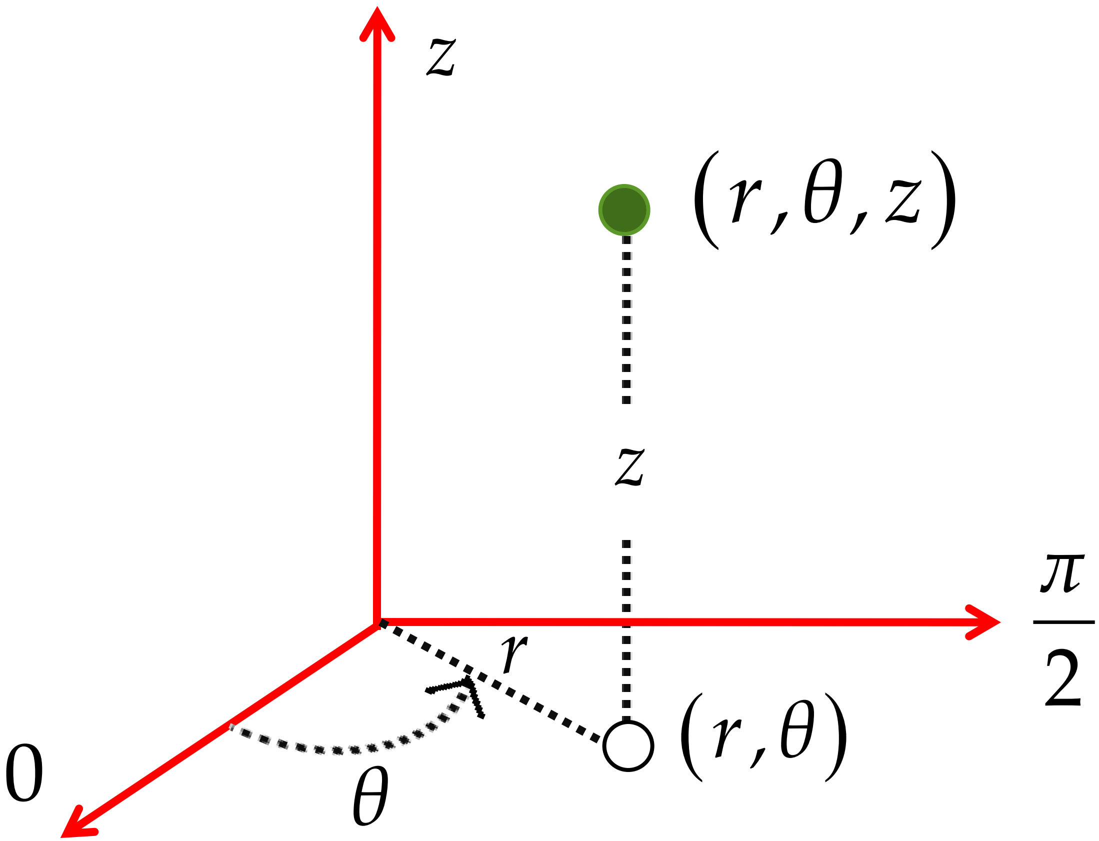
Triple Integrals in Cylindrical Coordinates

Chapter 4 Section 4

4.4 Triple Integral in Cylindrical Coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = z$$

$$\tan\theta = \frac{y}{x}$$
$$x^2 + y^2 = r^2$$

Exercise

EVALUATE the triple integral.

a. $\int_0^{\pi}\int_0^{\frac{\theta}{\pi}}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\, r\,dr\, d\theta$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}}\left(\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\right) r\,dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}}\left(\frac{z^2}{2}\right)_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} r\,dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 4\left(4-r^2\right) r\,dr\, d\theta$$

Exercise

EVALUATE the triple integral.

$$\int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 4\left(4-r^2\right) r\,dr\,d\theta \quad = \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} \left(16r-4r^3\right) dr\,d\theta$$

$$= \int_0^{\pi} \left(8r^2-r^4\right)_0^{\theta/\pi} d\theta \quad = \int_0^{\pi} \left(\frac{8\theta^2}{\pi^2}-\frac{\theta^4}{\pi^4}\right) d\theta$$

$$= \left(\frac{8\theta^3}{3\pi^2}-\frac{\theta^5}{5\pi^4}\right)_0^{\pi} \quad = \left(\frac{8\pi^3}{3\pi^2}-\frac{\pi^5}{5\pi^4}\right) \quad = \frac{37\pi}{15}$$

Exercise

EVALUATE the triple integral.

b. $$\int_0^{2\pi}\int_0^3\int_{\frac{r^2}{3}}^{\sqrt{18-r^2}} dz\, r dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^3\left(\int_{\frac{r^2}{3}}^{\sqrt{18-r^2}} dz\right) r dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^3\left(\sqrt{18-r^2}-\frac{r^2}{3}\right) r dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^3\left(r\sqrt{18-r^2}-\frac{r^3}{3}\right) dr\, d\theta$$

Exercise

EVALUATE the triple integral.

$$\int_0^{2\pi}\int_0^3\left(r\sqrt{18-r^2}-\frac{r^3}{3}\right)dr\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{-\left(18-r^2\right)^{3/2}}{3}-\frac{r^4}{12}\right)\Bigg|_0^3 d\theta \qquad =\int_0^{2\pi}\left(18\sqrt{2}-\frac{63}{4}\right)d\theta$$

$$=2\pi\left(18\sqrt{2}-\frac{63}{4}\right) \qquad =36\pi\sqrt{2}-\frac{63\pi}{2}$$

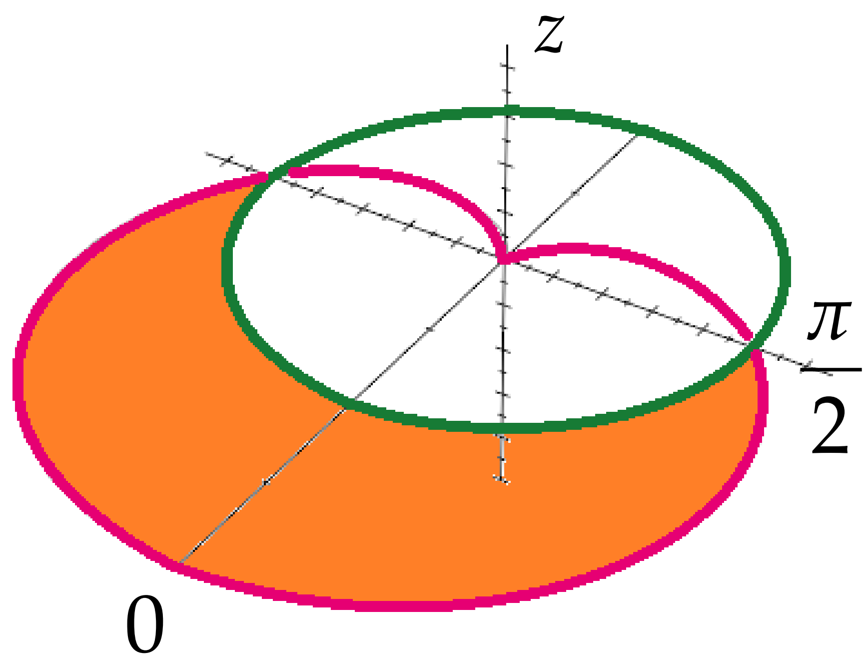
Exercise **SET-UP** a triple integral that gives the volume of the described solid.

a. Base is the region in the *xy*-plane that lies inside the cardioid

$$r = 1 + \cos\theta$$

and outside the circle

$r = 1$ and whose top lies in the plane $z = 4$



$$V = 2\int_0^{\frac{\pi}{2}} \int_1^{1+\cos\theta} \int_0^4 dz\, r\, dr\, d\theta$$

Exercise

SET-UP a triple integral that gives the volume of the described solid.

b. Enclosed by the cylinder

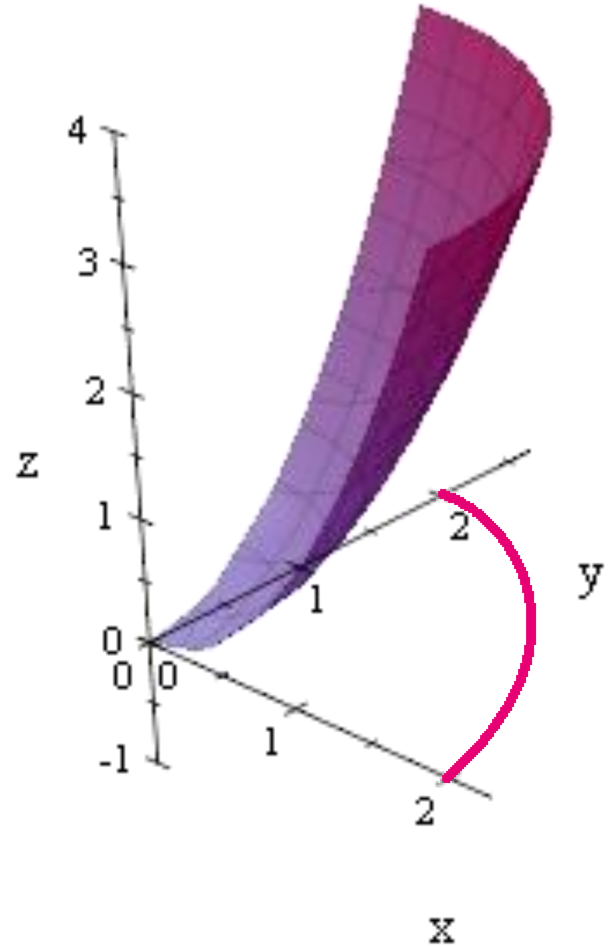
$$x^2 + y^2 = 4$$

above by

$$z = x^2 + y^2$$

below by the *xy*-plane

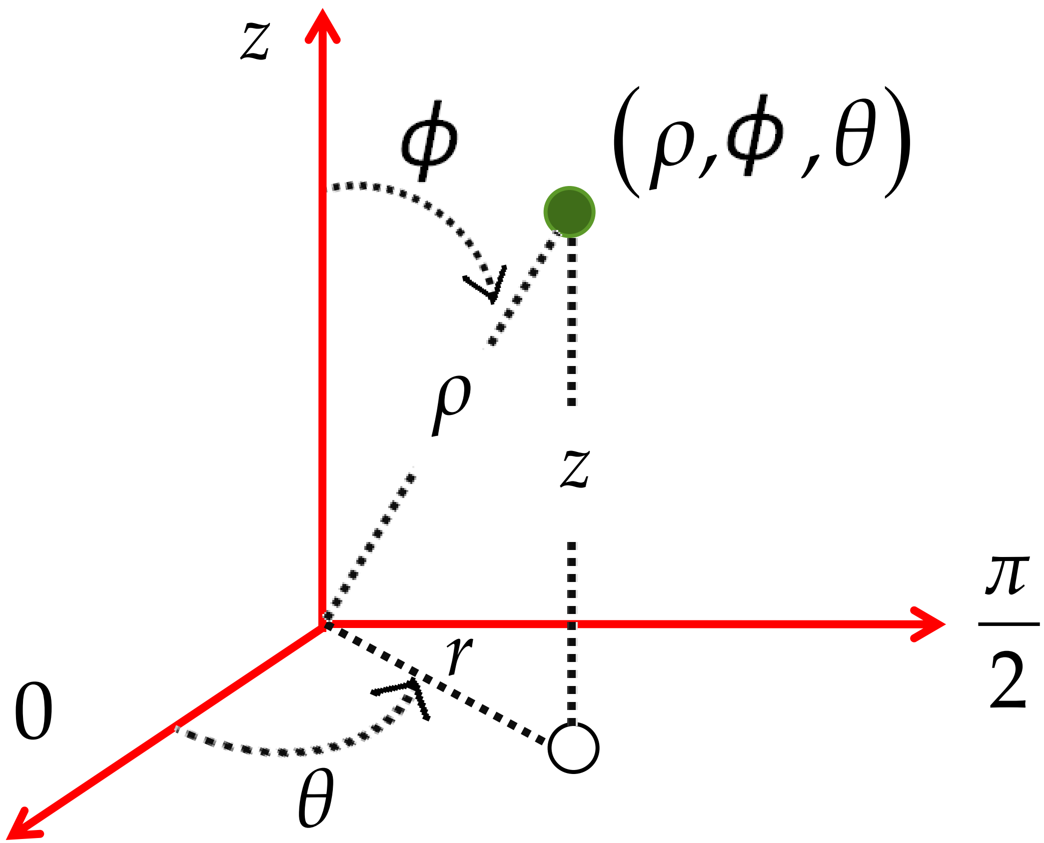
$$V = 4\int_0^{\frac{\pi}{2}}\int_0^2\int_0^{r^2} dz\, r\, dr\, d\theta$$



Triple Integrals in Spherical Coordinates

Chapter 4 Section 5

4.5 Triple Integral in Spherical Coordinates



ρ

is the distance from P to the origin

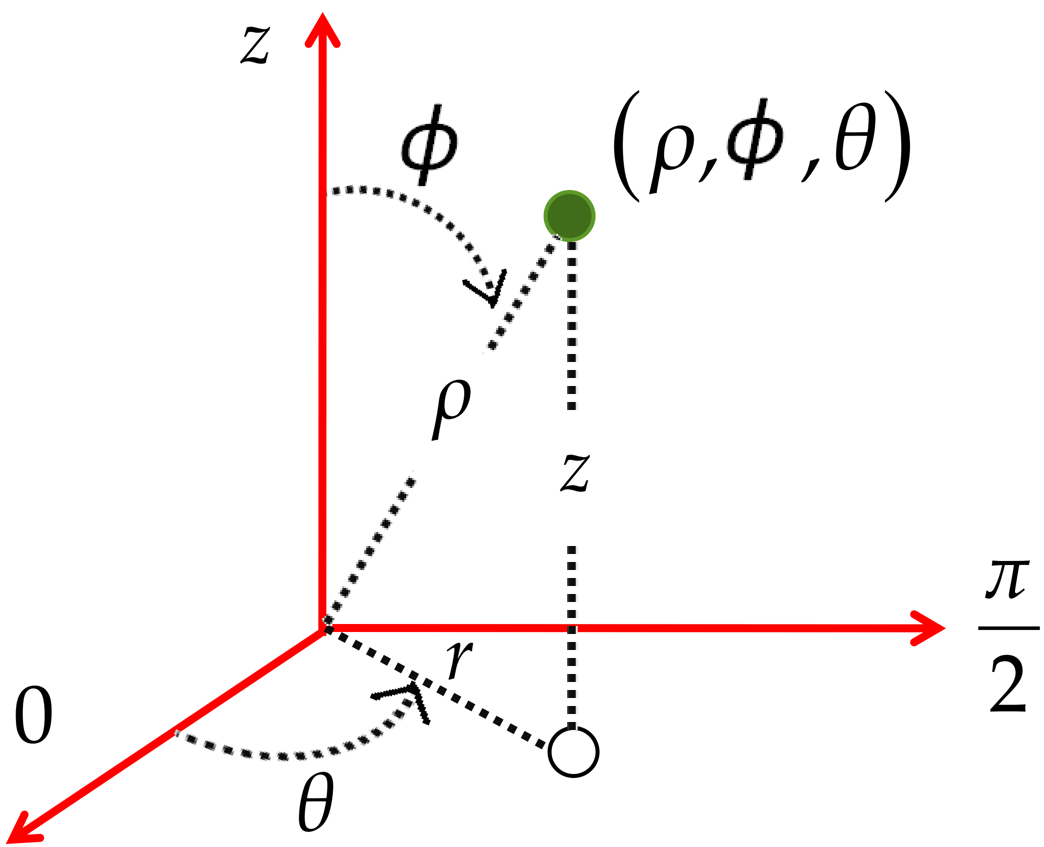
θ

is the angle from cylindrical coordinates

ϕ

is the angle OP makes with the positive z-axis

4.5 Triple Integral in Spherical Coordinates



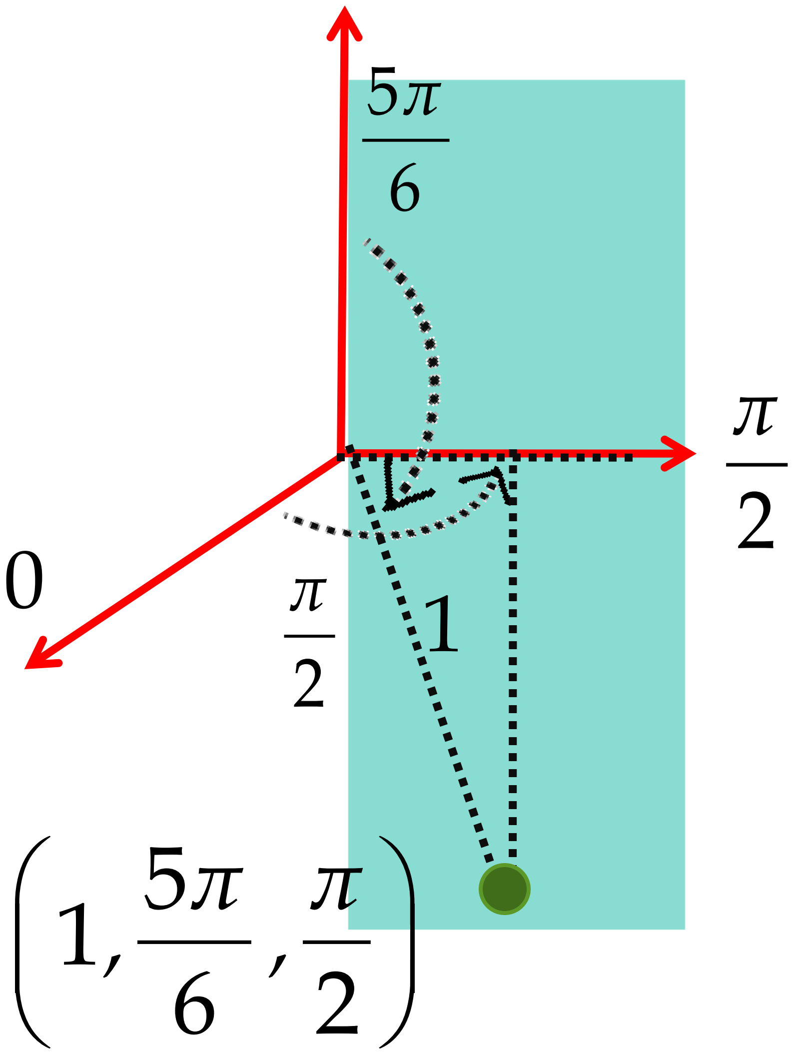
$$\rho=\sqrt{r^2+z^2}$$
$$=\sqrt{x^2+y^2+z^2}$$

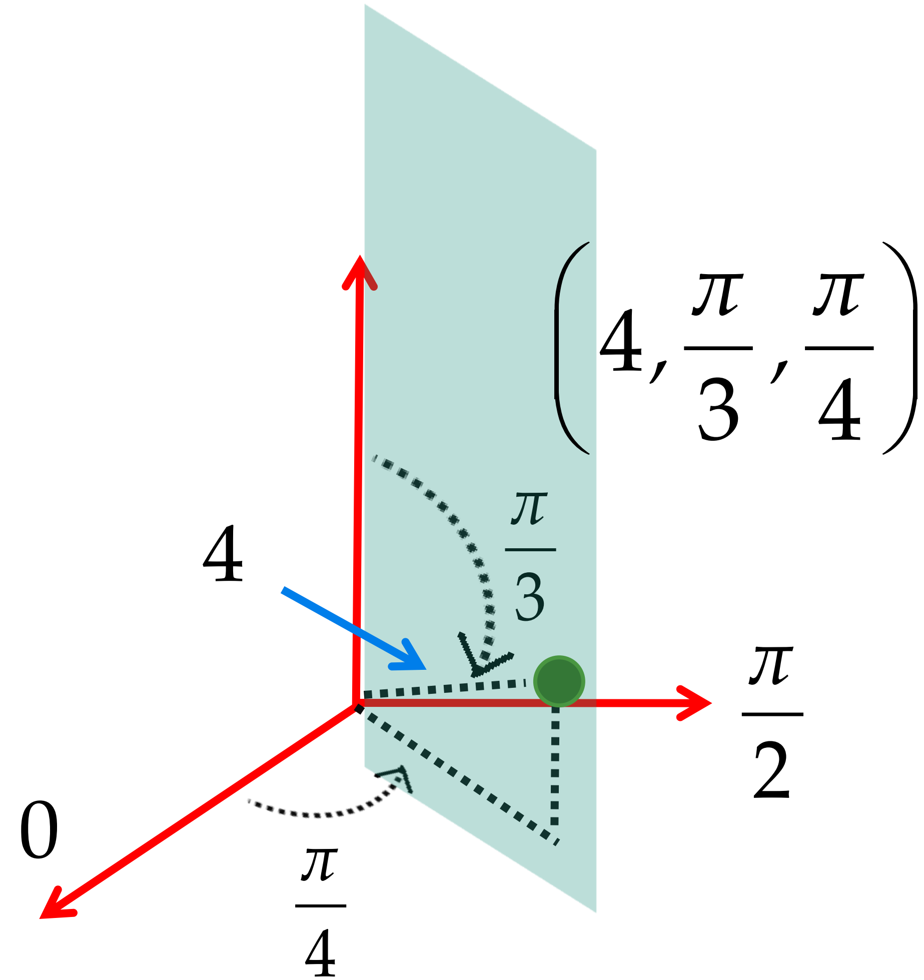
$$r=\rho\sin\phi$$
$$z=\rho\cos\phi$$

$$x=\rho\sin\phi\cos\theta$$
$$y=\rho\sin\phi\sin\theta$$

Exercise

Locating a point (ρ, ϕ, θ)





Exercise

Find a spherical coordinate equation for the given Cartesian coordinate equation

a. $z=\sqrt{x^2+y^2}$

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi}$$

$$\rho\cos\phi=\rho\sin\phi$$

$$\cos\phi=\sin\phi$$

$$\phi=\frac{\pi}{4}$$

$$x=\rho\sin\phi\cos\theta$$

$$y=\rho\sin\phi\sin\theta$$

$$z=\rho\cos\phi$$

Exercise

Find a spherical coordinate equation for the given Cartesian coordinate equation

b. $x^2+y^2+(z-1)^2=1 \Rightarrow r^2+(\rho\cos\phi-1)^2=1$

$$(\rho\sin\phi)^2+\left(\rho^2\cos^2\phi-2\rho\cos\phi+1\right)=1$$

$$\rho^2\sin^2\phi+\left(\rho^2\cos^2\phi-2\rho\cos\phi+1\right)=1$$

$$\rho^2-2\rho\cos\phi=0$$

$$\rho(\rho-2\cos\phi)=0$$

$$\rho=2\cos\phi$$

$$x=\rho\sin\phi\cos\theta$$

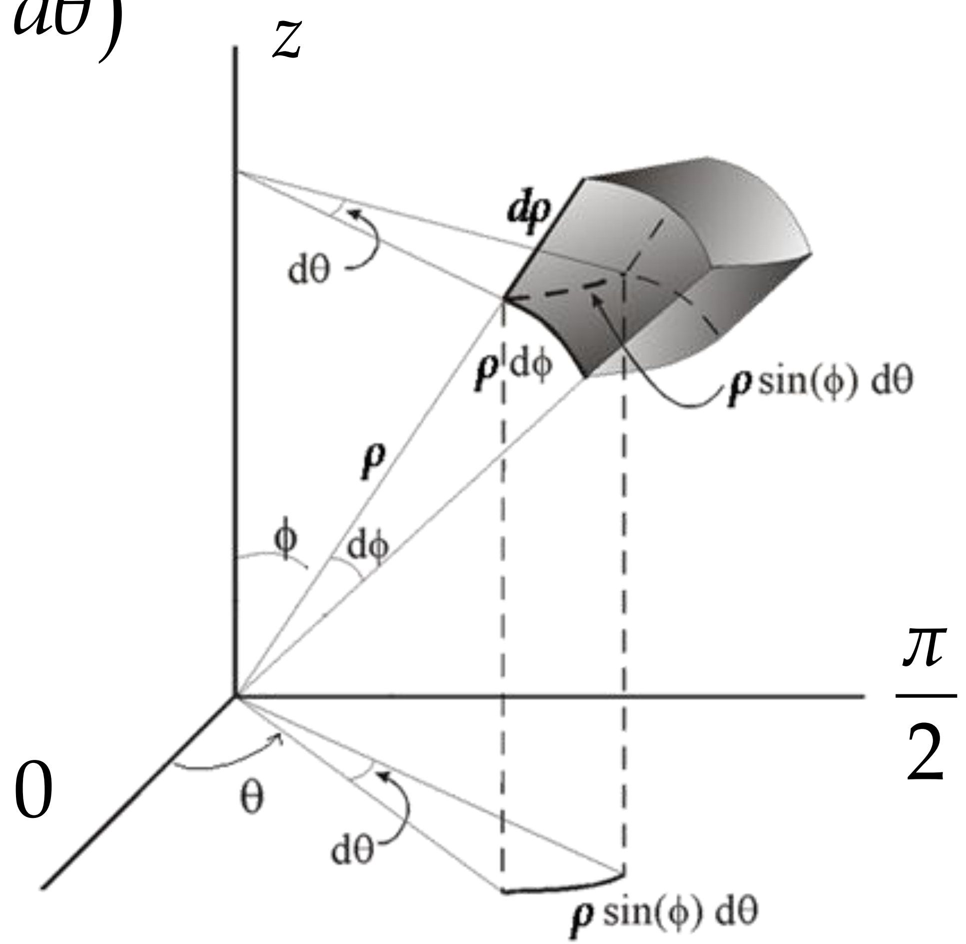
$$y=\rho\sin\phi\sin\theta$$

$$z=\rho\cos\phi$$

Volume in Spherical Coordinates

$$dV = d\rho\,(\rho\, d\phi)(\rho \sin\phi\, d\theta)$$

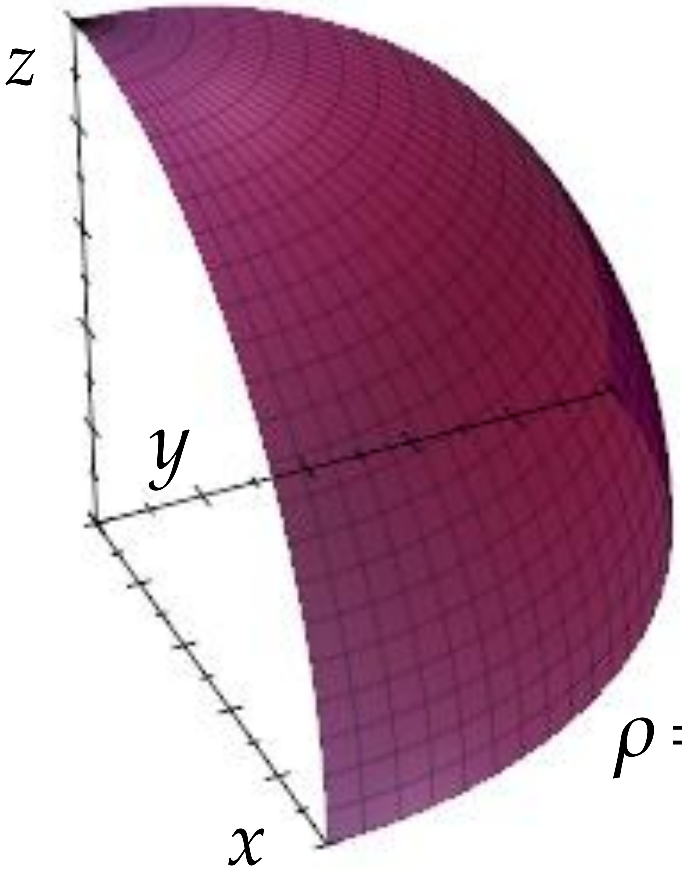
$$= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$



Exercise .

SET-UP and **EVALUATE** a triple integral in spherical coordinates that gives the volume of the solid.

a. Sphere of radius r



$$V = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{r} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{4\pi}{3} r^3$$

Exercise.

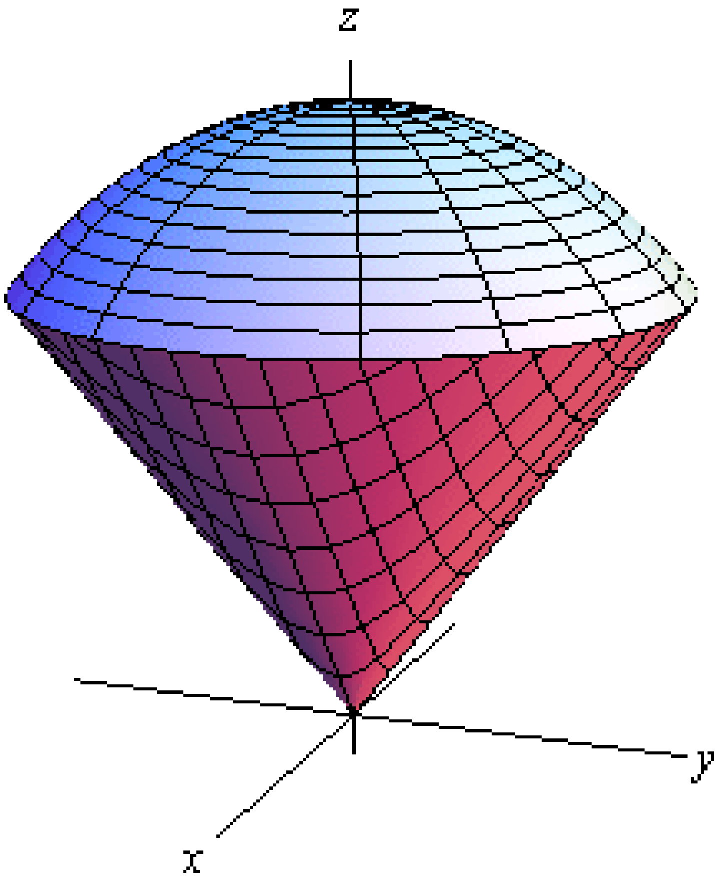
SET-UP and **EVALUATE** a triple integral in spherical coordinates that gives the volume of the solid.

b. Bounded above by

$$f(x,y)=\sqrt{2-x^2-y^2}$$

but bounded below by

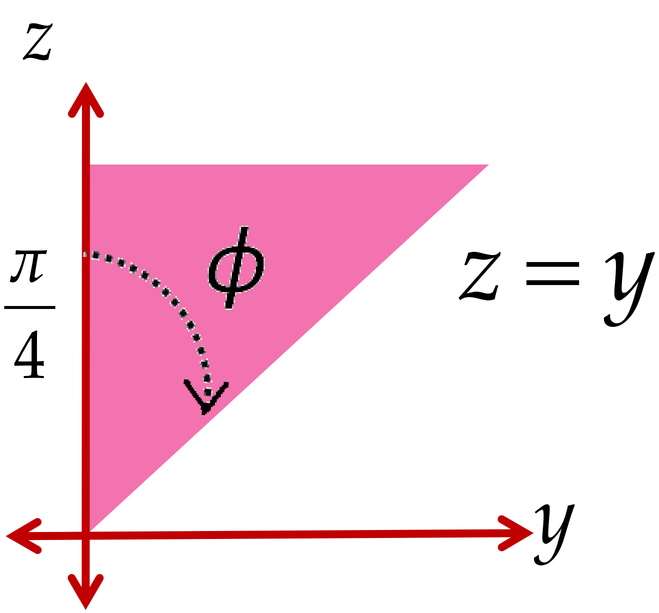
$$g(x,y)=\sqrt{x^2+y^2}$$

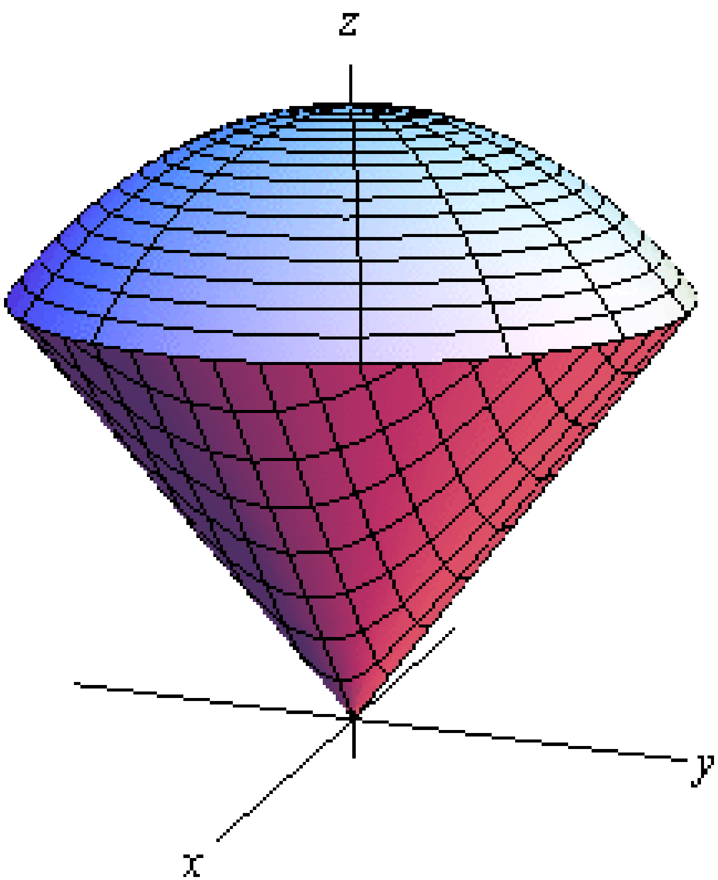


Exercise

SET-UP and **EVALUATE** a triple integral in spherical coordinates that gives the volume of the solid.

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$





Exercise.

SET-UP and **EVALUATE** a triple integral in spherical coordinates that gives the volume of the solid.

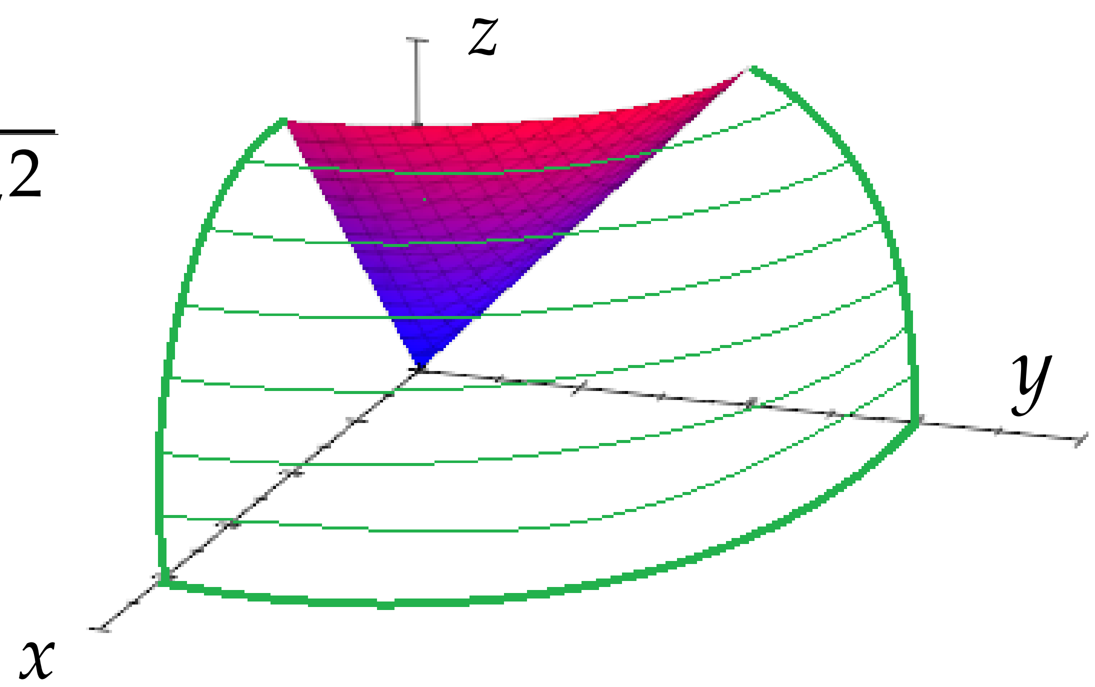
c. Bounded by the graph of

$$f(x,y)=\sqrt{2-x^2-y^2}$$

and

$$g(x,y)=\sqrt{x^2+y^2}$$

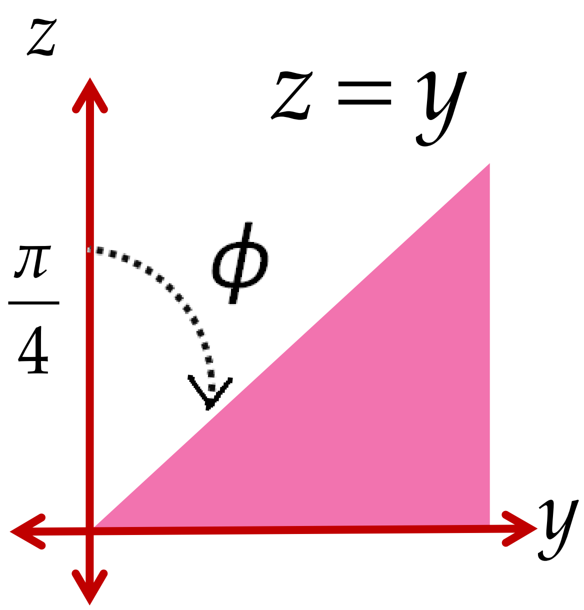
and the *xy*-plane

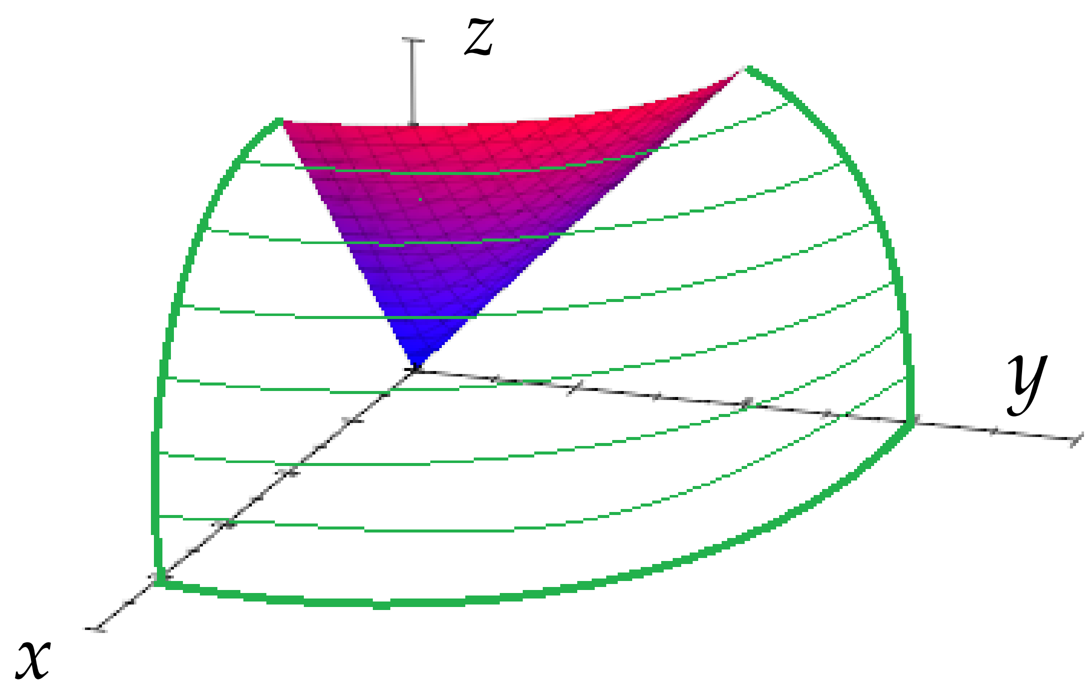


Exercise

SET-UP and **EVALUATE** a triple integral in spherical coordinates that gives the volume of the solid.

$$V = 4\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$





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