Chapter 2
Section 2

Limits and Continuity of Functions of More Than One Variable

RECALL:

If $J\left(x_1, x_2, \ldots, x_n\right)$ **and** $P\left(y_1, y_2, \ldots, y_n\right)$ **are two points in** R^n**, then**

$$d\left(J, P\right) = \left|\overline{JP}\right| = \left\|J - P\right\| =$$

$$\sqrt{\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2 + \ldots + \left(x_n - y_n\right)^2}$$

Definition.

If A is a point in R^n and r is a positive number, then the **open ball** $B(A;r)$ with

CENTER: A RADIUS: r

is the set of all points P in R^n such that

$$\|P-A\|<r$$

i.e.,

$$B(A;r)=\left\{P\in R^n \,\middle|\, \|P-A\|<r\right\}.$$

Definition.

If A is a point in R^n and r is a positive number, then the *closed ball* $B[A;r]$ with

CENTER: A RADIUS: r

is the set of all points P in R^n such that

$$\|P-A\| \leq r$$

i.e.,

$$B[A;r] = \left\{ P \in R^n \,\middle|\, \|P-A\| \leq r \right\}.$$

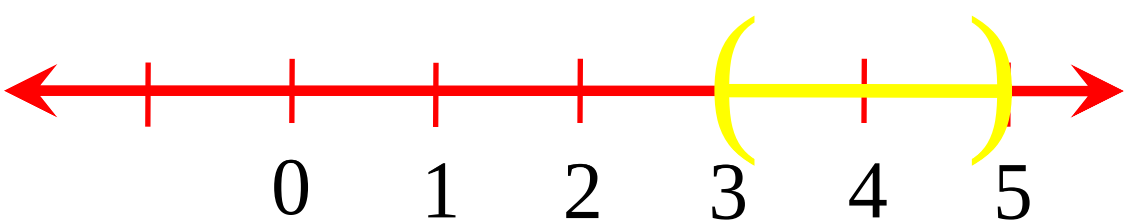
Illustrations.

1. $B(4;1) = \left\{ P \in R^1 \middle| \left\| P - 4 \right\| < 1 \right\}$

$$= \left\{ x \in R \middle| \left| x - 4 \right| < 1 \right\}$$

$$= \left\{ x \in R \middle| -1 < x - 4 < 1 \right\}$$

$$= \left\{ x \in R \middle| 3 < x < 5 \right\} = (3,5)$$



Illustrations.

2. $$B((-1,3);2)=\left\{P\in R^2 \,\middle|\, \left\|P-(-1,3)\right\|<2\right\}$$

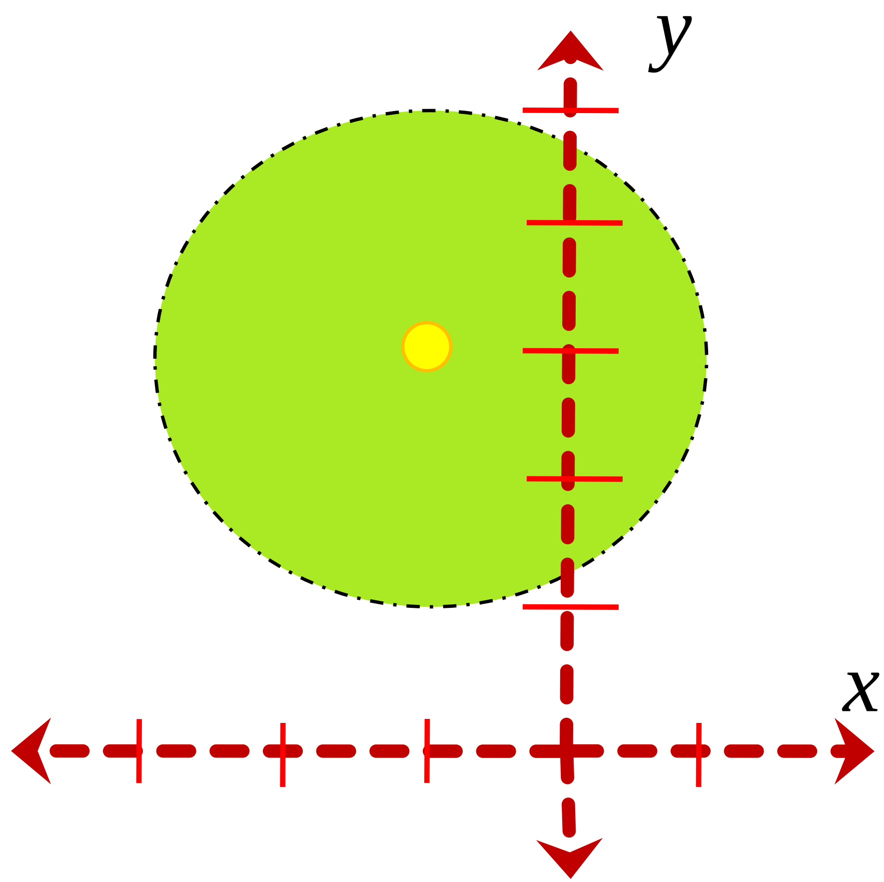
$$=\left\{(x,y)\in R^2 \,\middle|\, \left\|(x,y)-(-1,3)\right\|<2\right\}$$

$$=\left\{(x,y)\in R^2 \,\middle|\, \sqrt{(x+1)^2+(y-3)^2}<2\right\}$$

$$=\left\{(x,y)\in R^2 \,\middle|\, (x+1)^2+(y-3)^2<4\right\}$$

Illustrations.

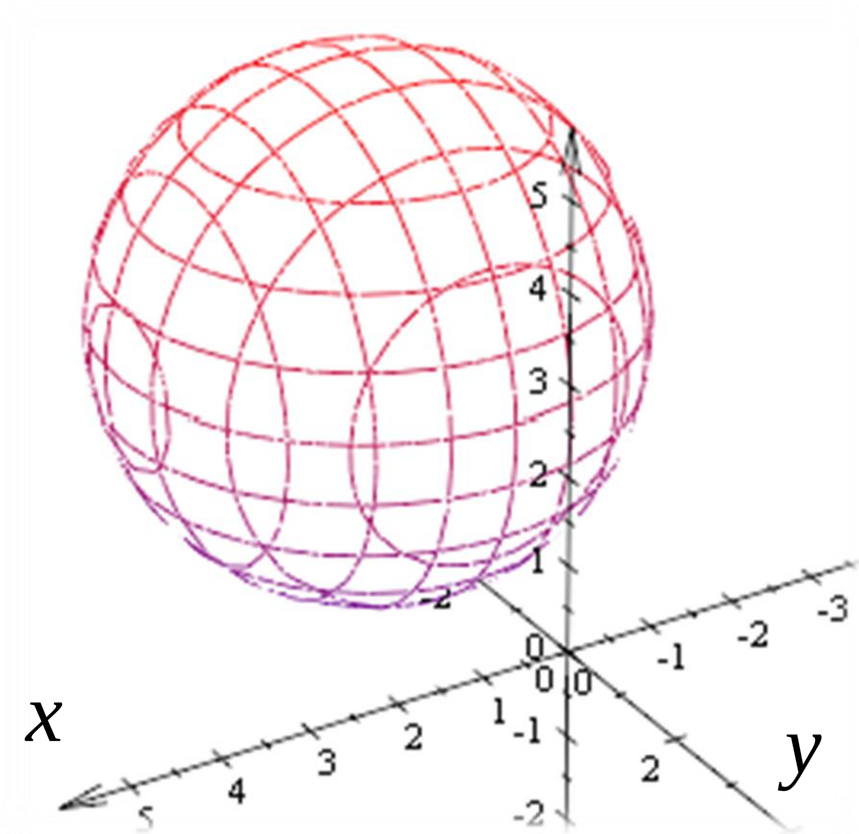
2. $B\left(\left(-1,3\right);2\right)=\left\{\left(x,y\right)\in R^{2}\,\middle|\,\left(x+1\right)^{2}+\left(y-3\right)^{2}<4\right\}$



Illustrations.

3. $B\left[(0,-4,2);3\right]$

$$=\left\{(x,y,z)\in R^3 \,\middle|\, x^2+(y+4)^2+(z-2)^2\le 9\right\}$$



Definition.

Let f be a function of n variables defined on some open ball $B(A;r)$, except possibly at the point A itself.

The ***limit*** of $f(P)$ as P approaches A is L, written as

$$\lim_{P\to A} f(P) = L$$

if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that if

$$0 < \|P - A\| < \delta \quad \Rightarrow \quad |f(P) - L| < \varepsilon$$

Definition.

Let f be a function of n variables defined on some open ball $B\left(\left(x_0, y_0\right); r\right)$, except possibly at the point $\left(x_0, y_0\right)$ itself.

The ***limit*** of $f\left(x, y\right)$ as $\left(x, y\right)$ approaches $\left(x_0, y_0\right)$ is L , written as

$$\lim_{(x,y)\to(x_0,y_0)} f\left(x, y\right) = L$$

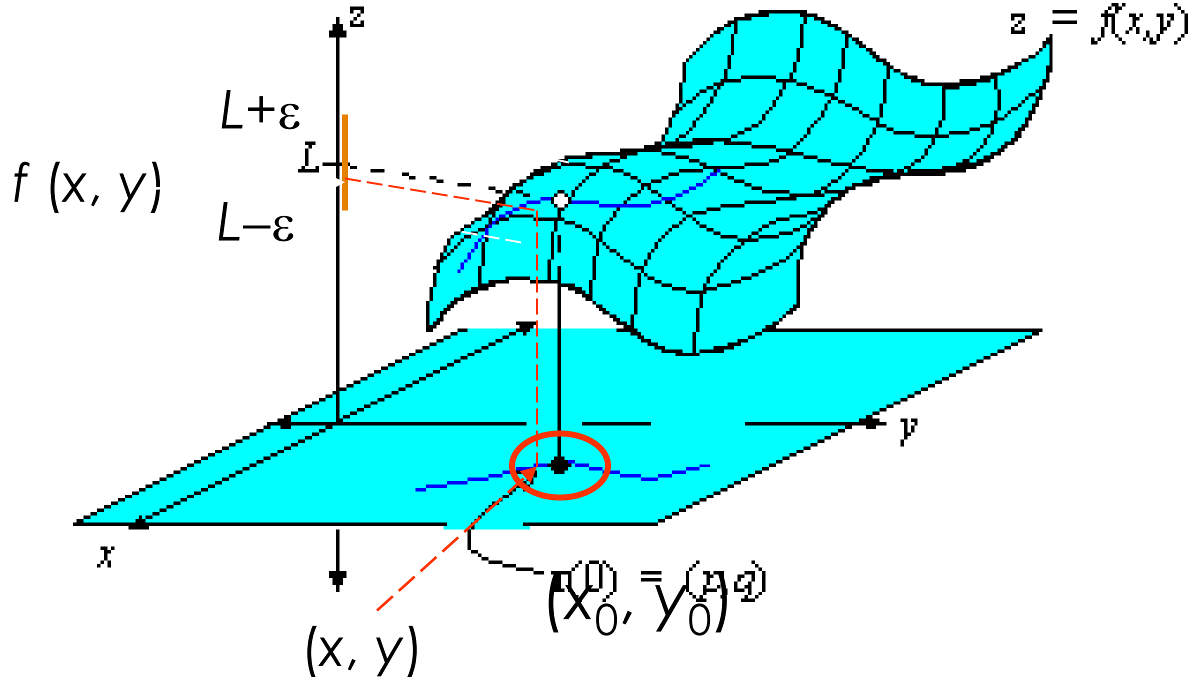
if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that if

$$0 < \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2} < \delta \quad \Longrightarrow \quad \left|f\left(x, y\right) - L\right| < \varepsilon$$

Definition.

if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that if

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon$$



Example.

Show that $$\lim_{(x,y)\to(0,4)} (6x+y)=4 \quad .$$

Proof.

Given any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{7}$ so that

if $0 < \sqrt{x^2+(y-4)^2} < \delta = \frac{\varepsilon}{7}$, then

$$|x| < \frac{\varepsilon}{7} \text{ and } |y-4| < \frac{\varepsilon}{7}$$

Example.

$$\left| f\left(x, y\right) - L \right| = \left| 6x + y - 4 \right|$$

$$\leq 6\left| x \right| + \left| y - 4 \right|$$

$$< 6\left(\frac{\varepsilon}{7} \right) + \left(\frac{\varepsilon}{7} \right) = \varepsilon$$

$$\lim_{(x,y)\to(0,4)} \left(6x + y \right) = 4$$

(end)

Assignment.

Show that $$\lim_{(x,y)\to(2,-5)} (3x-y) = 11 \quad .$$

Show that $$\lim_{(x,y)\to(1,-1)} \left(y+x^2\right) = 2 \quad .$$

Limit Theorems.

Evaluate the following limits using the basic limit theorems.

1. $$\lim_{(x,y)\to(1,-2)} x\sin^2(y\pi)$$

2. $$\lim_{(x,y)\to(-1,0)} Arc\tan(y-x)$$

3. $$\lim_{(x,y)\to(4,4)} \frac{x+y}{2\sqrt{xy}}$$

4. $$\lim_{(x,y)\to(2,1)} \frac{x-2y}{x^2-4y^2}$$

Theorem.

Suppose that the function f is defined at all points in an open disk having its center at (x_0, y_0), except possibly at (x_0, y_0) itself, and let

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If S is any curve in R^2 which passes through the point (x_0, y_0), then

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in S}} f(x,y)$$

exists and always has the value L.

Theorem.

If a function *f* has different limits as (x, y) approaches (x_0, y_0) through two distinct curves having (x_0, y_0) as an ***accumulation point***, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

does not exist.

Definition.

accumulation point

An accumulation point *P* of a set *S* is a point such that every open ball around it contains at least one point of *A* different from *P*.

Example.

Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x}{x^2+y^2}$ **does not exist.**

Solution:

Let $S_1 = \{(x,0) | x \in R\}$

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in S_1}} \frac{4x}{x^2+y^2} = \lim_{x\to 0} \frac{4x}{x^2} = \lim_{x\to 0} \frac{4}{x} = +\infty$$

Example.

Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x}{x^2+y^2}$ **does not exist.**

Solution:

Let $S_2 = \{(0,y) | y \in R\}$

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in S_2}} \frac{4x}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = 0$$

Example.

Since

$$\lim_{(x,y)\to(0,0)} \frac{4x}{x^2+y^2} \neq \lim_{(x,y)\to(0,0)} \frac{4x}{x^2+y^2}$$

$$(x,y)\in S_1 \qquad\qquad (x,y)\in S_2$$

$$\lim_{(x,y)\to(0,0)} \frac{4x}{x^2+y^2}$$ **does not exist.**

Example.

Show that the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{\left(x^4+2y^2\right)^2}$ **does not exist.**

Solution:

Let $S_1 = \{(x,0) \mid x \in R\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_1}} \frac{x^4y^2}{\left(x^4+2y^2\right)^2} = \lim_{x\to 0} \frac{0}{\left(x^4+0\right)^2} = 0$$

Example.

Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{\left(x^4+2y^2\right)^2}$ **does not exist.**

Solution:

Let $S_2 = \{(0,y) \mid y \in R\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_2}} \frac{x^4y^2}{\left(x^4+2y^2\right)^2} = \lim_{y\to 0} \frac{0}{\left(0+2y^2\right)^2} = 0$$

Example.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_3}} \frac{x^4y^2}{\left(x^4+2y^2\right)^2} = \lim_{x\to 0}\frac{x^4\left(x^2\right)^2}{\left(x^4+2\left(x^2\right)^2\right)^2}$$

$$= \lim_{x\to 0}\frac{x^8}{9x^8} = \frac{1}{9}$$

Thus, $\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{\left(x^4+2y^2\right)^2}$ **does not exist.**

Exercise.

Show that the following functions does not have a limit as $(x, y) \to (0, 0)$ **.**

$$f(x, y) = \frac{x^3 - y^3}{x^3 + y^3}$$

$$F(x, y) = \frac{x^2 y^2}{3x^4 + y^4}$$

$$g(x, y) = \frac{x^2 + y}{y}$$

$$G(x, y) = \frac{x^3 y}{x^6 + y^2}$$

$$h(x, y) = \frac{xy}{|xy|}$$

$$H(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Definition.

Suppose that f is a function of n variables and A is a point in R^n. Then f is said to be **continuous** *at the point* A if and only if the following three conditions are satisfied:

i. $f(A)$ exists

ii. $\lim_{P \to A} f(P)$ exists

iii. $\lim_{P \to A} f(P) = f(A)$

Recall.

Theorem.

Polynomial functions are continuous everywhere.

Theorem.

A rational function is continuous at each point in its domain.

Theorem.

Circular functions are continuous at each point in its domain.

Recall.

Definition.

If a function f is discontinuous at a point A, the discontinuity is said to be ***removable*** if

$$\lim_{P \to A} f(P) \quad \text{exists.}$$

Definition.

If a function f is discontinuous at a point A, the discontinuity is said to be ***essential*** if

$$\lim_{P \to A} f(P) \quad \text{does not exist.}$$

Assignment.

At what points (x,y) in the plane are the functions continuous?

1. $f\left(x,y\right)=\dfrac{x-y}{2-\cos y}$

2. $g\left(x,y\right)=\dfrac{e^{x^2+y^2}}{x^2-y^2}$

3. $h\left(x,y\right)=\dfrac{x^2-1}{x^2-3x+2}$

END