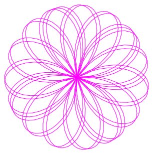
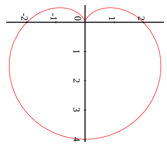
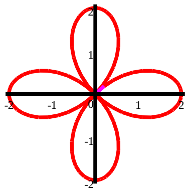
CHAPTER 4







POLAR COORDINATES

Objectives:

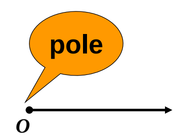
At the end of the chapter, you should be able to

1. plot polar points,
2. find the polar coordinates of a cartesian point and vice-versa,
3. sketch polar curves and
4. find the area of a polar region.

OUTLINE

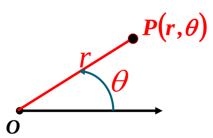
4.1 The polar coordinate system

A polar coordinate system consists of a horizontal ray called the *polar axis* (0-axis, 2π *axis*).



The initial point of the polar axis is called the *pole* (*O*).

A point *P* on a polar plane has coordinates (r,θ)



where *r* is the distance of the point from the pole,

and θ is the measure of the angle which the ray *OP* makes with the polar axis.

How do we plot a polar point (r,θ)?

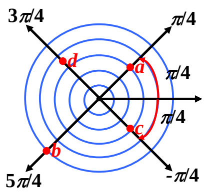
1. **We first locate the θ-axis.**
2. **a. If $r > 0$, the point is plotted along the θ-axis.**

   b. If $r < 0$, the point is plotted on the opposite side of the θ-axis.

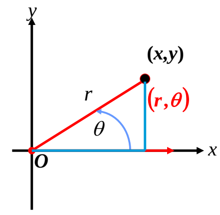
Example 4.1.1 **Plot the following polar points.**

a. $(3, \pi/4)$ *b.* $(-5, \pi/4)$ *c.* $(3, -\pi/4)$ *d.* $(-3.5, -\pi/4)$

Solution:



How are the cartesian coordinates (x,y) and polar coordinates (r,θ) of a point related?



$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}, x \neq 0$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

Example 4.1.2 **Find the cartesian coordinates of the given polar point.**

a. $(6, \pi/4)$ *b.* $(4, 2\pi/3)$

Solution:

a. $(6, \pi/4)$

$$x = r\cos\theta = 6 \cdot \cos\frac{\pi}{4} = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$$

$$y = r\sin\theta = 6 \cdot \sin\frac{\pi}{4} = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$$

Answer. $\left(3\sqrt{2}, 3\sqrt{2}\right)$

b. $(4, 2\pi/3)$

$$x = r\cos\theta = 4 \cdot \cos\frac{2\pi}{3} = 4 \cdot \frac{-1}{2} = -2$$

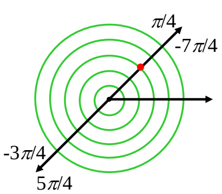
$$y = r\sin\theta = 4 \cdot \sin\frac{2\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Answer. $\left(-2, 2\sqrt{3}\right)$

While the cartesian coordinates of any point are unique, the polar coordinates of any point are not unique.

Consider the point with polar coordinates

$(3, \pi/4)$



The same point has polar coordinates

$(3, -7\pi/4)$, $(-3, 5\pi/4)$

$(-3, -3\pi/4)$, etc.

Example 4.1.3 **Find a set of polar coordinates (r,θ) of the cartesian point (-3,3) such that $-2\pi \leq \theta \leq 2\pi$ and**

a. $r > 0$ **and** $\theta > 0$ **c.** $r < 0$ **and** $\theta > 0$

b. $r > 0$ **and** $\theta < 0$ **d.** $r < 0$ **and** $\theta < 0$

Solution:

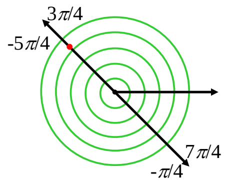
$$x^2 + y^2 = r^2$$

$$r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{18} = \pm 3\sqrt{2}$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = \frac{3\pi}{4}$$ **(since (-3,3) $\in$ QII)**

a. $r > 0$ and $\theta > 0$
b. $r > 0$ and $\theta < 0$
c. $r < 0$ and $\theta > 0$
d. $r < 0$ and $\theta < 0$



Answers:

a. $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$ c. $\left(-3\sqrt{2}, \frac{7\pi}{4}\right)$

b. $\left(3\sqrt{2}, \frac{-5\pi}{4}\right)$ d. $\left(-3\sqrt{2}, \frac{-\pi}{4}\right)$

Example 4.1.4 **Find a set of polar coordinates (r,θ) of the cartesian point (-1,-2) such that $-2\pi \le \theta \le 2\pi$ and**

a. $r > 0$ and $\theta > 0$ c. $r < 0$ and $\theta > 0$
b. $r > 0$ and $\theta < 0$ d. $r < 0$ and $\theta < 0$

Solution:

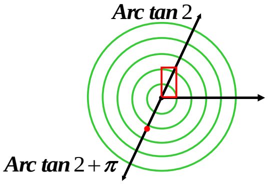
$$x^2 + y^2 = r^2$$

$$r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{5}$$

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \tan\theta = \frac{-2}{-1} = 2 \quad \Rightarrow \theta = \pi + \operatorname{Arc}\tan 2$$

(since (-1,-2) $\in$ QIII)

a. $r > 0$ and $\theta > 0$
b. $r > 0$ and $\theta < 0$
c. $r < 0$ and $\theta > 0$
d. $r < 0$ and $\theta < 0$



Answers:

a. $\left(\sqrt{5}, \operatorname{Arc}\tan 2 + \pi\right)$ c. $\left(-\sqrt{5}, \operatorname{Arc}\tan 2\right)$

b. $\left(\sqrt{5}, \operatorname{Arc}\tan 2 - \pi\right)$ d. $\left(-\sqrt{5}, \operatorname{Arc}\tan 2 - 2\pi\right)$

Example 4.1.5 **Find a polar equation of a curve whose cartesian equation is given by**

a. $x^2 + y^2 = 9$ b. $y = \sqrt{3}x$

Solution:

a. $x^2 + y^2 = 9$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 9$$

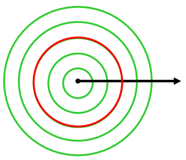
$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$$

$$r^2 = 9$$

$$r = \pm 3$$

$$x^2 + y^2 = 9$$



The same graph is given by

$r = 3$ or $r = -3.$

$(3,\theta), \theta \in R$ $(-3,\theta), \theta \in R$

b. $y = \sqrt{3}x$

$$r\sin\theta = \sqrt{3} \cdot r\cos\theta$$

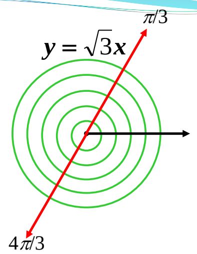
$$r\sin\theta - \sqrt{3} \cdot r\cos\theta = 0$$

$$r\left(\sin\theta - \sqrt{3}\cos\theta\right) = 0$$

$$r = 0 \text{ or } \sin\theta - \sqrt{3}\cos\theta = 0$$

$$r = 0 \text{ or } \tan\theta = \sqrt{3}$$

$$r = 0 \text{ or } \theta = \frac{\pi}{3} \text{ or } \theta = \frac{4\pi}{3}$$



The same graph is given by

$$\theta=\frac{\pi}{3}\ \text{ or }\ \theta=\frac{4\pi}{3}.$$

4.2 Graphs of polar equations

The graph of a cartesian equation consists of all points (x,y) that satisfies the given equation.

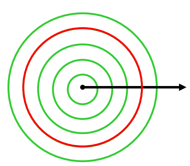
The graph of a polar equation consists of all points (r,θ) that satisfies the given equation.

A. Circles centered at the pole

$$r=\pm a,\ \text{where}\ a\neq 0$$

Illustration:

$$r=4$$



The same circle is given by $r=-4$.

B. Circles tangent to the pole

$$r=a\cos\theta,\ \text{where}\ a\neq 0$$

$$r=a\sin\theta,\ \text{where}\ a\neq 0$$

If

$$r=a\cos\theta$$

$$\sqrt{x^2+y^2}=a\cdot\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2 \quad \boxed{\text{an equation of a circle}}$$

$$C\left(\frac{a}{2},0\right), r=\left|\frac{a}{2}\right|$$

Illustrations:

$r=4\cos\theta$

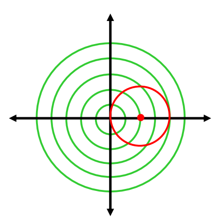
$a = 4$

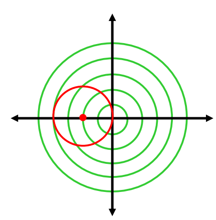
$C(2,0), r=2$

$r=-4\cos\theta$

$a = -4$

$C(-2,0), r=2$





If

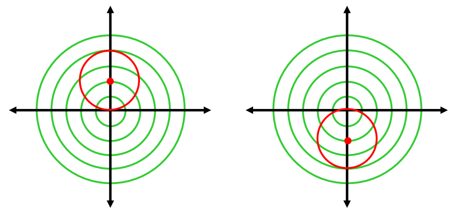
$$r = a\sin\theta$$
$$\sqrt{x^2+y^2} = a\cdot\frac{y}{\sqrt{x^2+y^2}}$$
$$x^2+y^2 = ay$$
$$x^2+y^2-ay=0$$
$$x^2+\left(y-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$$

equation of a circle

$$C\left(0,\frac{a}{2}\right), r=\left|\frac{a}{2}\right|$$

Illustrations:

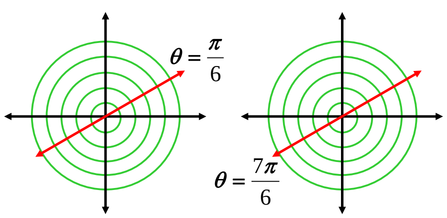
$r = 4\sin\theta$	$r = -4\sin\theta$
$a = 4$	$a = -4$
$C(0,2), r = 2$	$C(0,-2), r = 2$



C.i. Lines through the pole

$$\theta = a, a \in R$$

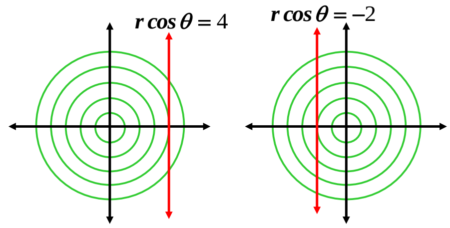
Illustrations:



C.ii. Vertical lines

$$r\cos\theta = a, a \in R$$
$$x = a, a \in R$$

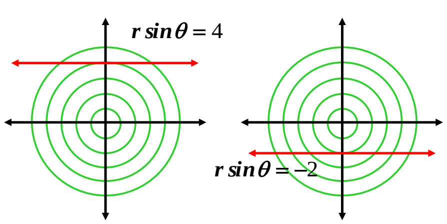
Illustrations:



C.iii. horizontal lines

$$r\sin\theta = a, a \in R$$
$$y = a, a \in R$$

Illustrations:



Symmetry of a polar curve

a. The graph of a polar equation is **symmetric with respect to the x-axis** when an equivalent polar equation is obtained when (r,θ) is replaced by either

$(r,-\theta)$ or $(-r,\pi-\theta)$.

Illustration:

a. The graph of $r = 5 + 5\cos\theta$

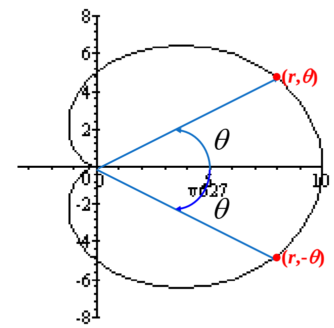
is symmetric with respect to the *x*-axis.

When (r,θ) is replaced by $(r,-\theta)$, the equation becomes

$$r = 5 + 5\cos(-\theta)$$

$$r = 5 + 5\cos\theta.$$

$r = 5 + 5\cos\theta$



b. The graph of a polar equation is symmetric with respect to the *y*-axis when an equivalent polar equation is obtained when (r,θ) is replaced by either

$(-r,-\theta)$ or $(r,\pi - \theta)$.

Illustration:

b. The graph of $r = 3 + 3\sin\theta$

is symmetric with respect to the *y*-axis.

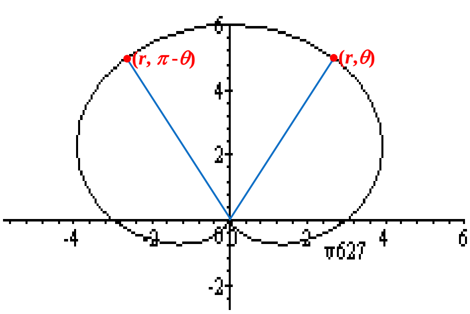
When (r,θ) is replaced by $(r, \pi - \theta)$, the equation becomes

$$r = 3 + 3\sin(\pi - \theta)$$

$$r = 3 + 3\cancelto{0}{\sin\pi}\cos\theta - \cancelto{-1}{\cos\pi}\sin\theta$$

$$r = 3 + 3\sin\theta$$

$r = 3 + 3\sin\theta$



D. Limacons

$$r = a \pm b\cos\theta \quad or \quad r = a \pm b\sin\theta,$$

where $a \neq 0$ and $b \neq 0$.

Types of Limacons

1. **Limacon with a loop** $\left|\frac{a}{b}\right| < 1$
2. **Cardioid** $\left|\frac{a}{b}\right| = 1$
3. **Limacon with a dent** $1 < \left|\frac{a}{b}\right| < 2$
4. **Convex Limacon** $\left|\frac{a}{b}\right| \geq 2$

Illustration. Sketch the graph of

$$r = 2 + 3\cos\theta.$$

Solution:

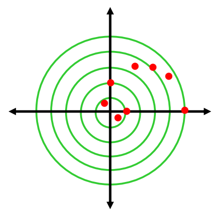
$$a = 2, b = 3$$

$$\left|\frac{a}{b}\right| = \left|\frac{2}{3}\right| \Rightarrow 0 < \frac{2}{3} < 1$$

The graph is a limacon with a loop which is symmetric with respect to the *x*-axis.

$$r = 2 + 3\cos\theta$$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	-1/2	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$3\cos\theta$	3	$3\sqrt{3}/2$	$3\sqrt{2}/2$	3/2	0	-3/2	$-3\sqrt{2}/2$	$-3\sqrt{3}/2$	-3
$2+3\cos\theta$	5	4.6	4.1	3.5	2	.5	.12	-.6	-1

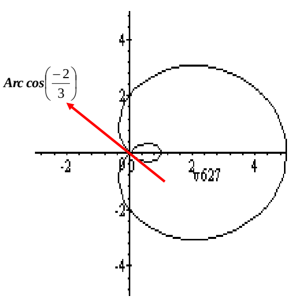


$$r = 2 + 3\cos\theta$$

$$\frac{dr}{d\theta} = -3\sin\theta$$

$$\frac{dr}{d\theta} < 0 \text{ when } \theta \in (0, \pi)$$

$$r = 2 + 3\cos\theta$$



$$2 + 3\cos\theta = 0$$

$$\cos\theta = \frac{-2}{3}$$

$$\theta = Arc\cos\left(\frac{-2}{3}\right)$$

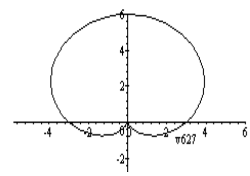
Illustration. Sketch the graph of

$$r = 3 + 3\sin\theta.$$

Solution:

$$a = 3, b = 3$$

$$\left|\frac{a}{b}\right| = \left|\frac{3}{3}\right| = 1$$



The graph is a cardioid which is symmetric with respect to the *y*-axis.

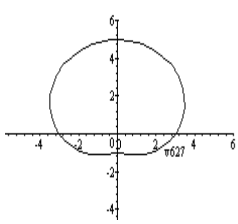
Illustration. Sketch the graph of

$$r = 3 + 2\sin\theta.$$

Solution:

$$a = 3, b = 2$$

$$\left|\frac{a}{b}\right| = \left|\frac{3}{2}\right| \Rightarrow 1 < \left|\frac{a}{b}\right| < 2$$



The graph is a limacon with a dent which is symmetric with respect to the *y*-axis.

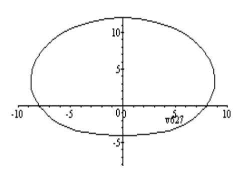
Illustration. Sketch the graph of

$$r = 8 + 4\sin\theta.$$

Solution:

$$a = 8, b = 4$$

$$\left|\frac{a}{b}\right| = \left|\frac{8}{4}\right| = 2$$



The graph is a convex limacon which is symmetric with respect to the *y*-axis.

E. Roses

$$r = a\cos(n\theta) \quad or \quad r = a\sin(n\theta)$$

where $n \geq 2$.

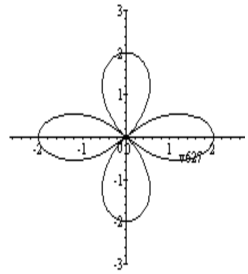
1. If n is odd, then the rose has n congruent leaves.
2. If n is even, then the rose has $2n$ congruent leaves.

Illustration. **Sketch the graph of** $r = 2\cos 2\theta$.

Solution:

$a = 2, n = 2$

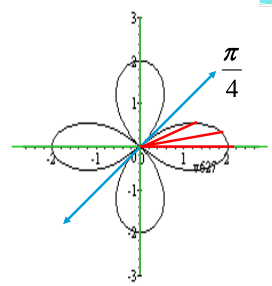
The graph is a rose with 4 leaves and which is symmetric with respect to the *x*-axis and *y*-axis.



$$r = 2\cos 2\theta$$

$\cos 2\theta = 1$	$\cos 2\theta = 0$
$2\theta = 0, 2\pi, \ldots$	$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
$\theta = 0, \pi, \ldots$	$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

$$\frac{2\pi}{4} = \frac{\pi}{2} = 90^\circ$$



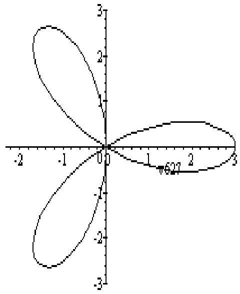
Half a leaf can be generated by considering the interval $\left[0, \frac{\pi}{4}\right]$.

Illustration. **Sketch the graph of** $r = 3\cos 3\theta$.

Solution:

$a = 3, n = 3$

The graph is a rose with 3 leaves and which is symmetric with respect to the *x*-axis.

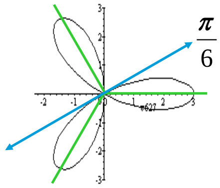


$$r = 3\cos 3\theta$$

$\cos 3\theta = 1$	$\cos 3\theta = 0$
$3\theta = 0, 2\pi, \ldots$	$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
$\theta = 0, \frac{2\pi}{3}, \ldots$	$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \ldots$

$$\frac{2\pi}{3} = 120^\circ$$

$r = 3\cos 3\theta$



Half a leaf can be generated by considering the interval $\left[0, \frac{\pi}{6}\right]$.

F. Lemniscates

$r^2 = a\sin(2\theta)$ *or* $r^2 = a\cos(2\theta)$

where $a \neq 0$.

The graph of a lemniscate is a figure 8.

$r^2 = a\sin(2\theta)$

$r = \pm\sqrt{a\sin(2\theta)}$

If $a > 0$

$\sin(2\theta) \geq 0$

$0 \leq 2\theta \leq \pi$

$0 \leq \theta \leq \frac{\pi}{2}$

If $a < 0$

$\sin(2\theta) \leq 0$

$\pi \leq 2\theta \leq 2\pi$

$\frac{\pi}{2} \leq \theta \leq \pi$

$r^2 = a\cos(2\theta)$

$r = \pm\sqrt{a\cos(2\theta)}$

If $a > 0$

$\cos(2\theta) \geq 0$

$\frac{-\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$

$\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$

If $a < 0$

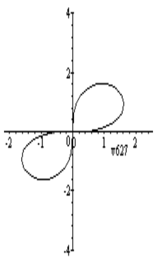
$\cos(2\theta) \leq 0$

$\frac{\pi}{2} \leq 2\theta \leq \frac{3\pi}{2}$

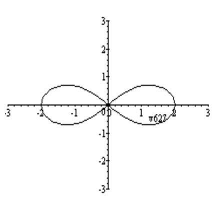
$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

Illustrations:

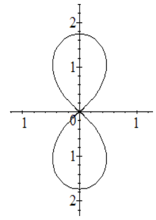
$r^2 = 4\sin 2\theta$



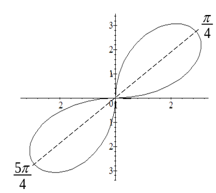
$r^2 = 4\cos 2\theta$



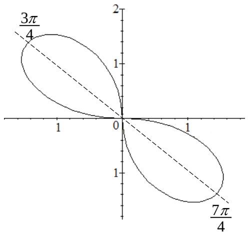
$r^2 = -3\cos 2\theta$



$r^2 = 16\sin 2\theta$



$$r^2 = -2\sin 2\theta$$



G. Spirals

Spiral of Archimedes

$$r = \theta,\ \ \theta \ge 0$$

Logarithmic spiral

$$r = \log\theta,\ \theta > 0$$

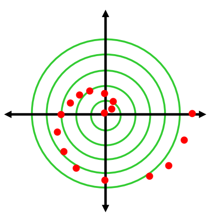
Exponential spiral

$$r = e^{\theta},\ \theta \ge 0$$

Illustrations:

1. $r = \theta,\ \ \theta \ge 0$

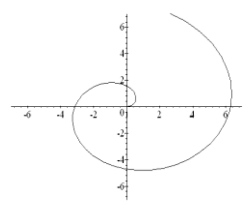
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	0	.52	.79	1.05	1.57	2.09	2.35	2.62	3.1

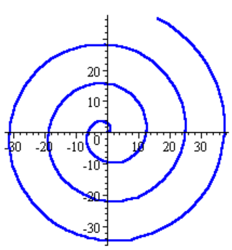


$$\frac{dr}{d\theta} = 1 > 0$$

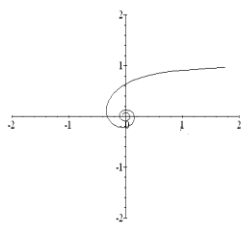
As θ increases, r increases.

$$r = \theta,\ \ \theta \ge 0$$



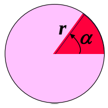


2. $r = \dfrac{1}{\theta}, \theta > 0$



4.3 Area of polar regions

We recall…



the area of a sector of a circle of radius r and which subtends a central angle of α radians is

$$\frac{1}{2}r^2\alpha.$$

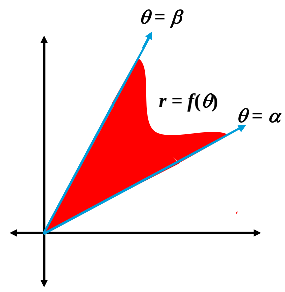
Let R be the region enclosed by the graph of

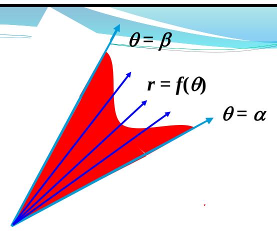
$$r = f(\theta)$$

and the lines given by

$$\theta = \alpha \text{ and } \theta = \beta,$$

where f is continuous and non-negative on the interval $[\alpha, \beta]$.





Subdivide the closed interval $[\alpha, \beta]$ into n sub-intervals by choosing $(n-1)$ intermediate numbers $\theta_1, \theta_2, \dots, \theta_{n-1}$, where

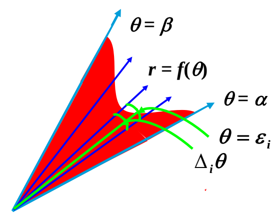
$$\alpha = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n = \beta.$$

Denote the ith sub-interval by I_i so that

$$I_1 = [\theta_0, \theta_1]$$
$$I_2 = [\theta_1, \theta_2]$$
$$I_3 = [\theta_2, \theta_3]$$
$$I_i = [\theta_{i-1}, \theta_i]$$
$$I_n = [\theta_{n-1}, \theta_n]$$

For each $i = 1, 2, \dots, n$, choose a number

$$\varepsilon_i \in I_i.$$



Construct a sector of radius

$$f(\varepsilon_i).$$

The area of the ith sector is

$$\frac{1}{2}r^2\alpha = \frac{1}{2}[f(\varepsilon_i)]^2 \cdot \Delta_i\theta$$

The sum of areas of the n sectors is

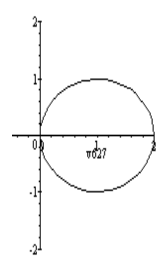
$$\sum_{i=1}^{n} \frac{1}{2}[f(\varepsilon_i)]^2 \cdot \Delta_i\theta = \frac{1}{2}\sum_{i=1}^{n} [f(\varepsilon_i)]^2 \cdot \Delta_i\theta$$

The area of the region is

$$\frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n} [f(\varepsilon_i)]^2 \cdot \Delta_i\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta$$

Illustration: Find the area of the region enclosed by the graph of $r = 2\cos\theta$.

Solution:



The graph is symmetric with respect to the x-axis.

We may consider the area of the region above or below the x-axis.

$$A=2\left[\frac{1}{2}\int_{0}^{\pi/2}(2\cos\theta)^2\,d\theta\right]$$
$$=\int_{0}^{\pi/2}(2\cos\theta)^2\,d\theta=4\int_{0}^{\pi/2}\cos^2\theta\,d\theta$$
$$=4\int_{0}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta=2\int_{0}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$=2\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big]_{0}^{\pi/2}$$
$$=2\left(\frac{\pi}{2}+\frac{1}{2}\cancelto{0}{\sin 2\cdot\frac{\pi}{2}}\right)-2\left(0+\frac{1}{2}\cancelto{0}{\sin 2\cdot 0}\right)=\pi$$

The area of the region is graph is π square units.

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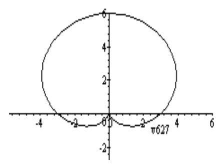
Illustration: **Find the area of the region enclosed by the graph of**

$$r=3+3\sin\theta.$$

Solution:



The graph is symmetric with respect to the *y*-axis.

We may consider the area of the region to the right or to the left of the *y*-axis.

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$$A=2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}(3+3\sin\theta)^2\,d\theta\right]$$
$$=\int_{-\pi/2}^{\pi/2}(3+3\sin\theta)^2\,d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\left(9+18\sin\theta+\sin^2\theta\right)d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\left(9+18\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\left(\frac{19}{2}+18\sin\theta-\frac{1}{2}\cos 2\theta\right)d\theta$$

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$$=\int_{-\pi/2}^{\pi/2}\left(\frac{19}{2}+18\sin\theta-\frac{1}{2}\cos 2\theta\right)d\theta$$
$$=\frac{19}{2}\theta-18\cos\theta-\frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2}$$
$$=\left(\frac{19}{2}\cdot\frac{\pi}{2}-18\cancelto{0}{\cos\frac{\pi}{2}}-\frac{1}{4}\cancelto{0}{\sin 2\cdot\frac{\pi}{2}}\right)$$
$$-\left(\frac{19}{2}\cdot\frac{-\pi}{2}-18\cancelto{0}{\cos\frac{-\pi}{2}}-\frac{1}{4}\cancelto{0}{\sin 2\cdot\frac{-\pi}{2}}\right)$$
$$=\frac{19}{4}\pi+\frac{19}{4}\pi=\frac{19}{2}\pi$$

The area of the region is graph is $\frac{19}{2}\pi$ square units.

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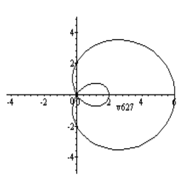
Illustration: **Find the area of the region enclosed by the loop of the graph of**

$$r=2+4\cos\theta.$$

Solution:



The graph is symmetric with respect to the *x*-axis.

We may consider the area of the region to the above below the *x*-axis.

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$r=2+4\cos\theta=0$	$r=2+4\cos\theta=-2$
$4\cos\theta=-2$	$4\cos\theta=-4$
$\cos\theta=\frac{-1}{2}$	$\cos\theta=-1$
$\theta=\frac{2\pi}{3}$	$\theta=\pi$

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$$A = 2\left[\frac{1}{2}\int_{2\pi/3}^{\pi} (2+4\cos\theta)^2\,d\theta\right]$$
$$= \int_{2\pi/3}^{\pi} (2+4\cos\theta)^2\,d\theta$$
$$= \int_{2\pi/3}^{\pi} \left(4+16\cos\theta+16\cos^2\theta\right)d\theta$$
$$= \int_{2\pi/3}^{\pi} \left(4+16\cos\theta+16\left(\frac{1+\cos 2\theta}{2}\right)\right)d\theta$$
$$= \int_{2\pi/3}^{\pi} \left(4+16\cos\theta+8(1+\cos 2\theta)\right)d\theta$$
$$= \int_{2\pi/3}^{\pi} \left(12+16\cos\theta+8\cos 2\theta\right)d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(12+16\cos\theta+8\cos 2\theta\right)d\theta$$
$$= 12\theta+16\sin\theta+4\sin 2\theta\Big]_{2\pi/3}^{\pi}$$
$$= \left(12\pi+16\cancelto{0}{\sin\pi}+4\cancelto{0}{\sin 2\pi}\right)$$
$$-\left(12\cdot\frac{2\pi}{3}+16\sin\left(\frac{2\pi}{3}\right)+4\sin\left(2\cdot\frac{2\pi}{3}\right)\right)$$
$$= 12\pi-8\pi-16\cdot\frac{\sqrt{3}}{2}-4\cdot\frac{-\sqrt{3}}{2} = 4\pi-4\sqrt{3}$$

The area of the region enclosed by the loop is $\left(4\pi-4\sqrt{3}\right)$ square units.

Area between two polar curves

If R is the region enclosed by the graphs of

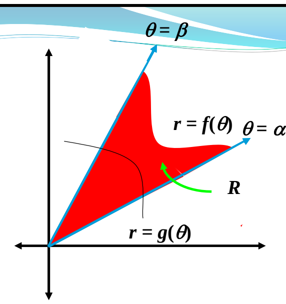
$$r = f(\theta) \text{ and } r = g(\theta)$$

on $[\alpha,\beta]$, where f and g are continuous and non-negative and

$$f(\theta) \geq g(\theta)$$

for each θ in $[\alpha,\beta]$, then the area of R is given by

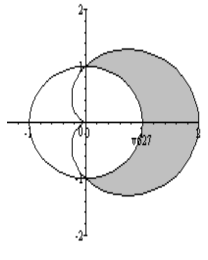
$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)^2 - g(\theta)^2\right]d\theta.$$



$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)^2\right]d\theta - \frac{1}{2}\int_{\alpha}^{\beta} \left[g(\theta)^2\right]d\theta$$
$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)^2 - g(\theta)^2\right]d\theta.$$

Illustration. **Find the area of the region inside the graph of $r = 1+\cos\theta$ but outside the graph of $r = 1$.**

Solution:



$$1+\cos\theta = 1$$
$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)^2 - g(\theta)^2\right]d\theta.$$
$$A = 2\left[\frac{1}{2}\int_{0}^{\pi/2} \left((1+\cos\theta)^2-1^2\right)d\theta\right]$$
$$= \int_{0}^{\pi/2} \left((1+\cos\theta)^2-1^2\right)d\theta$$
$$= \int_{0}^{\pi/2} \left(2\cos\theta+\cos^2\theta\right)d\theta$$
$$= \int_{0}^{\pi/2} \left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \int_{0}^{\pi/2} \left(2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \Big]_0^{\pi/2}$$

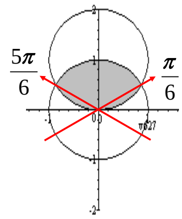
$$= \left(2\sin\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{4}\cancelto{0}{\sin} 2\cdot\frac{\pi}{2}\right) - 2\cancelto{0}{\sin} 0 + \frac{1}{2}\cdot 0 + \frac{1}{4}\cancelto{0}{\sin} 2\cdot 0$$

$$= 2 + \frac{\pi}{4}$$

The area of the region is $\left(2+\frac{\pi}{4}\right)$ **square units.**

Illustration. **Find the area of the region common to the regions enclosed by the graphs of** $r = 1$ and $r = 2\sin\theta$.

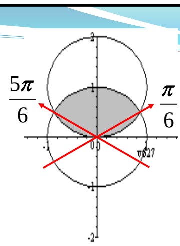
Solution:



$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$



$$A = A_1 + A_2$$

$$A_1 = 2\left[\frac{1}{2}\int_0^{\pi/6} (2\sin\theta)^2\, d\theta\right]$$

$$A_2 = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 1^2\, d\theta\right]$$

The end !!!

$r = 2\sin(2.15\theta)$

$0 \le \theta \le 16\pi$