CHAPTER 0

Review of MATH 36

1

When do we say that a function f is continuous at a number a ?

i. $f(a)$ exists.

ii. $\lim_{x \to a} f(x)$ exists.

iii. $\lim_{x \to a} f(x) = f(a)$

How do we solve for the derivative of a function f ?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f is discontinuous at a

$\Longrightarrow$ **f is NOT differentiable at a.**

Rules of Differentiation

If u and v are differentiable functions of x, then

$$D_x(u+v) = D_x u + D_x v$$

$$D_x(uv) = uD_x v + vD_x u$$

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Rules of Differentiation

If u and v are differentiable functions of x, then

$$D_x\left(\frac{u}{v}\right) = \frac{vD_x u - uD_x v}{v^2}$$

$$D_x u^n = nu^{n-1} D_x u,$$

n is any real number.

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The Antiderivative.

If $F'(x) = f(x)$, then

$$\int f(x)\, dx = F(x) + C$$

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The Integral.

If $y=f(x)$ is continuous on an interval (a,b) partitioned into sub-intervals I_i's such that $\xi_i \in I_i$,

then

$$\lim_{n\to+\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x = \int_a^b f(x)\,dx$$

if the limit exists.

Review of MATH 36

Let u be a differentiable function of x.

$D_x(\sin u) =$ __________

$D_x(\cos u) =$ __________

$D_x(\tan u) =$ __________

$D_x(\cot u) =$ __________

$D_x(\sec u) =$ __________

$D_x(\csc u) =$ __________

Review of MATH 36

Let u be a differentiable function of x.

$$D_x(\sin u) = \cos u \cdot D_x u$$
$$D_x(\cos u) = -\sin u \cdot D_x u$$
$$D_x(\tan u) = \sec^2 u \cdot D_x u$$
$$D_x(\cot u) = -\csc^2 u \cdot D_x u$$
$$D_x(\sec u) = \sec u \tan u \cdot D_x u$$
$$D_x(\csc u) = -\csc u \cot u \cdot D_x u$$

Review of MATH 36

$$\int \cos u\,du = \sin u + C$$
$$\int \sin u\,du = -\cos u + C$$
$$\int \sec^2 u\,du = \tan u + C$$
$$\int \csc^2 u\,du = -\cot u + C$$
$$\int \sec u \tan u\,du = \sec u + C$$
$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int \tan x\,dx = ???$$
$$\int \cot x\,dx = ???$$
$$\int \sec x\,dx = ???$$
$$\int \csc x\,dx = ???$$

CHAPTER 1

Derivatives of and Integrals Involving Transcendental Functions

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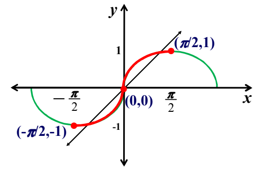
Chapter objectives:

At the end of the chapter, you should be able to

1. **find the derivatives of transcendental functions,**
2. **evaluate integrals involving transcendental functions,**
3. **use logarithmic differentiation appropriately,**
4. **evaluate limits of functions with indeterminate forms,**
5. **solve applied problems.**

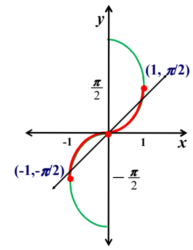
1.1 Derivatives of and integrals yielding inverse trigonometric functions

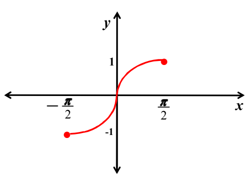


$y = \sin x$

$Domain = R$

$Range = [-1,1]$

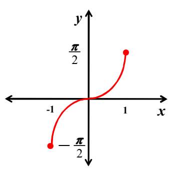




$y = f(x) = \sin x$

$Domain = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$Range = [-1,1]$



$y = f^{-1}(x) = Arc\sin x$

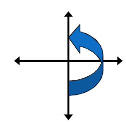
$Domain = [-1,1]$

$Range = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$y = Arc\sin x \;\Leftrightarrow\; \underline{\sin y = x}$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\Rightarrow \cos y\,\frac{dy}{dx} = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\cos y \ge 0$.



$$\cos y = \sqrt{1-\sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d(Arc\sin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Theorem 1.1.a **If u is a differentiable function of x,**

$$\frac{d(Arc\sin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

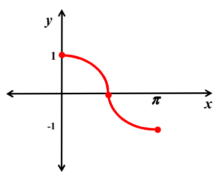
Illustrations:

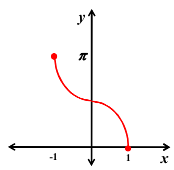
1. If $y = Arc\sin(3x)$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}}\cdot 3 = \frac{3}{\sqrt{1-9x^2}}.$$

2. If $y = Arc\sin\left(x^4\right)$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x^4\right)^2}}\cdot 4x^3 = \frac{4x^3}{\sqrt{1-x^8}}.$$





$y = f(x) = \cos x$

$Domain = [0, \pi]$

$Range = [-1, 1]$

$y = f^{-1}(x) = Arc\cos x$

$Domain = [-1, 1]$

$Range = [0, \pi]$

$$\frac{d(Arc\cos x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Theorem 1.1.b If u is a differentiable function of x,

$$\frac{d(Arc\cos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

Proof: Exercise

Illustrations:

1. If $y = Arc\cos(2x)$, then

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{-2}{\sqrt{1-(2x)^2}}$$

2. If $y = Arc\cos(2x^3)$, then

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x^3)^2}} \cdot 6x^2 = \frac{-6x^2}{\sqrt{1-4x^6}}$$

3. If $y = Arc\sin(2x) + Arc\cos(2x)$, then

$$\frac{dy}{dx} = \frac{d(Arc\sin(2x))}{dx} + \frac{d(Arc\cos(2x))}{dx}$$

$$= \frac{2}{\sqrt{1-(2x)^2}} + \frac{-2}{\sqrt{1-(2x)^2}}$$

$$= \mathbf{0}$$

4. If $y = \underline{\sqrt{1-4x^2}}\,\underline{Arc\cos(2x)}$, then

$$\frac{dy}{dx} = \sqrt{1-4x^2}\,\frac{d(Arc\cos(2x))}{dx} + Arc\cos(2x)\frac{d\left(\sqrt{1-4x^2}\right)}{dx}$$

$$= \cancel{\sqrt{1-4x^2}} \cdot \frac{-2}{\cancel{\sqrt{1-(2x)^2}}} + Arc\cos(2x) \cdot \frac{1}{\cancel{2}\sqrt{1-4x^2}} \cdot -\cancelto{4}{8}x$$

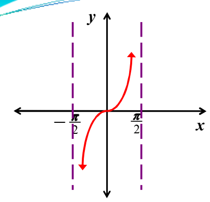
$$= -2 - Arc\cos(2x) \cdot \frac{4x}{\sqrt{1-4x^2}}$$

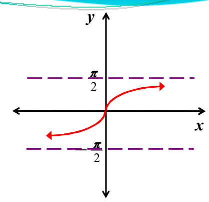
$$= -2 - \frac{4x Arc\cos(2x)}{\sqrt{1-4x^2}}$$

$$= -2 - \frac{4x\,Arc\cos(2x)}{\sqrt{1-4x^2}}$$

$$= \frac{-2\sqrt{1-4x^2} - 4x\,Arc\cos(2x)}{\sqrt{1-4x^2}} \cdot \frac{\sqrt{1-4x^2}}{\sqrt{1-4x^2}}$$

$$= \frac{-\left[2\sqrt{1-4x^2} + 4x\,Arc\cos(2x)\right]\sqrt{1-4x^2}}{1-4x^2}$$





$y = f(x) = \tan x$

$Domain = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$Range = R$

$y = f^{-1}(x) = Arc\tan x$

$Domain = R$

$Range = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$y = Arc\tan x \Leftrightarrow \tan y = x$ and $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\Rightarrow \sec^2 y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(Arc\tan x)}{dx} = \frac{1}{1+x^2}$$

Theorem 1.1.c **If u is a differentiable function of x,**

$$\frac{d(Arc\tan u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

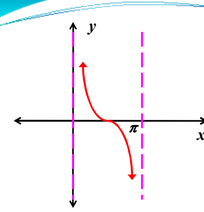
Illustrations:

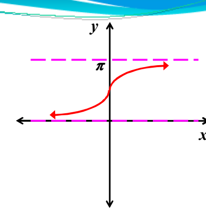
1. If $y = Arc\tan(5x)$, then

$$\frac{dy}{dx} = \frac{1}{1+(5x)^2} \cdot 5 = \frac{5}{1+25x^2}.$$

2. If $y = Arc\tan(2x^5)$, then

$$\frac{dy}{dx} = \frac{1}{1+(2x^5)^2} \cdot 10x^4 = \frac{10x^4}{1+4x^{10}}.$$





$y = f(x) = \cot x$

$Domain = (0, \pi)$

$Range = R$

$y = f^{-1}(x) = Arc\cot x$

$Domain = R$

$Range = (0, \pi)$

$$\frac{d(Arc\cot x)}{dx}=\frac{-1}{1+x^2}$$

Theorem 1.1.c **If u is a differentiable function of x,**

$$\frac{d(Arc\cot u)}{dx}=\frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

Proof: Exercise

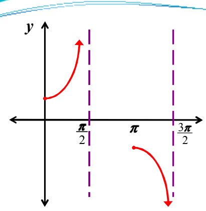
Illustrations:

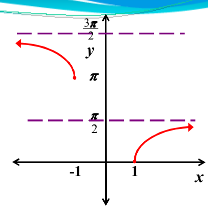
1. If $y = Arc\cot(5x)$, then

$$\frac{dy}{dx}=\frac{-1}{1+(5x)^2}\cdot 5=\frac{-5}{1+25x^2}.$$

2. If $y = Arc\cot(2x^5)$, then

$$\frac{dy}{dx}=\frac{-1}{1+(2x^5)^2}\cdot 10x^4=\frac{-10x^4}{1+4x^{10}}.$$





$y = f(x) = \sec x$ | $y = f^{-1}(x) = Arc\sec x$

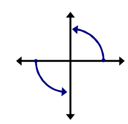
$Domain = [0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$ | $Domain = [1,\infty)\cup(-\infty,-1]$

$Range = [1,\infty)\cup(-\infty,-1]$ | $Range = [0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$

$y = Arc\sec x \Leftrightarrow \sec y = x$ and $y\in[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$.

$$\Rightarrow \sec y\tan y\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec y\tan y}$$



$$1+\tan^2 y=\sec^2 y \quad \Rightarrow \tan^2 y=\sec^2 y-1$$

$$\Rightarrow \tan y=\sqrt{\sec^2 y-1}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d(Arc\sec x)}{dx}=\frac{1}{x\sqrt{x^2-1}}$$

Theorem 1.1.e **If u is a differentiable function of x,**

$$\frac{d(Arc\sec u)}{dx}=\frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

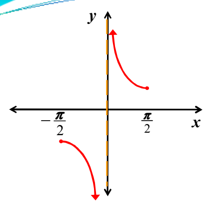
Illustrations:

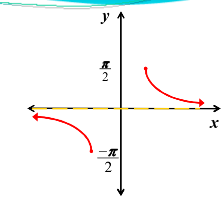
1. If $y = Arc\sec(3x)$, then

$$\frac{dy}{dx}=\frac{1}{\cancel{3}x\sqrt{(3x)^2-1}}\cdot\cancel{3}=\frac{1}{x\sqrt{9x^2-1}}.$$

2. If $y = Arc\sec(x^2)$, then

$$\frac{dy}{dx}=\frac{1}{x^{\cancel{2}}\sqrt{(x^2)^2-1}}\cdot 2\cancel{x}=\frac{2}{x\sqrt{x^4-1}}.$$





$y = f(x) = \csc x$ $\qquad$ $y = f^{-1}(x) = Arc\csc x$

$Domain = [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ $\qquad$ $Domain = [1, \infty) \cup (-\infty, -1]$

$Range = [1, \infty) \cup (-\infty, -1]$ $\qquad$ $Range = [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

$$\frac{d(Arc\csc x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Theorem 1.1.e **If u is a differentiable function of x,**

$$\frac{d(Arc\csc u)}{dx} = \frac{-1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Proof: Exercise

Illustrations:

1. If $y = Arc\csc(3x)$, then

$$\frac{dy}{dx} = \frac{-1}{\cancel{3}x\sqrt{(3x)^2-1}} \cdot \cancel{3} = \frac{-1}{x\sqrt{9x^2-1}}.$$

2. If $y = Arc\csc(x^2)$, then

$$\frac{dy}{dx} = \frac{-1}{x^{\cancel{2}}\sqrt{(x^2)^2-1}} \cdot 2\cancel{x} = \frac{-2}{x\sqrt{x^4-1}}.$$

Theorem 1.1 (Derivatives of Inverse Trigo. Functions)

If u is a differentiable function of x,

a. $\frac{d(Arc\sin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ $\qquad$ b. $\frac{d(Arc\cos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

c. $\frac{d(Arc\tan u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$ $\qquad$ d. $\frac{d(Arc\cot u)}{dx} = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$

e. $\frac{d(Arc\sec u)}{dx} = \frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$ $\qquad$ f. $\frac{d(Arc\csc u)}{dx} = \frac{-1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Illustration:

If $y = \frac{Arc\tan x}{Arc\sin x}$

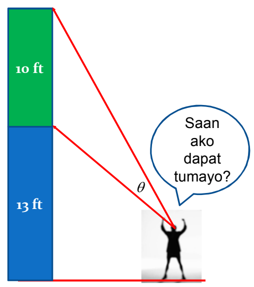
$$y' = \frac{Arc\sin x \cdot D_x(Arc\tan x) - Arc\tan x \cdot D_x(Arc\sin x)}{(Arc\sin x)^2}$$

$$y' = \frac{Arc\sin x \cdot \left(\frac{1}{1+x^2}\right) - (Arc\tan x) \cdot \left(\frac{1}{\sqrt{1-x^2}}\right)}{(Arc\sin x)^2}$$

$$y' = \frac{\sqrt{1-x^2}\,Arc\sin x - (1+x^2)\,Arc\tan x}{(1+x^2)\sqrt{1-x^2}\,(Arc\sin x)^2}.$$

Example of an applied problem:

A statue 10 ft high is standing on a base 13 feet high. If an observer's eye is 5 feet above the ground, how far should he stand from the base so that the angle between his lines of sight to the top and bottom of the statue is a maximum?



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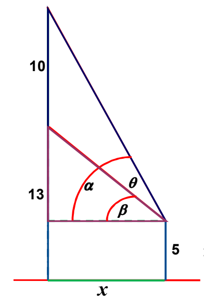

Find x so that θ is a maximum.

$$\cot\alpha = \frac{x}{18} \Rightarrow \alpha = \mathrm{Arc}\cot\left(\frac{x}{18}\right)$$

$$\cot\beta = \frac{x}{8} \Rightarrow \beta = \mathrm{Arc}\cot\left(\frac{x}{8}\right)$$

$$\theta = \alpha - \beta$$

$$\Rightarrow \theta = \mathrm{Arc}\cot\left(\frac{x}{18}\right) - \mathrm{Arc}\cot\left(\frac{x}{8}\right)$$

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$$\theta = \mathrm{Arc}\cot\left(\frac{x}{18}\right) - \mathrm{Arc}\cot\left(\frac{x}{8}\right)$$

$$\frac{d\theta}{dx} = D_x\left(\mathrm{Arc}\cot\left(\frac{x}{18}\right)\right) - D_x\left(\mathrm{Arc}\cot\left(\frac{x}{8}\right)\right)$$

$$= -\frac{1}{1+(x/18)^2}\cdot\frac{1}{18} - -\frac{1}{1+(x/8)^2}\cdot\frac{1}{8}$$

$$= -\frac{(18)^2}{(18)^2+x^2}\cdot\frac{1}{18} + \frac{(8)^2}{(8)^2+x^2}\cdot\frac{1}{8}$$

$$= -\frac{18}{(18)^2+x^2} + \frac{8}{8^2+x^2}$$

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$$\frac{d\theta}{dx} = 0 \Rightarrow -\frac{18}{(18)^2+x^2} + \frac{8}{8^2+x^2} = 0$$

$$\Rightarrow \frac{8}{8^2+x^2} = \frac{18}{(18)^2+x^2}$$

$$\Rightarrow \frac{4}{8^2+x^2} = \frac{9}{(18)^2+x^2}$$

$$\Rightarrow 4\left((18)^2+x^2\right) = 9\left(8^2+x^2\right)$$

$$\Rightarrow 4\left((18)^2+x^2\right) = 9\left(8^2+x^2\right)$$

$$\Rightarrow 4(18)^2+4x^2 = 9\cdot 8^2+9x^2$$

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$$4(18)^2+4x^2 = 9\cdot 8^2+9x^2$$

$$4(18)^2-9\cdot 8^2 = 5x^2$$

$$720 = 5x^2$$

$$x^2 = \frac{720}{5} = 144$$

$$x = \sqrt{144} = 12$$

Thus, the observer should stand 12 ft.from the base.

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Theorem 1.2 (Integrals Yielding Inverse Trigo. Functions)

a. $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \mathrm{Arc}\sin u + C$

b. $\displaystyle\int \frac{du}{1+u^2} = \mathrm{Arc}\tan u + C$

c. $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \mathrm{Arc}\sec u + C$

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Illustrations:

1. Evaluate $\int \frac{x^2dx}{\sqrt{1-x^6}}$.

Solution:

$$\int \frac{x^2dx}{\sqrt{1-x^6}} = \int \frac{x^2dx}{\sqrt{1-\left(x^3\right)^2}} = \int \frac{\frac{du}{3}}{\sqrt{1-u^2}}$$

$$\boxed{\begin{gathered} u = x^3 \\ du = 3x^2dx \\ \frac{du}{3} = x^2dx \end{gathered}}$$

$$= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}Arc\sin u + C$$

$$\boxed{= \frac{1}{3}Arc\sin\left(x^3\right) + C}$$

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2. Evaluate $\int \frac{\cos(2x)dx}{1+\sin^2(2x)}$.

Solution:

$$\int \frac{\cos(2x)dx}{1+\sin^2(2x)} = \int \frac{\cos(2x)dx}{1+\left[\sin(2x)\right]^2}$$

$$\boxed{\begin{gathered} u = \sin(2x) \\ du = 2\cos(2x)dx \\ \frac{du}{2} = \cos(2x)dx \end{gathered}}$$

$$= \int \frac{\frac{du}{2}}{1+u^2} = \frac{1}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}Arc\tan u + C$$

$$\boxed{= \frac{1}{2}Arc\tan\left(\sin(2x)\right) + C}$$

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3. Evaluate $\int \frac{dx}{x\sqrt{x^4-1}}$.

Solution:

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{xdx}{x^2\sqrt{\left(x^2\right)^2-1}}$$

$$\boxed{\begin{gathered} u = x^2 \\ du = 2xdx \\ \frac{du}{2} = xdx \end{gathered}}$$

$$= \int \frac{\frac{du}{2}}{u\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}}$$

$$= \frac{1}{2}Arc\sec u + C$$

$$\boxed{= \frac{1}{2}Arc\sec\left(x^2\right) + C}$$

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Theorem 1.3 (Integrals Yielding Inverse Trigo. Functions)

a. $\int \frac{du}{\sqrt{a^2-u^2}} = Arc\sin\left(\frac{u}{a}\right) + C$

b. $\int \frac{du}{a^2+u^2} = \frac{1}{a}Arc\tan\left(\frac{u}{a}\right) + C$

c. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}Arc\sec\left(\frac{u}{a}\right) + C$

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a. $\int \frac{du}{\sqrt{a^2-u^2}} = Arc\sin\left(\frac{u}{a}\right) + C$

Proof:

$$d\left(Arc\sin\left(\frac{u}{a}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\cdot\frac{1}{a}\cdot du = \frac{1}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\cdot\frac{1}{a}\cdot du$$

$$= \frac{\cancel{a}}{\sqrt{a^2-u^2}}\cdot\frac{1}{\cancel{a}}\cdot du$$

$$= \frac{1}{\sqrt{a^2-u^2}}\cdot du$$

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Illustration: Evaluate $\int \frac{dx}{\sqrt{9-4x^2}}$.

Solution:

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{9-4x^2}}\cdot dx$$

$$= \int \left(\frac{1}{\sqrt{a^2-u^2}}\right)\left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{2}Arc\sin\left(\frac{u}{a}\right) + C$$

$$\boxed{= \frac{1}{2}Arc\sin\left(\frac{2x}{3}\right) + C}$$

$$\boxed{\begin{gathered} a = 3 \\ u = 2x \\ \Rightarrow du = 2dx \\ \Rightarrow \frac{du}{2} = dx \end{gathered}}$$

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Another solution:

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{9\left(1-\frac{4x^2}{9}\right)}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\frac{4x^2}{9}}}$$

$$u = \frac{2x}{3}$$
$$\Rightarrow du = \tfrac{2}{3}dx$$
$$\Rightarrow \tfrac{3}{2}du = dx$$

$$= \frac{1}{\cancel{3}}\int \frac{\cancel{\frac{3}{2}}du}{\sqrt{1-u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2} Arc\,sin(u) + C$$

$$= \frac{1}{2} Arc\,sin\left(\frac{2x}{3}\right) + C$$

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b. $\int \frac{du}{a^2+u^2} = \frac{1}{a} Arc\,tan\left(\frac{u}{a}\right) + C$

Proof:

$$d\left(\frac{1}{a} Arc\,tan\left(\frac{u}{a}\right)\right) = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{u}{a}\right)^2}\cdot\frac{1}{a}\cdot du$$

$$= \frac{1}{a}\cdot\frac{1}{1+\left(\frac{u}{a}\right)^2}\cdot\frac{1}{a}\cdot du$$

$$= \frac{1}{\cancel{a^2}}\cdot\frac{\cancel{a^2}}{a^2+u^2}\cdot du = \frac{1}{a^2+u^2}\cdot du$$

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Illustration:

Evaluate $\int \frac{3dx}{25+16x^2}$.

Solution:

$$\int \frac{3dx}{25+16x^2}$$

$$= 3\int \left(\frac{1}{a^2+u^2}\right)\left(\frac{du}{4}\right)$$

$$= \frac{3}{4}\int \frac{du}{a^2+u^2} = \frac{3}{4}\cdot\frac{1}{a} Arc\,tan\left(\frac{u}{a}\right) + C$$

$$\boxed{= \frac{3}{20} Arc\,tan\left(\frac{4x}{5}\right) + C}$$

$$a = 5$$
$$u = 4x$$
$$\Rightarrow du = 4dx$$
$$\Rightarrow \frac{du}{4} = dx$$

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c. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} Arc\,sec\left(\frac{u}{a}\right) + C$

Proof:

$$d\left(\frac{1}{a} Arc\,sec\left(\frac{u}{a}\right)\right) = \frac{1}{\cancel{a}}\cdot\frac{1}{\frac{u}{\cancel{a}}}\cdot\frac{1}{\sqrt{\left(\frac{u}{a}\right)^2-1}}\cdot\frac{1}{a}\cdot du$$

$$= \frac{1}{u}\cdot\frac{\cancel{a}}{\sqrt{u^2-a^2}}\cdot\frac{1}{\cancel{a}}\cdot du$$

$$= \frac{1}{u\sqrt{u^2-a^2}}\cdot du$$

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Illustration:

Evaluate $\int \frac{dx}{(x-2)\sqrt{x^2-4x-5}}$.

Solution:

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x-5}}$$

$$= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-9}}$$

$$= \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} Arc\,sec\left(\frac{u}{a}\right) + C$$

$$\boxed{= \frac{1}{3} Arc\,sec\left(\frac{x-2}{3}\right) + C}$$

$$a = 3$$
$$u = x-2$$
$$\Rightarrow du = dx$$

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Review of logarithms

$$f(x) = a^x \Rightarrow f^{-1}(x) = log_a\, x$$

1. $log_a 1 = 0$ 2. $log_a a = 1$

If $c > 0$ and $d > 0$,

3. $log_a cd = log_a c + log_a d$
4. $log_a \frac{c}{d} = log_a c - log_a d$
5. $log_a c^r = r\, log_a c$

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If $a = e$, $\log_a x = \log_e x = \ln x$.

a. $\ln 1 = 0$

b. $\ln e = 1$

c. $\ln(cd) = \ln c + \ln d$

d. $\ln \frac{c}{d} = \ln c - \ln d$

e. $\ln c^r = r \ln c$

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The Fundamental Theorem of Calculus

Let g be a function of t which is continuous on the closed interval $[a,x]$. If

$$f(x) = \int_a^x g(t)\,dt$$

then

$$f'(x) = g(x).$$

Illustration:

If $f(x) = \int_a^x t^2\,dt$,

then $g(t) = t^2$.

$$\Rightarrow f(x) = \left.\frac{t^3}{3}\right]_a^x$$

$$= \frac{x^3}{3} - \frac{a^3}{3}$$

$$\Rightarrow f'(x) = x^2$$

$$= g(x)$$

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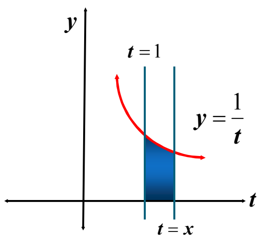
1.2 Derivatives of and integrals yielding logarithmic functions

Definition 1.2.1

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

$$\frac{d(\ln x)}{dx} = \frac{1}{x}$$



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Theorem 1.2.1

If $y = \ln u$, then $\frac{dy}{dx} = \frac{1}{u} \cdot D_x u$.

Illustrations:

1. If $y = \ln(x^4 + 5x^2 - 2)$

$$\frac{dy}{dx} = \frac{1}{x^4 + 5x^2 - 2} \cdot (4x^3 + 10x)$$

$$\boxed{= \frac{4x^3 + 10x}{x^4 + 5x^2 - 2}}$$

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2. If $y = \ln(\sin(2x))$

$$\frac{dy}{dx} = \frac{1}{\sin(2x)} \cdot 2\cos(2x)$$

$$= \frac{2\cos(2x)}{\sin(2x)}$$

$$\boxed{= 2\cot(2x)}$$

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3. If $y = \ln\sqrt{x^4 + 5x^2 - 2}$ then $y = \ln(x^4 + 5x^2 - 2)^{1/2}$

so that $y = \frac{1}{2}\ln(x^4 + 5x^2 - 2)$.

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x^4 + 5x^2 - 2} \cdot (4x^3 + 10x)$$

$$\boxed{= \frac{2x^3 + 5x}{x^4 + 5x^2 - 2}}$$

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4. If $y = ln^3(x^4 + 5x^2 - 2)$,

then $y = [ln(x^4 + 5x^2 - 2)]^3$

$$\frac{dy}{dx} = 3 \cdot [ln(x^4 + 5x^2 - 2)]^2 \cdot \frac{1}{x^4 + 5x^2 - 2} \cdot (4x^3 + 10x)$$

$$\boxed{= \frac{(12x^3 + 30x)[ln(x^4 + 5x^2 - 2)]^2}{x^4 + 5x^2 - 2}}$$

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Theorem 1.2.2

If $y = log_a u$, then $\frac{dy}{dx} = \frac{1}{u \ln a} \cdot \frac{du}{dx}$.

Proof:

Suppose $y = log_a u$. Then

$$a^y = u \Rightarrow y \ln a = \ln u$$

$$\Rightarrow \ln a \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u \ln a} \cdot \frac{du}{dx}.$$

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Illustrations:

1. If $y = log_3(x^4 + 5x^2 - 2)$

$$\frac{dy}{dx} = \frac{1}{(x^4 + 5x^2 - 2)\ln 3} \cdot (4x^3 + 10x)$$

$$\boxed{= \frac{4x^3 + 10x}{(x^4 + 5x^2 - 2)\ln 3}}$$

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2. If $y = log_5(sec(3x))$

$$\frac{dy}{dx} = \frac{1}{(\cancel{sec(3x)})\ln 5} \cdot 3\,\cancel{sec(3x)}\,tan(3x)$$

$$\boxed{= \frac{3\,tan(3x)}{\ln 5}}$$

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Theorem 1.2.3 $\int \frac{1}{u} du = \ln|u| + C$

Proof:

If $u > 0$ then $\ln|u| = \ln u$ so that

$$d(\ln|u|) = d(\ln u) = \frac{1}{u}.$$

If $u < 0$ then $\ln|u| = \ln(-u)$ so that

$$d(\ln|u|) = d(\ln(-u)) = \frac{1}{-u} \cdot (-1) = \frac{1}{u}.$$

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Illustrations:

1. $\int \frac{3x^2 - 6}{x^3 - 6x} dx = \int \frac{1}{x^3 - 6x} \cdot (3x^2 - 6) dx$

$$\boxed{\begin{aligned} u &= x^3 - 6x \\ du &= (3x^2 - 6)\, dx \end{aligned}}$$

$$= \int \frac{1}{u} \cdot du = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|x^3 - 6x| + C$$

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2. $\displaystyle\int \frac{\sec^2 x}{4+\tan x}dx = \int \frac{1}{4+\tan x}\cdot \sec^2 x\, dx$

$\boxed{\begin{aligned} u &= 4+\tan x \\ du &= \sec^2 x\, dx \end{aligned}}$

$= \displaystyle\int \frac{1}{u}\cdot du$

$= \displaystyle\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|4+\tan x| + C$

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Theorem 1.2.4 $\displaystyle\int \tan u\, du = \ln|\sec u| + C$

Proof:

$\boxed{\begin{aligned} t &= \cos u \\ dt &= -\sin u\, du \\ -dt &= \sin u\, du \end{aligned}}$

$\displaystyle\int \tan u\, du = \int \frac{\sin u}{\cos u}du$

$= \displaystyle\int \frac{-dt}{t}$

$= -\displaystyle\int \frac{dt}{t} = -\ln|t| + C$

$= -\ln|\cos u| + C = \ln|\cos u|^{-1} + C$

$= \ln|\sec u| + C$

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Illustration:

$\displaystyle\int 6\tan(3x)dx$

$\boxed{\begin{aligned} u &= 3x \\ du &= 3\, dx \\ \frac{du}{3} &= dx \end{aligned}}$

$= 6\displaystyle\int \tan(3x)dx$

$= 6\displaystyle\int \tan u\cdot \frac{du}{3}$

$= 2\displaystyle\int \tan u\cdot du$

$= 2\ln|\sec u| + C = 2\ln|\sec(3x)| + C$

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Theorem 1.2.5 $\displaystyle\int \cot u\, du = \ln|\sin u| + C$

Proof: (Exercise)

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Illustration:

$\displaystyle\int x\cot(3x^2)dx$

$\boxed{\begin{aligned} u &= 3x^2 \\ du &= 6x\, dx \\ \frac{du}{6} &= x dx \end{aligned}}$

$= \displaystyle\int \cot(3x^2)\cdot x dx$

$= \displaystyle\int \cot u\cdot \frac{du}{6}$

$= \displaystyle\frac{1}{6}\int \cot u\cdot du$

$= \displaystyle\frac{1}{6}\ln|\sin u| + C = \frac{1}{6}\ln|\sin(3x^2)| + C$

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Theorem 1.2.6 $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$

Proof:

$\boxed{\begin{aligned} t &= \sec u + \tan u \\ dt &= (\sec u\tan u + \sec^2 u)du \end{aligned}}$

$\displaystyle\int \sec u\, du$

$= \displaystyle\int \sec u\cdot \frac{\sec u+\tan u}{\sec u+\tan u}du$

$= \displaystyle\int \frac{(\sec^2 u+\sec u\tan u)du}{\sec u+\tan u}$

$= \displaystyle\int \frac{dt}{t} = \ln|t| + C = \ln|\sec u + \tan u| + C$

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Illustration:

$$\int csc^2 x \, sec(cot\, x)\, dx$$
$$= \int sec(cot\, x) \cdot csc^2 x dx$$
$$= \int sec\ u \cdot -du$$
$$= -\int sec\ u\, du$$
$$= -\ln|sec\ u + tan\ u| + C$$
$$= -\ln|sec\,(cot\ x) + tan\,(cot\ x)| + C$$

$$\boxed{\begin{aligned} u &= cot\, x \\ du &= -csc^2 x\, dx \\ -du &= csc^2 xdx \end{aligned}}$$

Theorem 1.2.7 $\int csc\, u\, du = \ln|csc\, u - cot\, u| + C$

Proof: **(Exercise)**

Illustration:

$$\int cos(2x) csc(sin(2x))\, dx = \int csc(sin(2x)) \cdot cos(2x) dx$$

$$\boxed{\begin{aligned} u &= sin(2x) \\ du &= 2\,cos(2x) dx \\ \frac{du}{2} &= cos(2x) dx \end{aligned}}$$

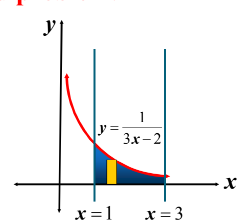
$$= \int csc\ u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int csc\ u\, du$$
$$= \frac{1}{2}\ln|csc\ u - cot\ u| + C$$
$$= \frac{1}{2}\ln|csc\,(sin\,(2x)) - cot\,(sin\,(2x))| + C$$

Example of an applied problem:

Find the area of the region bounded by the graphs of

$$y = \frac{1}{3x-2},$$
$$x = 1,$$
$$x = 3,$$

and the x-axis.



$$A = \int_1^3 \frac{1}{3x-2}\, dx$$

$$A = \int_1^3 \frac{dx}{3x-2}$$
$$= \frac{1}{3}\ln|3x-2|\Big|_1^3$$
$$= \frac{1}{3}(\ln 7 - \cancel{\ln 1}^{\,0})$$
$$= \frac{1}{3}\ln 7$$

The area of the region is (ln7)/3 square units.

Theorem 1.2.1

If $y = \ln u$, then $\frac{dy}{dx} = \frac{1}{u} \cdot D_x u$.

Theorem 1.2.2

If $y = log_a u$, then $\frac{dy}{dx} = \frac{1}{u \ln a} \cdot \frac{du}{dx}$.

Theorem 1.2.3 $\int \frac{1}{u} du = \ln|u| + C$

Theorem 1.2.4 $\int \tan u\,du = \ln|\sec u| + C$

Theorem 1.2.5 $\int \cot u\,du = \ln|\sin u| + C$

Theorem 1.2.6

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

Theorem 1.2.7

$$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

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Review of exponents

1. $a^0 = 1$
2. $a^1 = a$
3. $a^x \cdot a^y = a^{x+y}$
4. $\frac{a^x}{a^y} = a^{x-y}$
5. $\frac{1}{a^x} = a^{-x}$
6. $(a^x)^y = a^{xy}$

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1.3 Derivatives of exponential functions

$$y = a^x$$

$$\Rightarrow \log_a y = x$$

$$\Rightarrow \frac{d(\log_a y)}{dx} = \frac{d(x)}{dx}$$

$$\Rightarrow \frac{1}{y\ln a} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = y\ln a \quad \Rightarrow \boxed{\frac{dy}{dx} = a^x \ln a.}$$

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Illustrations:

1. $y = 2^x$

$$\frac{dy}{dx} = 2^x \ln 2$$

2. $y = 3^x$

$$\frac{dy}{dx} = 3^x \ln 3$$

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Theorem 1.3.1 $\frac{d(a^u)}{dx} = a^u \cdot \ln a \cdot \frac{du}{dx}$

Illustrations:

1. If $y = 3^{x^4+5x^2-2}$

$$\frac{dy}{dx} = 3^{x^4+5x^2-2} \cdot \ln 3 \cdot (4x^3 + 10x)$$

$$\boxed{= 3^{x^4+5x^2-2}(\ln 3)(4x^3 + 10x)}$$

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2. If $y = 5^{\tan(2x)}$

$$\frac{dy}{dx} = 5^{\tan(2x)} \cdot \ln 5 \cdot (2\sec^2(2x))$$

$$\boxed{= 5^{\tan(2x)}(2\ln 5)\sec^2(2x)}$$

3. If $y = 7^{\ln x^2}$, then $y = 7^{2\ln x}$.

$$\frac{dy}{dx} = 7^{2\ln x} \cdot \ln 7 \cdot \frac{2}{x} \boxed{= \frac{7^{2\ln x}\ln 49}{x}}$$

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Corollary 1.3.2 $\dfrac{d(e^u)}{dx} = e^u \cdot \dfrac{du}{dx}$

Illustrations:

1. If $y = e^{x^4+5x^2-2}$

$$\frac{dy}{dx} = e^{x^4+5x^2-2} \cdot (4x^3+10x)$$

$$\boxed{= (4x^3+10x)e^{x^4+5x^2-2}}$$

2. If $y = e^{Arcsin(3x)}$

$$\frac{dy}{dx} = e^{Arcsin(3x)} \cdot \frac{1}{\sqrt{1-(3x)^2}} \cdot 3$$

$$\boxed{= \frac{3e^{Arc\,sin\,(3x)}}{\sqrt{1-9x^2}}}$$

3. If y is a function of x such that

$$e^{xy} = e^x + e^y,$$

$$\frac{d(e^{xy})}{dx} = \frac{d(e^x+e^y)}{dx},$$

$$e^{xy} \cdot (x \cdot y' + y) = e^x + e^y \cdot y'$$

$$xe^{xy}y' + e^{xy}y = e^x + e^y y'$$

$$(xe^{xy} - e^y)y' = e^x - e^{xy}y$$

$$y' = \frac{e^x - e^{xy}y}{xe^{xy} - e^y}$$

Theorem 1.3.3 $\displaystyle\int a^u du = \frac{a^u}{lna} + C$

where a is a positive constant.

Illustration:

$$\int x\, 10^{24x^2} dx = \int 10^{24x^2} \cdot xdx$$

$$\boxed{\begin{aligned} u &= 24x^2 \\ du &= 48x\ dx \\ \frac{du}{48} &= xdx \end{aligned}}$$

$$= \int (10^u)\left(\frac{du}{48}\right)$$

$$= \frac{1}{48}\int 10^u\, du$$

$$= \frac{1}{48} \cdot \frac{10^u}{ln10} + C$$

$$\boxed{= \frac{10^{24x^2}}{48\,ln10} + C}$$

Illustration:

$$\int sin\,x\ 4^{cos\,x} dx = \int 4^{cos\,x} \cdot sin\,xdx$$

$$\boxed{\begin{aligned} u &= cos\,x \\ du &= -sin\,x\ dx \end{aligned}}$$

$$= \int (4^u)(-du)$$

$$= -\int 4^u\, du$$

$$= -\frac{4^u}{ln4} + C$$

$$\boxed{= -\frac{4^{cosx}}{ln4} + C}$$

Corollary 1.3.4 $\int e^u du = e^u + C$

Illustration:

$$\int e^{15x} dx = \int \left(e^u\right) \cdot \frac{du}{15} = \frac{1}{15}\int e^u du = \frac{e^u}{15} + C = \frac{e^{15x}}{15} + C$$

$$\boxed{\begin{aligned} u &= 15x \\ du &= 15dx \\ \frac{du}{15} &= dx \end{aligned}}$$

Illustration:

$$\int \frac{e^{Arc\tan(2x)}}{1+4x^2}dx = \int e^{Arc\tan(2x)} \cdot \frac{dx}{1+4x^2} = \int e^u \cdot \frac{du}{2} = \frac{1}{2}e^u + C = \frac{1}{2}e^{Arctan(2x)} + C$$

$$\boxed{\begin{aligned} u &= Arc\tan(2x) \\ du &= \frac{2}{1+4x^2}dx \\ \frac{du}{2} &= \frac{dx}{1+4x^2} \end{aligned}}$$

Example of an applied problem:

In a certain culture, the rate of increase or decrease of the population is proportional to the present population.

Let

***P* be the population at any time *t* and**

***A* be the initial population.**

Since the rate of increase/decrease of the population is proportional to the present population,

$$\frac{dP}{dt} \approx P$$

$$\Rightarrow \frac{dP}{dt} = kP \quad \text{for some constant } k.$$

$$\Rightarrow \frac{dP}{P} = kdt$$

$$\Rightarrow \int \frac{dP}{P} = \int kdt$$

$$\Rightarrow \int \frac{dP}{P} = \int kdt \quad \Rightarrow \ln P = kt + C$$

$$\Rightarrow e^{\ln P} = e^{kt+C}$$

$$\Rightarrow P = e^{kt} \cdot e^C$$

$$\Rightarrow P = e^{kt} \cdot A$$

$$\boxed{\Rightarrow P = Ae^{kt}.}$$

Example:

In a certain culture, Bacteria M is growing in proportion to the amount present. Initially, there were 10,000 bacteria present and the amount triples in 12 min. Estimate to the nearest minute how long it will take until 100,000 bacteria are present.

$t = 0$	$t = 12$	$t = ?$
10,000	**30,000**	**100,000**

$$P = Ae^{kt}$$
$$P = 10{,}000e^{kt}$$
$$30{,}000 = 10{,}000e^{k(12)}$$
$$3 = e^{k(12)}$$
$$P = 10{,}000e^{k(12)\frac{t}{12}}$$
$$P = 10{,}000(3)^{\frac{t}{12}}$$
$$100{,}000 = 10{,}000(3)^{\frac{t}{12}}$$

$$10 = (3)^{\frac{t}{12}}$$
$$\ln 10 = \tfrac{t}{12}\ln 3$$
$$12\ln 10 = t\cdot\ln 3$$
$$t = \frac{12\ln 10}{\ln 3}$$
$$\approx 25.16$$

There will be 100,000 present approximately after 25 minutes.

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1.4 Logarithmic Differentiation

We recall that if

$$f(x) = [g(x)]^n$$

where n is a constant, then

$$f'(x) = n[g(x)]^{n-1}g'(x).$$

This formula does not apply if a function is raised to a *non-constant* function as in the following:

$$x^{x^2},\ (2x+1)^{3x},\ \sin(3x)^{2x}$$

In such cases, we use logarithmic differentiation.

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Illustrations:

1. If $y = x^x$, $x>0$, $\ln y = \ln(x^x)$

$$\Rightarrow \ln y = x\cdot\ln x$$
$$\frac{d(\ln y)}{dx} = \frac{d(x\cdot\ln x)}{dx}$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x$$
$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 1+\ln x$$
$$\frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x)$$

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2. If $y = x^{\sin(3x)}$, $x>0$,

$$\ln y = \ln\left(x^{\sin(3x)}\right) \Rightarrow \ln y = \sin(3x)\cdot\ln x$$
$$\frac{d(\ln y)}{dx} = \frac{d(\sin(3x)\cdot\ln x)}{dx}$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin(3x)\cdot\frac{1}{x} + (\ln x)3\cos(3x)$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{\sin(3x)+3x(\ln x)\cos(3x)}{x}$$

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$$\frac{dy}{dx} = y\left(\frac{\sin(3x)+3x(\ln x)\cos(3x)}{x}\right)$$
$$\frac{dy}{dx} = x^{\sin(3x)}\left(\frac{\sin(3x)+3x(\ln x)\cos(3x)}{x}\right)$$

We may also use logarithmic differentiation if we want to differentiate products and/or quotients of several functions.

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3. If $y = \dfrac{(2x-1)^2(x^2+1)^3}{\sqrt{x^3+1}}$,

$$\ln y = \ln\left(\frac{(2x-1)^2(x^2+1)^3}{\sqrt{x^3+1}}\right)$$
$$\ln y = \ln\left[(2x-1)^2(x^2+1)^3\right] - \ln\sqrt{x^3+1}$$
$$\ln y = \ln(2x-1)^2 + \ln(x^2+1)^3 - \ln(x^3+1)^{1/2}$$
$$\ln y = 2\ln(2x-1) + 3\ln(x^2+1) - \frac{1}{2}\ln(x^3+1)$$

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$$\ln y = 2\ln(2x-1)+3\ln(x^2+1) - \frac{1}{2}\ln(x^3+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{2x-1}\cdot 2 + 3\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{2}\cdot\frac{1}{x^3+1}\cdot 3x^2$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{4}{2x-1}+\frac{6x}{x^2+1}-\frac{3x^2}{2(x^3+1)}$$

$$\frac{dy}{dx} = y\left(\frac{4}{2x-1}+\frac{6x}{x^2+1}-\frac{3x^2}{2(x^3+1)}\right)$$

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1.5 Hyperbolic Functions

Definitions.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

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$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

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Illustrations:

1. $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$
2. $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$
3. $\tanh 0 = \dfrac{\sinh 0}{\cosh 0} = \dfrac{0}{2} = 0$

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x}}$$

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Fundamental Identities

a. $\cosh^2 x - \sinh^2 x = 1$

b. $1 - \tanh^2 x = \operatorname{sech}^2 x$

c. $\coth^2 x - 1 = \operatorname{csch}^2 x$

(b) and (c) follow directly from (a) by dividing both sides of (a) by $\cosh^2 x$ and then by $\sinh^2 x$.

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a. $\cosh^2 x - \sinh^2 x = 1$

Proof:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4}$$

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$$\cosh^2 x - \sinh^2 x$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$$= \frac{4}{4}$$

$$= 1.$$

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Reduction Formulas

a. $\sinh(-x) = -\sinh x$

b. $\cosh(-x) = \cosh x$

c. $\tanh(-x) = -\tanh x$

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a. $\sinh(-x) = -\sinh x$

Proof:

$$\begin{aligned} \sinh(-x) &= \frac{e^{-x} - e^{--x}}{2} \\ &= \frac{e^{-x} - e^{x}}{2} \\ &= -\left(\frac{e^{x} - e^{-x}}{2}\right) \\ &= -\sinh x \end{aligned}$$

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b. $\cosh(-x) = \cosh x$

Proof:

$$\begin{aligned} \cosh(-x) &= \frac{e^{-x} + e^{--x}}{2} \\ &= \frac{e^{-x} + e^{x}}{2} \\ &= \frac{e^{x} + e^{-x}}{2} \\ &= \cosh x \end{aligned}$$

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c. $\tanh(-x) = -\tanh x$

Proof:

$$\begin{aligned} \tanh(-x) &= \frac{\sinh(-x)}{\cosh(-x)} \\ &= \frac{-\sinh x}{\cosh x} \\ &= -\tanh x \end{aligned}$$

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Addition Formulas

a. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

b. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

c. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

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a1. $sinh(x-y) = sinh\,x\,cosh\,y - cosh\,x\,sinh\,y$

Proof:

By definition, $sinh(x-y) = \dfrac{e^{x-y} - e^{-(x-y)}}{2}$.

$$sinhx\,coshy - coshx\,sinhy$$

$$= \frac{(e^x - e^{-x})}{2}\frac{(e^y + e^{-y})}{2} - \frac{(e^x + e^{-x})}{2}\frac{(e^y - e^{-y})}{2}$$

$$= \frac{(\cancel{e^x e^y} + e^x e^{-y} - e^{-x}e^y - \cancel{e^{-x}e^{-y}}) - (\cancel{e^x e^y} - e^x e^{-y} + e^{-x}e^y - \cancel{e^{-x}e^{-y}})}{4}$$

$$sinhx\,coshy - coshx\,sinhy$$

$$= \frac{2e^x e^{-y} - 2e^{-x}e^y}{4}$$

$$= \frac{2e^{x-y} - 2e^{-x+y}}{4}$$

$$= \frac{e^{x-y} - e^{-x+y}}{2}$$

$$= \frac{e^{x-y} - e^{-(x-y)}}{2}$$

$$= sinh(x-y)$$

Since $sinh(x-y) = sinh\,x\,cosh\,y - cosh\,x\,sinh\,y$,

$$sinh(x+y) = sinh(x-(-y))$$

$$= sinh\,x\,cosh(-y) - cosh\,x\,sinh(-y)$$

$$= sinhx\,coshy - coshx(-sinhy)$$

$$= sinhx\,coshy + coshx\,sinhy$$

The proof of (b) is left to you as an exercise.

c1. $tanh(x-y) = \dfrac{tanhx - tanhy}{1 - tanhx\,tanhy}$

Proof:

$$tanh(x-y)$$

$$= \frac{sinh(x-y)}{cosh(x-y)}$$

$$= \left(\frac{sinhx\,\cancel{coshy} - \cancel{coshx}\,sinhy}{\cancel{coshx\,coshy} - sinhx\,sinhy}\right) \cdot \frac{\frac{1}{\cancel{coshx\,coshy}}}{\frac{1}{\cancel{coshx\,coshy}}}$$

$$= \frac{tanhx - tanhy}{1 - tanhx\,tanhy}$$

Since $tanh(x-y) = \dfrac{tanh\,x - tanh\,y}{1 - tanh\,x\,tanh\,y}$,

$$tanh(x+y) = tanh(x-(-y))$$

$$= \frac{tanh\,x - tanh(-y)}{1 - tanh\,x\,tanh(-y)}$$

$$= \frac{tanh\,x - (-tanh\,y)}{1 - tanh\,x(-tanh\,y)}$$

$$= \frac{tanh\,x + tanh\,y}{1 + tanh\,x\,tanh\,y}$$

Double Number Formulas

a. $sinh(2x) = 2\,sinh\,x\,cosh\,x$

b. $cosh(2x) = cosh^2 x + sinh^2 x$

c. $tanh(2x) = \dfrac{2\,tanh^2 x}{1 + tanh^2 x}$

The proofs will make use of the addition formulas and are left to you as exercises.

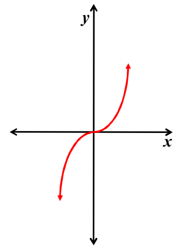
The hyperbolic sine function

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$Domain(f) = R$$

$$Range(f) = R$$

$$f'(x) = \cosh x$$



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Theorem 1.5.1 **If u is a differentiable of x,**

$$\frac{d(\sinh u)}{dx} = \cosh u \cdot \frac{du}{dx}.$$

Illustrations:

1. If $y = \sinh(x^2)$,

$$\frac{dy}{dx} = \cosh(x^2) \cdot 2x = 2x\cosh(x^2)$$

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2. If $y = \sinh(3^{2x})$,

$$\frac{dy}{dx} = \cosh(3^{2x}) \cdot 3^{2x} \cdot \ln 3 \cdot 2$$

$$= 2(\ln 3)(3^{2x})\cosh(3^{2x})$$

3. If $y = \sinh^3(x^3)$, then $y = [\sinh(x^3)]^3$.

$$\frac{dy}{dx} = 3[\sinh(x^3)]^2 \cosh(x^3) \cdot 3x^2$$

$$= 9x^2 \sinh^2(x^3) \cdot \cosh(x^3)$$

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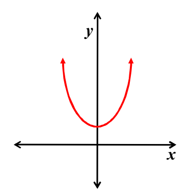
The hyperbolic cosine function

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$Domain(f) = R$$

$$Range(f) = [1, +\infty)$$

$$f'(x) = \sinh x$$



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Theorem 1.5.2 **If u is a differentiable of x,**

$$\frac{d(\cosh u)}{dx} = \sinh u \cdot \frac{du}{dx}.$$

Illustrations:

1. If $y = \cosh(x^2)$,

$$\frac{dy}{dx} = \sinh(x^2) 2x = 2x \sinh(x^2)$$

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2. If $y = \cosh(Arc\sin(5x))$,

$$\frac{dy}{dx} = \sinh(Arc\sin(5x)) \frac{1}{\sqrt{1-(5x)^2}} \cdot 5$$

$$= \frac{5\sinh(Arc\sin(5x))}{\sqrt{1-25x^2}}$$

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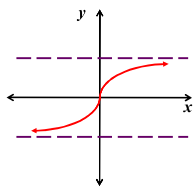
The hyperbolic tangent function

$$f(x)=\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$Domain(f)=R$

$Range(f)=(-1,1)$

$f'(x)=\operatorname{sech}^{2}x$



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$$y=\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\frac{dy}{dx}=\frac{\left(e^{x}+e^{-x}\right)D_x\left(e^{x}-e^{-x}\right)-\left(e^{x}-e^{-x}\right)D_x\left(e^{x}+e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$$

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$$=\frac{\left(e^{x}+e^{-x}\right)^{2}-\left(e^{x}-e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{4\cosh^{2}x-4\sinh^{2}x}{\left(e^{x}+e^{-x}\right)^{2}}=\frac{4\cancelto{1}{\left(\cosh^{2}x-\sinh^{2}x\right)}}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}=\left(\frac{2}{e^{x}+e^{-x}}\right)^{2}=\operatorname{sech}^{2}x$$

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Theorem 1.5.3 **If u is a differentiable of x,**

$$\frac{d(\tanh u)}{dx}=\operatorname{sech}^{2}u\cdot\frac{du}{dx}.$$

Illustrations:

1. If $y=\tanh(3x)$, then

$$\frac{dy}{dx}=\sec h^{2}(3x)\cdot 3=3\sec h^{2}(3x).$$

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2. If $y=\tanh(\sin(3x))$, then

$$\frac{dy}{dx}=\sec h^{2}(\sin(3x))\cdot\cos(3x)\cdot 3$$

$$=3\cos(3x)\cdot\sec h^{2}(\sin(3x)).$$

3. If $y=\tanh(2x)+\sinh(3x)+\cosh(5x)$, then

$$\frac{dy}{dx}=\sec h^{2}(2x)\cdot 2+\cosh(3x)\cdot 3+\sinh(5x)\cdot 5$$

$$=2\sec h^{2}(2x)+3\cosh(3x)+5\sinh(5x).$$

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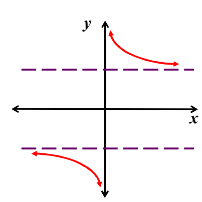
The hyperbolic cotangent function

$$f(x)=\coth x=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$Domain(f)=R-\{0\}$

$Range(f)=(1,+\infty)\cup(-\infty,-1)$

$f'(x)=-\operatorname{csch}^{2}x$



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Theorem 1.5.4 **If u is a differentiable of x,**

$$\frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \cdot \frac{du}{dx}.$$

Illustrations:

1. If $y = \coth(3x)$, then

$$\frac{dy}{dx} = -\csc h^2(3x) \cdot 3 = -3\csc h^2(3x).$$

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2. If $y = \coth(\cos(3x))$, then

$$\frac{dy}{dx} = -\csc h^2(\cos(3x)) \cdot -\sin(3x)\cdot 3$$
$$= 3\sin(3x)\cdot \csc h^2(\cos(3x)).$$

3. If $y = \coth(3x)\tan(5x)$, then

$$\frac{dy}{dx} = \coth(3x)\cdot \frac{d(\tan(5x))}{dx} + \tan(5x)\cdot \frac{d(\coth(3x))}{dx}$$
$$= \coth(3x)\cdot \sec^2(5x)\cdot 5 + \tan(5x)\cdot -\csc h^2(3x)\cdot 3$$
$$= 5\coth(3x)\cdot \sec^2(5x) - 3\tan(5x)\cdot \csc h^2(3x).$$

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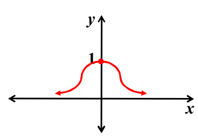
The hyperbolic secant function

$$f(x) = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$Domain(f) = R$

$Range(f) = (0,1]$

$f'(x) = -\operatorname{sech} x \tanh x$



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If $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$ then $y = 2\left(e^x + e^{-x}\right)^{-1}$

so that

$$\frac{dy}{dx} = 2(-1)\left(e^x + e^{-x}\right)^{-2} \cdot \left(e^x - e^{-x}\right)$$
$$= -\frac{2}{\left(e^x + e^{-x}\right)} \cdot \frac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)}$$
$$= -\operatorname{sech} x \tanh x.$$

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Theorem 1.5.5 **If u is a differentiable of x,**

$$\frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \cdot \frac{du}{dx}.$$

Illustrations:

1. If $y = \sec h\left(2^{3x}\right)$, then

$$\frac{dy}{dx} = -\sec h\left(2^{3x}\right)\tanh\left(2^{3x}\right) \cdot 2^{3x} \cdot \ln 2 \cdot 3$$
$$= -2^{3x}(\ln 8)\sec h\left(2^{3x}\right)\tanh\left(2^{3x}\right)$$

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2. If $y = \sec h^2\left(1 - 3x^2\right)$, then

$$y = \left[\sec h\left(1 - 3x^2\right)\right]^2$$

so that

$$\frac{dy}{dx} = 2\sec h\left(1 - 3x^2\right) \cdot -\sec h\left(1 - 3x^2\right)\tanh\left(1 - 3x^2\right) \cdot -6x$$
$$= 12x\sec h^2\left(1 - 3x^2\right)\tanh\left(1 - 3x^2\right).$$

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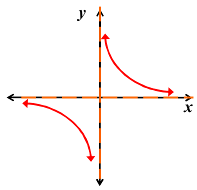
The hyperbolic cosecant function

$$f(x)=cschx=\frac{2}{e^{x}-e^{-x}}$$

$$Domain(f)=R-\{0\}$$

$$Range(f)=R-\{0\}$$

$$f'(x)=-cschx\,cothx$$



Theorem 1.5.6 If u is a differentiable of x,

$$\frac{d(cschu)}{dx}=-cschu\,cothu\cdot\frac{du}{dx}.$$

Illustrations:

1. If $y=csc\,h(ln\,x)$, then

$$\frac{dy}{dx}=-csc\,h(ln\,x)coth(ln\,x)\cdot\frac{1}{x}$$

$$=\frac{-csc\,h(ln\,x)coth(ln\,x)}{x}.$$

2. If $y=\dfrac{x^2}{4+csc\,h(3x)}$, then

$$\frac{dy}{dx}=\frac{(4+csc\,ch(3x))\cdot D_x(x^2)-x^2\cdot D_x(4+csc\,h(3x))}{(4+csc\,h(3x))^2}$$

$$=\frac{(4+csc\,ch(3x))\cdot 2x-x^2-3\,csc\,h(3x)coth(3x)}{(4+csc\,h(3x))^2}$$

$$=\frac{2x(4+csc\,ch(3x))+3x^2\,csc\,h(3x)coth(3x)}{(4+csc\,h(3x))^2}$$

Theorem 1.5.1-1.5.6

1. $\dfrac{d(sinhu)}{dx}=coshu\cdot\dfrac{du}{dx}.$
2. $\dfrac{d(coshu)}{dx}=sinhu\cdot\dfrac{du}{dx}.$
3. $\dfrac{d(tanhu)}{dx}=sech^2u\cdot\dfrac{du}{dx}.$
4. $\dfrac{d(cothu)}{dx}=-csch^2u\cdot\dfrac{du}{dx}.$
5. $\dfrac{d(sechu)}{dx}=-sechutanhu\cdot\dfrac{du}{dx}.$
6. $\dfrac{d(cschu)}{dx}=-cschucothu\cdot\dfrac{du}{dx}.$

Theorem 1.5.7

a. $\int coshudu=sinhu+C$

b. $\int sinhudu=coshu+C$

c. $\int sech^2udu=tanhu+C$

d. $\int csch^2udu=-cothu+C$

e. $\int sechutanhudu=-sechu+C$

f. $\int cschucothudu=-cschu+C$

Illustrations:

1. $\int x\,sinh(x^2)dx=\int sinh(x^2)\cdot xdx$

$$\boxed{\begin{aligned}u&=x^2\\ du&=2xdx\\ \frac{du}{2}&=xdx\end{aligned}}$$

$$=\int sinhu\cdot\frac{du}{2}$$

$$=\frac{1}{2}\int sinhu\,du$$

$$=\frac{1}{2}coshu+C$$

$$\boxed{=\frac{1}{2}cosh(x^2)+C}$$

2. $\int \frac{csc\,h^2(\sqrt{x})}{\sqrt{x}}dx \;=\; \int csc\,h^2(\sqrt{x})\frac{1}{\sqrt{x}}\,dx$

$$\boxed{\begin{aligned} u &= \sqrt{x} = x^{1/2} \\ du &= \frac{1}{2\sqrt{x}}dx \\ \Rightarrow 2du &= \frac{1}{\sqrt{x}}dx \end{aligned}}$$

$= \int csc\,h^2 u \cdot 2du$

$= 2\int csc\,h^2 u du$

$= -2\coth u + C$

$= -2\coth(\sqrt{x}) + C$

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3. $\int \sin x \cosh(\cos x)dx = \int \cosh(\cos x)\cdot \sin xdx$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin xdx \\ -du &= \sin xdx \end{aligned}}$$

$= \int \cosh u \cdot -du$

$= -\int \cosh u du$

$= -\sinh u + C$

$\boxed{= -\sinh(\cos x) + C}$

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4. $\int \frac{1}{\cosh^2(1+3x)}dx = \int sec\,h^2(1+3x)dx$

$$\boxed{\begin{aligned} u &= 1+3x \\ du &= 3dx \\ \frac{du}{3} &= dx \end{aligned}}$$

$= \int sec\,h^2 u \cdot \frac{du}{3}$

$= \frac{1}{3}\int sec\,h^2 u du$

$= \frac{1}{3}\tanh u + C$

$= \frac{1}{3}\tanh(1+3x) + C$

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Steps to find the inverse of a function

1. **interchange x and y.**
2. **solve for y to find the inverse.**

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Illustration: Show that

$$Arcsinhx = \ln\left(x + \sqrt{x^2+1}\right),\ \forall x \in R$$

Proof:

$y = \sinh x \Rightarrow y = \frac{e^x - e^{-x}}{2}$

$x = \frac{e^y - e^{-y}}{2}$

$2x = e^y - \frac{1}{e^y}$

$2x = \frac{e^{2y}-1}{e^y}$

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$2xe^y = e^{2y} - 1$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \frac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2(1)}$

$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^y = x \pm \sqrt{x^2+1}$

Since $e^y > 0$,

$e^y = x + \sqrt{x^2+1}.$

$\ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$

$y = f^{-1}(x) = Arcsinhx$

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1.6 Inverse Hyperbolic Functions

Definitions. (Inverse hyperbolic functions)

$$Arc\sinh x = \ln\left(x+\sqrt{x^2+1}\right),\ \forall x \in R$$

$$Arc\cosh x = \ln\left(x+\sqrt{x^2-1}\right),\ x \ge 1$$

$$Arc\tanh x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ -1 < x < 1$$

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$$Arccoth x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ x^2 > 1$$

$$Arcsech x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ 0 < x \le 1$$

$$Arccsch x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right),\ x \ne 0$$

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$$y = Arcsinh x$$

$$\Rightarrow y = \ln\left(x+\sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot\cancel{2}x\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}}\left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

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Theorem 1.6.1

a. $\dfrac{d(Arcsinh u)}{dx} = \dfrac{1}{\sqrt{u^2+1}}\cdot\dfrac{du}{dx}$

b. $\dfrac{d(Arccosh u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\cdot\dfrac{du}{dx}$

c. $\dfrac{d(Arctanh u)}{dx} = \dfrac{1}{1-u^2}\cdot\dfrac{du}{dx}$

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d. $\dfrac{d(Arccoth u)}{dx} = \dfrac{1}{1-u^2}\cdot\dfrac{du}{dx}$

e. $\dfrac{d(Arcsech u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\cdot\dfrac{du}{dx}$

f. $\dfrac{d(Arccsch u)}{dx} = \dfrac{-1}{u\sqrt{1+u^2}}\cdot\dfrac{du}{dx}$

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1.7 Indeterminate forms

Let f and g be functions. The function $\dfrac{f}{g}$ has the *indeterminate form* $\dfrac{0}{0}$ as x approaches a number a if

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0.$$

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Illustration:

Let $f(x) = x^2 - 9$ and $g(x) = x - 3$.

Since $\lim\limits_{x\to 3} f(x) = 0$ **and** $\lim\limits_{x\to 3} g(x) = 0$, $\frac{f}{g}$

has the indeterminate form $\frac{0}{0}$ **as** x **approaches the number 3.**

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Theorem 1.7.1 L' Hospital's Rule 1

Let f and g be functions which are differentiable on an open interval I containing a, except possibly at the number a. Suppose that for all $x \neq a$ in I, $g'(x) \neq 0$.

If $\frac{f}{g}$ has the indeterminate form $\frac{0}{0}$ as x approaches a, and $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

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Illustrations:

1. Evaluate $\lim\limits_{x\to 3} \frac{x^2-9}{x-3}$, it if exists.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 3} \frac{\frac{d(x^2-9)}{dx}}{\frac{d(x-3)}{dx}} = \lim_{x\to 3} \frac{2x}{1} = \lim_{x\to 3} 2x = 6.$$

Thus, $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = 6.$

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2. Evaluate $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta}$, it if exists.

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{\theta\to 0} \frac{\frac{d(1-\cos\theta)}{d\theta}}{\frac{d(\theta)}{d\theta}} = \lim_{\theta\to 0} \frac{--\sin\theta}{1}$$

$$= \lim_{\theta\to 0} \sin\theta = 0.$$

Thus, $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$

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Remarks:

L'Hospital's Rule 1 may be applied when evaluating limits

as $x \to \infty$

or

as $x \to -\infty$

as long as the ratio of functions has the indeterminate form 0/0 and the differentiability condition is satisfied.

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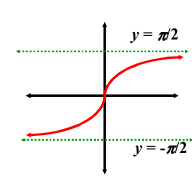
Illustration: Evaluate $\lim\limits_{x\to\infty} \frac{\frac{\pi}{2} - \text{Arc tan } x}{\frac{1}{x}}$, it if exists.

$$\lim_{x\to\infty} \frac{\frac{\pi}{2} - \text{Arc tan } x}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{1+x^2}$$

$$= 1.$$



$y = \text{Arc tan } x$

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Let f and g be functions. The function $\dfrac{f}{g}$ has the indeterminate form $\dfrac{\infty}{\infty}$ as x approaches a if

$$\lim_{x\to a} f(x)=\infty \quad \text{and} \quad \lim_{x\to a} g(x)=\infty.$$

Remark:

In the definition given above, a could be a real number or $\pm\infty$.

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Theorem 1.7.2 L' Hospital's Rule 2

Let f and g be functions which are differentiable on an open interval I containing a, except possibly at the number a. Suppose that for all $x \neq a$ in I, $g'(x) \neq 0$.

If $\dfrac{f}{g}$ has the indeterminate form $\dfrac{\pm\infty}{\pm\infty}$ as x approaches a, and $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

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Illustration: Evaluate $\displaystyle\lim_{x\to+\infty}\frac{x}{9^x}$, if it exists.

$$\lim_{x\to+\infty}\frac{x}{9^x} \quad \left(\frac{+\infty}{+\infty}\right)$$

$$= \lim_{x\to+\infty}\frac{D_x(x)}{D_x(9^x)}$$

$$= \lim_{x\to\infty}\frac{1}{9^x \ln 9} = 0$$

Thus, $\displaystyle\lim_{x\to+\infty}\frac{x}{9^x} = 0.$

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List of indeterminate forms

$\dfrac{f}{g}$	$\dfrac{0}{0}, \dfrac{+\infty}{+\infty}, \dfrac{+\infty}{-\infty}, \dfrac{-\infty}{+\infty}, \dfrac{-\infty}{-\infty}$
f^g	$0^0, \infty^0, 1^\infty$
$f \cdot g$	$0 \cdot \pm\infty$
$f - g$	$\infty - \infty$

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Remarks:

- L'Hospital's Rule 1 applies only when the indeterminate form is 0/0.
- L'Hospital's Rule 2 applies only when the indeterminate form is $\pm\infty/\pm\infty$.
- Limits of functions have indeterminate forms not mentioned above can be evaluated by using a theorem provided in MATH 36 or by rewriting the function so that the rewritten function takes any of the following forms: 0/0, $\pm\infty/\pm\infty$
- L'Hospital's Rule can be applied repetitively.

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Illustration: Evaluate $\displaystyle\lim_{x\to0^+} x^x$, if it exists.

$$\lim_{x\to0^+} x^x \quad (0^0)$$

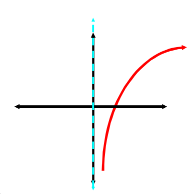
Let $y = x^x$.

$$\ln y = \ln(x^x)$$

$$\ln y = x \ln x$$

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+} x\ln x \quad (0\cdot -\infty)$$

$$= \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} \quad \left(\frac{-\infty}{+\infty}\right)$$



$y = \ln x$

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$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -\lim_{x\to 0^+} x = 0.$$

$$\lim_{x\to 0^+} \ln y = 0 \Rightarrow e^{\lim\limits_{x\to 0^+} \ln y} = e^0$$

$$\Rightarrow \lim_{x\to 0^+} e^{\ln y} = e^0$$

$$\Rightarrow \lim_{x\to 0^+} y = e^0 = 1 \quad \Rightarrow \lim_{x\to 0^+} x^x = 1.$$

Illustration: Evaluate $\lim\limits_{x\to\infty} \left(1+\frac{3}{x}\right)^x$, if it exists.

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x \quad \left(1^\infty\right)$$

Let $y = \left(1+\frac{3}{x}\right)^x$.

$$\ln y = \ln\left(1+\frac{3}{x}\right)^x$$

$$\ln y = x \ln\left(1+\frac{3}{x}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\frac{3}{x}\right) \quad (\infty \cdot 0)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\frac{3}{x}\right) \quad (\infty \cdot 0)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot \frac{\cancel{-}3}{\cancel{x^2}}}{\frac{\cancel{-}1}{\cancel{x^2}}}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{3}{1+\frac{3}{\cancel{x}}_{\,0}} = 3$$

$$\lim_{x\to\infty} \ln y = 3$$

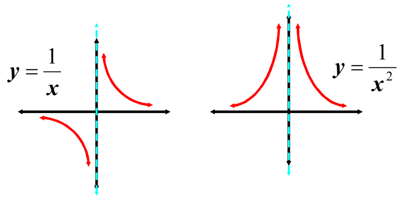
$$e^{\lim\limits_{x\to\infty} \ln y} = e^3$$

$$\lim_{x\to\infty} e^{\ln y} = e^3$$

$$\lim_{x\to\infty} y = e^3 \Rightarrow \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = e^3$$

Illustration: Evaluate $\lim\limits_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{x^2}\right)$, if it exists.

$$\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{x^2}\right) \quad (\infty - \infty)$$



$$\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{x^2}\right) = \lim_{x\to 0^+} \frac{x-1}{x^2} \quad \left(\frac{-1}{0^+}\right)$$

$$= -\infty.$$

The

End!

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