Triple Integrals in Rectangular Coordinates

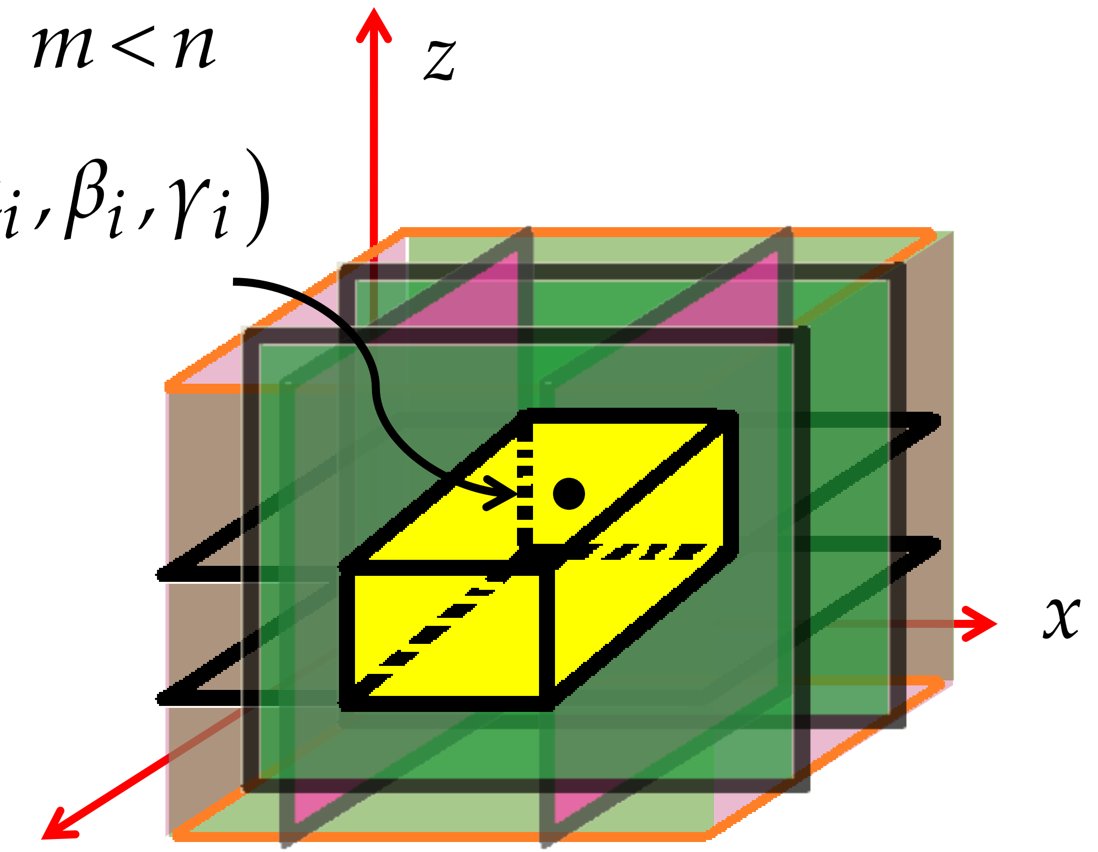
Chapter 4 Section 3

4.3 Triple Integral in Rectangular Coordinates

Let S be the solid which is bounded by the planes given by $x = a, x = b,\ y = c, y = d\ , z = m, z = n$

where $a < b,\ c < d,\ m < n$

(a_i, β_i, γ_i)



The volume of this sub-region is

$$V_i = \Delta x_i \Delta y_i \Delta z_i$$

4.3 Triple Integral in Rectangular Coordinates

Now, obtain $\sum_{i=1}^{n} f(a_i, \beta_i, \gamma_i)\Delta_i x \Delta_i y \Delta z_i$

If $\lim_{n\to\infty} \sum_{i=1}^{n} f(a_i, \beta_i, \gamma_i)\Delta_i x \Delta_i y \Delta z_i$ exists, then this limit is called the ***triple integral*** of f over the solid S.

Triple Integral of f over S

In symbols,

$$\iiint_S f(x, y, z)\, dV$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(a_i, \beta_i, \gamma_i)\, \Delta_i x \Delta_i y \Delta z_i$$

Triple Integral of f over S

REMARKS:

- $dV = dx\,dydz = dy\,dxdz = dz\,dxdy = \ldots$

- Triple integrals have the same kind of domain additivity property that single and double integrals have.

- Triple integrals are evaluated as iterated integrals.

Exercise. **SET-UP** then **EVALUATE** the triple integral over the described solid.

$$\iiint_S 12xze^y \, dydxdz$$

S: $-1 \le x \le 1,\ 0 \le y \le \ln 2,\ 1 \le z \le 2$

$$\int_0^{\ln 2}\int_1^2\int_{-1}^1 12xze^y \, dxdzdy = 0$$

$$\int_{-1}^1\int_0^{\ln 2}\int_1^2 12xze^y \, dzdydx = \int_{-1}^1\int_0^{\ln 2} 12xe^y \int_1^2 z\,dzdydx$$

$$= \int_{-1}^1\int_0^{\ln 2} 12xe^y \left(\frac{z^2}{2}\right)_1^2 dydx$$

Exercise. **SET-UP** then **EVALUATE** the triple integral over the described solid.

$$\int_{-1}^{1}\int_{0}^{ln 2} 18xe^{y}\,dydx \quad = \int_{-1}^{1}\int_{0}^{ln 2} 18xe^{y}\,dydx$$

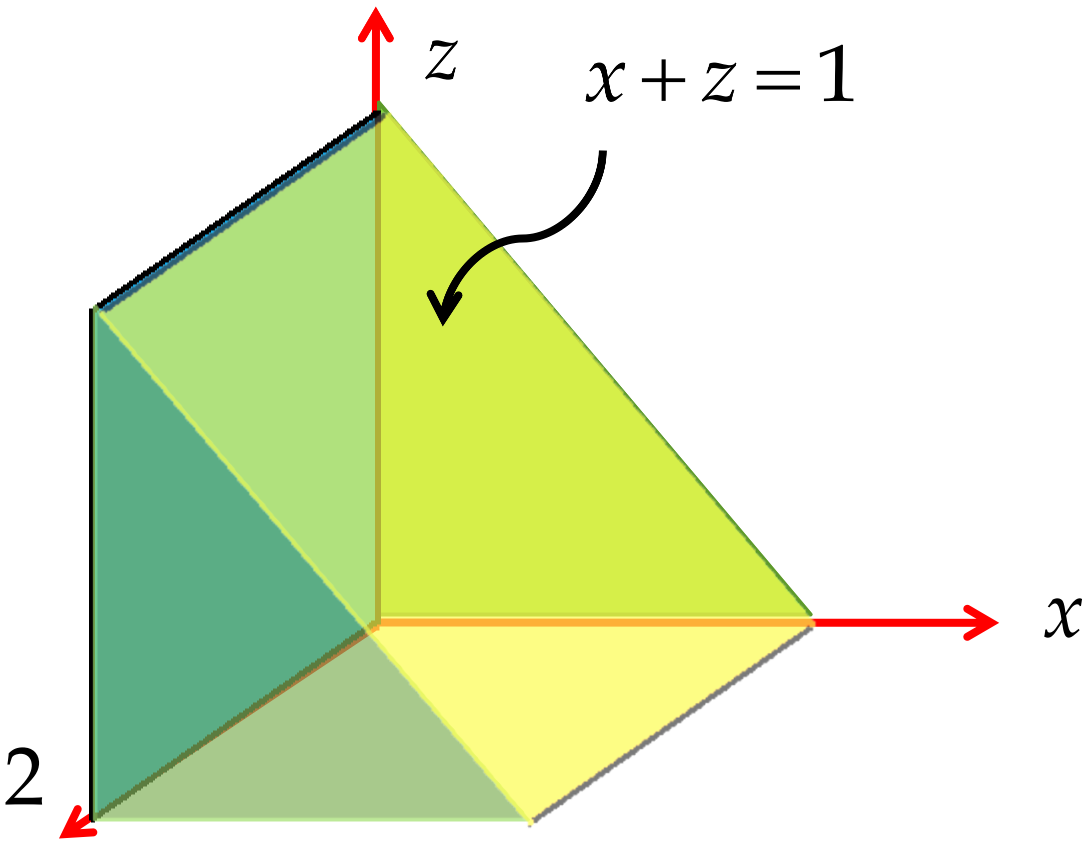
$$= \int_{-1}^{1} 18x \int_{0}^{ln 2} e^{y}\,dydx$$

$$= \int_{-1}^{1} 18x\left(e^{y}\right)_{0}^{ln 2} dx = \int_{-1}^{1} 18x dx$$

$$= \left(9x^{2}\right)_{-1}^{1} \quad = 0$$

Using Different Orders of Integration

Each of the following integrals gives the volume of the solid shown below.



$$\int_0^2\int_0^1\int_0^{1-x} dzdxdy$$

$$\int_0^1\int_0^2\int_0^{1-z} dxdydz$$

$$\int_0^1\int_0^{1-x}\int_0^2 dydzdx$$

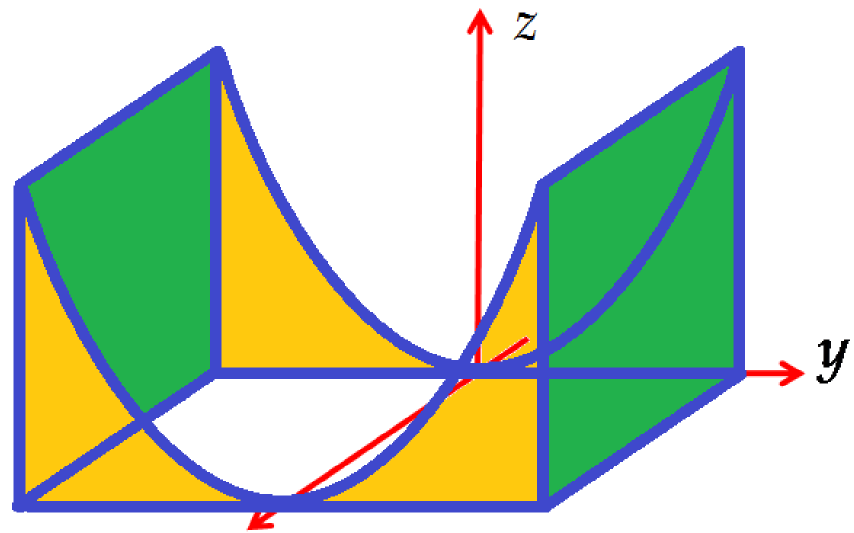
$$\int_0^1\int_0^{1-z}\int_0^2 dydxdz$$

Exercise. **SET-UP** a triple integral that gives the volume of the described solid.

a. Between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1$ and $y = 1$

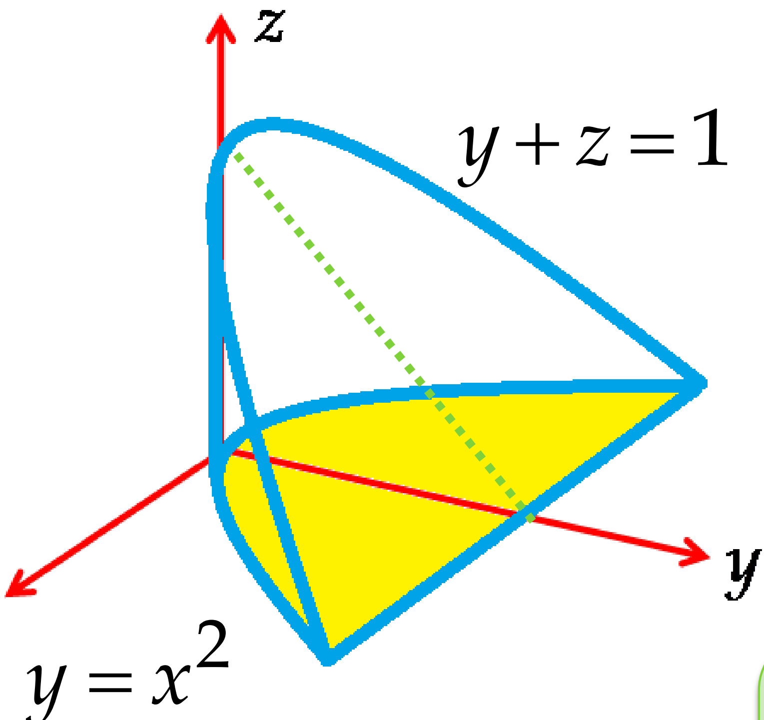
$$V = 2\int_0^1 \int_0^1 \int_0^{y^2} dz\,dx\,dy$$

$$V = 2\int_0^1 \int_{\sqrt{z}}^1 \int_0^1 dx\,dy\,dz$$



Exercise. **SET-UP** a triple integral that gives the volume of the described solid.

b.



$$V = 2\int_0^1 \int_0^{\sqrt{y}} \int_0^{1-y} dz\,dx\,dy$$

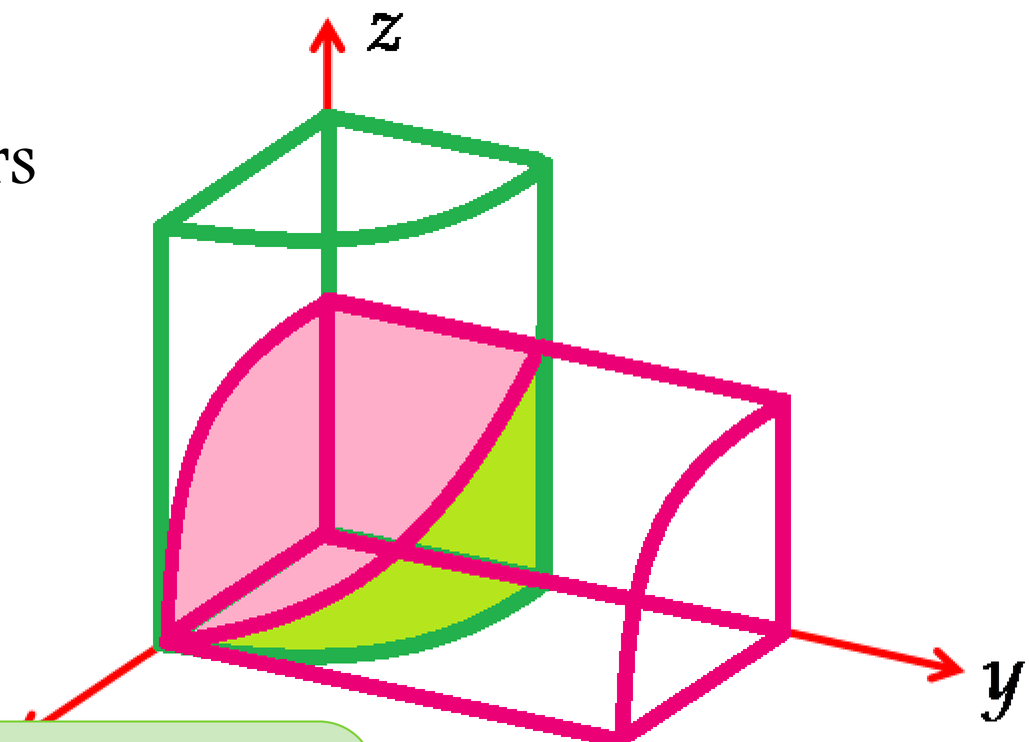
$$V = 2\int_0^1 \int_0^{1-z} \int_0^{\sqrt{y}} dx\,dy\,dz$$

Exercise. **SET-UP** a triple integral that gives the volume of the described solid.

c. Region common to the interior of the cylinders

$x^2 + y^2 = 1$ and

$x^2 + z^2 = 1$



$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$$

Assign. Do as indicated.

1. Evaluate: $\displaystyle\int_{-1}^{0}\int_{e}^{2e}\int_{0}^{\pi/3} y\ln z\tan x\,dx\,dz\,dy$

2. SET-UP a triple integral that will give the volume of the solid in the 1st octant bounded below by the $xy-$plane, above by the plane $z=y$, and laterally by the cylinder $y^2=x$ and the plane $x=1$.

TC7: pp. 1115-1116, items 5 and 19 of Exercise 13.5

END