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POWER SERIES

Notion

Polynomial:

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

✓ **WELL-BEHAVED**

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

Power series

Power series about a

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_{n-1}(x-a)^{n-1} + c_n(x-a)^n + \ldots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Power series

Power series about a

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Power series about $a = 0$

$$\sum_{n=0}^{\infty} c_n x^n$$

GOAL

Given $\sum_{n=0}^{\infty} c_n (x-a)^n$

Determine ALL values of x **for which** $\sum_{n=0}^{\infty} c_n (x-a)^n$ **is convergent.**

Interval of convergence

If $\sum_{n=0}^{+\infty} c_n (x-a)^n$ **converges for all values of** x **such that** $|x-a| < R$ **,**

$(a-R, a+R)$**: interval of convergence**

R **: radius of convergence**

WARNING!

RATIO TEST

$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad u_n = c_n (x-a)^n$$

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

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Where is $\sum_{n=0}^{\infty} c_n (x-a)^n$ convergent?

Values of x where

$$L = \lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

TEST OVER BOUNDARIES OF INTERVALS!

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Example. Determine the interval of convergence.

$$\sum_{n=0}^{+\infty} x^n$$

interval of convergence: $(-1,1)$

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MOREOVER

GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

if $|r|<1$ or when the series is convergent.

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REMARK:

$\sum_{n=0}^{+\infty} x^n$ is in the form of a geometric series.

$$\sum_{n=0}^{+\infty} x^n = \sum_{n=1}^{+\infty} x^{n-1} = \frac{1}{1-x}$$

if the series is convergent.
(if $x \in (-1,1)$)

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Example. Determine the interval of convergence.

$$\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n+1}$$

interval of convergence: $(-1,1]$

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Solution: $\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n+1}$

Apply Ratio Test:

Let $u_n = (-1)^{n+1} \frac{x^n}{n+1}$

$$u_{n+1} = (-1)^{n+2} \frac{x^{n+1}}{n+2}$$

Solutions (continued)

$$\boxed{\frac{u_{n+1}}{u_n}} = \left| \frac{(-1)^{n+2} \frac{x^{n+1}}{n+2}}{(-1)^{n+1} \frac{x^n}{n+1}} \right|$$

$$= \left| -\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right|$$

$$= \left| x \cdot \frac{n+1}{n+2} \right| \boxed{= \frac{n+1}{n+2} \cdot |x|}$$

Solutions (continued)

$$\lim_{n \to +\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to +\infty} \left(\frac{n+1}{n+2} |x| \right)$$

$$= |x| \cdot \underbrace{\lim_{n \to +\infty} \frac{n+1}{n+2}}_{1}$$

$$= |x|$$

The series is convergent when

$$|x| < 1 \Rightarrow -1 < x < 1$$

Solutions (continued)

$$\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n+1}$$

If $x = 1$, $\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n+1}$

which is convergent using AST.

Solutions (continued)

$$\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n+1}$$

If $x = -1$, $\sum_{n=0}^{+\infty} \frac{1}{n+1}$

which is divergent when

compared to $\sum_{n=0}^{+\infty} \frac{1}{n}$.

Solutions (continued)

Thus, the interval of

convergence of $\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n+1}$

is $(-1, 1]$.

Example. Determine ALL values of x where the given is convergent.

$$\sum_{n=1}^{+\infty} \frac{(x+2)^n}{n^2}$$

Solution:

$$u_n = \frac{(x+2)^n}{n^2} \qquad u_{n+1} = \frac{(x+2)^{n+1}}{(n+1)^2}$$

Solutions (continued)

$$\boxed{\frac{u_{n+1}}{u_n}} = \left| \frac{\frac{(x+2)^{n+1}}{(n+1)^2}}{\frac{(x+2)^n}{n^2}} \right|$$

$$= \left| \frac{(x+2)\cdot n^2}{(n+1)^2} \right|$$

$$= \frac{n^2}{(n+1)^2} \cdot |x+2|$$

Solutions (continued)

$$\lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} \cdot |x+2|$$

$$= |x+2| \cdot \overset{1}{\cancel{\lim_{n\to+\infty} \frac{n^2}{(n+1)^2}}}$$

$$= |x+2|$$

Solutions (continued)

The series is convergent when

$$|x+2| < 1 \Rightarrow -1 < x+2 < 1$$

$$\Rightarrow -1-2 < x < 1-2$$

$$\Rightarrow -3 < x < -1$$

Solutions (continued)

$$\sum_{n=1}^{+\infty} \frac{(x+2)^n}{n^2}$$

If $x=-3$, $\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2}$ which is convergent.

If $x=-1$, $\sum_{n=0}^{+\infty} \frac{1}{n^2}$ which is convergent.

Solutions (continued)

Thus, the interval of convergence of $\sum_{n=1}^{+\infty} \frac{(x+2)^n}{n^2}$ is $[-3,-1]$.

REMARK

Given a power series $\sum_{n=0}^{+\infty} c_n(x-a)^n$.

Its interval of convergence can be :

i. an interval of the form (a-R , a+R)

ii. ($-\infty$, $+\infty$) convergent for every real number

iii. { a } convergent only at a single number

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Example. Show that the given is always convergent.

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Solution:

$$u_n = \frac{x^n}{n!} \qquad u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

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Solutions (continued)

$$\frac{u_{n+1}}{u_n} = \left| \frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}} \right|$$

$$= \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \left| \frac{x}{n+1} \right| = \frac{1}{n+1} \cdot |x|$$

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Solutions (continued)

$$\lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to+\infty} \left(\frac{1}{n+1} \cdot |x| \right)$$

$$= |x| \cdot \lim_{n\to+\infty} \frac{1}{n+1} \to 0$$

$$= 0$$

For any value of x,

$$\lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0$$

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Solutions (continued)

For any value of x,

$$\lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0 < 1$$

Thus, $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ is CONVERGENT for any value of x.

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Differentiating Power Series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$\Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} c_n\, n (x-a)^{n-1}$$

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Integrating Power Series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$\Rightarrow \int_0^x f(t)dt = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

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