1.3

INFINITE SERIES of CONSTANT TERMS

Definition.

Let $\{u_n\}$ be a sequence of real numbers and

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Then the sequence $\{s_n\}$ is called an *infinite series*.

NOTATION: $\{s_n\}$, $\sum_{n=1}^{\infty} u_n$

Definition.

In an infinite series $\sum_{n=1}^{\infty} u_n$,

$u_1, u_2, \ldots, u_n, \ldots$ are called the *terms* of the infinite series

$s_1, s_2, \ldots, s_n, \ldots$ are called the *partial sums* of the infinite series

$\{s_n\}$ is the *sequence of partial sums* defining the infinite series

Example 1. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

The *first four terms* of the series are

$$u_1 = \frac{1}{2} \qquad u_2 = \frac{1}{4} \qquad u_3 = \frac{1}{8} \qquad u_4 = \frac{1}{16}$$

The *first four partial sums* of the series are

$$s_1 = \frac{1}{2} \qquad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \qquad s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

Remarks:

- If $\{s_n\}$ is the sequence of partial sums defining the infinite series $\sum_{n=1}^{\infty} u_n$

  Then for $n \geq 2,\ s_n = s_{n-1} + u_n$.

- Our main concern on infinite series is to determine whether the series converges or not.

Definitions.

Consider an infinite series $\sum_{n=1}^{\infty} u_n$ and $\{s_n\}$

be the sequence of partial sums defining the infinite series.

If $\lim_{n \to +\infty} s_n$ exists and is equal to S ,

Then: $\sum_{n=1}^{\infty} u_n$ is *convergent*

S is the *sum* of the infinite series

Definitions.

If $\lim_{n \to +\infty} s_n$ does not exist,

Then:

$$\sum_{n=1}^{\infty} u_n$$ is *divergent*

and it does not have a sum.

Can you add an infinite number of terms and have a finite sum?

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$$

$$= 1$$

Prove. $$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots=1$$

PROOF.

$$s_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{n-1}}+\frac{1}{2^n}$$

$$\frac{1}{2}s_n=\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$$

$$s_n-\frac{1}{2}s_n=\frac{1}{2}-\frac{1}{2^{n+1}}\quad\Rightarrow\quad\frac{1}{2}s_n=\frac{1}{2}-\frac{1}{2^{n+1}}$$

PROOF. (cont.)

$$\frac{1}{2}s_n = \frac{1}{2} - \frac{1}{2^{n+1}} \quad \Rightarrow \quad s_n = 1 - \frac{1}{2^n}$$

Now,

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(1 - \frac{1}{2^n}\right) = 1 \qquad \boxed{0}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}$$ is convergent and its sum is 1 .

Example 2. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)(3n+2)}$

Let $u_n = \dfrac{1}{(3n-1)(3n+2)}$

Recall that

$$\frac{1}{(3n-1)(3n+2)} = \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

Example 2. (cont.)

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(3n+2)}$$

Now,

$$s_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$$

$$s_n = \left[\frac{1}{3(2)} - \cancel{\frac{1}{3(5)}}\right] + \left[\cancel{\frac{1}{3(5)}} - \cancel{\frac{1}{3(8)}}\right] + \left[\cancel{\frac{1}{3(8)}} - \cancel{\frac{1}{3(11)}}\right]$$

$$+ \ldots + \left[\cancel{\frac{1}{3(3n-4)}} - \cancel{\frac{1}{3(3n-1)}}\right] + \left[\cancel{\frac{1}{3(3n-1)}} - \frac{1}{3(3n+2)}\right]$$

Example 2. (cont.)

So, $s_n = \dfrac{1}{3(2)} - \dfrac{1}{3(3n+2)}$

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(\frac{1}{6} - \underbrace{\frac{1}{3(3n+2)}}_{\boxed{0}}\right) = \frac{1}{6}$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)(3n+2)}$

is convergent and its sum is $\dfrac{1}{6}$.

Convergent	Divergent
$\sum_{n=1}^{\infty}\left[\frac{k_1}{f(n)}-\frac{k_2}{f(n+1)}\right]$	
$\sum_{n=1}^{\infty} ar^{n-1} \quad , \lvert r \rvert < 1$	$\sum_{n=1}^{\infty} ar^{n-1} \quad , \lvert r \rvert \geq 1$
$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad , p > 1$	$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad , p \leq 1$
$\sum_{n=1}^{\infty} \frac{1}{n!}$	$\sum_{n=1}^{\infty} a_n$ for which $\lim_{n \to +\infty} a_n \neq 0$

Theorem.

If $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n \to +\infty} u_n = 0$.

PROOF.

Suppose $\sum_{n=1}^{\infty} u_n$ is convergent with sum S.

Then $\lim_{n \to +\infty} s_n = S$.

Recall: $s_n = s_{n-1} + u_n$

Thus, $\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} \left(s_n - s_{n-1} \right) = 0$ ∎

Remark:

- ✓ If $\lim\limits_{n \to +\infty} u_n \neq 0$, then $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

BUT

- ✓ If $\lim\limits_{n \to +\infty} u_n = 0$, then $\sum\limits_{n=1}^{\infty} u_n$ is either

  divergent or convergent.

Examples. Explain why each series is divergent.

$$\sum_{n=1}^{\infty} \frac{3n}{n+2} \qquad \sum_{n=1}^{\infty} \frac{5-n^2}{2+3n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{1+e^{-n}} \qquad \sum_{n=1}^{\infty} n \sin\left(\frac{\pi}{n}\right)$$

$$\sum_{n=1}^{\infty} n^2 \qquad \sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right) \qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

Theorem.

If $\sum_{n=1}^{\infty} u_n$ is convergent and $\{s_n\}$ is the sequence of partial sums defining the series, then for each $\varepsilon > 0$, there exists a number N such that if R and T are natural numbers such that $R > N$ and $T > N$, then

$$\left| s_R - s_T \right| < \varepsilon .$$

Theorem. The *harmonic series* is divergent.

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

PROOF. Here we use the previous theorem with

$$R=2n,\ T=n$$

$$s_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

$$s_{2n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\frac{1}{n+1}+\ldots+\left(\frac{1}{2n}\right)$$

PROOF. (cont.)

$$\left|s_{2n}-s_n\right| = \left|\frac{1}{n+1}+\ldots+\frac{1}{2n}\right| = \frac{1}{n+1}+\ldots+\frac{1}{2n}$$

$$\geq \frac{1}{2n}+\ldots+\frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

That is, no number N exists such that

$\left|s_{2n}-s_n\right|<\varepsilon$ when $\varepsilon=\dfrac{1}{2}$.

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. ∎

Theorem. The *geometric series* $\sum_{n=1}^{\infty} ar^{n-1}$

where a and r are constants and $a \neq 0$.

1. converges to $\dfrac{a}{1-r}$ if $|r| < 1$;

2. diverges if $|r| \geq 1$.

PROOF. $$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n$$

PROOF. (cont.)

$$s_n - rs_n = a - ar^n \quad \Rightarrow \quad s_n(1-r) = a\left(1-r^n\right)$$

$$s_n = \frac{a\left(1-r^n\right)}{1-r} \quad \Rightarrow \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{a\left(1-r^n\right)}{1-r}$$

If $|r| < 1$, then $\lim\limits_{n\to+\infty} s_n = \dfrac{a}{1-r}$.

If $|r| \geq 1$, then $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} ar^{n-1} \neq 0$

Thus, theorem holds. ▨

Examples. Determine if the geometric series is convergent. If it does, find its sum.

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{5}\right)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{1}{3}(4)^{n-1}$$

$$\sum_{n=1}^{\infty} \frac{2^{n}}{5^{n-1}} \qquad \sum_{n=1}^{\infty} \frac{e^{n-1}}{2^{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{3^{n-2}}{\pi^{n}} \qquad \sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{3}{7}\right)^{n-1}$$

4 Theorems about Infinite Series.

Consider two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

1. If they differ only in a finite number of terms, then either both series converge or both diverge.

4 Theorems about Infinite Series.

2a. If the series $\sum_{n=1}^{\infty} a_n$ is convergent and its sum is S, then the series $\sum_{n=1}^{\infty} ca_n$ is also convergent and its sum is cS for each constant c.

4 Theorems about Infinite Series.

2b. If the series $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} d a_n$ is also divergent for each nonzero constant d .

4 Theorems about Infinite Series.

3. The sum or difference of 2 convergent series is also convergent.

4. The sum or difference of a convergent series and a divergent series is divergent.

Examples. Determine whether the series is convergent or divergent. Explain why.

$$\sum_{n=1}^{\infty} \frac{1}{n+3} \qquad \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{n} \right]$$

$$\sum_{n=1}^{\infty} \left[\frac{3n}{n+2} + \frac{1}{2^n} \right] \qquad \sum_{n=1}^{\infty} \frac{3}{n}$$

$$\sum_{n=1}^{\infty} \left[\frac{2^n}{5^{n-1}} + \frac{1}{2}\left(-\frac{3}{7} \right)^{n-1} \right]$$

END