3.7 Directional Derivatives and Gradients

Notion

A <u>directional derivative</u> is a rate of change towards a particular direction.

Directional derivative

Let $z = f(x,y)$ *and* $u = \cos\theta\, i + \sin\theta\, j$ *be a unit vector in* θ*-direction.*

Directional derivative:

$$D_u f(x,y)$$
$$= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Directional derivative

$D_u f(x_o, y_o)$ *is the rate of change of* f *at the point* (x_o, y_o) *in the direction of* u *.*

Example.

Consider $f(x,y) = 4 - 2x^2 - y^2$ *.*

Solve for $D_u f(1,1)$ *in the direction*

a. $\theta = \dfrac{\pi}{3}$ b. $\theta = \pi$

Solution

$f(x,y) = 4 - 2x^2 - y^2$

$$D_u f(x,y)$$
$$= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$
$$= -4x\cos\theta - 2y\sin\theta$$

Solution

$$D_u f(x,y) = -4x\cos\theta - 2y\sin\theta$$

in the direction $\theta = \frac{\pi}{3}$ *,*

$$D_u f(x,y) = -2x - \sqrt{3}y$$

in the direction $\theta = \pi$ *,*

$$D_u f(x,y) = 4x$$

Illustration

in the direction $\theta = \frac{\pi}{3}$ *,*

$$D_u f(x,y) = -2x - \sqrt{3}y$$

$$D_u f(1,1) = -2 - \sqrt{3} < 0$$

Hence, it is a descent in the direction of $\frac{\pi}{3}$ *.*

Illustration

in the direction $\theta = \pi$ *,*

$$D_u f(x,y) = 4x$$

$$D_u f(1,1) = 4 > 0$$

Hence, it is an ascent in the direction of π *.*

Gradient

Gradient:
(direction of steepest ascent)

$$\nabla f(x,y)$$

$$= f_x(x,y)i + f_y(x,y)j$$

$$= \left\langle f_x(x,y)\ ,\ f_y(x,y) \right\rangle$$

Gradient

$$D_u f(x,y) = \underbrace{U \cdot \nabla f}_{\text{dot product}}(x,y)$$

where U *is a unit vector!*

Example.

Consider $f(x,y) = \sqrt{x^2 + y^2}$ *.*

Determine $D_u f(4,-3)$ *in the direction of* $\langle 12, -5 \rangle$*.*

Solution:

$$U = \left\langle \frac{12}{13}, \frac{-5}{13} \right\rangle$$

$$f_x(x,y) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2 + y^2}}$$

Solution (continued)

$$f_x(x,y)=\frac{x}{\sqrt{x^2+y^2}} \quad f_y(x,y)=\frac{y}{\sqrt{x^2+y^2}}$$

$$f_x(4,-3)=\frac{4}{5} \qquad f_y(4,-3)=\frac{-3}{5}$$

$$\nabla f(4,-3)=\left\langle \frac{4}{5},\frac{-3}{5}\right\rangle$$

This is the direction of the greatest rate of change!

Solution (continued)

$$U=\left\langle \frac{12}{13},\frac{-5}{13}\right\rangle \quad \nabla f(4,-3)=\left\langle \frac{4}{5},\frac{-3}{5}\right\rangle$$

$$D_u f(4,3)=U\cdot\nabla f(4,3)$$

$$=\frac{63}{65}$$

The function is <u>increasing</u> in the given direction!

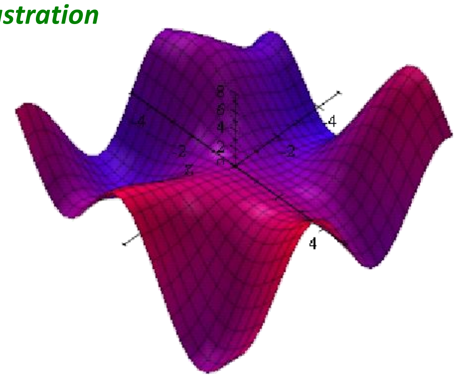
Example.

The density of a rectangular plate at a point in the xy-plane is given by

$$\rho(x,y)=x\cos y+y\sin x$$

Determine the change in density at the origin in the direction of $\langle -3,-4\rangle$.

Illustration



Solution

$$\rho(x,y)=x\cos y+y\sin x \quad U=\left\langle \frac{-3}{5},\frac{-4}{5}\right\rangle$$

$$\rho_x(x,y)=\cos y+y\cos x \qquad \rho_x(0,0)=1$$

$$\rho_y(x,y)=-x\sin y+\cos x \qquad \rho_y(0,0)=1$$

$$D_u\rho(0,0)=U\cdot\nabla\rho(0,0)$$

$$=\left\langle \frac{-3}{5},\frac{-4}{5}\right\rangle\cdot\langle 1,1\rangle=\frac{-7}{5}$$

Conclusion

$$D_u\rho(0,0)=-\frac{7}{5}$$

Hence, from the origin, the density is <u>decreasing</u> in the direction $\langle -3,-4\rangle$.

Supplement

If a surface S is given by the equation $F(x,y,z)=0$, then $\nabla F(P_0)$ is a normal vector to the surface at the point P_0.

Supplement

Consider a surface S is given by $F(x,y,z)=0$.

The tangent plane to the surface at the point (x_0,y_0,z_0) on the surface is given by

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

where $\nabla F(x_0,y_0,z_0)=\langle a,b,c\rangle$.

Supplement

Consider a surface S is given by $F(x,y,z)=0$.

The normal line to the surface at the point (x_0,y_0,z_0) is given by

$$x=x_0+t\cdot a$$
$$y=y_0+t\cdot b$$
$$z=z_0+t\cdot c$$

where $\nabla F(x_0,y_0,z_0)=\langle a,b,c\rangle$.

QUIZ: 7 minutes

Let $f(x,y)=x\ln y-3x^4y$

a. Find $\nabla f(-1,1)$

b. Solve for $D_U f(-1,1)$ given that $U=\frac{3}{5}i+\frac{4}{5}j$

END