OTHER OPERATIONS ON VECTORS

Dot and Cross Product; Scalar and Vector Projection

Dot product

Consider vectors

$A = \langle a_1, a_2 \rangle \qquad B = \langle b_1, b_2 \rangle$

DOT PRODUCT

$$A \cdot B = \underbrace{a_1 b_1 + a_2 b_2}_{\text{scalar}}$$

Dot product

Consider vectors

$A = \langle a_1, a_2, a_3 \rangle$

$B = \langle b_1, b_2, b_3 \rangle$

DOT PRODUCT

$$A \cdot B = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{scalar}}$$

Example 1.

Evaluate the following:

1. $\langle 2, -3 \rangle \cdot \langle -2, 4 \rangle$
2. $\langle 1, 2, -3 \rangle \cdot \langle 3, 2, 4 \rangle$

Solutions:

1. $\langle 2, -3 \rangle \cdot \langle -2, 4 \rangle$

$$= 2 \cdot (-2) + (-3) \cdot 4$$
$$= -4 + (-12) = -16$$

Solutions

2. $\langle 1, 2, -3 \rangle \cdot \langle 3, 2, 4 \rangle$

$$= 1 \cdot 3 + 2 \cdot 2 + (-3) \cdot 4$$
$$= 3 + 4 + (-12)$$
$$= -5$$

Angle between vectors

Given nonzero vectors A and B.

$$A \cdot B = \|A\| \|B\| \cos\theta_{AB}$$

where θ_{AB} is the smallest nonnegative angle in radian measure between the vectors

Angle between vectors

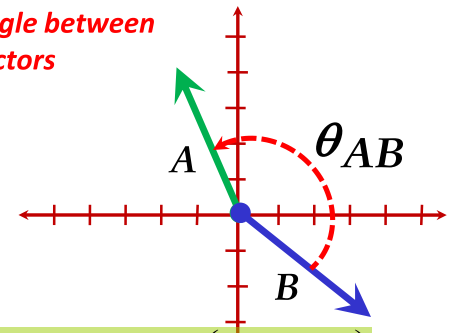
$$A \cdot B = \|A\|\|B\| \cos\theta_{AB}$$

$$\Rightarrow \cos\theta_{AB} = \frac{A \cdot B}{\|A\|\|B\|}$$

$$\Rightarrow \theta_{AB} = Arc\cos\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$$

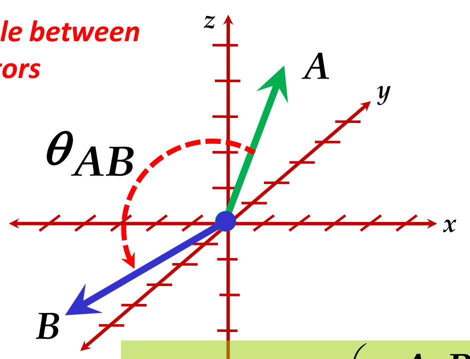
since $0 \le \theta_{AB} \le \pi$

Angle between vectors



$$\theta_{AB} = Arc\cos\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$$

Angle between vectors



$$\theta_{AB} = Arc\cos\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$$

Angle between vectors

If A and B are in the same direction, $\theta_{AB} = 0$.

If A and B are in opposite directions, $\theta_{AB} = \pi$.

Example 2.

Determine the angles between the following pairs of vectors.

1. $\langle 4, -5 \rangle$ and $\langle 5, -12 \rangle$
2. $\langle 2, -1, 2 \rangle$ and $\langle 3, -3, 0 \rangle$

Solutions:

1. $\langle 4, -5 \rangle$ and $\langle 5, -12 \rangle$

Let $A = \langle 4, -5 \rangle$

$B = \langle 5, -12 \rangle$

$A \cdot B = 80$

$\|A\| = \sqrt{41} \qquad \|B\| = 13$

Solutions (continued)

$$\theta_{AB} = Arc\cos\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$$

$$\Rightarrow \theta_{AB} = Arc\cos\left(\frac{80}{\sqrt{41 \cdot 13}}\right)$$

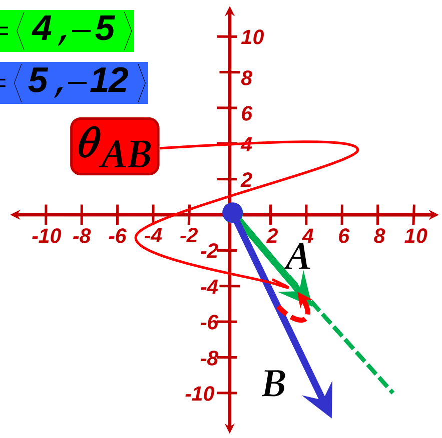
$$\Rightarrow \theta_{AB} = Arc\cos(0.96107)$$

$$\Rightarrow \theta_{AB} \approx 0.27995 \text{ radian}$$

$$\approx 16.04 \text{ degrees}$$

$A = \langle 4, -5 \rangle$

$B = \langle 5, -12 \rangle$



Solutions:

2. $\langle 2, -1, 2 \rangle$ and $\langle 3, -3, 0 \rangle$

Let $A = \langle 2, -1, 2 \rangle$

$B = \langle 3, -3, 0 \rangle$

$A \cdot B = 9$

$\|A\| = 3 \quad \|B\| = 3\sqrt{2}$

Solutions (continued)

$$\theta_{AB} = Arc\cos\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$$

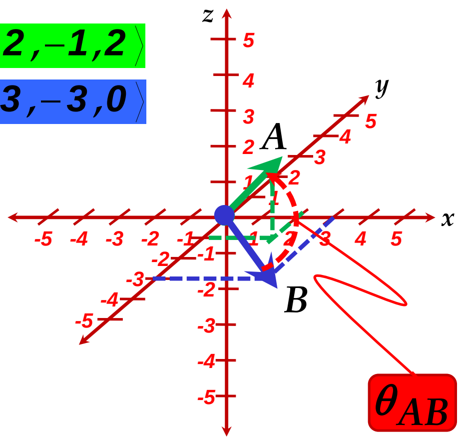
$$\Rightarrow \theta_{AB} = Arc\cos\left(\frac{9}{3 \cdot 3\sqrt{2}}\right)$$

$$\Rightarrow \theta_{AB} = Arc\cos\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta_{AB} = Arc\cos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$A = \langle 2, -1, 2 \rangle$

$B = \langle 3, -3, 0 \rangle$



Dot product and orthogonality

Two non-zero vectors are orthogonal (or perpendicular with each other) if and only if their dot product is 0 (zero).

Illustration

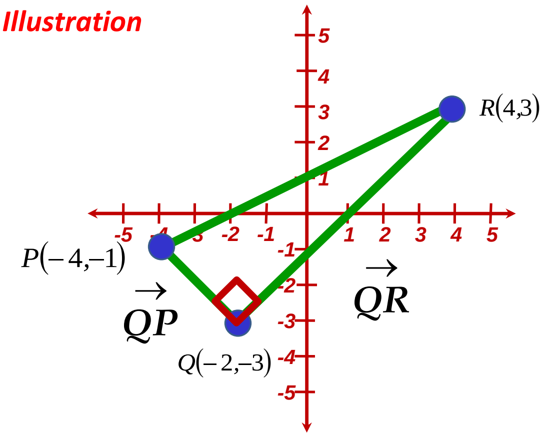
Show that the line segements joining points P (-4,-1), Q(-2,-3) and R(4,3) form a right triangle

Solution

$\vec{QP} = \langle -2, 2\rangle$ $\qquad$ $\vec{QR} = \langle 6, 6\rangle$

$$\vec{QP} \bullet \vec{QP} = -12 + 12 = 0$$

Illustration



Solution

Since $\vec{QR} \cdot \vec{QP} = \mathbf{0}$,
$\vec{QR}$ and $\vec{QP}$ are orthogonal.

Hence, sides QR and QP are perpendicular.

Thus, ΔPQR is a right triangle.

Supplement

If A, B and C are vectors, and c is a scalar,

i. $A \cdot B = B \cdot A$ **(commutativity)**

ii. $A \cdot (B + C) = A \cdot B + A \cdot C$ **(distributivity)**

iii. $c(A \cdot B) = (cA) \cdot B$

iv. $O \cdot A = \mathbf{0}$, $O = \langle \mathbf{0}, \mathbf{0} \rangle$

v. $A \cdot A = \| A \|^{2}$

Cross product

Consider vectors

$A = \langle a_1, a_2, a_3 \rangle \qquad B = \langle b_1, b_2, b_3 \rangle$

CROSS PRODUCT

$$A \times B = \underbrace{\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}_{\text{vector}}$$

Cross product

$$A \times B = \begin{vmatrix} +i & -j & +k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By cofactor expansion

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

REVIEW

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Example

1. If $A = \langle 2, -1, 2 \rangle$ and $B = \langle 3, -3, 0 \rangle$, evaluate $A \times B$.

Solution:

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 3 & -3 & 0 \end{vmatrix}$$

Solution (continued)

$$A \times B = \begin{vmatrix} +i & -j & +k \\ 2 & -1 & 2 \\ 3 & -3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ -3 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} k$$

Solution (continued)

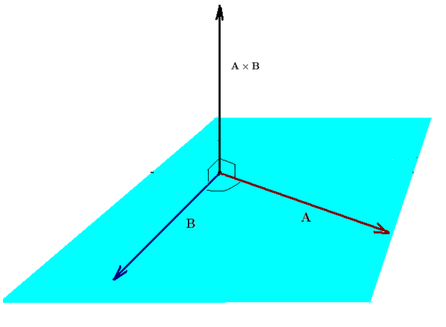
$$A \times B = \begin{vmatrix} -1 & 2 \\ -3 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} k$$

$$= [0 - (-6)]i - (0 - 6)j + [(-6) - (-3)]k$$

$$= 6i + 6j - 3k = \langle 6, 6, -3 \rangle$$

Geometrically

If A and B are nonzero vectors, $A \times B$ is a vector orthogonal to both A and B.



Verification

$A = \langle 2, -1, 2 \rangle \quad B = \langle 3, -3, 0 \rangle$

$A \times B = \langle 6, 6, -3 \rangle$

$$A \cdot (A \times B) = 2 \cdot 6 + (-1) \cdot 6 + 2 \cdot (-3)$$
$$= 12 - 6 - 6 = 0$$

Hence, A and $A \times B$ are orthogonal.

$$B \cdot (A \times B) = 3 \cdot 6 + (-3) \cdot 6 + 0 \cdot (-3)$$
$$= 18 - 18 + 0 = 0$$

Hence, B and $A \times B$ are orthogonal.

Example 3

2. If $M = \langle -2, 0, 4 \rangle$ and $N = \langle 0, 2, 1 \rangle$, evaluate $M \times N$.

Solution:

$$M \times N = \begin{vmatrix} i & j & k \\ -2 & 0 & 4 \\ 0 & 2 & 1 \end{vmatrix}$$

Solution (continued)

$$M \times N = \begin{vmatrix} +i & -j & +k \\ -2 & 0 & 4 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} k$$

$$= -8i + 2j - 4k = \langle -8, 2, -4 \rangle$$

Exercise

Consider the points

$P(2, 3, 0)$

$Q(0, 5, -1)$

$R(1, 0, 3)$

Determine a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

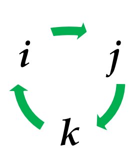
Supplement

i. If A is a vectors in the three-dimensional space,

$$A \times A = O \qquad O \times A = O \times A = O$$

ii. $i \times i = O \quad j \times j = O \quad k \times k = O$

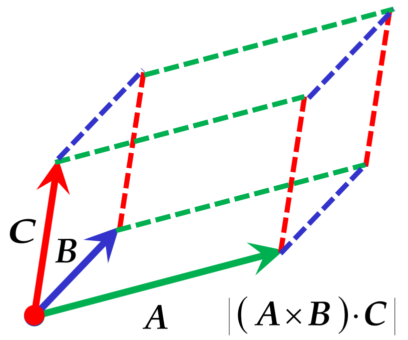
$$i \times j = k \qquad j \times i = -k$$
$$j \times k = i \qquad k \times j = -i$$
$$k \times i = j \qquad i \times k = -j$$



Supplement

iii. $A \times B = -(B \times A)$

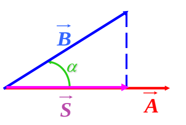
iv. If A, B and C are vectors identifying nonparallel sides of some parallelepiped, then $|(A \times B) \cdot C|$ is the volume of the parallelepiped.



$\left|(A \times B) \cdot C\right|$

: the volume of the parallelepiped

If $\vec{A}$ and $\vec{B}$ are non-zero vectors and α is the angle between them, the *scalar projection of $\vec{B}$ onto $\vec{A}$* is defined to be $\left\|\vec{B}\right\| \cos\alpha$.



$$\cos\alpha = \frac{\left\|\vec{S}\right\|}{\left\|\vec{B}\right\|}$$

$$\left\|\vec{B}\right\| \cos\alpha = \left\|\vec{S}\right\|$$

The scalar projection of a vector $\vec{B}$ onto the vector $\vec{A}$ is $\dfrac{\vec{A}.\vec{B}}{\left\|\vec{A}\right\|}$.

$$\boxed{\cos\alpha = \frac{\vec{A} \cdot \vec{B}}{\left\|\vec{A}\right\|\left\|\vec{B}\right\|} \rightarrow \left\|\vec{B}\right\| \cos\alpha = \frac{\vec{A} \cdot \vec{B}}{\left\|\vec{A}\right\|}.}$$

Let $\vec{A} = \langle 3,4 \rangle$ and $\vec{B} = \langle 1,2 \rangle$.
Find the scalar projection of

a. $\vec{B}$ onto $\vec{A}$ b. $\vec{A}$ onto $\vec{B}$

solution:

a. $\dfrac{\vec{A} \cdot \vec{B}}{\left\|\vec{A}\right\|} = \dfrac{\langle 3,4 \rangle \cdot \langle 1,2 \rangle}{\sqrt{3^2+4^2}} = \dfrac{3 \cdot 1 + 4 \cdot 2}{\sqrt{25}} = \dfrac{11}{5}.$

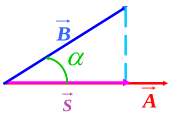
b. $\dfrac{\vec{A} \cdot \vec{B}}{\left\|\vec{B}\right\|} = \dfrac{\langle 3,4 \rangle \cdot \langle 1,2 \rangle}{\sqrt{1^2+2^2}} = \dfrac{3 \cdot 1 + 4 \cdot 2}{\sqrt{5}} = \dfrac{11}{\sqrt{5}} = \dfrac{11\sqrt{5}}{5}.$

Remark:

Dot products are used to compute for vector projections!

The vector projection of a vector $\vec{B}$ onto a non-zero vector $\vec{A}$ is

$$\frac{\vec{A} \cdot \vec{B}}{\left\|\vec{A}\right\|^2}\left(\vec{A}\right) .$$



$$u_{\vec{A}} = \frac{1}{\left\|\vec{A}\right\|} \cdot \vec{A}$$

$$\vec{S} = \left(\frac{\vec{A}.\vec{B}}{\left\|\vec{A}\right\|}\right) u_{\vec{A}}$$

$$\left\|\vec{S}\right\| = \left\|\vec{B}\right\| \cos\alpha \qquad \vec{S} = \left\|\vec{B}\right\| \cos\alpha \cdot \vec{U}_A$$

Since, $\left\|\vec{B}\right\| \cos\alpha = \dfrac{\vec{A}\cdot\vec{B}}{\left\|\vec{A}\right\|}$ and $\vec{U}_A = \dfrac{1}{\left\|\vec{A}\right\|}\cdot\vec{A}$

$$\vec{S} = \left(\frac{\vec{A}\cdot\vec{B}}{\left\|\vec{A}\right\|}\right)\left(\frac{1}{\left\|\vec{A}\right\|}\left(\vec{A}\right)\right) = \frac{\vec{A}\cdot\vec{B}}{\left\|\vec{A}\right\|^2}\left(\vec{A}\right).$$

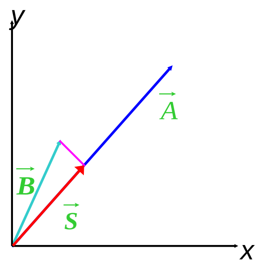
Example. Let $\vec{A} = \langle 3,4\rangle$ and $\vec{B} = \langle 1,2\rangle$.

Find the vector projection of $\vec{B}$ onto $\vec{A}$.

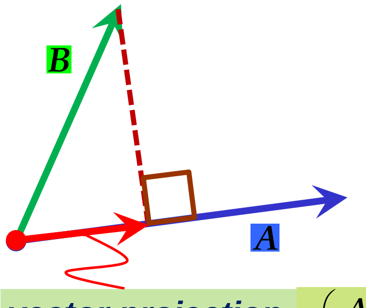
Draw the position representations of $\vec{A}$ and $\vec{B}$ and the vector projection of $\vec{B}$ onto $\vec{A}$.

solution:

$$\vec{A}\cdot\vec{B} = \langle 3,4\rangle\cdot\langle 1,2\rangle$$
$$= 3+8 = 11$$
$$\left\|\vec{A}\right\| = \sqrt{3^2+4^2}$$
$$= \sqrt{25} = 5$$
$$\frac{\vec{A}\cdot\vec{B}}{\left\|\vec{A}\right\|^2}\left(\vec{A}\right) = \frac{11}{5^2}\langle 3,4\rangle$$
$$= \left\langle \frac{33}{25}, \frac{44}{25}\right\rangle$$

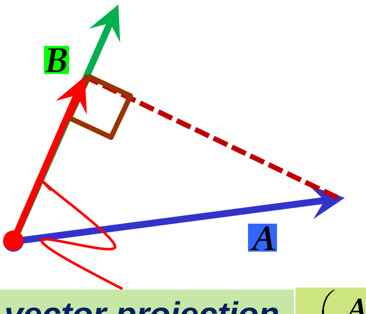


PROJECTION



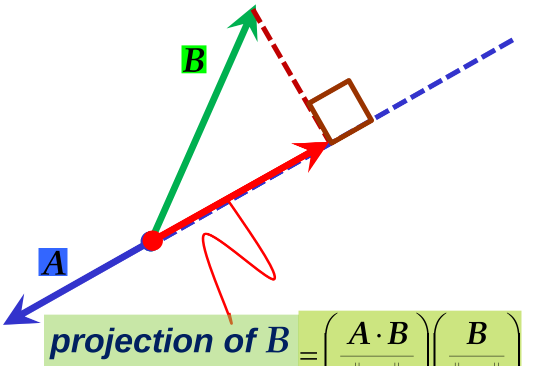
vector projection of B onto A $= \left(\dfrac{A\cdot B}{\|A\|}\right)\left(\dfrac{A}{\|A\|}\right)$

PROJECTION



vector projection of A onto B $= \left(\dfrac{A\cdot B}{\|B\|}\right)\left(\dfrac{B}{\|B\|}\right)$

PROJECTION



projection of B onto A $= \left(\dfrac{A\cdot B}{\|B\|}\right)\left(\dfrac{B}{\|B\|}\right)$

END