3.8

Extrema of Functions of More Than One Variable

CASES

1. **Relative extrema of a function.**
   Solve for critical points
   Then, use second-derivative test.

2. **Constrained-optimization.**
   Use the Lagrange method!

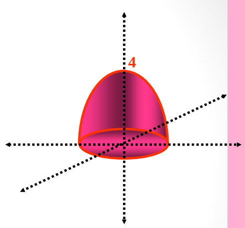
Extrema of functions of more than one variable

A function f of two variables x and y is said to have a *relative maximum value* at the point (x_0, y_0) if there exists an open disk

$$B((x_0, y_0); r)$$

such that for all (x, y) in B

$$f(x_0, y_0) \geq f(x, y).$$



$f(x, y) = 4 - x^2 - y^2.$

Thus, f has a relative maximum value of 4 at $(0,0)$.

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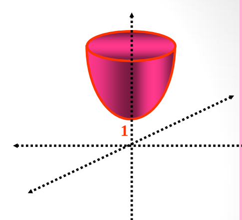

$f(x, y) = 1 + x^2 + y^2.$

Thus, f has a relative minimum value of 1 at $(0,0)$.

A function f of two variables x and y is said to have an *absolute maximum value* on its domain D in the xy-plane if there is some point (x_0, y_0) in D such that for all (x, y) in D,

$$f(x_0, y_0) \geq f(x, y).$$

In this case, $f(x_0, y_0)$ is the *absolute maximum value of* f on D.

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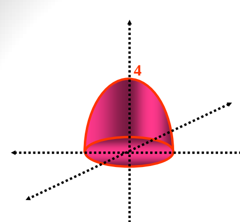
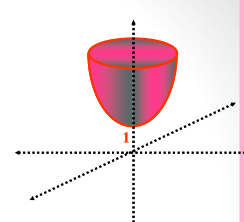


$f(x, y) = 4 - x^2 - y^2.$

f has an absolute maximum value of 4 at $(0,0)$.

f has no absolute minimum value.



$f(x, y) = 1 + x^2 + y^2.$

f has an absolute minimum value of 1 at $(0,0)$.

f has no absolute maximum value.

Remarks:

A maximum or minimum value is also called an *extremum*.

If the domain of a function f is either an open disk or the entire xy plane, an absolute extremum must also be a relative extremum.

Theorem If $f(x,y)$ exist at all points in some open disk $B((x_0,y_0);r)$ and if f has a relative extremum at (x_0,y_0), then if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist, $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$.

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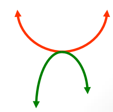
Remarks:

1. If

$$f_x(x_0,y_0)=0 \text{ and } f_y(x_0,y_0)=0$$

then f does not necessarily have a relative extremum value at (x_0,y_0).

2. A critical point at which there is no relative extrema is called a *saddle point* of the graph of the function.



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If $f(x,y)$ exists at all points in some open disk $B((x_0,y_0);r)$, then (x_0,y_0) is a *critical point of f* if one of the following conditions holds:

1. $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$.
2. $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist.

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Example Find the critical points of the indicated function.

a. $f(x,y)=x^3+y^2-6x^2+6y-1$

b. $f(x,y)=e^{xy}$

c. $f(x,y)=x^3-3x^2y-y^2$

solution:

a. $f(x,y)=x^3+y^2-6x^2+6y-1$

$f_x(x,y)=3x^2-12x=0$

$\Rightarrow 3x(x-4)=0$

$\Rightarrow x=0$ or $x=4$

$f_y(x,y)=2y+6=0 \quad \Rightarrow y=-3$

ans. $(0,-3)$ and $(4,-3)$

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b. $f(x,y)=e^{xy}$

$f_x(x,y)=ye^{xy}=0$

$\Rightarrow ye^{xy}=0$

$\Rightarrow y=0$

$f_y(x,y)=xe^{xy}=0$

$\Rightarrow xe^{xy}=0$

$\Rightarrow x=0$

ans. $(0,0)$

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c. $f(x,y)=x^3-3x^2y-y^2$

$f_x(x,y)=3x^2-6xy=0$

$\Rightarrow 3x(x-2y)=0$

$\Rightarrow 3x=0$ or $x-2y=0$

$\Rightarrow x=0$ or $x=2y$

$f_y(x,y)=-\cancel{3x^2}-2y=0$

Case 1. $x=0$.

$-2y=0 \Rightarrow y=0$

Case 2. $x=2y$.

$-3(2y)^2-2y=0$

$-12y^2-2y=0$

$-2y(6y+1)=0$

Case 2.1. $y=0$.

$x=2\cancel{y}$

$x=0$

Case 2.2. $y=-1/6$.

$x=2(\frac{-1}{6})=\frac{-1}{3}$

ans. $(0,0)$ and $(\frac{-1}{3},\frac{-1}{6})$

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Theorem (Second Derivative Test) Let f be a function of two variables such that f and its first and second-order partial derivatives are continuous on some open disk $B((a,b);r)$.
Suppose further that $f_x(a,b)=0$ and $f_y(a,b)=0$.
Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$. Then

a. f has a relative minimum value at (a,b)

if $D(a,b)>0$ and $f_{xx}(a,b)>0$ (or $f_{yy}(a,b)>0$).

b. f has a relative maximum value at (a,b)

if $D(a,b)>0$ and $f_{xx}(a,b)<0$ (or $f_{yy}(a,b)<0$).

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c. $f(a,b)$ is not a relative extremum, but the graph of f has a saddle point at $(a,b,f(a,b))$

if $D(a,b)<0$.

d. No conclusion regarding relative extrema can be made if $D(a,b)=0$.

Remark: If $f_{xy}(a,b)=f_{yx}(a,b)$, the expression $D(a,b)$ is the value of

$$\begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

which is called the *Hessian* (or *discriminant*) of f.

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Example Let $f(x,y)=x^3+y^2-6x^2+6y-1$.
Determine the relative extrema of f, if there is any and locate any saddle point of the graph of f.

solution: $f_x(x,y)=3x^2-12x$ $f_y(x,y)=2y+6$

Critical point	$f_{xx}(x,y)=6x-12$	$f_{yy}(x,y)=2$	$f_{xy}(x,y)=0$	$f_{xx}f_{yy}-(f_{xy})^2$	*conclusion*
$(0,-3)$	-12	2	0	-24	*f has no relative extremum*
$(4,-3)$	12	2	0	24	*f has a relative minimum*

f has a relative minimum value of

$f(4,-3)=4^3+(-3)^2+6(4)^2+6(-3)-1=-42$

$(0,-3,-10)$ *is a saddle point of the graph of f.*

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Example Let $f(x,y)=e^{xy}$.
Determine the relative extrema of f, if there is any and locate any saddle point of the graph of f.

solution: $f_x(x,y)=ye^{xy}$ $f_y(x,y)=xe^{xy}$

Critical point	$f_{xx}(x,y)=y^2e^{xy}$	$f_{yy}(x,y)=x^2e^{xy}$	$f_{xy}(x,y)=e^{xy}(xy+1)$	$f_{xx}f_{yy}-(f_{xy})^2$	*Conclusion*
$(0,0)$	0	0	1	-1	*f has no relative extremum*

$(0,0,1)$ *is a saddle point of the graph of f.*

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Example Let $f(x,y)=x^3-3x^2y-y^2$.
Determine the relative extrema of f, if there is any and locate any saddle point of the graph of f.

solution: $f_x(x,y)=3x^2-6xy$ $f_y(x,y)=-3x^2-2y$

Critical point	$f_{xx}(x,y)=6x-6y$	$f_{yy}(x,y)=-2$	$f_{xy}(x,y)=-6x$	$f_{xx}f_{yy}-(f_{xy})^2$	*conclusion*
$(0,0)$	0	-2	0	0	*No conclusion can be made.*
$\left(\frac{-1}{3},\frac{-1}{6}\right)$	-1	-2	2	-2	*f has no relative extremum*

$\left(\frac{-1}{3},\frac{-1}{6},\frac{-1}{108}\right)$ *is a saddle point of the graph of f.*

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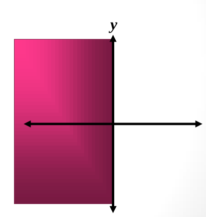
Absolute Extrema on a Closed and Bounded Region

A region is *bounded* if it is a sub-region of a closed disk or closed ball.

1. Let $S=\{(x,y)\in R^2 : x\le 0\}$
Then S is not bounded since no closed disk can contain S.



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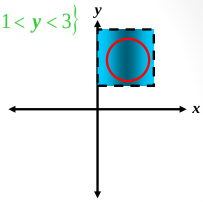
2. Let

$R = \{(x, y) \in R^2 : 0 < x < 2,\ 1 < y < 3\}$

Then R is bounded.



The *boundary* of a region R is the set of all points P for which every open ball centered at P contains a point in R and a point not in R.

A *closed* region is one that contains its boundary.

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Theorem 2.9.3. (Extreme Value Theorem or EVT)

Let R be a closed bounded region in the xy-plane and let f be a continuous function on R. Then f has an absolute maximum and an absolute minimum value on R.

Remarks:

If a function f satisfies the EVT, then an absolute extremum occurs either at a critical point of f in the interior of R or at a boundary point of R.

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LAGRANGE MULTIPLIERS

Theorem Suppose f and g are functions of x and y with continuous first partial derivatives. If f has a relative extremum value at the point $(x_0, y_0,)$ subject to the constraint

$$g(x, y) = 0$$

and $\nabla g(x_0, y_0) \neq 0$, then there is a constant λ such that

$$\nabla f(x_0, y_0) + \lambda \nabla g(x_0, y_0) = 0.$$

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Steps in finding the relative extrema of a function f of x and y subject to the constraint

$$g(x, y) = 0$$

using the method of Lagrange multipliers:

1. Form the auxiliary function F of three variables and where

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

2. Set the first partial derivatives of F to zero and obtain the system of equations

$$F_x(x, y, \lambda) = 0$$
$$F_y(x, y, \lambda) = 0$$
$$F_\lambda(x, y, \lambda) = 0$$

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3. Find the critical points of F by solving the system of equations in step 2.

4. Among the first two coordinates of a critical point of F are the values of x and y that give the desired relative extrema.

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Example If $f(x, y) = xy$ use Lagrange multipliers to find the relative extrema of f subject to the constraint

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

solution:

$$f(x, y) = xy \qquad g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$$

$$F(x, y, \lambda) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$$

$$F_x(x, y, \lambda) = y + \frac{x}{4}\lambda \qquad \Rightarrow y + \frac{x}{4}\lambda = 0 \qquad (1)$$

$$F_y(x, y, \lambda) = x + y\lambda \qquad \Rightarrow x + y\lambda = 0 \qquad (2)$$

$$F_\lambda(x, y, \lambda) = \frac{x^2}{8} + \frac{y^2}{2} - 1 \Rightarrow \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \qquad (3)$$

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From (1),

$$\lambda = \frac{-4y}{x}$$

From (2),

$$x + y \cdot \left(\frac{-4y}{x}\right) = 0$$

$$x^2 - 4y^2 = 0$$

From (3),

$$\frac{4y^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{y^2}{2} + \frac{y^2}{2} - 1 = 0$$

$$y + \frac{x}{4}\lambda = 0 \quad \textbf{(1)}$$
$$x + y\lambda = 0 \quad \textbf{(2)}$$
$$\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \quad \textbf{(3)}$$

$$y^2 - 1 = 0$$

$$y = \pm 1$$

$$x^2 - 4 = 0$$

$$x = \pm 2$$

Critical point	$f(x, y) = xy$
(2,1)	2
(2,-1)	-2
(-2,1)	-2
(-2,-1)	2

Hence, *f* has a relative maximum value of 2 at (2,1) and (-2,-1)
while *f* has a relative minimum value of –2 at (2,-1) and (-2,1).

Example **Use Lagrange multipliers to find the shortest distance from the point** $(1,-1,-1)$ **to the plane given by** $x + 4y + 3z = 2$ **.**

solution:

We want to minimize

$$w(x, y, z) = \sqrt{(x-1)^2 + (y+1)^2 + (z+1)^2}$$

subject to the constraint

$$g(x, y, z) = x + 4y + 3z - 2 = 0$$

We want to minimize

$$f(x, y, z) = (x-1)^2 + (y+1)^2 + (z+1)^2$$

$$F(x, y, z, \lambda) = (x-1)^2 + (y+1)^2 + (z+1)^2 + \lambda(x + 4y + 3z - 2)$$

$$F_x(x, y, z, \lambda) = 2(x-1)^2 + \lambda \quad \Rightarrow 2(x-1)^2 + \lambda = 0$$

$$F_y(x, y, z, \lambda) = 2(y+1)^2 + 4\lambda \quad \Rightarrow 2(y+1)^2 + 4\lambda = 0$$

$$F_z(x, y, z, \lambda) = 2(z+1)^2 + 3\lambda \quad \Rightarrow 2(z+1)^2 + 3\lambda$$

$$F_\lambda(x, y, z, \lambda) = x + 4y + 3z - 2 \quad \Rightarrow x + 4y + 3z - 2 = 0$$

The only critical point of *f* is $\left(\frac{17}{13}, \frac{3}{13}, \frac{-1}{13}\right)$ **.**

$$w\left(\tfrac{17}{13}, \tfrac{3}{13}, \tfrac{-1}{13}\right) = \sqrt{\left(\tfrac{17}{13} - 1\right)^2 + \left(\tfrac{3}{13} + 1\right)^2 + \left(\tfrac{-1}{13} + 1\right)^2}$$

$$= \sqrt{\left(\tfrac{4}{13}\right)^2 + \left(\tfrac{16}{13}\right)^2 + \left(\tfrac{12}{13}\right)^2} = \frac{\sqrt{416}}{13}.$$