Partial Derivatives

Chapter 2 Section 3

Definition.

Let f be a function of two variables x and y. The ***partial derivative of f with respect to x*** is the function, denoted by

$$D_1 f \qquad f_1 \qquad f_x \qquad \frac{\partial f}{\partial x}$$

such that its value at any point (x,y) in the domain of f is given by

$$D_1 f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

if this limit exists.

Example.

Find $D_1 f(x,y)$: **a.** $f(x,y) = 4x - xy^3 + 1$

$$D_1 f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\left(4 - y^3\right)(x+h) + 1\right] - \left[\left(4 - y^3\right)x + 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left(4 - y^3\right)h}{h} = \lim_{h \to 0} \left(4 - y^3\right) = 4 - y^3$$

Example.

b. $f(x,y) = x^2 - 5y$

$$D_1 f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^2 - 5y\right] - \left[x^2 - 5y\right]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x+h) = 2x$$

Definition.

Let f be a function of two variables x and y. The ***partial derivative of f with respect to y*** is the function, denoted by

$$D_2 f \qquad f_2 \qquad f_y \qquad \frac{\partial f}{\partial y}$$

such that its value at any point (x,y) in the domain of f is given by

$$D_2 f(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

if this limit exists.

Example. Find $\dfrac{\partial f}{dy}$: $f(x,y) = 2xy^2 + 5y$

$$\frac{\partial f}{dy} = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\left[2x(y+h)^2 + 5(y+h)\right] - \left[2xy^2 + 5y\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[2x\left(y^2 + 2yh + h^2\right) + 5y + 5h\right] - \left[2xy^2 + 5y\right]}{h}$$

$$= \lim_{h\to 0} \frac{4xyh + 2xh^2 + 5h}{h} = 4xy + 5$$

REMARKS.

1. $$D_1 f\left(x_0, y_0\right) = \lim_{h\to 0} \frac{f\left(x_0 + h, y_0\right) - f\left(x_0, y_0\right)}{h}$$

2. $$D_1 f\left(x_0, y_0\right) = \lim_{x\to x_0} \frac{f\left(x, y_0\right) - f\left(x_0, y_0\right)}{x - x_0}$$

3. $$D_2 f\left(x_0, y_0\right) = \lim_{h\to 0} \frac{f\left(x_0, y_0 + h\right) - f\left(x_0, y_0\right)}{h}$$

4. $$D_2 f\left(x_0, y_0\right) = \lim_{y\to y_0} \frac{f\left(x_0, y\right) - f\left(x_0, y_0\right)}{y - y_0}$$

Example. $f\left(x,y\right)=2xy^{2}+5y$. Find $f_{2}\left(-1,3\right)$

$$f_{2}\left(x_{0},y_{0}\right)=\lim_{y\to y_{0}}\frac{f\left(x_{0},y\right)-f\left(x_{0},y_{0}\right)}{y-y_{0}}$$

$$f_{2}\left(-1,3\right)=\lim_{y\to 3}\frac{\left[-2y^{2}+5y\right]-\left[-2(3)^{2}+5(3)\right]}{y-3}$$

$$=\lim_{y\to 3}\frac{\left[-2y^{2}+5y\right]+3}{y-3}\qquad=\lim_{y\to 3}\frac{\left(y-3\right)\left(-2y-1\right)}{y-3}$$

$$=\lim_{y\to 3}\left(-2y-1\right)=-7$$

Definition.

Let $P\left(x_1, x_2, \ldots, x_n\right)$ be a point in R^n and f be a function of n variables $x_1, x_2, \ldots, x_n$.

The ***partial derivative of f with respect to x_k*** is the function, denoted by $D_k f$ such that its function value at any point P in the domain of f is given by

$$D_k f\left(P\right) = \lim_{h \to 0} \frac{f\left(x_1, x_2, \ldots, x_k + h, \ldots, x_n\right) - f\left(P\right)}{h}$$

if this limit exists.

Example. Let $f(x,y)=3x^2y-xy^4+7$

Find $f_1(x,y)$ and $f_2(x,y)$.

Solution.

Differentiate with respect to x ;

Treating y as a constant. $f_1(x,y)=6xy-y^4$

Differentiate with respect to y ;

Treating x as a constant.

$$f_2(x,y)=3x^2-4xy^3$$

Example. Let $g\left(x,y,z\right)=z^4e^{-x^2-y^2}$

Find $g_1\left(x,y,z\right), g_2\left(x,y,z\right)$ and $g_3\left(x,y,z\right)$.

Solution.

$$g_1\left(x,y,z\right)=z^4e^{-x^2-y^2}\cdot D_x\left(-x^2-y^2\right) \quad \left(-2x\right)$$

$$g_2\left(x,y,z\right)=z^4e^{-x^2-y^2}\cdot D_x\left(-x^2-y^2\right) \quad \left(-2y\right)$$

$$g_3\left(x,y,z\right)=4z^3e^{-x^2-y^2}$$

Example. Let $f\left(x,y\right)=2^{-x}\,Arc\tan\left(y^{2}+4x\right)$

Find $f_{1}\left(x,y\right)$ and $f_{2}\left(x,y\right)$.

Solution.

$$f_{2}\left(x,y\right)=2^{-x}\cdot D_{y}\left(Arc\tan\left(y^{2}+4x\right)\right)$$

$$=2^{-x}\cdot\frac{1}{1+\left(y^{2}+4x\right)^{2}}\cdot D_{y}\left(y^{2}+4x\right)$$

Example. Let $f(x,y) = 2^{-x} Arc\tan\left(y^2+4x\right)$

Find $f_1(x,y)$ and $f_2(x,y)$.

Solution.

PRODUCT RULE!

$$f_1(x,y) = Arc\tan\left(y^2+4x\right)\cdot D_x\left(2^{-x}\right)$$

$$f_1(x,y) = Arc\tan\left(y^2+4x\right)\cdot \left(2^{-x}\right)\left(-\ln 2\right)$$

$$+\ 2^{-x}\cdot D_x\left(Arc\tan\left(y^2+4x\right)\right)$$

$$+\ 2^{-x}\cdot \frac{1}{1+\left(y^2+4x\right)^2}\cdot D_x\left(y^2+4x\right)$$

Example. Let $h(x,y)=\sin(y)\ln\left(y^2-xy\right)$

Find $h_1(x,y)$ and $h_2(x,y)$.

Solution.

$$h_1(x,y)=\sin(y)\cdot\frac{1}{y^2-xy}\cdot(-y)$$

PRODUCT RULE!

$$h_2(x,y)=\sin(y)\cdot\frac{1}{y^2-xy}\cdot(2y-x)$$

$$+\ln\left(y^2-xy\right)\cdot\cos(y)$$

Example. Let $g\left(x,y,z\right)=\dfrac{\ln\left(yz\right)}{\tan\left(xz\right)}$

Find $g_1\left(x,y,z\right), g_2\left(x,y,z\right)$ and $g_3\left(x,y,z\right)$.

Solution. ***QUOTIENT RULE!***

$$\frac{\partial g}{\partial x}=\ln\left(yz\right)\cdot\frac{-1}{\tan^2\left(xz\right)}\cdot\left(\sec^2 xz\right)\left(z\right)$$

$$\frac{\partial g}{\partial y}=\frac{1}{\tan\left(xz\right)}\cdot\frac{1}{yz}\cdot z$$

Example. Let $g(x,y,z)=\dfrac{\ln(yz)}{\tan(xz)}$

Solution. ***QUOTIENT RULE!***

$$\frac{\partial g}{\partial z}=\frac{\tan(xz)\cdot D_z(\ln yz)-\ln(yz)\cdot D_z(\tan xz)}{\tan^2(xz)}$$

$$\frac{\partial g}{\partial z}=\frac{\tan(xz)\cdot\frac{1}{z}-\ln(yz)\cdot x\sec^2(xz)}{\tan^2(xz)}$$

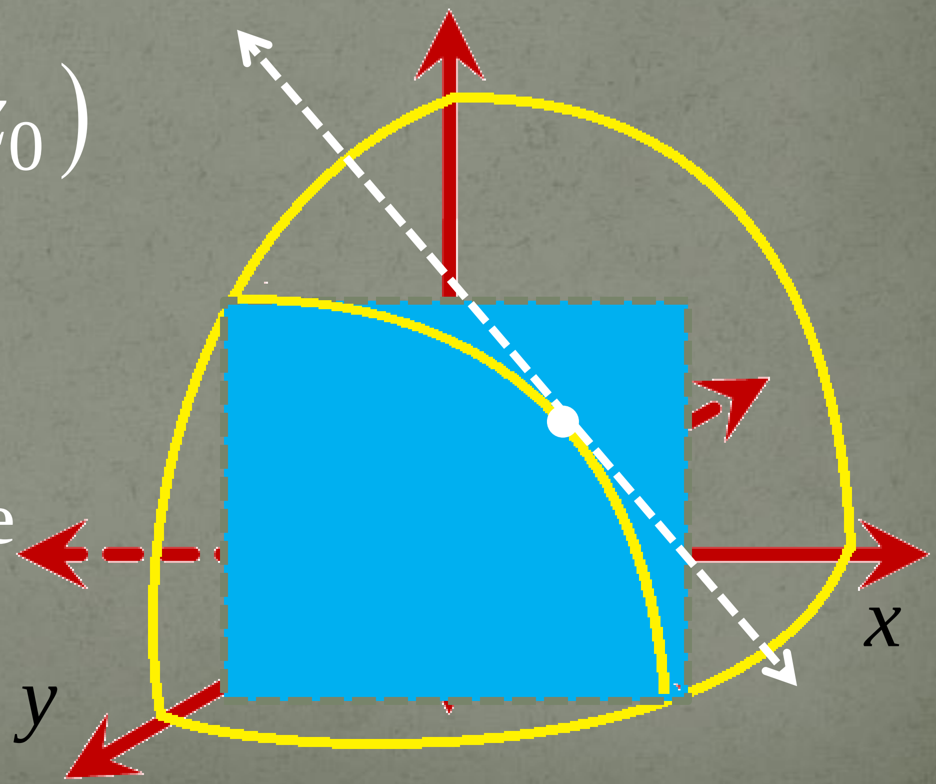
Geometric Interpretation.

Let S be a surface given by $z = f(x, y)$.

Consider a plane given by $y = y_0$ and which intersects S at a curve C.

Suppose $P_0(x_0, y_0, z_0)$ is on C.

$f_1(x, y)$: slope of the tangent line to the curve C at P_o.



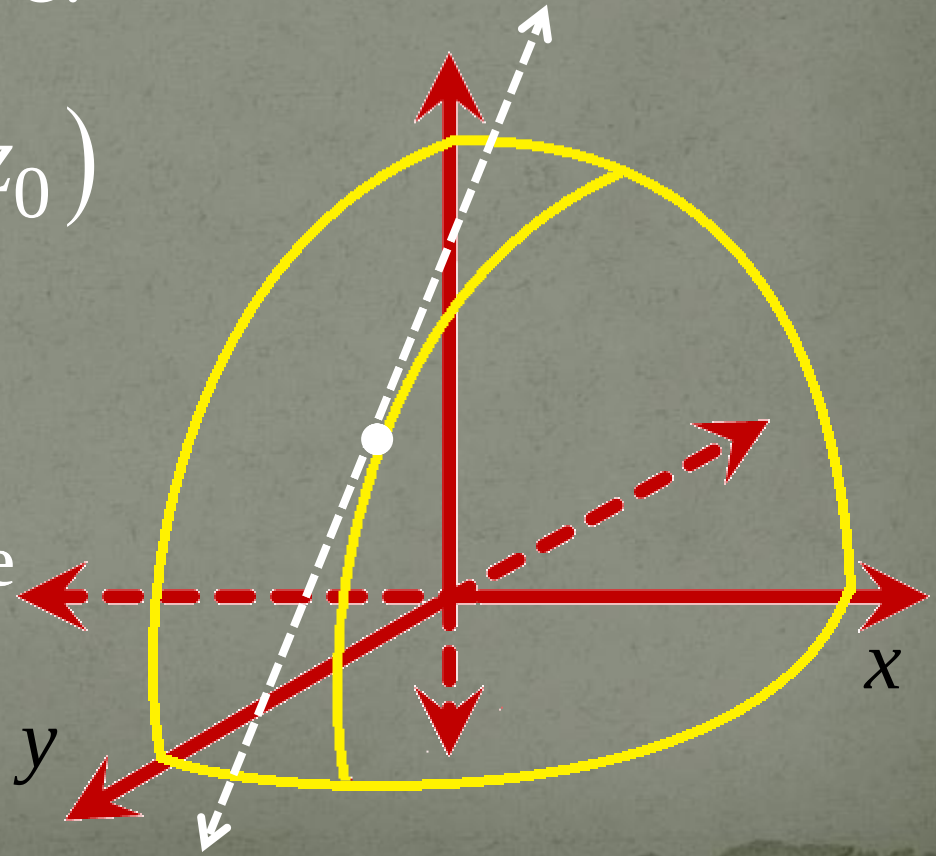
Geometric Interpretation.

Let S be a surface given by $z = f(x, y)$.

Consider a plane given by $x = x_0$ and which intersects S at a curve C.

Suppose $P_0(x_0, y_0, z_0)$ is on C.

$f_2(x, y)$: slope of the tangent line to the curve C at P_o.



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