1.6

POWER SERIES

Definition.

A power series in $f(x)$ is a series of the form

$$c_0 + c_1 f(x) + c_2 \left[f(x)\right]^2 + \ldots + c_n \left[f(x)\right]^n + \ldots$$

$$= \sum_{n=0}^{+\infty} c_n \left[f(x)\right]^n$$

where the c_n's are constants.

In this section, we study how to find the value(s) of x so that a power series is convergent.

Definition.

A power series in x about a is a series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

$$= \sum_{n=0}^{+\infty} c_n (x-a)^n$$ where the c_n's are constants.

In this section, we study how to find the value(s) of x so that a power series is convergent.

Definitions

The interval for which a power series is convergent is called the

Interval of convergence

and half the length of this interval is called the

radius of convergence.

Theorem.

Consider a power series $\sum_{n=0}^{+\infty} c_n (x-a)^n$.

Then exactly one of the ff holds:

i. It converges only to $x = a$

I.O.C: $[a, a]$ ***R.O.C:*** 0

ii. It absolutely converges for all $x \in \mathbb{R}$

I.O.C: $(-\infty, +\infty)$ ***R.O.C:*** ∞

Theorem. (cont…)

iii. There exists $R > 0$ such that it is absolutely convergent for all

$$x \in (a - R, a + R)$$

R.O.C: R

and divergent for all

$$x \in (-\infty, a - R) \cup (a + R, +\infty)$$

We have to test convergence at the endpoints of the interval.

How to find the interval of convergence:

1. Apply the ***RATIO TEST*** or the ***ROOT TEST***.

2. Test convergence at the endpoints using tests other than the two stated above.

Examples. Find all values of x so that the given power series is convergent.

1. $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

Using *Root Test*, we'll have $L = \lim_{n \to +\infty} \sqrt[n]{\left| x^n \right|} = |x|$

***Recall*: To conclude convergence using this test,** $L < 1$.

So, $|x| < 1 \implies -1 < x < 1$

1. $\sum_{n=0}^{\infty} x^{n} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$

Now, we'll test convergence at the endpoints.

If $x = 1$ then $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1$.

NOTE: This series is divergent by the nth term test.

If $x = -1$ then $\sum_{n=0}^{\infty} x^n = -\sum_{n=0}^{\infty} 1$.

NOTE: This series is also divergent by the nth term test.

Thus, power series defined by $\sum_{n=0}^{\infty} x^n$

has

Interval of Convergence: $(-1,1)$

Radius of Convergence: 1

Examples. Find all values of x so that the given power series is convergent.

2. $$\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)5^n}$$

Using *Ratio Test*:

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{|x+3|}{5}$$

***Recall*: To conclude convergence using this test,** $L < 1$.

So, $\frac{|x+3|}{5} < 1 \quad \Rightarrow -8 < x < 2$

2. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+3)^n}{(n+1)5^n}$

Now, we'll test convergence at the endpoints.

If $x=-8$ **then**

$$\sum_{n=0}^{\infty}\frac{(x+3)^n}{(n+1)5^n}=\sum_{n=0}^{\infty}\frac{(-5)^n}{(n+1)5^n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$$

NOTE: This series is convergent by the alternating series test.

2. $\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)5^n}$

Now, we'll test convergence at the endpoints.

If $x = 2$ **then**

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)5^n} = \sum_{n=0}^{\infty} \frac{(5)^n}{(n+1)5^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

NOTE: The harmonic series is divergent.

Thus, power series defined by $$\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)5^n}$$

has

Interval of Convergence: $[-8, 2)$

Radius of Convergence: 5

Examples. Find all values of x so that the given power series is convergent.

3. $\sum_{n=0}^{\infty} n!(2x+1)^n$

Using *Ratio Test*:

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= |2x+1| \lim_{n\to+\infty} (n+1) = +\infty$$

Thus, power series defined by $$\sum_{n=0}^{\infty} n!(2x+1)^n$$

has

Interval of Convergence: $$\left[-\frac{1}{2},-\frac{1}{2}\right]$$

Radius of Convergence: $$0$$

Examples. Find all values of x so that the given power series is convergent.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{\left(x+\sqrt{3}\right)^{2n}}{(2n+1)!}$ **Using *Ratio Test*:**

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \left(x+\sqrt{3}\right)^2 \lim_{n\to+\infty} \frac{1}{(2n+3)(2n+2)}$$

$$= 0 \quad < 1$$

Thus, power series defined by $$\sum_{n=0}^{\infty} \frac{\left(x+\sqrt{3}\right)^{2n}}{(2n+1)!}$$

has

Interval of Convergence: $$\left(-\infty, +\infty\right)$$

Radius of Convergence: $$\infty$$

Definition.

A power series in x about a is a series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

$$= \sum_{n=0}^{+\infty} c_n (x-a)^n$$ where the c_n's are constants.

In this section, we study how to find the value(s) of x so that a power series is convergent.

Now, we study how to write the power series expansion of a function.

1.7

Differentiation of POWER SERIES

Term-by-Term Differentiation

A power series can be differentiated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

$$\sum_{n=1}^{+\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + n c_n x^{n-1} + \ldots$$

Theorem.

If the power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the

power series

$$f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

also has R as its radius of convergence.

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n \quad \Longrightarrow \quad f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

$$\Longrightarrow f''(x) = \sum_{n=2}^{+\infty} n(n-1) c_n x^{n-2}$$

$$\Longrightarrow f'''(x) = \sum_{n=3}^{+\infty} n(n-1)(n-2) c_n x^{n-3}$$

$$\vdots$$

Example. Find a series expansion for $f'(x)$ and $f''(x)$

if $f(x) = \dfrac{1}{1-x}$ $, -1 < x < 1$

SOL'N.

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{+\infty} x^n$$

$$, -1 < x < 1$$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \sum_{n=1}^{+\infty} nx^{n-1}$$

$$, -1 < x < 1$$

Example. Find a series expansion for $f'(x)$ and $f''(x)$

if $f(x)=\dfrac{1}{1-x}$ $,-1<x<1$

SOL'N.

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots=\sum_{n=1}^{+\infty}nx^{n-1}$$

$$,-1<x<1$$

$$f''(x)=\frac{2}{(1-x)^3}=2+6x+\ldots+n(n-1)x^{n-2}+\ldots$$

$$=\sum_{n=2}^{+\infty}n(n-1)x^{n-2}\quad ,-1<x<1$$

Example. **Obtain a series expansion for** $\dfrac{1}{(1+3x)^2}$ **and give its validity.**

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r} \quad , -1 < r < 1$$

SOL'N. $a = 1 \quad , r = -3x$

$$-1 < 3x < 1 \quad \Rightarrow \quad -\frac{1}{3} < x < \frac{1}{3}$$

$$g(x) = \frac{1}{1+3x} = \sum_{n=1}^{+\infty} (-3x)^{n-1} = \sum_{n=1}^{+\infty} (-1)^{n-1} 3^{n-1} x^{n-1}$$

$$g'(x) = \frac{-3}{(1+3x)^2} = \sum_{n=2}^{+\infty} (-1)^{n-1} 3^{n-1} (n-1) x^{n-2}$$

Example. Obtain a series expansion for $\dfrac{1}{(1+3x)^2}$

and give its validity.

SOL'N. Since, $$\frac{-3}{(1+3x)^2} = \sum_{n=2}^{+\infty}(-1)^{n-1}3^{n-1}(n-1)x^{n-2}$$

We'll have

$$\frac{1}{(1+3x)^2} = \frac{1}{-3}\sum_{n=2}^{+\infty}(-1)^{n-1}3^{n-1}(n-1)x^{n-2}$$

$$= \sum_{n=2}^{+\infty}(-1)^{n-2}3^{n-2}(n-1)x^{n-2} \quad ,-\frac{1}{3} < x < \frac{1}{3}$$

Example. **Obtain a series expansion for** $\dfrac{1}{(7-2x)^2}$ **and give its validity.**

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r} \quad ,-1<r<1$$

SOL'N. $a=1 \quad ,r=\dfrac{2}{7}x$

$$-1<\frac{2}{7}x<1 \quad \Rightarrow \quad -\frac{7}{2}<x<\frac{7}{2}$$

$$g(x)=\frac{1}{1-\frac{2}{7}x} = \frac{7}{7-2x} = \sum_{n=1}^{+\infty}\left(\frac{2}{7}x\right)^{n-1} = \sum_{n=1}^{+\infty}\left(\frac{2}{7}\right)^{n-1} x^{n-1}$$

$$g'(x)=\frac{14}{(7-2x)^2} = \sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$$

Example. Obtain a series expansion for $\dfrac{1}{(7-2x)^2}$ and give its validity.

SOL'N. Since, $$\frac{14}{(7-2x)^2} = \sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$$

We'll have

$$\frac{1}{(7-2x)^2} = \frac{1}{14}\sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$$

$$= \sum_{n=2}^{+\infty}\frac{(n-1)2^{n-2}x^{n-2}}{7^n} \quad ,-\frac{7}{2} < x < \frac{7}{2}$$

Integration of POWER SERIES

Term-by-Term Integration

A power series can be integrated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$
$$+ c_n x^n + \ldots$$

$$\sum_{n=0}^{+\infty} \frac{c_n x^{n+1}}{n+1} + C = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots$$
$$+ c_n \frac{x^{n+1}}{n+1} + \ldots$$

Theorem.

If the power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the

power series

$$\int f(x)\,dx = \sum_{n=0}^{+\infty} c_n \frac{x^{n+1}}{n+1} + C$$

also has R as its radius of convergence.

Example. **Obtain a series expansion for** $\ln(2+x)$ **and give its validity.**

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r} \quad ,-1<r<1$$

SOL'N. $a=1 \quad ,r=-\frac{1}{2}x$

$$-1<\frac{1}{2}x<1 \quad \Rightarrow \quad -2<x<2$$

$$g(x)=\frac{1}{1+\frac{1}{2}x} = \frac{2}{2+x} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}x\right)^{n-1} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1} x^{n-1}$$

$$g(t)=\frac{2}{2+t} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1} t^{n-1}$$

Example. **Obtain a series expansion for** $\ln(2+x)$

and give its validity.

SOL'N.

$$\int_0^x \frac{2\,dt}{2+t} = \int_0^x \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1} t^{n-1}dt$$

$$\left(2\ln(2+t)\right]_0^x = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1}\left(\frac{t^n}{n}\right]_0^x$$

$$2\ln(2+x) - 2\ln 2 = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1}\frac{x^n}{n}$$

$$\ln(2+x) = \frac{1}{2}\sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1}\frac{x^n}{n} + \ln 2 \quad ,-2<x<2$$

END