APPLICATIONS OF PARTIAL DERIVATIVES

Chapter 3

DIFFERENTIABILITY AND THE TOTAL DIFFERENTIAL

Chapter 3 Section 1

OUTLINE

- Actual change in $f(x,y)$

  Vs.

  Approximate change in $f(x,y)$

- When is $f(x,y)$ differentiable?

- Relationship between continuity and differentiability of $f(x,y)$

Definition.

If f is a function of two variables x and y, the ***increment of f at the point*** (x_0, y_0), denoted by $\Delta f(x_0, y_0, \Delta x, \Delta y)$ is given by

$$\Delta f\left(x_0, y_0, \Delta x, \Delta y\right) = f\left(x_0 + \Delta x, y_0 + \Delta y\right) - f\left(x_0, y_0\right)$$

For brevity, we may denote $\Delta f(x_0, y_0, \Delta x, \Delta y)$ by $\Delta f(x_0, y_0)$.

Example: If $f(x, y) = 2x^2 + 5xy - 4y^2$, find $\Delta f(x_0, y_0)$.

$$\Delta f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= 2(x_0 + \Delta x)^2 + 5(x_0 + \Delta x)(y_0 + \Delta y) - 4(y_0 + \Delta y)^2$$
$$-\left(2x_0^2 + 5x_0 y_0 - 4y_0^2\right)$$

$$= 2\left(x_0^2 + 2x_0\Delta x + \Delta^2 x\right) + 5\left(x_0 y_0 + y_0\Delta x + x_0\Delta y + \Delta x\Delta y\right)$$
$$-4\left(y_0^2 + 2y_0\Delta y + \Delta^2 y\right) - 2x_0^2 - 5x_0 y_0 + 4y_0^2$$

$$= 2x_0^2 + 4x_0\Delta x + 2\Delta^2 x + 5x_0 y_0 + 5y_0\Delta x + 5x_0\Delta y + 5\Delta x\Delta y$$
$$-4y_0^2 - 8y_0\Delta y - 4\Delta^2 y - 2x_0^2 - 5x_0 y_0 + 4y_0^2$$

$$= 4x_0\Delta x + 2\Delta^2 x + 5y_0\Delta x + 5x_0\Delta y + 5\Delta x\Delta y - 8y_0\Delta y - 4\Delta^2 y.$$

Definition.

If f is a function of two variables x and y, and the increment of f at the point (x_0, y_0) can be written as

$$\Delta f(x_0, y_0) = D_1 f(x_0, y_0)\Delta x + D_2 f(x_0, y_0)\Delta y + \in_1 \Delta x + \in_2 \Delta y$$

where $\in_1$ and $\in_2$ are functions of Δx and Δy such that $\in_1 \to 0$ and $\in_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$, then f is said to be ***differentiable*** at the point (x_0, y_0).

Example: If $f(x, y) = 2x^2 + 5xy - 4y^2$, show that f is differentiable in R^2.

Solution:

We need to show that

$$\Delta f(x_0, y_0) = D_1 f(x_0, y_0)\Delta x + D_2 f(x_0, y_0)\Delta y + \in_1 \Delta x + \in_2 \Delta y$$

where $\in_1$ and $\in_2$ are functions of Δx and Δy such that $\in_1 \to 0$ and $\in_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

$$\Delta f\left(x_0, y_0\right) = f\left(x_0 + \Delta x, y_0 + \Delta y\right) - f\left(x_0, y_0\right)$$

$$= 4x_0\Delta x + 2\Delta^2 x + 5y_0\Delta x + 5x_0\Delta y + 5\Delta x\Delta y - 8y_0\Delta y - 4\Delta^2 y.$$

If $f(x, y) = 2x^2 + 5xy - 4y^2$, then

$$\Rightarrow D_1 f\left(x, y\right) = 4x + 5y \quad \Rightarrow D_1 f\left(x_0, y_0\right) = 4x_0 + 5y_0$$

$$\Rightarrow D_2 f\left(x, y\right) = 5x - 8y \quad \Rightarrow D_2 f\left(x_0, y_0\right) = 5x_0 - 8y_0$$

Theorem.

If a function f of two variables is differentiable at a point, then it is continuous at that point.

(Differentiability implies continuity.)

Remarks:

1) The converse of this theorem is not always true. A function which is continuous at a point may not be differentiable at that point.

2) But if a function is not continuous at a point then it is not differentiable at that point.

Theorem.

Let f be a function of x and y such that $D_1 f$ and $D_2 f$ exist on an open disk $B(P_0; r)$. If $D_1 f$ and $D_2 f$ are continuous at P_0, then f is differentiable at P_0.

Example: Show that the indicated function is differentiable at all points in its domain.

a. $f(x,y)=\dfrac{3x}{x^2+y^2}$

The domain of f is the set of points in R^2 except (0,0).

$$D_1 f(x,y)=\frac{(x^2+y^2)\cdot 3-3x(2x)}{(x^2+y^2)^2}=\frac{3x^2+3y^2-6x^2}{(x^2+y^2)^2}=\frac{3y^2-3x^2}{(x^2+y^2)^2}$$

$$D_2 f(x,y)=\frac{(x^2+y^2)\cdot 0-3x\cdot 2y}{(x^2+y^2)^2}=\frac{-6xy}{(x^2+y^2)^2}$$

Since $D_1 f$ and $D_2 f$ are rational functions, each is continuous at every point in R^2 except at (0,0). Hence f is differentiable at all points in its domain.

b. $g(x,y) = y \ln x$

Solution:

The domain of g is $\{(x,y) \in R^2 : x > 0\}$.

$$g_x(x,y) = \frac{y}{x} \quad \text{and} \quad g_y(x,y) = \ln x$$

Since g_x is continuous at the point (x, y) whenever $x \neq 0$, and g_y is continuous at the point (x, y) whenever $x > 0$, then g is differentiable at all points in its domain.

Example: Let $f(x,y)=\begin{cases}\dfrac{3x^2y^2}{x^4+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$

Show that $D_1 f(0,0)$ and $D_2 f(0,0)$ exist but f is not differentiable at (0, 0).

solution:

$$D_1 f(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}$$

$$=\lim_{x\to 0}\frac{0-0}{x}$$

$$=\lim_{x\to 0}\frac{0}{x}$$

$$=0$$

$$D_2 f(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}$$

$$=\lim_{y\to 0}\frac{0-0}{y}$$

$$=\lim_{y\to 0}\frac{0}{y}$$

$$=0$$

Take S_1 as the set of points on the y- axis and S_2 as the set of all points on the line given by $y = x$.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_1}} \frac{3x^2y^2}{x^4+y^4} = \lim_{y\to 0}\frac{0}{y^4} = \lim_{y\to 0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_2}} \frac{3x^2y^2}{x^4+y^4} = \lim_{x\to 0}\frac{3x^4}{2x^4} = \lim_{x\to 0}\frac{3}{2} = \frac{3}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^4+y^4} \quad \text{does not exist.}$$

Therefore f is discontinuous at (0,0) and hence f is not differentiable at (0,0).

Definition.

If f is a function of two variables x and y, and f is differentiable at (x, y), the ***total differential of f*** is the function ***df*** such that

$$df(x, y, \Delta x, \Delta y) = D_1 f(x, y)\Delta x + D_2 f(x, y)\Delta y$$

Example: If $f(x, y) = 2x^2 + 5xy - 4y^2$,

Find: a. $df(2,-1,\Delta x,\Delta y)$ b. $df(2,-1,-0.01,0.02)$

solution:

$$df(x,y,\Delta x,\Delta y) = D_1 f(x,y)\Delta x + D_2 f(x,y)\Delta y$$

$$df(x,y,\Delta x,\Delta y) = (4x+5y)\Delta x + (5x-8y)\Delta y$$

$$df(2,-1,\Delta x,\Delta y) = (4\cdot 2+5(-1))\Delta x + (5\cdot 2-8(-1))\Delta y$$

$$= (8-5)\Delta x + (10+8)\Delta y$$

$$= 3\Delta x + 18\Delta y$$

$$df(2,-1,-0.01,0.02) = 3(-0.01) + 18(0.02)$$

$$= -0.03 + 0.36 = 0.33$$

Remarks: If $z = f(x, y)$, then

$$1.\quad dz = D_1 f(x,y)\Delta x + D_2 f(x,y)\Delta y$$

$$2.\quad dz = D_1 f(x,y)dx + D_2 f(x,y)dy$$

$$3.\quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Definition.

If f is a function of n variables $x_{1,} x_2,.., x_n$, and A is the point $\left(a_{1,} a_2,.., a_n\right)$, the ***increment of f*** at A is given by

$$\Delta f\left(A\right)= f\left(a_1+\Delta x_1, a_2+\Delta x_2,...., a_n+\Delta x_n\right)- f\left(A\right)$$

Definition.

If f is a function of n variables $x_{1,} x_2,.., x_n$, and the increment of f at the point A can be written as

$$\Delta f(A) = D_1 f(A)\Delta x_1 + D_2 f(A)\Delta x_2 + \ldots + D_n f(A)\Delta x_n$$
$$+ \in_1 \Delta x_1 + \in_2 \Delta x_2 + \ldots + \in_n \Delta x_n$$

where

$$\in_1 \to 0, \in_2 \to 0, \ldots, \in_n \to 0, \text{ as } (\Delta x_1, \Delta x_2, \ldots, \Delta x_n) \to (0,0,\ldots,0),$$

then f is said to be ***differentiable*** at A.

Definition.

If f is a function of n variables $x_{1,} x_2, .., x_n$, and f is differentiable at a point A, the ***total differential of f*** is given by

$$df\left(A, \Delta x_1, \Delta x_2, ..., \Delta x_n\right) = D_1 f\left(A\right)\Delta x_1 + D_2 f\left(A\right)\Delta x_2 + ... + D_n f\left(A\right)\Delta x_n .$$

Remark:

If w is a function of n variables $x_{1,} x_2, .., x_n$ then

$$dw = \frac{\partial w}{\partial x_1} dx_1 + \frac{\partial w}{\partial x_2} dx_2 + ... + \frac{\partial w}{\partial x_n} dx_n$$

$$\Delta f(A) = f(a_1 + \Delta x_1, a_2 + \Delta x_2, \ldots., a_n + \Delta x_n) - f(A)$$

$$\Delta f(A) = D_1 f(A)\Delta x_1 + D_2 f(A)\Delta x_2 + \ldots + D_n f(A)\Delta x_n$$
$$+ \in_1 \Delta x_1 + \in_2 \Delta x_2 + \ldots + \in_n \Delta x_n$$

$$df(A, \Delta x_1, \Delta x_2, \ldots., \Delta x_n) = D_1 f(A)\Delta x_1 + D_2 f(A)\Delta x_2 + \ldots + D_n f(A)\Delta x_n.$$

When f is differentiable and Δx_i is “small” for $i = 1$ to n, then $\Delta f \approx df$

END