Primitives and Geometric Objects

CMSC 161: Interactive Computer Graphics

2nd Semester 2014-2015

Institute of Computer Science

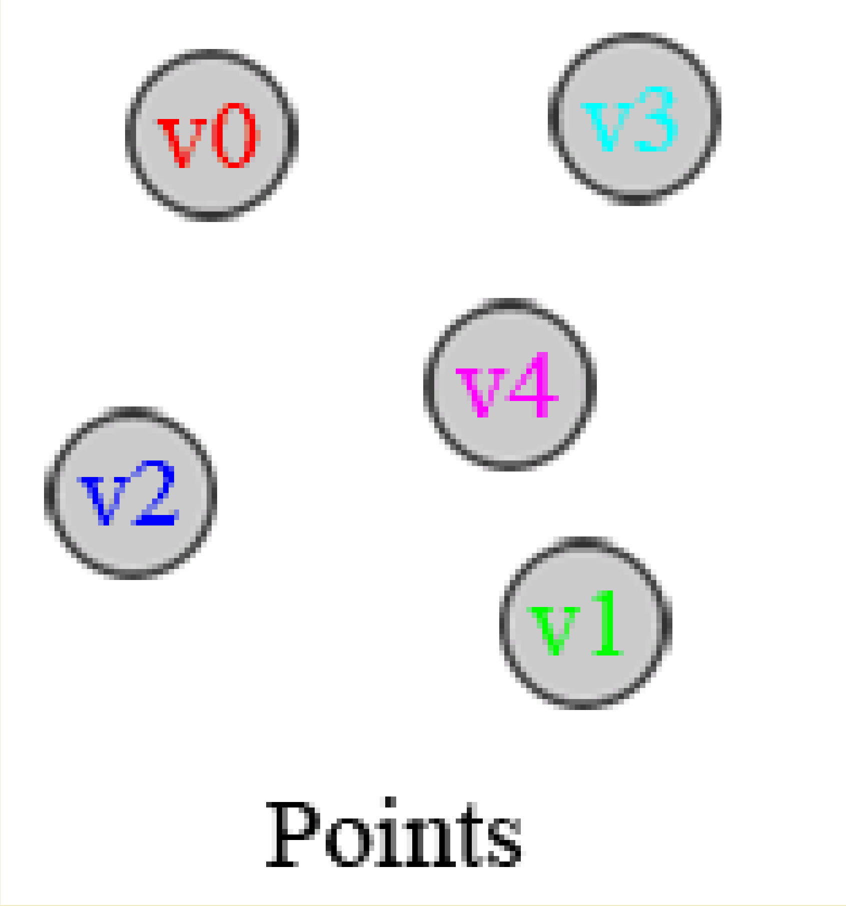
University of the Philippines – Los Baños

Lecture by James Carlo Plaras

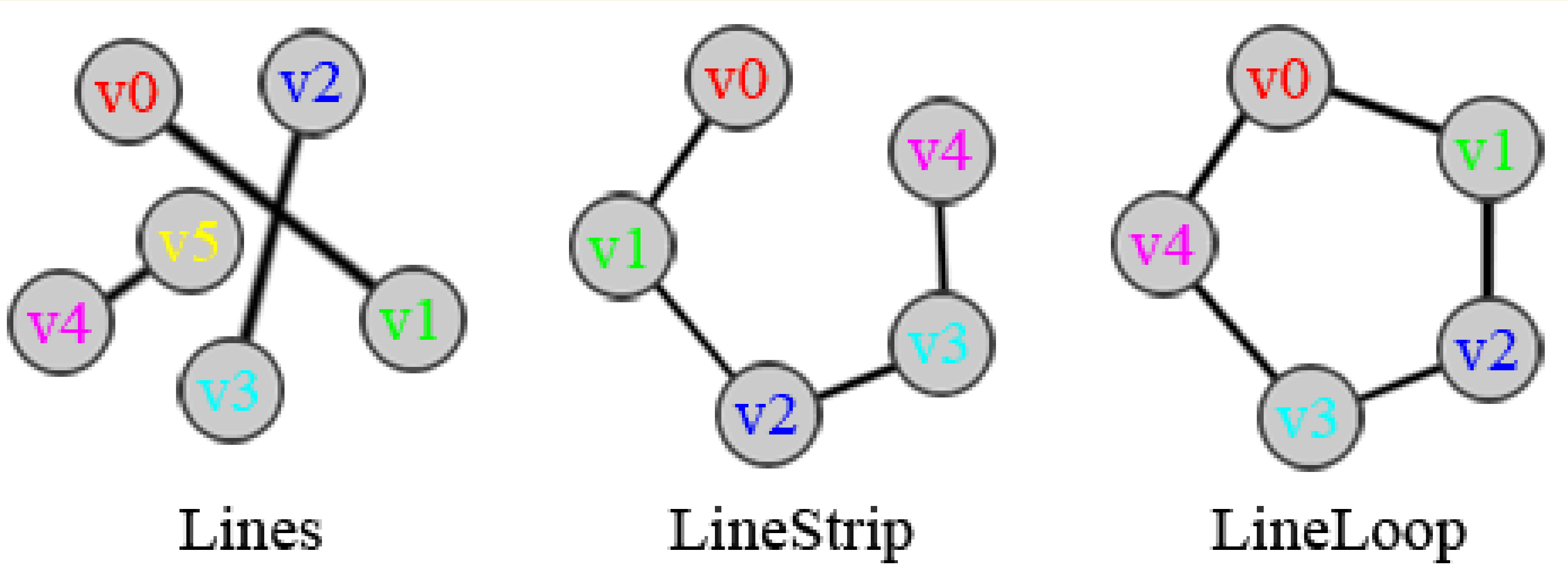
Complex graphics can be represented by using only primitives

PRIMITIVES

Points

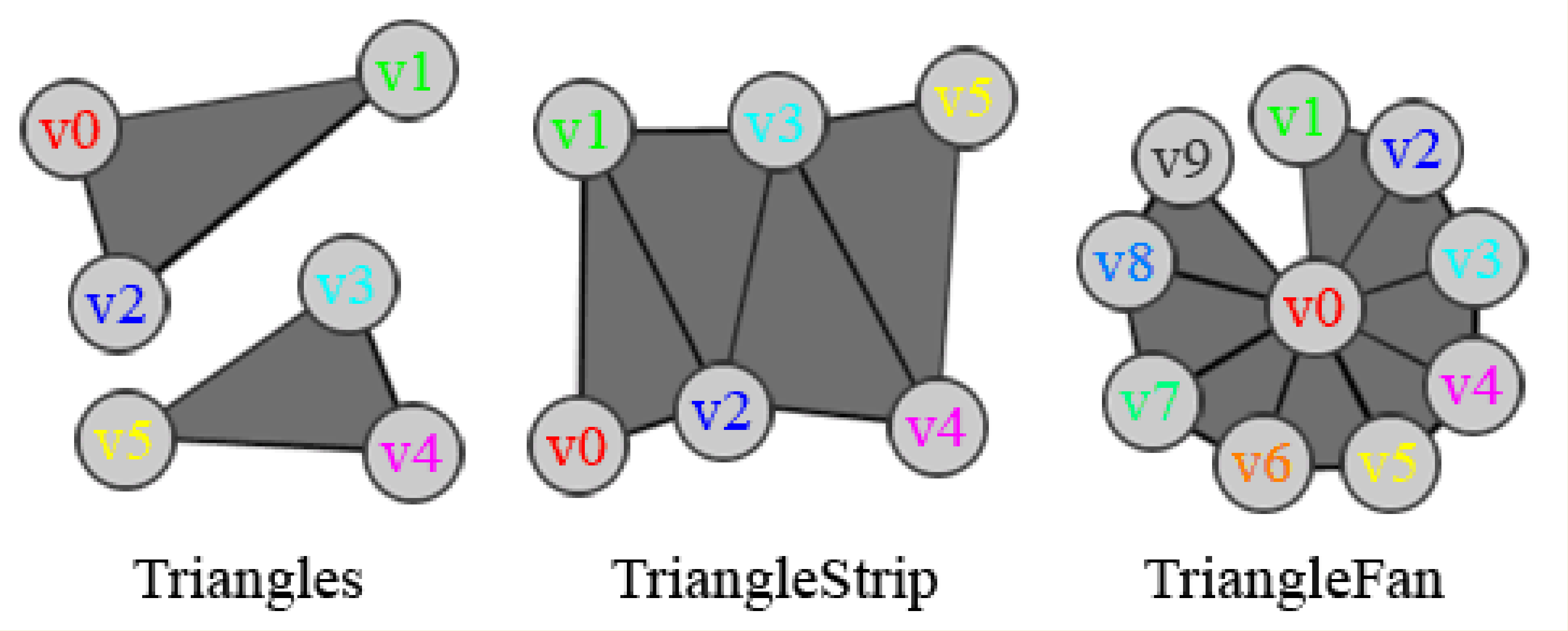


Lines



v0
v2
v5
v4
v3
v1
Lines
v0
v4
v1
v3
v2
LineStrip
v0
v1
v4
v2
v3
LineLoop

Triangle



v0
v1
v2
v3
v4
v5
v1
v3
v5
v0
v2
v4
v9
v1
v2
v8
v3
v0
v7
v4
v6
v5
Triangles
TriangleStrip
TriangleFan

Polygons?

Polygon Triangulation

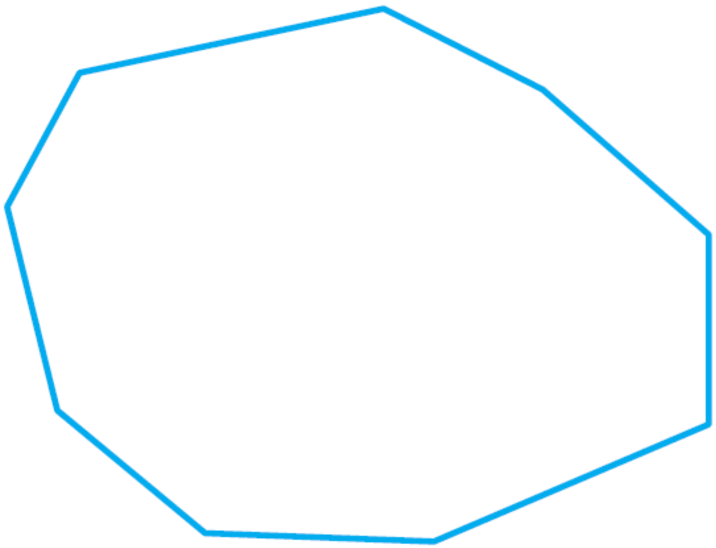
Tessellation using triangles

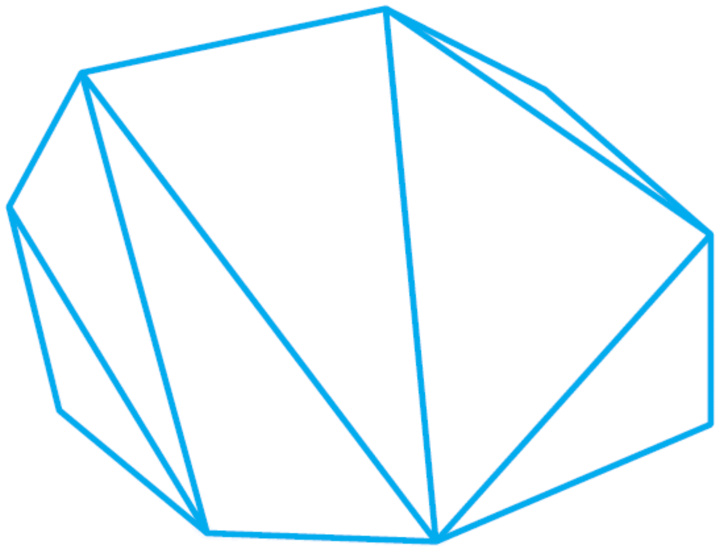
breaking down of a polygon to geometric shapes

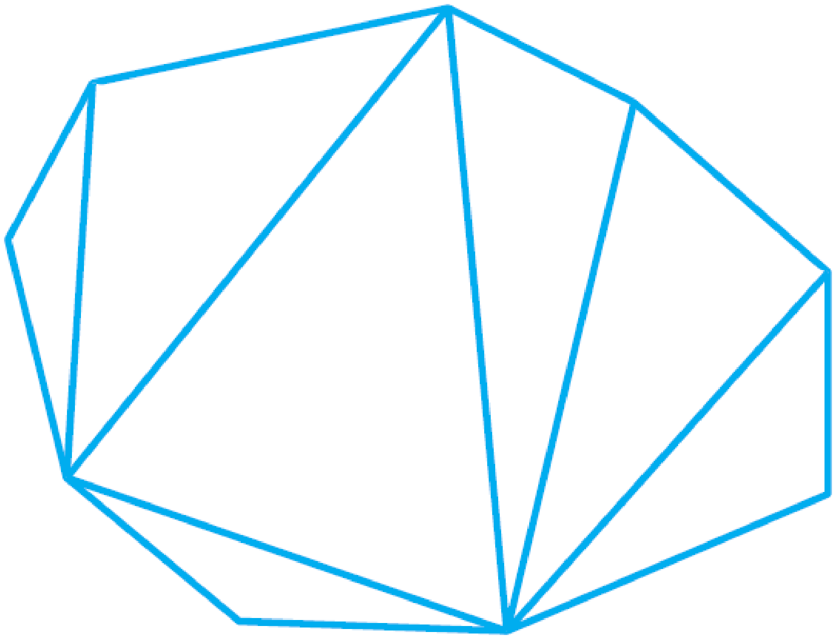
Polygon Triangulation

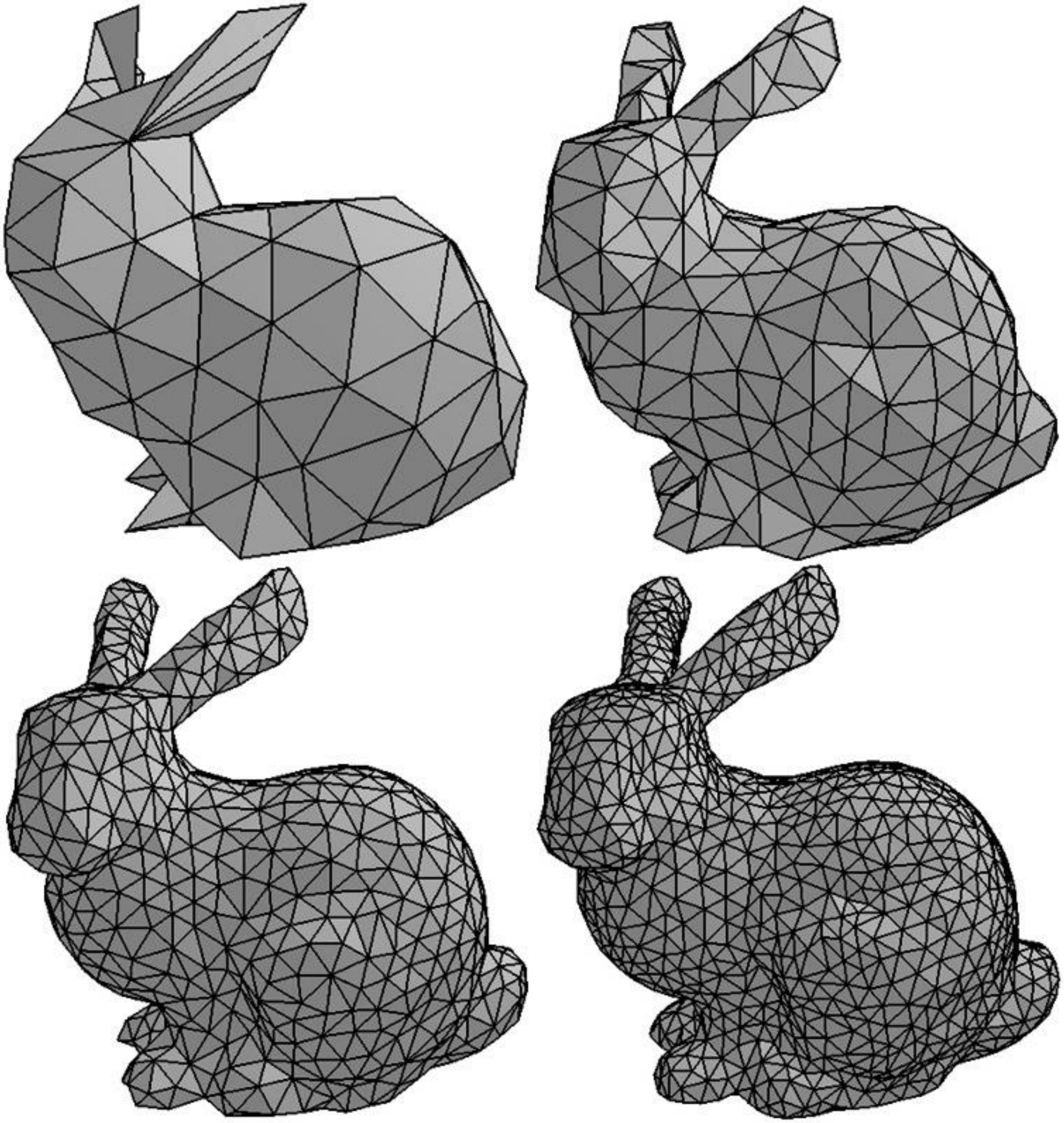
Minimum triangulation

Convex polygon with n edges has minimum of n-2 triangle tessellation









GEOMETRIC OBJECTS IN COMPUTER GRAPHICS

Point

Location in space

Point

In Euclidean geometry

Known as Cartesian coordinates

(x, y) is a point in 2D space

(x, y, z) is a point in 3D space

Point

In Projective Geometry

known as Homogenous Coordinates

$$(x, y, z, w) = \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$$

Point Convention

$$P = (3,5)$$

$$Q = (-2,1)$$

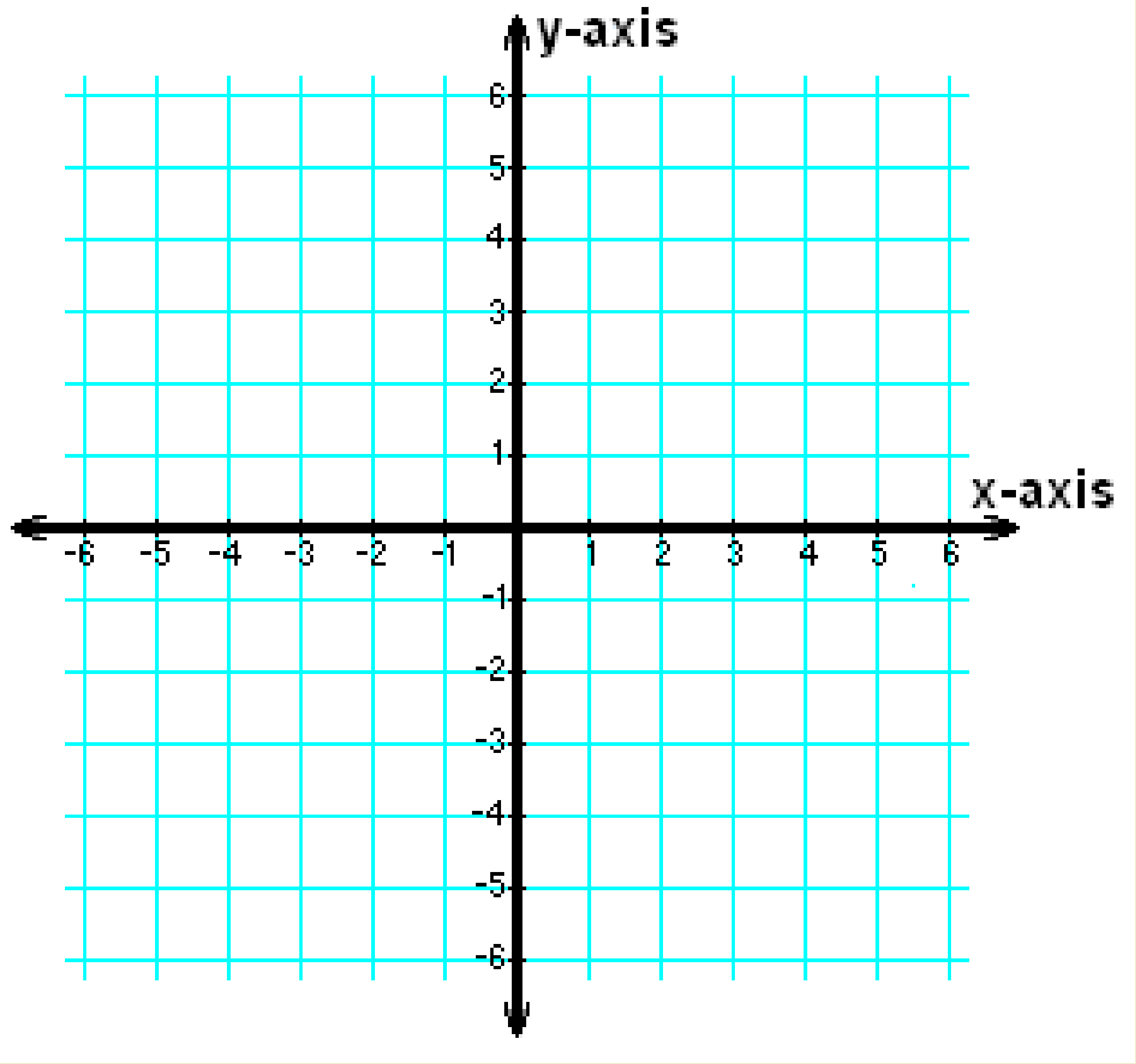
$$S = (3,2,7)$$

$$T = (9.3,0.0,-1.5)$$

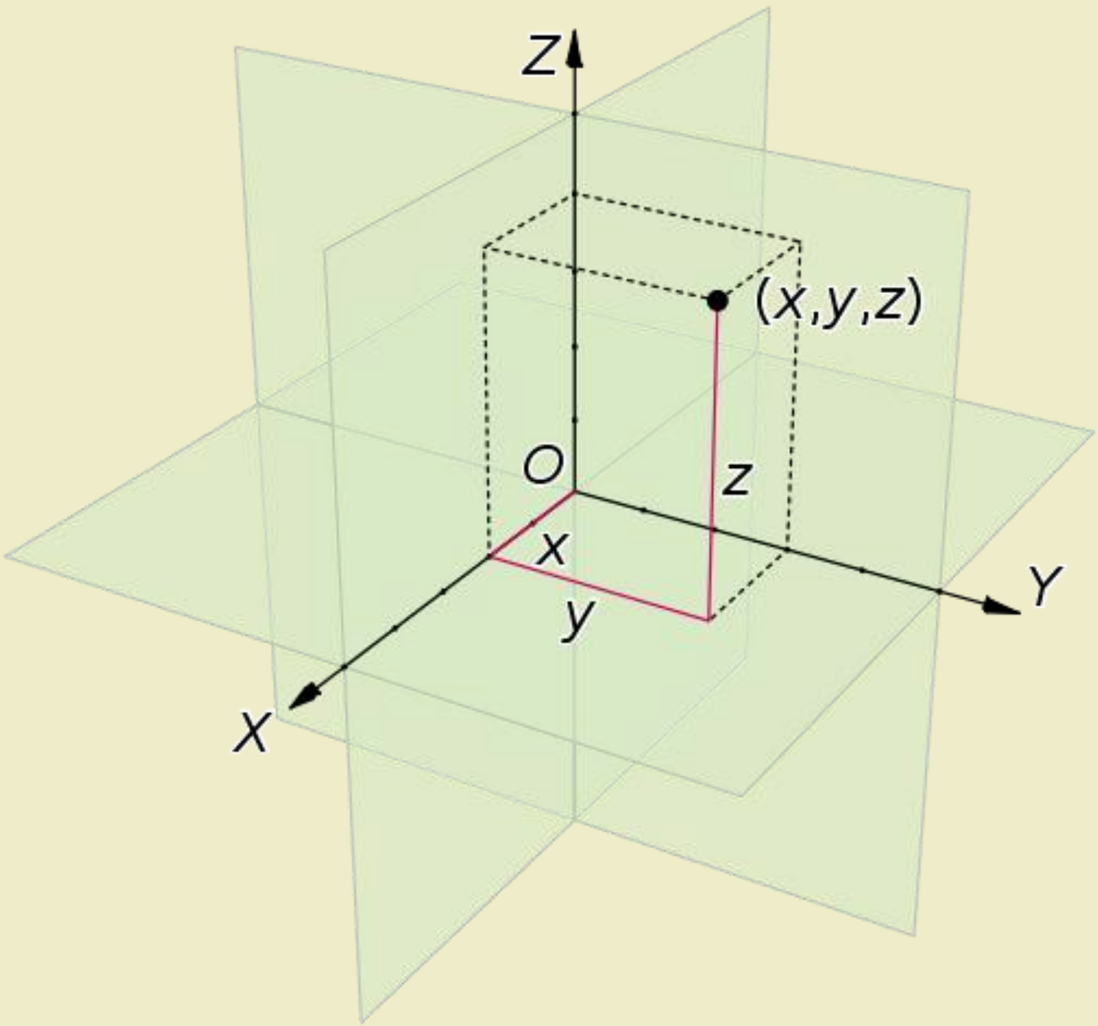
$$A = (1,3,5,1) \rightarrow (1,3,5)$$

$$B = (2,6,4,2) \rightarrow (1,3,2)$$

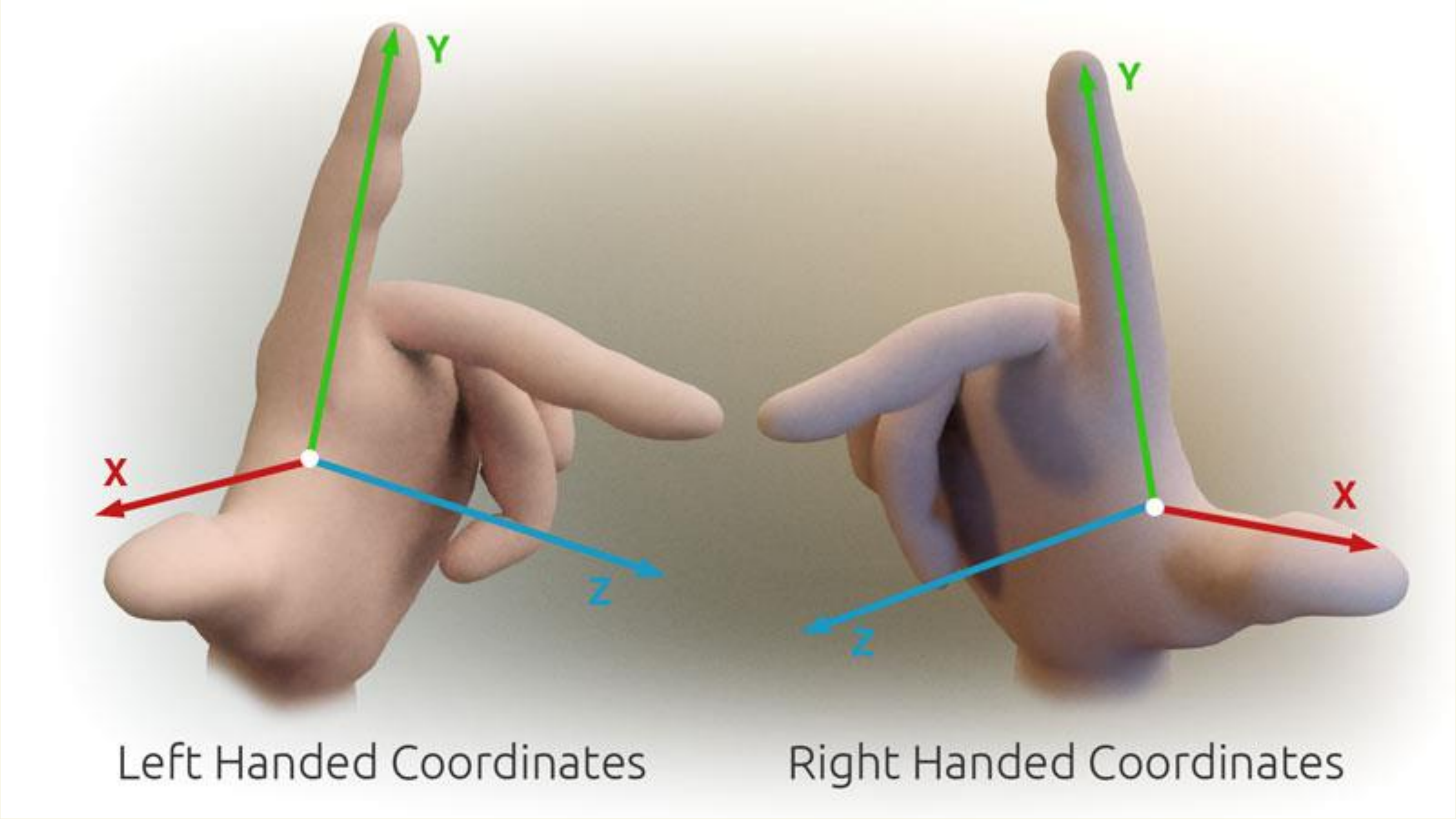
Cartesian Coordinate System

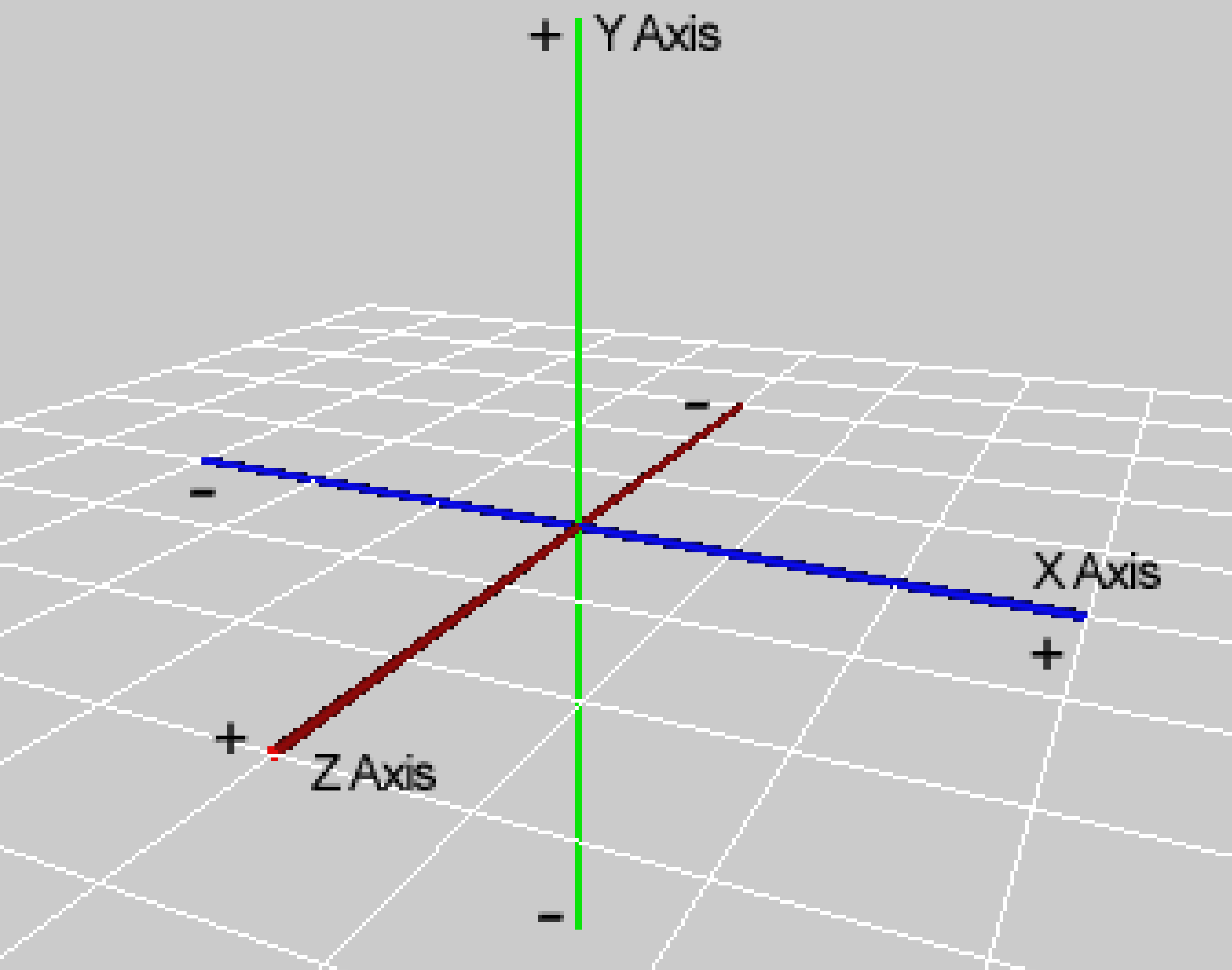


Cartesian Coordinate System



Right-hand or Left-hand





+
Y Axis
-
-
X Axis
+
+
Z Axis
-

SCALARS AND VECTORS

Scalar

Real numbers to specify quantities

Distance between two points

Scalar Convention

$$\alpha = 10$$

$$\beta = 3.1416$$

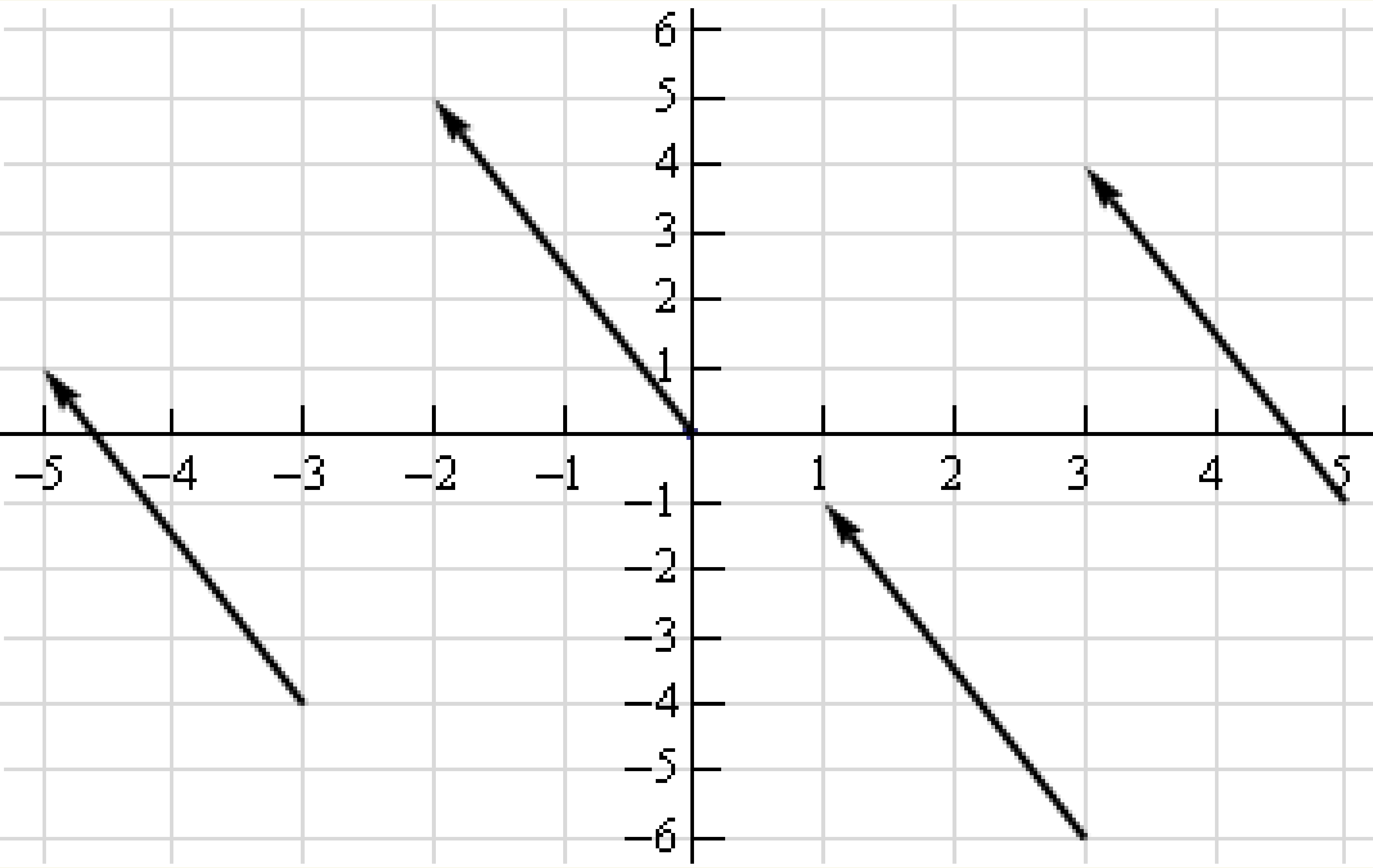
$$\varepsilon = 0.000010$$

Vector

Direction + Magnitude

Direction: orientation in space

Magnitude: length



6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−5
−4
−3
−2
−1
1
2
3
4
5

Vector Conventions

$$u = < -2, 5 >$$
$$v = < 3, -1, 4 >$$
$$u = [\,-2 \quad 5\,]^T$$
$$v = [3 \quad -1 \quad 4\,]^T$$
$$u = -2i + 5j$$
$$v = 3i - j + 4k$$

Angles

Figure formed by two vectors sharing a common endpoint

Amount of turn from one vector to another vector

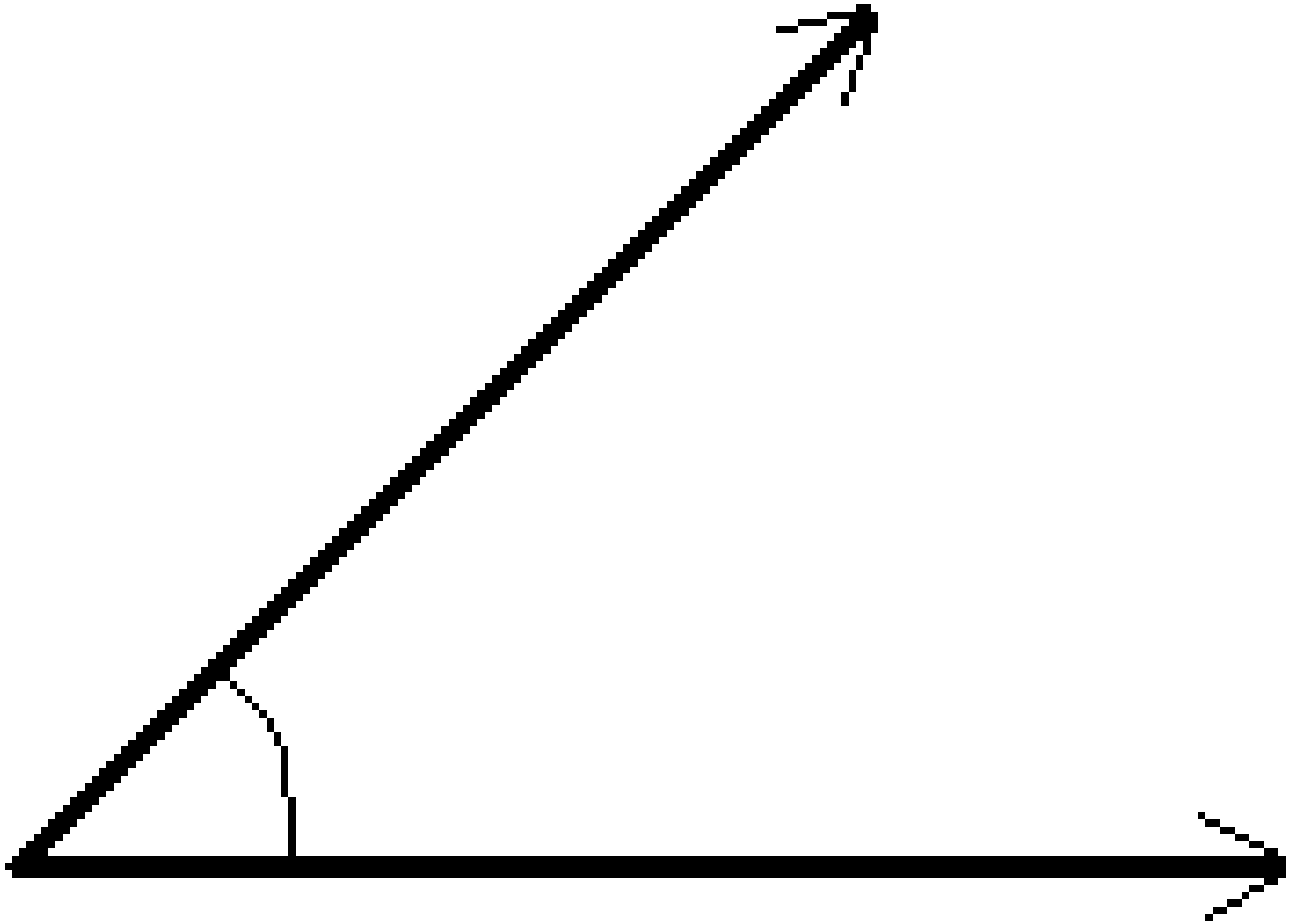
Angles

May be in degrees or in radians

Common trigonometric functions

$$\sin\theta$$

$$\cos\theta$$



Homogeneous Point

$$H = (1,1,1,1)$$

$$H = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Homogenous Vector

$$z = < 1, 1, 1, 0 >$$

$$z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Matrices

$$\begin{bmatrix} 1 & \cdots & n \\ \vdots & \ddots & \vdots \\ m & \cdots & mn \end{bmatrix}$$

Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 1 \end{bmatrix}$$

MATHEMATICAL OPERATIONS IN COMPUTER GRAPHICS

Common Operations in CG

Scalar-vector multiplication

Vector-vector addition

Inverse vectors

Zero vector

Vector Magnitude

Unit Vector

Point-vector addition

Point-point subtraction

Dot Product

Determinant of a Matrix

Cross Product

Matrix Transposition

Matrix Multiplication

Scalar-vector multiplication

$$\alpha(u)$$

$$3\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

Vector-vector addition

$$u + v$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Inverse vectors and Zero Vector

$$-u$$

$$-\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

Inverse vectors and Zero Vector

$$u + -u = 0$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Magnitude of a vector

$$|u|$$

$$|u| = \sqrt{x^2 + y^2 + z^2}$$

$$\left|\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \approx 3.74$$

Unit vector

$$\hat{u} = \frac{1}{|u|} u$$

$$\widehat{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \frac{1}{3.74} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \approx 0.27 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 0.27 \\ 0.54 \\ 0.81 \end{bmatrix}$$

Unit Vector

$$|\hat{u}| = 1$$

Point-vector addition

$$Q = P + u$$

$$(3, 4, 5) + < 2, 2, 2 > = (5,6,7)$$

$$\begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 1 \end{bmatrix}$$

Point-point subtraction

$$u = P - Q$$

$$(3, 3, 3) - (1, 1, 1) = < 2, 2, 2 >$$

$$\begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Dot product of two vectors

$$u \cdot v = u_x v_x + u_y v_y + u_z v_z$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1(4) + 2(5) + 3(6)$$

$$= 4 + 10 + 18 = 32$$

Determinant of a matrix (2 x 2)

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\right) = 3(2) - 4(1) = 6 - 4 = 2$$

Determinant of a matrix (3 x 3)

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)$$

$$= a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Determinant of a matrix (3 x 3)

$$= a\,\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\,\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\,\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

Cross product of two vectors

$$u \times v = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

Cross product of two vectors

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}2(6)-3(5)\\3(4)-1(6)\\1(5)-2(4)\end{bmatrix}$$

$$= \begin{bmatrix}12-15\\12-6\\5-8\end{bmatrix} = \begin{bmatrix}-3\\6\\-3\end{bmatrix}$$

Matrix Transposition

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

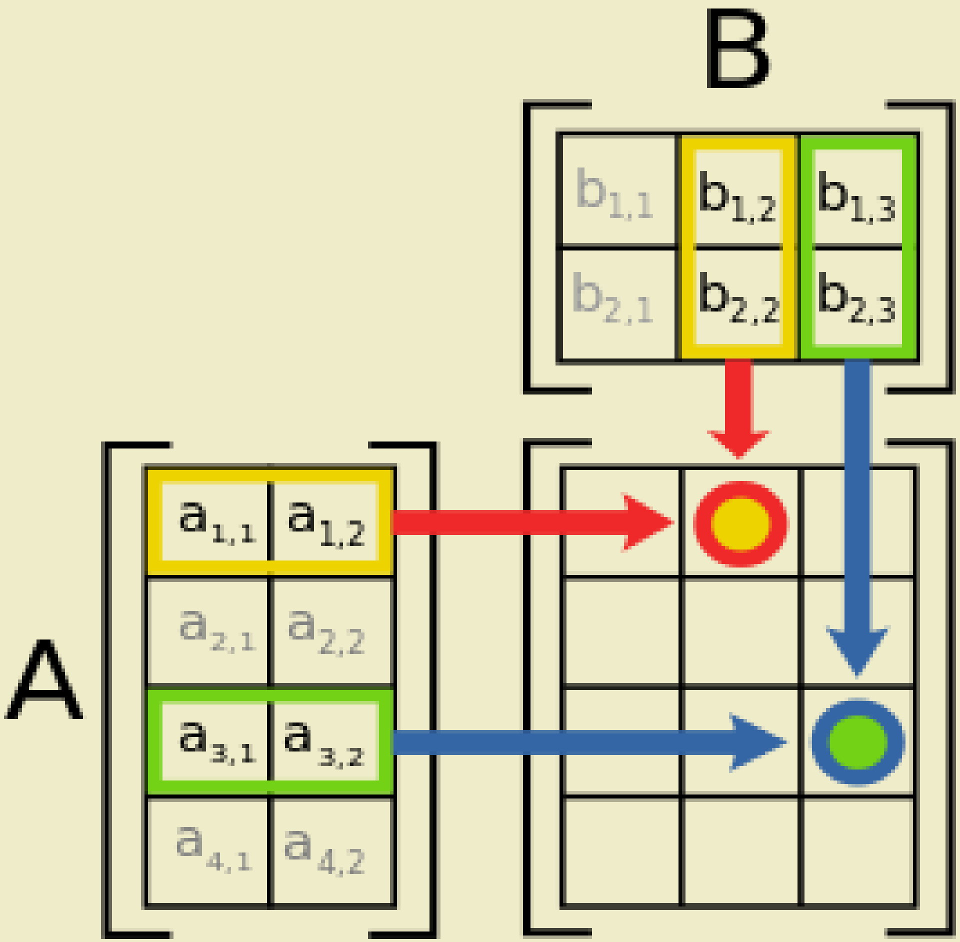
$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Matrix Multiplication

Two matrices ($A \ and\ B$)
can only be multiplied ($A \times B$) if

A is a $n \times m$ matrix
B is a $m \times p$ matrix

Matrix Multiplication



Matrix Multiplication

$$\begin{bmatrix} 3 & 1 & 7 \\ -2 & 1 & 7 \\ 4 & 5 & -9 \end{bmatrix} \times \begin{bmatrix} 1 & 7 & 8 \\ 2 & -8 & 6 \\ 9 & 4 & 7 \end{bmatrix}$$

FRAMES AND COORDINATE SYSTEMS

Vector Representation

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

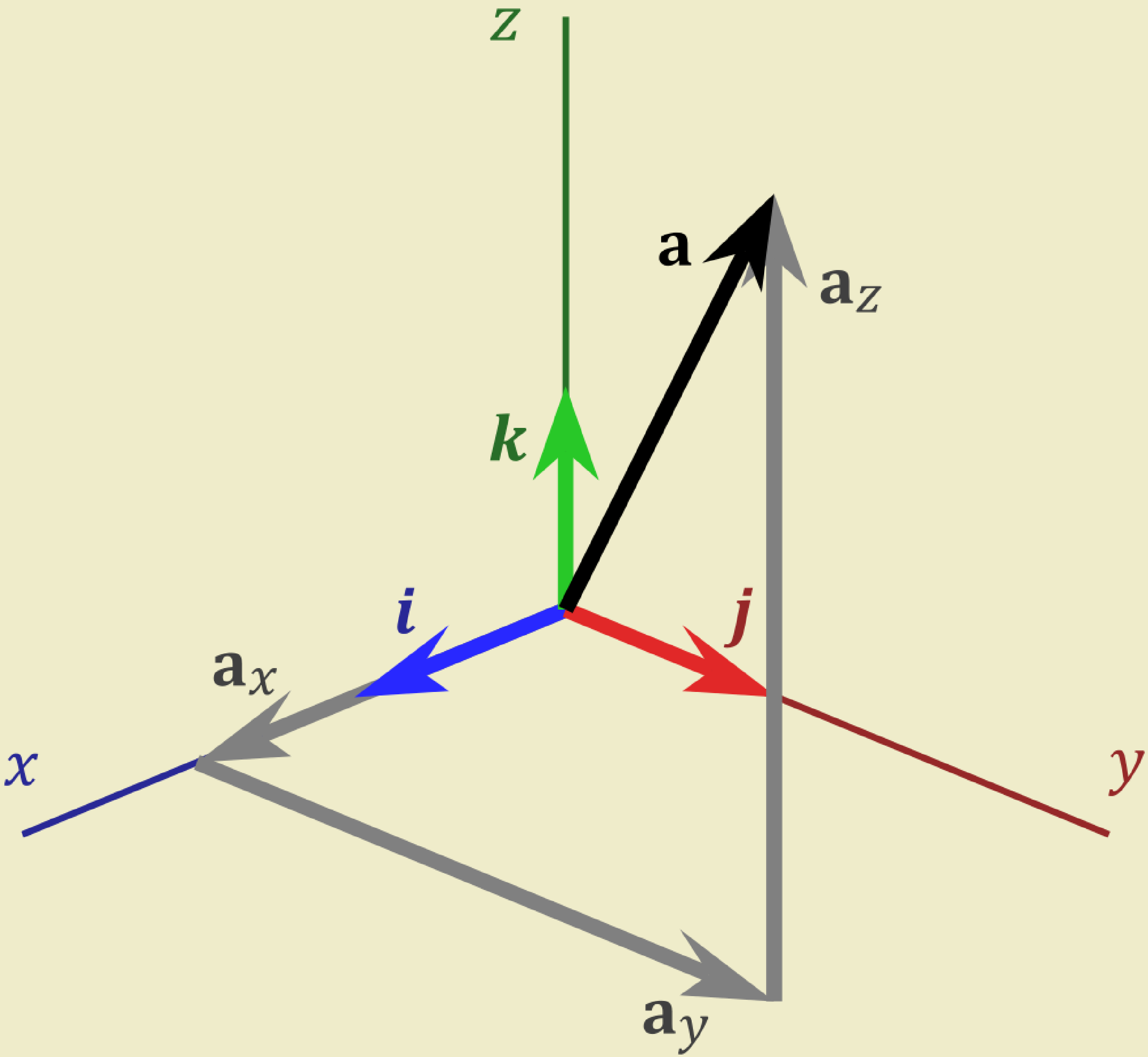
$$a = 1i + 2j + 3k$$

Vector Representation

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the components of the vector a

Vector Representation

$\begin{bmatrix} i & j & k \end{bmatrix}$ is the basis of vector a



z
a
a_z
k
i
j
a_x
x
y
a_y

Point Representation

What if I want to represent a point $\mathrm{Q} = (1,2,3)$ using the basis $[i \quad j \quad k]$?

Point Representation Problem

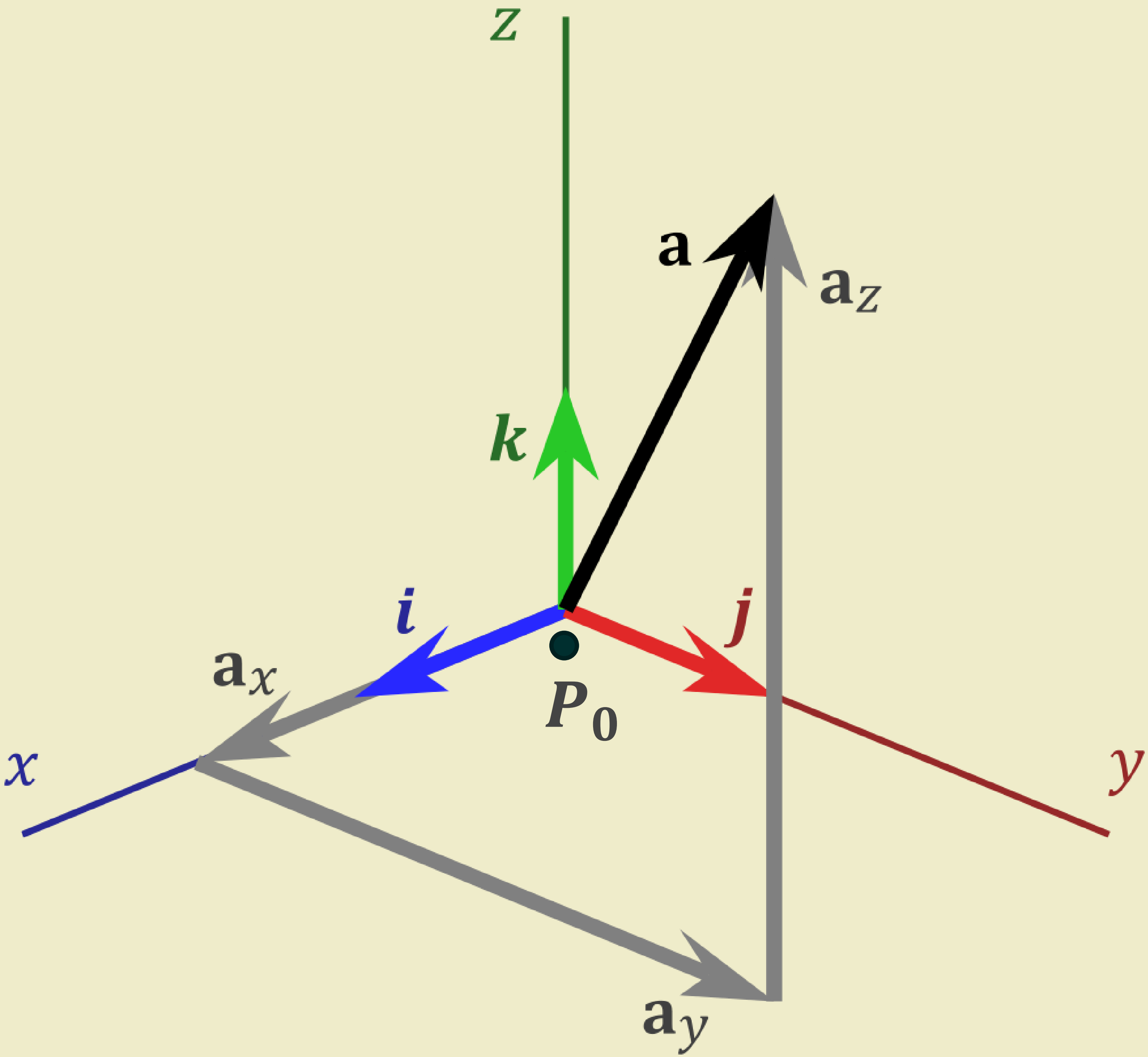
If we represent $Q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then our vector

$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has the same representation as Q

Homogeneous Representation

Let $[i \quad j \quad k]$ be the basis vectors

P_0 be the common starting point of the basis vectors (origin point)

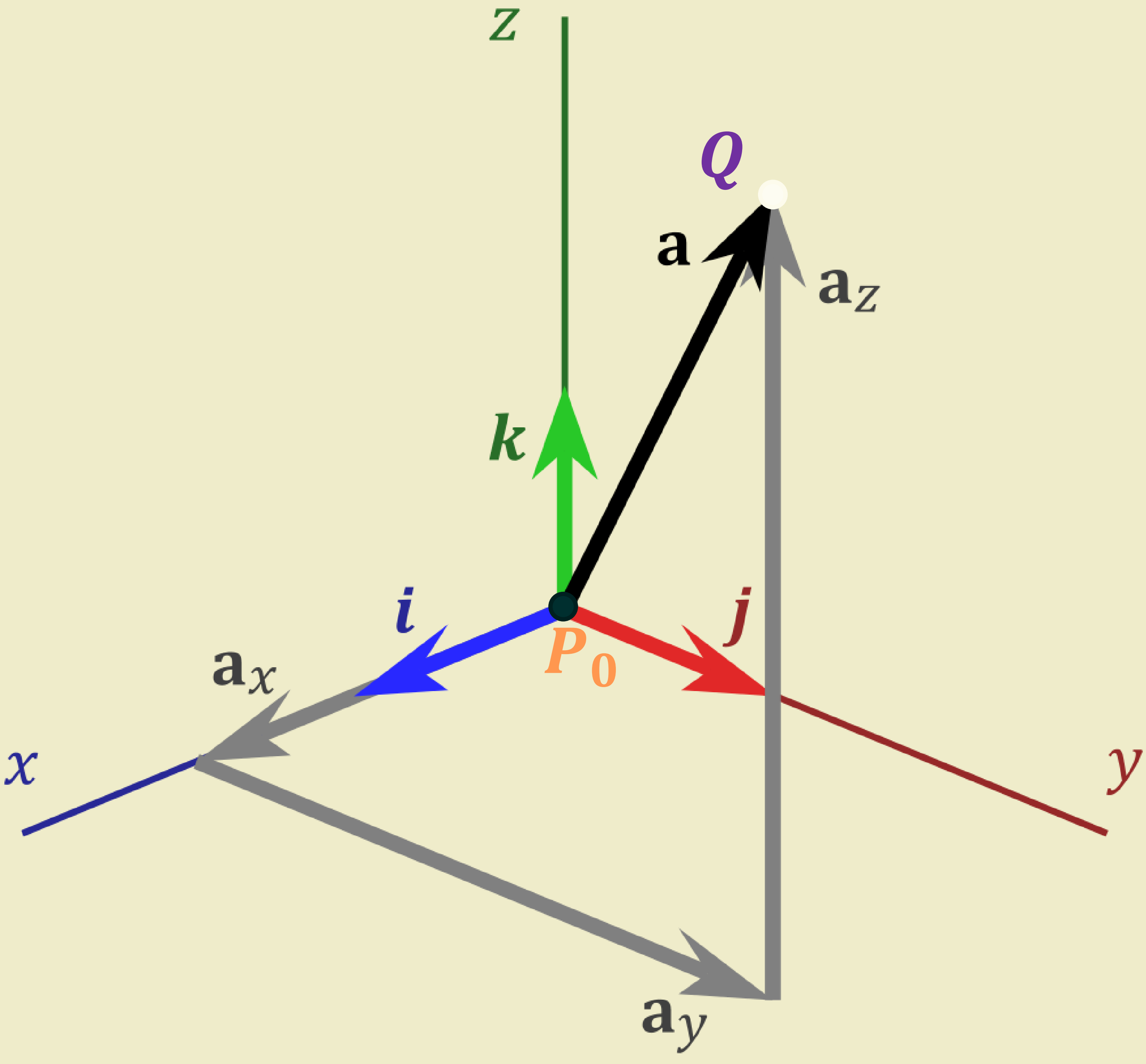


z
a
$\mathbf{a}_z$
k
i
j
$\mathbf{a}_x$
P_0
x
y
$\mathbf{a}_y$

Homogeneous Representation: Point

$$Q = (1i + 2j + 3k) + 1P_0$$

$$Q = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

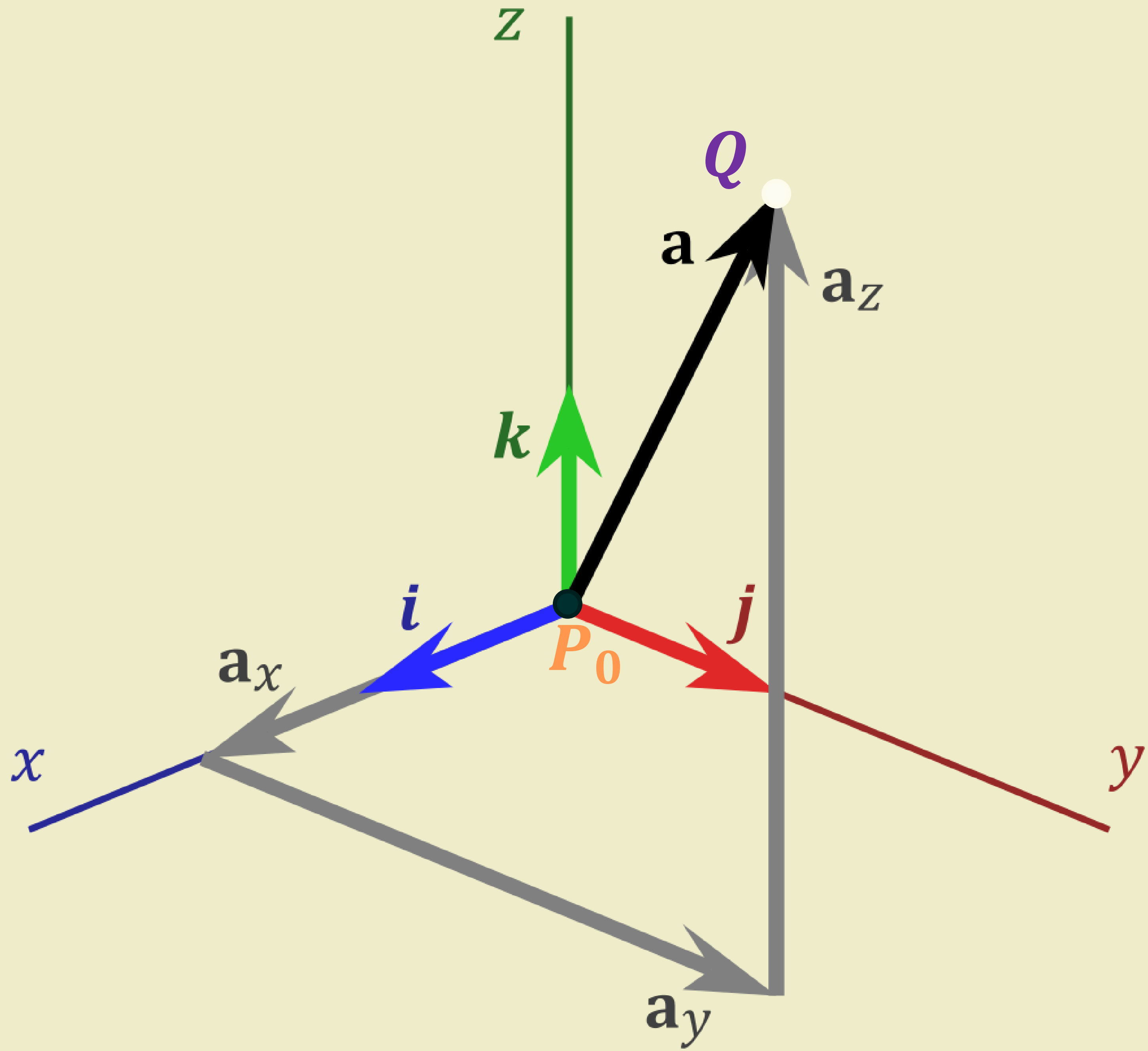


z
Q
a
a_z
k
i
j
P_0
a_x
x
y
a_y

Homogeneous Representation: Vector

$$a = 1i + 2j + 3k + 0P_0$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$



z
Q
a
a_z
k
i
j
P_0
a_x
x
y
a_y

Frame

Basis Vectors + Origin Point

Standard Frame

Let i, j, k be the basis vectors and P_0 be the origin point

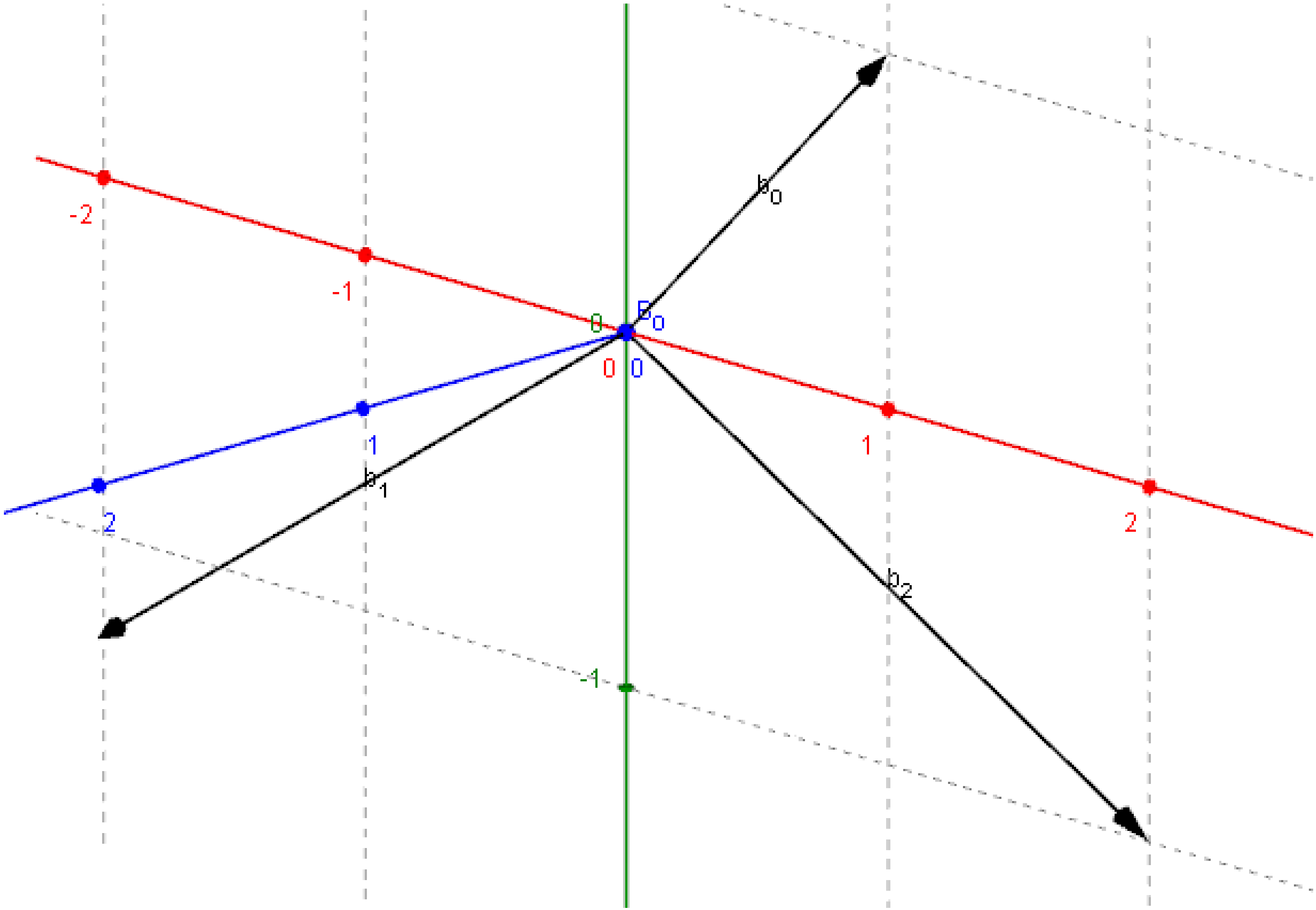
$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathrm{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathrm{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Non-standard Frames

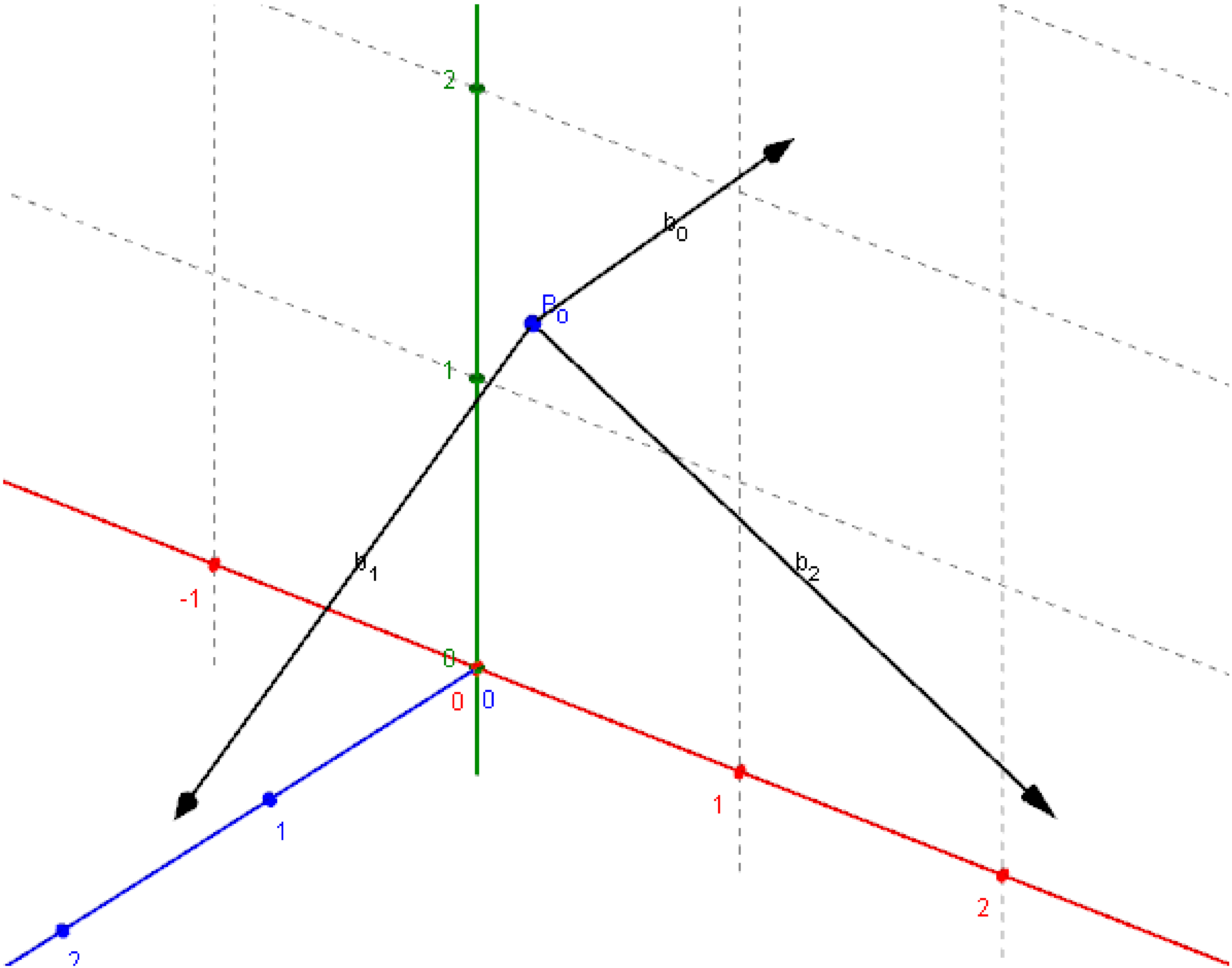
Let b_0, b_1, b_2 be the basis vectors and B_0 be the origin point of our new frame M_1

Non-standard Frames

$$b_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathrm{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \mathrm{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$



b_0
b_1
b_2
P_0
-2
-1
0
1
2
0
1
2
0
-1



2
1
0
b_0
P_0
b_1
b_2
-1
0
0
1
1
2

Non-Standard Frames

$$M_1 = [b_0 \quad b_1 \quad b_2 \quad B_0]$$

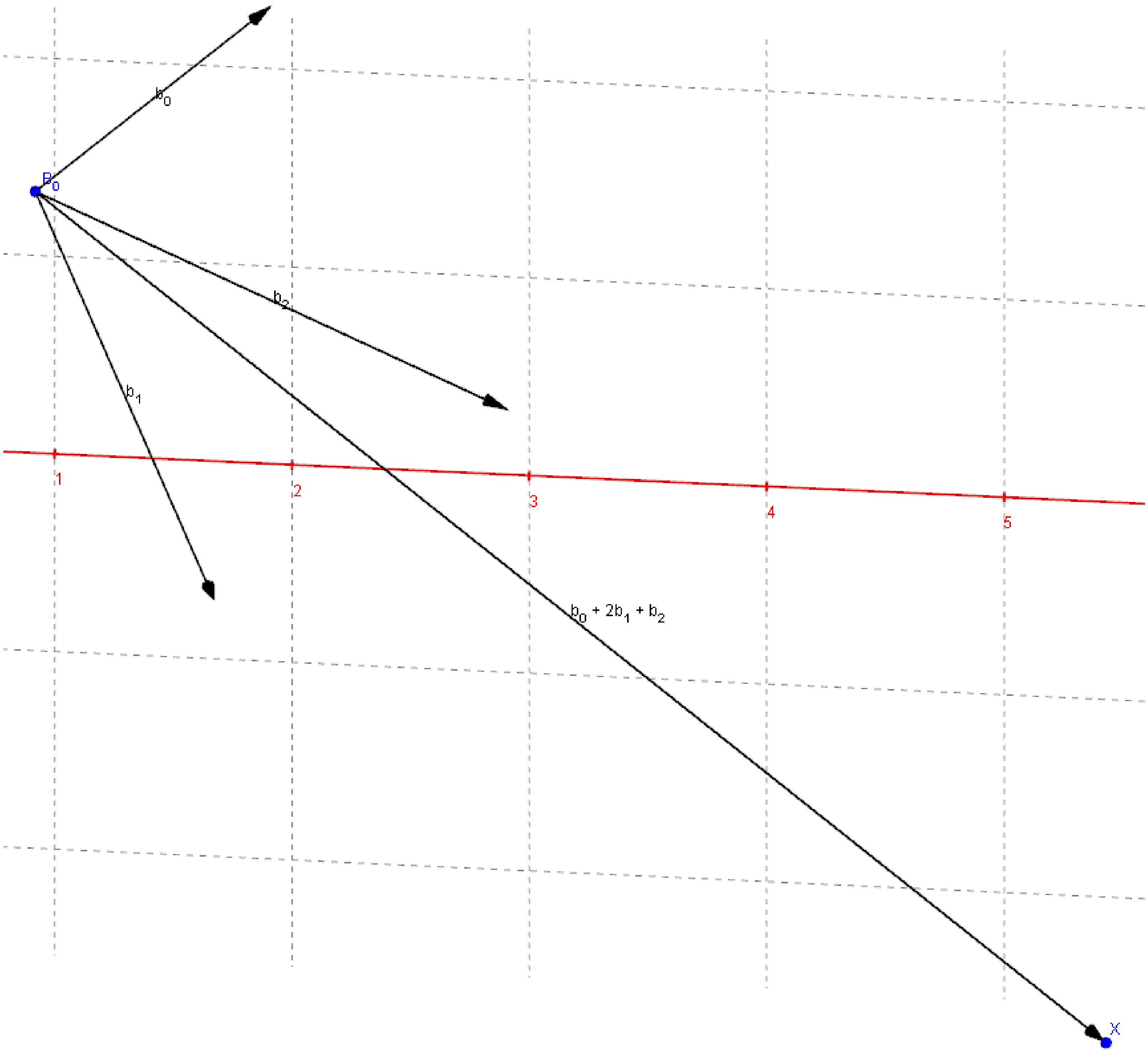
Non-standard Frames

$$M_1 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representations in Non-standard Frames

Let $X_{M_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ be a point represented in M_1

$$X_{M_1} = 1b_0 + 2b_1 + 1b_2 + 1B_0$$



b_0
B_0
b_2
b_1
1
2
3
4
5
$b_0 + 2b_1 + b_2$
X

Change of Frame: Non-standard to Standard

What if we want X_{M_1} be represented in Standard Frame?

Change of Frame: Non-standard to Standard

$$X_s = M_1 X_{M_1}$$

Change of Frame: Non-standard to Standard

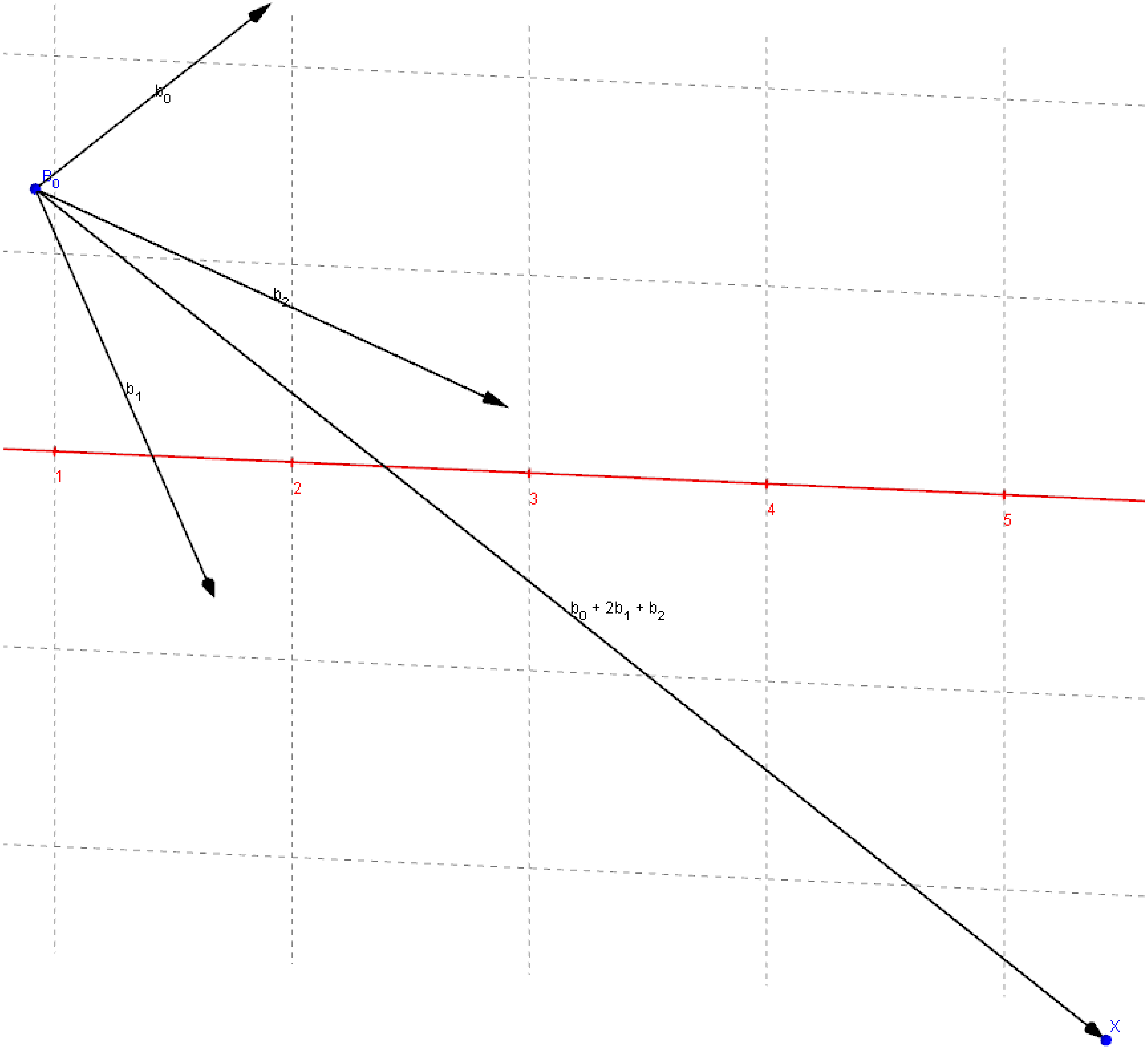
$$X_s = \begin{bmatrix} b_0 & b_1 & b_2 & B_0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Change of Frame: Non-standard to Standard

$$X_s = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

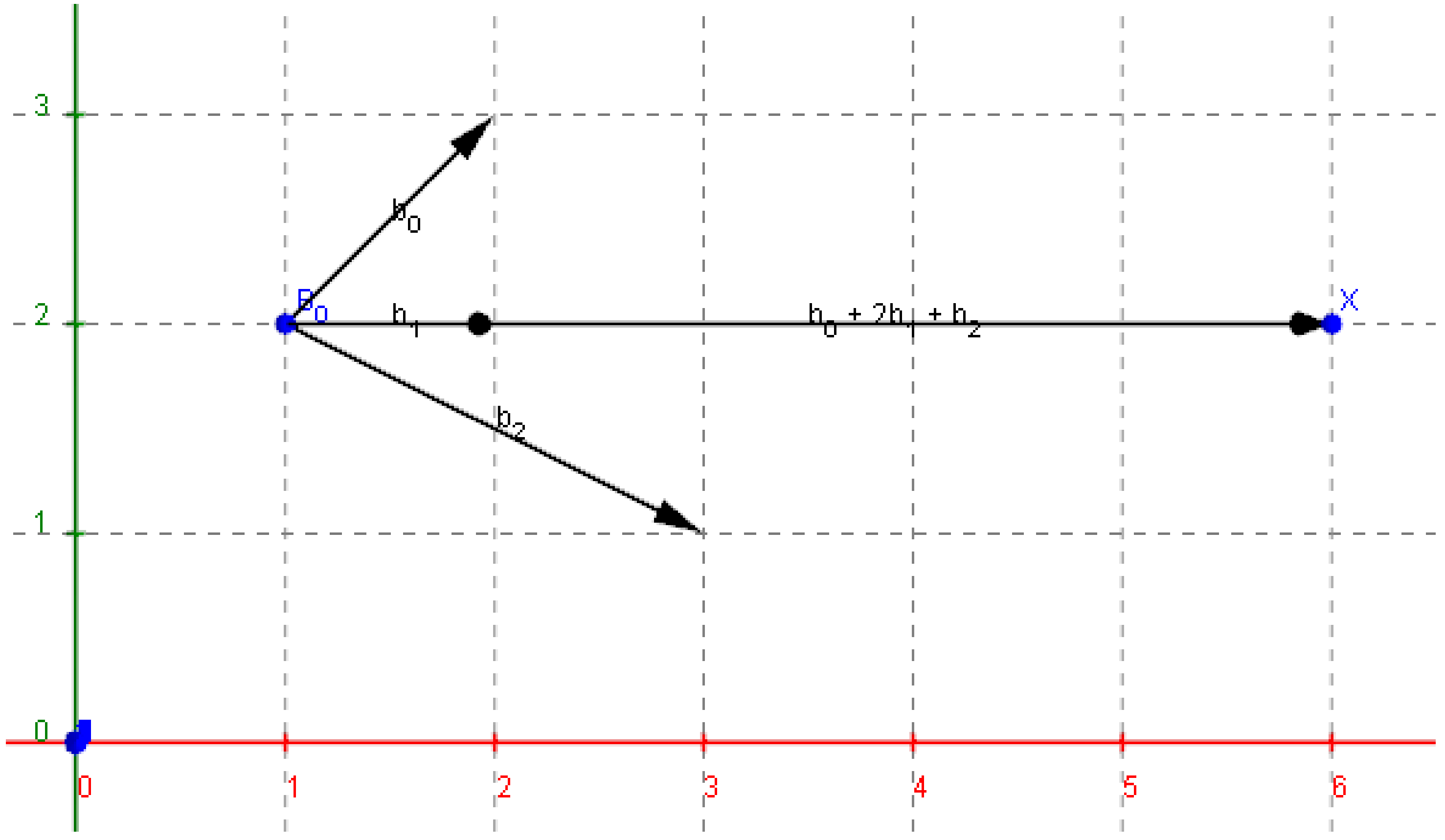
Change of Frame: Non-standard to Standard

$$X_s = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

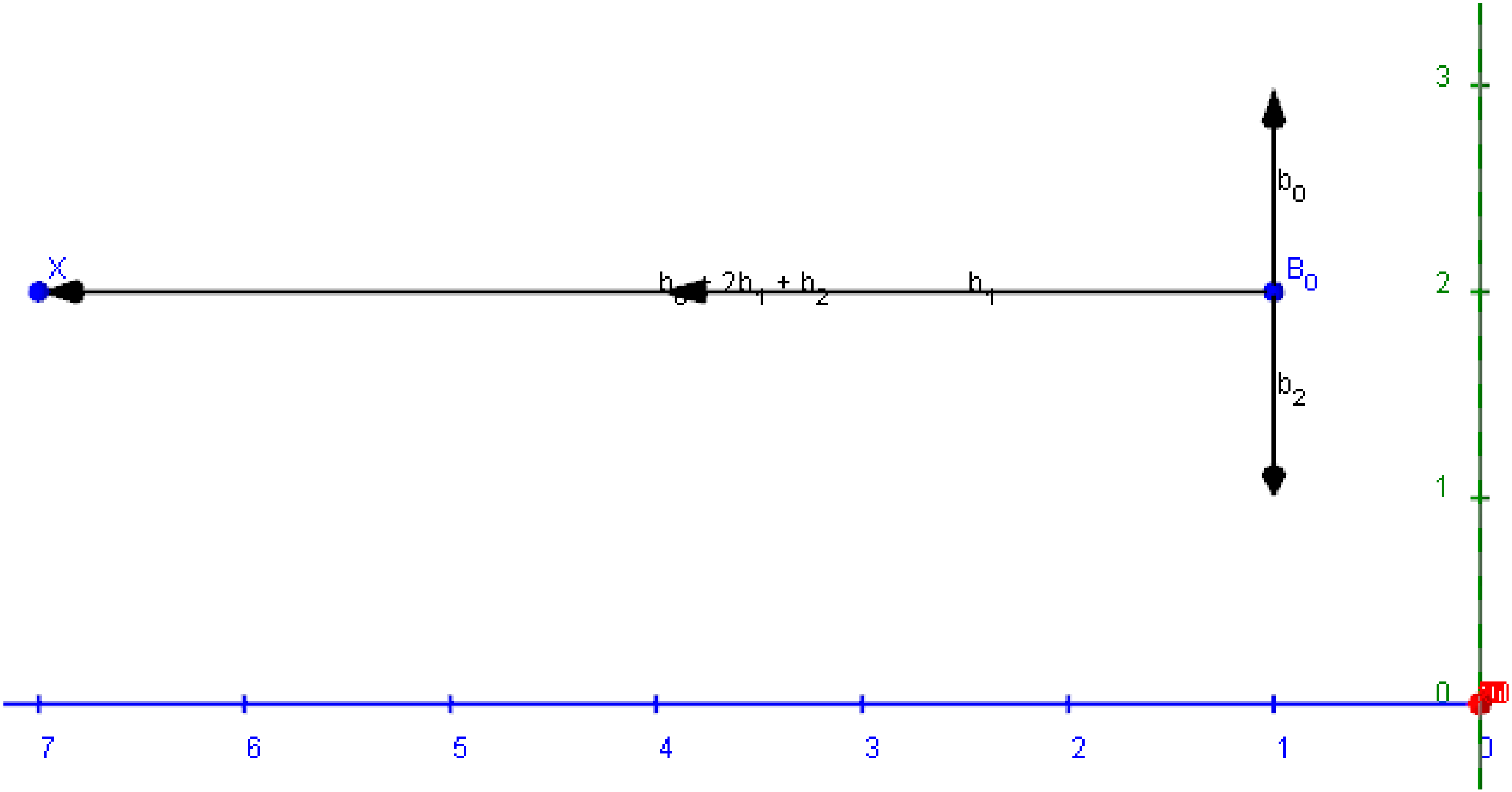


b_0
B_0
b_2
b_1
1
2
3
4
5
$b_0 + 2b_1 + b_2$
X

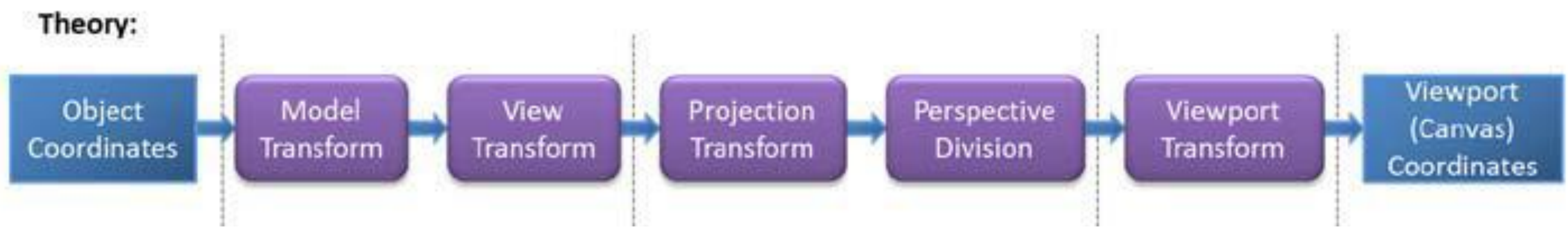
XY - Plane



YZ - Plane



Change of Frame: Application in Computer Graphics



References

Books

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- SALOMON, D. 2011. The Computer Graphics Manual. Vol. 1. Springer. Northridge, CA.
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Images

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Webpages

- http://www.songho.ca/math/homogeneous/homogeneous.html