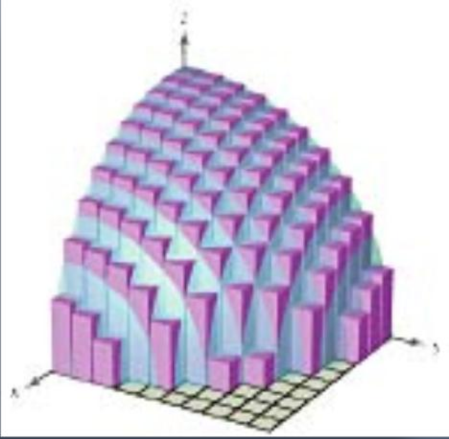
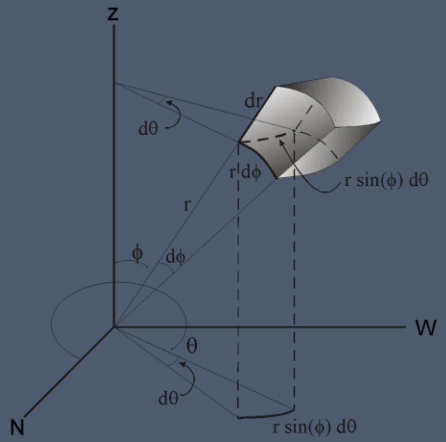
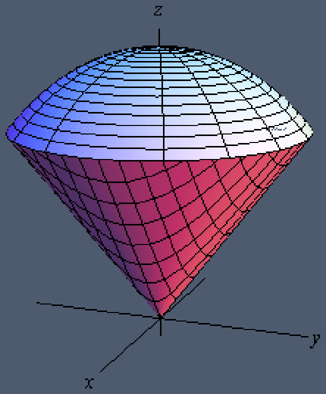
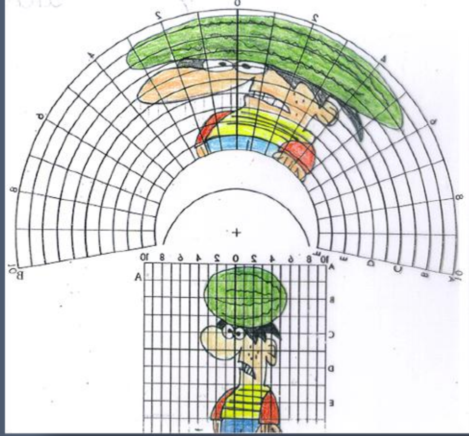




$\{r, \theta, z\}$





CHAPTER 4

Multiple Integration

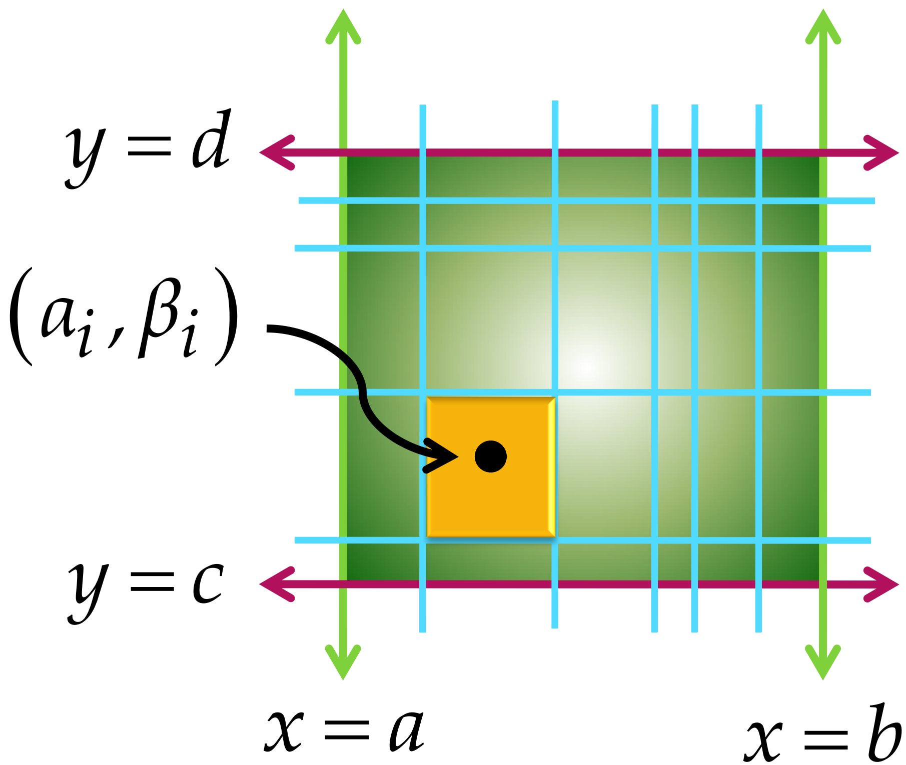
Topics to be discussed:

- Double integration
  - In rectangular coordinates, In polar coordinates
- Triple integration
  - In rectangular coordinates, In cylindrical coordinates, In spherical coordinates
- Applications
  - Area, Volume

4.1 Double Integral in Rectangular Coordinates

Let R be a region in the plane which is bounded by $x = a, x = b, y = c$ and $y = d$, where $a < b$ and $c < d$.

$$Area_{R_i} = \Delta_i x \cdot \Delta_i y$$



3.1 Double Integral in Rectangular Coordinates

Now, obtain $\sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y$

If $\lim_{n \to \infty} \sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y$ exists, then this limit is called the ***double integral*** of f over the region R.

Double Integral of f over R

In symbols,

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(a_i,\beta_i)\Delta_i x \Delta_i y$$

Double Integral of f over R

REMARKS:

- $dA = dx\,dy = dy\,dx$
- Double integrals have the same kind of domain additivity property that single integrals have.
- Double integrals are evaluated as iterated integrals.

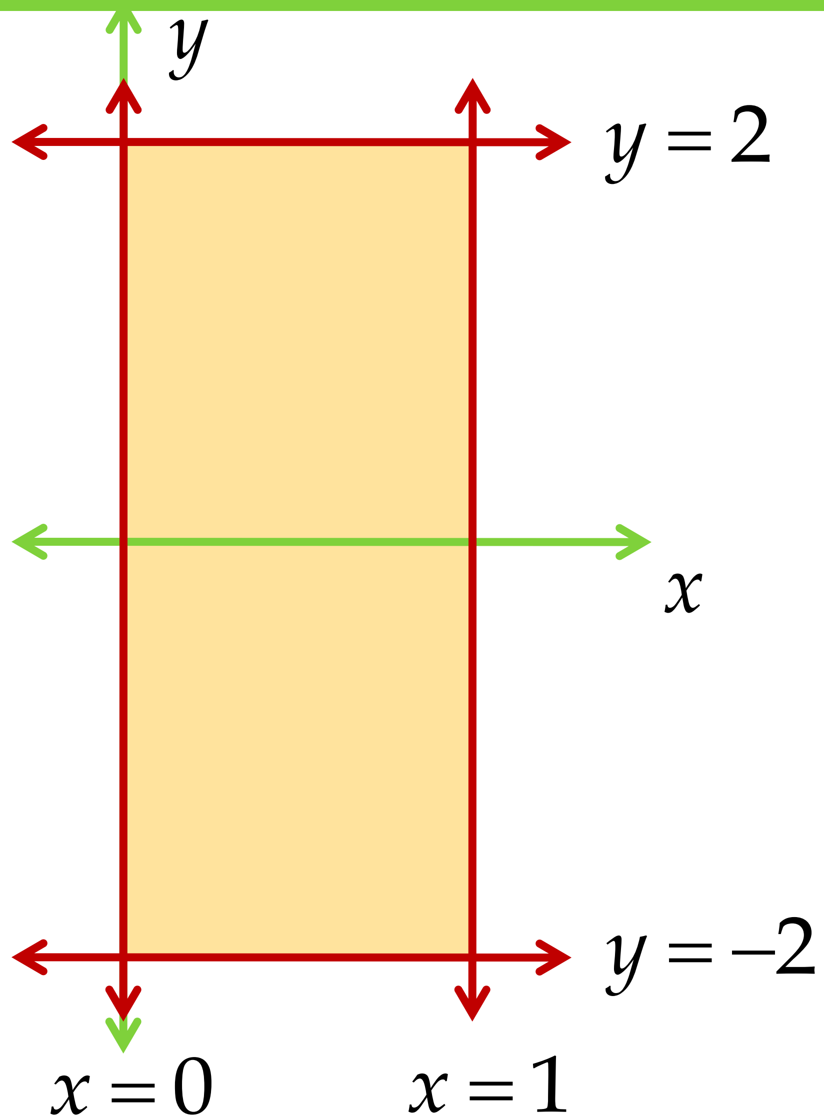
Evaluating Double Integrals

$$\int_{-2}^{2}\int_{0}^{1}(4-xy)\,dx\,dy \quad = \int_{-2}^{2}\left[\int_{0}^{1}(4-xy)\,dx\right]dy$$

$$= \int_{-2}^{2}\left[4x-\frac{1}{2}x^2y\right]_0^1 dy \quad = \int_{-2}^{2}\left(4-\frac{y}{2}\right)dy$$

$$= \left(4y-\frac{y^2}{4}\right)_{-2}^{2} = (8-1) - (-8-1) \quad = 16$$

Region of Integration



$$\int_{-2}^{2}\int_{0}^{1}(4 - xy)\,dx\,dy$$

$$= \int_{0}^{1}\int_{-2}^{2}(4 - xy)\,dy\,dx$$

$$= 16$$

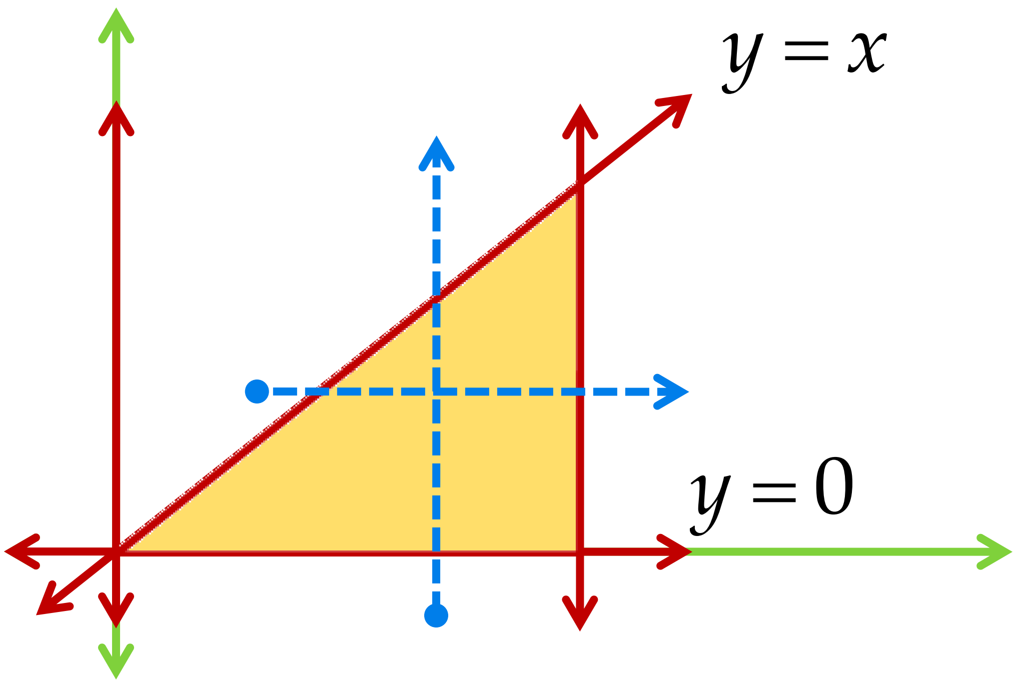
Evaluating Double Integrals

$$\int_0^{\pi}\int_0^{x}\frac{\sin x}{x}\,dy\,dx = \int_0^{\pi}\left(\int_0^{x}\frac{\sin x}{x}\,dy\right)dx$$

$$= \int_0^{\pi}\left(y\,\frac{\sin x}{x}\right)_0^{x}dx = \int_0^{\pi}\sin x\,dx$$

$$= \left(-\cos x\right]_0^{\pi} = 2$$

Region of Integration

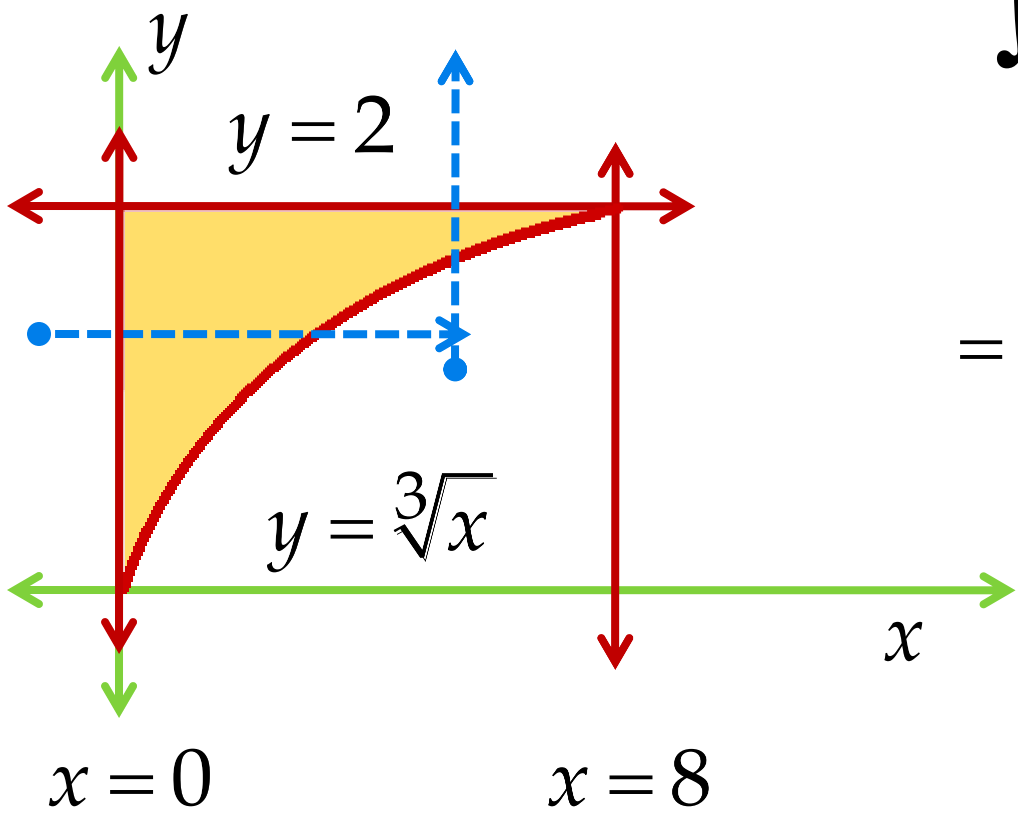


$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= 2$$

$$= \int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$$

Evaluating Double Integrals



$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$$

$$=\int_0^2 \int_0^{y^3} \frac{dx\,dy}{y^4+1}$$

Evaluating Double Integrals

$$\int_0^2 \int_0^{y^3} \frac{dx\,dy}{y^4+1} = \int_0^2 \left(\int_0^{y^3} \frac{dx}{y^4+1} \right) dy$$

$$= \int_0^2 \frac{y^3}{y^4+1}\,dy \quad = \frac{1}{4} ln\left(y^4+1\right)\Big|_0^2$$

$$= \frac{1}{4} ln(17)$$

Exercise.

Sketch the region of integration. Write an integral with the order of integration reversed. Then evaluate both integrals.

1. $\displaystyle\int_0^2 \int_0^{4-x} 2x\,dy\,dx \qquad = \frac{32}{3}$

2. $\displaystyle\int_0^1 \int_{x^2}^{x} \sqrt{x}\,dy\,dx \qquad = \frac{4}{35}$

3. $\displaystyle\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy \qquad = \frac{4}{3}$

4. $\displaystyle\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy \qquad = \frac{9}{2}$

Assignment.

Verify

1. $$\int_{-1}^{1}\int_{4}^{9}\left(x^2+\frac{3x}{\sqrt{y}}\right)dydx=\frac{10}{3}$$

2. $$\int_{0}^{1}\int_{\sqrt{y}}^{2-\sqrt{y}} xydxdy \;=\frac{1}{5}$$

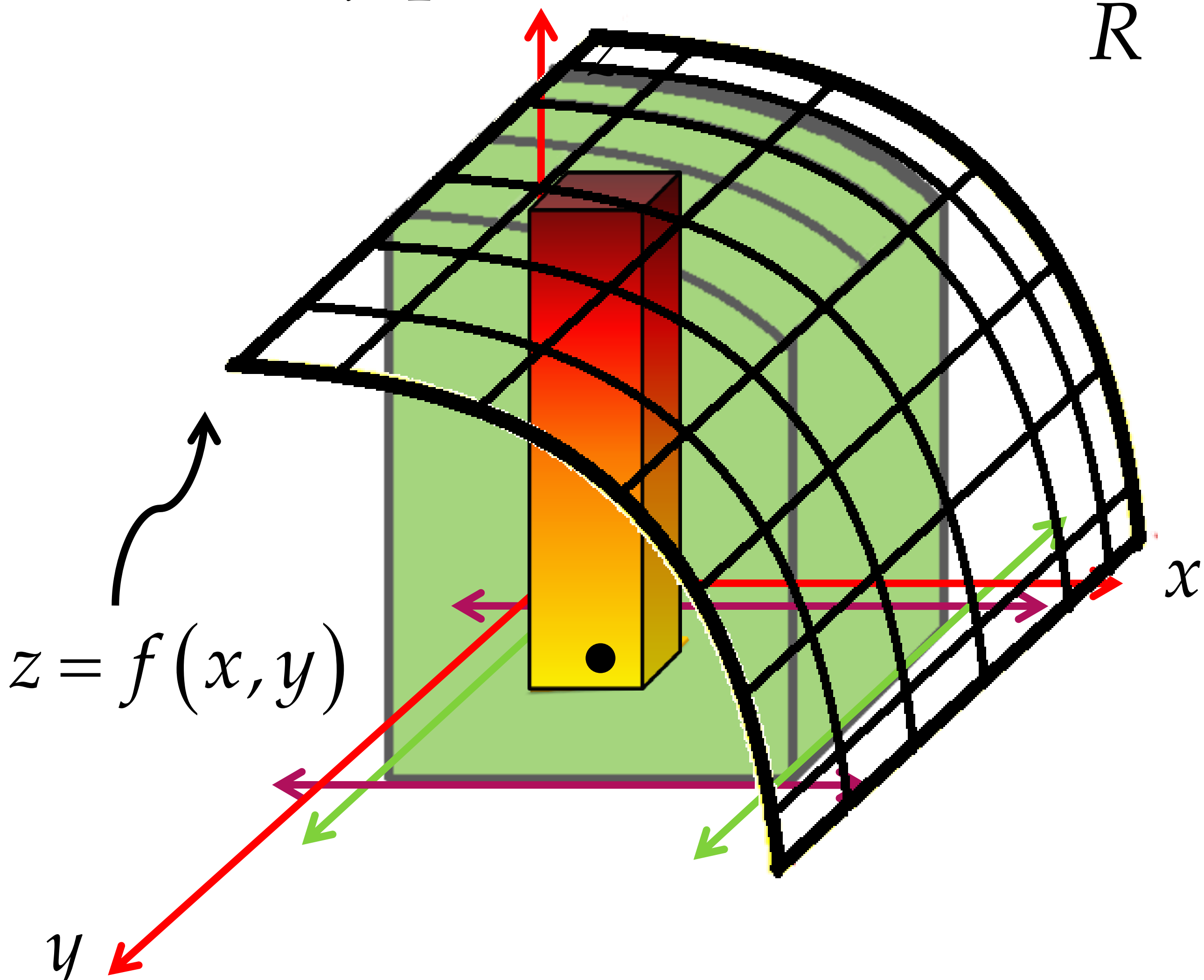
Application of Double Integrals

REMARKS:

- If $f(x,y) \geq 0$ for all (x,y) in a region ***R***, the double integral of f over ***R*** is the volume of the solid whose base is ***R*** and whose height at a point (x,y) in ***R*** is $f(x,y)$.

- If $f(x,y) = 1$, the double integral of f over a region ***R*** is just the area of ***R***.

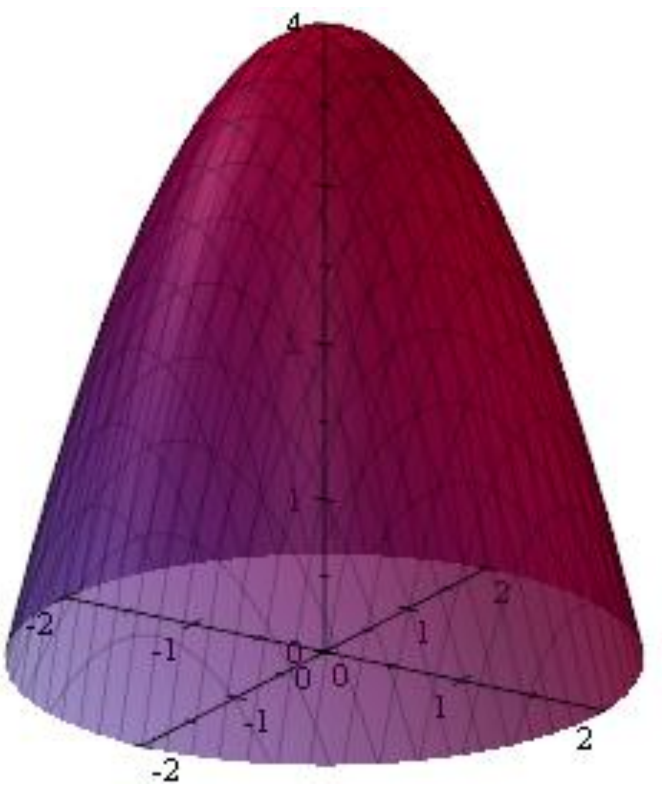
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} f(a_i, \beta_i)\Delta_i x \Delta_i y = \iint_R f(x,y)\,dA$$

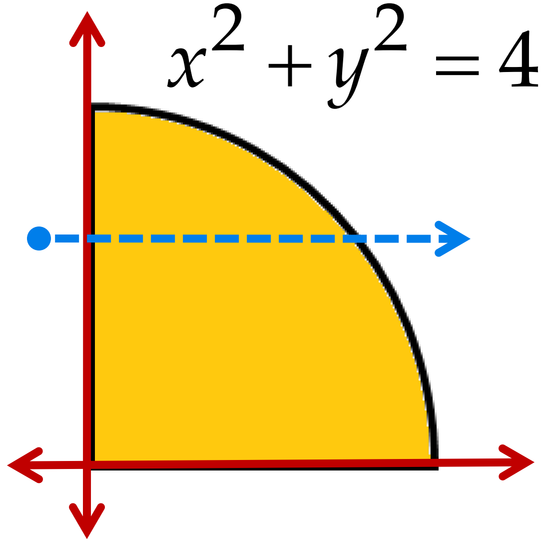


Exercise. **SET-UP** the double integral which gives the volume of the solid described.

a. Solid in the first octant bounded by

$$f(x,y) = 4 - x^2 - y^2$$



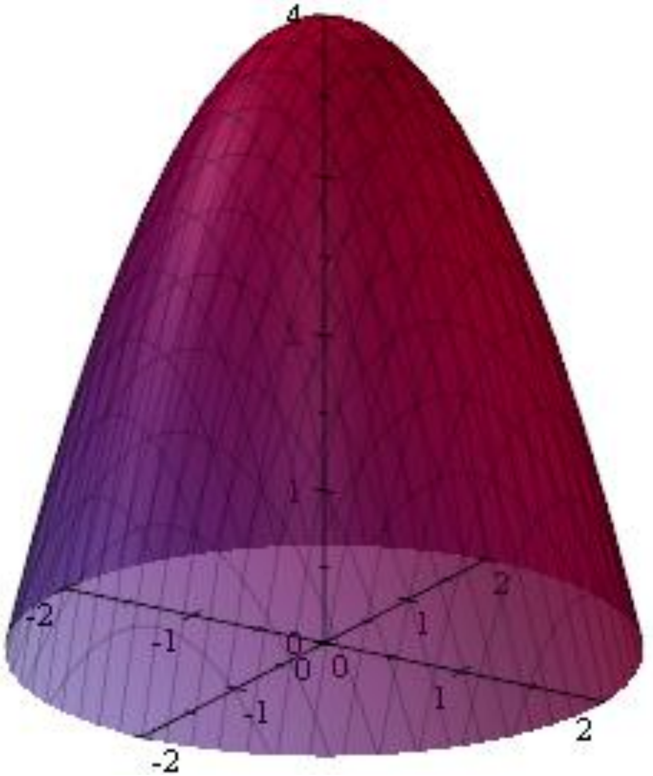


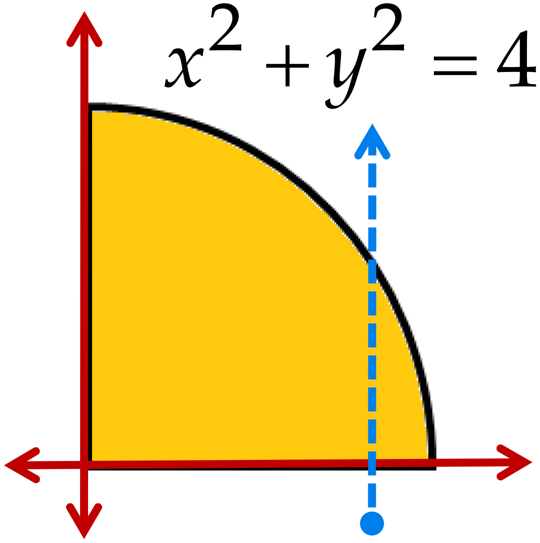
$$V = \int_0^2 \int_0^{\sqrt{4-y^2}} \left(4 - x^2 - y^2\right) dx\, dy$$

Exercise. **SET-UP** the double integral which gives the volume of the solid described.

a. Solid in the first octant bounded by

$$f(x,y)=4-x^2-y^2$$





$$V=\int_0^2\int_0^{\sqrt{4-x^2}}\left(4-x^2-y^2\right)dy\,dx$$

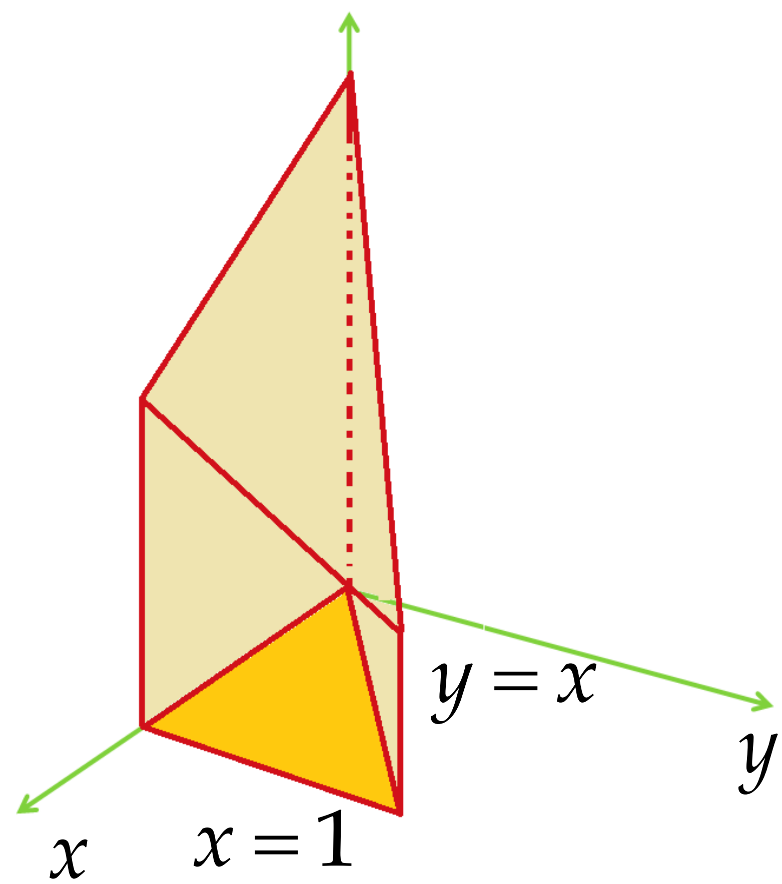
Exercise. **SET-UP** the double integral which gives the volume of the solid described.

b. Prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis, and the lines

$$x = 1, y = x$$

and whose top lies in the plane

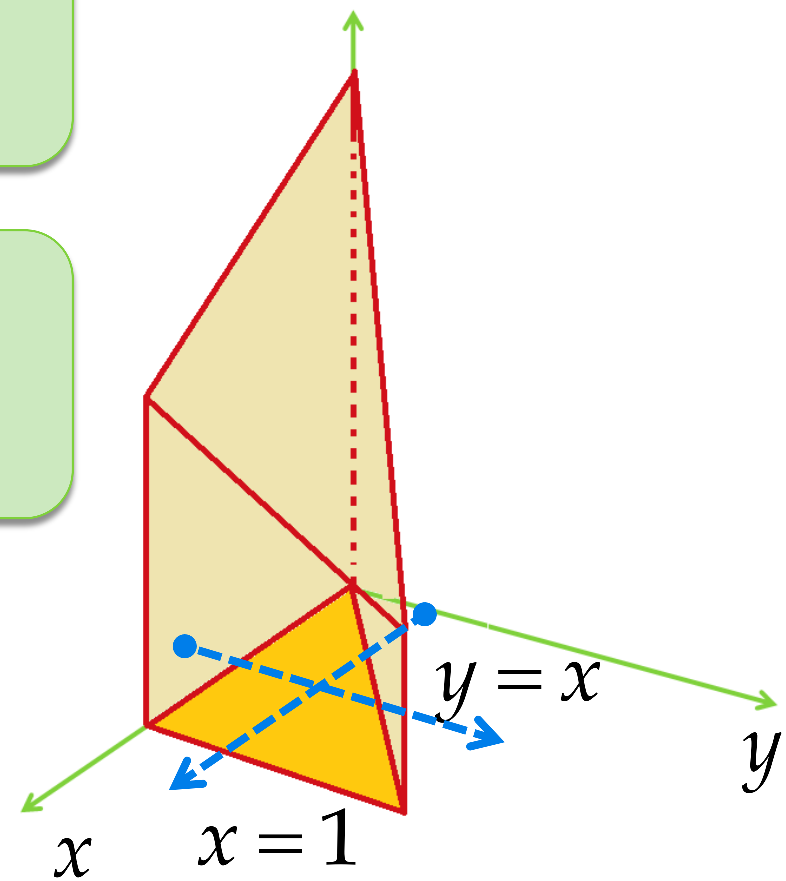
$$f(x, y) = 3 - x - y$$



Exercise. **SET-UP** the double integral which gives the volume of the solid described.

$$V=\int_0^1\int_0^x (3-x-y)\,dy\,dx$$

$$V=\int_0^1\int_y^1 (3-x-y)\,dx\,dy$$



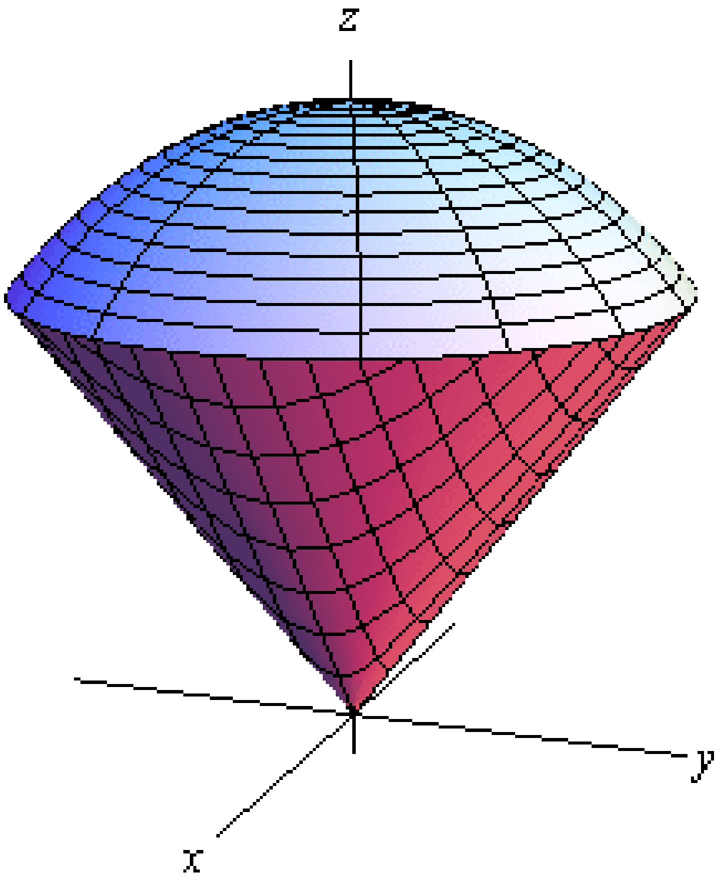
Exercise. **SET-UP** the double integral which gives the volume of the solid described.

c. Bounded above by

$$f(x,y)=\sqrt{2-x^2-y^2}$$

but bounded below by

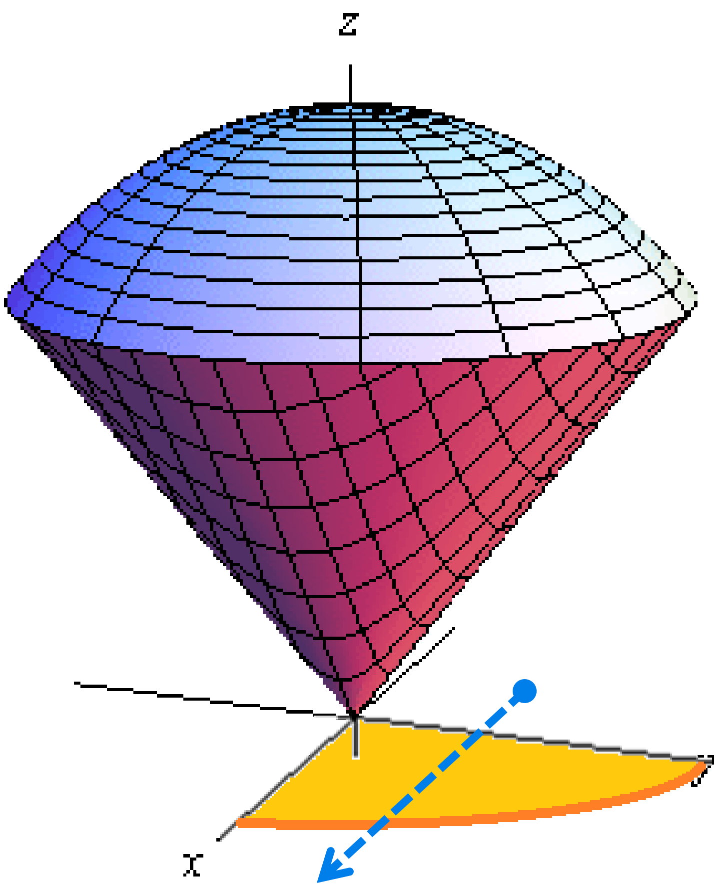
$$f(x,y)=\sqrt{x^2+y^2}$$



Exercise. **SET-UP** the double integral which gives the volume of the solid described.

$$V = 4\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{2-x^2-y^2}\,dx\,dy$$

$$-4\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\,dx\,dy$$



$x^2+y^2=1$

END