1.8

Differentiation of POWER SERIES

Term-by-Term Differentiation

A power series can be differentiated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

$$\sum_{n=1}^{+\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + n c_n x^{n-1} + \ldots$$

Theorem.

If the power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the power series

$$f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

also has R as its radius of convergence.

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n \implies f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

$$\implies f''(x) = \sum_{n=2}^{+\infty} n(n-1) c_n x^{n-2}$$

$$\implies f'''(x) = \sum_{n=2}^{+\infty} n(n-1)(n-2) c_n x^{n-3}$$

$$\vdots$$

Example. Find a series expansion for $f'(x)$ and $f''(x)$

if $f(x)=\dfrac{1}{1-x}$ $,-1<x<1$

SOL'N.

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+\ldots=\sum_{n=0}^{+\infty}x^n$$

$$,-1<x<1$$

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots=\sum_{n=1}^{+\infty}nx^{n-1}$$

$$,-1<x<1$$

Example. Find a series expansion for $f'(x)$ and $f''(x)$

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SOL'N.

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots=\sum_{n=1}^{+\infty}nx^{n-1}$$

$$,-1<x<1$$

$$f''(x)=\frac{2}{(1-x)^3}=2+6x+\ldots+n(n-1)x^{n-2}+\ldots$$

$$=\sum_{n=2}^{+\infty}n(n-1)x^{n-2}\qquad,-1<x<1$$

Example. **Obtain a series expansion for** $\dfrac{1}{\left(1+3x\right)^2}$

and give its validity.

$$\sum_{n=1}^{+\infty} ar^{\,n-1} = \frac{a}{1-r} \quad , -1 < r < 1$$

SOL'N. $a = 1 \quad , r = -3x$

$$-1 < 3x < 1 \quad \Rightarrow \quad -\frac{1}{3} < x < \frac{1}{3}$$

$$g\left(x\right) = \frac{1}{1+3x} = \sum_{n=1}^{+\infty}\left(-3x\right)^{n-1} = \sum_{n=1}^{+\infty}\left(-1\right)^{n-1} 3^{n-1} x^{n-1}$$

$$g'\left(x\right) = \frac{-3}{\left(1+3x\right)^2} = \sum_{n=2}^{+\infty}\left(-1\right)^{n-1} 3^{n-1}\left(n-1\right)x^{n-2}$$

Example. Obtain a series expansion for $\dfrac{1}{(1+3x)^2}$ and give its validity.

SOL'N. Since, $\dfrac{-3}{(1+3x)^2} = \sum_{n=2}^{+\infty}(-1)^{n-1}3^{n-1}(n-1)x^{n-2}$

We'll have

$$\frac{1}{(1+3x)^2} = \frac{1}{-3}\sum_{n=2}^{+\infty}(-1)^{n-1}3^{n-1}(n-1)x^{n-2}$$

$$= \sum_{n=2}^{+\infty}(-1)^{n-2}3^{n-2}(n-1)x^{n-2} \quad , -\frac{1}{3} < x < \frac{1}{3}$$

Example. **Obtain a series expansion for** $\dfrac{1}{(7-2x)^2}$ **and give its validity.**

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r} \quad , -1 < r < 1$$

SOL'N. $a = 1 \quad , r = \dfrac{2}{7}x$

$$-1 < \frac{2}{7}x < 1 \quad \Rightarrow \quad -\frac{7}{2} < x < \frac{7}{2}$$

$$g(x) = \frac{1}{1-\frac{2}{7}x} = \frac{7}{7-2x} = \sum_{n=1}^{+\infty}\left(\frac{2}{7}x\right)^{n-1} = \sum_{n=1}^{+\infty}\left(\frac{2}{7}\right)^{n-1} x^{n-1}$$

$$g'(x) = \frac{14}{(7-2x)^2} = \sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$$

Example. Obtain a series expansion for $\dfrac{1}{(7-2x)^2}$ and give its validity.

SOL'N. Since, $\dfrac{14}{(7-2x)^2} = \displaystyle\sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$

We'll have

$$\frac{1}{(7-2x)^2} = \frac{1}{14}\sum_{n=2}^{+\infty}\left(\frac{2}{7}\right)^{n-1}(n-1)x^{n-2}$$

$$= \sum_{n=2}^{+\infty}\frac{(n-1)2^{n-2}x^{n-2}}{7^n} \quad -\frac{7}{2} < x < \frac{7}{2}$$

Integration of POWER SERIES

Term-by-Term Integration

A power series can be integrated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

$$\sum_{n=0}^{+\infty} \frac{c_n x^{n+1}}{n+1} + C = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots + c_n \frac{x^{n+1}}{n+1} + \ldots$$

Theorem.

If the power series

$$f(x)=\sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the

power series

$$\int f(x)dx=\sum_{n=0}^{+\infty} c_n \frac{x^{n+1}}{n+1}+C$$

also has R as its radius of convergence.

Example. Obtain a series expansion for $\ln(2+x)$ and give its validity.

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r} \quad ,-1<r<1$$

SOL'N. $a=1 \quad ,r=-\dfrac{1}{2}x$

$$-1<\frac{1}{2}x<1 \quad \Rightarrow \quad -2<x<2$$

$$g(x)=\frac{1}{1+\frac{1}{2}x} = \frac{2}{2+x} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}x\right)^{n-1} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1} x^{n-1}$$

$$g(t)=\frac{2}{2+t} = \sum_{n=1}^{+\infty}\left(-\frac{1}{2}\right)^{n-1} t^{n-1}$$

Example. **Obtain a series expansion for** $\ln(2+x)$ **and give its validity.**

SOL'N.

$$\int_0^x \frac{2\,dt}{2+t} = \int_0^x \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} t^{n-1} dt$$

$$\left(2\ln(2+t)\right]_0^x = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \left(\frac{t^n}{n}\right]_0^x$$

$$2\ln(2+x) - 2\ln 2 = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \frac{x^n}{n}$$

$$\ln(2+x) = \frac{1}{2}\sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \frac{x^n}{n} + \ln 2 \quad , -2 < x < 2$$

END