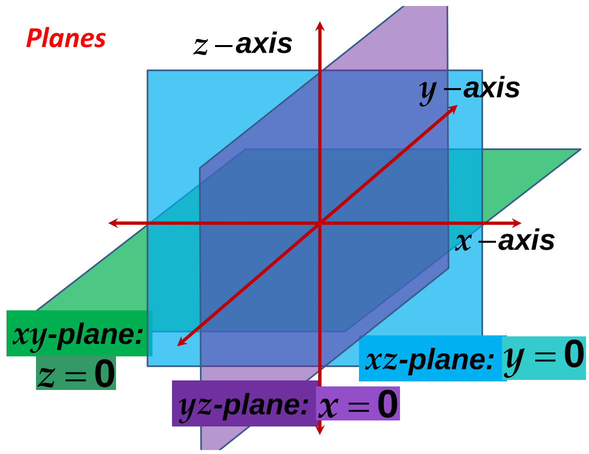
2.5
LINES and PLANES
in $\mathbb{R}^3$

Planes

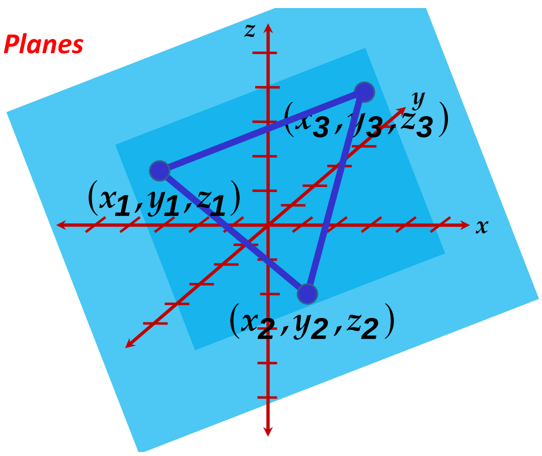


Planes

A plane can be uniquely determined by any of the following:

- ✓ three non-collinear points
- ✓ a line and a point not on the line
- ✓ two lines with one point of intersection
- ✓ two parallel lines

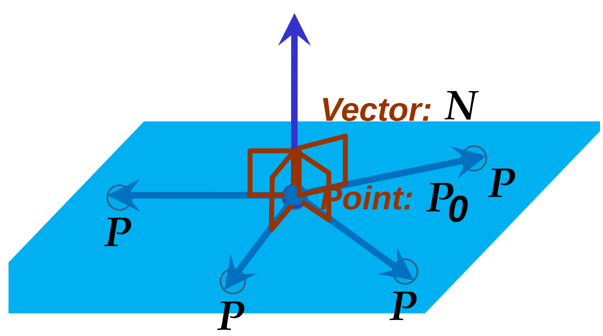
Planes



Planes

If N is a given non-zero vector and P_0 is a point, then the set of all points P for which $\overrightarrow{P_0P}$ and N are orthogonal is a **PLANE** through P_0 and having N as a normal vector.

Planes



Equation of a plane in 3D

Point on the plane:

$$P_0(x_0, y_0, z_0)$$

Normal vector to the plane:

$$N = \langle a, b, c \rangle$$

Standard equation of the plane:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Equation of a plane in 3D

General equation of a plane:

$$\boxed{ax + by + cz + d = 0}$$

if a, b and c are not all zero, $\langle a, b, c \rangle$ is a normal vector to the plane

Remark

Two planes are parallel if their normal vectors are parallel,

Two planes are perpendicular if their normal vectors are orthogonal,

Remark

Parallel planes

normal vectors are scalar multiples of each other

Perpendicular planes

dot product of normal vectors is zero

Example. Determine the equation of the given plane.

1. plane Ω through the point $(4, -2, -3)$ and perpendicular to the vector $\langle 2, 1, 3 \rangle$

Solution:

$$P_0(4, -2, -3)$$

$$N = \langle 2, 1, 3 \rangle$$

Solution (continued)

$$P_0(4, -2, -3) \quad N = \langle 2, 1, 3 \rangle$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Rightarrow 2(x - 4) + 1(y + 2) + 3(z + 3) = 0$$

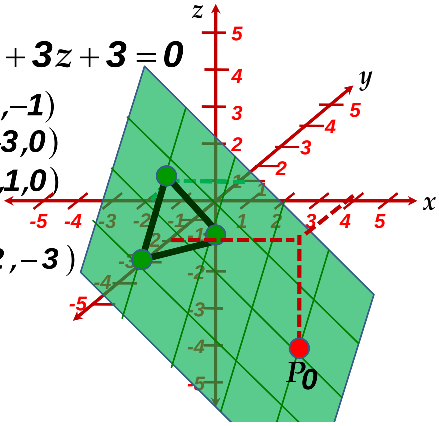
$$\Rightarrow \boxed{2x + y + 3z + 3 = 0}$$

Graph

$$2x+y+3z+3=0$$

$P_1(0,0,-1)$
$P_2(0,-3,0)$
$P_3(-2,1,0)$

$P_0(4,-2,-3)$



Example. Determine the equation of the given plane.

2. plane K through the point $(-1,7,4)$ and parallel to the plane $M: 2x-y+3z-5=0$

Solution:

$P_0(-1,7,4)$

$N_K = N_M = \langle 2,-1,3\rangle$

Solution (continued)

$P_0(-1,7,4)\quad N_K=\langle 2,-1,3\rangle$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\Rightarrow 2(x+1)-1(y-7)+3(z-4)=0$$

$$\Rightarrow \boxed{2x-y+3z-3=0}$$

Graph

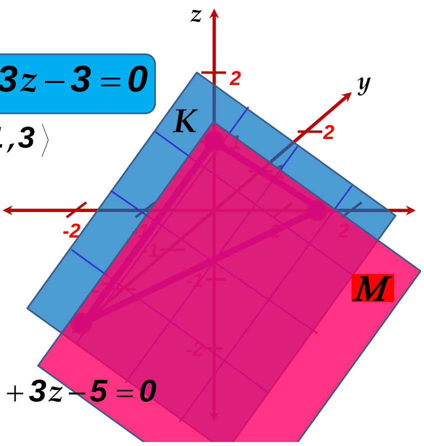
$$\boxed{2x-y+3z-3=0}$$

x-intercept: $y=0; z=0$	$\frac{3}{2}$	$P_1\left(\frac{3}{2},0,0\right)$
y-intercept: $x=0; z=0$	-3	$P_2(0,-3,0)$
z-intercept: $x=0; y=0$	1	$P_3(0,0,1)$

Graph

$$\boxed{2x-y+3z-3=0}$$

$N=\langle 2,-1,3\rangle$



$M: 2x-y+3z-5=0$

Example. Determine the equation of the given plane.

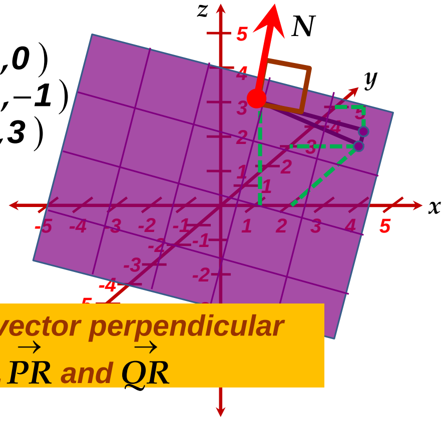
3. plane Γ containing the points $P(2,3,0)$, $Q(0,5,-1)$ and $R(1,0,3)$

Solution:

normal vector ? ? ?

Solution

$P(2,3,0)$
$Q(0,5,-1)$
$R(1,0,3)$



N is a vector perpendicular to $\vec{PQ}$, $\vec{PR}$ and $\vec{QR}$

Solution (continued)

N is a vector perpendicular to $\vec{PQ}$, $\vec{PR}$ and $\vec{QR}$

$P(2,3,0)\ Q(0,5,-1)\ R(1,0,3)$

$\vec{PQ} = \langle -2,2,-1 \rangle \quad \vec{PR} = \langle -1,-3,3 \rangle$

$$N = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ -1 & -3 & 3 \end{vmatrix} = \langle 3,7,8 \rangle$$

Equation of a plane in 3D

Point on the plane:

$$P_0(x_0, y_0, z_0)$$

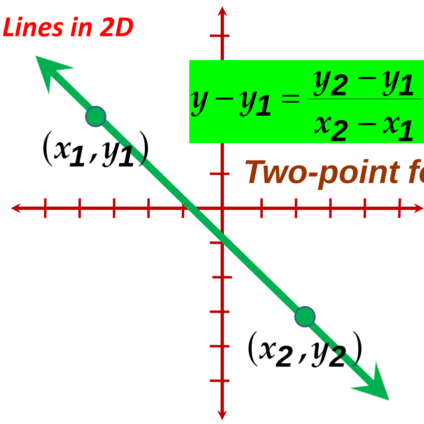
Normal vector to the plane:

$$N = \langle a,b,c \rangle$$

Standard equation of the plane:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Lines in 2D



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Two-point form

Lines in 2D

Point-slope form $y - y_1 = m(x - x_1)$

Slope-intercept form $y = mx + b$

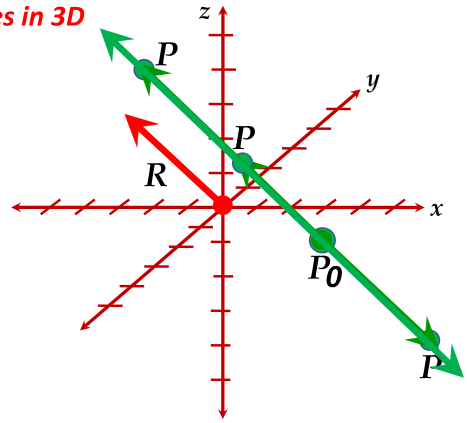
Intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$

General equation $Ax + By + C = 0$

Lines in 3D

If R is a given non-zero vector and P_0 is a point, then the set of all points P for which $\vec{P_0P}$ is parallel to R is a LINE through P_0 and parallel to R.

Lines in 3D



Lines in 3D

Let L be a line that contains the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $R = \langle a, b, c \rangle$.

Using t as a parameter,

PARAMETRIC EQUATIONS of L

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Lines in 3D

Let L be a line that contains the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $R = \langle a, b, c \rangle$.

SYMMETRIC EQUATIONS of L

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example. Determine the parametric and symmetric equations of the given line.

1. ***line L through the point $(1, -2, 3)$ and is parallel to the vector $\langle -2, 4, 5 \rangle$***

Solution:

$$P_0(1, -2, 3) \quad R = \langle -2, 4, 5 \rangle$$

Solution (continued)

$$P_0(1, -2, 3) \quad R = \langle -2, 4, 5 \rangle$$

PARAMETRIC EQUATIONS of L

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

$$\boxed{x = 1 - 2t \quad y = -2 + 4t \quad z = 3 + 5t}$$

Solution (continued)

$$P_0(1, -2, 3) \quad R = \langle -2, 4, 5 \rangle$$

SYMMETRIC EQUATIONS of L

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\boxed{\frac{x - 1}{-2} = \frac{y + 2}{4} = \frac{z - 3}{5}}$$

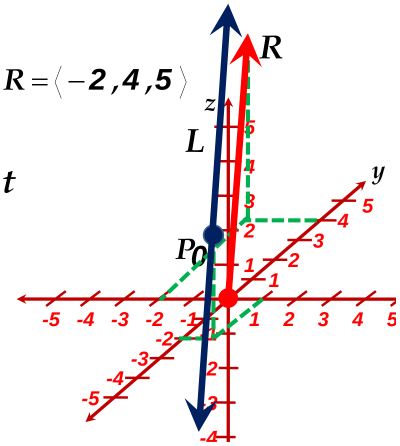
Graph

$P_0(1,-2,3)\quad R=\langle -2,4,5\rangle$

$x=1-2t$

$y=-2+4t$

$z=3+5t$



Using parametric equations

$x=1-2t\quad y=-2+4t\quad z=3+5t$

At $t=0$, $x=1\ \ y=-2\ \ z=3\ \ (1,-2,3)$

At $t=2$, $x=-3\ \ y=6\ \ z=13\ \ (-3,6,13)$

At $t=-1$, $x=3\ \ y=-6\ \ z=-2\ \ (3,-6,-2)$

At $t=\frac{1}{2}$, $x=-1\ \ y=0\ \ z=\frac{11}{2}\ \ \left(-1,0,\frac{11}{2}\right)$

Example. Determine the parametric and symmetric equations of the given line.

2. line M through the points $P(2,3,0)$ and $Q(4,5,-1)$

Solution:

M is parallel to vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ}=\langle 2,2,-1\rangle$

Solution (continued)

$Q(4,5,-1)\quad \overrightarrow{PQ}=\langle 2,2,-1\rangle$

PARAMETRIC EQUATIONS of M

$x=4+2t\quad y=5+2t\quad z=-1-1t$

SYMMETRIC EQUATIONS of M

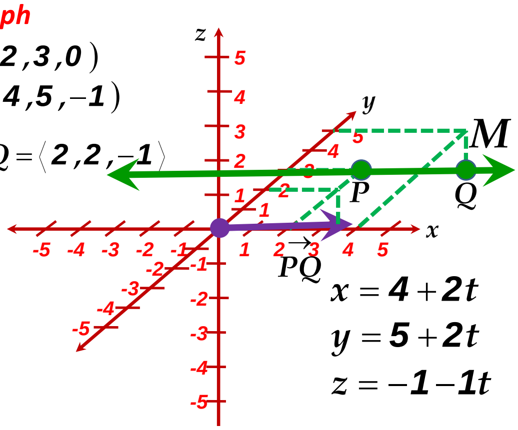
$$\frac{x-4}{2}=\frac{y-5}{2}=\frac{z+1}{-1}$$

Graph

$P(2,3,0)$

$Q(4,5,-1)$

$\overrightarrow{PQ}=\langle 2,2,-1\rangle$



$x=4+2t$

$y=5+2t$

$z=-1-1t$

END