# 1.5

# ALTERNATING SERIES

# Recall:

An infinite series of the form

$$\sum_{n=1}^{\infty}(-1)^n u_n = -u_1 + u_2 - u_3 + \dots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n+1} u_n = u_1 - u_2 + u_3 - \dots$$

where $u_n > 0 \ , \forall n \in N$

is called an *alternating series*.

**Examples.** Which of the ff. are alternating series?

$$\sum_{n=1}^{\infty}(-1)^{n+2}\frac{7}{n^{20}}$$

$$\sum_{n=1}^{\infty}(-1)^{2n}\frac{1}{1+\ln n}$$

$$\sum_{n=1}^{\infty}(-1)^{2n+1}e^{-n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}$$

$$\sum_{n=1}^{\infty}\left(-\frac{7}{8}\right)^{n-1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n-2^{n}}$$

# More Examples.

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n} = -\frac{1}{\ln 2}+\frac{1}{\ln 3}-\frac{1}{\ln 4}+\frac{1}{\ln 5}-\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2+2n+1} = \frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n}\sec h(n)$$

$$= -\sec h(1)+\sec h(2)-\sec h(3)+\ldots$$

# **Theorem.** *Alternating Series Test*

The alternating series

$$\sum_{n=1}^{\infty} (-1)^n u_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

***converges*** if both conditions are satisfied:

i. $u_n > u_{n+1}$ for all $n \in N$

ii. $\lim\limits_{n \to +\infty} u_n = 0$

**Examples.** Use the alternating series test to show that the series is convergent

1. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$

***Note:*** $u_n = \frac{1}{\ln n}$

i. Let $n \in N$ Then $\ln(n+1) > \ln n$

$$\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$$

ii. $\lim_{n\to+\infty}\frac{1}{\ln n} = 0$

Thus, $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ is convergent by AST.

**Examples.** Use the alternating series test to show that the series is convergent

2. $$\sum_{n=1}^{\infty} (-1)^n \sec h(n)$$

***Note:***

$$u_n = \sec h(n)$$

$$u_n = \frac{2e^n}{1+e^{2n}}$$

i. Let

$$f(x) = \frac{e^x}{1+e^{2x}}$$

Then $$f'(x) = \frac{e^x\left(1-e^{2x}\right)}{\left(1+e^{2x}\right)^2} < 0$$

**Examples.** Use the alternating series test to show that the series is convergent

2. $$\sum_{n=1}^{\infty}(-1)^n \sec h(n)$$

***Note:***

$$u_n = \sec h(n)$$

$$u_n = \frac{2e^n}{1+e^{2n}}$$

ii. $$\lim_{n\to+\infty}\frac{2e^n}{1+e^{2n}} = 0$$

Thus, $\sum_{n=1}^{\infty}(-1)^n \sec h(n)$ is convergent by AST. ▨

## Definition.

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be

a. ***absolutely convergent*** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

b. ***conditionally convergent*** if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

# Examples:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

***is conditionally convergent.***

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$$

***is absolutely convergent.***

# Theorem.

An infinite series that is absolutely convergent are convergent.

**Examples:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{5^{n-1}}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+2}$$

***are both convergent since both are absolutely convergent.***

# Other tests for convergence:

1. Absolute Ratio Test

2. Root Test

# Absolute Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a series for which $u_n \neq 0$ and let $L = \lim_{n \to +\infty} \left| \frac{u_{n+1}}{u_n} \right|$.

- **If $L < 1$ , then the given series is absolutely convergent.**
- **If $L > 1$ or if $L = \infty$ , then the given series is divergent.**
- **If $L = 1$ , test fails.**

# Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a series for which $u_n \neq 0$

and let $L = \lim_{n \to +\infty} \sqrt[n]{|u_n|}$ .

- **If $L < 1$ , then the given series is absolutely convergent.**

- **If $L > 1$ or if $L = \infty$ , then the given series is divergent.**

- **If $L = 1$ , test fails.**

**Examples.** Use either the ratio or root test to determine if the series is convergent.

1. $$\sum_{n=1}^{\infty} \frac{n^{20}}{10^{n}}$$

**Use: Ratio Test**

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to+\infty} \frac{(n+1)^{20}}{10^{n+1}} \cdot \frac{10^n}{n^{20}}$$

$$= \lim_{n\to+\infty} \frac{1}{10} \left( \frac{n+1}{n} \right)^{20} = \frac{1}{10} < 1$$

Thus, $\sum_{n=1}^{\infty} \frac{n^{20}}{10^n}$ is absolutely convergent. ▨

**Examples.** Use either the ratio or root test to determine if the series is convergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

**Use:** ***Ratio Test***

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to+\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}$$

$$= \lim_{n\to+\infty} \frac{3}{n+1} = 0 < 1$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent.

**Examples.** Use either the ratio or root test to determine if the series is convergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$

***Use:* Root Test**

$$L = \lim_{n\to+\infty} \sqrt[n]{|u_n|} = \lim_{n\to+\infty} \sqrt[n]{\left|\left(\frac{3}{2}\right)^n\right|}$$

$$= \lim_{n\to+\infty} \frac{3}{2} = \frac{3}{2} > 1$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$ is divergent. ▨

# END