Higher Order Partial Derivatives

Second-Order Partial Derivative

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$\dfrac{\partial^2 f}{\partial x^2}$ f_{xx}	$\dfrac{\partial^2 f}{\partial y^2}$ f_{yy}	$\dfrac{\partial^2 f}{\partial y \partial x}$ f_{xy}	$\dfrac{\partial^2 f}{\partial x \partial y}$ f_{yx}

The defining equations are

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Let $w = xy^2 + x^2y^3 + x^3y^4$

Solution.

$$\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$$

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x^2} = 2y^3 + 6xy^4$$

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$$

Let $w = xy^2 + x^2y^3 + x^3y^4$

Solution.

$$\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial y^2} = 2x + 6x^2y + 12x^3y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$$

Let $f(x,y)=e^{x}-3\sin(xy)+Arc\tan(y)$

Find all of its second order partial derivatives.

Solution.

$$f_x(x,y)= e^{x}-3y\cos(xy)$$

$$f_{xx}(x,y)= e^{x}+3y^{2}\sin(xy)$$

$$f_{xy}(x,y)= -3\cos(xy)+3xy\sin(xy)$$

Let $f\left(x,y\right)=e^{x}-3\sin\left(xy\right)+Arc\tan\left(y\right)$

Solution.

$$f_{y}\left(x,y\right)=\ -3x\cos\left(xy\right)+\frac{1}{1+y^{2}}$$

$$f_{yy}\left(x,y\right)=\ 3x^{2}\sin\left(xy\right)+\frac{-1}{\left(1+y^{2}\right)^{2}}\left(2y\right)$$

$$f_{yx}\left(x,y\right)=\ -3\cos\left(xy\right)+3xy\sin\left(xy\right)$$

Theorem. **The Mixed Derivative Theorem**

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} Are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

(¼ paper)

Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ given

$$f(x, y) = 2^x + x \ln y + y \ln x$$

Remark.

There is no theoretical limit to how many times we can differentiate a function as long as the derivatives exist.

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = f_{xyx}$$

$$\frac{\partial^4 f}{\partial y^2 \partial x^2} = f_{xxyy}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{yxx}$$

$$\frac{\partial^4 f}{\partial x^3 \partial y} = f_{yxxx}$$

Exercise.

Let $f(x,y)=\cos\left(x^2-y^2\right)$. Find $\dfrac{\partial^3 f}{\partial x^2 \partial y} = f_{yxx}$

Solution.

$$f_y(x,y)= 2y\sin\left(x^2-y^2\right)$$

$$f_{yx}(x,y)= 4xy\cos\left(x^2-y^2\right)$$

$$f_{yxx}(x,y)= 4y\cos\left(x^2-y^2\right)$$
$$-8x^2y\sin\left(x^2-y^2\right)$$

END