CMSC 170

Introduction to Artificial Intelligence

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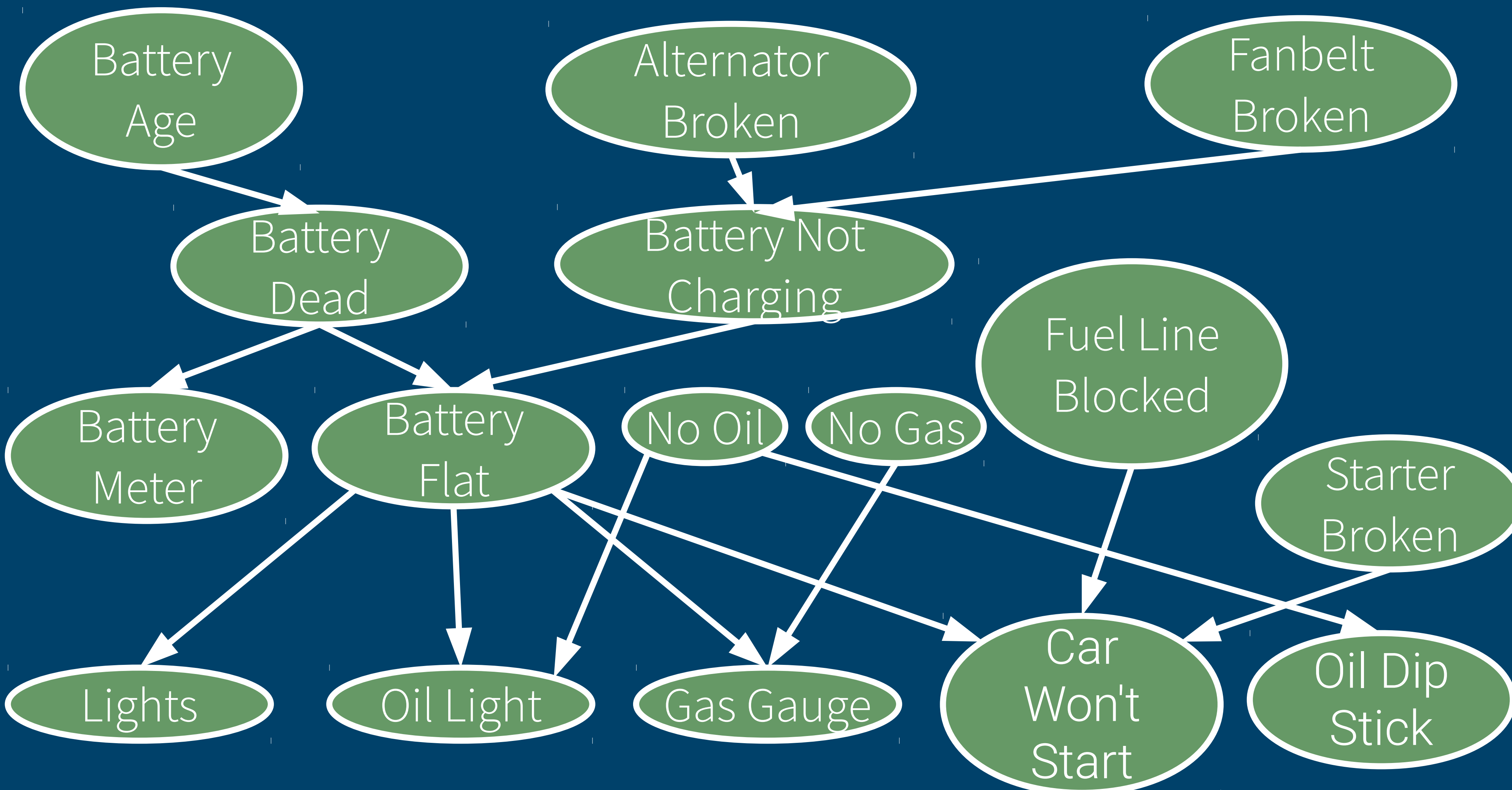
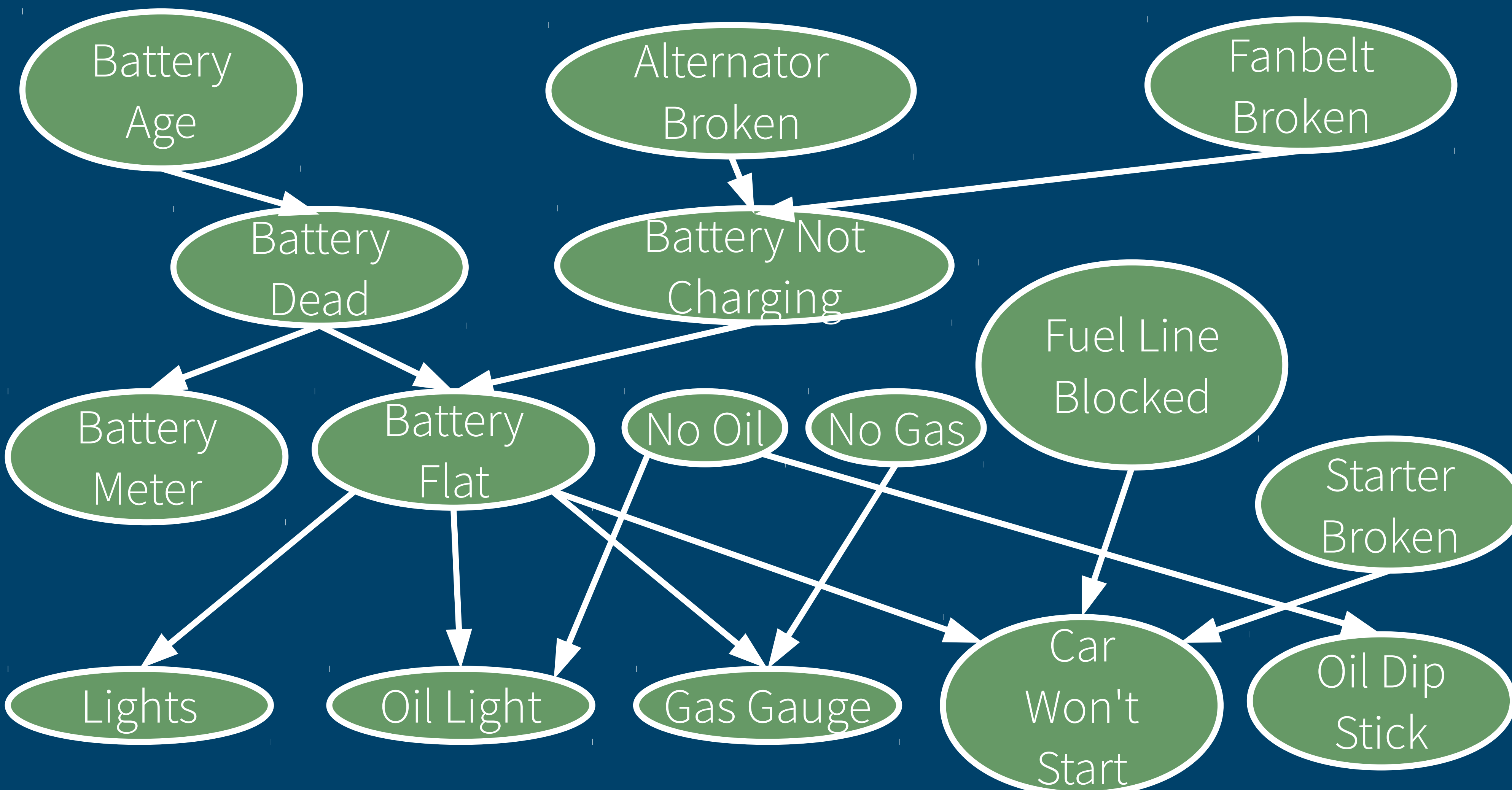
2nd Semester AY 2014-2015

PROBABILITY IN AI

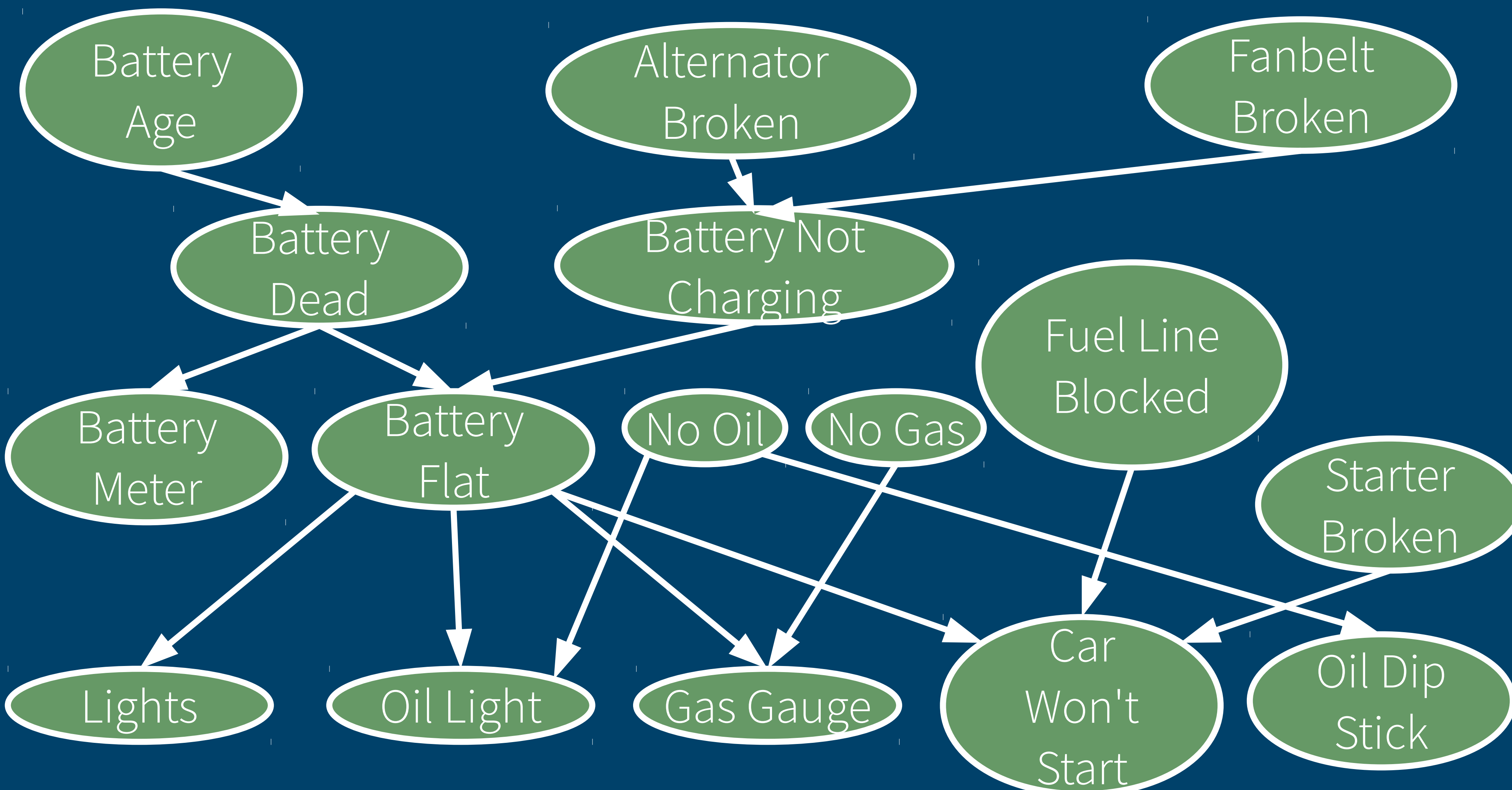
BAYES NETWORKS



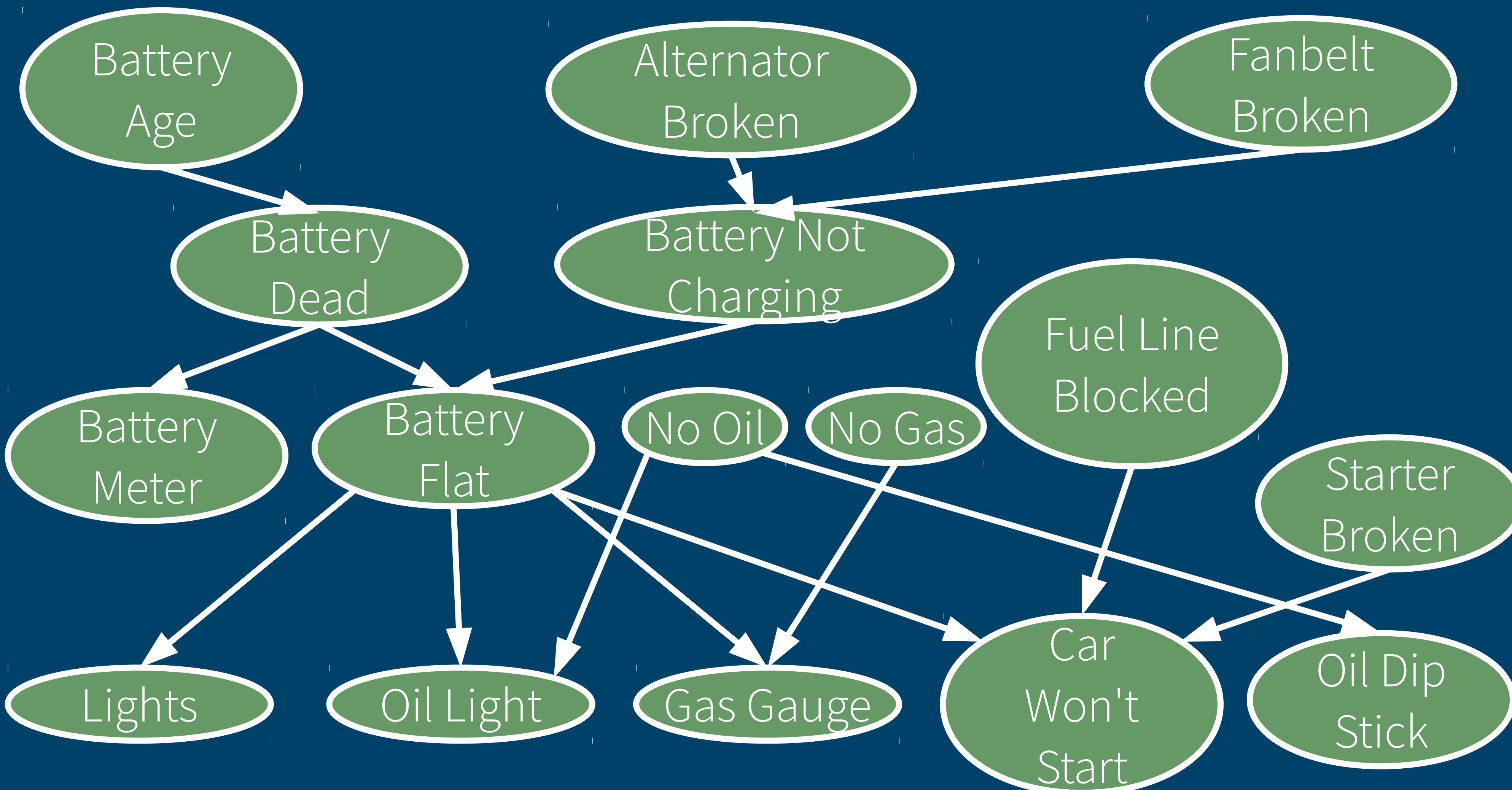
Composed of **nodes** corresponding to **known** or **unknown** events.



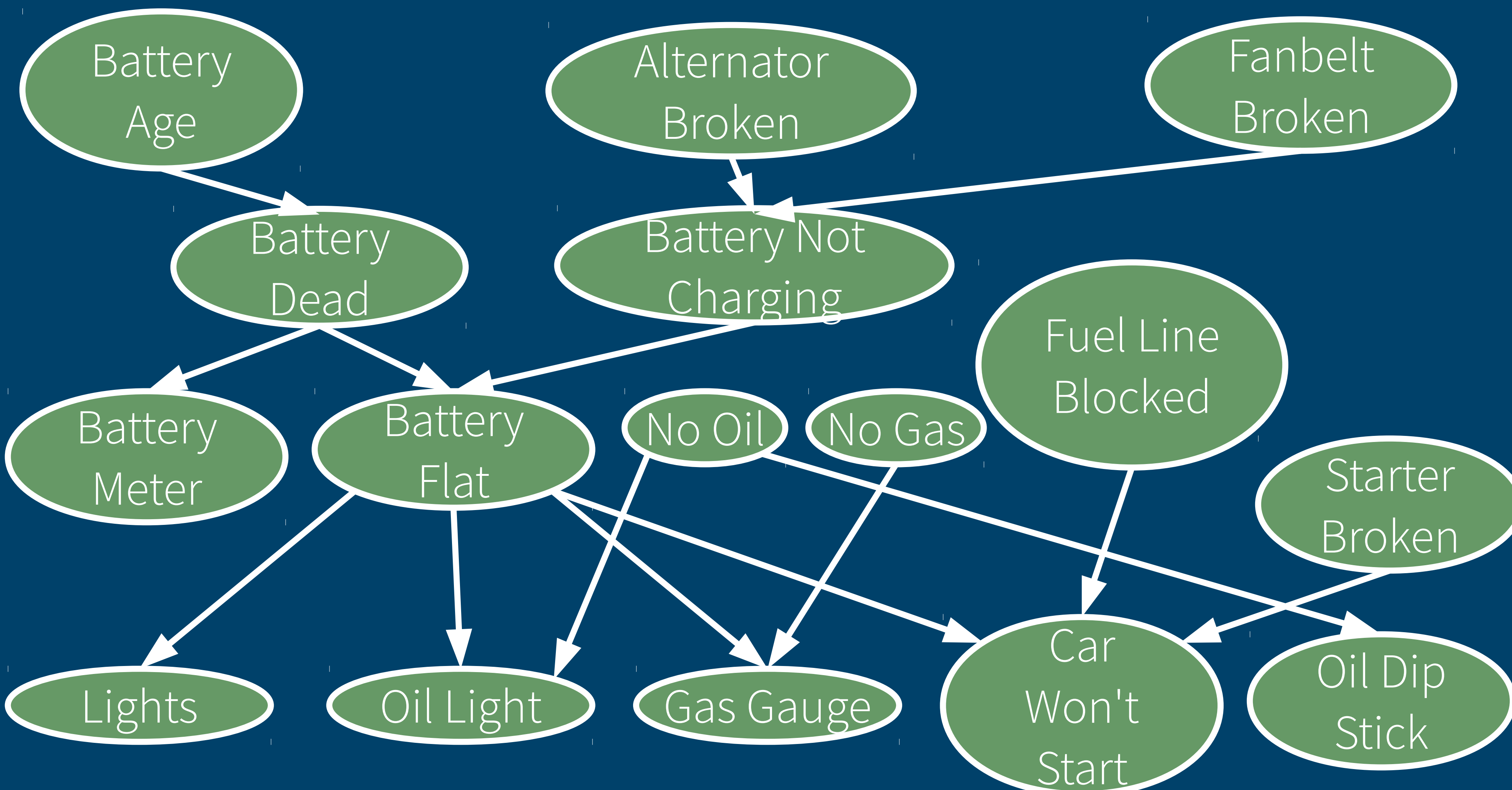
Each **event** is represented by a **random variable**.



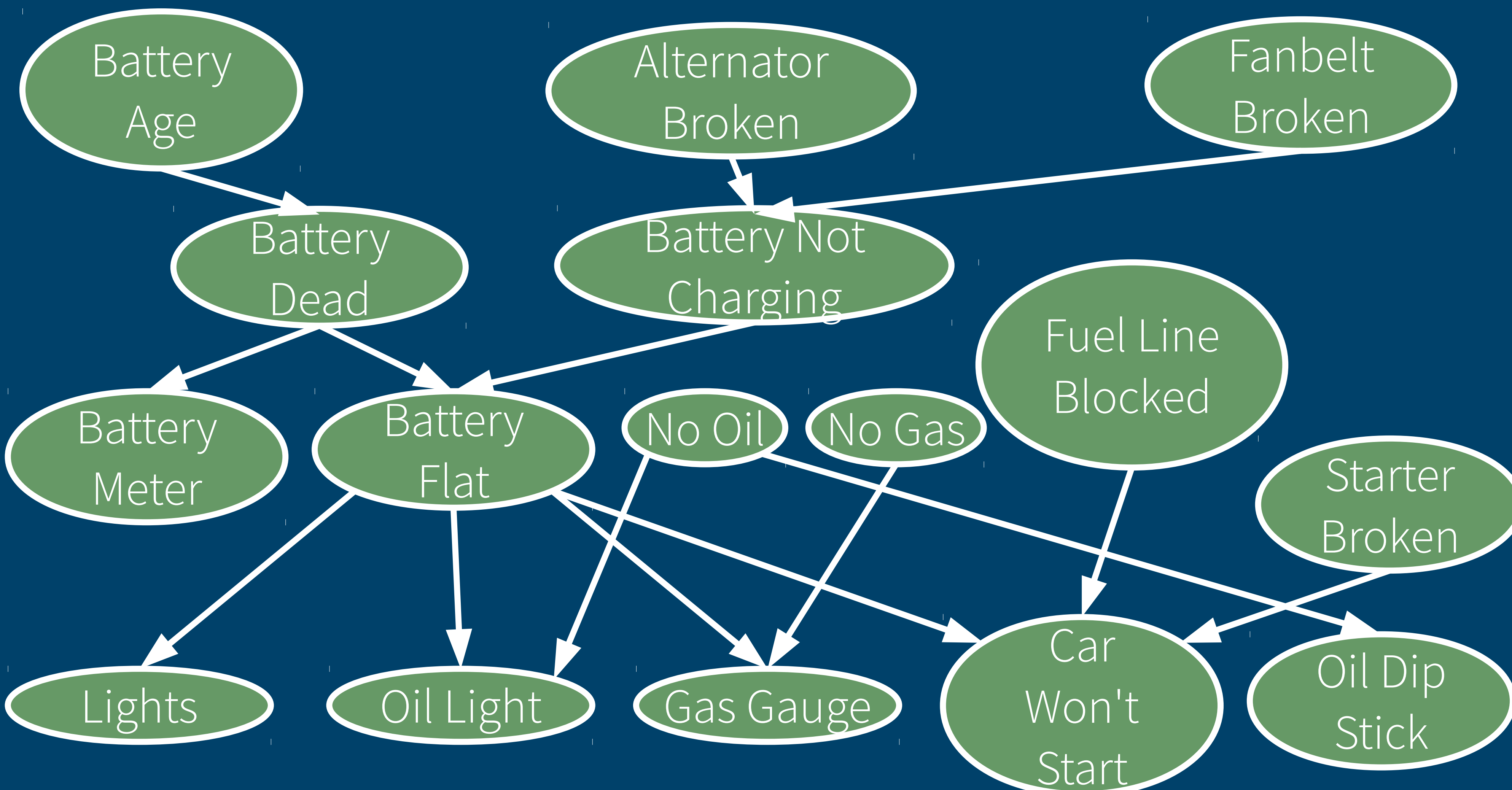
Nodes are connected by **arcs**.



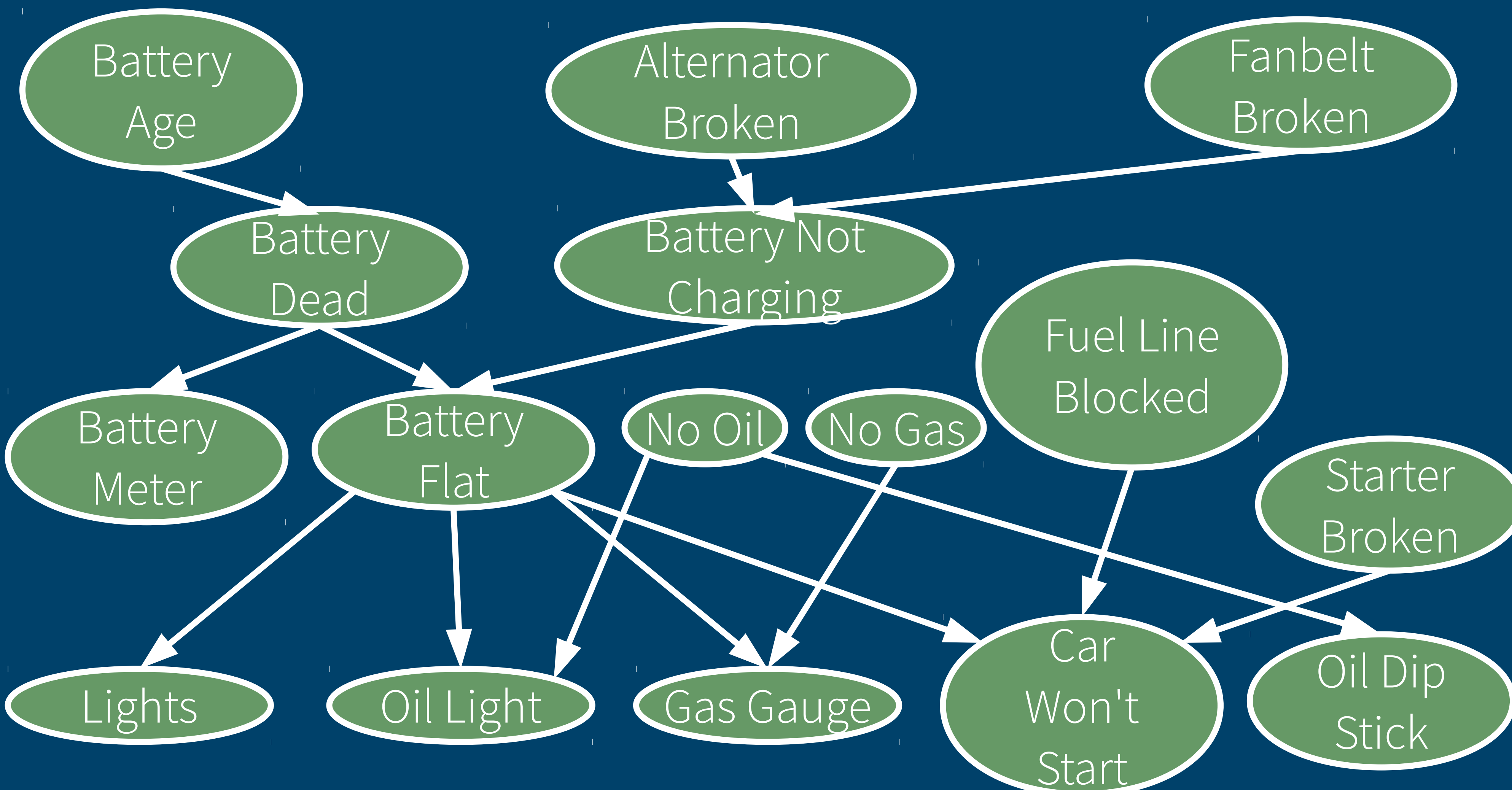
The **child** of an arc is **influenced by** its **parent**.



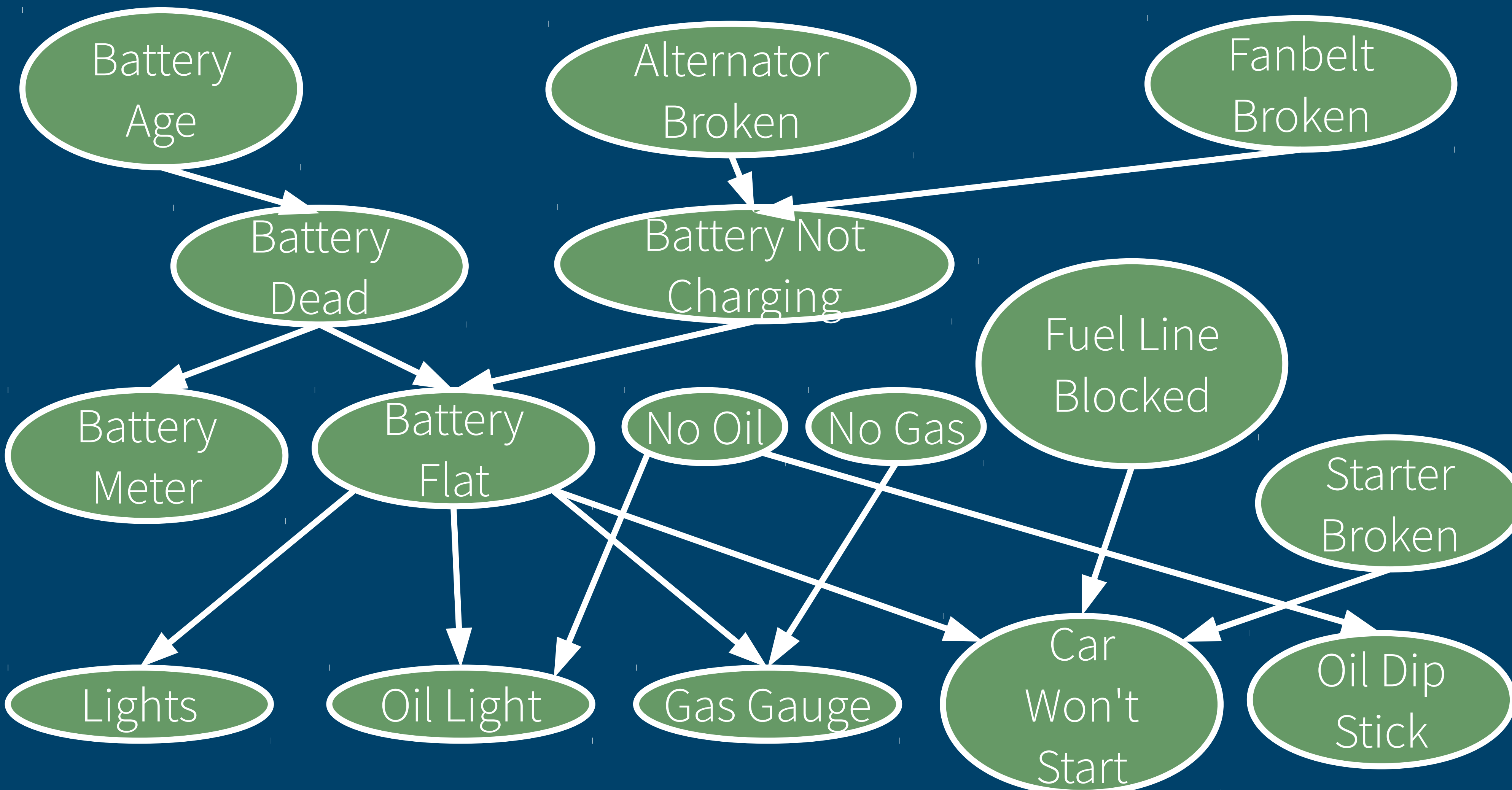
Parents may **influence** their **children** in a **probabilistic** way.



Now, if **each random variable** is **binary**, how many states are there in this state space?



Answer: 2^{16} states.



Bayes Networks

A **complex representation** of a **very large joint probability distribution of variables.**

When **events are observed**, they can be used to **compute** the **probability** of other, **unknown events**.

For the purposes of our discussion, assume that **all random variables** are **binary** unless otherwise specified.

Bayes networks are used extensively in smart computer systems.

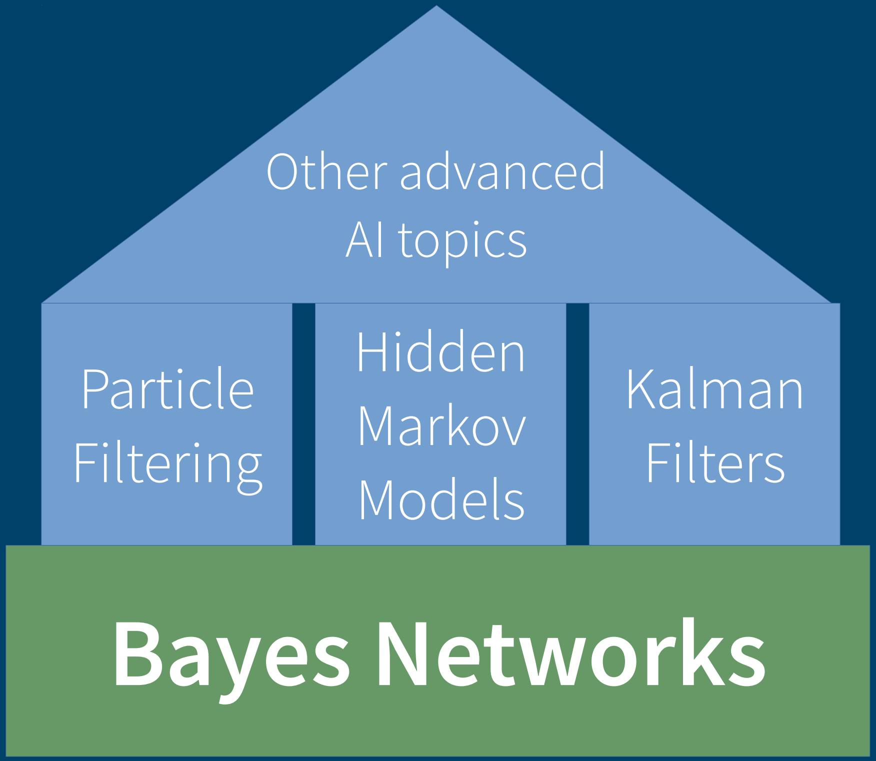
Bayes Networks

Diagnostics

Prediction

Machine Learning

Bayes networks are also the building blocks for more advanced AI topics.



PROBABILITY

Probability

The **cornerstone of Artificial Intelligence**, it is used to **express uncertainty** using a **value** from **0 to 1, inclusive**.

QUIZ (1/4)

1. Say we have a **fair coin** where the **probability of getting heads** is **0.5**; what is the probability of getting tails?

ANSWER

1. Say we have a **fair coin** where the **probability of getting heads** is **0.5**; what is the probability of getting tails?

$$P(Heads) = 0.5$$

$$P(Tails) = 1 - P(Heads) = 0.5$$

QUIZ (1/4)

2. Say we have a loaded **coin** where the **probability of getting heads** is **0.25**; what is the probability of getting tails?

ANSWER

2. Say we have a loaded **coin** where the **probability of getting heads** is **0.25**; what is the probability of getting tails?

$$P(Heads) = 0.25$$

$$P(Tails) = 1 - P(Heads) = 0.75$$

QUIZ (1/4)

3. What is the probability of getting three consecutive heads, $P(H, H, H)$, given $P(H) = 0.25$?

ANSWER

Each **event** of **getting heads** is **independent** of the others; thus, we just **multiply** the **probability** for **each coin flip** resulting in **heads**.

$$\begin{aligned} P(H,H,H) &= P(H)P(H)P(H) \\ &= 0.5 \times 0.5 \times 0.5 \\ &= 0.5^3 \\ &= 0.125 \end{aligned}$$

QUIZ (1/4)

4. Say xi is the ith fair coin flip and each xi can be either heads or tails. What is the probability that a succesion of four coin flips have the same outcome, that is,

$$P\left(x_1 = x_2 = x_3 = x_4\right)$$

ANSWER

There are four ways to get the same outcome: all flips are heads and all flips are tails. We simply add their probabilities:

$$
\begin{aligned}
P(x_1 = x_2 = x_3 = x_4) &= P(H,H,H,H) + P(T,T,T,T) \\
&= 0.5^4 + 0.5^4 \\
&= 0.0625 + 0.0625 \\
&= 0.125
\end{aligned}
$$

QUIZ (1/4)

5. What is the probability that, out of four fair coin flips, at least three turn out to be heads, that is

$$P(x_{1,}\, x_{2,}\, x_{3,}\, x_4\, has \geqslant 3\, H\,'s)$$

ANSWER

There are five ways to have more than 3 H's: HHHH, HHHT, THHH, HHTH, HTHH, THHH. Each has a probability of 0.0625 or 1/16.

$$
\begin{aligned}
P(x_{1,} x_{2,} x_{3,} x_4 \, has \geqslant 3 \, H's) &= 5 \times 0.0625 \\
&= 0.3125
\end{aligned}
$$

REVIEW OF PROBABILITY THEORY

Complementary Probability

$$P(\neg A) = 1 - P(A)$$

Independence

$$P(X, Y) = P(X)\,P(Y)$$

Joint Probability

Marginals

if X is independent of Y.

$$(X \perp Y)$$

Dependence $P(X|Y)$

The probability of X given Y [already happened].

Total Probability

$$P(Y) = \sum_i P(Y|X=i) \times P(X=i)$$

Probability Negation

$$P(\neg X \mid Y) = 1 - P(X \mid Y)$$

Dependence

$$P(A,B) = P(B|A)\,P(A)$$

Joint Probability

QUIZ (1/4)

$$P(x_1 = H) = 0.5 \qquad P(x_2 = H \mid x_1 = H) = 0.9$$

$$P(x_2 = T \mid x_1 = T) = 0.8$$

What is $P(X_2 = H)$?

ANSWER

$$
\begin{aligned}
P(x_2 = H) &= P(x_2 = H \mid x_1 = H) \times P(x_1 = H) + \\
&\quad P(x_2 = H \mid x_1 = T) \times P(x_1 = T) \\
&= 0.9 \times 0.5 + (1 - 0.8) \times 0.5 \\
&= 0.45 + 0.1 \\
&= 0.55
\end{aligned}
$$

QUIZ (1/4)

Say, a day can be either sunny or rainy. Given the following probabilities:

$$P(D_i = \text{sunny}) = 0.9$$

$$P(D_{i+1} = \text{sunny} | D_i = \text{sunny}) = 0.8$$

What is $P(D_2 = \text{rainy} | D_1 = \text{sunny})$?

ANSWER

$$
\begin{aligned}
P(D_2 = \text{rainy} \mid D_1 = \text{sunny}) \\
&= 1 - P(D_2 = \text{sunny} \mid D_1 = \text{sunny}) \\
&= 1 - 0.8 \\
&= 0.2
\end{aligned}
$$

QUIZ (1/4)

Say, a day can be either sunny or rainy. Given the following probabilities:

$$P(D_i = \text{sunny}) = 0.9$$

$$P(D_{i+1} = \text{sunny} | D_i = \text{rainy}) = 0.6$$

What is $P(D_2 = \text{rainy} | D_1 = \text{rainy})$?

ANSWER

$$
\begin{aligned}
P(D_2 = \text{rainy} \mid D_1 = \text{rainy}) \\
&= 1 - P(D_2 = \text{sunny} \mid D_1 = \text{rainy}) \\
&= 1 - 0.6 \\
&= 0.4
\end{aligned}
$$

ANSWER

$$
\begin{aligned}
P(D_2 = \text{rainy} \mid D_1 = \text{rainy}) \\
&= 1 - P(D_2 = \text{sunny} \mid D_1 = \text{rainy}) \\
&= 1 - 0.6 \\
&= 0.4
\end{aligned}
$$

QUIZ (1/4)

Given:

$$P(D_1 = \text{sunny}) = 0.9$$

$$P(D_{i+1} = \text{sunny} | D_i = \text{sunny}) = 0.8$$

$$P(D_{i+1} = \text{sunny} | D_i = \text{rainy}) = 0.6$$

What is $P(D_2 = \text{sunny})$?

What is $P(D_3 = \text{sunny})$?

ANSWER

$$
\begin{aligned}
P(D_2 = \text{sunny}) \\
&= P(D_2 = \text{sunny} \mid D_1 = \text{sunny}) \times P(D_1 = \text{sunny}) \\
&\quad + P(D_2 = \text{sunny} \mid D_1 = \text{rainy}) \times P(D_1 = \text{rainy}) \\
&= 0.8 \times 0.9 + 0.6 \times 0.1 \\
&= 0.78
\end{aligned}
$$

ANSWER

$$P(D_2 = \text{sunny})$$
$$= P(D_2 = \text{sunny} | D_1 = \text{sunny}) \times P(D_1 = \text{sunny})$$
$$+ P(D_2 = \text{sunny} | D_1 = \text{rainy}) \times P(D_1 = \text{rainy})$$
$$= 0.8 \times 0.9 + 0.6 \times 0.1$$
$$= 0.78$$

$$P(D_3 = \text{sunny})$$
$$= P(D_3 = \text{sunny} | D_2 = \text{sunny}) \times P(D_2 = \text{sunny})$$
$$+ P(D_3 = \text{sunny} | D_2 = \text{rainy}) \times P(D_2 = \text{rainy})$$
$$= 0.8 \times 0.78 + 0.6 \times 0.22$$
$$= 0.756$$

BAYES' RULE

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

BAYES' RULE

Likelihood Prior

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Posterior

Marginal

Likelihood

Bayes' rule inverts a **diagnostic relationship**, $P(A|B)$, to a **causal relationship** $P(B|A)$.

Remember, **B is known**, but **we want to know A**.

The **marginal likelihood** is often **expanded** using **total probability**, giving:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$= \frac{P(B|A)\,P(A)}{\sum_a P(B|A=a)\,P(A=a)}$$

QUIZ (1/4)

Given that the probability of cancer (C) is 0.01, and we have a test (T) for cancer that has a 0.9 probability of being positive (+) if the patient has cancer, and 0.2 probability of being positive (+) if the patient does not have cancer.

Compute the following:

$$P(C) = ?$$
$$P(\neg C) = ?$$
$$P(T = + | C) = ?$$
$$P(T = - | C) = ?$$
$$P(T = + | \neg C) = ?$$
$$P(T = - | \neg C) = ?$$

ANSWERS

$$P(C) = 0.01$$
$$P(\neg C) = 0.99$$
$$P(T = + | C) = 0.9$$
$$P(T = - | C) = 0.1$$
$$P(T = + | \neg C) = 0.2$$
$$P(T = - | \neg C) = 0.8$$

QUIZ (1/4)

What, then, are the following joint probabilities?

$$P(T{=}+, C) = ? \quad P(T{=}+, \neg C) = ?$$

$$P(T{=}-, C) = ? \quad P(T{=}-, \neg C) = ?$$

ANSWERS

$$P(T{=}+, C) = 0.9 \times 0.01 = 0.009$$

$$P(T{=}\text{-}, C) = 0.1 \times 0.01 = 0.001$$

$$P(T{=}+, \neg C) = 0.2 \times 0.99 = 0.198$$

$$P(T{=}\text{-}, \neg C) = 0.8 \times 0.99 = 0.792$$

Note that the sum of all the joint probabilities is 1.

QUIZ (1/4)

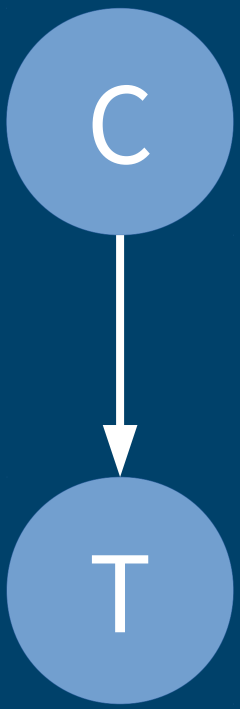
What is the probability of having cancer if the test result is positive?

$$P(C|T=+)=?$$

ANSWER

$$
\begin{aligned}
P(C \mid T=+) &= \frac{P(T=+ \mid C)\,P(C)}{P(T=+)} \\
&= \frac{0.9 \times 0.01}{P(T=+ \mid C)\,P(C) + P(T=+ \mid \neg C)\,P(\neg C)} \\
&= \frac{0.009}{0.9 \times 0.01 + 0.2 \times 0.99} \\
&= 0.04347826087
\end{aligned}
$$

Cause & Effect: The **presence of cancer affects** whether the **test result** is positive or not.



$P(C)$

Actual presence of cancer; Unknown/unobserved event; cause

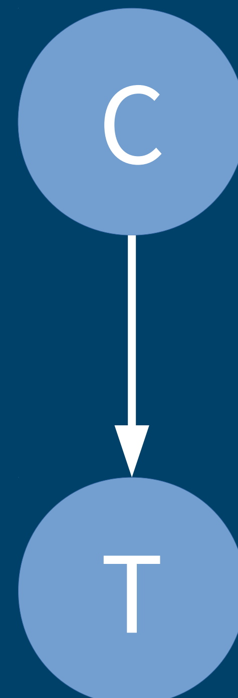
$P(T=+|C)$

$P(T=+|\neg C)$

Test result; Known/observed event; effect

Causal Relationship: Given the presence/absence of cancer, what will the test result be?

$$P(T=+|C)$$
$$P(T=+|\neg C)$$



$P(C)$

Actual presence of cancer; Unknown/unobserved event; cause

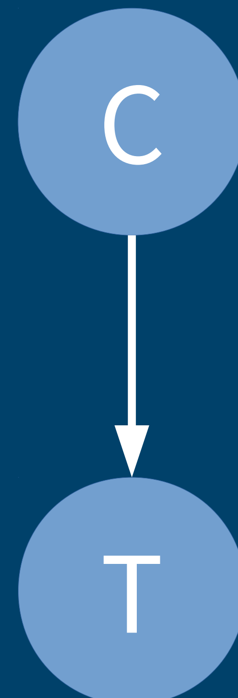
$P(T=+|C)$
$P(T=+|\neg C)$

Test result; Known/observed event; effect

Diagnostic Relationship:

Given the test result, does the patient have cancer?

$$P(C \mid T{=}+)$$
$$P(C \mid T{=}-)$$



$P(C)$

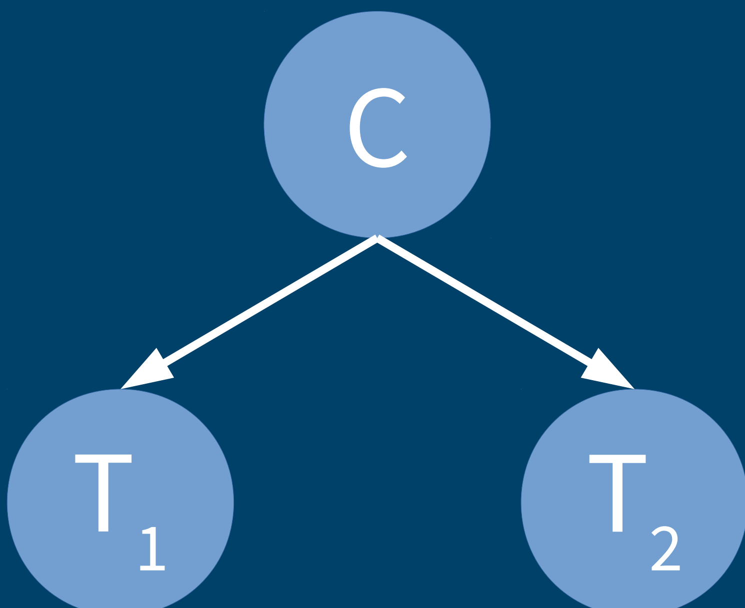
Actual presence of cancer; Unknown/unobserved event; cause

$P(T{=}+ \mid C)$
$P(T{=}+ \mid \neg C)$

Test result; Known/observed event; effect

EXAMPLE

What if there are two tests?



$$P(C) = 0.01$$

$$P(T = + \mid C) = 0.9$$

$$P(T = - \mid \neg C) = 0.8$$

(same probability for both tests)

How to compute:

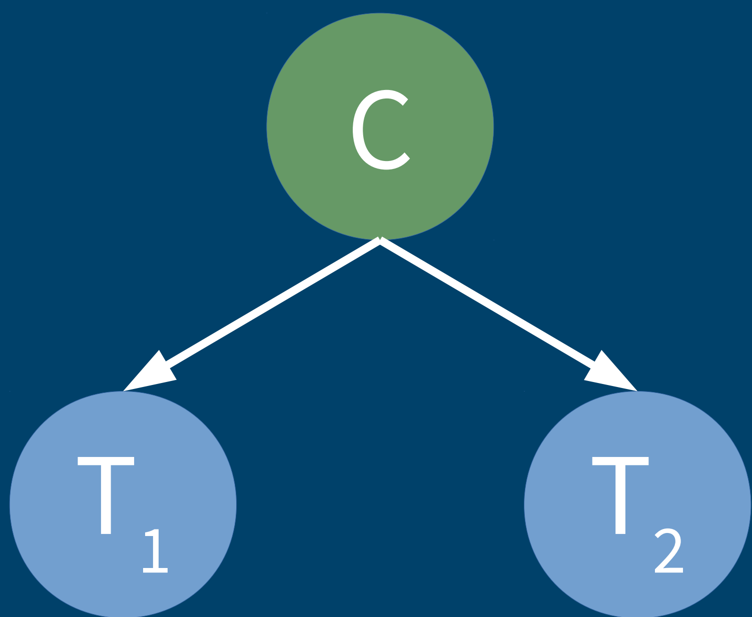
$$P(C \mid T_1 = +, T_2 = +) = ?$$

To do this, we need to assume that T_1 and T_2 are

conditionally independent.

Conditional Independence

$$P(T_2|C, T_1) = P(T_2|C)$$



If we **know the outcome of** C, then T_1 **and** T_2 **become independent** of each other, that is:

$$T_1 \perp T_2 | C$$

ANSWER

$$P(C|++) = \frac{P(++|C)\,P(C)}{P(++)}$$

$$= \frac{P(T_1=+|C)\,P(T_2=+|C)\,P(C)}{P(++)}$$

$$= \frac{0.9 \times 0.9 \times 0.1}{P(++|C)\,P(C) + P(++|\neg C)\,P(\neg C)}$$

Note: $++ \rightarrow T_1 = +, T_2 = +$

ANSWER

$$
\begin{aligned}
P(C|++) &= \frac{0.9 \times 0.9 \times 0.1}{P(++|C)\,P(C) + P(++|\neg C)\,P(\neg C)} \\
&= \frac{0.9 \times 0.9 \times 0.1}{0.9 \times 0.9 \times 0.01 + 0.2 \times 0.2 \times 0.99} \\
&= 0.1698113208
\end{aligned}
$$

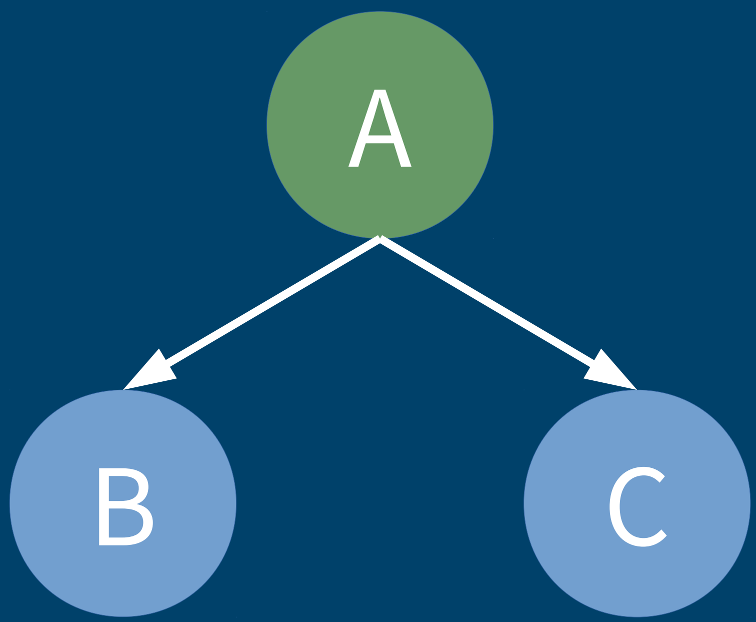
Note: $++ \rightarrow T_1 = +, T_2 = +$

EXAMPLE

What about $P(C|T_1 = +, T_2 = -)$?

$$P(C|+-) = \frac{P(+-|C)\,P(C)}{P(+-)}$$

$$= \frac{P(T_1 = +|C)\,P(T_2 = -|C)\,P(C)}{P(+-)}$$



Remember,

$$B \perp C | A$$

but,

$$B \neg \perp C$$

EXAMPLE

How do we compute $P(T_2 = + | T_1 = +)$?

$$P(+_2|+_1) = P(+_2|+_{1,} C) P(C|+_1)$$
$$+ P(+_2|+_{1,} \neg C) P(\neg C|+_1)$$

We use **total probability** because we know that T_1 and T_2 are conditionally independent given C.

EXAMPLE

$$P(+_2|+_1) = P(+_2|+_{1,}C)P(C|+_1) + P(+_2|+_{1,}\neg C)P(\neg C|+_1)$$

Since T1 and T2 are conditionally independent given C...

$$P(+_2|+_{1,}C) = P(+_2|C)$$

$$P(+_2|+_{1,}\neg C) = P(+_2|\neg C)$$

EXAMPLE

$$
\begin{aligned}
P(+_2 | +_1) &= P(+_2 | C)\,P(C | +_1) \\
&\quad + P(+_2 | \neg C)\,P(\neg C | +_1) \\
&= 0.9 \times 0.0434\ldots + 0.2 \times (1 - 0.0434\ldots) \\
&= 0.2304347826
\end{aligned}
$$

EXAMPLE

This is higher than $T_2 = +$ alone.

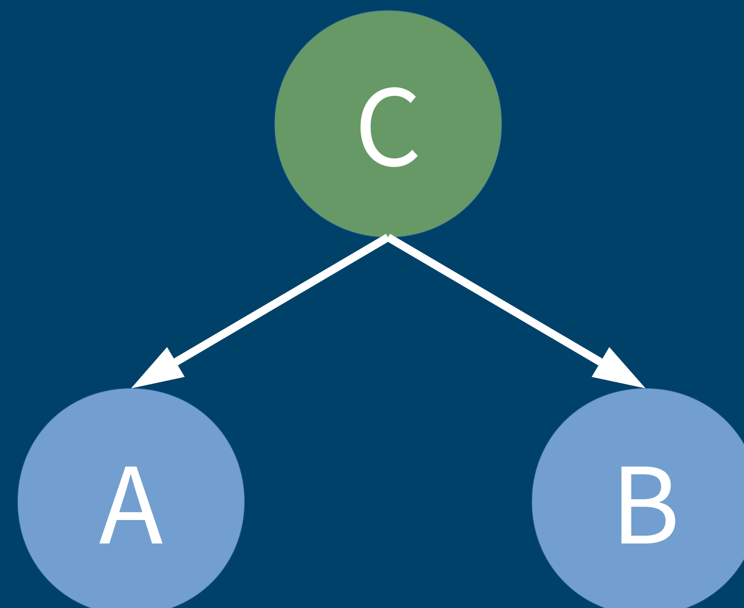
$$
\begin{aligned}
P(T_2{=}+) \\
&= P(+|C)\,P(C) + P(+|\neg C)\,P(\neg C) \\
&= 0.9 \times 0.01 + 0.2 \times 0.99 \\
&= 0.207
\end{aligned}
$$

RECALL

Independence:

$$A \perp B$$

Conditional Independence



$$A \perp B \,|\, C$$

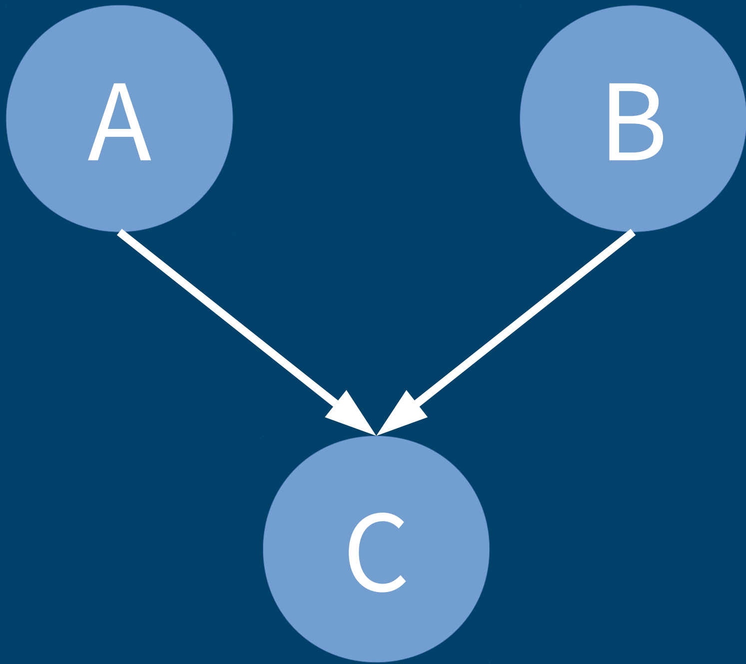
Absolute independence does not imply conditional independence.

$$A \perp B \nRightarrow A \perp B \mid C$$

Conditional independence does not imply absolute independence.

$$A \perp B \mid \mathrm{C} \not\Rightarrow A \perp B$$

What if we have a Bayes network with the following format?

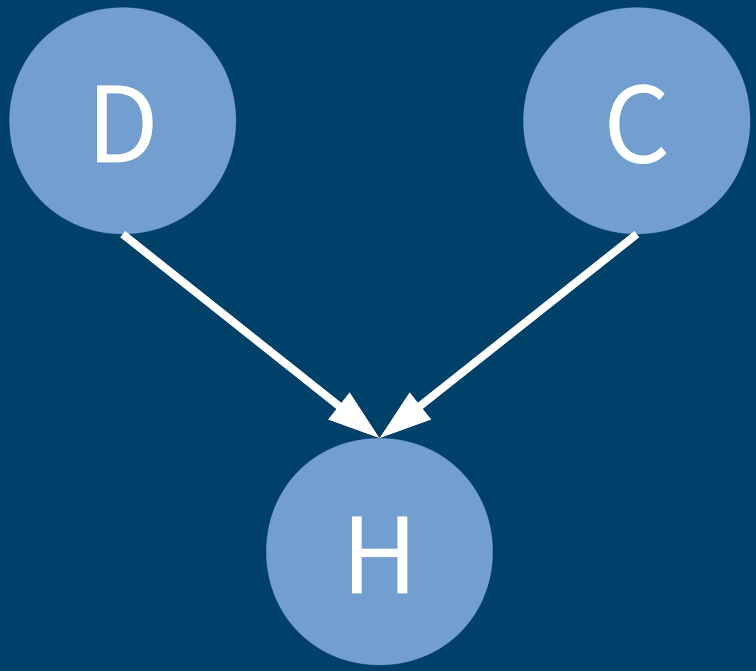


EXAMPLE

D = I won my DotA 2 match

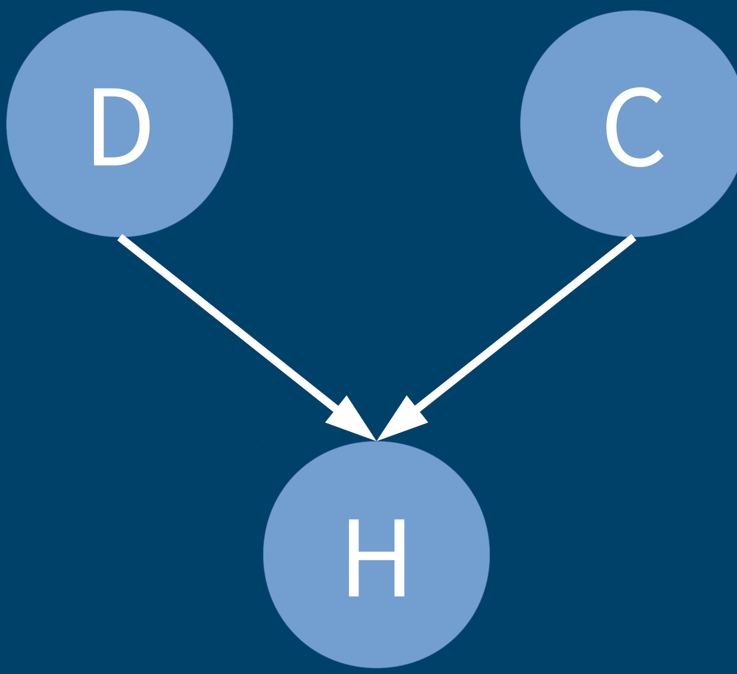
C = I ate a cake

H = I am happy



EXAMPLE

$$P(D) = 0.7$$
$$P(C) = 0.01$$
$$P(H|D, C) = 1$$
$$P(H|\neg D, C) = 0.7$$
$$P(H|D, \neg C) = 0.9$$
$$P(H|\neg D, \neg C) = 0.1$$



EXAMPLE

What is the probability that I've eaten a cake given that it is sunny, that is $P(C \mid S)$?

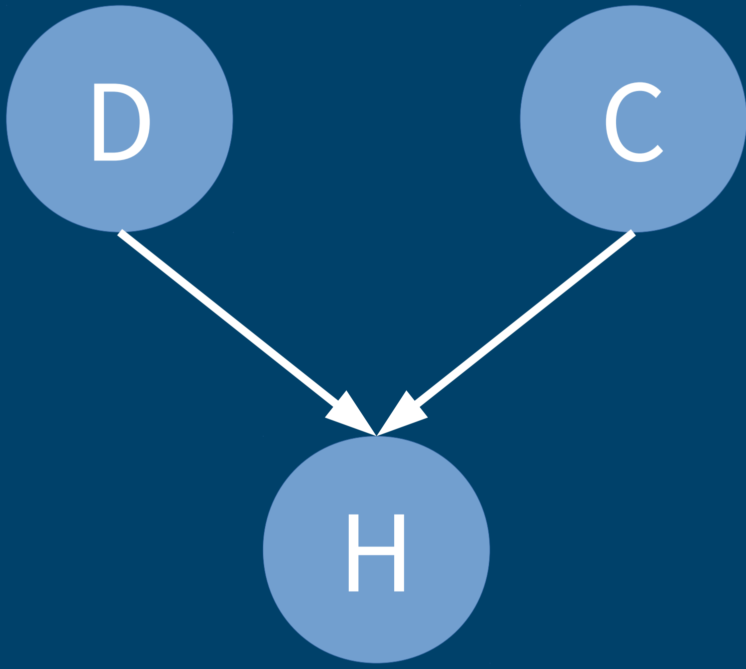
ANSWER

What is the probability that I've eaten a cake given that I won my DotA 2 match, that is $P(C \mid D)$?

$$P(C|D) = P(C) = 0.01$$

WHY?

Eating cake and **winning in DotA 2** are **independent** of each other even though my happiness is dependent on both of them.



EXAMPLE

What about the probability of eating cake given that I am happy and I won my DotA 2 match, that is, $P(C \mid H, D)$?

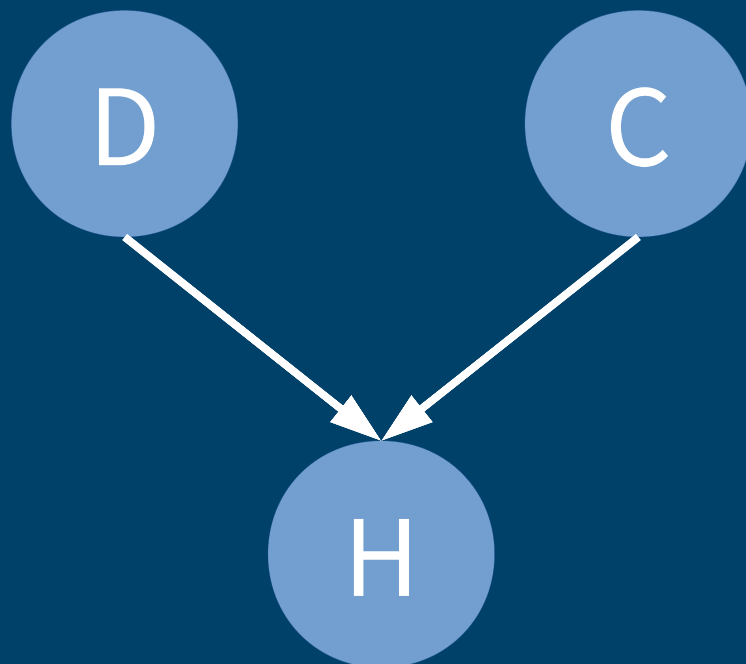
EXAMPLE

$$P(C|H,D)=\frac{P(H|C,D)\,P(C|D)}{P(H|S)}$$

$$=\frac{P(H|C,D)\,P(C)}{P(H|C,D)\,P(C)+P(H|\neg C,D)\,P(\neg C)}$$

RECALL…

$$P(D) = 0.7$$
$$P(C) = 0.01$$
$$P(H|D,C) = 1$$
$$P(H|\neg D,C) = 0.7$$
$$P(H|D,\neg C) = 0.9$$
$$P(H|\neg D,\neg C) = 0.1$$



EXAMPLE

$$P(C|H,D) = \frac{P(H|C,D)\,P(C|D)}{P(H|S)}$$

$$= \frac{P(H|C,D)\,P(C)}{P(H|C,D)\,P(C) + P(H|\neg C,D)\,P(\neg C)}$$

$$= \frac{1 \times 0.01}{1 \times 0.01 + 0.9 \times 0.99}$$

$$= 0.0110987791$$