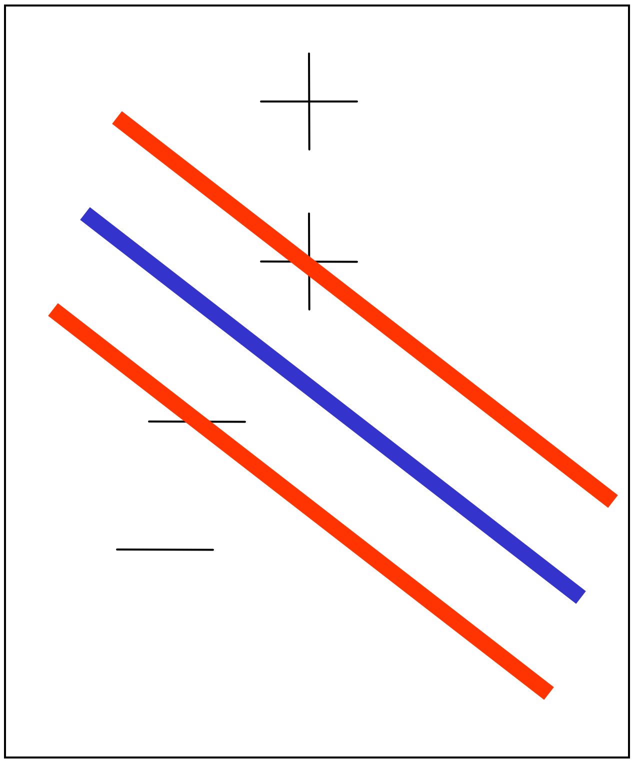
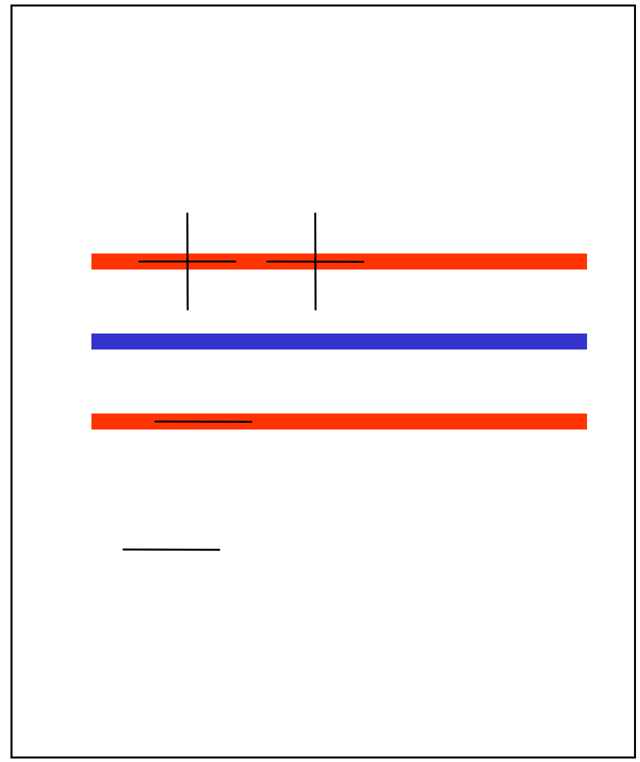
Foundations: fortunate choices

- Unusual choice of separation strategy:
  > Maximize "street" between groups
- Attack maximization problem:
  > Lagrange multipliers + hairy mathematics
- New problem is a quadratic minimization:
  > Susceptible to fancy numerical methods
- Result depends on dot products only
  > Enables use of kernel methods.

Key idea: find widest separating "street"





Classifier form is given and constrained

- Classify as plus if:

$$f(x) = \mathbf{w} \cdot \mathbf{u} + b > 0$$

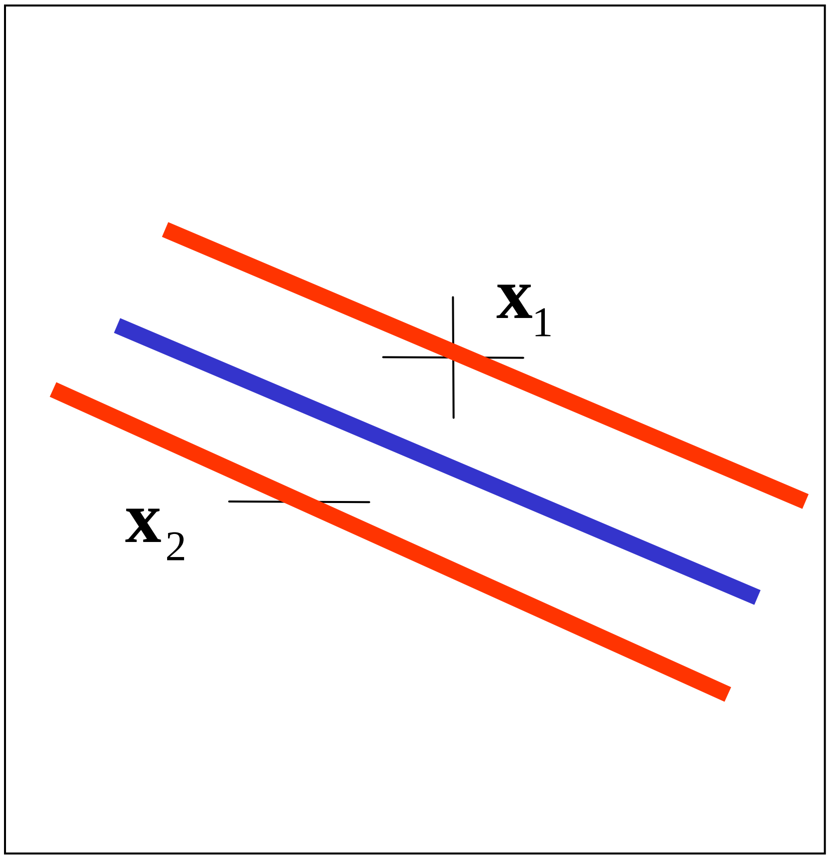
- Then, constrain, for all plusses:

$$f(x) = \mathbf{w} \cdot \mathbf{x}_{+} + b \geq 1$$

- And for all minuses

$$f(x) = \mathbf{w} \cdot \mathbf{x}_{-} + b \leq -1$$

Distance between street's gutters



- The constraints require:

$$\mathbf{w}\cdot\mathbf{x}_1 + b = +1$$

$$\mathbf{w}\cdot\mathbf{x}_2 + b = -1$$

- So, subtracting:

$$\mathbf{w}\cdot(\mathbf{x}_1 - \mathbf{x}_2) = 2$$

- Dividing by the length of **w** produces the distance between the lines:

$$\frac{\mathbf{w}}{\|\mathbf{w}\|}\cdot(\mathbf{x}_1 - \mathbf{x}_2) = \frac{2}{\|\mathbf{w}\|}$$

From maximizing to minimizing…

- So, to maximize the width of the street, you need to “wiggle” w until the length of w is minimum, *while still honoring constraints*:

$$\frac{2}{\|\mathbf{w}\|} = \text{separation}$$

- Alternatively, you can “wiggle” to minimize the following, *while still honoring constraints*

$$\frac{1}{2}\|\mathbf{w}\|^2$$

…while honoring constraints

- Remember, the minimization is constrained
- You can write the constraints as:

$$y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1$$

Where y*i* is 1 for plusses and –1 for minuses.

Dependence on dot products

- After some hairy mathematics, you get to the following problem:

Maximize $$\sum_{i=1}^{l} a_i - \frac{1}{2}\sum_{i,j=1}^{l} a_i a_j y_i y_j \mathbf{x}_i \cdot \mathbf{x}_j$$

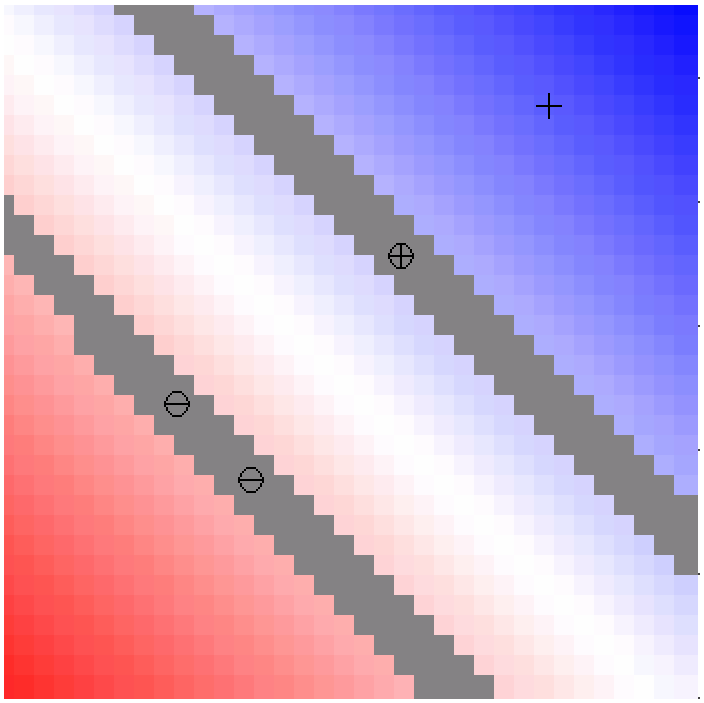
Subject to $$\sum_{i}^{l} a_i y_i = 0_i \quad \text{and} \quad a_i \geq 0$$

Then check sign of $$f(x) = \mathbf{w} \cdot \mathbf{u} + b = (\sum_{i,j=1}^{l} a_i y_i \mathbf{x}_i \cdot \mathbf{u}) + b$$

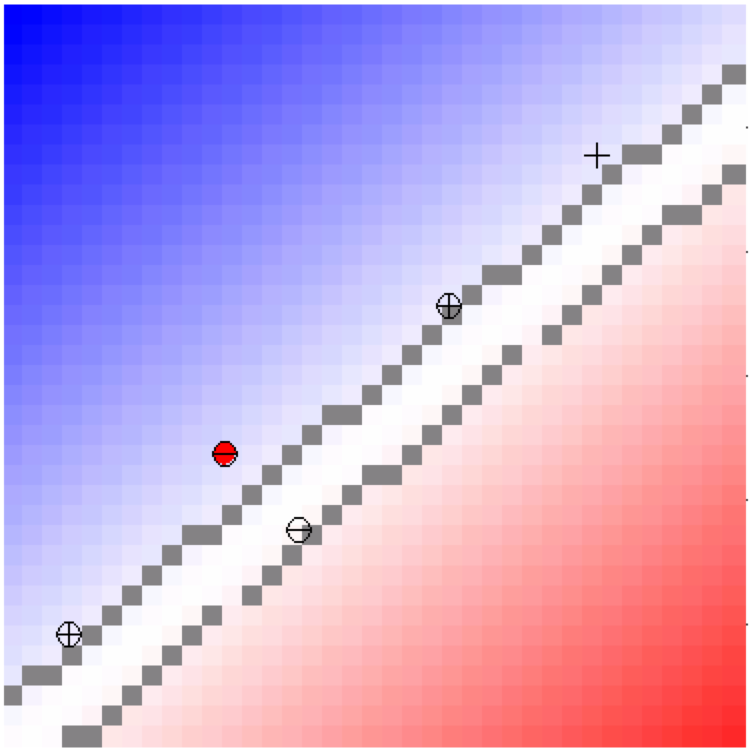
Key to importance

- Learning depends only on dot products of sample pairs.
- Recognition depends only on dot products of unknown with samples.
- Exclusive reliance on dot products enables approach to problems in which samples cannot be separated by a straight line.

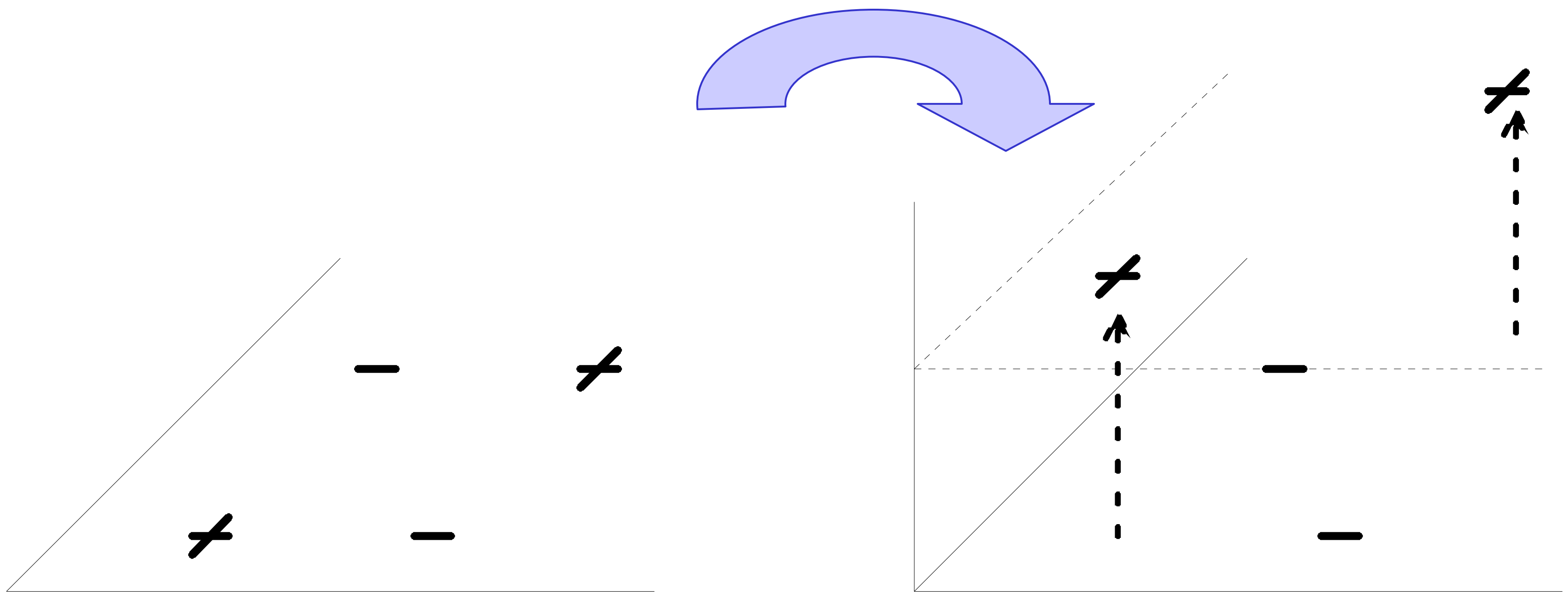
Example



Another example



Not separable?
Try another space!



Problem starts here, 2D

Dot products computed here, 3D

What you need

- To get into the high-dimensional space, you use

$\Phi(\mathbf{x}_1)$

- To optimize, you need

$\Phi(\mathbf{x}_1)\cdot\Phi(\mathbf{x}_2)$

- To use, you need

$\Phi(\mathbf{x}_1)\cdot\Phi(\mathbf{u})$

- So, all you need is a way to compute dot products in high-dimensional space as a function of vectors in original space!

What you don't need

- Suppose dot products are supplied by

$$\Phi(\mathbf{x}_1) \cdot \Phi(\mathbf{x}_2) = K(\mathbf{x}_1, \mathbf{x}_2)$$

- Then, all you need is

$$K(\mathbf{x}_1, \mathbf{x}_2)$$

- You don't need

$$\Phi(\mathbf{x}_1)$$

Standard choices

- No change

$$\Phi(\mathbf{x}_1)\cdot\Phi(\mathbf{x}_2) = K(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{x}_1\cdot\mathbf{x}_2$$

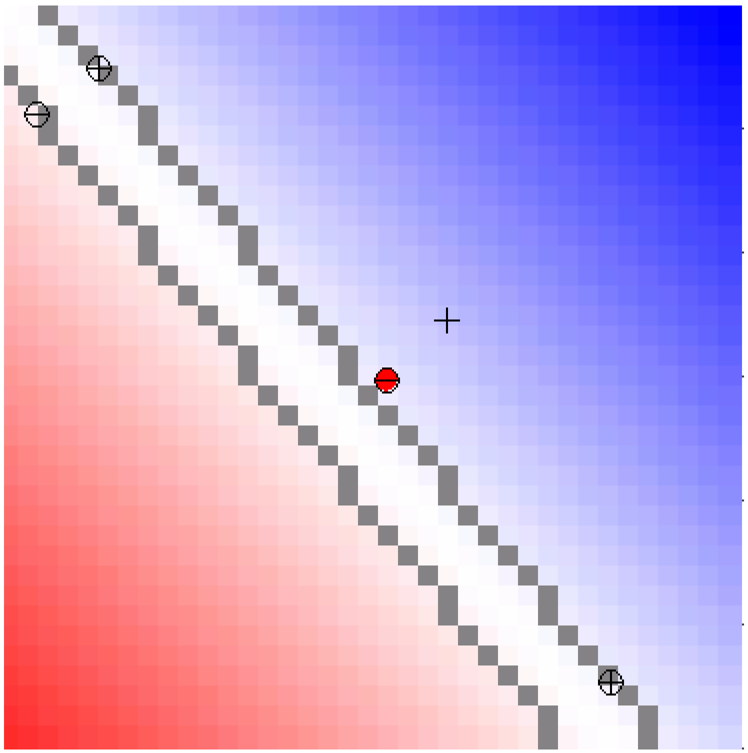
- Polynomial

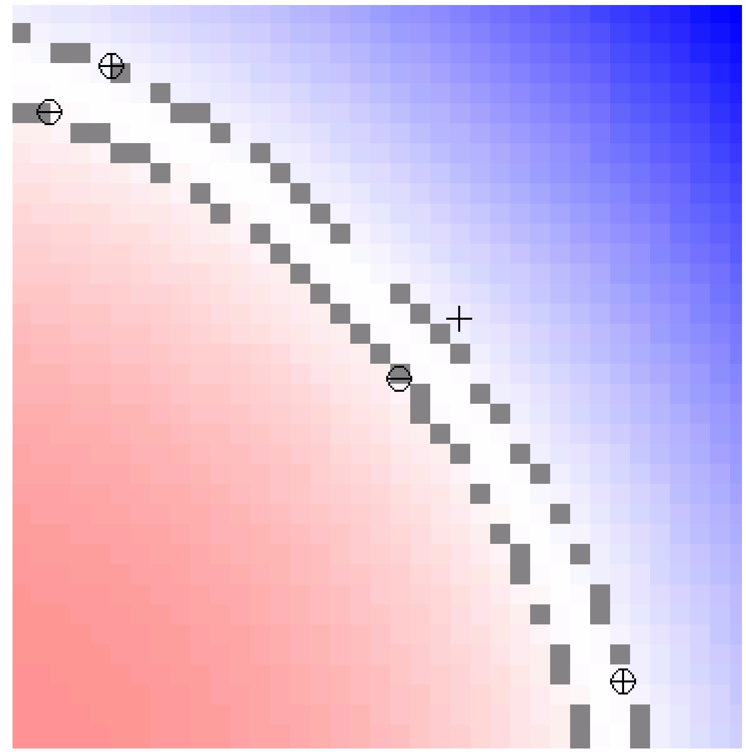
$$K(\mathbf{x}_1,\mathbf{x}_2) = (\mathbf{x}_1\cdot\mathbf{x}_2)^n$$

- Radial basis function

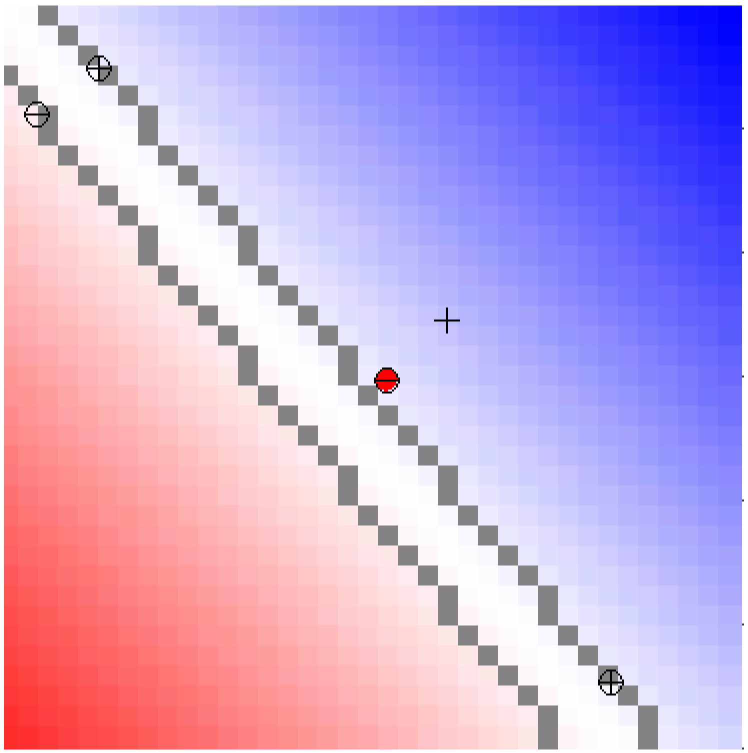
$$K(\mathbf{x}_1,\mathbf{x}_2) = e\frac{-\|\mathbf{x}_1-\mathbf{x}_2\|^2}{2\sigma^2}$$

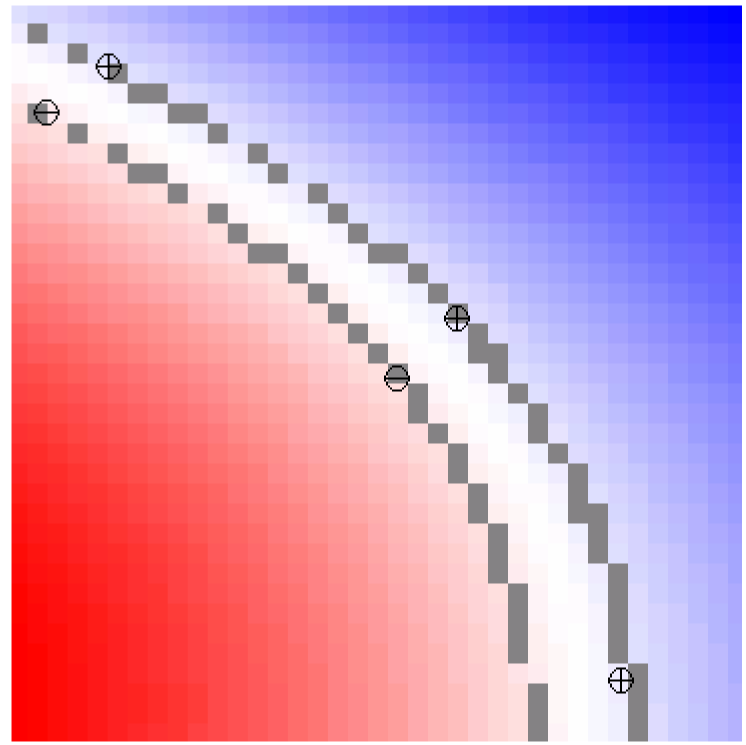
Polynomial Kernel



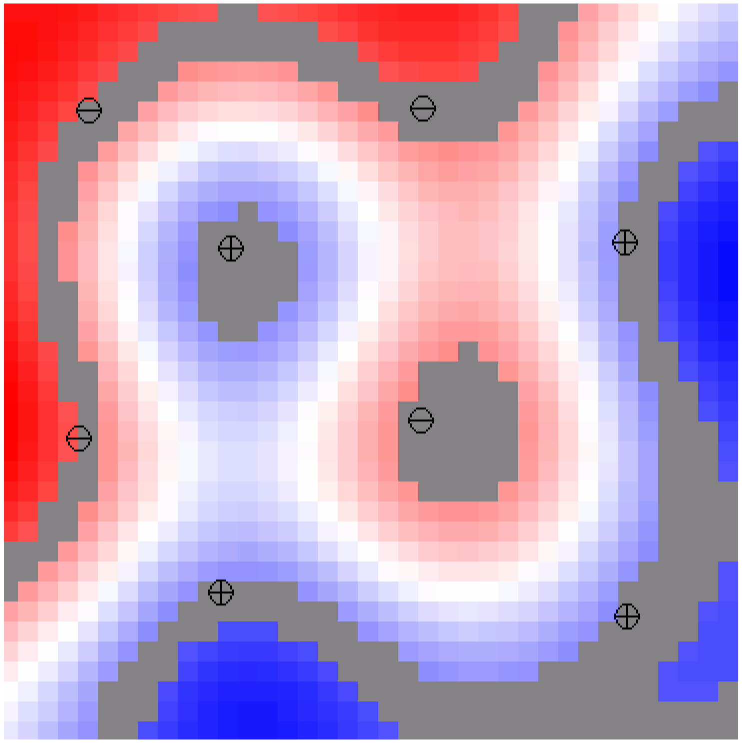


Radial-basis kernel





Another radial-basis example



Aside: about the hairy mathematics

- Step 1: Apply method of Lagrange multipliers

Minimize $\frac{1}{2}\|\mathbf{w}\|^2$ subject to constraints $y_i(\mathbf{x}_i \cdot \mathbf{w} + b) \geq 1$

yields

Find places where $L = \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{i=1}^{l} a_i (y_i(\mathbf{x}_i \cdot \mathbf{w} + b) - 1)$

has zero derivatives

Aside: about the hairy mathematics

Step 2: remember how to differentiate vectors

$$\frac{\partial \|\mathbf{w}\|^2}{\partial \mathbf{w}} = 2\mathbf{w} \quad \text{and} \quad \frac{\partial \mathbf{x} \cdot \mathbf{w}}{\partial \mathbf{w}} = \mathbf{x}$$

Step 3: find derivatives of the Lagrangian L

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{w} - \sum_{i=1}^{l} a_i y_i \mathbf{x}_i = 0$$

$$\frac{\partial L}{\partial b} = \sum_{i=1}^{l} a_i y_i = 0$$

Aside: about the hairy mathematics

- Step 4: do the algebra

$$L_{\text{Dual}} = \sum_{i=1}^{l} a_i - \frac{1}{2} \sum_{i,j=1}^{l} a_i a_j y_i y_j \mathbf{x}_i \cdot \mathbf{x}_j$$

- Step 5: do more mathematics, obtaining

$$\sum_{i=1}^{l} a_i y_i = 0$$

$$0 \le a_i \le C$$

Summary

- Quadratic minimization depends on only on dot products of sample vectors
- Recognition depends only on dot products of unknown vector with sample vectors
- Reliance on only dot products key to remaining magic