CMSC 141 Automata and Language Theory

Context-Free Languages

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Non-Regular Languages

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- There are many other non-regular languages that can be very useful
- We need something more powerful than finite automata that can express non-regular languages

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- A more powerful machine needs more (theoretically infinite) memory
- One simple storage we can use is a *stack*

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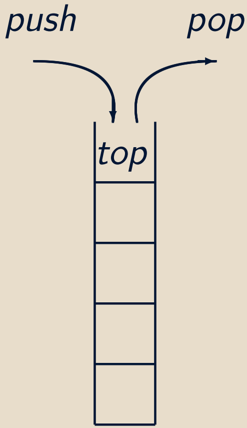

Pushdown Automata (PDA)

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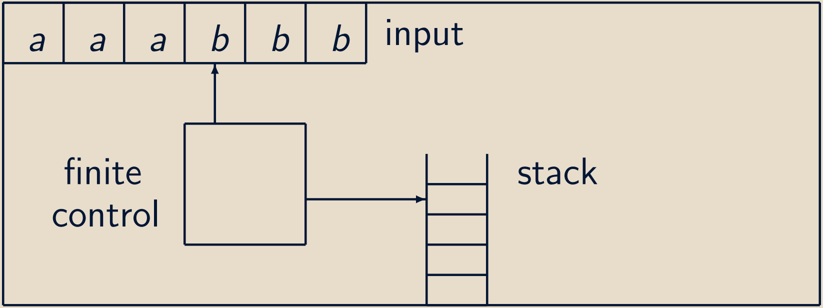

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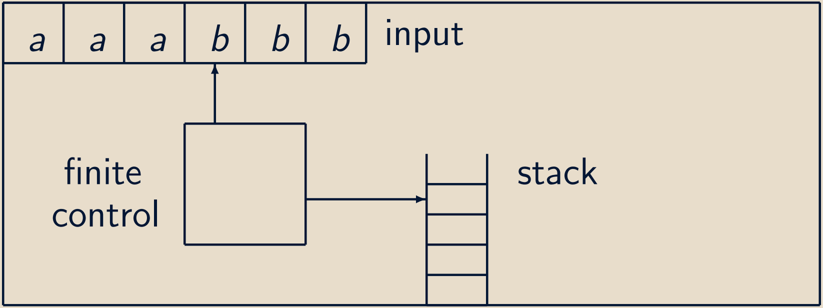

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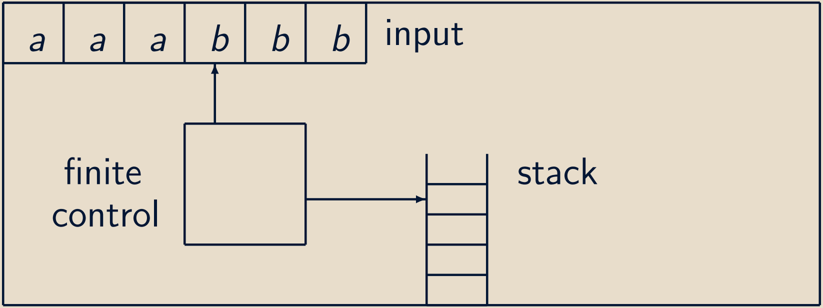

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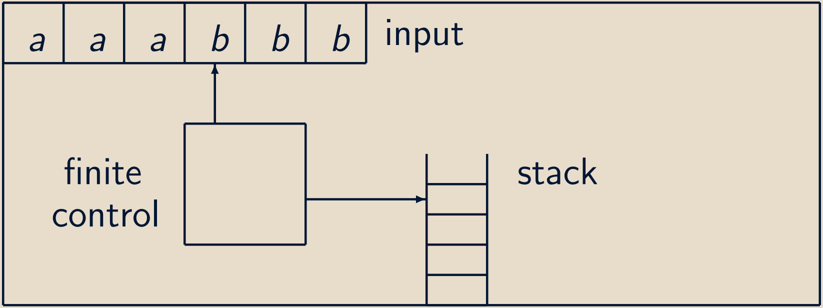

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The class of languages PDAs recognize are called Context-Free Languages (CFL)

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Bottom of the stack

Often times, a special marker Z is placed at the bottom of the stack

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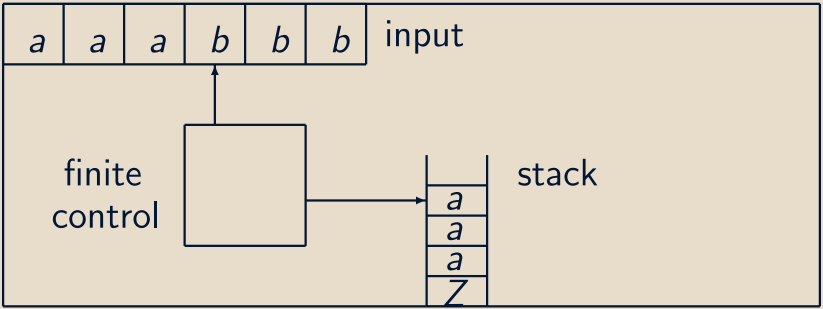

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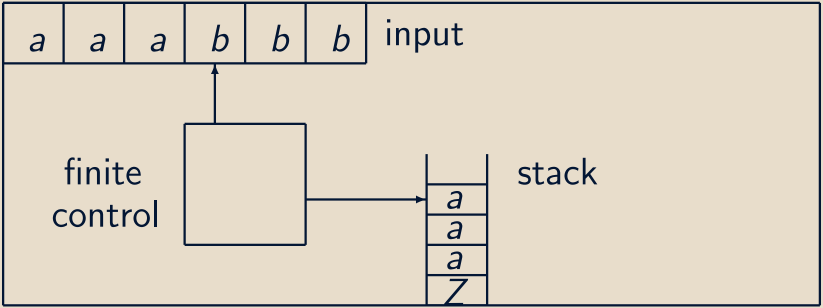

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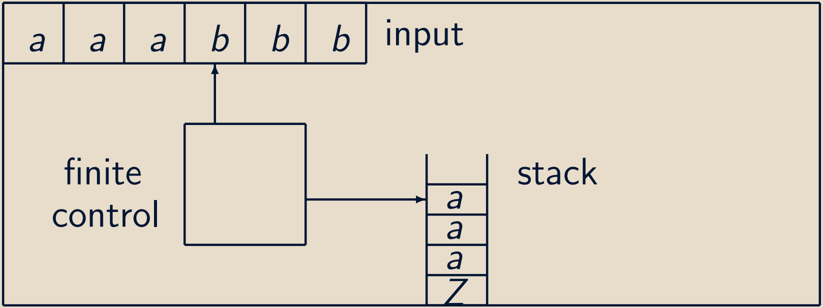

- If (a, Z) or (a, a), then push a
- If (b, a), then pop

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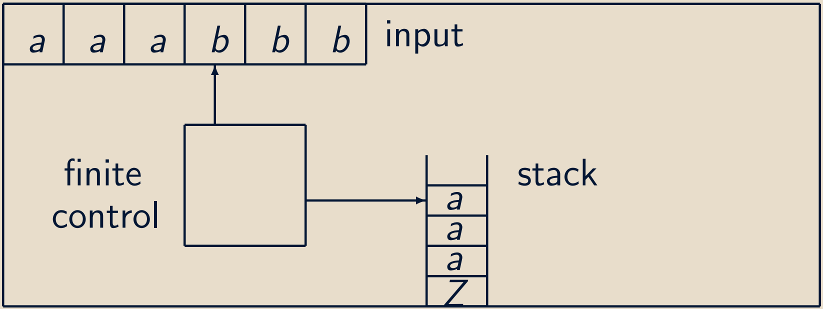

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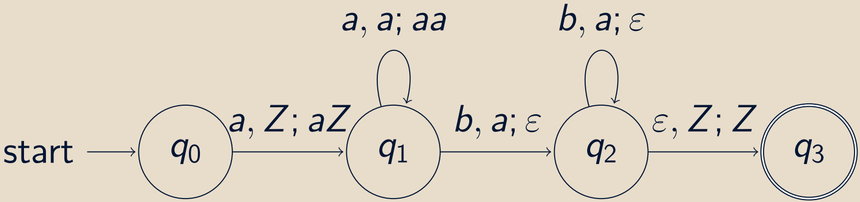

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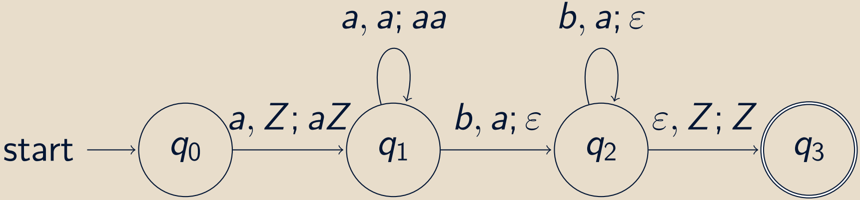
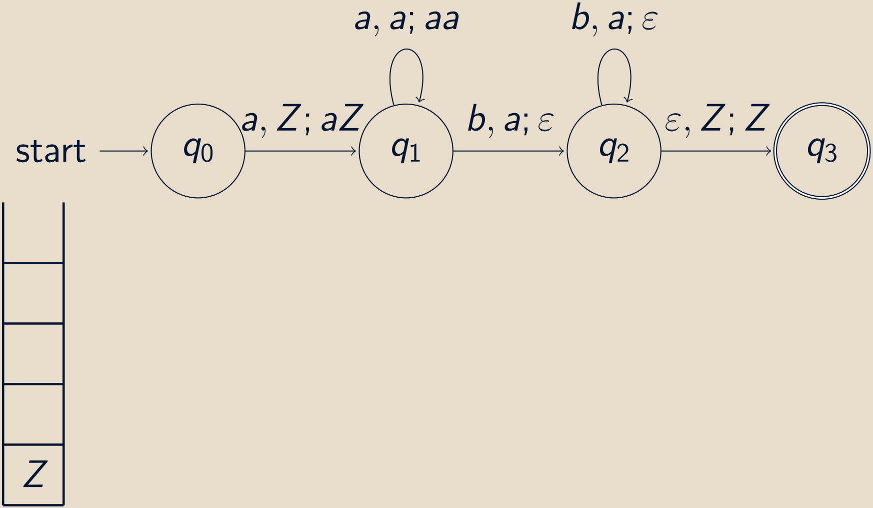


SYNTAX

(current symbol on tape,
symbol on top of the stack;
replacement symbols for the top)

PDA for $\{a^n b^n : n > 0\}$

`aaabbb`



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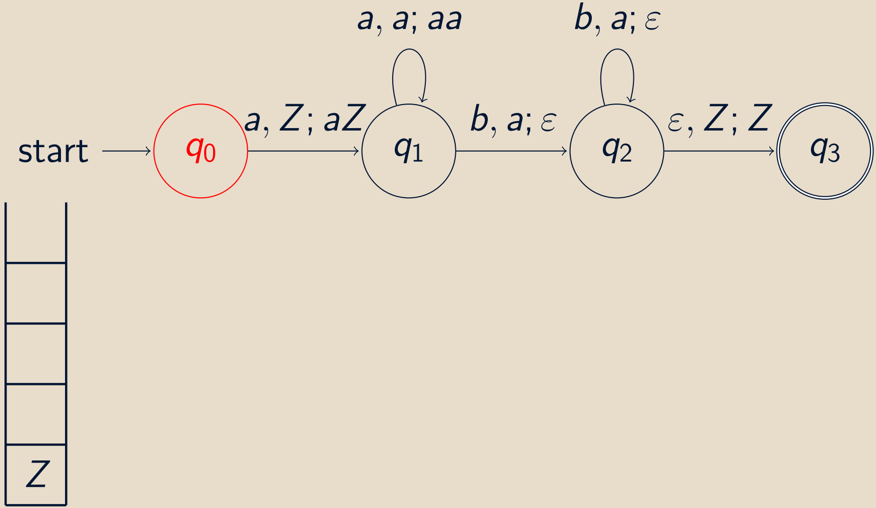

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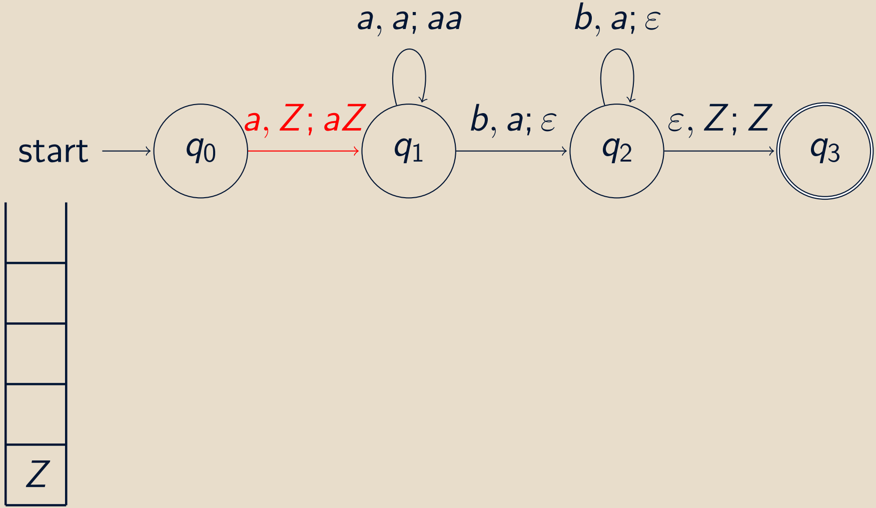

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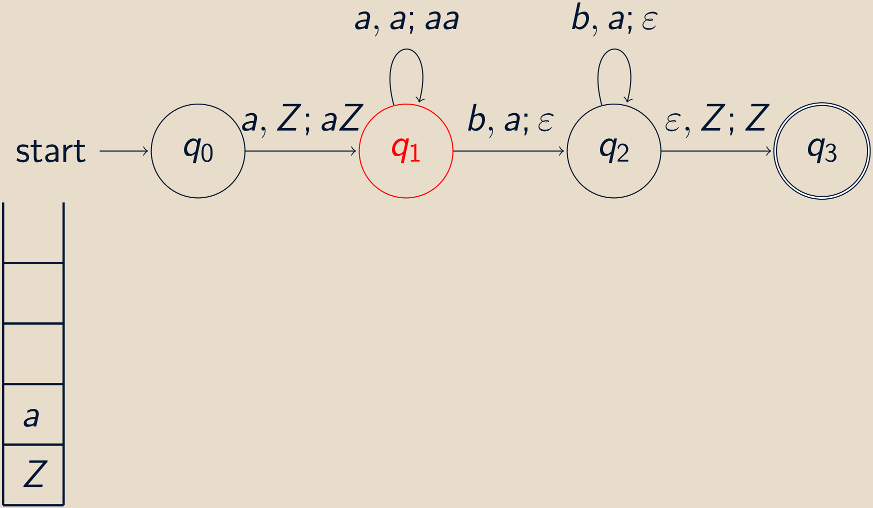

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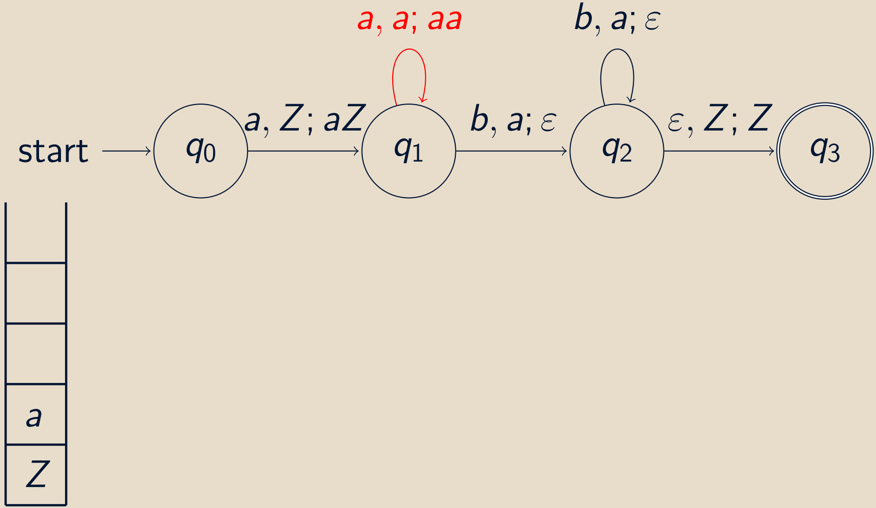

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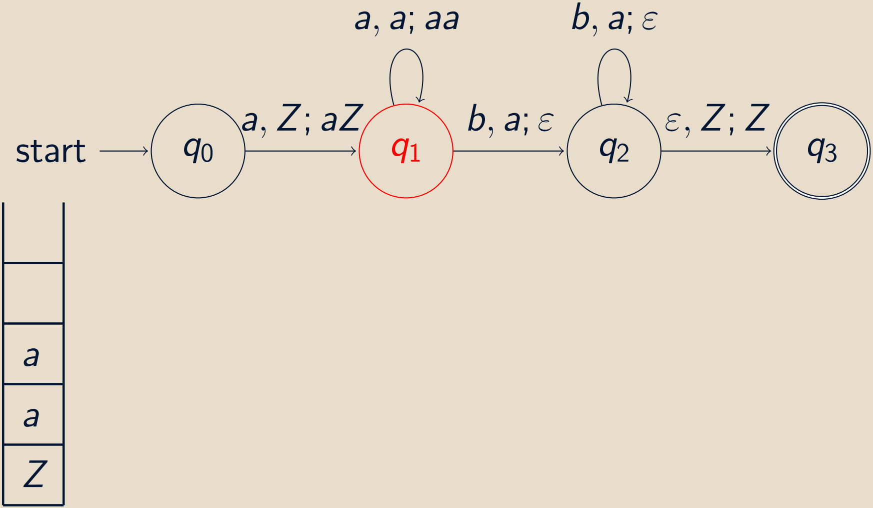

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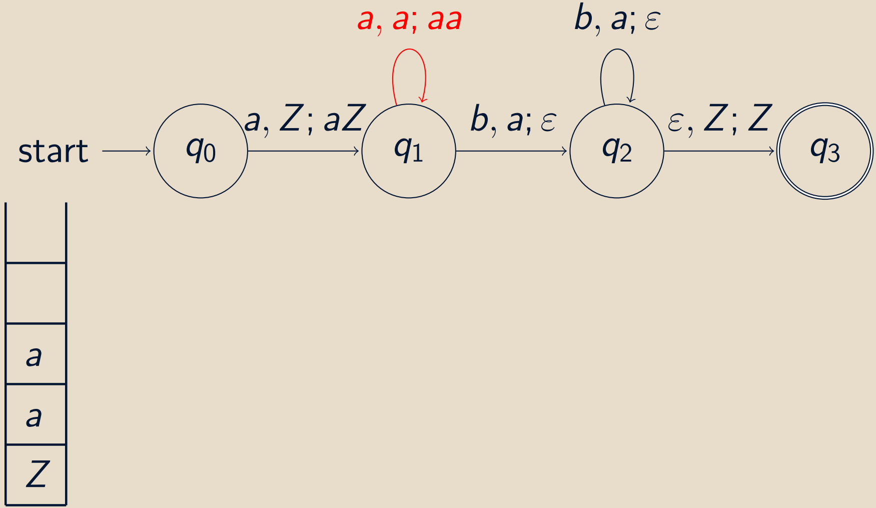

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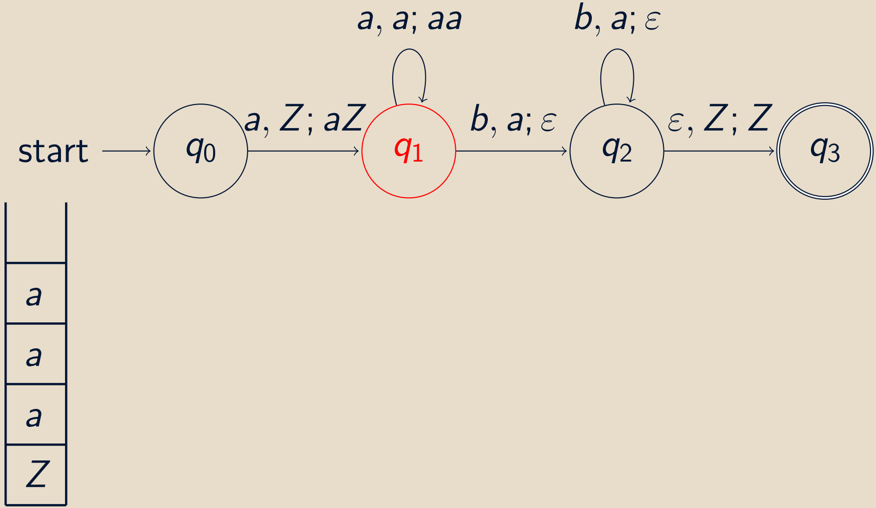

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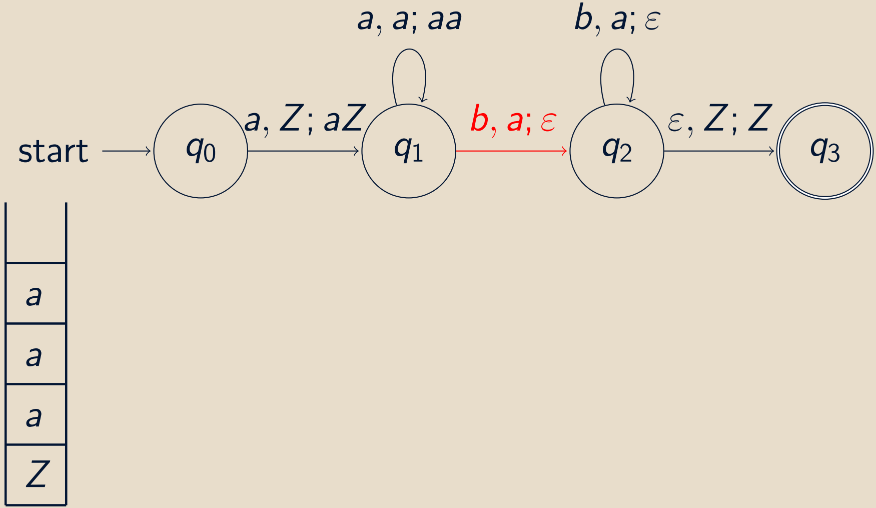

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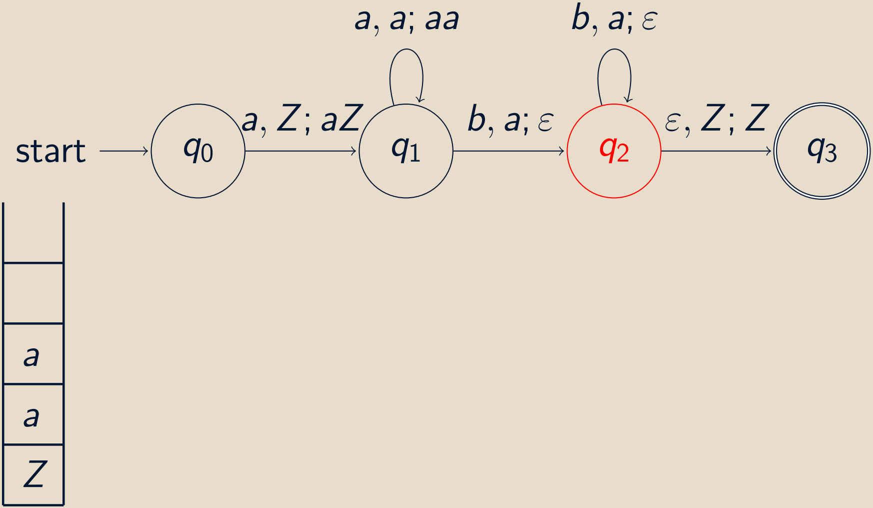

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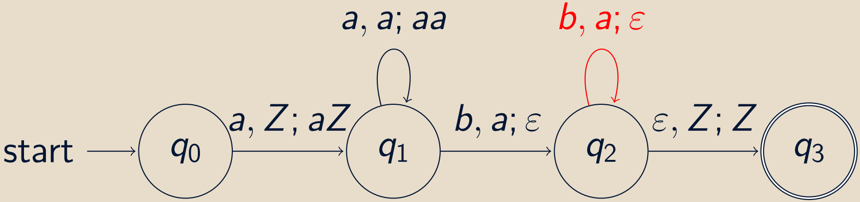

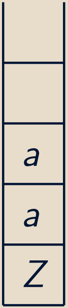

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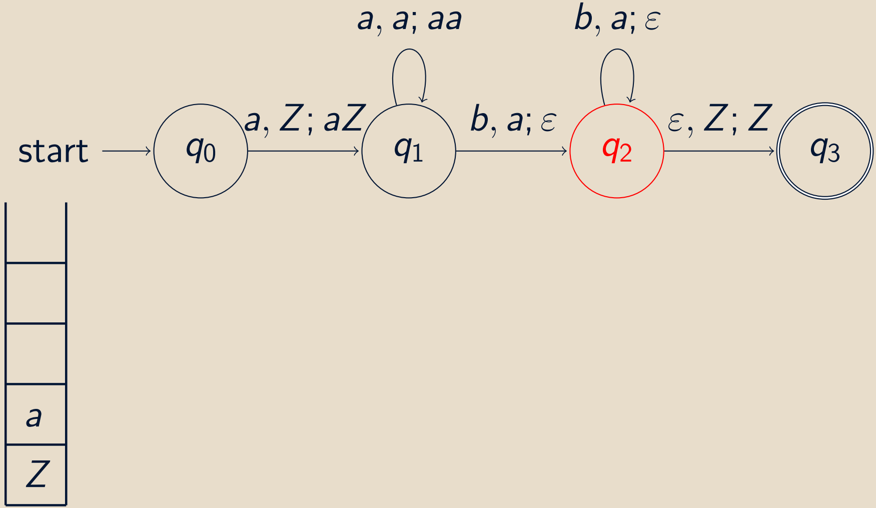

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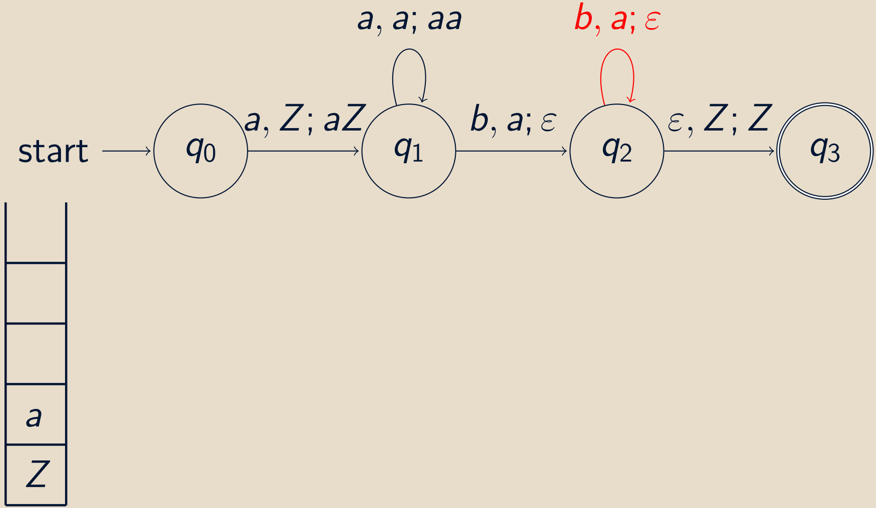

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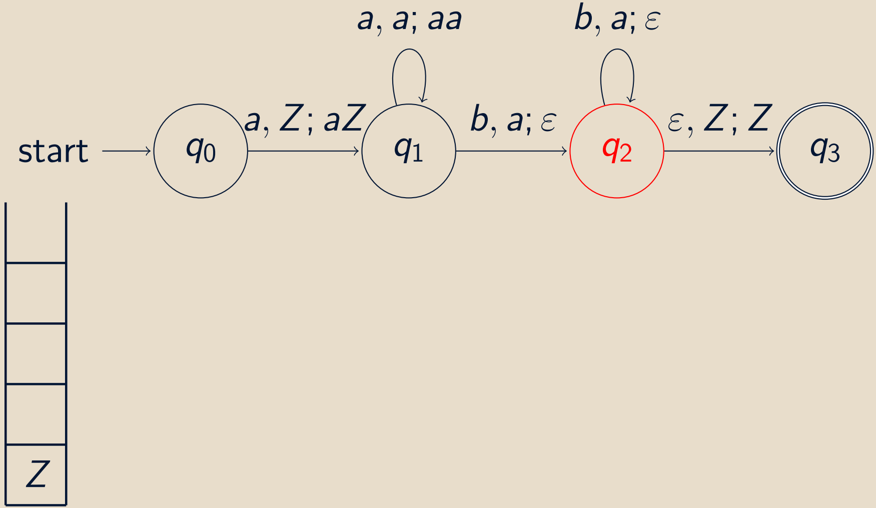

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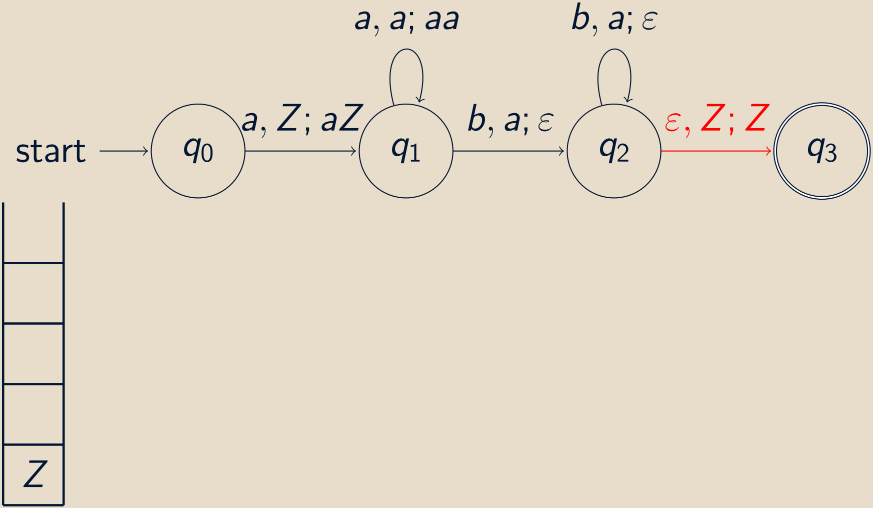

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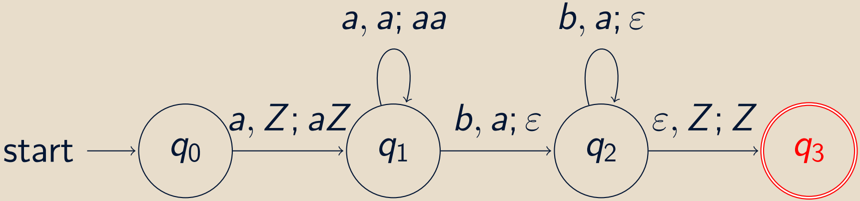

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- $F \rightarrow$ set of final/accepting states

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The output is a finite set of pairs (p, γ), where p is the new state, and γ is the string of stack symbols to replace X

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(current state, remaining input, stack contents)

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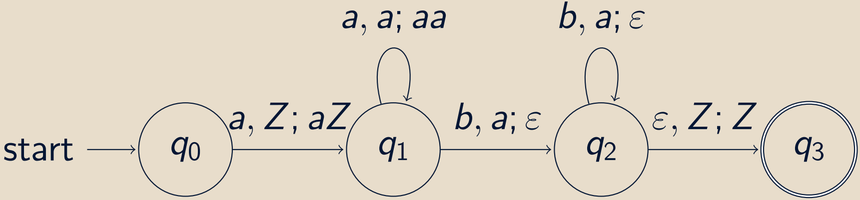

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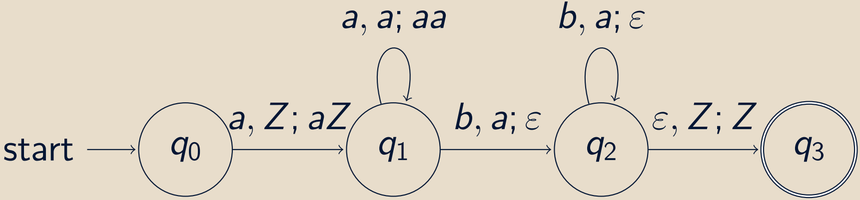

Execution of aabb

$(q_0, aabb, Z) \vdash (q_1, abb, aZ) \vdash (q_1, bb, aaZ) \vdash$
$(q_2, b, aZ) \vdash (q_2, \varepsilon, Z) \vdash (q_3, \varepsilon, Z)$

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- The two forms of acceptance in PDAs are shown to be equivalent. That is one can be converted to the other

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- balance pair of parentheses $=$
  $\{\texttt{(), (()), ()(), ((())), (())(), ...}\}$

References

- Previous slides on CMSC 141
- M. Sipser. Introduction to the Theory of Computation. Thomson, 2007.
- J.E. Hopcroft, R. Motwani and J.D. Ullman. Introduction to Automata Theory, Languages and Computation. 2nd ed, Addison-Wesley, 2001.
- E.A. Albacea. Automata, Formal Languages and Computations, UPLB Foundation, Inc. 2005
- JFLAP, www.jflap.org
- Various online LaTeX and Beamer tutorials