Have you ever asked . . .

How CALCULATORS compute

$$\ln 2$$

1.5

Some Tests of Convergence of Infinite Series

LIMIT COMPARISON TEST

Let $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ be two series of positive terms and $L = \lim\limits_{x\to\infty} \frac{u_n}{v_n}$

a. If L>0, then the two series either both converge or both diverge.

b. If L=0 and $\sum_{n=1}^{+\infty} v_n$ is convergent, then $\sum_{n=1}^{+\infty} u_n$ is also convergent

c. If L= ∞ and $\sum_{n=1}^{+\infty} v_n$ is divergent , then $\sum_{n=1}^{+\infty} u_n$ is also divergent

ILLUSTRATION $\sum_{n=1}^{+\infty} \frac{n}{4n^4-3}$

Solution

$$let\ u_n = \frac{n}{4n^4-3} \qquad v_n = \frac{1}{n^3}$$

$$\lim_{x\to\infty} \frac{u_n}{v_n} = \lim_{x\to\infty}\left(\frac{n}{4n^4-3}\right)\left(\frac{n^3}{1}\right) = \frac{1}{4} > 0$$

Since $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ is convergent, then $\sum_{n=1}^{+\infty} \frac{n}{4n^4-3}$ is also convergent.

ILLUSTRATION $\sum_{n=1}^{+\infty} \frac{5n^3}{3n^4+n}$

Solution

$$let\ u_n = \frac{5n^3}{3n^4+n} \qquad v_n = \frac{1}{n}$$

$$\lim_{x\to\infty} \frac{u_n}{v_n} = \lim_{x\to\infty}\left(\frac{5n^3}{3n^4+n}\right)\left(\frac{n}{1}\right) = \frac{5}{3} > 0$$

Since $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{+\infty} \frac{5n^3}{3n^4+n}$ is also divergent.

Alternating series

If $u_n > 0$ for each natural number n, then the following are alternating series

$$\sum_{n=1}^{\infty} (-1)^n u_n = -u_1 + u_2 - u_3 + \ldots$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - \ldots$$

Example

1. $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ alternating harmonic series

2. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{3^{n-1}}$

Alternating Series Test

Given $\sum_{n=1}^{\infty} (-1)^n u_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$

If i. $u_n > u_{n+1}$ for each n

ii. $\lim_{n \to +\infty} u_n = 0$

then the series is convergent.

Illustration: Show convergence.
Conclude properly.

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{3^{n-1}}$$

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{3^{n-1}}$$

Solution

$u_n = \frac{2}{3^{n-1}}$ since $u_n = \frac{2}{3^{n-1}} > \frac{2}{3^n} = u_{n+1}$

and $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2}{3^{n-1}} = 0$

By AST, $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{3^{n-1}}$ is convergent.

ASIDE . . . (other approach)

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{3^{n-1}} = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{2}{3^{n-1}}$$

$$= \sum_{n=1}^{+\infty} 2 \cdot \left(-\frac{1}{3}\right)^{n-1}$$

$$= \frac{a}{1-r} = \frac{3}{2}$$

Absolute convergence

If $\sum_{n=1}^{\infty} |u_n|$ is convergent,

then $\sum_{n=1}^{\infty} u_n$ is also convergent.

Ratio test

$\sum_{n=1}^{\infty} u_n$ is an infinite series where u_n is nonzero.

$$L = \lim_{n \to +\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

Ratio test

If $L < 1$, the series is (absolutely) convergent.

If $L > 1$ or $L = \infty$, the series is divergent.

If $L = 1$, NO CONCLUSION on convergence or divergence of the series.

USE OTHER TESTS.

Example. Test for convergence.

$$\sum_{n=1}^{+\infty} \frac{1}{n!}$$

Example. Test for convergence.

$$\sum_{n=1}^{+\infty} (-1)^n \frac{2^n}{n!}$$

Example. Test for convergence.

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{e^n}{n^2}$$

Example. Test for convergence.

$$\sum_{n=1}^{+\infty} (-1)^n \frac{n}{n^2+1}$$

OTHER TESTS

- ✓ INTEGRAL TESTS
- ✓ COMPARISON TEST
- ✓ ROOT TEST

END