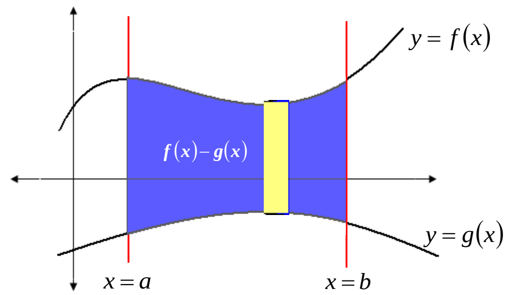
$$f(x) \geq g(x)$$

for all $x$ in $[a, b]$.

Then the area of the region bounded by the graph of the graphs of $f$ and $g$ and the lines given by $x = a$ and $x = b$ is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$



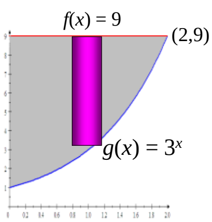



$$A = \int_a^b [f(x) - g(x)]dx$$



**Example 3.1.5** Set-up the definite integral that gives the area of the region $R$ bounded by the graphs of $y = 3^x, y = 9$ and the $y$-axis.

*Solution:*




$$A(R) = \int_a^b (f(x) - g(x))dx$$

$$= \int_0^2 (9 - 3^x)dx$$



**Example 3.1.6** Set-up the definite integral that gives the area of the region $R$ bounded by the graphs of $y = x^2 + 1$ and $y = 2x + 4$.

*Solution:*

$$x^2 + 1 = 2x + 4$$

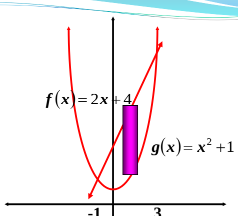
$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1$$

POI: (3,10) and (-1,2)






$$A(R) = \int_a^b (f(x) - g(x))dx$$

$$= \int_{-1}^3 [(2x + 4) - (x^2 + 1)]dx$$



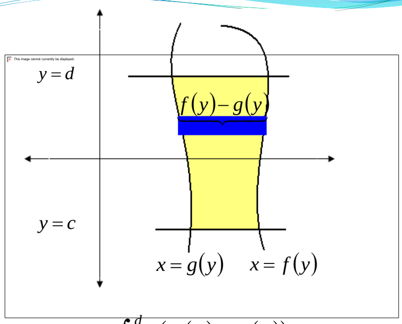
**Theorem 3.1.2** Let $f$ and $g$ be functions of $y$ which are continuous on $[c, d]$ and

$$f(y) \geq g(y)$$

for all $y$ in $[c, d]$.

Then the area of the region bounded by the graph of the graphs of $f$ and $g$ and the lines given by $y = c$ and $y = d$ is given by

$$A = \int_c^d (f(y) - g(y))dy.$$




$$A = \int_c^d \left(f(y) - g(y)\right)dy.$$



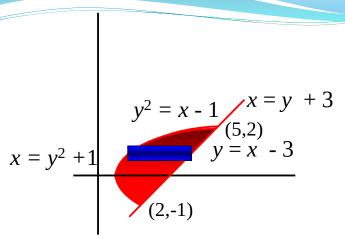
**Example 3.1.7** Set-up the definite integral that gives the area of the region bounded by the graphs of $y^2 = x - 1$ and $y = x - 3$.

***Solution:***

$$y^2 + 1 = y + 3$$
$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$
$$y = 2 \text{ or } y = -1$$

**POI: (5,2) and (2,-1)**






$$A = \int_c^d \left(f(y) - g(y)\right)dy = \int_{-1}^{2} \left((y+3) - (y^2+1)\right)dy$$
$$= \int_{-1}^{2} \left(y + 2 - y^2\right)dy.$$



## 3.2 Volume of a Solid of Revolution

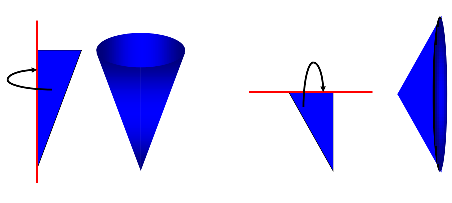
***Solid of Revolution***

A *solid of revolution* is obtained when a region in the *xy*-plane is rotated about a fixed line in the plane.

The fixed line is called the *axis of symmetry or axis of revolution* of the resulting solid.

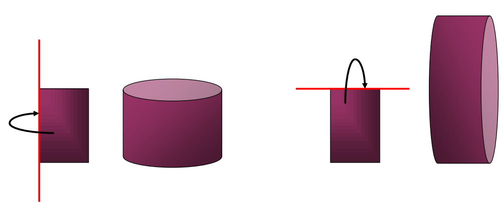
We shall only consider vertical or horizontal line as axis of revolution.



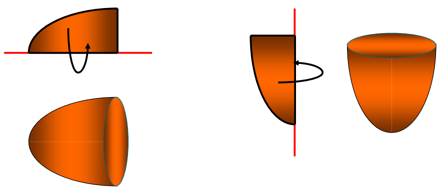


**Rotate a *triangular region* about one of its legs to obtain a *conical solid.***



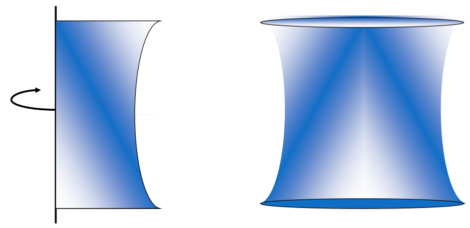


**Rotate a *rectangular region* about one of its sides to obtain a *right-circular cylindrical solid*.**



Rotate a ***parabolic region*** about its axis of symmetry to obtain a ***parabolic cylindrical solid***.







## Methods of finding the volume of a solid of revolution

**1. Circular Disk Method**

**- a rectangular element is perpendicular to the axis of revolution.**

**2. Cylindrical Shell**

**- a rectangular element is parallel to the axis of revolution.**



## 3.2.1 Circular Disk Method

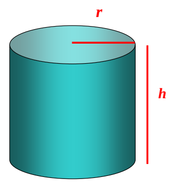
**We derive the formula for finding the volume of a solid formed when a plane region is revolved about the *x*-axis using the circular disk method.**

**The method may be applied even when the axis of revolution is no longer the *x*-axis but with a slight modification.**



**We recall that the volume of a solid enclosed by a right circular cylinder of radius *r* and altitude *h* is**

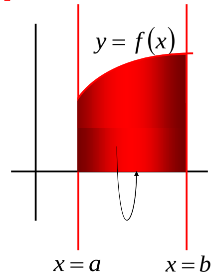
$$V = \pi r^2 h.$$



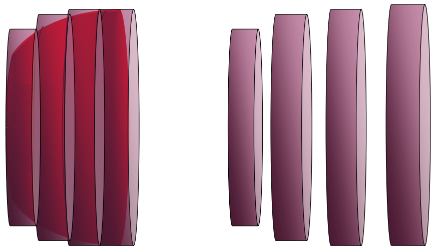



**Let the function *f* be continuous on the closed interval [*a*,*b*] and assume that $f(x) \geq 0$ for all *x* in [*a*,*b*].**

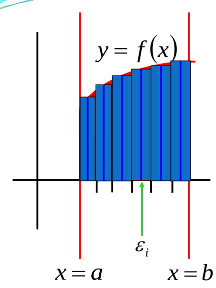
**Let *R* be the region bounded by the graph of $y = f(x)$, the *x*-axis and the lines given by $x = a$ and $x = b$.**



**Suppose *R* is revolved around the *x*-axis.**








**When the *i*th rectangular region is revolved about the *x*-axis, a circular disk is formed.**
**Its volume is**

$$V_i = \pi r^2 h = \pi\left[f(\varepsilon_i)\right]^2 \Delta_i x$$

**The sum of volumes of the *n* disks is**

$$\sum_{i=1}^{n} V_i = \pi \sum_{i=1}^{n} \left[f(\varepsilon_i)\right]^2 \Delta_i x.$$



**Thus, the volume of the solid formed when the region *R* is revolved about the *x*-axis is**

$$V = \lim_{n\to\infty} \pi \sum_{i=1}^{n} \left[f(\varepsilon_i)\right]^2 \Delta_i x$$

**or equivalently,**

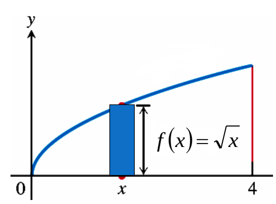
$$V = \pi \int_a^b \left[f(x)\right]^2 dx.$$



**Example 3.2.1**

**Consider the region bounded by the graphs of** $y = \sqrt{x}$ **,** $x = 4$ **and the *x*-axis.**

***Solution:***



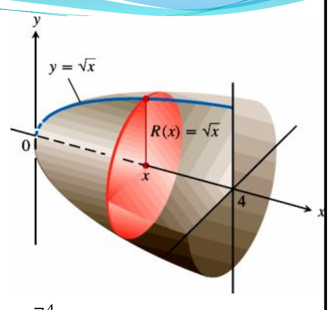

**If the region is revolved about the *x*-axis, a solid is formed.**

**Using the Disk Method, the radius of the disk is given by**

$$f(x) = \sqrt{x}$$



- Using the Disk Method, the volume of the solid generated is




$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx$$

$$= \pi \int_0^4 x\,dx \quad = \left.\frac{\pi x^2}{2}\right]_0^4 = \frac{\pi \cdot 4^2}{2} = 8\pi \text{ cu. units}$$



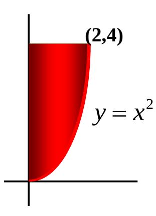
**Example 3.2.2**

**Consider the region bounded by the graphs of**

$y = x^2$ **and** $y = 4$

**in the first quadrant.**

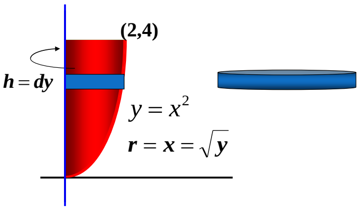
**Set up the definite integral which gives the volume of the solid formed when the given region is revolved about**




**1.** $x = 0$ **2.** $x = 2$ **3.** $x = -1$

**4.** $y = 0$ **5.** $y = 4$ **6.** $y = -1$

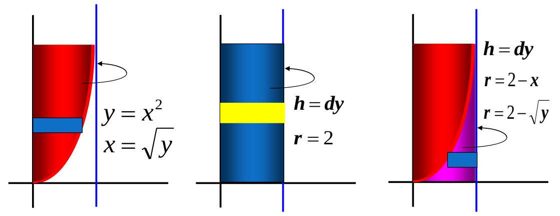
**1. $x = 0$**




$$V = \pi\int_0^4 \left(\sqrt{y}\right)^2 dy$$



**2. $x = 2$**

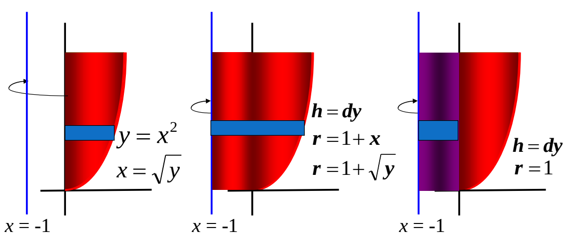



$$V_1 = \pi\int_0^4 2^2\, dy \qquad V_2 = \pi\int_0^4 \left(2 - \sqrt{y}\right)^2 dy$$

$$\boxed{V = V_1 - V_2}$$



**3. $x = -1$**

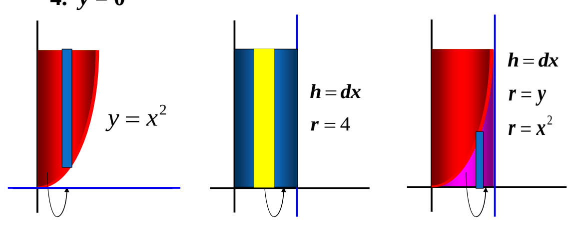



$$V_1 = \pi\int_0^4 \left(1 + \sqrt{y}\right)^2 dy \qquad V_2 = \pi\int_0^4 1^2\, dy$$

$$\boxed{V = V_1 - V_2}$$



**4. $y = 0$**

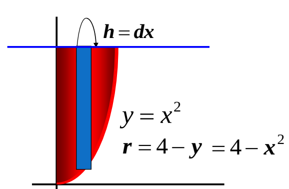



$$V_1 = \pi\int_0^2 4^2\, dx \qquad V_2 = \pi\int_0^2 \left(x^2\right)^2 dx$$

$$\boxed{V = V_1 - V_2}$$



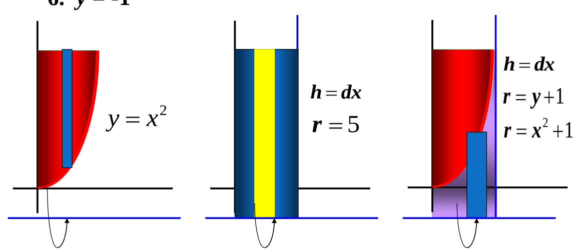
**5. $y = 4$**




$$V = \pi\int_0^2 \left(4 - x^2\right)^2 dx$$



**6. $y = -1$**

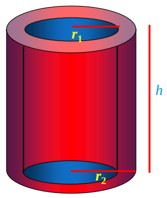



$$V_1 = \pi\int_0^2 5^2\, dx \qquad V_2 = \pi\int_0^2 \left(x^2 + 1\right)^2 dx$$

$$\boxed{V = V_1 - V_2}$$

## 3.2.2 Cylindrical Shell Method

- A cylindrical shell is a solid contained between two cylinders having the same altitude and axis.
- The Cylindrical Shell method involves taking rectangular elements which are parallel to the axis of revolution.




$$V = \pi r^2 h$$

$$V = \pi r_2^2 h - \pi r_1^2 h$$

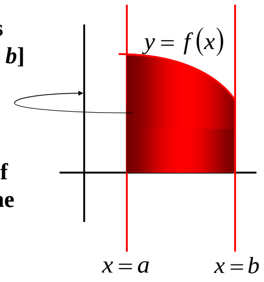
$$V = \pi\left(r_2^2 - r_1^2\right)h$$



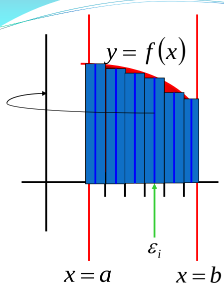
**Theorem 3.2.2** Let the function $f$ be continuous on the closed interval $[a, b]$ and assume that $f(x) \geq 0$ for all $x$ in $[a, b]$.

Let $R$ be the region bounded by the graph of $y = f(x)$, the $x$-axis and the lines given by $x = a$ and $x = b$.

Suppose $R$ is revolved around the $y$-axis.






Let $\varepsilon_i = \dfrac{x_i + x_{i-1}}{2}$.

When the $i$th rectangular region is revolved about the $y$-axis, a circular shell is formed.

Its volume is

$$\begin{aligned} V_i &= \pi\left(r_2^2 - r_1^2\right)h \\ &= \pi\left(x_i^2 - x_{i-1}^2\right)f(\varepsilon_i) \\ &= \pi\left(x_i - x_{i-1}\right)\left(x_i + x_{i-1}\right)f(\varepsilon_i) \\ &= \pi\Delta_i x 2\varepsilon_i f(\varepsilon_i) \\ &= 2\pi\varepsilon_i f(\varepsilon_i)\Delta_i x \end{aligned}$$



The sum of volumes of the $n$ shells is

$$\sum_{i=1}^{n} 2\pi\varepsilon_i f(\varepsilon_i)\Delta_i x = 2\pi\sum_{i=1}^{n} \varepsilon_i f(\varepsilon_i)\Delta_i x.$$

Thus, the volume of the solid formed when the region $R$ is revolved about the $y$-axis is

$$V = 2\pi \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon_i f(\varepsilon_i)\Delta_i x$$

or equivalently,

$$V = 2\pi\int_a^b x f(x)\,dx.$$

Length of a rectangular element

Width of a rectangular element

Distance from the axis of revolution

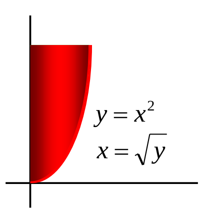


**Example 3.2.3**

Consider the region bounded by the graphs of $y = x^2$, $y = 4$ and the $y$-axis in the first quadrant.

***Solution:***

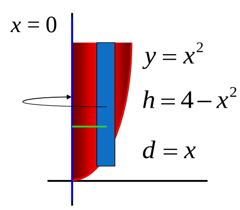
Using Shell Method, set up the definite integral which gives the volume of the solid formed when the given region is revolved about




| | |
|---|---|
| 1. $x = 0$ | 4. $y = 0$ |
| 2. $x = -1$ | 5. $y = 5$ |
| 3. $x = 3$ | 6. $y = -2$ |

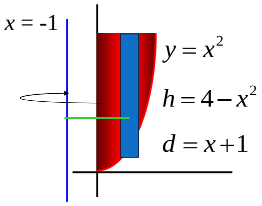


**1. $x = 0$**




$$V = 2\pi\int_0^2 x\left(4 - x^2\right)dx.$$

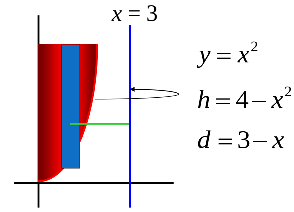
**2. $x = -1$**




$$V = 2\pi \int_0^2 (x+1)(4-x^2)\,dx.$$



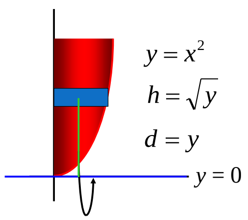
**3. $x = 3$**




$$V = 2\pi \int_0^2 (3-x)(4-x^2)\,dx.$$



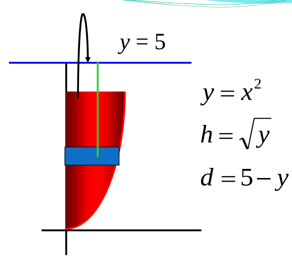
**4. $y = 0$**




$$V = 2\pi \int_0^4 y\sqrt{y}\,dy.$$



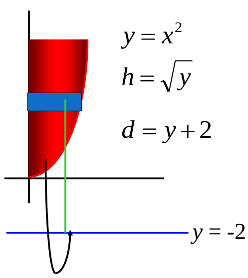
**5. $y = 5$**




$$V = 2\pi \int_0^4 (5-y)\sqrt{y}\,dy.$$



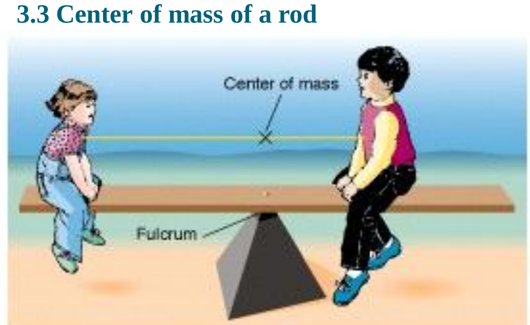
**6. $y = -2$**




$$V = 2\pi \int_0^4 (y+2)\sqrt{y}\,dy.$$

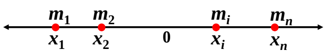


## 3.3 Center of mass of a rod



Consider a horizontal rod, of negligible weight and thickness, placed on the *x-axis.*

On the rod is a system of $n$ particles located at points $x_1, x_2, \ldots, x_n$.






The **total mass of the system** is

$$M = m_1 + m_2 + \ldots + m_n = \sum_{i=1}^{n} m_i .$$

The **moment of mass of the *ith* particle with respect to the origin** is

$$m_i x_i .$$

The **total moment of mass of the system with respect to the origin** is

$$M_0 = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n = \sum_{i=1}^{n} m_i x_i .$$



The **center of mass of the system** is at

$$\bar{x} = \frac{M_0}{M} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} .$$

$$\Rightarrow \left(\sum_{i=1}^{n} m_i\right)\bar{x} = \sum_{i=1}^{n} m_i x_i .$$



**Example 3.3.1** Find the center of mass of the system of 5 particles with masses 3,4,2,1 and 6 kgms located at -2,-1,0,3 and 4 respectively.

***Solution:***

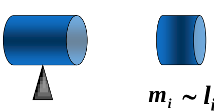
$$\bar{x} = \frac{\sum_{i=1}^{5} m_i x_i}{\sum_{i=1}^{5} m_i} = \frac{(3)(-2)+(3)(-1)+(3)(0)+(3)(3)+(3)(4)}{3+4+2+1+6}$$

$$= \frac{12}{16} = \frac{3}{4}.$$

The center of mass of the system is 0.75 to the right of the origin.



A rod is said to be ***homogeneous*** if the mass of a part of it is proportional to the length of that part.



$$m_i \sim l_i$$

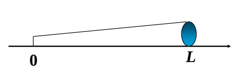
Linear density

$$m_i = k l_i$$

The center of mass of a homogeneous rod is at its center.



Consider a **non-homogeneous** rod, that is the linear density varies along the rod. Let $L$ meters be the length of the rod and place the rod on the *x-axis* so that an endpoint of the rod coincides with the origin and the other endpoint is at $L$.




Suppose the linear density at point $x$ is $\rho(x)$ where $\rho$ is continuous on $[0, L]$.

Consider a partition of $[0, L]$ into $n$ sub-intervals and let the $i$th subinterval be

$$I_i = [x_{i-1}, x_i].$$

Let $\varepsilon_i$ be a number in $I_i$.

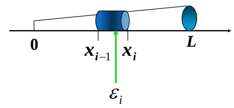
The length of the $i$th subinterval is

$$\Delta_i x.$$

An approximation to the mass of the part of the rod in the $i$th subinterval is

$$m_i = \rho(\varepsilon_i)\Delta_i x.$$






$$m_i = \rho(\varepsilon_i)\Delta_i x.$$

An approximation to the mass of the rod is

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \rho(\varepsilon_i)\Delta_i x$$

The mass of the rod is

$$M = \lim_{n\to\infty}\sum_{i=1}^{n} \rho(\varepsilon_i)\Delta_i x = \int_0^L \rho(x)dx$$



An approximation to the **moment of mass of the part of the rod in the $i$th subinterval with respect to the origin** is

$$m_i x_i = (\rho(\varepsilon_i)\Delta_i x)\cdot\varepsilon_i = \varepsilon_i\rho(\varepsilon_i)\Delta_i x.$$

An approximation to the **moment of mass of the rod with respect to the origin** is

$$\sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} \varepsilon_i\rho(\varepsilon_i)\Delta_i x.$$



The **moment of mass of the rod with respect to the origin** is

$$M_0 = \lim_{n\to\infty}\sum_{i=1}^{n} \varepsilon_i\rho(\varepsilon_i)\Delta_i x = \int_0^L x\rho(x)dx.$$

The **center of mass of the rod** is at

$$\bar{x} = \frac{M_0}{M} = \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx}.$$



**Example 3.3.2** Find the center of mass of a rod that is 20 cm. long and the linear density on the rod at a point $x$ centimeters from one end is $(3x + 2)$ gm/cm.

*Solution:*

$$\bar{x} = \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx} = \frac{\int_0^{20} x(3x+2)dx}{\int_0^{20}(3x+2)dx} = \frac{\int_0^{20}(3x^2+2x)dx}{\int_0^{20}(3x+2)dx}$$

$$= \frac{\left. x^3+x^2\right]_0^{20}}{\left.\frac{3}{2}x^2+2x\right]_0^{20}} = \frac{20^3+20^2}{\frac{3}{2}20^2+2\cdot 20} = \frac{(20^2+20)\cancel{20}}{\left(\frac{3}{2}20+2\right)\cancel{20}} = \frac{420}{32} = \frac{105}{8}$$



**Example 3.3.3** A rod is 12 cm. long and the linear density at a point on the rod is a linear function of the distance of the point from the left endpoint of the rod. If the linear density at the left end is 3 gm/cm and at the right end is 4 gm/cm, find the center of mass of the rod.

*Solution:*

Let $x$ be the distance from the left endpoint of a point on the rod and $\rho(x)$ be the linear density at $x$. Then for some constants $a$ and $b$,

$$\rho(x) = ax + b.$$

Since $\rho(0) = 3$ and $\rho(12) = 4$, and

$$\rho(x) = ax + b,$$

$$\rho(0) = 3 \quad \Rightarrow \boldsymbol{b} = 3$$

$$\Rightarrow \rho(x) = \boldsymbol{a}x + 3$$

$$\rho(12) = 4 \Rightarrow 12\boldsymbol{a} + 3 = 4$$

$$\Rightarrow \boldsymbol{a} = \frac{1}{12}$$

$$\Rightarrow \rho(x) = \frac{1}{12}x + 3$$



$$\bar{x} = \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx} = \frac{\int_0^{12} x\left(\frac{1}{12}x + 3\right)dx}{\int_0^{12}\left(\frac{1}{12}x + 3\right)dx}$$

$$= \frac{\int_0^{12}\left(\frac{1}{12}x^2 + 3x\right)dx}{\int_0^{12}\left(\frac{1}{12}x + 3\right)dx} = \frac{\left[\frac{1}{36}x^3 + \frac{3}{2}x^2\right]_0^{12}}{\left[\frac{1}{24}x^2 + 3x\right]_0^{12}}$$

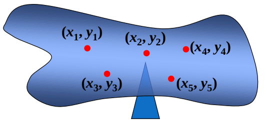


$$= \frac{\frac{1}{36}12^3 + \frac{3}{2}12^2}{\frac{1}{24}12^2 + 3\cdot 12} = \frac{\left(\frac{1}{36}12 + \frac{3}{2}\right)12^{\cancel{2}}}{\left(\frac{1}{24}12 + 3\right)\cancel{12}}$$

$$= \frac{\left(\frac{1}{3} + \frac{3}{2}\right)12}{\left(\frac{1}{2} + 3\right)} = \frac{4 + 18}{\frac{7}{2}} = \frac{22}{\frac{7}{2}} = \frac{44}{7}$$



## 3.4 Centroid of a Plane Region




**Consider a system of $n$ particles located at the points $(x_1, y_1)$, $(x_2, y_2)$ …, $(x_n, y_n)$ in the $xy$ plane and let their masses be $m_1$, $m_2$, .., $m_n$, respectively. Imagine the particles as supported by a sheet of negligible weight and thickness.**



**Let the $i$th particle, located at $(x_i, y_i)$ have mass $m_i$.**

**The total mass of the system is**

$$M = \sum_{i=1}^{n} m_i.$$

**The moment of mass of the system with respect to the $x$-axis is**

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

**The moment of mass of the system with respect to the $y$-axis is**

$$M_y = \sum_{i=1}^{n} m_i x_i.$$



**The center of mass of the system is at**

$$(\bar{x}, \bar{y})$$

**where**

$$\bar{x} = \frac{M_y}{M} = \frac{\sum_{i=1}^{n} m_i x_i}{M} \quad \textbf{and}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\sum_{i=1}^{n} m_i y_i}{M}.$$

**Example 3.4.1** Find the center of mass of the system of 4 particles with masses 4,2,3,6 and which are located at the points (-2,-3), (1,2), (3,-2) and (4,-3).

***Solution:***

$$M = \sum_{i=1}^{4} m_i = 4+2+3+6=15$$

$$M_x = \sum_{i=1}^{4} m_i y_i = 4(-3)+2(2)+3(-2)+6(3)= 4$$

$$M_y = \sum_{i=1}^{4} m_i x_i = 4(-2)+2(1)+3(3)+6(4)=27$$



The center of mass of the system is at

$$(\bar{x}, \bar{y})$$

where

$$\bar{x} = \frac{M_y}{M} = \frac{27}{15} = \frac{9}{5} \text{ and}$$

$$\bar{y} = \frac{M_x}{M} = \frac{4}{15}.$$



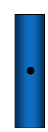
A lamina is said to be ***homogeneous*** if the mass of a portion of it is proportional to the area of the part. Thus, if $M$ is the mass of the region $R$ and $A$ is its area, then there is a constant $k$ such that

$$M = kA.$$

Area density



So if we have a homogeneous rectangular region, its mass $M_i$ is given by



$$M_i = k(lw).$$

Also, its *centroid or center of mass* is at its center.

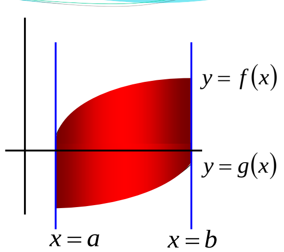


Let $R$ be the region bounded above and below, respectively by the graphs of

$$y = f(x),\ y = g(x)$$

and the lines given by

$$x = a \text{ and } x = b.$$





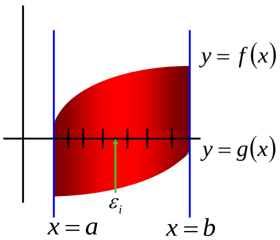

The area of the region is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

Its mass is given by

$$M = k\int_a^b [f(x) - g(x)]dx.$$

As we did in the first 2 sections, we subdivide **[*a*, *b*] into *n*** subintervals and we choose a number $\varepsilon_i$ in each subinterval.




**This time, we choose $\varepsilon_i$ as the abscissa of the midpoint of each sub-interval.**

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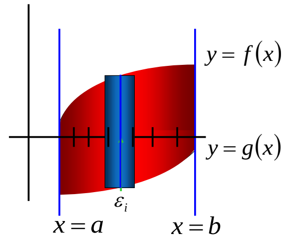
**Then we construct a rectangle whose width is $\Delta_i x$ and whose length is $f(\varepsilon_i) - g(\varepsilon_i)$.**

**The area of the *i*th rectangular region is**

$$[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x.$$

**Thus, the mass of the *i*th rectangular region is**

$$k[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x.$$

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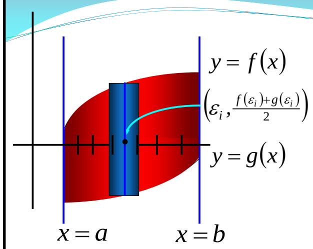


**The centroid of the *i*th rectangular region is at $\left(\varepsilon_i, \frac{f(\varepsilon_i)+g(\varepsilon_i)}{2}\right)$.**

**The *moment of mass* $m_{i,y}$ with respect to the *y* axis of the *i*th rectangular region is the product of its mass and the distance of its centroid from the *y*-axis. Thus,**

$$m_{i,y} = \varepsilon_i \cdot k[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x$$
$$= k\varepsilon_i[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x.$$

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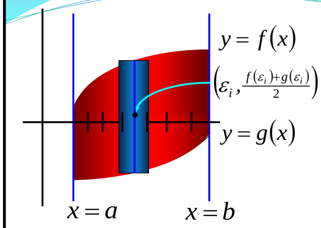


**The *moment of mass* $m_{i,x}$ with respect to the *x*- axis of the *i*th rectangular region is the product of its mass and the distance of its centroid from the *x*-axis. Thus,**

$$m_{i,x} = \frac{f(\varepsilon_i)+g(\varepsilon_i)}{2} \cdot k[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x$$
$$= \frac{k}{2}\left[[f(\varepsilon_i)]^2 - [g(\varepsilon_i)]^2\right]\Delta_i x.$$

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**The *total moment of mass* $M_{i,y}$ with respect to the *y* axis of the *n* rectangular regions is given by**

$$M_{i,y} = \sum_{i=1}^{n} m_{i,y} = \sum_{i=1}^{n} k\varepsilon_i[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x.$$

**The *moment of mass* $M_y$ with respect to the *y* axis of region *R* is given by**

$$M_y = \lim_{n\to\infty} M_{i,y} = \lim_{n\to\infty}\sum_{i=1}^{n} m_{i,y}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} k\varepsilon_i[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x$$
$$= k\lim_{n\to\infty}\sum_{i=1}^{n} \varepsilon_i[f(\varepsilon_i) - g(\varepsilon_i)]\Delta_i x$$
$$= k\int_a^b x[f(x) - g(x)]dx$$

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**The *total moment of mass* $M_{i,x}$ with respect to the *x* axis of the *n* rectangular regions is given by**

$$M_{i,x} = \sum_{i=1}^{n} m_{i,x} = \sum_{i=1}^{n} \frac{k}{2}\left[[f(\varepsilon_i)]^2 - [g(\varepsilon_i)]^2\right]\Delta_i x.$$

**The *moment of mass* $M_x$ with respect to the *x* axis of region *R* is given by**

$$M_x = \lim_{n\to\infty} M_{i,x} = \lim_{n\to\infty}\sum_{i=1}^{n} m_{i,x}$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{k}{2}\left[[f(\varepsilon_i)]^2 - [g(\varepsilon_i)]^2\right]\Delta_i x$$
$$= \frac{k}{2}\lim_{n\to\infty}\sum_{i=1}^{n} \left[[f(\varepsilon_i)]^2 - [g(\varepsilon_i)]^2\right]\Delta_i x$$
$$= \frac{k}{2}\int_a^b \left[[f(x)]^2 - [g(x)]^2\right]dx$$

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**If $(\bar{x},\bar{y})$ is the centroid of the plane region $R$ whose mass is $M$, then**

$$\bar{x}=\frac{M_y}{M} \quad \textbf{and} \quad \bar{y}=\frac{M_x}{M}.$$

**Since**

$$M=k\int_a^b [f(x)-g(x)]dx,$$

$$M_y=k\int_a^b x[f(x)-g(x)]dx,$$

$$M_x=\frac{k}{2}\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx$$

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$$\bar{x}=\frac{M_y}{M}=\frac{\cancel{k}\int_a^b x[f(x)-g(x)]dx}{\cancel{k}\int_a^b [f(x)-g(x)]dx}=\frac{\int_a^b x[f(x)-g(x)]dx}{\int_a^b [f(x)-g(x)]dx}$$

$$\bar{y}=\frac{M_x}{M}=\frac{\frac{\cancel{k}}{2}\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx}{\cancel{k}\int_a^b [f(x)-g(x)]dx}=\frac{\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx}{2\int_a^b [f(x)-g(x)]dx}$$

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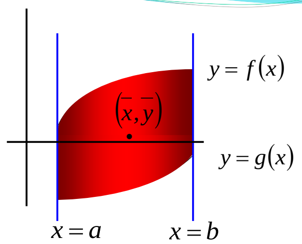
Summary:




$$\bar{x}=\frac{\int_a^b x[f(x)-g(x)]dx}{\int_a^b [f(x)-g(x)]dx} \qquad \bar{y}=\frac{\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx}{2\int_a^b [f(x)-g(x)]dx}$$

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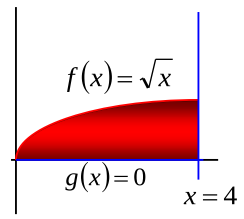
**Example 3.4.2**

Find the centroid of the region bounded by the graphs of $y=\sqrt{x}$, $x=4$ and the $x$-axis.

***Solution:***

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$$\bar{x}=\frac{\int_a^b x[f(x)-g(x)]dx}{\int_a^b [f(x)-g(x)]dx}=\frac{\int_0^4 x\left[\sqrt{x}-0\right]dx}{\int_0^4 \left[\sqrt{x}-0\right]dx}=\frac{\int_0^4 x^{3/2}dx}{\int_0^4 x^{1/2}dx}$$

$$=\frac{\left.\frac{x^{5/2}}{5/2}\right]_0^4}{\left.\frac{x^{3/2}}{3/2}\right]_0^4}=\frac{\left.\frac{2x^{5/2}}{5}\right]_0^4}{\left.\frac{2x^{3/2}}{3}\right]_0^4}=\frac{\frac{2\cdot 4^{5/2}}{5}}{\frac{2\cdot 4^{3/2}}{3}}=\frac{\frac{64}{5}}{\frac{16}{3}}$$

$$=\frac{64}{5}\cdot\frac{3}{16}=\frac{12}{5}.$$

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$$\bar{y}=\frac{\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx}{2\int_a^b [f(x)-g(x)]dx}=\frac{\int_0^4 \left[\left(\sqrt{x}\right)^2-0^2\right]dx}{2\int_0^4 \left[\sqrt{x}-0\right]dx}$$

$$=\frac{\int_0^4 xdx}{2\int_0^4 x^{1/2}dx}$$

$$=\frac{\left.\frac{x^2}{2}\right]_0^4}{2\cdot\frac{16}{3}}=\frac{\frac{4^2}{2}}{2\cdot\frac{16}{3}}=\frac{8}{\frac{32}{3}}=8\cdot\frac{3}{32}=\frac{3}{4}.$$

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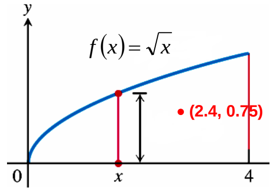
The centroid of the region bounded by the graphs of $y=\sqrt{x}$ , $x=4$ and the $x$-axis is at **(2.4, 0.75).**

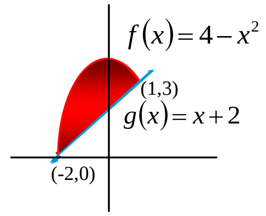





**Example 3.4.3**

Find the centroid of the region bounded by the graphs of $y=4-x^2$ and $y=x+2$ .

***Solution:***




$$x+2=4-x^2$$
$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$
$$x=-2 \text{ or } x=1.$$



$$\bar{x}=\frac{\int_a^b x[f(x)-g(x)]dx}{\int_a^b [f(x)-g(x)]dx}=\frac{\int_{-2}^1 x[(4-x^2)-(x+2)]dx}{\int_{-2}^1 [(4-x^2)-(x+2)]dx}$$

$$=\frac{\int_{-2}^1 x(2-x^2-x)dx}{\int_{-2}^1 (2-x^2-x)dx}=\frac{\int_{-2}^1 (2x-x^3-x^2)dx}{\int_{-2}^1 (2-x^2-x)dx}$$

$$=\frac{\left.x^2-\frac{x^4}{4}-\frac{x^3}{3}\right]_{-2}^1}{\left.2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^1}=\frac{\left(1-\frac{1}{4}-\frac{1}{3}\right)-\left((-2)^2-\frac{(-2)^4}{4}-\frac{(-2)^3}{3}\right)}{\left(2-\frac{1}{3}-\frac{1}{2}\right)-\left(2(-2)-\frac{(-2)^3}{3}-\frac{(-2)^2}{2}\right)}$$



$$=\frac{\left(1-\frac{1}{4}-\frac{1}{3}\right)-\left(2-4+\frac{8}{3}\right)}{\left(2-\frac{1}{3}-\frac{1}{2}\right)-\left(-4+\frac{8}{3}-2\right)}$$

$$=\frac{1-\frac{1}{4}-\frac{1}{3}+2-\frac{8}{3}}{2-\frac{1}{3}-\frac{1}{2}+6-\frac{8}{3}}$$

$$=\frac{-\frac{1}{4}}{5-\frac{1}{2}}=\frac{-\frac{1}{4}}{\frac{9}{2}}=-\frac{1}{4}\cdot\frac{2}{9}=\frac{-1}{18}$$



$$\bar{y}=\frac{\int_a^b \left[[f(x)]^2-[g(x)]^2\right]dx}{2\int_a^b [f(x)-g(x)]dx}=\frac{\int_{-2}^1 \left[(4-x^2)^2-(x+2)^2\right]dx}{2\int_{-2}^1 [(4-x^2)-(x+2)]dx}$$

$$=\frac{\int_{-2}^1 [(16-8x^2+x^4)-(x^2+4x+4)]dx}{2\int_{-2}^1 (2-x^2-x)dx}$$

$$=\frac{\int_{-2}^1 (12-9x^2+x^4-4x)dx}{2\int_{-2}^1 (2-x^2-x)dx}$$



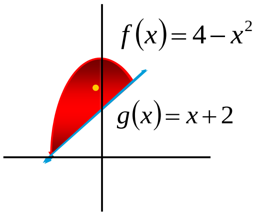
$$=\frac{\int_{-2}^1 (12-9x^2+x^4-4x)dx}{2\int_{-2}^1 (2-x^2-x)dx}=\frac{\left.12x-3x^3+\frac{x^5}{5}-2x^2\right]_{-2}^1}{2\left(2x-\frac{x^3}{3}-\frac{x^2}{2}\right)\Big]_{-2}^1}$$

$$=\frac{\left(12-3+\frac{1}{5}-2\right)-\left(12(-2)-3(-2)^3+\frac{(-2)^5}{5}-2(-2)^2\right)}{2\left[\left(2-\frac{1}{3}-\frac{1}{2}\right)-\left(2(-2)-\frac{(-2)^3}{3}-\frac{(-2)^2}{2}\right)\right]}$$

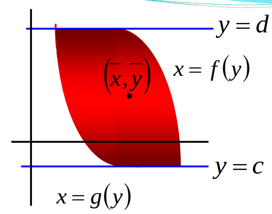
$$=\frac{7+\frac{1}{5}-\left(-\cancel{24}+\cancel{24}-\frac{32}{5}-8\right)}{2\left[\left(2-\frac{1}{3}-\frac{1}{2}\right)-\left(-4+\frac{8}{3}-2\right)\right]}=\frac{15+\frac{33}{5}}{2\left(5-\frac{1}{2}\right)}=\frac{\frac{108}{5}}{9}=\frac{12}{5}=2.4$$

The centroid of the region bounded by the graphs of $y = 4 - x^2$ and $y = x + 2$ is at

**(-1/18, 2.4).**





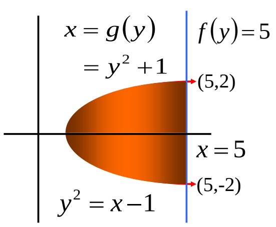

$$\bar{x} = \frac{\int_c^d \left[(f(y))^2 - (g(y))^2\right] dx}{2\int_c^d [f(y) - g(y)]dy} \qquad \bar{y} = \frac{\int_c^d y[[f(y)] - [g(y)]]dy}{\int_c^d [f(y) - g(y)]dy}$$



**Example 3.4.4**

Find the centroid of the region bounded by the graphs of $y^2 = x - 1$ and $x = 5$ .

***Solution:***






The centroid of the region is at $(\bar{x}, \bar{y})$ where,

$$\bar{x} = \frac{\int_c^d \left[(f(y))^2 - (g(y))^2\right] dx}{2\int_c^d [f(y) - g(y)]dy} = \frac{\int_{-2}^{2} \left[(5)^2 - (y^2 + 1)^2\right] dx}{2\int_{-2}^{2} (5 - (y^2 + 1))dy}$$

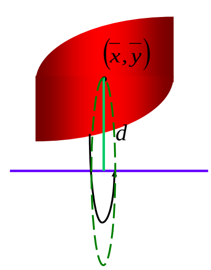
$$\bar{y} = \frac{\int_c^c y[[f(y)] - [g(y)]]dy}{\int_c^c [f(y) - g(y)]dy} = \frac{\int_{-2}^{2} y(5 - (y^2 + 1))dx}{\int_{-2}^{2} (5 - (y^2 + 1))dy}$$



## Theorem of Pappus

**When a plane region *R* is revolved about a line which does not intersect the region except possibly at its boundaries, the volume of the solid formed is equal to the *area of the region* times the *distance traveled by the centroid* of the region.**






**Let *A* be the area of the region.**

**The distance traveled by the centroid of the region when the region is revolved about the given line is $2\pi d$.**

**The volume of the solid of revolution is $A*(2\pi d)$.**

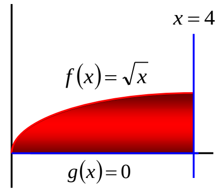
**Example 3.4.5**

Use the Theorem of Pappus to find the volume of the solid formed when the region bounded by the graphs of $y = \sqrt{x}$, $x = 4$ and the $x$-axis is revolved about the indicated line.

**a.** $y = 0$ **b.** $y = 6$ **c.** $x = 0$ **d.** $x = -2$



***Solution:***

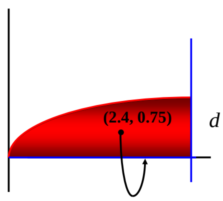



**From Example 3.4.2, the centroid of the region is at (2.4, 0.75).**

**Also from Example 3.4.2, the area of the region is 16/3 sq. units.**



a. About $y = 0$.

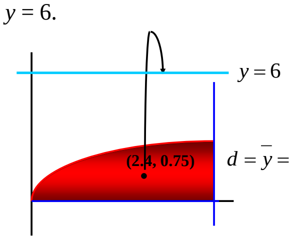



$d = \bar{y} = 0.75$

$$V = \frac{16}{3} \cdot 2\pi(0.75)$$
$$= \frac{16}{3} \cdot 2\pi \cdot \frac{3}{4}$$
$$= 8\pi \text{ cubic units.}$$



b. About $y = 6$.

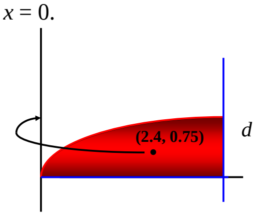



$d = \bar{y} = 6 - \frac{3}{4} = \frac{21}{4}$

$$V = \frac{16}{3} \cdot 2\pi\left(\frac{21}{4}\right)$$
$$= 56\pi \text{ cubic units.}$$



c. About $x = 0$.

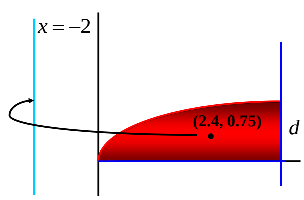



$d = \bar{x} = 2.4 = \frac{12}{5}$

$$V = \frac{16}{3} \cdot 2\pi \cdot \frac{12}{5}$$
$$= \frac{128\pi}{5} \text{ cubic units.}$$



d. About $x = -2$.




$d = \bar{x} + 2 = \frac{12}{5} + 2 = \frac{22}{5}$

$$V = \frac{16}{3} \cdot 2\pi\left(\frac{22}{5}\right)$$
$$= \frac{704\pi}{15} \text{ cubic units.}$$

## 3.5 Center of Mass of a Solid of Revolution

**In this section, we only consider finding the center of mass of a solid of revolution formed when a region whose boundary includes the *x*-axis is revolved about the *x*-axis or when a region whose boundary includes the *y*-axis is revolved about the *y*-axis.**



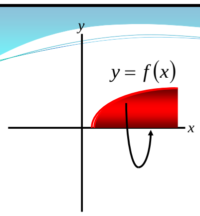
**Remarks:**

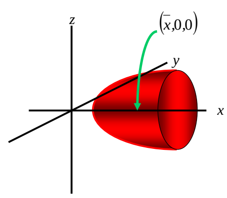
**In a solid of revolution, the center of mass lies along the axis of revolution, that is,**

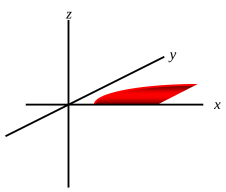
**i.** **if the solid of revolution *S* is generated by revolving a region *R* about the *x*-axis, then the centroid of *S* is located at** $(\bar{x},0,0)$ **.**

**ii.** **if *S* is generated by revolving a region *R* about the *y*-axis, then the centroid of *S* is located at** $(0,\bar{y},0)$ **.**



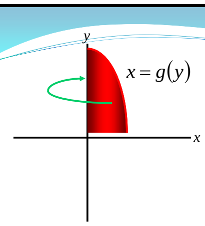


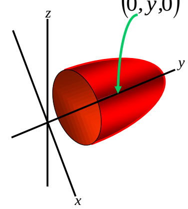



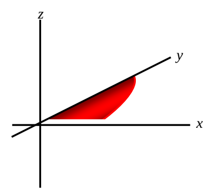




$$\bar{x}=\frac{\int_a^b x[f(x)]^2\,dx}{\int_a^b [f(x)]^2\,dx}$$






$$\bar{y}=\frac{\int_c^d y[g(y)]^2\,dy}{\int_c^d [g(y)]^2\,dy}$$



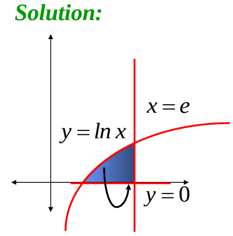
**Example 3.5.1** Find the center of mass of the solid formed when the region bounded by the graphs of

$$\boldsymbol{y=\ln x \;\;, x=e}$$

and the *x*-axis in the first quadrant is revolved about the *x*-axis.



***Solution:***




The centroid of the solid is at $(\bar{x},0,0)$ where,

$$\bar{x}=\frac{\int_1^e x[\ln x]^2\,dx}{\int_1^e [\ln x]^2\,dx}$$

$$\int x[\ln x]^2 dx = \int u\,dv$$

$$\boxed{u=[\ln x]^2,\ dv = x\,dx \quad du = 2\ln x\cdot\frac{1}{x}dx,\ v=\frac{x^2}{2}}$$

$$= uv - \int v\,du$$

$$= [\ln x]^2\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}dx$$

$$= \frac{x^2[\ln x]^2}{2} - \int x\ln x\,dx$$



$$\boxed{u=\ln x,\ dv = x\,dx \quad du = \frac{1}{x}dx,\ v=\frac{x^2}{2}}$$

$$\int x\ln x\,dx$$

$$= \int u\,dv = uv - \int v\,du$$

$$= \ln x\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}dx$$

$$= \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln x}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + C$$

$$= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$



$$\int x[\ln x]^2 dx = \frac{x^2[\ln x]^2}{2} - \int x\ln x\,dx$$

$$\int x\ln x\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$

$$\int x[\ln x]^2 dx = \frac{x^2[\ln x]^2}{2} - \left(\frac{x^2\ln x}{2} - \frac{x^2}{4}\right) + C$$

$$\int x[\ln x]^2 dx = \frac{x^2[\ln x]^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4} + C$$



$$\int_1^e x[\ln x]^2 dx = \left.\frac{x^2[\ln x]^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^e$$

$$= \left(\frac{e^2[\cancelto{1}{\ln e}]^2}{2} - \frac{e^2\cancelto{1}{\ln e}}{2} + \frac{e^2}{4}\right) - \left(\frac{[\cancelto{0}{\ln 1}]^2}{2} - \frac{\cancelto{0}{\ln 1}}{2} + \frac{1}{4}\right)$$

$$= \frac{e^2}{4} - \frac{1}{4} = \frac{e^2-1}{4}$$



$$\int [\ln x]^2 dx = \int u\,dv$$

$$\boxed{u=[\ln x]^2,\ dv = dx \quad du = 2\ln x\cdot\frac{1}{x}dx,\ v=x}$$

$$= uv - \int v\,du$$

$$= [\ln x]^2\cdot x - \int x\cdot 2\ln x\cdot\frac{1}{x}dx$$

$$= x[\ln x]^2 - 2\int \ln x\,dx$$

$$= x[\ln x]^2 - 2(x\ln x - x) + C$$

$$= \mathbf{x[\ln x]^2 - 2x\ln x + 2x + C}$$



$$\int_1^e [\ln x]^2 = x[\ln x]^2 - 2x\ln x + 2x\Big]_1^e$$

$$\int_1^e [\ln x]^2 = \left(e[\cancelto{1}{\ln e}]^2 - 2e\cancelto{1}{\ln e} + 2e\right) - \left([\cancelto{0}{\ln 1}]^2 - 2\cancelto{0}{\ln 1} + 2\right)$$

$$\int_1^e [\ln x]^2 = e - 2e + 2e - 2 = e - 2$$

The center of mass of the solid is at $(\bar{x},0,0)$ where,

$$\bar{x} = \frac{\int_1^e x[\ln x]^2 dx}{\int_1^e [\ln x]^2 dx} = \frac{\frac{e^2-1}{4}}{e-2} = \frac{e^2-1}{4(e-2)} \approx 2.24$$

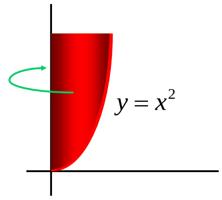
**Example 3.5.2** Find the center of mass of the solid formed when the region bounded by the graphs of

$$y = x^2 \quad , \quad y = 4$$

and the $y$-axis in the first quadrant is revolved about the $y$-axis.



***Solution:***



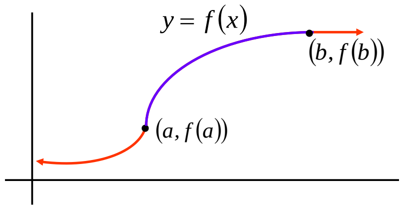

The centroid of the region is at $(0, \bar{y}, 0)$ where,

$$\bar{y} = \frac{\int_c^d y[g(y)]^2\,dx}{\int_c^d [g(y)]^2\,dx} = \frac{\int_0^4 y\left[\sqrt{y}\right]^2 dx}{\int_0^4 \left[\sqrt{y}\right]^2 dx} = \frac{\int_0^4 y^2\,dy}{\int_0^4 y\,dy} = \frac{\left.\frac{y^3}{3}\right]_0^4}{\left.\frac{y^2}{2}\right]_0^4} = \frac{\frac{64}{3}}{8} = \frac{8}{3}.$$

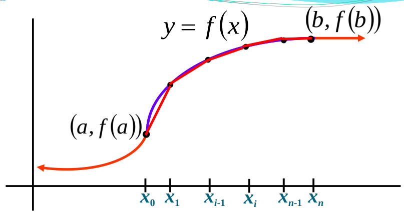


## 3.6 Length of Arc of a Curve

**Let the function $f$ and its derivative $f'$ be continuous on the closed interval $[a, b]$.**






**Subdivide the closed interval $[a, b]$ into $n$ sub-intervals by choosing $(n - 1)$ intermediate numbers $x_1, x_2, \ldots, x_{n-1}$ .**



**The distance between $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ is**

$$d_i = \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2}$$

$$= \sqrt{(f(x_i) - f(x_{i-1}))^2 + (\Delta_i x)^2}$$

$$= \sqrt{\left[\frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta_i x)^2} + 1\right](\Delta_i x)^2}$$

$$= \sqrt{\left[\left(\frac{f(x_i) - f(x_{i-1})}{\Delta_i x}\right)^2 + 1\right](\Delta_i x)^2}$$



$$d_i = \sqrt{1 + \left(\boxed{\frac{f(x_i) - f(x_{i-1})}{\Delta_i x}}\right)^2} \cdot (\Delta_i x)$$

**By the Mean-Value Theorem, there exists an $\varepsilon_i$ in $[x_{i-1}, x_i]$ such that**

$$f(\varepsilon_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$\boxed{f(\varepsilon_i)} = \frac{f(x_i) - f(x_{i-1})}{\Delta_i x}$$

$$d_i = \sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)$$

**An approximation to the length of the curve from (*a*, *f*(*a*)) to (*b*, *f*(*b*)) is**

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x).$$



**The length of the curve from (*a*, *f*(*a*)) to (*b*, *f*(*b*)) is**

$$L = \lim_{n\to\infty}\sum_{i=1}^{n} d_i = \lim_{n\to\infty}\sum_{i=1}^{n} \sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x).$$

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$



**Example 3.6.1** Find the length of the curve given by $y = e^x$ from the point where $x = 1$ to the point where $x = 2$ .

***Solution:***

$$f(x) = e^x \to f'(x) = e^x$$

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_1^2 \sqrt{1+\left[e^x\right]^2}\,dx$$

$$= \int_1^2 \sqrt{1+e^{2x}}\,dx$$



$$\int\sqrt{1+e^{2x}}\,dx = \int y\cdot\frac{ydy}{y^2-1}$$

$$y = \sqrt{1+e^{2x}}$$
$$y^2 = 1+e^{2x}$$
$$2ydy = 2e^{2x}dx$$
$$ydy = e^{2x}dx$$
$$ydy = \left(y^2-1\right)dx$$
$$\frac{ydy}{y^2-1} = dx$$

$$= \int\frac{y^2dy}{y^2-1}$$

$$= \int\left(1+\frac{1}{y^2-1}\right)dy$$

$$= \int\left(1+\frac{1}{2(y-1)}-\frac{1}{2(y+1)}\right)dy$$

$$= y+\frac{1}{2}ln\left|y-1\right|-\frac{1}{2}ln\left|y+1\right|+C$$



$$\int\sqrt{1+e^{2x}}\,dx = y+\frac{1}{2}ln\left|\frac{y-1}{y+1}\right|+C$$

$$\int\sqrt{1+e^{2x}}\,dx = y+\frac{1}{2}ln\left|\frac{\sqrt{1+e^{2x}}-1}{\sqrt{1+e^{2x}}+1}\right|+C$$

$$\int_1^2\sqrt{1+e^{2x}}\,dx = y+\frac{1}{2}ln\left|\frac{\sqrt{1+e^{2x}}-1}{\sqrt{1+e^{2x}}+1}\right|\Bigg]_1^2$$



$$= \left(2+\frac{1}{2}ln\left|\frac{\sqrt{1+e^4}-1}{\sqrt{1+e^4}+1}\right|\right)-\left(1+\frac{1}{2}ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right|\right)$$

$$= 1+\frac{1}{2}ln\left|\frac{\sqrt{1+e^4}-1}{\sqrt{1+e^4}+1}\right|-\frac{1}{2}ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right|$$

$$= 1+\frac{1}{2}ln\left|\left(\frac{\sqrt{1+e^4}-1}{\sqrt{1+e^4}+1}\right)\cdot\left(\frac{\sqrt{1+e^2}+1}{\sqrt{1+e^2}-1}\right)\right|$$

**Example 3.6.2** Find the length of the curve given by $y=\sqrt{4-x^2}$ on the interval $[-2,2]$.

***Solution:***

$$f(x)=\sqrt{4-x^2} \rightarrow f'(x)=\frac{-x}{\sqrt{4-x^2}}$$

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_{-2}^{2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$=\int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}\,dx$$



$$=2\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$$ **(Improper of type DB and DA)**

$$=2\int_{-2}^{0} \frac{1}{\sqrt{4-x^2}}\,dx+2\int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$$

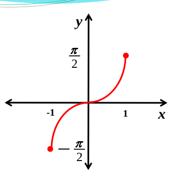
$$=2\lim_{a\to -2^+}\int_{a}^{0} \frac{1}{\sqrt{4-x^2}}\,dx+2\lim_{b\to 2^-}\int_{0}^{b} \frac{1}{\sqrt{4-x^2}}\,dx$$

$$=2\lim_{a\to -2^+} Arc\ sin\left(\tfrac{x}{2}\right)\Big]_a^0+2\lim_{b\to 2^-} Arc\ sin\left(\tfrac{x}{2}\right)\Big]_0^b$$



$$=2\lim_{a\to -2^+}\left(\overset{0}{\cancel{Arc\ sin(0)}}-Arc\ sin\left(\tfrac{a}{2}\right)\right)$$

$$+2\lim_{b\to 2^-}\left(Arc\ sin\left(\tfrac{b}{2}\right)-\overset{0}{\cancel{Arc\ sin\, 0}}\right)$$




$$=-2\lim_{a\to -2^+} Arc\ sin\left(\tfrac{a}{2}\right)$$

$$+2\lim_{b\to 2^-} Arc\ sin\left(\tfrac{b}{2}\right)$$

$$=-2\left(\tfrac{-\pi}{2}\right)+2\left(\tfrac{\pi}{2}\right)$$

$$=2\pi$$



**Let the function $g$ and its derivative $g'$ be continuous on the closed interval $[c,d]$. Then the length of the curve $x=g(y)$ from the point $(g(c),c)$ to the point $(g(d),d)$ is given by**

$$L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy$$



**Example 3.6.3** Find the length of the curve given by $xy=1$ from the point where $y=1$ to the point where $y=2$.

***Solution:***

$$xy=1 \rightarrow x=g(y)=\frac{1}{y} \rightarrow g'(y)=\frac{-1}{y^2}$$

$$L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy$$

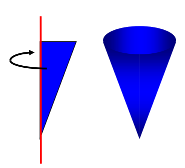
$$L=\int_1^2 \sqrt{1+\left(\frac{-1}{y^2}\right)^2}\,dy=\int_1^2 \sqrt{1+\frac{1}{y^4}}\,dy$$

**The evaluation of the integral is left to you as an exercise.**

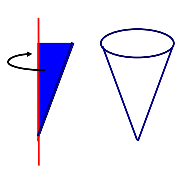


## 3.7 Area of a surface of revolution

**When a region is revolved about a line, a solid is formed.**

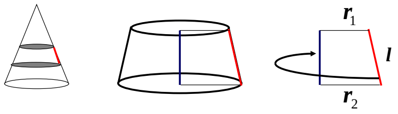


**When an arc is revolved about a line, a surface is formed.**



From Euclidean Geometry, the surface area of a frustum of a cone is

$$SA = \pi\, l(r_1 + r_2).$$



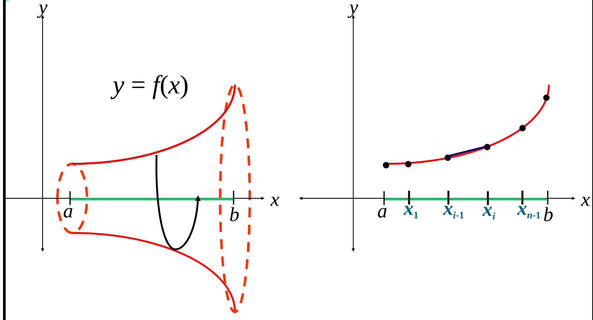

The frustum is generated by revolving the red segment about the blue one.



Let the function $f$ be continuous on the closed interval $[a, b]$, differentiable on the open interval $(a, b)$ and $f(x) \geq 0$ for each $x$ in $[a, b]$.

Let $C$ be the arc of the graph of $f$ on $[a, b]$. We wish to find the area of the surface formed when $C$ is revolved about the $x$-axis.








The length of the $i$th segment on the curve is

$$d_i = \sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)$$

With $r_1 = f(x_{i-1})$ , $r_2 = f(x_i)$ and

$$l = \sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x),$$

The area of the $i$th frustum is $\boxed{SA = \pi\, l(r_1 + r_2)}$

$$(SA)_i = \pi\underline{\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)}\underline{\left(f(x_{i-1}) + f(x_i)\right)}$$



$$= \pi\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)\cdot 2f(\overline{x_i}),$$

$$f(\overline{x_i}) = \frac{f(x_{i-1}) + f(x_i)}{2}$$

$$= 2\pi\, f(\overline{x_i})\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)$$

$$\approx 2\pi\, f(\varepsilon_i)\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)$$

The area of the surface of revolution is approximately equal to

$$\sum_{i=1}^{n} 2\pi\, f(\varepsilon_i)\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x).$$



The area of the surface of revolution is equal to

$$SA = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\, f(\varepsilon_i)\sqrt{1+\left(f(\varepsilon_i)\right)^2}\cdot(\Delta_i x)$$

$$SA = 2\pi\int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$SA = 2\pi\int_a^b d\ \underline{l\ ds}$$

Element of arclength

Distance of a point of the curve from the axis of revolution

**Example 3.7.a** Find the area of the surface formed when the arc of $y = e^x$ from the point where $x = 1$ to the point where $x = 2$ is revolved about the given indicated line.

| | |
|---|---|
| a. $y = 0$ | d. $x = 0$ |
| b. $y = 9$ | e. $x = 5$ |
| c. $y = -2$ | f. $x = -1$ |

***Solution:***

From Example 3.6.1, the length of the arc is given by

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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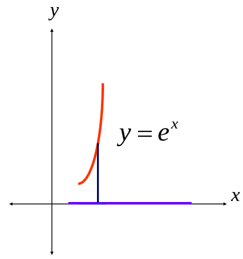
**a. $y = 0$**




$$SA = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= 2\pi\int_1^2 e^x\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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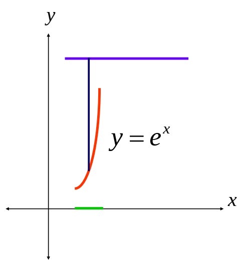
**b. $y = 9$**




$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+e^{2x}}\,dx$$

$$= 2\pi\int_1^2 \left(9-e^x\right)\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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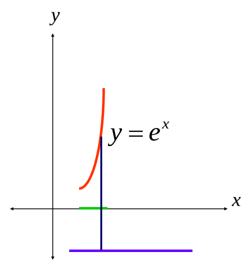
**c. $y = -2$**




$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+e^{2x}}\,dx$$

$$= 2\pi\int_1^2 \left(2+e^x\right)\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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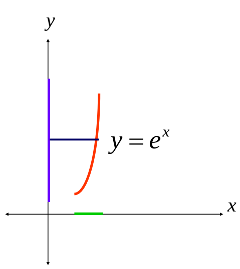
**d. $x = 0$**




$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+e^{2x}}\,dx$$

$$= 2\pi\int_1^2 x\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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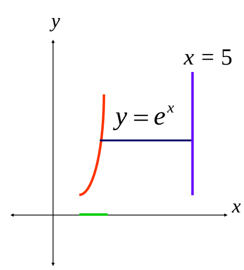
**e. $x = 5$**




$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+e^{2x}}\,dx$$

$$= 2\pi\int_1^2 \left(5-x\right)\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$

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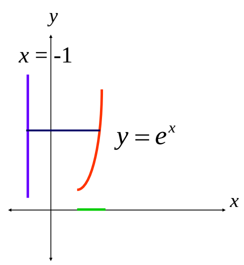
**f. $x = -1$**




$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+e^{2x}}\,dx$$

$$= 2\pi\int_1^2 (1+x)\sqrt{1+e^{2x}}\,dx$$

$$L = \int_1^2 \sqrt{1+e^{2x}}\,dx.$$



**Example 3.7.b** **Find the area of the surface formed when the arc of $y = e^x$ from the point where $x = 1$ to the point where $x = 2$ is revolved about the given indicated line.**

**a. $y = 0$** **d. $x = 0$**

**b. $y = 9$** **e. $x = 5$**

**c. $y = -2$** **f. $x = -1$**

***Solution:***

$$y = e^x \Rightarrow \ln y = x \qquad \Rightarrow \frac{1}{y}dy = dx$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{y}$$



$y = e^x$

$\Rightarrow$ when $x = 1$, $y = e$

$\Rightarrow$ when $x = 2$, $y = e^2$

**The length of the arc is given by**

$$L = \int_e^{e^2} \sqrt{1+\left(\frac{1}{y}\right)^2}\,dy$$

$$L = \int_e^{e^2} \sqrt{1+\frac{1}{y^2}}\,dy$$



**Example 3.7.c** **Find the area of the surface formed when the arc of $x = \frac{1}{y}$ from the point where $y = 1$ to the point where $y = 2$ is revolved about the given indicated line.**

**a. $x = 0$** **d. $y = 0$**

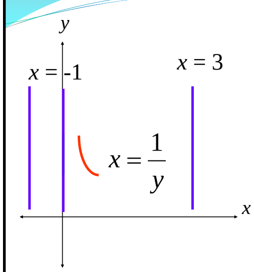
**b. $x = 3$** **e. $y = 3$**

**c. $x = -1$** **f. $y = -1$**

***Solution:***

$$x = \frac{1}{y} \qquad \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$$






$$L = \int_1^2 \sqrt{1+\frac{1}{y^4}}\,dy$$

a. $x = 0$

$$SA = 2\pi\int_c^d g(y)\sqrt{1+[g'(y)]^2}\,dy$$

$$= 2\pi\int_1^2 \frac{1}{y}\sqrt{1+\frac{1}{y^4}}\,dy$$

b. $x = 3$

$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+\frac{1}{y^4}}\,dy$$

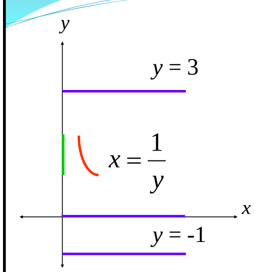
$$= 2\pi\int_1^2 \left(3-\frac{1}{y}\right)\sqrt{1+\frac{1}{y^4}}\,dy$$

c. $x = -1$

$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+\frac{1}{y^4}}\,dy$$

$$= 2\pi\int_1^2 \left(1+\frac{1}{y}\right)\sqrt{1+\frac{1}{y^4}}\,dy$$






$$L = \int_1^2 \sqrt{1+\frac{1}{y^4}}\,dy$$

c. $y = 0$

$$SA = 2\pi\int_c^d y\sqrt{1+[g'(y)]^2}\,dy$$

$$= 2\pi\int_1^2 y\sqrt{1+\frac{1}{y^4}}\,dy$$

d. $y = 3$

$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+\frac{1}{y^4}}\,dy$$

$$= 2\pi\int_1^2 (3-y)\sqrt{1+\frac{1}{y^4}}\,dy$$

e. $y = -1$

$$SA = 2\pi\int_1^2 \underline{?}\ \sqrt{1+\frac{1}{y^4}}\,dy$$

$$= 2\pi\int_1^2 (1+y)\sqrt{1+\frac{1}{y^4}}\,dy$$

The End

of Chapter 3

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