3.2
Limits and Continuity
of Functions of More Than One Variable

LIMITS
of Functions of More Than One Variable

REVIEW

Limit of Function of a Single Variable

Let f be defined on some open interval containing a, except possibly at a. Then, $\lim_{x \to a} f(x) = L$

if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

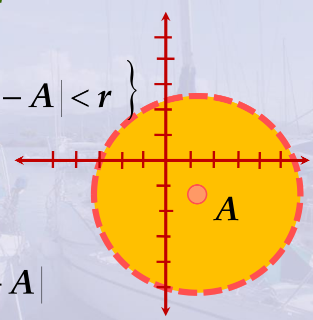
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

Open and closed balls

Open disk/ball

$B(A;r)$

$= \{P \in R^n \mid |P - A| < r\}$



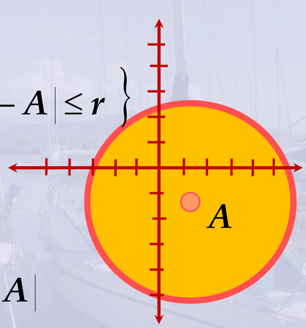
$d(P,A) = |P - A|$

Open and closed balls

Open disk/ball

$B[A;r]$

$= \{P \in R^n \mid |P - A| \le r\}$



$d(P,A) = |P - A|$

Limit of a Function of More Than One Variable

Let f be defined on some open ball $B(A;r)$, except possibly at A itself. Then, $\lim_{P \to A} f(P) = L$

if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $0 < |P - A| < \delta$, then $|f(P) - L| < \varepsilon$

Limit of a Function of More Than One Variable

if $0 < |P - A| < \delta$, then $|f(P) - L| < \varepsilon$

If $f : R^n \to R$,

$|P - A|$ is distance in R^n, and

$|f(P) - L|$ is distance in R.

For $f : R^2 \to R$,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$

Illustration

Consider $z = f(x,y) = x^2 + y^2$.

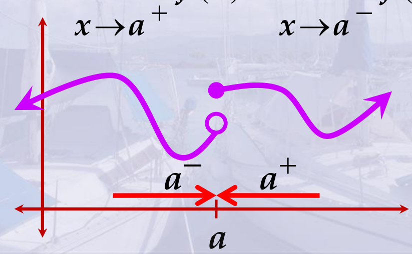
$$\lim_{(x,y)\to(2,1)} \left(x^2 + y^2\right) = \boxed{5}$$

f is defined on any open ball centered at $(2,1)$.

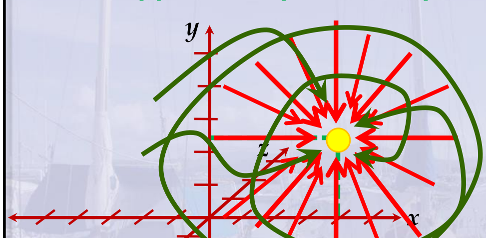
REVIEW (One-sided limits)

$\lim\limits_{x\to a} f(x)$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$



How to approach a point on a plane



There are infinitely many ways to approach a point on the plane, even "more" in space.

WARNING ! ! !

The next procedure is used to show non-existence of a limit.
DO NOT use it to show existence.

Limits through specific sets

Let A be an **accumulation point** *of a set S.*

$$\lim_{\substack{P \to A \\ P \text{ in } S}} f(P)$$ *is the limit of f when f is restricted to S.*

Simplification: limits through specific curves

Using limits through specific curves

Let S be a continuous curve through a point A.

$$\lim_{\substack{P \to A \\ P \text{ in } S}} f(P) = L$$ *if through the curve S, the values of f approach L.*

To evaluate, restrict f to S.

If the limit exists . . .

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

S is a continuous curve through (x_0,y_0).

$$\Longrightarrow \lim_{\substack{(x,y)\to(x_0,y_0) \\ P \text{ in } S}} f(x,y) = L$$

To show non-existence . . .

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ P \text{ in } S_1}} f(x,y) = L_1 \qquad L_1 \neq L_2$$

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ P \text{ in } S_2}} f(x,y) = L_2$$

$$\Longrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

DOES NOT EXIST

Example. Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

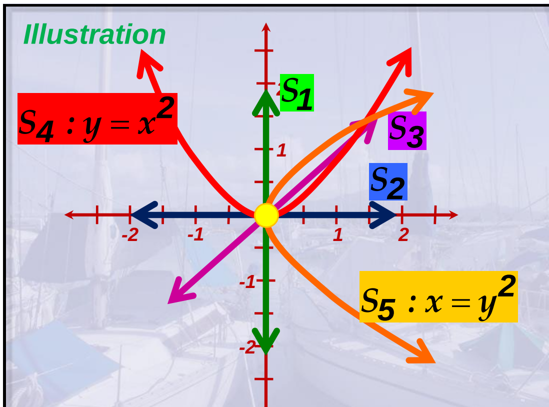
Solution:

$$S_1 = \{(x,y) \mid x = 0\}$$

$$S_2 = \{(x,y) \mid y = 0\}$$

$$S_3 = \{(x,y) \mid y = x\}$$

Illustration



Solution

$S_1 = \{(x,y) \mid x = 0\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ P\ in\ S_1}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{0 - y^2}{0 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} (-1) = -1$$

Solution

$S_2 = \{(x,y) \mid y = 0\}$

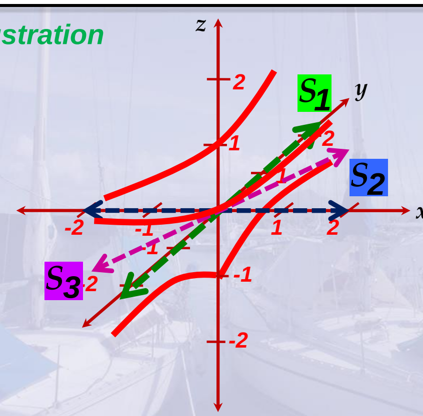
$$\lim_{\substack{(x,y)\to(0,0)\\ P\ in\ S_2}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 - 0}{x^2 + 0} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

Solution

$S_3 = \{(x,y) \mid y = x\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ P\ in\ S_3}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 - x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{0}{2x^2} = \lim_{x\to 0} 0 = 0$$

Illustration



Solution (continued)

Since $$\lim_{\substack{(x,y)\to(0,0)\\ P\ in\ S_1}} \frac{x^2 - y^2}{x^2 + y^2} \neq \lim_{\substack{(x,y)\to(0,0)\\ P\ in\ S_2}} \frac{x^2 - y^2}{x^2 + y^2},$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$ **DOES NOT EXIST.**

Example. Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy + y^3}{x^2 + y^2}$$

Solution:

$S_1 = \{(x,y) \mid x = 0\}$

$S_2 = \{(x,y) \mid y = 0\}$

$S_3 = \{(x,y) \mid y = x\}$

Solution

$S_1 = \{(x, y) \mid x = 0\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_1}} \frac{xy - y^3}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{0 - y^3}{0 + y^2}$$

$$= \lim_{y\to 0} \frac{-y^3}{y^2}$$

$$= \lim_{y\to 0} (-y) = 0$$

Solution

$S_2 = \{(x, y) \mid y = 0\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_2}} \frac{xy - y^3}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{0 - 0}{x^2 + 0}$$

$$= \lim_{x\to 0} \frac{0}{x^2}$$

$$= \lim_{x\to 0} 0 = 0$$

Solution

$S_3 = \{(x, y) \mid y = x\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_3}} \frac{xy - y^3}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 - x^3}{x^2 + x^2}$$

$$= \lim_{x\to 0} \frac{x^2(1 - x)}{2x^2}$$

$$= \lim_{x\to 0} \frac{(1 - x)}{2} = \frac{1}{2}$$

Solution (continued)

Since $$\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_1}} \frac{xy - y^3}{x^2 + y^2} \neq \lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_3}} \frac{xy - y^3}{x^2 + y^2},$$

$$\lim_{(x,y)\to(0,0)} \frac{xy + y^3}{x^2 + y^2}$$ **DOES NOT EXIST.**