

1.8

*Differentiation of
POWER SERIES*

Term-by-Term Differentiation

A power series can be differentiated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\ + c_n x^n + \dots$$

$$\sum_{n=1}^{+\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots \\ + n c_n x^{n-1} + \dots$$

Theorem.

If the power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the

power series

$$f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

also has R as its radius of convergence.

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n \implies f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

$$\implies f''(x) = \sum_{n=2}^{+\infty} n(n-1) c_n x^{n-2}$$

$$\implies f'''(x) = \sum_{n=2}^{+\infty} n(n-1)(n-2) c_n x^{n-3}$$

⋮

Example. Find a series expansion for $f'(x)$ and $f''(x)$

if $f(x) = \frac{1}{1-x}$, $-1 < x < 1$

SOL'N.

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots = \sum_{n=0}^{+\infty} x^n$$

$$, -1 < x < 1$$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots = \sum_{n=1}^{+\infty} nx^{n-1}$$

$$, -1 < x < 1$$

Example. Find a series expansion for $f'(x)$ and $f''(x)$

if $f(x) = \frac{1}{1-x}$, $-1 < x < 1$

SOL'N.

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots = \sum_{n=1}^{+\infty} nx^{n-1}$$

$$, -1 < x < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + \dots + n(n-1)x^{n-2} + \dots$$

$$= \sum_{n=2}^{+\infty} n(n-1)x^{n-2}, -1 < x < 1$$

Example. Obtain a series expansion for $\frac{1}{(1+3x)^2}$

and give its validity.

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}, \quad -1 < r < 1$$

SOL'N. $a = 1$, $r = -3x$

$$-1 < 3x < 1 \quad \Rightarrow \quad -\frac{1}{3} < x < \frac{1}{3}$$

$$g(x) = \frac{1}{1+3x} = \sum_{n=1}^{+\infty} (-3x)^{n-1} = \sum_{n=1}^{+\infty} (-1)^{n-1} 3^{n-1} x^{n-1}$$

$$g'(x) = \frac{-3}{(1+3x)^2} = \sum_{n=2}^{+\infty} (-1)^{n-1} 3^{n-1} (n-1) x^{n-2}$$

Example. Obtain a series expansion for $\frac{1}{(1+3x)^2}$ and give its validity.

SOL'N. Since, $\frac{-3}{(1+3x)^2} = \sum_{n=2}^{+\infty} (-1)^{n-1} 3^{n-1} (n-1) x^{n-2}$

We'll have

$$\begin{aligned} \frac{1}{(1+3x)^2} &= \frac{1}{-3} \sum_{n=2}^{+\infty} (-1)^{n-1} 3^{n-1} (n-1) x^{n-2} \\ &= \sum_{n=2}^{+\infty} (-1)^{n-2} 3^{n-2} (n-1) x^{n-2}, -\frac{1}{3} < x < \frac{1}{3} \end{aligned}$$

Example. Obtain a series expansion for $\frac{1}{(7-2x)^2}$

and give its validity.

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}, \quad -1 < r < 1$$

SOL'N. $a=1, r=\frac{2}{7}x$

$$-1 < \frac{2}{7}x < 1 \Rightarrow -\frac{7}{2} < x < \frac{7}{2}$$

$$g(x) = \frac{1}{1 - \frac{2}{7}x} = \frac{7}{7-2x} = \sum_{n=1}^{+\infty} \left(\frac{2}{7}x\right)^{n-1} = \sum_{n=1}^{+\infty} \left(\frac{2}{7}\right)^{n-1} x^{n-1}$$

$$g'(x) = \frac{14}{(7-2x)^2} = \sum_{n=2}^{+\infty} \left(\frac{2}{7}\right)^{n-1} (n-1) x^{n-2}$$

Example. Obtain a series expansion for $\frac{1}{(7-2x)^2}$ and give its validity.

SOL'N. Since,
$$\frac{14}{(7-2x)^2} = \sum_{n=2}^{+\infty} \left(\frac{2}{7}\right)^{n-1} (n-1)x^{n-2}$$

We'll have

$$\begin{aligned} \frac{1}{(7-2x)^2} &= \frac{1}{14} \sum_{n=2}^{+\infty} \left(\frac{2}{7}\right)^{n-1} (n-1)x^{n-2} \\ &= \sum_{n=2}^{+\infty} \frac{(n-1)2^{n-2}x^{n-2}}{7^n} \quad -\frac{7}{2} < x < \frac{7}{2} \end{aligned}$$

The background of the slide features a complex, multi-layered pattern of colorful waveforms and peaks. The colors include yellow, green, pink, and light blue. The patterns are somewhat abstract, resembling a combination of sine waves and sharp, pointed peaks, creating a vibrant and textured backdrop for the text.

Integration of POWER SERIES

Term-by-Term Integration

A power series can be integrated term by term at each interior point of its interval of convergence.

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\ + c_n x^n + \dots$$

$$\sum_{n=0}^{+\infty} \frac{c_n x^{n+1}}{n+1} + C = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \dots \\ + c_n \frac{x^{n+1}}{n+1} + \dots$$

Theorem.

If the power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

has R as its radius of convergence, then the
power series

$$\int f(x) dx = \sum_{n=0}^{+\infty} c_n \frac{x^{n+1}}{n+1} + C$$

also has R as its radius of convergence.

Example. Obtain a series expansion for $\ln(2+x)$

and give its validity.

$$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}, \quad -1 < r < 1$$

SOL'N. $a=1$, $r=-\frac{1}{2}x$

$$-1 < \frac{1}{2}x < 1 \quad \Rightarrow \quad -2 < x < 2$$

$$g(x) = \frac{1}{1 + \frac{1}{2}x} = \frac{2}{2+x} = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}x\right)^{n-1} = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} x^{n-1}$$

$$g(t) = \frac{2}{2+t} = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} t^{n-1}$$

Example. Obtain a series expansion for $\ln(2+x)$

and give its validity.

SOL'N.
$$\int_0^x \frac{2dt}{2+t} = \int_0^x \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} t^{n-1} dt$$

$$\left(2\ln(2+t)\right)\Big|_0^x = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \left[\frac{t^n}{n}\right]_0^x$$

$$2\ln(2+x) - 2\ln 2 = \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \frac{x^n}{n}$$

$$\ln(2+x) = \frac{1}{2} \sum_{n=1}^{+\infty} \left(-\frac{1}{2}\right)^{n-1} \frac{x^n}{n} + \ln 2, \quad -2 < x < 2$$



END