Tangent Planes and Normals to Surfaces

Chapter 3 Section 4

RECALL:

Equation of a line in R^3 :

a point on the line: $P_0\left(x_0, y_0, z_0\right)$

a parallel vector: $\langle a, b, c \rangle$

Thus, the symmetric equations of this line are

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

RECALL:

Equation of a plane in R^3 :

a point on the plane: $P_0(x_0, y_0, z_0)$

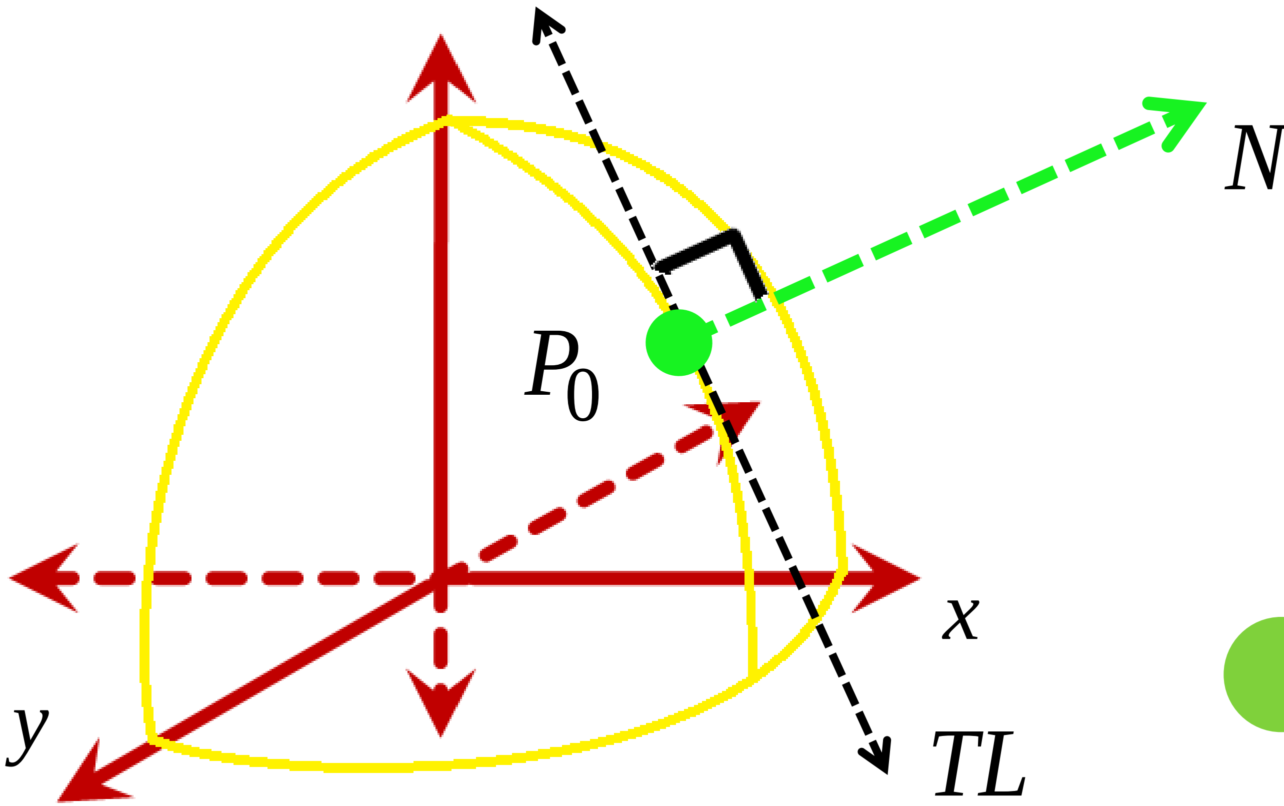
a normal vector: $\langle a, b, c \rangle$

Thus, the equation of this plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Definition.

A vector orthogonal to a tangent vector of every curve C through a point P_0 on a surface S is called a ***normal vector to S at P_0***.



Theorem.

If an equation of surface S is $F(x, y, z) = 0$ and F is differentiable and F_x, F_y, F_z are not all are equal to zero at the point $P_0(x_0, y_0, z_0)$ on S, then

$$\nabla F(x_0, y_0, z_0)$$

is a **normal vector to** S **at** P_0.

Theorem.

If an equation of surface S is $F(x, y, z) = 0$ and F is differentiable and F_x, F_y, F_z are not all are equal to zero at the point $P_0(x_0, y_0, z_0)$ on S, then the **tangent plane of *S* at the point *P₀*** having

$$\nabla F(x_0, y_0, z_0)$$

as a normal vector.

Example. Find equations for the tangent plane at the point P on the given surface.

1. $x^2 + 2xy - y^2 + z^2 = 7$

$P(1, -1, 3)$

2. $z = \ln\left(x^2 + y^2\right)$

$P(0, 0, 0)$

Example. Find equations for the tangent plane at the point P on the given surface.

3. $$x^2 e^{-2y} + Arc\tan(x+z) - z^2 = 0$$

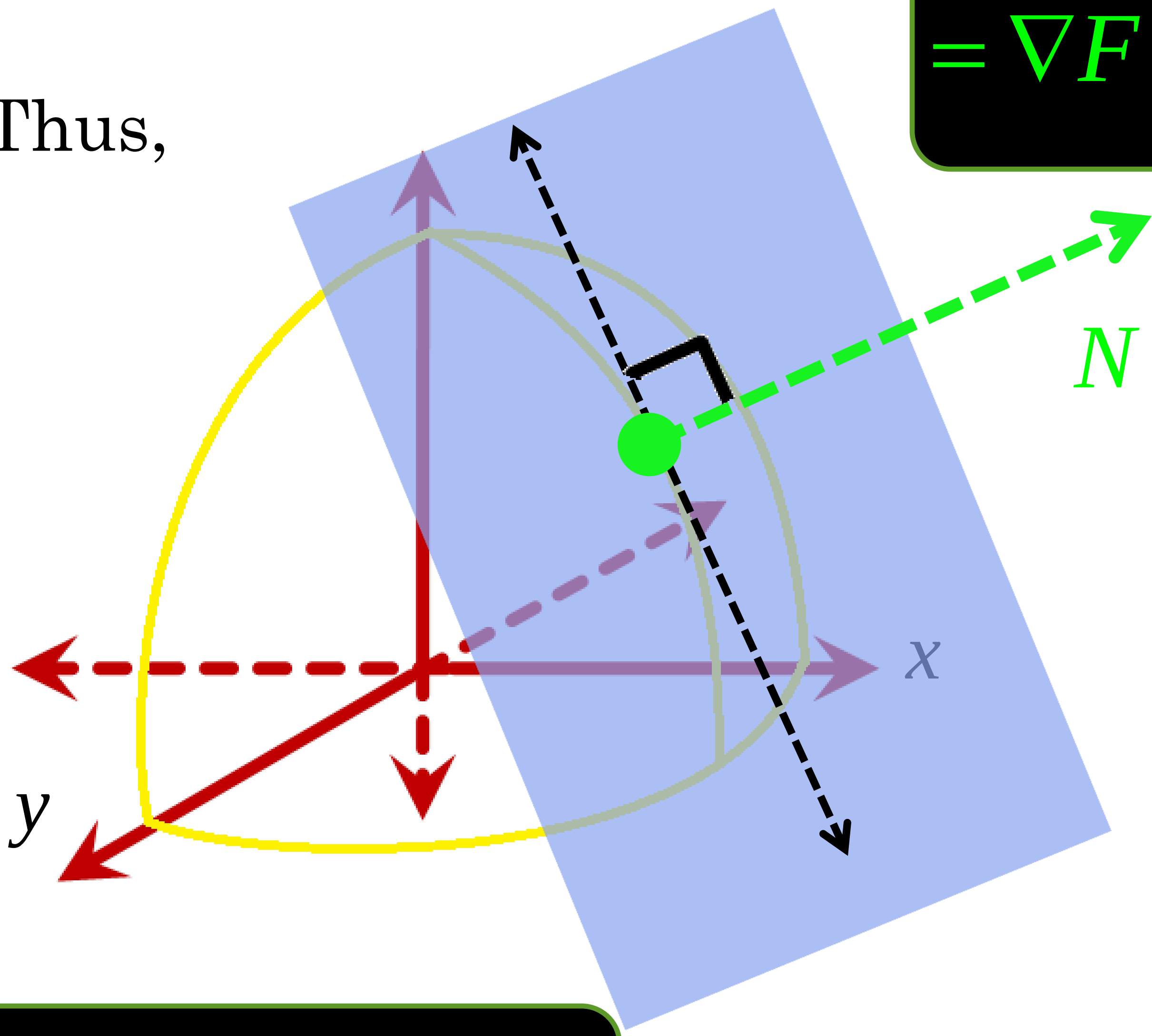
$$P(-1,0,1)$$

4. $$\cos(\pi x) - x^2 y + e^{xz} + yz = 4$$

$$P(0,1,2)$$

Thus,

$$= \nabla F\left(x_0, y_0, z_0\right)$$



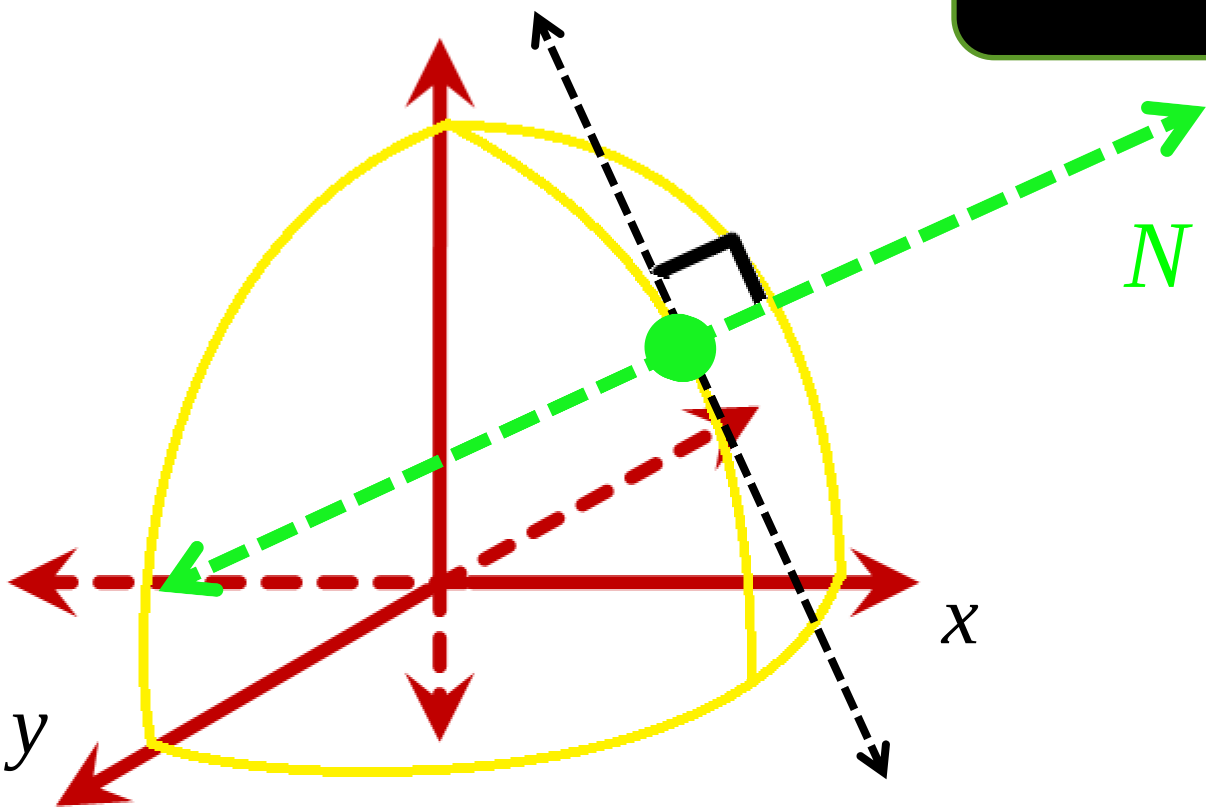
$$F\left(x, y, z\right) = 0$$

Remark.

The equation of this tangent plane is given by

$$F_x\left(x-x_0\right)+F_y\left(y-y_0\right)+F_z\left(z-z_0\right)=0$$

$$= \nabla F\left(x_0, y_0, z_0\right)$$



$$F\left(x, y, z\right) = 0$$

Definition.

The ***normal line to a surface S at the point P_0*** on S is the line through P_0 having as a set of direction numbers the components of any normal vector to S at P_0.

Remark.

If an equation of surface S is $F(x, y, z) = 0$ then the symmetric equations of the normal line to S at $P_0(x_0, y_0, z_0)$ are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example. Find equations for the normal line at the point P on the given surface.

1. $$x^2 + 2xy - y^2 + z^2 = 7$$

$$P(1, -1, 3)$$

2. $$z = \ln\left(x^2 + y^2\right)$$

$$P(1, 0, 0)$$

Example. Find equations for the normal line at the point P on the given surface.

3. $$x^2 e^{-2y} + \operatorname{Arc}\tan(x+z) - z^2 = 0$$

$$P(-1,0,1)$$

4. $$\cos(\pi x) - x^2 y + e^{xz} + yz = 4$$

$$P(0,1,2)$$

Remark.

Let $P_0(x_0, y_0, z_0)$ be a point on the curve of intersection C of two surfaces having equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$.

Now, we consider the tangent line to C at P_0 .

If $\nabla F(P_0)$ and $\nabla G(P_0)$ are not parallel, then the components of $\nabla F(P_0) \times \nabla G(P_0)$ serve as the ***direction numbers*** for the this tangent line.

Definition.

If two surfaces have a common tangent plane at a point, then the two surfaces are said to be tangent at that point.

Remark.

If $F(x,y,z)=0$ and $G(x,y,z)=0$ are tangent at the point $P_0(x_0,y_0,z_0)$,

then for some constant k,

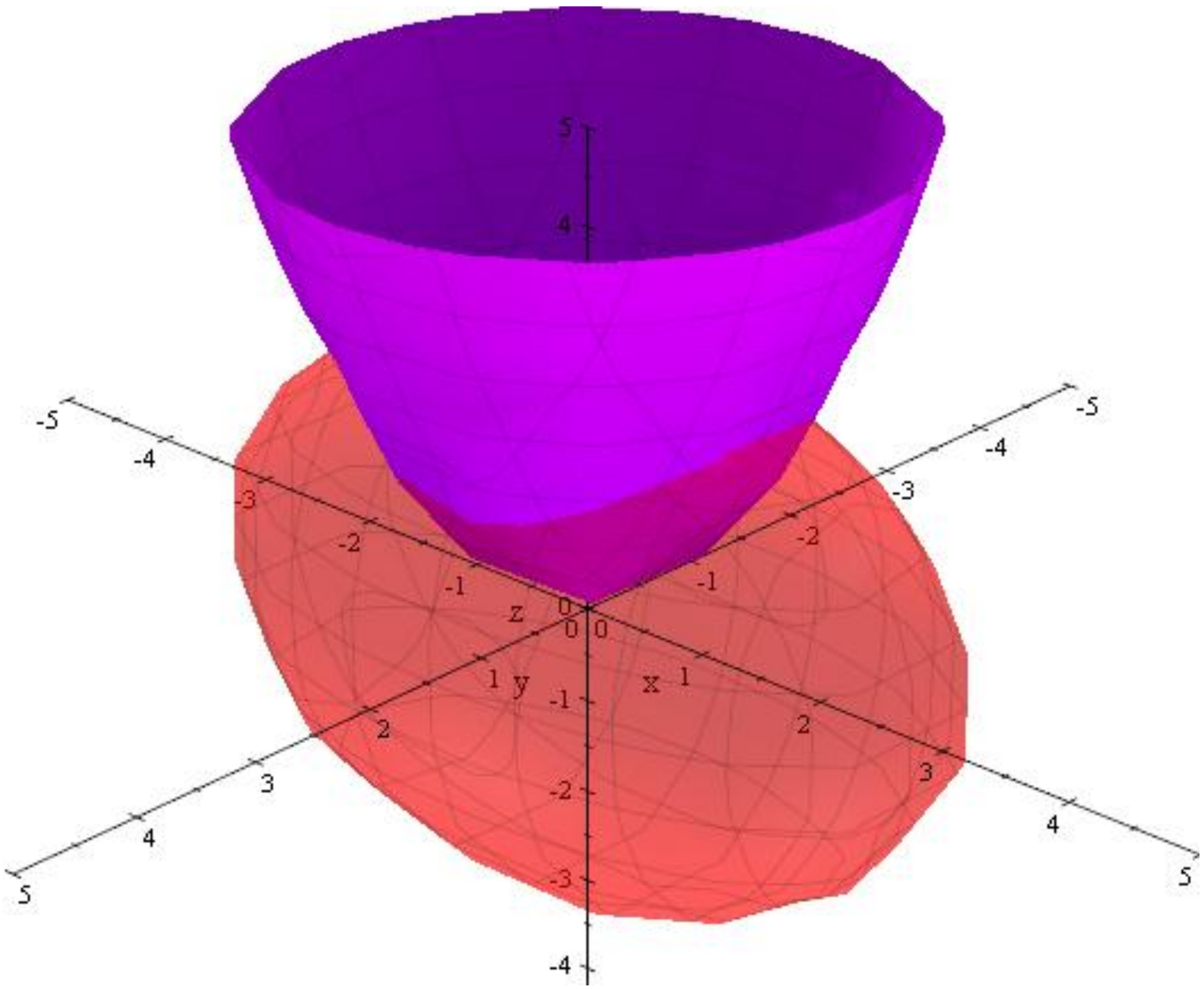
$$\nabla F(x_0,y_0,z_0)=k\nabla G(x_0,y_0,z_0)$$

Example. Find the parametric equations for the tangent line to the curve of intersection of the paraboloid and the ellipsoid whose equations are given below at the given point P.

$$z = x^2 + y^2$$

$$4x^2 + y^2 + z^2 = 9$$

$$P(-1,1,2)$$



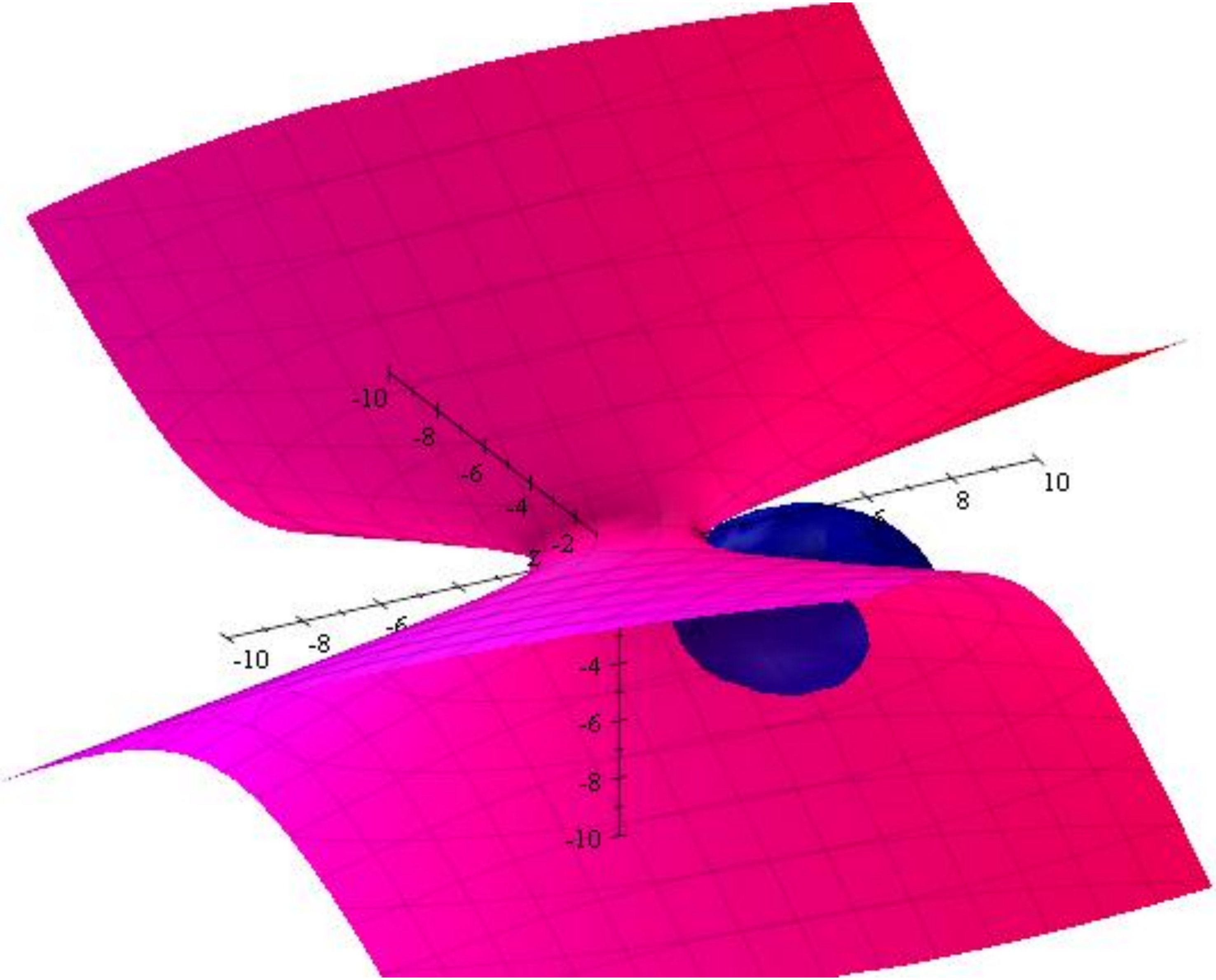
z
y
x

Example. Show that the given surfaces are tangent at the given point:

$$F(x,y,z) = x^2 + 4y^2 - 4z^2 - 4 = 0$$

$$G(x,y,z) = x^2 + y^2 + z^2 - 6x - 6y + 2z + 10 = 0$$

$$P(2,1,1)$$



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END