UNIT 1

INFINITE SERIES

OBJECTIVES

By the end of the unit, you must be able to:

- ✓ find the limit of a sequence
- ✓ test an infinite series for convergence
- ✓ establish sum of convergent infinite series
- ✓ obtain a power series expansion of a function

1.1

LIMIT OF A SEQUENCE

NOTIONS

What is a sequence?

- a list of objects arranged by a particular order

What is a sequence?

- a finite or infinite list

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32} \quad \frac{1}{64} \quad \frac{1}{128} \quad \cdots$$

What is a sequence?

- most common form: a list of "numbers" following some pattern

1 1 2 3 5 8 13

1 1 2 3 5 8 13 21 34 55 89 · · ·

FIBONACCI SEQUENCE

What is a sequence?

- representation: as a set where the elements (or terms) follow an order

$$\frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad \cdots \quad \frac{n}{n+1} \quad \cdots$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \quad n??? \quad \cdots$$

Definition

What is a sequence?

- a function whose domain is the set of natural numbers

$$1, 2, 3, 4, 5, \ldots$$

- elements of the range are called as the terms

Notations

$$\{a_n\}_{n=1}^{\infty} \qquad \{a_n\} \qquad \{f(n)\}$$

Example

Sequence: $\left\{\frac{1}{n}\right\}$

$$a_n = \frac{1}{n} \quad \text{or} \quad f(n) = \frac{1}{n}$$

Graphical representation of a sequence

Graph of a sequence $\{a_n\}$

the set of isolated points (n, a_n) on the plane

Example

$$\left\{\frac{1}{n}\right\}$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \quad n \quad \cdots$$

$$(1,1) \quad \left(2,\tfrac{1}{2}\right) \quad \left(3,\tfrac{1}{3}\right) \quad \left(4,\tfrac{1}{4}\right) \quad \left(5,\tfrac{1}{5}\right) \quad \left(1,\tfrac{1}{n}\right)$$

Example

$$\left\{\frac{(-1)^n}{n}\right\}$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \quad n \quad \cdots$$

$$(1,-1)\left(2,\tfrac{1}{2}\right)\left(3,-\tfrac{1}{3}\right)\left(4,\tfrac{1}{4}\right)\left(5,-\tfrac{1}{5}\right)\left(1,\frac{(-1)^n}{n}\right)$$

GOAL

GIVEN A SEQUENCE $\{a_n\}$.

✓ BEHAVIOR OF a_n AS $n \to +\infty$

LIMIT OF A SEQUENCE: $\lim\limits_{n \to +\infty} a_n$

Definition

$\lim\limits_{n \to +\infty} a_n = L$ if and only if for every $\varepsilon > 0$, there exists an $N > 0$ such that if $n > N$, then $|a_n - L| < \varepsilon$.

$$|a_n - L| < \varepsilon \quad \Rightarrow -\varepsilon < a_n - L < \varepsilon$$
$$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$$

Tale of the tail

$\lim\limits_{n \to +\infty} a_n = L$

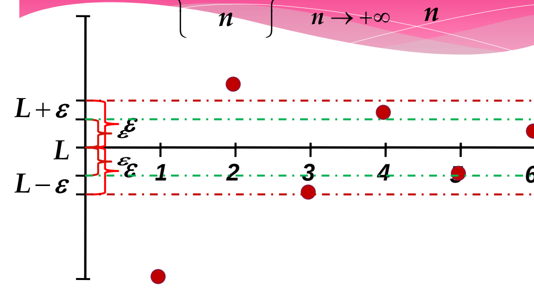
$$a_1 \quad a_2 \quad a_3 \quad \cdots \quad \underset{N}{} \quad \overbrace{\underbrace{a_n \quad a_{n+1} \quad a_{n+2} \quad \cdots}_{\text{TAIL!}}}^{n > N}$$

TAIL!

"near" the limit L

Illustration

$\left\{\dfrac{(-1)^n}{n}\right\}$ $\quad \lim\limits_{n \to +\infty} \dfrac{(-1)^n}{n} = 0$



Simplification

IF $n \to +\infty$ AS $a_n \to L$,

THEN $\lim\limits_{n \to +\infty} a_n = L$.

(HITCH!) Theorem

Let f be defined for every positive integer

IF $\lim\limits_{x \to +\infty} f(x) = L$,

THEN $\lim\limits_{n \to +\infty} f(n) = L$

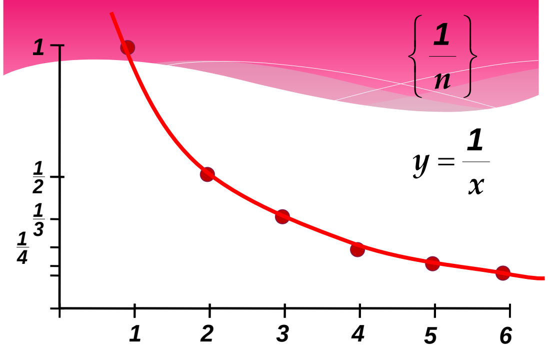
NOTE

$f(n)$ is over natural numbers

While $f(x)$ is over $[1, +\infty)$.

Illustration

$$\left\{\frac{1}{n}\right\}$$



Example 1. Determine the limit of $\left\{\dfrac{2n-1}{4-3n}\right\}$

Solution:

Let $f(x)=\dfrac{2x-1}{4-3x}$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{2x-1}{4-3x} = \frac{-2}{3}$$

Thus, the limit of the sequence $\left\{\dfrac{2n-1}{4-3n}\right\}$ is $\dfrac{-2}{3}$.

Example 2. Determine the limit of $\left\{\dfrac{n^2+1}{n+2}\right\}$

Solution:

Let $f(x)=\dfrac{x^2+1}{x+2}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{x^2+1}{x+2} = \lim_{x\to+\infty}\frac{\frac{x^2+1}{x}}{\frac{x+2}{x}}$$

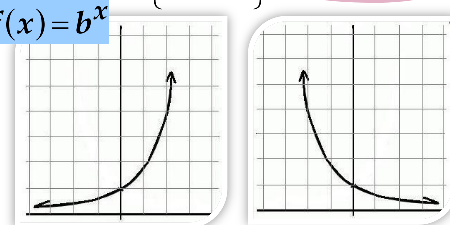
$$= \lim_{x\to+\infty}\frac{x+\frac{1}{x}}{1+\frac{2}{x}} = +\infty$$

Thus, the limit of the sequence $\left\{\dfrac{n^2+1}{n+2}\right\}$ is $=+\infty$

Example 3. Determine the limit of

$$\left\{\left(\frac{1}{2}\right)^n\right\}$$

$y=f(x)=b^x$



$b>1$ $\quad$ $0<b<1$

Example 4. Determine the limit of $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$

Solution:

Let $f(x)=\left(1+\dfrac{2}{x}\right)^x$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^x \qquad \left(1^\infty\right)$$

INDETERMINATE

Let $y=\left(1+\dfrac{2}{x}\right)^x$

$$\Rightarrow \ln y = x\ln\left(1+\frac{2}{x}\right)$$

$$\Rightarrow \ln y = x\ln\left(1+\frac{2}{x}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$\overset{LHR}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\left(\frac{-2}{x^2}\right)}{\frac{-1}{x^2}} = 2$$

Thus, the limit of the sequence $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$ is e^2

Example 5. Determine the limit of

$$\left\{(-1)^n\right\}$$

ANSWER

Does not exist!

Dilemmas

Let n be a natural number.

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$$

$n!$ CANNOT be converted to a form $x!$

Dilemma

$$\lim_{n\to+\infty}\frac{1}{n!} = ???$$

AS $n\to+\infty$, $n!\to+\infty$.

Hence, $\displaystyle\lim_{n\to+\infty}\frac{1}{n!} = 0$.

Convergence / Divergence

If the limit of $\{a_n\}$ exists, then $\{a_n\}$ is <u>convergent</u>.

Else, the sequence is <u>divergent</u>.

Also, if $\displaystyle\lim_{n\to+\infty} a_n = L$, the sequence converges to L.

Example

Since $\displaystyle\lim_{n\to+\infty}\frac{2n-1}{4-3n} = -\frac{2}{3}$, $\left\{\dfrac{2n-1}{4-3n}\right\}$ is convergent.

Since $\displaystyle\lim_{n\to+\infty}\frac{n^2+1}{n+2} = +\infty$ $\left\{\dfrac{n^2+1}{n+2}\right\}$ is divergent.