CMSC 141 Automata and Language Theory
Context-Free Languages

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Closure Properties for CFL

Like regular languages, CFLs are close under union, concatenation, and Kleene star

Closure Under Union

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Proof

Given two grammars for two context-free languages, with start symbols S and T. Rename the variables to ensure that the two grammars will not share any variable. Then construct a grammar for the union of the two languages by taking all the rules of both grammars and adding a new start state Z with rules $Z \to S \mid T$

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$L_1 \cup L_2 \Rightarrow$

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- $|vxy| \leq p$

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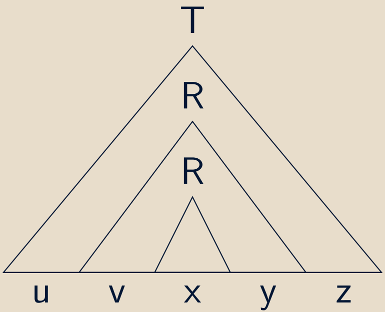

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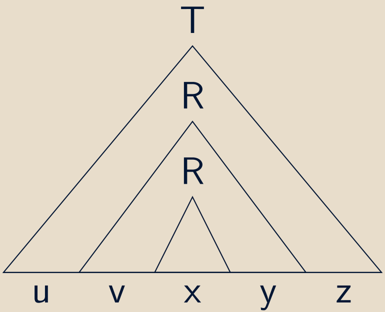

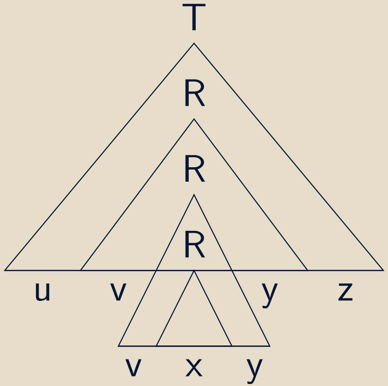

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  Can be proven using proof by contradiction and pumping lemma

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- One of the simplest non-CFL is $L = \{a^n b^n c^n : n > 0\} = \{abc, aabbcc, ...\}$
  Can be proven using proof by contradiction and pumping lemma
- How can we still extend PDAs? 2 stacks??

Some applications of CFL

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- Compiler Design/Programming Language Design

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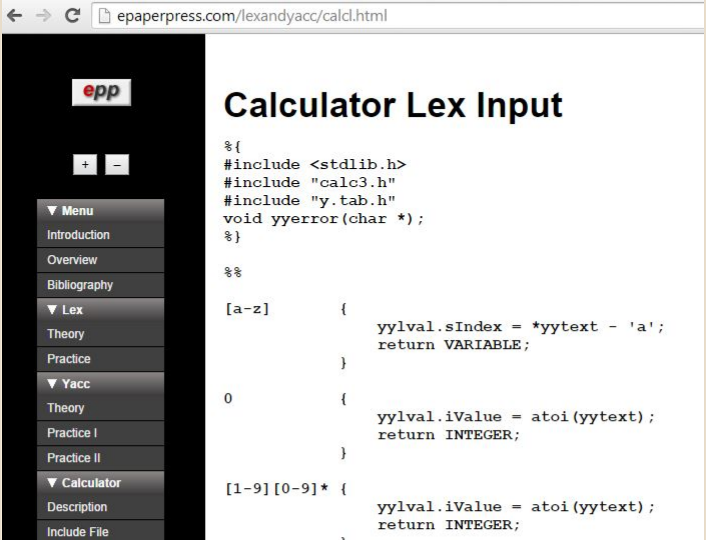
- Compiler Design/Programming Language Design
- Lindenmayer Systems (L-Systems)

Can be used in tools like Lex/Flex and Yacc/Bison

From http://epaperpress.com/lexandyacc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Include File

Calculator Lex Input

```
%{
#include <stdlib.h>
#include "calc3.h"
#include "y.tab.h"
void yyerror(char *);
%}

%%

[a-z]       {
                yylval.sIndex = *yytext - 'a';
                return VARIABLE;
            }

0           {
                yylval.iValue = atoi(yytext);
                return INTEGER;
            }

[1-9][0-9]* {
                yylval.iValue = atoi(yytext);
                return INTEGER;
```

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From http://epaperpress.com/lexandyacc

epaperpress.com/lexandyacc/calcy.html

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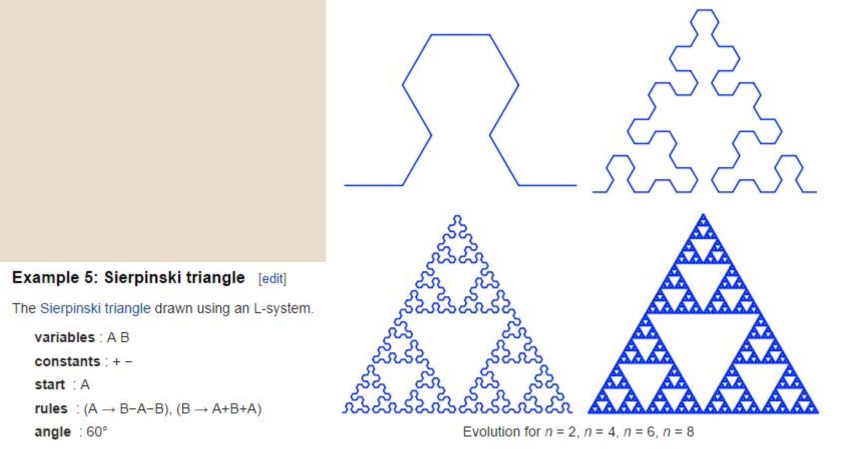
```
%nonassoc UMINUS

%type <nPtr> stmt expr stmt_list

%%

program:
        function                { exit(0); }
        ;

function:
          function stmt         { ex($2); freeNode($2); }
        | /* NULL */
        ;

stmt:
          ';'                               { $$ = opr(';', 2, NULL, NULL); }
        | expr ';'                          { $$ = $1; }
        | PRINT expr ';'                    { $$ = opr(PRINT, 1, $2); }
        | VARIABLE '=' expr ';'             { $$ = opr('=', 2, id($1), $3); }
        | WHILE '(' expr ')' stmt           { $$ = opr(WHILE, 2, $3, $5); }
        | IF '(' expr ')' stmt %prec IFX { $$ = opr(IF, 2, $3, $5); }
        | IF '(' expr ')' stmt ELSE stmt { $$ = opr(IF, 3, $3, $5, $7); }
        | '{' stmt_list '}'                 { $$ = $2; }
        ;

stmt_list:
          stmt                  { $$ = $1; }
        | stmt_list stmt        { $$ = opr(';', 2, $1, $2); }
        ;

expr:
          INTEGER               { $$ = con($1); }
        | VARIABLE              { $$ = id($1); }
        | '-' expr %prec UMINUS { $$ = opr(UMINUS, 1, $2); }
        | expr '+' expr         { $$ = opr('+', 2, $1, $3); }
```

Lindenmayer Systems

grammar-like structures for drawing fractals


Example 5: Sierpinski triangle [edit]

The Sierpinski triangle drawn using an L-system.

variables : A B

constants : + −

start : A

rules : (A → B−A−B), (B → A+B+A)

angle : 60°

Evolution for n = 2, n = 4, n = 6, n = 8

From Wikipedia

References

- Previous slides on CMSC 141
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- JFLAP, www.jflap.org
- Various online LaTeX and Beamer tutorials