Obtaining a Function from its Gradients

Chapter 3 Section 3

Theorem.

Suppose M and N are functions of 2 variables x and y on an open disk $B((x_0, y_0); r)$ in R^2 and M_y, N_x are continuous on B.

Then the vector $\langle M(x,y), N(x,y) \rangle$

is a ***gradient*** on B if and only if

$$M_y(x,y) = N_x(x,y)$$

for all point (x,y) in B.

Remark:

Suppose $\langle M(x,y), N(x,y)\rangle$ is a gradient on B.

Then there is a function f for which

$$f_x(x,y) = M(x,y)$$

$$f_y(x,y) = N(x,y)$$

for all point (x,y) in B.

OUR AIM: ***Find the function f.***

Example.

Show that the vector $\left\langle 4xy + y, 2x^2 + \sin y \right\rangle$ is not a gradient.

Solution.

$$M = 4xy + y \quad \Rightarrow \quad M_y = 4x + 1$$

$$N = 2x^2 + \sin y \quad \Rightarrow \quad N_x = 4x$$

Since $M_y \neq N_x$, $\left\langle 4xy + y, 2x^2 + \sin y \right\rangle$

is not a gradient.

Theorem.

Suppose M and N are functions of 2 variables x and y on an open disk $B\left(\left(x_0, y_0\right); r\right)$ in R^2 and M_y, N_x are continuous on B.

Then the differential equation

$$M\left(x, y\right) dx + N\left(x, y\right) dy = 0$$

is ***exact*** on B if and only if

$$M_y\left(x, y\right) = N_x\left(x, y\right)$$

for all point $\left(x, y\right)$ in B.

Remark:

If the D. E. $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact,

Its general solution is given by $f(x,y)+c$

where

$$\left.\begin{aligned} f_x(x,y) &= M(x,y) \\ f_y(x,y) &= N(x,y) \end{aligned}\right. \quad , \; c \in R$$

Example.

Solve $\left(4xy+1\right)dx+\left(2x^2+\cos y\right)dy=0$

Solution.

$$M\left(x,y\right)=4xy+1 \quad \Longrightarrow \quad M_y\left(x,y\right)=4x$$

$$N\left(x,y\right)=2x^2+\cos y \quad \Longrightarrow \quad N_x\left(x,y\right)=4x$$

Since $M_y=N_x$, the D.E. is exact.

Example.

Solve $\left(4xy+1\right)dx+\left(2x^2+\cos y\right)dy=0$

Now, $f_x\left(x,y\right)=4xy+1$

$$\Rightarrow f\left(x,y\right)=2x^2y+x+g\left(y\right)$$

$$\Rightarrow f_y\left(x,y\right)=2x^2+g'\left(y\right)$$

So, $g'\left(y\right)=\cos y \quad \Rightarrow \quad g\left(y\right)=\sin y+c$

Thus, $f\left(x,y\right)=2x^2y+x+\sin y+c$ is the solution to the given D.E.

Example.

Solve $y^2 + \left(2xy - e^{-2y}\right) y' = 0$

Solution.

$$y^2 dx + \left(2xy - e^{-2y}\right) dy = 0$$

$$M\left(x, y\right) = y^2 \quad \Longrightarrow \quad M_y\left(x, y\right) = 2y$$

$$N\left(x, y\right) = 2xy - e^{-2y} \quad \Longrightarrow \quad N_x\left(x, y\right) = 2y$$

Since $M_y = N_x$, the D.E. is exact.

Example.

Solve $y^2 + \left(2xy - e^{-2y}\right) y' = 0$

Now, $f_x\left(x, y\right) = y^2$

$$\Rightarrow \quad f\left(x, y\right) = xy^2 + g\left(y\right)$$

$$\Rightarrow \quad f_y\left(x, y\right) = 2xy + g'\left(y\right)$$

So, $g'\left(y\right) = -e^{-2y} \quad \Rightarrow \quad g\left(y\right) = \dfrac{e^{-2y}}{2} + c$

Thus, $f\left(x, y\right) = xy^2 + \dfrac{e^{-2y}}{2} + c$ is the solution to the given D.E.

Exercise. Solutions to Exact D.E.

Solve the D.E.'s that are exact.

1. $$\left(ye^{xy} - x\right)dx + \left(xe^{xy} - y\right)dy = 0$$

2. $$y' = -\frac{2xy}{x^2 + y^2}$$

3. $$\left(yArc\tan x\right)y' + \left(\frac{2x}{1+x^2}\right)y = 0$$

Theorem.

Suppose M, N and R are functions of 3 variables x, y and z on an open disk $B\left(\left(x_0, y_0, z_0\right); r\right)$ in R^3 and $M_y, M_z, N_x, N_z, R_x, R_y$ are continuous on B.

Then the vector

$$\left\langle M\left(x, y, z\right), N\left(x, y, z\right), R\left(x, y, z\right)\right\rangle$$

is a ***gradient*** on B if and only if

$$M_y\left(x, y, z\right) = N_x\left(x, y, z\right)$$

$$M_z\left(x, y, z\right) = R_x\left(x, y, z\right)$$

$$N_z\left(x, y, z\right) = R_y\left(x, y, z\right)$$

for all point $\left(x, y, z\right)$ in B.

Theorem.

Suppose M, N and R are functions of 3 variables x, y and z on an open disk $B\left(\left(x_0, y_0, z_0\right); r\right)$ in R^3 and $M_y, M_z, N_x, N_z, R_x, R_y$ are continuous on B.

Then the differential equation

$$M\left(x, y, z\right)dx + N\left(x, y, z\right)dy + R\left(x, y, z\right)dz = 0$$

is ***exact*** on B if and only if

$$M_y\left(x, y, z\right) = N_x\left(x, y, z\right)$$

$$M_z\left(x, y, z\right) = R_x\left(x, y, z\right)$$

$$N_z\left(x, y, z\right) = R_y\left(x, y, z\right)$$

for all point $\left(x, y, z\right)$ in B.

END