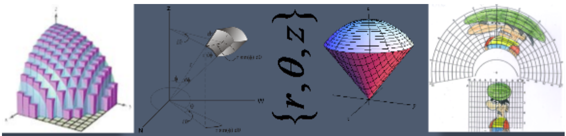


CHAPTER 4

Multiple Integration

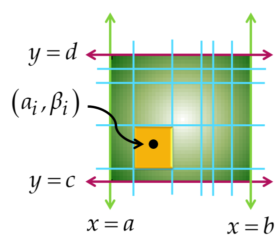
Topics to be discussed:

- Double integration
  - In rectangular coordinates, In polar coordinates
- Triple integration
  - In rectangular coordinates, In cylindrical coordinates, In spherical coordinates
- Applications
  - Area, Volume

4.1 Double Integral in Rectangular Coordinates

Let R be a region in the plane which is bounded by $x=a, x=b, y=c$ and $y=d$, where $a<b$ and $c<d$.

$Area_{R_i} = \Delta_i x \cdot \Delta_i y$



4.1 Double Integral in Rectangular Coordinates

Now, obtain $\sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y$

If $\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y$ exists, then this limit is called the ***double integral*** of f over the region R.

Double Integral of *f* over *R*

In symbols,

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y$$

Double Integral of *f* over *R*

REMARKS:

- $dA = dx\,dy = dy\,dx$
- Double integrals have the same kind of domain additivity property that single integrals have.
- Double integrals are evaluated as iterated integrals.

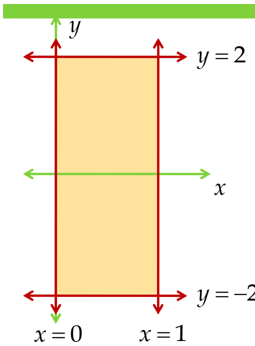
Evaluating Double Integrals

$$\int_{-2}^{2}\int_{0}^{1}(4-xy)\,dx\,dy \;=\int_{-2}^{2}\left[\int_{0}^{1}(4-xy)\,dx\right]dy$$

$$=\int_{-2}^{2}\left[4x-\frac{1}{2}x^2y\right]_0^1 dy \;=\int_{-2}^{2}\left(4-\frac{y}{2}\right)dy$$

$$=\left(4y-\frac{y^2}{4}\right)_{-2}^{2} =(8-1)\,-(-8-1) \;=16$$

Region of Integration



$$\int_{-2}^{2}\int_{0}^{1}(4-xy)\,dx\,dy$$

$$=\int_{0}^{1}\int_{-2}^{2}(4-xy)\,dy\,dx$$

$$=16$$

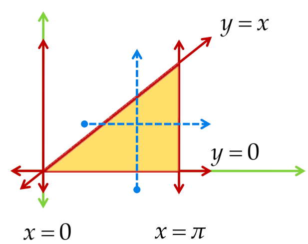
Evaluating Double Integrals

$$\int_0^{\pi}\int_0^{x}\frac{\sin x}{x}\,dy\,dx \;=\int_0^{\pi}\left(\int_0^{x}\frac{\sin x}{x}\,dy\right)dx$$

$$=\int_0^{\pi}\left(y\frac{\sin x}{x}\right)_0^{x}dx \;=\int_0^{\pi}\sin x\,dx$$

$$=(-\cos x]_0^{\pi} \;=2$$

Region of Integration

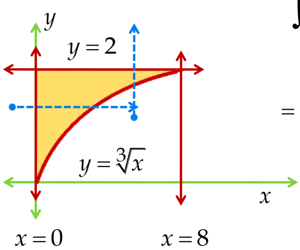


$$\int_0^{\pi}\int_0^{x}\frac{\sin x}{x}\,dy\,dx$$

$$=2$$

$$=\int_0^{\pi}\int_y^{\pi}\frac{\sin x}{x}\,dx\,dy$$

Evaluating Double Integrals



$$\int_0^{8}\int_{\sqrt[3]{x}}^{2}\frac{dy\,dx}{y^4+1}$$

$$=\int_0^{2}\int_0^{y^3}\frac{dx\,dy}{y^4+1}$$

Evaluating Double Integrals

$$\int_0^{2}\int_0^{y^3}\frac{dx\,dy}{y^4+1} \;=\int_0^{2}\left(\int_0^{y^3}\frac{dx}{y^4+1}\right)dy$$

$$=\int_0^{2}\frac{y^3}{y^4+1}dy \;=\frac{1}{4}ln\left(y^4+1\right)\Big|_0^2$$

$$=\frac{1}{4}ln(17)$$

Exercise.

Sketch the region of integration. Write an integral with the order of integration reversed. Then evaluate both integrals.

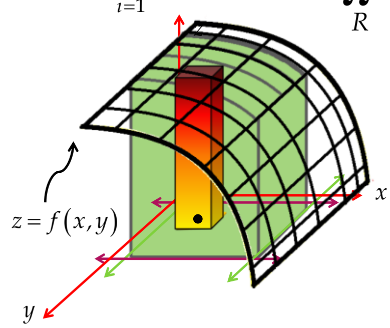
1. $\int_0^2 \int_0^{4-x} 2x\,dy\,dx \qquad = \frac{32}{3}$
2. $\int_0^1 \int_{x^2}^{x} \sqrt{x}\,dy\,dx \qquad = \frac{4}{35}$
3. $\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy \qquad = \frac{-4}{3}$
4. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy \qquad = \frac{9}{2}$

Application of Double Integrals

REMARKS:

- If $f(x,y) \geq 0$ for all (x,y) in a region $\boldsymbol{R}$, the double integral of f over $\boldsymbol{R}$ is the volume of the solid whose base is $\boldsymbol{R}$ and whose height at a point (x,y) in $\boldsymbol{R}$ is $f(x,y)$.
- If $f(x,y) = 1$, the double integral of f over a region $\boldsymbol{R}$ is just the area of $\boldsymbol{R}$

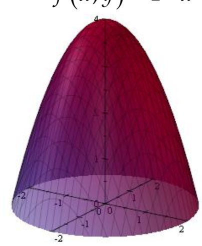
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y = \iint_R f(x,y)\,dA$$

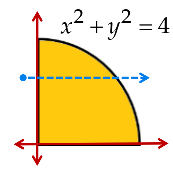


Exercise.

SET-UP the double integral which gives the volume of the solid described.

a. Solid in the first octant bounded by $f(x,y) = 4 - x^2 - y^2$



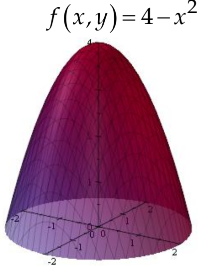


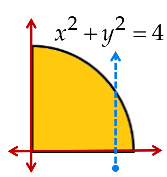
$$V = \int_0^2 \int_0^{\sqrt{4-y^2}} \left(4 - x^2 - y^2\right) dx\,dy$$

Exercise.

SET-UP the double integral which gives the volume of the solid described.

a. Solid in the first octant bounded by $f(x,y) = 4 - x^2 - y^2$

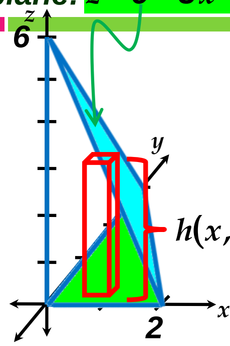




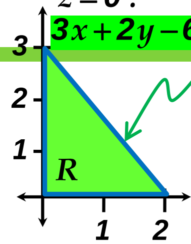
$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\,dx$$

b. Set-up the iterated integral that solves for the volume of the tetrahedron bounded by the plane $3x + 2y + z - 6 = 0$ and the coordinate planes.

plane: $z = 6 - 3x - 2y$



$z = 0$: $3x + 2y - 6 = 0$



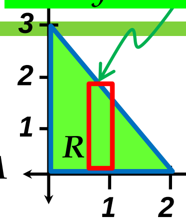
$h(x,y) = 6 - 3x - 2y$

$z = 0$: $3x + 2y - 6 = 0$

$$V = \iint_R h(x,y)\,dA$$

$$= \iint_R (6 - 3x - 2y)\,dA$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} (6 - 3x - 2y)\,dy\,dx$$

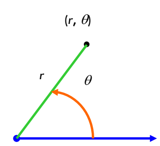


Recall:

In polar form, a point has coordinates

(r, θ),

where r is the directed distance of the point from the pole and θ is the radian measure of the angle which the terminal side of θ makes with the positive side of the x-axis, also known as the *polar axis*.

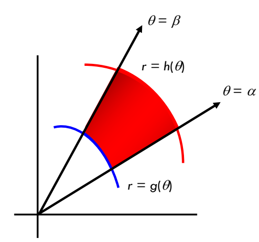


$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

4.2 The Double Integral in Polar Coordinates



$$A = \frac{1}{2}\int_\alpha^\beta \left[h^2(\theta) - g^2(\theta)\right] d\theta$$

$$= \frac{1}{2}\int_\alpha^\beta r^2 \Big]_{g(\theta)}^{h(\theta)} d\theta$$

$$= \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\,dr\,d\theta$$

In polar coordinates,

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} g(r,\theta)\,r\,dr\,d\theta$$

$x = r\cos\theta$

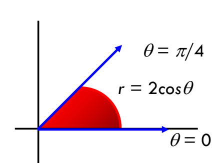
$y = r\sin\theta$

Example 1

Consider the region R bounded by the graphs of $r = 2\cos\theta$, $\theta = 0$ and $\theta = \pi/4$.

Set-up $\iint_R (y+2)\,dA$

Solution:



$$A = \frac{1}{2}\int_\alpha^\beta \left[h^2(\theta) - g^2(\theta)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[2\cos(\theta)\right]^2 d\theta$$

$$= \int_0^{\pi/4} \int_0^{2\cos\theta} r\,dr\,d\theta$$

$$\iint_R (y+2)\,dA = \int_0^{\pi/4} \int_0^{2\cos\theta} \qquad r\,dr\,d\theta$$

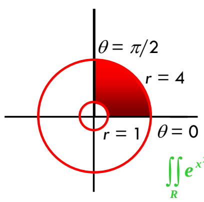
Example 2

Consider the region R in the first quadrant enclosed by the graphs of

$$x^2+y^2=16 \quad \text{and} \quad x^2+y^2=1\ .$$

Set-up $\iint\limits_R e^{x^2+y^2}\,dA$.

Solution:



$$A=\frac{1}{2}\int_\alpha^\beta\left[h^2(\theta)-g^2(\theta)\right]d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[4^2-1^2\right]d\theta$$

$$=\int_0^{\pi/2}\int_1^4 r\,dr\,d\theta$$

$$\iint\limits_R e^{x^2+y^2}\,dA=\int_0^{\pi/2}\int_1^4 \qquad r\,dr\,d\theta$$

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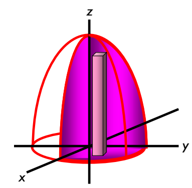
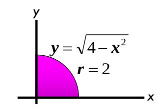
Example 3

Set-up the double integral in polar coordinates which gives the volume of the solid in the first octant which is enclosed by the graph of

$$z=4-x^2-y^2.$$

Solution:





$$A=\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta$$

$$h=4-x^2-y^2$$

$$=4-r^2$$

$$V=\int_0^{\pi/2}\int_0^2\left(4-r^2\right)r\,dr\,d\theta$$

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Exercise 2

a. Use double integration to solve the following:

1. Find the area inside the circle $r=2\sqrt{3}\sin\theta$ which is outside the circle $r = 3$.

   Ans. $\frac{1}{2}\left(3\sqrt{3}-\pi\right)$ sq. units

2. Find the area inside the circle $r=6$ which lies to the righ of the parabola $r=3\sec^2\left(\frac{\theta}{2}\right)$.

   Ans. $(18\pi - 24)$ sq. units

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3. Find the first-quadrant area bounded by the curve $r=2\tan\theta$ and the lines $r=\sqrt{2}\sec\theta$ and $\theta=0$.

   Ans. $\left(\frac{\pi}{2}-1\right)$ sq. units

4. Find the area enclosed by the cardioid $r=a(1+\cos\theta)$.

   Ans. $\frac{3\pi a^2}{2}$ sq. units

5. Find the area enclosed by the lemniscate $r^2=a^2\cos(2\theta)$.

   Ans. a^2 sq. units

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b. Use polar coordinates to evaluate the following. Sketch the region of integration.

6. $\int_0^3\int_0^{\sqrt{9-x^2}}\left(x^2+y^2\right)dy\,dx$ ***Ans.*** $\frac{81\pi}{8}$

7. $\int_0^{\frac{\sqrt{2}}{2}}\int_y^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\,dx\,dy$ ***Ans.*** $\frac{\pi}{12}$

c. Use polar coordinates to find the volume of the indicated solid. Sketch the solid.

8. Enclosed by the ellipsoid given by

   $9x^2+9y^2+z^2=9.$ ***Ans.*** 4π cu. units

9. Bounded by the paraboloid $z=4-x^2-y^2,$ the cylinder $x^2+y^2=1$ and the xy-plane.

   Ans. $\frac{7\pi}{2}$ cu. units

10. Enclosed by the sphere of radius a.

    Ans. $\frac{4\pi a^3}{3}$ cu. units

END