CMSC 57 Discrete Mathematical Structures in Computer Science II
Exercise 5: Recurrence Relation

1. Find a recurrence relation and initial condition that generates a sequence that begins with the given sequence: 3, 6, 9, 15, 24, 39, …

   A_n = nth element

$$A_n = \begin{cases} 3 & n=0 \\ 6 & n=1 \\ A_{n-1} + A_{n-2} & n>1 \end{cases}$$

2. The population of Prontera increases 5 percent per year. In 2013, the population was 10,000. What was the population in 1990?

   P_n = population of Prontera in n years
   Initial Condition: $P_0 = 10000$
   Recurrence Relation: $P_n = 1.05P_{n-1}$

   Solve for the Recurrence Relation:

$$P_n = 1.05P_{n-1}$$
$$P_n = (1.05)(1.05P_{n-2})$$
$$P_n = (1.05)(1.05)(1.05P_{n-3})$$
$$P_n = (1.05)^k P_{n-k} \qquad \text{//let k=n}$$
$$P_n = (1.05)^n P_0$$
$$P_n = (1.05)^n\, 10000$$

   Solve for population in 1990. Solution 1:
   Year 2013: $P_0 = 10000$ ➔ $P_{2013} = 10000$

$$P_{2013} = P_{2013-23}(1.05)^{23}$$
$$P_{2013} = P_{1990}(1.05)^{23}$$
$$P_{1990} = \frac{P_{2013}}{(1.05)^{23}}$$
$$P_{1990} = \frac{10000}{(1.05)^{23}}$$
$$P_{1990} = 3255.71 \quad \text{OR} \quad P_{1990} \approx 3256$$

   Solution 2:

$$P_n = 10000(1.05)^n$$
$$P_{1990} = 10000(1.05)^{-23}$$
$$P_{1990} = 3255.71 \quad \text{OR} \quad P_{1990} \approx 3256$$

3. The number of fastfood chains in the Philippines is 500 at time n = 0, and 550 at time n =1, and that the increase from time n-1 to time n is five times the increase from time n-2 to time n-1. Write a recurrence relation and an initial condition that defines the number of fastfood chains at time n and solve for the recurrence relation. What is the total number of food chains at n = 30?

   P_n = the number of fast food chains in the Philippines at time n
   Initial Condition: $F_0 = 500$ $F_1 = 550$
   Recurrence Relation: $F_n = 6F_{n-1} - 5F_{n-2}$

   Solve for the Recurrence Relation:

   1- Transform the R.R: $F_n - 6F_{n-1} + 5F_{n-2} = 0$
   2- Quadratic eqn: $x^2 - 6x + 5 = 0$
   3- Roots: $(x-5)(x-1) : x_1 = 5 \quad x_2 = 1$
   4- $x_1 \neq x_2$, use the form: $F_n = C_1X_1^n + C_2X_2^n$
      $F_n = C_15^n + C_21^n$
   5- Solve for C_1 and C_2

$500 = C_15^0 + C_21^0$	$550 = C_15^1 + C_21^1$
$500 = C_1 + C_2$	$550 = 5C_1 + 1C_2$
$500 - C_2 = C_1$	$550 = 5(500 - C_2) + C_2$
	$550 = 2500 - 5C_2 + C_2$
	$550-2500 = -4C_2$
	$-1950 = -4C_2$
	$C_2 = 487.5$
$500 - 487.5 = C_1$	
$12.5 = C_1$	

   Formula: $F_n = (12.5)5^n + (487.5)1^n$

   Total no. of food chains at n=30: $F_n = (12.5)5^{30} + (487.5)1^{30}$

4. Solve for the recurrence relation:

a.) $a_n = -3a_{n-1}$ $\quad a_0 = 2$

$a_n = -3a_{n-1}$
$a_n = -3 * (-3\, a_{n-2})$
$a_n = -3 * -3 * (-3\, a_{n-3})$
$a_n = -3^k\, a_{n-k}$
$a_n = -3^n\, a_{n-n}$ //let k=n
$a_n = -3^n\, a_0$
$a_n = -3^n(2)$

b.) $a_n = a_{n-1} + n$ $\quad a_0 = 0$

$a_n = a_{n-1} + n$
$a_n = a_{n-2} + (n-1) + n$
$a_n = a_{n-3} + (n-2) + (n-1) + n$

$a_n = a_{n-k} + \left(\sum_{i=0}^{k-1} (n-i) \right)$ //let k=n

$a_n = 0 + \left(\sum_{i=0}^{n-1} (n-i) \right)$

c.) $a_n = 6a_{n-1} – 8a_{n-2}$ $\quad a_0 = 1 \quad a_1=0$

$a_n = 6a_{n-1} – 8a_{n-2}$
$a_n - 6a_{n-1} + 8a_{n-2} = 0$
$x^2 – 6x + 8 = 0$
$(x -4)(x-2) = 0$ $\quad => \quad x_1 = 4 \quad x_2 = 2$

$a_n = C_1 4^n + C_2 2^n$

$1 = C_1 4^0 + C_2 2^0$
$1 = C_1 + C_2$
$1 - C_2 = C_1$

$1 – 2 = C_1$
$– 1 = C_1$

$0 = C_1 4^1 + C_2 2^1$
$0 = 4\, C_1 + 2C_2$
$0 = 4(1 - C_2) + 2C_2$
$0 = 4 - 4C_2 + 2C_2$
$-4 = - 2C_2$
$2 = C_2$

$a_n = (-1)4^n + (2)2^n$

d. $a_n = 2a_{n-1} + 8a_{n-2}$ $\quad a_0 = 4 \quad a_1=10$

$a_n = 2a_{n-1} + 8a_{n-2}$
$a_n - 2a_{n-1} - 8a_{n-2} = 0$
$x^2 – 2x – 8 = 0$
$(x -4)(x+2) = 0$ $\quad => \quad x_1 = 4 \quad x_2 = -2$

$a_n = C_1 4^n + C_2(-2)^n$

$4 = C_1 4^0 + C_2(-2)^0$
$4 = C_1 + C_2$
$4 - C_2 = C_1$

$4 – 1 = C_1$
$3 = C_1$

$10 = C_1 4^1 + C_2(-2)^1$
$10 = 4C_1 - 2C_2$
$10 = 4(4 - C_2) - 2C_2$
$10 = 16 - 4C_2 - 2C_2$
$-6 = - 6C_2$
$1 = C_2$

$a_n = (3)4^n + (1)(-2)^n$