UNIT 2

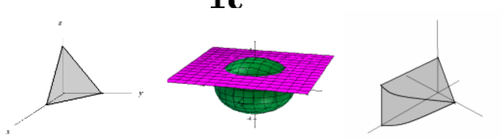
VECTORS, LINES and PLANES in SPACE

INTRODUCTION

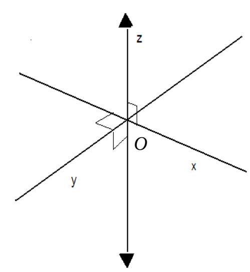
We live in a 3-dimensional world. The three dimensions of this world are length, width, and height. The houses we live in, the buildings that we work in, the tools we work with and the objects we create are described by 3-dimensional geometry, or space geometry.

2.1 THE 3-DIMENSIONAL SPACE

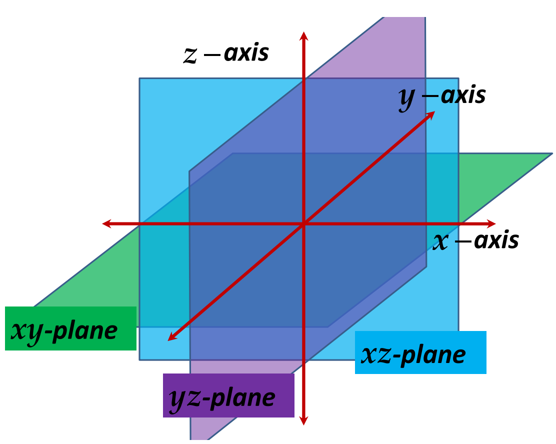
$\mathbf{R}^3$



2.1 THE 3-DIMENSIONAL SPACE $\mathbf{R}^3$



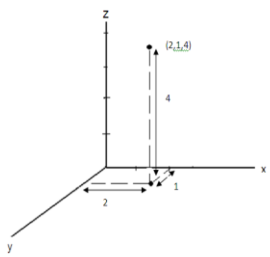
Consider three mutually perpendicular coordinate lines (x,y and z-axes) with a common point *O*. Let the zero points of these coordinate lines be located at *O*. Point *O* is called the **origin**.



The 3D space

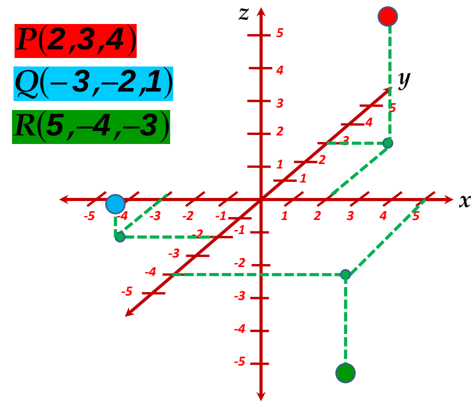
The set of all ordered triples of real numbers is called as the three-dimensional number space.

$$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$



TIPS: Plotting points in $\mathbf{R}^3$

Note that the coordinates of the ordered triple measures its directed distances from the three planes.



Distance and midpoint points: $P_1(x_1, y_1, z_1)$

$P_2(x_2, y_2, z_2)$

Distance: $|P_1P_2|$ ***or*** $d(P_1, P_2)$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Midpoint:

$$M_{P_1P_2}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example 1. Determine the distance between the given points and the midpoint of the segment joining them.

$P_1(-3, 2, 4)$ $\quad P_2(4, -3, -2)$

Solution:

$$\begin{aligned}|P_1P_2| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\\ &= \sqrt{(4-(-3))^2+(-3-2)^2+(-2-4)^2}\\ &= \sqrt{7^2+(-5)^2+(-6)^2}\end{aligned}$$

Solution (continued)

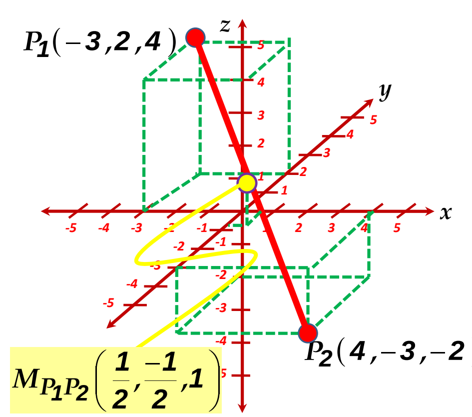
$$\begin{aligned}|P_1P_2| &= \sqrt{7^2+(-5)^2+(-6)^2}\\ &= \sqrt{49+25+36} = \sqrt{110}\end{aligned}$$

$P_1(-3, 2, 4)$ $\quad P_2(4, -3, -2)$

$$M_{P_1P_2}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$M_{P_1P_2}\left(\frac{-3+4}{2}, \frac{2+(-3)}{2}, \frac{4+(-2)}{2}\right)$$

$$M_{P_1P_2}\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$



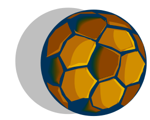
Definition. The **graph of an equation in** $\mathbf{R}^3$ is the set of all points (x, y, z) whose coordinates are numbers satisfying the equation.

SPHERES AND THEIR EQUATIONS

- A **sphere** is the set of all points in three-dimensional space equidistant from a fixed point. The fixed point is called the **center** of the sphere and the measure of the constant distance is called the **radius** of the sphere.

An **equation** of the sphere of radius *r* and center at (h, k, l) is

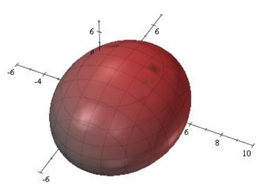
$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$



Illustrations

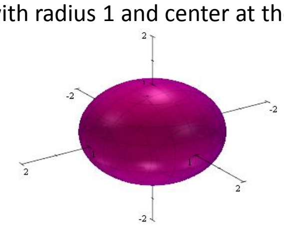
Sphere with radius 4 and center at $(2, -1, 1)$

$$(x-2)^2+(y+1)^2+(z-1)^2=16$$



Illustrations

Sphere with radius 1 and center at the origin



$$x^2+y^2+z^2=1$$

THEOREM 1

The graph of any second-degree equation in x, y and *z* of the form

$$\boxed{x^2+y^2+z^2+Ax+By+Cz+D=0}$$

is either a sphere, a point, or the empty set.

TIPS:

- To be able to determine the graph of any second degree equation of the form
$$(x-h)^2+(y-k)^2+(z-l)^2=c$$
where h, k, l, c are constants.

❖If $c>0$, the graph is a **sphere** with center (h, k, l) and radius $\sqrt{c}$

❖If $c=0$, the graph is the **point** (h, k, l)

❖If $c<0$, the graph is an **empty set**

Illustrations

In each of the following, determine the graph of the equation.

1. $x^2 - 2x + y^2 + 4y + z^2 = -10$
2. $x^2 - 4x + y^2 + 6y + z^2 - 2z = -5$
3. $x^2 - 4x + y^2 + 4y + z^2 - 6z + 17 = 0$

1. $x^2 - 2x + y^2 + 4y + z^2 = -10$

Solution

$$x^2 - 2x + y^2 + 4y + z^2 = -10$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 = -10 + 1 + 4$$

$$\Rightarrow (x-1)^2 + (y+2)^2 + z^2 = -5$$

Graph is an **empty set.**

2. $x^2 - 4x + y^2 + 6y + z^2 - 2z = -5$

Solution

$$x^2 - 4x + y^2 + 6y + z^2 - 2z = -5$$

$$\Rightarrow x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 2z + 1 = -5 + 4 + 9 + 1$$

$$\Rightarrow (x-2)^2 + (y+3)^2 + (z-1)^2 = 9$$

Graph is a **sphere** with center at (2,-3,1) and radius 3.

3. $x^2 - 4x + y^2 + 4y + z^2 - 6z + 17 = 0$

Solution

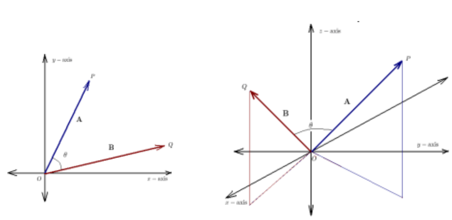
$$x^2 - 4x + y^2 + 4y + z^2 - 6z + 17 = 0$$

$$\Rightarrow x^2 - 4x + 4 + y^2 + 4y + 4 + z^2 - 6z + 9 = -17 + 4 + 4 + 9$$

$$\Rightarrow (x-2)^2 + (y+2)^2 + (z-3)^2 = 0$$

Graph is the **point** (2,-2,3).

2.2 VECTORS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$



DIFFERENCE BETWEEN A **VECTOR** AND A **SCALAR**

SCALAR is a quantity that has a magnitude but no direction

Examples of scalars:
mass, speed and natural numbers are common

DIFFERENCE BETWEEN A **VECTOR** AND A **SCALAR**

VECTOR is a quantity that has both magnitude and direction.

Examples of vectors

forces, velocity and acceleration.

Note that vectors arise naturally as physical quantities.

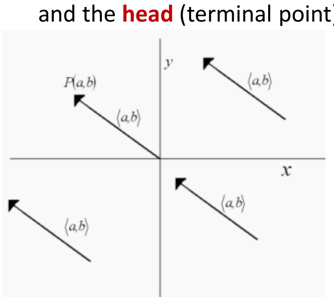
A **vector in** $\mathbf{R}^2$ is an ordered pair of real numbers a and b .

NOTATION: $\langle a, b \rangle$

The real numbers a and b are called the **components** of the vector.

GEOMETRIC INTERPRETATION OF VECTOR :

Arrow (directed line segment) in the xy-plane where the **tail** (initial point) is at the origin and the **head** (terminal point) is at point (a,b).



***NOTE:** Any equivalent directed line segment (line segments with same magnitude and direction) is also a **representation** of vector* $\langle a,b \rangle$

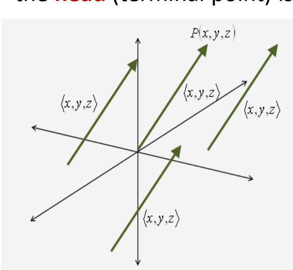
A **vector in** $\mathbf{R}^3$ is an ordered triple of real numbers x, y and z .

NOTATION: $\langle x, y, z \rangle$

The real numbers x, y and z are called the **components** of the vector.

GEOMETRIC INTERPRETATION OF VECTOR :

Arrow (directed line segment) in the space $\mathbf{R}^3$ where the **tail** (initial point) is at the origin and the **head** (terminal point) is at point (x,y,z).



***NOTE:** Any equivalent directed line segment (line segments with same magnitude and direction) is also a **representation** of vector* $\langle x,y,z \rangle$

Remark:

The particular representation of a vector that has its initial point at the origin is called the **POSITION REPRESENTATION** of the vector.

NOTE

Initial point: (x_i, y_i)

Terminal point: (x_t, y_t)

VECTOR COMPONENTS:

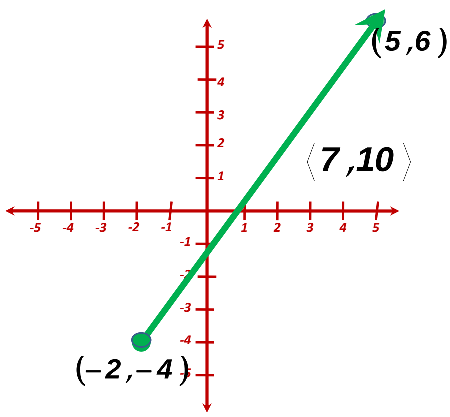
$$\langle x_t - x_i, y_t - y_i \rangle$$

Example 1

Determine the components of the vector with initial point at $(-2,-4)$ and terminal point at $(5,6)$.

Solution:

$$\langle x_t - x_i, y_t - y_i \rangle$$
$$= \langle 5-(-2), 6-(-4) \rangle$$
$$= \langle 7, 10 \rangle$$

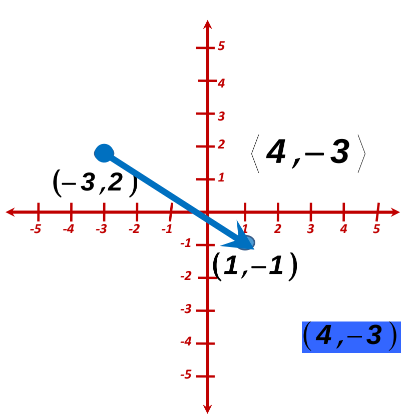


Example 2

Determine the components of the vector with initial point at $(-3,2)$ and terminal point at $(1,-1)$.

Solution:

$$\langle x_t - x_i, y_t - y_i \rangle$$
$$= \langle 1-(-3), -1-2 \rangle$$
$$= \langle 4, -3 \rangle$$



MAGNITUDE

- The **magnitude** of a vector $\mathbf{A}$, denoted by $\|\mathbf{A}\|$, is the length of any of its representations.

MAGNITUDE

MAGNITUDE FORMULA

Let $\mathbf{A}$ be a vector in $\mathbf{R}^2$ where $\mathbf{A} = \langle a_1, a_2 \rangle$, then $\|\mathbf{A}\| = \sqrt{a_1{}^2 + a_2{}^2}$.

Let $\mathbf{V}$ be a vector in $\mathbf{R}^3$ where $\mathbf{V} = \langle a_1, a_2, a_3 \rangle$ then $\|\mathbf{V}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$.

Illustration 1

The magnitude of vector $\mathbf{A} = \langle -5, -2 \rangle$ is

$$\|\mathbf{A}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

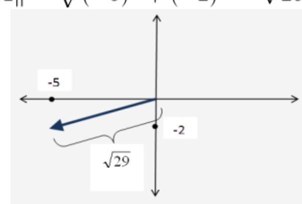
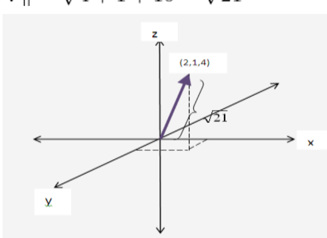


Illustration 2

The magnitude of vector $\mathbf{V} = \langle 2, 1, 4 \rangle$ is

$$\|\mathbf{V}\| = \sqrt{4 + 1 + 16} = \sqrt{21}$$



DEFINITIONS FOR VECTORS IN $\mathbf{R}^2$

If $\mathbf{A} = \langle a_1, a_2 \rangle$, $\mathbf{B} = \langle b_1, b_2 \rangle$ and c is a scalar, then

- **Sum of two vectors:**

$$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

- **Product of a scalar and a vector:**

$$c\mathbf{A} = \langle ca_1, ca_2 \rangle$$

NOTE: *$c\mathbf{A}$ is called a scalar multiple of $\mathbf{A}$*

DEFINITIONS FOR VECTORS IN $\mathbf{R}^2$

If $\mathbf{A} = \langle a_1, a_2 \rangle$, $\mathbf{B} = \langle b_1, b_2 \rangle$ and c is a scalar, then

- **Negative of a vector $\mathbf{A}$**

$$-\mathbf{A} = \langle -a_1, -a_2 \rangle$$

- **Vector difference:**

$$\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

DEFINITIONS FOR VECTORS IN $\mathbf{R}^3$

If $\mathbf{V} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{W} = \langle w_1, w_2, w_3 \rangle$ and c is a scalar, then

- **Sum of two vectors:**

$$\mathbf{V} + \mathbf{W} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

- **Product of a scalar and a vector:**

$$c\mathbf{V} = \langle cv_1, cv_2, cv_3 \rangle$$

NOTE: *$c\mathbf{V}$ is called a scalar multiple of $\mathbf{V}$*

DEFINITIONS FOR VECTORS IN $\mathbf{R}^3$

If $\mathbf{V} = \langle v_1, v_2, v_3 \rangle$,$\mathbf{W} = \langle w_1, w_2, w_3 \rangle$ and c is a scalar, then

- **Negative of a vector**

$$-\mathbf{V} = \langle -v_1, -v_2 - v_3 \rangle$$

- **Vector difference:**

$$\mathbf{V} - \mathbf{W} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$$

WARNING!!!

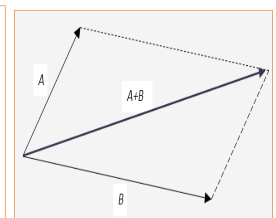
As defined above, we can only add and subtract vectors from the same dimension. That is, both vectors must be from only or from only.

GEOMETRIC INTERPRETATIONS

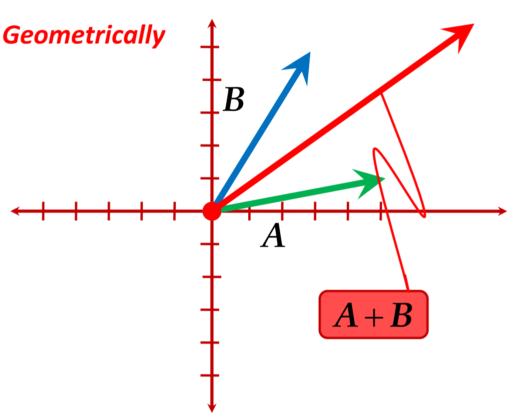
Consider vectors $\mathbf{A}$ and $\mathbf{B}$ from the same dimension. Move vector $\mathbf{B}$ so that its tail coincides with that of $\mathbf{A}$.

$\mathbf{A} + \mathbf{B}$

- Vector $\mathbf{A} + \mathbf{B}$ is the vector with this common tail and coinciding with the diagonal of the parallelogram that has $\mathbf{A}$ and $\mathbf{B}$ as sides.



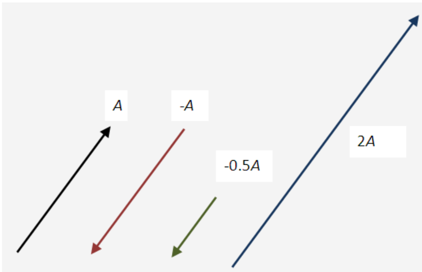
Geometrically



$c\mathbf{A}$

Vector $-\mathbf{A}$ is the vector with same magnitude but in opposite direction with vector $\mathbf{A}$.

If c is a scalar, $c\mathbf{A}$ is the vector with magnitude $|c||\mathbf{A}||$ and
In the same direction with $\mathbf{A}$ if c is positive
In the opposite direction with $\mathbf{A}$ if c is negative



SCALAR PRODUCT: cA

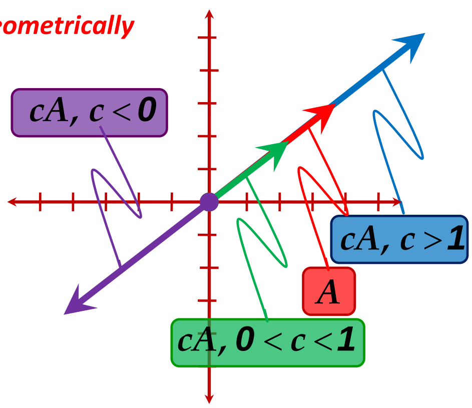
where c is a constant

If $c > 1$ ***,*** cA ***stretches*** A ***.***

If $0 < c < 1$***,*** cA ***shrinks*** A ***.***

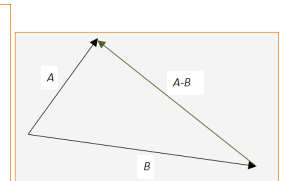
If $c < 0$ ***,*** cA ***reverses*** A ***.***

Geometrically



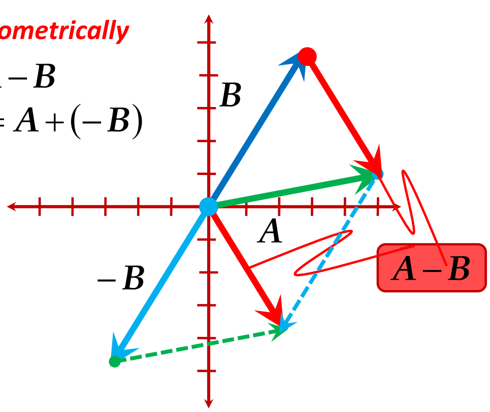
$$\mathbf{A} - \mathbf{B}$$

Vector $\mathbf{A} - \mathbf{B}$ is the vector with initial point coinciding with the head of vector $\mathbf{B}$ and terminal point coinciding with the head of vector $\mathbf{A}$.



Geometrically

$A - B$
$= A + (-B)$



DIRECTION OF A VECTOR

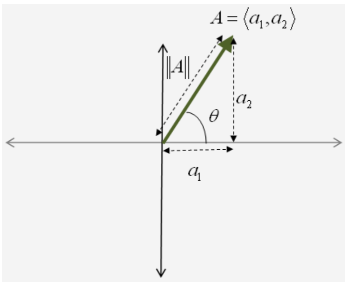
- The **direction** of a nonzero vector is the direction of any of its representation.

The direction angle of vector A in $\mathbf{R}^2$, $\boldsymbol{\theta}_{\boldsymbol{A}}$, is the measure of the angle formed by the vector with the positive x-axis in the counterclockwise direction.

$\mathbf{R}^2$

Consider vector $A=\langle a_1 ,a_2 \rangle$.

$\| A\| \; = \; \sqrt{a_1 + a_2}$



$$\cos\theta = \frac{a_1}{\|\mathbf{A}\|}$$

$$\sin\theta = \frac{a_2}{\|\mathbf{A}\|}$$

$$tan\,\theta \; = \; \frac{b}{a}$$

$$\mathbf{A} = \langle a_1, a_2\rangle = \langle \|\mathbf{A}\|\cos\theta, \|\mathbf{A}\|\sin\theta\rangle = \|\mathbf{A}\|\langle\cos\theta, \sin\theta\rangle$$

$\mathbf{R}^2$

TIPS:

- If the terminal point of vector in position representation is at the
  - 1st quadrant $\Rightarrow \theta = Arc\tan\left(\frac{a_2}{a_1}\right)$
  - 2nd or 3rd quadrant $\Rightarrow \theta = Arc\tan\left(\frac{a_2}{a_1}\right) + \pi$
  - 4th quadrant $\Rightarrow \theta = 2\pi + Arc\tan\left(\frac{a_2}{a_1}\right)$

ILLUSTRATION

Consider vector $\mathbf{A} = \langle\sqrt{3}, -3\rangle$. Now, since the terminal point of the position representation of this vector is at the 4th quadrant,

$$\Rightarrow \theta = 2\pi + Arc\tan\left(\frac{a_2}{a_1}\right) = 2\pi + Arc\tan\left(\frac{-3}{\sqrt{3}}\right) = 2\pi - \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{5\pi}{3}$$

MORE Examples

If $A=\langle -2,4 \rangle$ and $B=\langle 4,3 \rangle$, evaluate the following:

1. $A+B$
2. $A-B$
3. $2A$
4. $\frac{1}{2}A$
5. $2A-3B$

Also, determine the respective direction and magnitude.

Solutions

$A=\langle -2,4 \rangle \quad B=\langle 4,3 \rangle$

1. $A+B \;=\langle -2+4, 4+3 \rangle$
$\;=\langle 2,7 \rangle$

$\| A+B \| \;= \sqrt{53}$

$\theta_{A+B} \;= Arc\tan\frac{7}{2}$

Solutions

$A=\langle -2,4 \rangle \quad B=\langle 4,3 \rangle$

2. $A-B \;=\langle -2-4, 4-3 \rangle$
$\;=\langle -6,1 \rangle$

$\| A-B \| \;= \sqrt{37}$

$\theta_{A-B} \;= Arc\tan\frac{-1}{6} + \pi$

Solutions

$A = \langle -2, 4 \rangle \quad B = \langle 4, 3 \rangle$

3. $2A = \langle 2(-2), 2(4) \rangle$
$= \langle -4, 8 \rangle$

$\|2A\| = \sqrt{80} = 4\sqrt{5} = 2\|A\|$

$\theta_{2A} = Arctan(-2) + \pi = \theta_A$

Solutions

$A = \langle -2, 4 \rangle \quad B = \langle 4, 3 \rangle$

4. $\frac{1}{2}A = \left\langle \frac{1}{2}(-2), \frac{1}{2}(4) \right\rangle$
$= \langle -1, 2 \rangle$

$\left\| \frac{1}{2}A \right\| = \sqrt{5} = \frac{1}{2}\|A\|$

$\theta_{\frac{1}{2}A} = Arctan(-2) + \pi = \theta_A$

Solutions

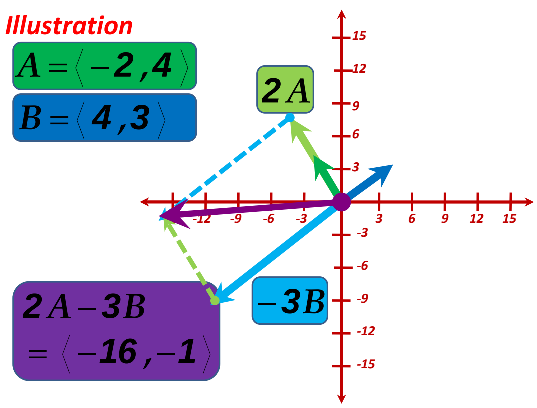
$A = \langle -2, 4 \rangle \quad B = \langle 4, 3 \rangle$

5. $2A - 3B = 2\langle -2, 4 \rangle - 3\langle 4, 3 \rangle$
$= \langle -4, 8 \rangle - \langle 12, 9 \rangle$
$= \langle -16, -1 \rangle$

$\|2A - 3B\| = \sqrt{257}$

$\theta_{2A-3B} = Arctan\frac{1}{16} + \pi$

Illustration

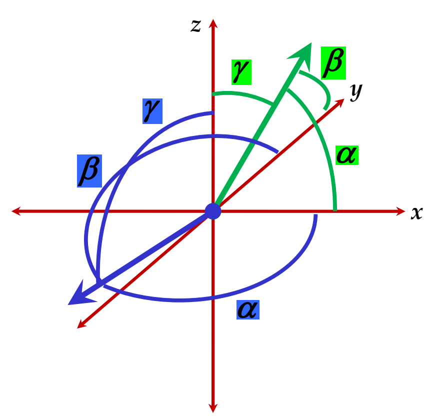


$\mathbb{R}^3$ **DIRECTION OF A VECTOR**

- The **direction** of a nonzero vector is the direction of any of its representation.

Direction angles of a non-zero vector A in $\mathbb{R}^3$:

smallest radian measure measured from the positive side of each axis



$\mathbf{R}^3$

MUST!!!

Initial point: (x_i, y_i, z_i)

Terminal point: (x_t, y_t, z_t)

VECTOR COMPONENTS:

$$\langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$$

Consider vector $A = \langle a, b, c \rangle$.

If α, β and γ are the direction angles,

$$\cos\alpha = \frac{a}{\|A\|} \quad \cos\beta = \frac{b}{\|A\|} \quad \cos\gamma = \frac{c}{\|A\|}$$

where $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Note

- If α, β, γ are called the direction angles of a nonzero vector $\mathbf{V}$ in $\mathbf{R}^3$, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the **direction cosines** of $\mathbf{V}$.

Example

Determine the components of the vector with initial point at $(-4, 5, 3)$ and terminal point at $(-2, 4, 5)$. Also, determine the magnitude and the cosines of the direction angles.

Solution:

initial: $(-4, 5, 3)$ *terminal:* $(-2, 4, 5)$

$$\langle x_t - x_i, y_t - y_i, z_t - z_i \rangle$$
$$= \langle -2 - (-4), 4 - 5, 5 - 3 \rangle$$
$$= \langle 2, -1, 2 \rangle$$

$$A\langle 2, -1, 2 \rangle$$

$$\|A\| = \sqrt{2^2 + (-1)^2 + 2^2}$$
$$= \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

Solution (continued)

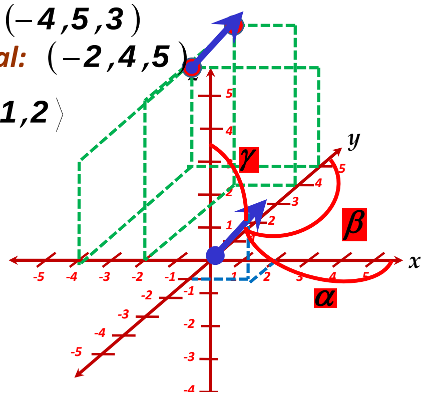
$$A\langle 2, -1, 2 \rangle \quad \|A\| = 3$$

$$\cos\alpha = \frac{a}{\|A\|} \quad \cos\beta = \frac{b}{\|A\|} \quad \cos\gamma = \frac{c}{\|A\|}$$

$$\cos\alpha = \frac{2}{3} \quad \cos\beta = \frac{-1}{3} \quad \cos\gamma = \frac{2}{3}$$

$$\alpha = \operatorname{Arc}\cos\frac{2}{3} \quad \beta = \operatorname{Arc}\cos\left(\frac{-1}{3}\right) \quad \gamma = \operatorname{Arc}\cos\frac{2}{3}$$

initial: $(-4,5,3)$
terminal: $(-2,4,5)$
$A\langle 2,-1,2\rangle$



Definition:

- **ZERO VECTOR**- Vector whose component are all zero.
  - Zero vector in $\mathbf{R}^2$ $\quad \mathbf{0}_2 = \langle 0,0\rangle$
  - Zero vector in $\mathbf{R}^3$ $\quad \mathbf{0}_3 = \langle 0,0,0\rangle$

Definition.

- **UNIT VECTOR** - Vector whose magnitude is equal to 1.

Unit vector $\quad \mathbf{R}^2$

A unit vector has a magnitude of 1.

$i=\langle 1,0\rangle$ ***: unit vector in the direction of positive x-axis***

$j=\langle 0,1\rangle$ ***: unit vector in the direction of positive y-axis***

Unit vector $\quad \mathbf{R}^2$

Given $A=\langle a,b\rangle$

$$A = ai+bj$$

or $A = a\langle 1,0\rangle + b\langle 0,1\rangle$

Unit vector $\quad \mathbf{R}^2$

Given $A=\langle a,b\rangle$.

Unit vector in the direction of A:

$$U_A = \left\langle \frac{a}{\|A\|}, \frac{b}{\|A\|} \right\rangle$$

$$= \langle \cos\theta_A, \sin\theta_A \rangle$$

Example

Determine a unit vector in the direction of $\langle 12,-5 \rangle$

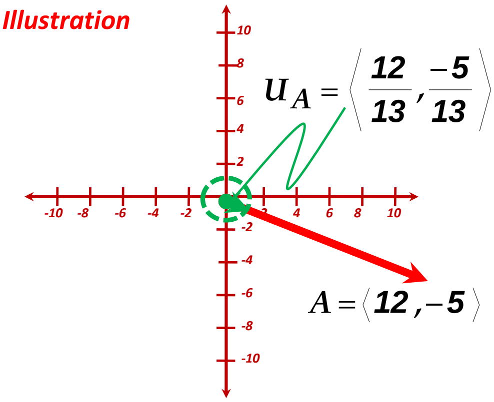
Solution:

Let $A=\langle 12,-5 \rangle$

$$\|A\|=\sqrt{12^2+(-5)^2}$$
$$=\sqrt{144+25}$$
$$=\sqrt{169}=13$$

$$U_A=\left\langle \frac{12}{13},\frac{-5}{13} \right\rangle$$

Illustration



Example

Determine a unit vector in the direction of the vector with a magnitude of 10 in the direction of $\frac{\pi}{6}$.

Solution: **Let** B **be the given vector.**

$$U_B=\langle \cos\theta_B,\sin\theta_B \rangle$$
$$=\left\langle \cos\frac{\pi}{6},\sin\frac{\pi}{6} \right\rangle=\left\langle \frac{\sqrt{3}}{2},\frac{1}{2} \right\rangle$$

Unit vectors $\mathbf{R}^3$

$i=\langle 1,0,0 \rangle$ **: unit vector in the direction of positive** x**-axis**

$j=\langle 0,1,0 \rangle$ **: unit vector in the direction of positive** y**-axis**

$k=\langle 0,0,1 \rangle$ **: unit vector in the direction of positive** z**-axis**

Unit vector $\mathbf{R}^3$

Given $A=\langle a,b,c \rangle$.

$$A=ai+bj+ck$$

$$U_A=\left\langle \frac{a}{\|A\|},\frac{b}{\|A\|},\frac{c}{\|A\|} \right\rangle$$
$$=\langle \cos\alpha,\cos\beta,\cos\gamma \rangle$$

Example. **Consider** $A=\langle -2,2,3 \rangle$ **and** $B=\langle 4,2,0 \rangle$ **. Determine the unit vector in the direction of** $3A-2B$.

Solution:

$$3A-2B=3\langle -2,2,3 \rangle-2\langle 4,2,0 \rangle$$
$$=\langle -6,6,9 \rangle-\langle 8,4,0 \rangle$$
$$=\langle -6-8,6-4,9-0 \rangle$$
$$=\langle -14,2,9 \rangle$$

Solution (continued)

$$3A-2B=\langle -14,2,9\rangle$$

$$U_{3A-2B}=\frac{3A-2B}{\|3A-2B\|}$$

$$=\frac{\langle -14,2,9\rangle}{\sqrt{(-14)^2+2^2+9^2}}$$

$$=\frac{\langle -14,2,9\rangle}{\sqrt{196+4+18}}$$

Solution (continued)

$$U_{3A-2B}=\frac{\langle -14,2,9\rangle}{\sqrt{196+4+18}}$$

$$=\frac{\langle -14,2,9\rangle}{\sqrt{218}}$$

$$=\left\langle \frac{-14}{\sqrt{218}},\frac{2}{\sqrt{218}},\frac{9}{\sqrt{218}}\right\rangle$$