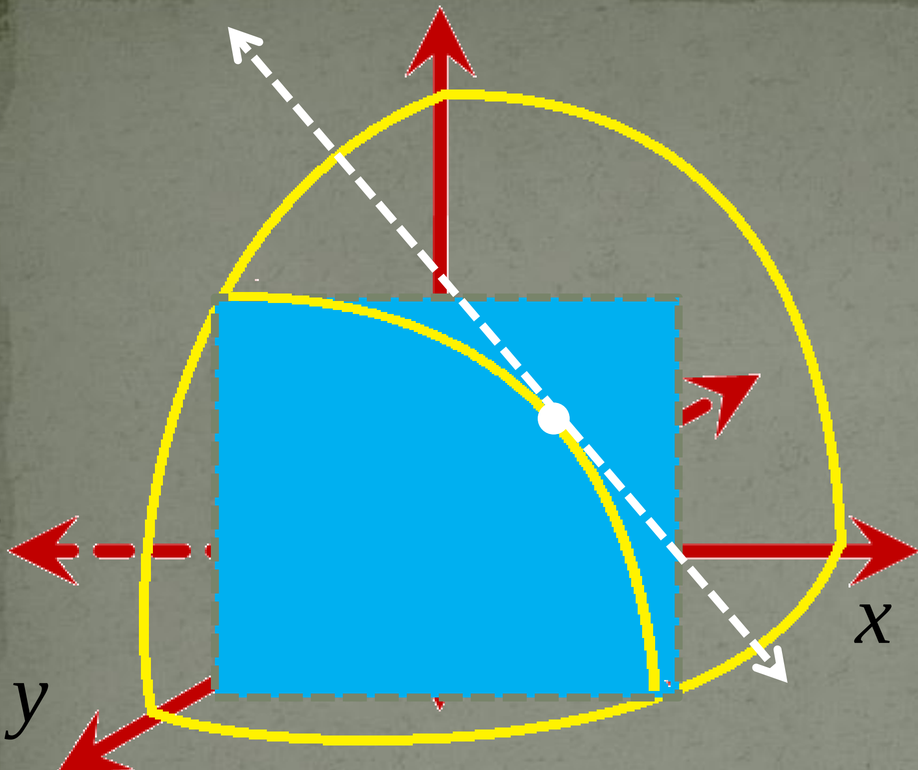
# Applications of Partial Differentiation

Chapter 3

# Chapter objectives:

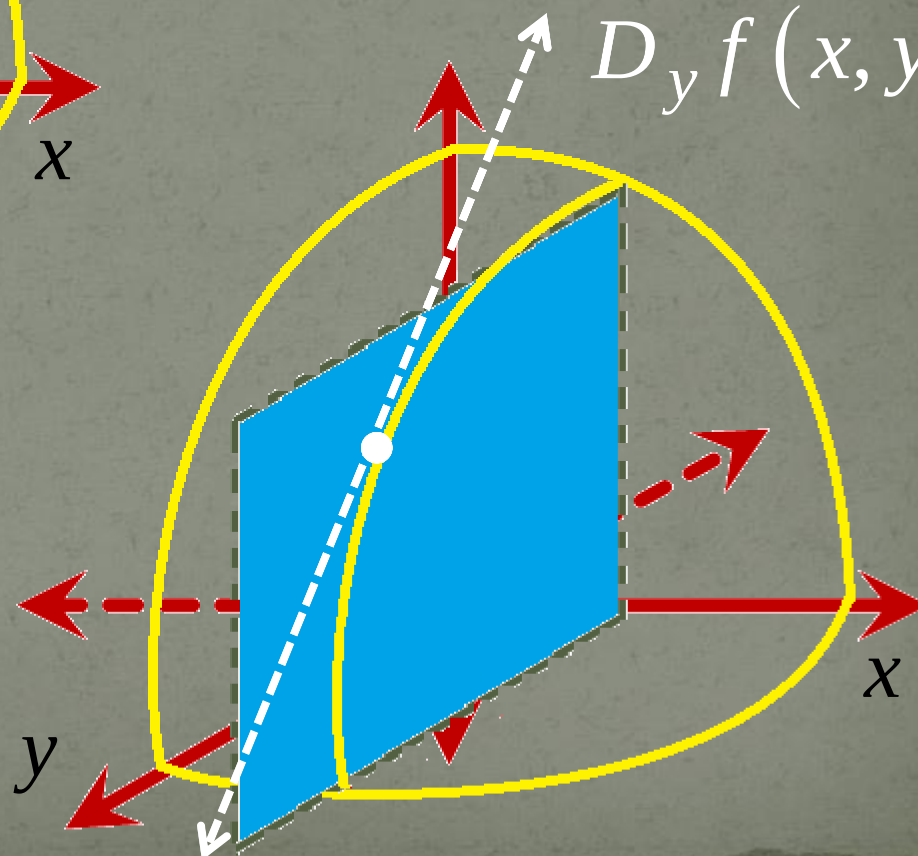
At the end of the chapter, you should be able to:

1. find and interpret ***directional derivatives*** and ***gradients***,
2. find an equation of a ***tangent plane*** and equations of a ***normal/tangent line*** to a surface,
3. find the relative extrema of a function of 2 or more variables,
4. use Lagrange multiplers to solve some ***optimization*** problems, and
5. obtain a function from its gradient.




$D_x f(x, y)$

$D_y f(x, y)$



# Directional Derivatives

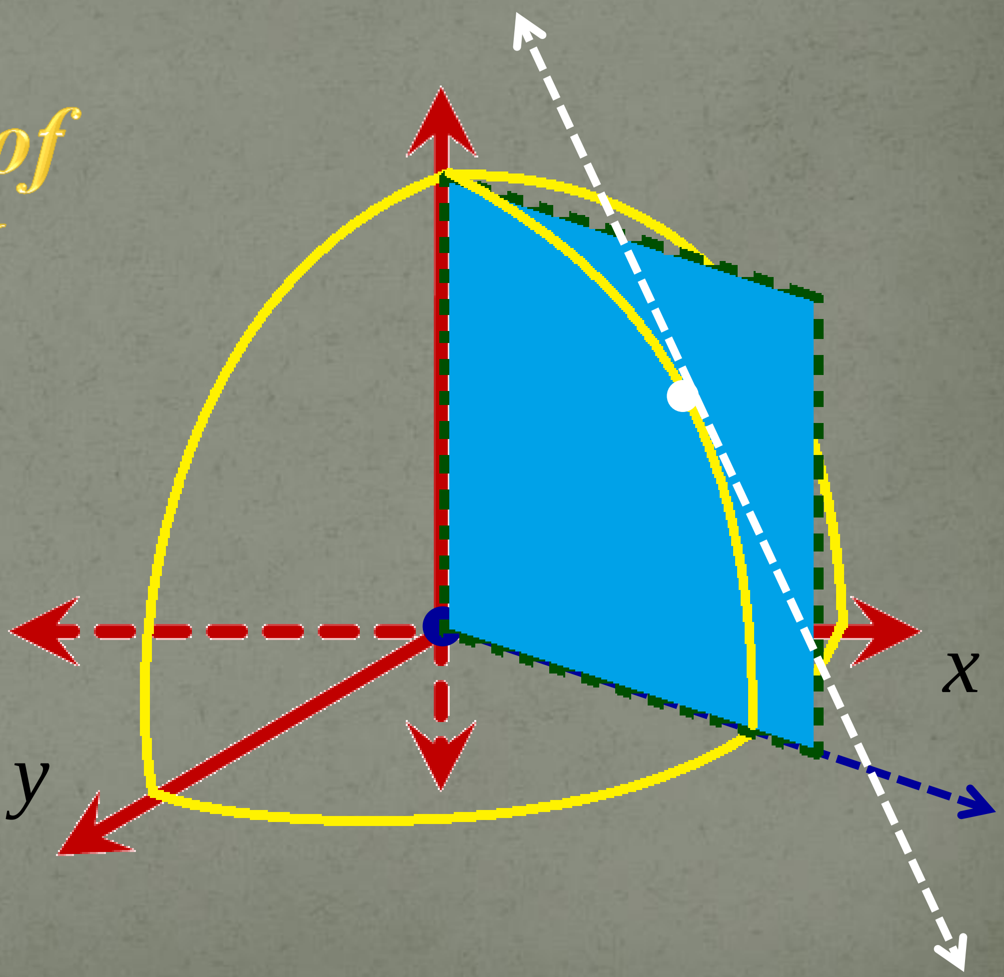
Let $f$ be a function of two variables $x$ and $y$. If $u$ is the unit vector $\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$, the ***derivative of f in the direction of u***, denoted by $D_u f$ is given by

$$D_u f\left(x,y\right)=\lim_{h\to 0}\frac{f\left(x+h\cos\theta, y+h\sin\theta\right)-f\left(x,y\right)}{h},$$

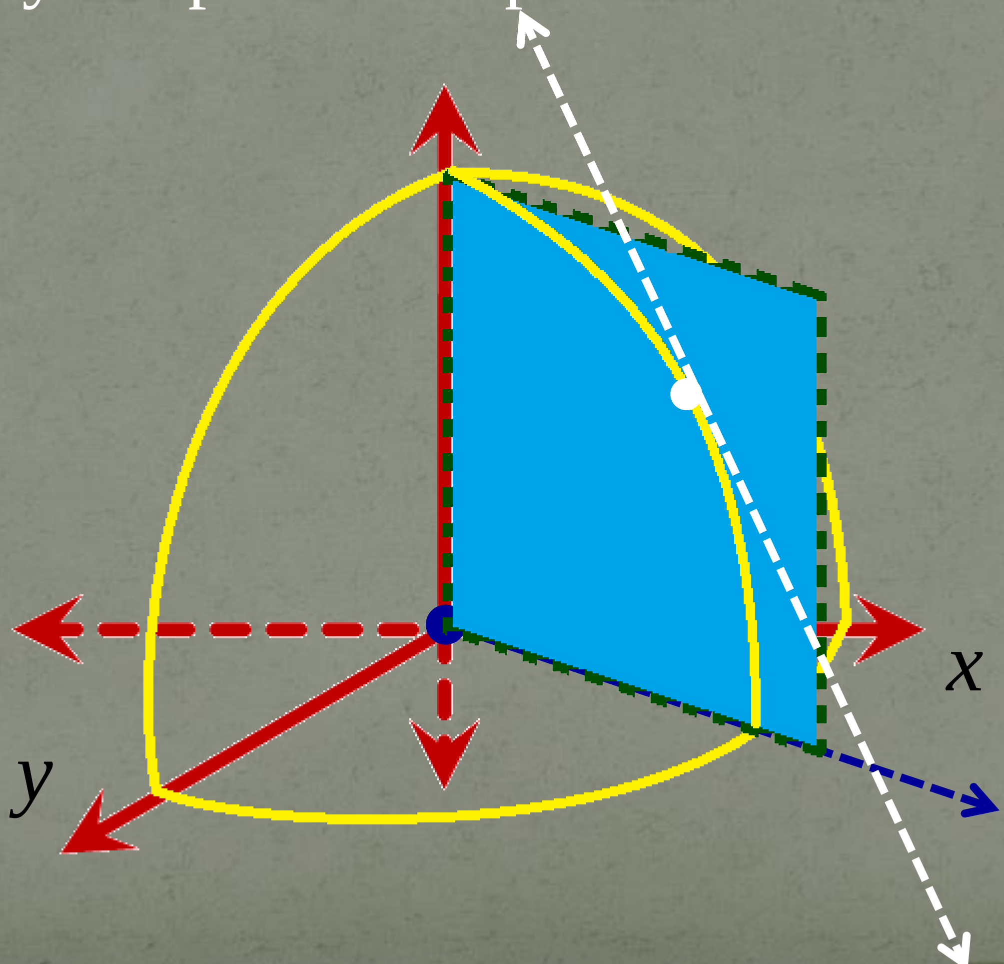
if this limit exists.

$$D_u f\left(x, y\right)$$

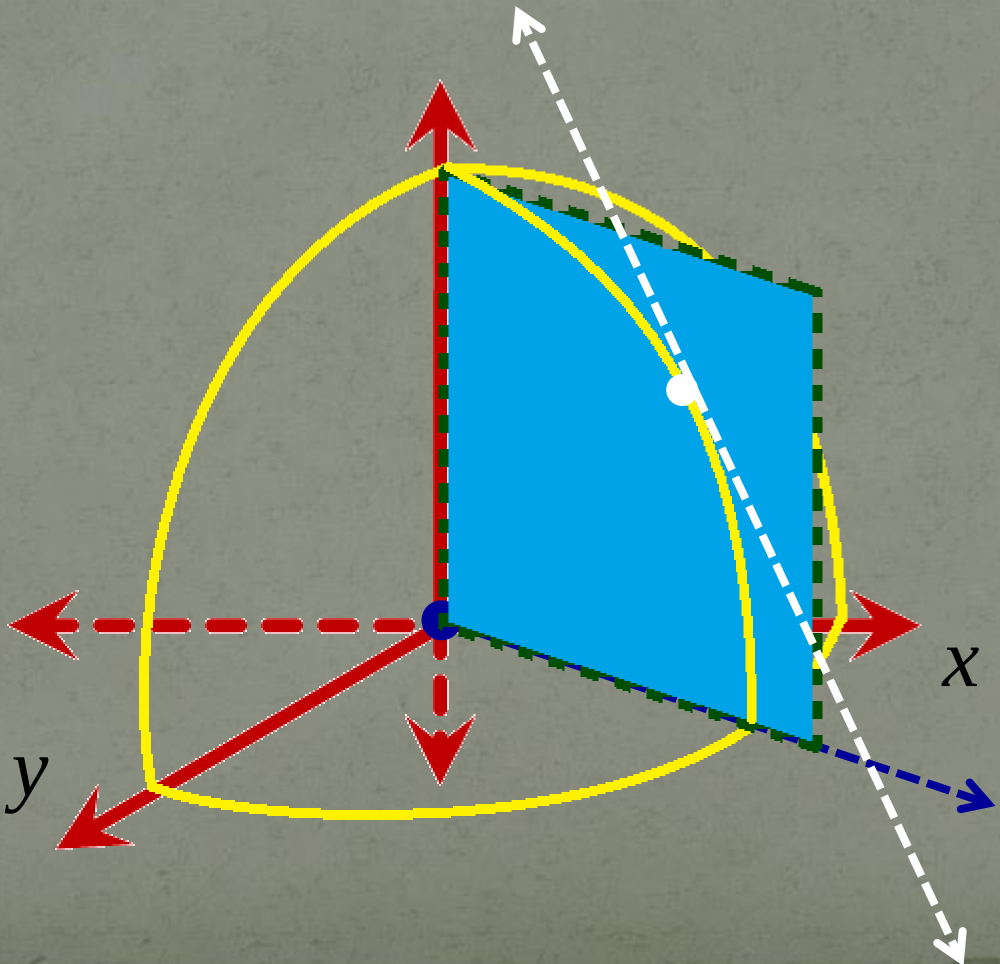
***Derivative of f in the direction of the unit vector U***

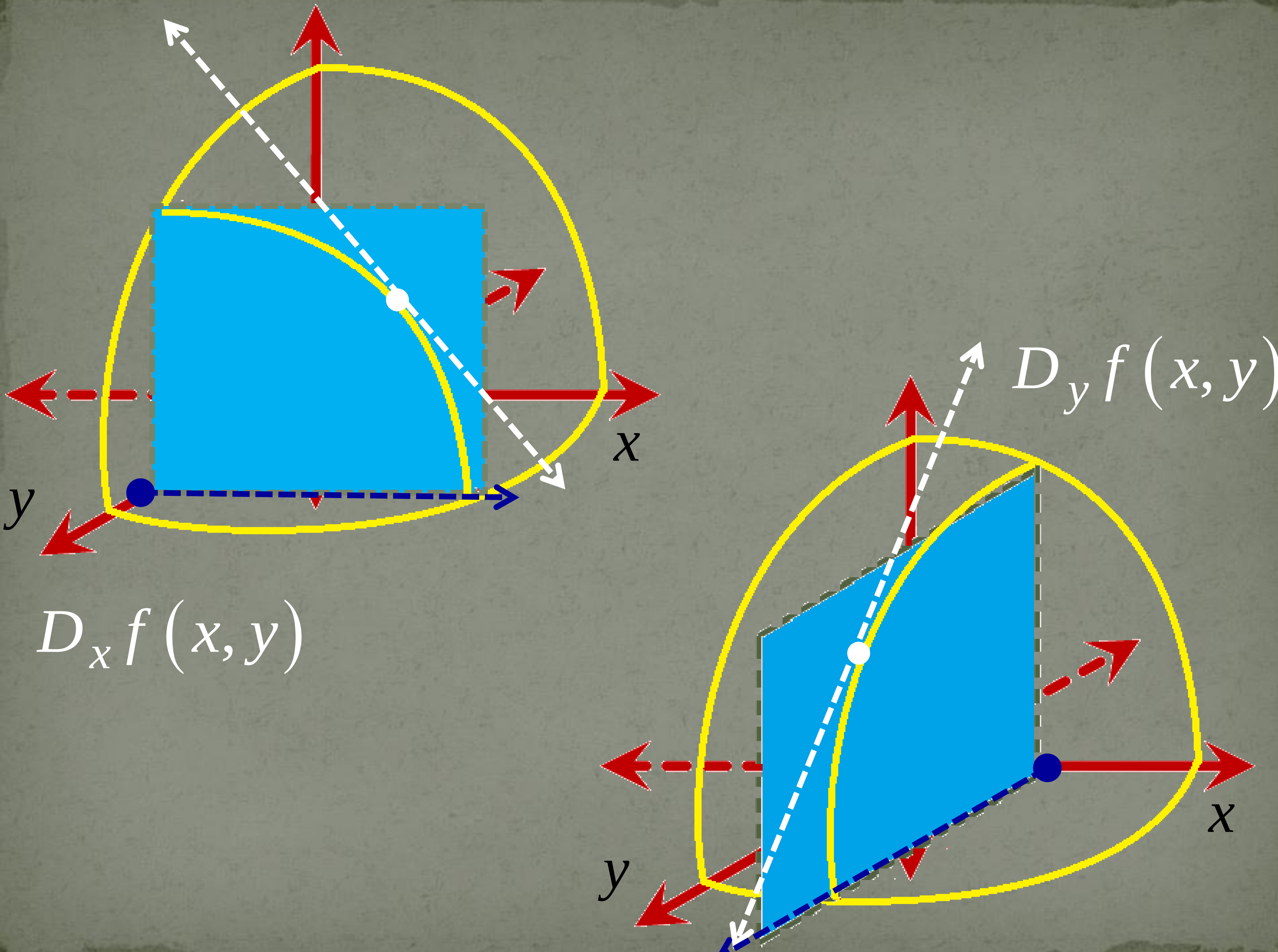


$D_u f\left(x_0, y_0\right)$ also gives the slope of the tangent line to the curve of intersection of the graph of *f* and the plane determined by the position representation of *u*.



$D_u f(x_0, y_0)$ gives the rate of change of the function $f$ at the point $(x_0, y_0)$ in the direction of the unit vector $u$.




x
y
$D_x f\left(x, y\right)$
$D_y f\left(x, y\right)$
x
y

**Example.** Find $D_u f(x,y)$ if

$$f(x,y) = 3xy^2 - x^2 \text{ and } u = \langle \cos\pi, \sin\pi \rangle$$

**Solution.**

$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x + h\cos\pi, y + h\sin\pi) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{f(x-h, y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\left[3(x-h)y^2 - (x-h)^2\right] - \left[3xy^2 - x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[3xy^2 - 3hy^2 - \left(x^2 - 2xh + h^2\right)\right] - \left[3xy^2 - x^2\right]}{h}$$

**Example.** Find $D_u f\left(x, y\right)$ if

$$f\left(x,y\right)=3xy^2-x^2 \quad \text{and} \quad u=\left\langle \cos\pi, \sin\pi \right\rangle$$

**Solution.**

$$D_u f\left(x, y\right)=\lim_{h\to 0}\frac{\left[3xy^2-3hy^2-\left(x^2-2xh+h^2\right)\right]-\left[3xy^2-x^2\right]}{h}$$

$$=\lim_{h\to 0}\frac{-3hy^2+2xh-h^2}{h}$$

$$=\lim_{h\to 0}\left(-3y^2+2x-h\right)$$

$$=-3y^2+2x$$

**Theorem.**

If $f$ is a differentiable function of $x$ and $y$, and $u = \langle \cos\theta, \sin\theta \rangle$ then

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle \cos\theta, \sin\theta \rangle$$

**Example.** Find $D_u f(x, y)$ using the previous theorem if

$$f(x,y) = 3xy^2 - x^2 \quad \text{and} \quad u = \langle \cos\pi, \sin\pi \rangle$$

**Solution.**

$$D_u f(x,y) = \left\langle f_x(x,y), f_y(x,y) \right\rangle \cdot \langle \cos\pi, \sin\pi \rangle$$

$$= \left\langle 3y^2 - 2x \ , \ 6xy \right\rangle \bullet \langle -1, \ 0 \rangle$$

$$= \left(3y^2 - 2x\right)(-1) + (6xy)(0)$$

$$= -3y^2 + 2x$$

$$\underbrace{D_u f(x,y)}_{\textit{directional derivative}} = \underbrace{\left\langle f_x(x,y), f_y(x,y) \right\rangle}_{\substack{\textit{gradient of } f \\ \nabla f,\, grad\ f}} \cdot \underbrace{\langle \cos\pi, \sin\pi \rangle}_{\substack{\textit{unit vector} \\ u}}$$

**(del *f*)**

**Thus,**

$$D_u f(x,y) = \nabla f(x,y) \bullet u$$

## **Exercise. Finding Directional Derivative**

Find the derivative of the function at $P_0$ in the direction of A.

$P_0(1,-1)$

**1.** $f(x,y) = 2xy - 3y^2$

$$A = \frac{1}{2}i + \frac{\sqrt{3}}{2}j$$

$P_0(5,5)$

**2.** $f(x,y) = x^2 + 2y^2$

$$A = \frac{4}{5}i + \frac{3}{5}j$$

## Exercise. Finding Directional Derivative

Find the derivative of the function at $P_0$ in the direction of A.

**3.** $f\left(x,y\right)=xe^{y}+\sqrt{y}\ln x$  $P_0\left(2,1\right)$

$A=i+j$

**4.** $f\left(x,y\right)=x-\dfrac{2y^2}{x}+Arc\sin y$  $P_0\left(-2,0\right)$

$A=3i-2j$

**Theorem.** Let $f$ be a function of two variables $x$ and $y$ and let $f$ be differentiable at the point $(x_0, y_0)$ where $\nabla f(x_0, y_0) \neq 0$.

Let $u$ be any unit vector so that $D_u f(x_0, y_0)$ is a function of $u$.

i. The ***maximum value*** of

$$D_u f(x, y) = u \cdot \nabla f(x, y)$$

is .

***Steepest ascent***

i. The ***minimum value*** of

$$D_u f(x, y) = u \cdot \nabla f(x, y)$$

is $-\left\|\nabla f(x_0, y_0)\right\|$.

***Steepest descent***

This value is attained when the direction of $u$ is <u>opposite</u> the direction of $\nabla f(x_0, y_0)$.

**Example.** If $f(x,y)=2^{-x}\sin y$, find the maximum value and the minimum value of $D_u f(-1,0)$

**Solution.**

$$\nabla f(x,y)=\left\langle f_x(x,y), f_y(x,y)\right\rangle$$

$$=\left\langle -2^{-x}\sin y\ln 2\ ,\ 2^{-x}\cos y\right\rangle$$

$$\nabla f(-1,0)=\left\langle 0\ ,\ \right\rangle \qquad \left\|\nabla f(-1,0)\right\|=\sqrt{0^2+2^2}=2$$

Thus, the maximum value of $D_u f(-1,0)$ is $2$.

Thus, the minimum value of $D_u f(-1,0)$ is $-2$.

## **Exercise.** **Directional of Most Rapid Increase or Decrease**

Find the directions which the functions increase and decrease most rapidly at $P_0$

**1.** $f(x,y) = x^2 y + e^{xy}\sin y$ $\quad P_0(1,0)$

**2.** $f(x,y) = \cos x \cos y$ $\quad P_0\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

**3.** $f(x,y) = y^2 e^{-2x}$ $\quad P_0(0,1)$

**Definition.**

Let $f$ be a function of three variables $x$, $y$ and $z$.

If $u$ is the unit vector $\cos\alpha\,\boldsymbol{i}+\cos\beta\,\boldsymbol{j}+\cos\gamma\,\boldsymbol{k}$

then the ***derivative of f in the direction of u***,

denoted by $D_u f$ is given by

$$D_u f\left(x,y,z\right)=\lim_{h\to 0}\frac{f\left(x+h\cos\alpha,y+h\cos\beta,z+h\cos\gamma\right)-f\left(x,y,z\right)}{h}$$

if this limit exists.

If $f$ is a function of three variables $x$, $y$ and $z$, and the first partial derivatives $f_x$, $f_y$, and $f_z$, exist, the ***gradient of $f$*** denoted by $\nabla f$ is given by

$$\nabla f(x,y,z)=\langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle.$$

**Example.** If $f(x,y,z)=x\sec yz-ye^{xz}$

$$\nabla f(x,y,z)=$$

$$\left\langle \sec yz-yze^{xz},\ xz\sec yz\tan yz-e^{xz},\ xy\sec yz\tan yz-yxe^{xz}\right\rangle$$

**Theorem.**

If $f$ is a differentiable function of $x$, $y$ and $z$, and

$$\boldsymbol{u} = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k}$$

then

$$D_u f(x,y,z) = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma.$$

**So,**

$$D_u f(x,y,z) = u \cdot \nabla f(x,y,z)$$

**Example.** Find the gradient of the indicated function at the point $P_0$ and use this to find $D_u f\left(P_0\right)$.

a. $f\left(x,y,z\right)=\cos\left(xy\right)+\sin\left(yz\right)$

$$P_0=\left(2,0,-3\right)\ ;\ \ u=\frac{1}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k}$$

**Solution:**

$$\nabla f\left(x,y,z\right)=\left\langle -y\sin\left(xy\right),-x\sin\left(xy\right)+z\cos\left(yz\right),y\cos\left(yz\right)\right\rangle$$

$$\nabla f\left(2,0,-3\right)=\left\langle 0,\quad -3,\quad 0\right\rangle$$

$$\nabla f(2,0,-3) = \langle 0,\quad -3,\quad 0 \rangle$$

$$P_0 = (2,0,-3)\ ;\quad u = \frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}$$

$$D_u f(2,0,-3) = u \cdot \nabla f(2,0,-3)$$

$$= \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \cdot \langle 0,-3,0 \rangle$$

$$= -2$$

b. $f(x, y, z) = x^2 + y^2 - 4xz;$

$u$ is the unit vector in the direction of $\overrightarrow{P_0P}$ where $P = (-6, 3, 4)$.

**Solution.**

$$\nabla f(x, y, z) = \langle 2x - 4z, 2y, -4x \rangle$$

$$\nabla f(3,\ 1,\ -2) = \langle 14,\ \ 2,\ \ -12 \rangle$$

$$\left\| \overrightarrow{P_0P} \right\| = 11$$

$$\Rightarrow u_{\overrightarrow{P_0P}} = \left\langle \frac{-9}{11}, \frac{2}{11}, \frac{6}{11} \right\rangle$$

$$\nabla f\left(3,\ 1,\ -2\right)=\langle 14,\ 2,\ -12\rangle$$

$$u_{\overrightarrow{P_0P}}=\left\langle \frac{-9}{11},\frac{2}{11},\frac{6}{11}\right\rangle$$

$$D_u f\left(3,1,-2\right)=u\cdot\nabla f\left(3,1,-2\right)$$

$$D_u f\left(3,1,-2\right)=\left\langle \frac{-9}{11},\frac{2}{11},\frac{6}{11}\right\rangle\cdot\langle 14,2,-12\rangle$$

$$=-\frac{126}{11}+\frac{4}{11}-\frac{72}{11}$$

$$=\frac{-194}{11}$$

END

**Example.** Let

$$f(x,y) = y^2 Arc\tan x,\ \ P_0 = \left(\sqrt{3},2\right) \text{ and } u = \left\langle \frac{-\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Find the gradient of $f$ at the point $P_0$ and use this to find the value of the directional derivative of $f$ in the direction of $u$.

**Solution.** $\nabla f(x,y) = \left\langle f_x(x,y), f_y(x,y) \right\rangle$

$$= \left\langle \frac{y^2}{1+x^2}, 2yArc\tan x \right\rangle$$

**So,** $\nabla f\left(\sqrt{3},2\right) = \left\langle 1, \frac{4\pi}{3} \right\rangle$

$$D_u f\left(\sqrt{3},2\right)=\nabla f\left(\sqrt{3},2\right)\bullet u$$

$$=\left\langle 1,\frac{4\pi}{3}\right\rangle\cdot\left\langle\frac{-\sqrt{3}}{2},\frac{1}{2}\right\rangle$$

$$=(1)\left(\frac{-\sqrt{3}}{2}\right)+\left(\frac{4\pi}{3}\right)\left(\frac{1}{2}\right)$$

$$=\frac{-\sqrt{3}}{2}+\frac{2\pi}{3}$$

**Example.** $f(x,y)=x^2y+e^{xy}$, $P_0=(0,-1)$ and $v=\langle 4,3\rangle$

Find the gradient of $f$ at the point $P_0$ and use this to find the value of the directional derivative of $f$ in the direction of vector $v$.

**Solution.**

$$\nabla f(x,y)=\left\langle 2xy+ye^{xy}, x^2+xe^{xy}\right\rangle$$

$$\nabla f(0,-1)=\langle -1,0\rangle$$

$$D_u f(0,-1)=\nabla f(0,-1)\bullet u$$

$$=\langle -1,0\rangle\bullet\left\langle \frac{4}{5},\frac{3}{5}\right\rangle=\frac{-4}{5}$$

**Example.** The density at any point of a rectangular plate in the *xy* plane is $\rho(x, y)$ kilograms per square meter where

$$\rho(x, y) = \frac{1}{\sqrt{x^2 + y^2 + 3}}.$$

Find the magnitude and direction of the greatest rate of change of $\rho$ at the point (3,2).

**Solution.** $\rho(x,y)=\dfrac{1}{\sqrt{x^2+y^2+3}}=\left(x^2+y^2+3\right)^{-1/2}$

$$\rho_x(x,y)=\frac{-1}{2}\left(x^2+y^2+3\right)^{-3/2}\cdot 2x=\frac{-x}{\left(x^2+y^2+3\right)^{3/2}}$$

$$\rho_y(x,y)=\frac{-1}{2}\left(x^2+y^2+3\right)^{-3/2}\cdot 2y=\frac{-y}{\left(x^2+y^2+3\right)^{3/2}}$$

$$\nabla\rho(x,y)=\left\langle -x\left(x^2+y^2+3\right)^{-3/2},-y\left(x^2+y^2+3\right)^{-3/2}\right\rangle$$

$$\nabla\rho(3,2)=\left\langle -3\left(3^2+2^2+3\right)^{-3/2},-2\left(3^2+2^2+3\right)^{-3/2}\right\rangle$$

$$=\left\langle\frac{-3}{64},\frac{-2}{64}\right\rangle=\left\langle\frac{-3}{64},\frac{-1}{32}\right\rangle$$

$$\nabla\rho(3,2)=\left\langle\frac{-3}{64},\frac{-1}{32}\right\rangle$$

$$\left\|\nabla\rho(3,2)\right\|=\sqrt{\left(\frac{-3}{64}\right)^2+\left(\frac{-1}{32}\right)^2}=\sqrt{\frac{9}{(64)^2}+\frac{1}{(32)^2}}$$

$$=\sqrt{\frac{9}{2^2(32)^2}+\frac{1}{(32)^2}}=\sqrt{\frac{9+4}{2^2(32)^2}}=\frac{\sqrt{13}}{64}$$

The magnitude of the greatest rate of change of $\rho$ at (3,2) is $\frac{\sqrt{13}}{64}$.

The direction $\theta$ of the greatest rate of change of $\rho$ at (3,2) is along the direction of the gradient vector . Thus since $\nabla\rho(3,2) = \left\langle \frac{-3}{64}, \frac{-1}{32} \right\rangle$, then

$$\tan\theta = \frac{\frac{-1}{32}}{\frac{-3}{64}} = \frac{2}{3}$$

so that

$$\theta = \pi + Arc\tan\left(\frac{2}{3}\right).$$