# 3.3 PARTIAL DERIVATIVES

NOTION

**Given:** $y = f(x)$

**Derivative:** $\frac{dy}{dx}$

**change in $y$ with respect to a change in $x$**

Partial derivative

**Given:** $z = f(x,y)$

**Partial derivative:** $\frac{\partial z}{\partial x} = f_x(x,y)$

**change in $z$ with respect to a change in $x$ keeping $y$ *fixed***

Partial derivatives

**Given:** $z = f(x,y)$

**Partial derivative:**

$$\frac{\partial z}{\partial x} = f_x(x,y)$$

$$= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Partial derivative

**Given:** $z = f(x,y)$

**Partial derivative:** $\frac{\partial z}{\partial y} = f_y(x,y)$

**change in $z$ with respect to a change in $y$ *keeping* $x$ *fixed***

Partial derivative

**Given:** $z = f(x,y)$

**Partial derivative:**

$$\frac{\partial z}{\partial y} = f_y(x,y)$$

$$= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

*Notations:*

$$f_x \qquad D_x f \qquad \frac{\partial f}{\partial x}$$

$$f_y \qquad D_y f \qquad \frac{\partial f}{\partial y}$$

$$f_{x_k} \qquad D_{x_k} f \qquad \frac{\partial f}{\partial x_k}$$

*Example.*

***Consider***

$$z = f(x,y) = x^2 + xy + 2x - y$$

***Use the definitions to solve for***
$f_x$ and $f_y$

Solution $\quad z = f(x,y) = x^2 + xy + 2x - y$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\begin{aligned} &f(x+h,y) - f(x,y) \\ &= (x+h)^2 + (x+h)y + 2(x+h) - y \\ &\qquad - \left(x^2 + xy + 2x - y\right) \\ &= \cancel{x^2} + 2xh + h^2 + \cancel{xy} + hy + \cancel{2x} + 2h - \cancel{y} \\ &\qquad - \cancel{x^2} - \cancel{xy} - \cancel{2x} + \cancel{y} \end{aligned}$$

Solution $\quad z = f(x,y) = x^2 + xy + 2x - y$

$$f(x+h,y) - f(x,y) = 2xh + h^2 + hy + 2h$$

$$\begin{aligned} f_x(x,y) &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + hy + 2h}{h} \\ &= \lim_{h \to 0} (2x + \cancel{h} + y + 2) \\ &= 2x + y + 2 \end{aligned}$$

Solution $\quad z = f(x,y) = x^2 + xy + 2x - y$

$$f(x,y+h) - f(x,y) = xh - h$$

$$\begin{aligned} f_y(x,y) &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \\ &= \lim_{h \to 0} \frac{xh - h}{h} \\ &= \lim_{h \to 0} (x - 1) \\ &= x - 1 \end{aligned}$$

*Observe!*

$$z = f(x,y) = x^2 + xy + 2x - y$$

$$f_x(x,y) = 2x + y + 2$$

$$f_y(x,y) = x - 1$$

*DO THIS!*

**Given:** $z = f(x_1, x_2, \ldots, x_n)$

**To solve for** $\frac{\partial f}{\partial x_k}$,

***use differentiation techniques for functions of a single variable (with $x_k$ as the variable) and treating other variables as constants.***

*In particular . . .*

**Given:** $w = f(x, y, z)$

**To solve for** $\frac{\partial f}{\partial x}$,

***use differentiation techniques for functions of a single variable (with $x$ as the variable) and treating $y$ and $z$ as constants.***

*Example.*

$$f(x,y) = x^2 + y^2 + 7x - 11y + 16$$

$$\frac{\partial f}{\partial x} = 2x + 0 + 7 + 0 + 0$$
$$= 2x + 7$$

$$\frac{\partial f}{\partial y} = 0 + 2y + 0 - 11 + 0$$
$$= 2y - 11$$

*Example.*

$$f(x,y) = x^3y^3 + 2xy - 3x^4 + 2y - 10$$

$$\frac{\partial f}{\partial x} = 3x^2y^3 + 2y - 12x^3$$

$$\frac{\partial f}{\partial y} = 3x^3y^2 + 2x + 2$$

*Example.*

$$f(x,y) = \sin(xy) + y \ln x$$

$$\frac{\partial f}{\partial x} = \cos(xy) \cdot y + y \cdot \frac{1}{x} = y\cos(xy) + \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \cos(xy) \cdot x + \ln x = x\cos(xy) + \ln x$$

*Example.*

$$f(x,y) = x\sin x + x\cos y + ye^y$$

$$\frac{\partial f}{\partial x} = x\cos x + \sin x + \cos y$$

$$\frac{\partial f}{\partial y} = -x\sin y + ye^y + e^y$$

*Example.*

$$f(x,y)=\frac{x-y}{x+y}$$

$$\frac{\partial f}{\partial x}=\frac{(x+y)\cdot 1-(x-y)\cdot 1}{(x+y)^2}$$

$$=\frac{2y}{(x+y)^2}$$

*Example.*

$$f(x,y)=\frac{x-y}{x+y}$$

$$\frac{\partial f}{\partial y}=\frac{(x+y)\cdot(-1)-(x-y)\cdot 1}{(x+y)^2}$$

$$=\frac{-2x}{(x+y)^2}$$

*Example.*

$$f(x,y,z)=\sin^2(x+y)+\cos^2(y-z)$$

$$\frac{\partial f}{\partial x}=2\sin(x+y)\cdot\cos(x+y)\cdot 1+0$$

$$\frac{\partial f}{\partial y}=2\sin(x+y)\cdot\cos(x+y)\cdot 1$$
$$+2\cos(y-z)\cdot(-\sin(y-z))\cdot 1$$

$$\frac{\partial f}{\partial z}=0+2\cos(y+z)\cdot(-\sin(y-z))\cdot(-1)$$

*Example.*

$$f(x,y,z)=e^{xyz}+2^x y-y\ln z$$

$$\frac{\partial f}{\partial x}=e^{xyz}\cdot yz+2^x\cdot\ln 2\cdot y$$

$$\frac{\partial f}{\partial y}=e^{xyz}\cdot xz+2^x-\ln z$$

$$\frac{\partial f}{\partial z}=e^{xyz}\cdot xy+0-\frac{y}{z}$$

*REVIEW*

**Given:** $y=f(x)$

$\frac{dy}{dx}=f'(a)$ ***: slope of the tangent line to the curve*** $y=f(x)$ ***at the point*** $P(a,f(a))$

*Geometrically*

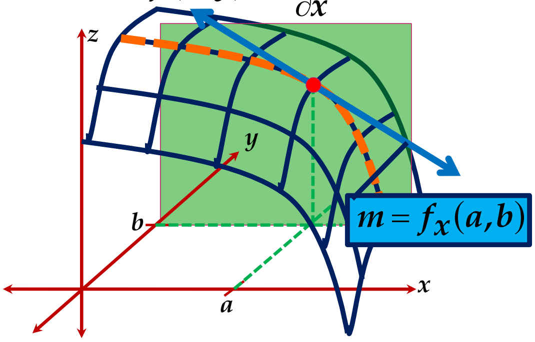
**Given:** $z=f(x,y)$

$\frac{\partial z}{\partial x}=f_x(a,b)$

***: slope of the tangent line to the curve of intersection of the surface*** $z=f(x,y)$ ***and the plane*** $y=b$ ***at the point*** $P(a,b,f(a,b))$

Geometrically

**Given:** $z = f(x,y)$ $\quad \dfrac{\partial z}{\partial x} = f_x(a,b)$




Geometrically

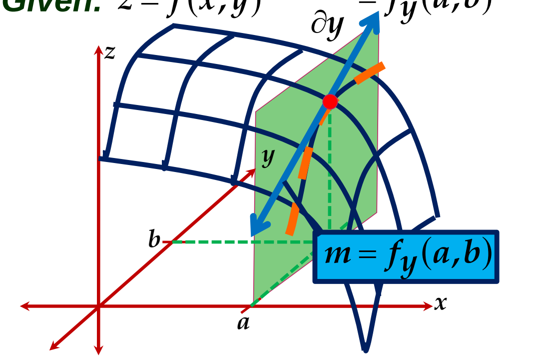
**Given:** $z = f(x,y)$

$$\frac{\partial z}{\partial y} = f_y(a,b)$$

**: slope of the tangent line to the curve of intersection of the surface $z = f(x,y)$ and the plane $x = a$ at the point $P(a, b, f(a,b))$**

Geometrically

**Given:** $z = f(x,y)$ $\quad \dfrac{\partial z}{\partial y} = f_y(a,b)$




*Example.* Determine the slope of the tangent line to the curve of intersection of the surface $z = x^2 - 2y^2 + 4$ **with the plane** $y = 2$ **at the point** $(1,2,-3)$.

**Solution:**

$z = f(x,y) = x^2 - 2y^2 + 4$

$y$ **is fixed at** $y = 2$.

**Solve for** $f_x(1,2)$.

Solution (continued)

$$f(x,y) = x^2 - 2y^2 + 4$$

$$\Rightarrow f_x(x,y) = 2x$$

$$\Rightarrow f_x(1,2) = 2$$

**Hence, the required slope is 2.**

*Example.* Determine the slope of the tangent line to the curve of intersection of the surface $z = x^2 - 2y^2 + 4$ **with the plane** $x = 1$ **at the point** $(1,2,-3)$.

**Solution:**

$z = f(x,y) = x^2 - 2y^2 + 4$

$x$ **is fixed at** $x = 1$.

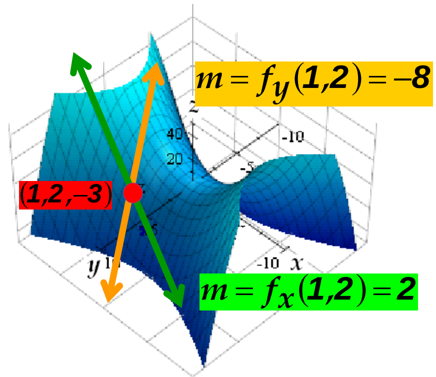
**Solve for** $f_y(1,2)$.

*Solution (continued)*

$$f(x,y)=x^2-2y^2+4$$
$$\Rightarrow f_y(x,y)=-4y$$
$$\Rightarrow f_y(1,2)=-8$$

***Hence, the required slope is -8***

*Illustration*




# END