Chain Rule for Functions of More Than One Variable

Chapter 2 Section 4

Theorem.

If u is a differentiable function of x and y defined by $u = f(x, y)$, where

$$x = F(r,s),\ y = G(r,s) \text{ and } \frac{\partial x}{\partial r}, \frac{\partial x}{\partial s}, \frac{\partial y}{\partial r}, \frac{\partial y}{\partial s}$$

all exist, then u is a function of r and s and

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Theorem

Suppose that u is a differentiable function of n variables $x_1, x_2, ..., x_n$ and each of these variables is in turn a function of m variables $y_1, y_2, ..., y_m$.

Suppose further that each of the partial derivatives,

$$\frac{\partial x_i}{\partial y_i} (i = 1, 2, ..., n;\ j = 1, 2, ..., m)$$

exists. Then u is a function of $y_1, y_2, ..., y_m$ and

$$\frac{\partial u}{\partial y_1} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_1} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial y_1}$$

$$\frac{\partial u}{\partial y_2} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_2} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial y_2}$$

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$$\frac{\partial u}{\partial y_m} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_m} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_m} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial y_m}$$

Example If $u = 3x^2 - 4y, \quad x = 6rs$ and $y = 4r^2 - 2s$

Find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= (6x)\cdot(6s) + (-4)\cdot(8r)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (6x)\cdot(6r) + (-4)\cdot(-2)$$

Example If $u = \tan xy, \quad x = 2r^3t^2$ and $y = 3tr$

Find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}\right)$$

$$=\left(y\sec^2 xy\right)\cdot\left(6r^2t^2\right) + \left(x\sec^2 xy\right)\cdot(3t)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$=\left(y\sec^2 xy\right)\cdot\left(4r^3t\right) + \left(x\sec^2 xy\right)\cdot(3r)$$

Example If $u = x^2 + y^2 + z^2$ where $x = r^2 \sin\varphi\cos\theta$
$y = 2r\sin\varphi\sin\theta$ and $z = r\cos\varphi\sin\theta$

Solution.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$=(2x)\cdot(2r\sin\varphi\cos\theta) + (2y)\cdot(2\sin\varphi\sin\theta)$$

$$+ (2z)\cdot(\cos\varphi\sin\theta)$$

Example If $u = x^2 + y^2 + z^2$ where $x = r^2 \sin\varphi\cos\theta$ $y = 2r\sin\varphi\sin\theta$ and $z = r\cos\varphi\sin\theta$

Solution.

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial \varphi}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial \varphi}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial \varphi}\right)$$

$$=(2x)\cdot\left(r^2\cos\varphi\cos\theta\right) + (2y)\cdot(2r\cos\varphi\sin\theta)$$

$$+ (2z)\cdot(-r\sin\varphi\sin\theta)$$

Example If $u = x^2 + y^2 + z^2$ where $x = r^2 \sin\varphi\cos\theta$ $y = 2r\sin\varphi\sin\theta$ and $z = r\cos\varphi\sin\theta$

Solution.

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial \theta}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial \theta}\right)$$

$$= (2x)\cdot\left(-r^2\sin\varphi\sin\theta\right) + (2y)\cdot\left(2r\sin\varphi\cos\theta\right)$$

$$+ (2z)\cdot\left(r\cos\varphi\cos\theta\right)$$

Remark:

If u is a differentiable function of n variables $x_1, x_2, ..., x_n$ and each of these variables is in turn a function of t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}.$$

Example Find the total derivative $\frac{du}{dt}$ given that

$$u = y\ln x + xe^{y}; \qquad x = \cos t; \qquad y = \sin t$$

Solution.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

$$= \left(\frac{y}{x} + e^{y}\right) \cdot (-\sin t) + \left(\ln x + xe^{y}\right) \cdot (\cos t)$$

Example Find the total derivative $\frac{du}{dt}$ given that

$$u = xy + yz + xz; \qquad x = 5^t; \qquad y = Arc\sin t; \qquad z = t$$

Solution.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial u}{\partial z}\left(\frac{dz}{dt}\right)$$

$$= (y+z)\cdot\left(5^t \ln 5\right) + (x+z)\cdot\left(\frac{1}{\sqrt{1-t^2}}\right) + (y+x)\cdot(1)$$

Recall. If $y = f(x)$ *and* $\tan(xy) - y4^x - x^2y^3 = 0$

Find $\dfrac{dy}{dx}$. ***IMPLICIT DIFFERENTIATION***

$$\Updownarrow$$

$$F(x,y)$$

$$x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy) - 4^x\frac{dy}{dx} - y4^x\ln 4$$

$$-3x^2y^2\frac{dy}{dx} - 2xy^3 = 0$$

$$\frac{dy}{dx} = \frac{-y\sec^2(xy) + y4^x\ln 4 + 2xy^3}{x\sec^2(xy) - 4^x - 3x^2y^2} = \frac{-F_x(x,y)}{F_y(x,y)}$$

Theorem

If f is a differentiable function of the single variable x such that $y = f(x)$ and f is defined implicitly by the equation $F(x, y) = 0$, then if F is differentiable and $F_y(x, y) \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

Theorem

If f is a differentiable function of x and y such that $F(x, y, z) = 0$ and f is defined implicitly by the equation $z = f(x, y)$, then if F is differentiable and $F_z(x, y, z) \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

Example. If $w = F\left(x, y, z\right)$ *and*

$$\frac{yz}{x^2 + w^2} - \ln\sqrt{z^2 - w} = ze^{-w}$$

Find $\dfrac{\partial w}{\partial z}$.

Solution. $F = \dfrac{yz}{x^2 + w^2} - \ln\sqrt{z^2 - w} - ze^{-w} = 0$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-\left[\dfrac{y}{x^2 + w^2} - \dfrac{1}{2}\left(\dfrac{2z}{z^2 - w}\right) - e^{-w}\right]}{\dfrac{-2wyz}{\left(x^2 + w^2\right)^2} - \dfrac{1}{2}\left(\dfrac{-1}{z^2 - w}\right) + ze^{-w}}$$

END