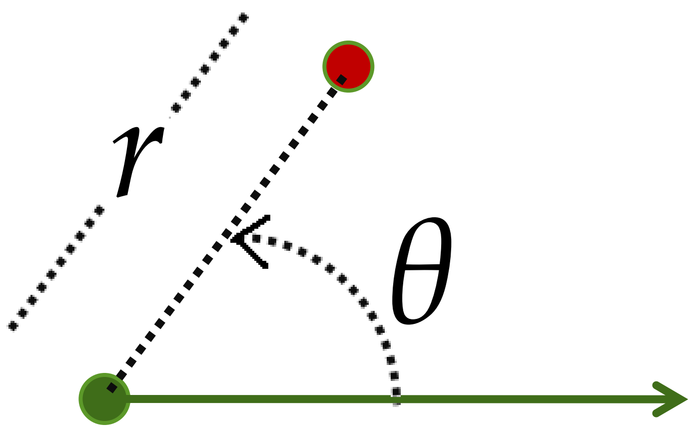
Double Integrals in Polar Coordinates

Chapter 4 Section 2

4.2 Double Integral in Polar Coordinates

REVIEW:

In polar coordinates, a point has coordinates (r, θ)

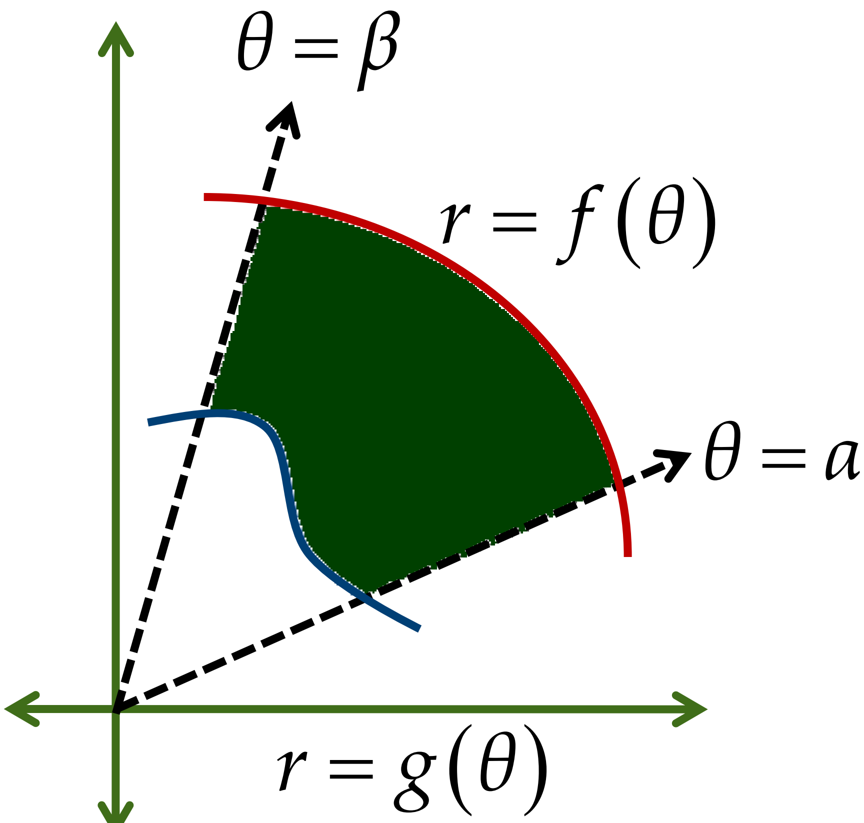


r directed distance of the point from the ***pole***

θ radian measure of the angle formed by the terminal side wide the ***polar axis***

4.2 Double Integral in Polar Coordinates

$$dA = r\,dr\,d\theta$$



$$\frac{1}{2}\int_{a}^{\beta}\left[f^{2}(\theta)-g^{2}(\theta)\right]d\theta$$

$$=\frac{1}{2}\int_{a}^{\beta} r^{2}\Big|_{g(\theta)}^{f(\theta)}\,d\theta$$

$$=\frac{1}{2}\int_{a}^{\beta}\int_{g(\theta)}^{f(\theta)} 2r\,dr\,d\theta$$

$$=\int_{a}^{\beta}\int_{g(\theta)}^{f(\theta)} r\,dr\,d\theta$$

Double Integral of *f* over *R*

In polar coordinates

$$\iint_R dA = \int_a^{\beta} \int_{g(\theta)}^{f(\theta)} r\,dr\,d\theta$$

$$x = r\cos\theta$$

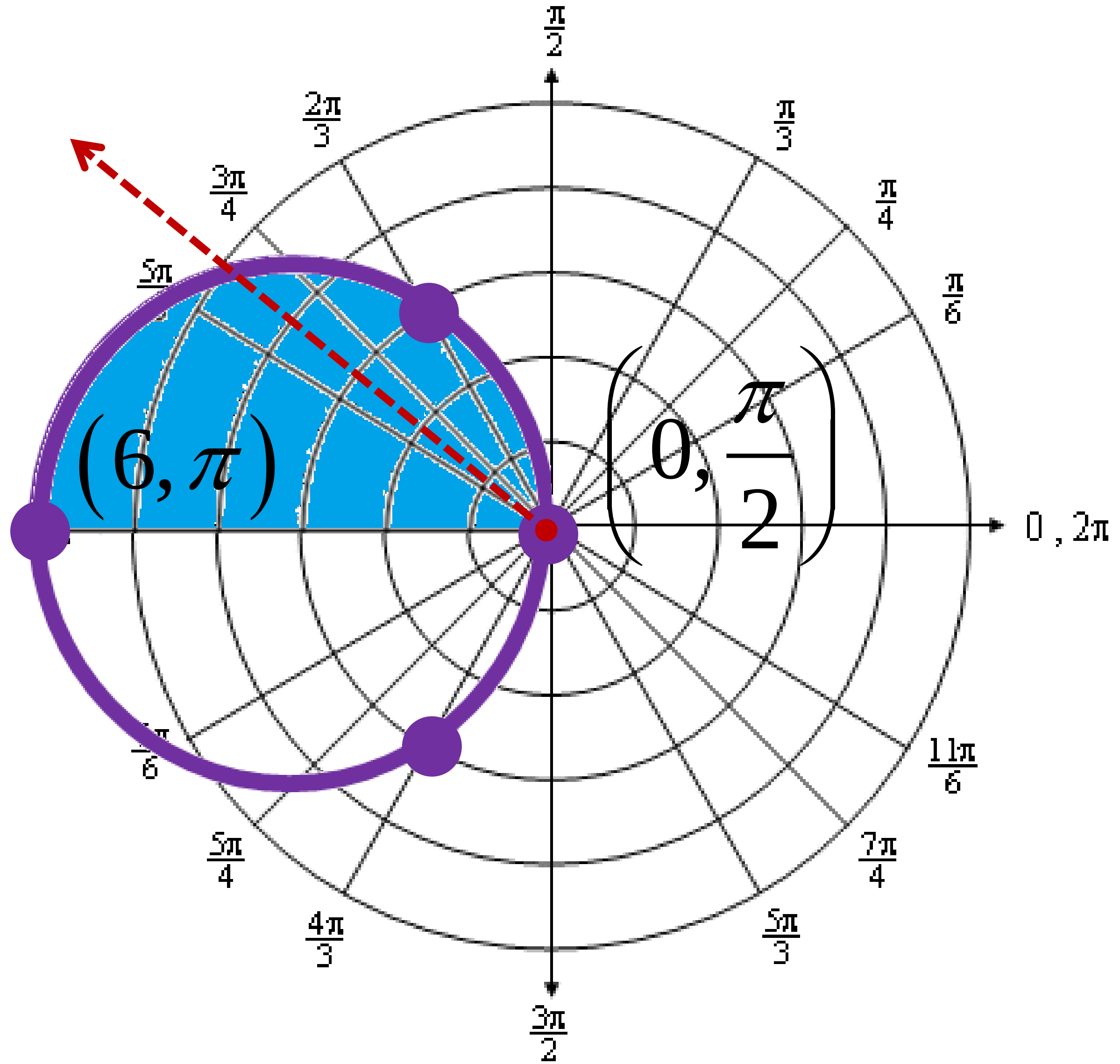
$$y = r\sin\theta$$

$$\frac{y}{x} = \tan\theta$$

Examples. **SET-UP** then **EVALUATE** the double integral which gives the area of the region

a. Inside the circle

$$r = -6\cos\theta$$



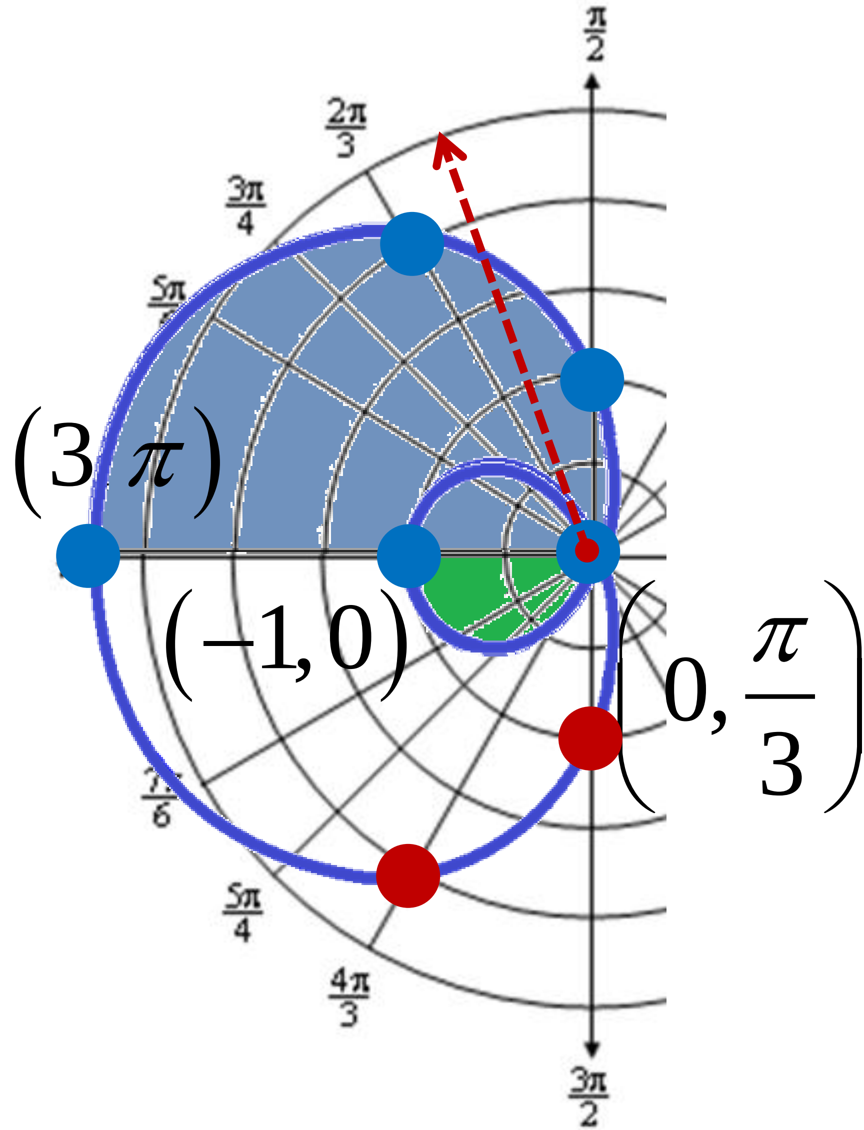
$$2\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{-6\cos\theta} r\,dr\,d\theta$$

$$= 9\pi$$

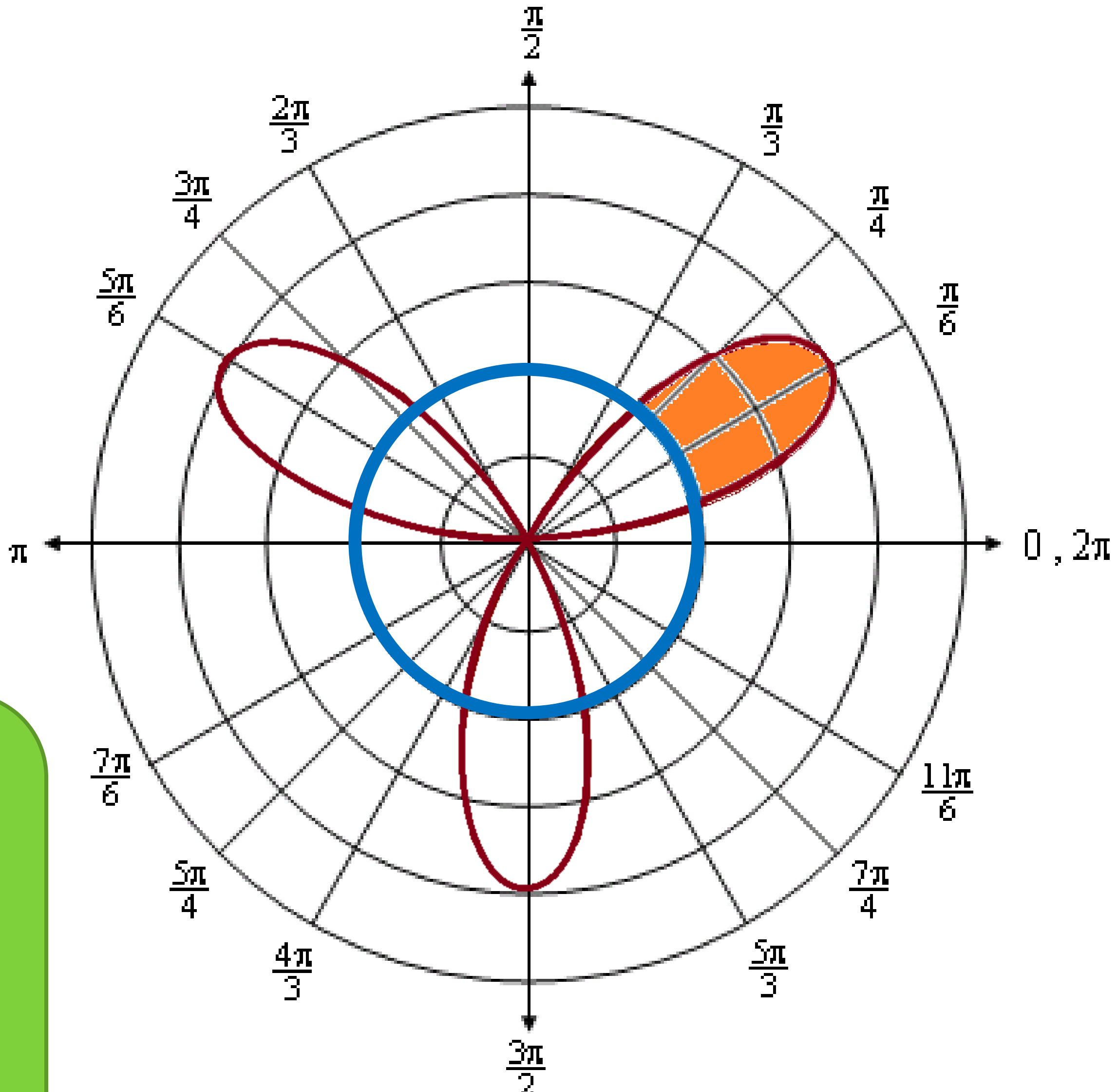
Examples. **SET-UP** the double integral which gives the area of the region

b. Inside the larger loop but outside the smaller loop of $r = 1 - 2\cos\theta$

$$2\int_{\frac{\pi}{3}}^{\pi}\int_{0}^{1-2\cos\theta} r\,dr\,d\theta - 2\int_{0}^{\frac{\pi}{3}}\int_{0}^{1-2\cos\theta} r\,dr\,d\theta$$



c. Inside $r = 4\sin 3\theta$ but outside $r = 2$

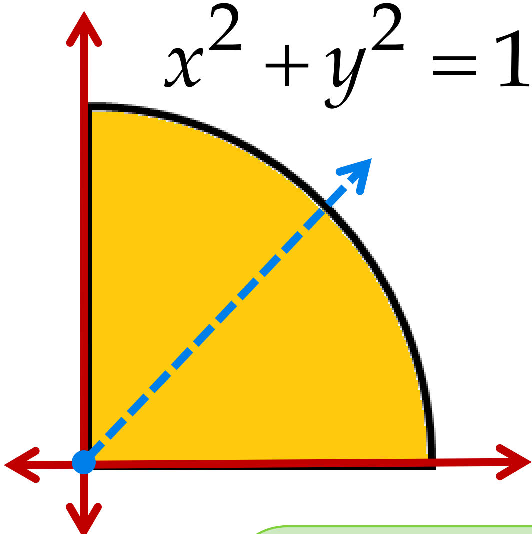


$$3\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}}\int_{2}^{4\sin 3\theta} r\,dr\,d\theta$$

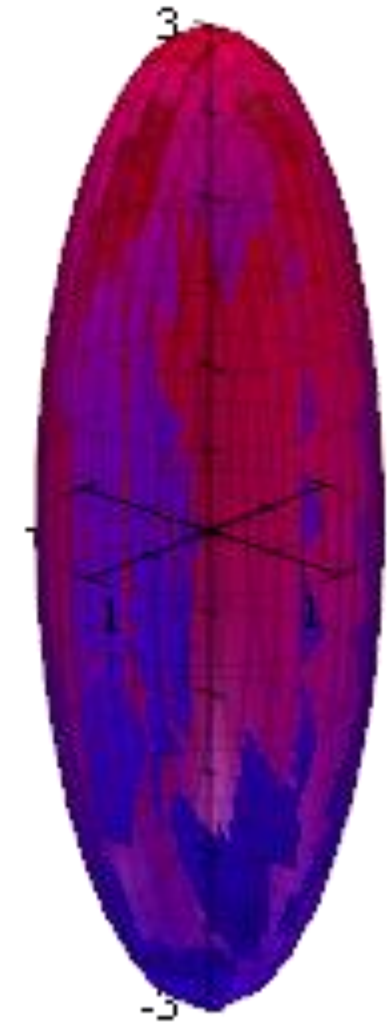
Exercise. **SET-UP** the double integral which gives the volume of the solid described.

a. Enclosed by $9x^2 + 9y^2 + z^2 = 9$

$x^2 + y^2 = 1$

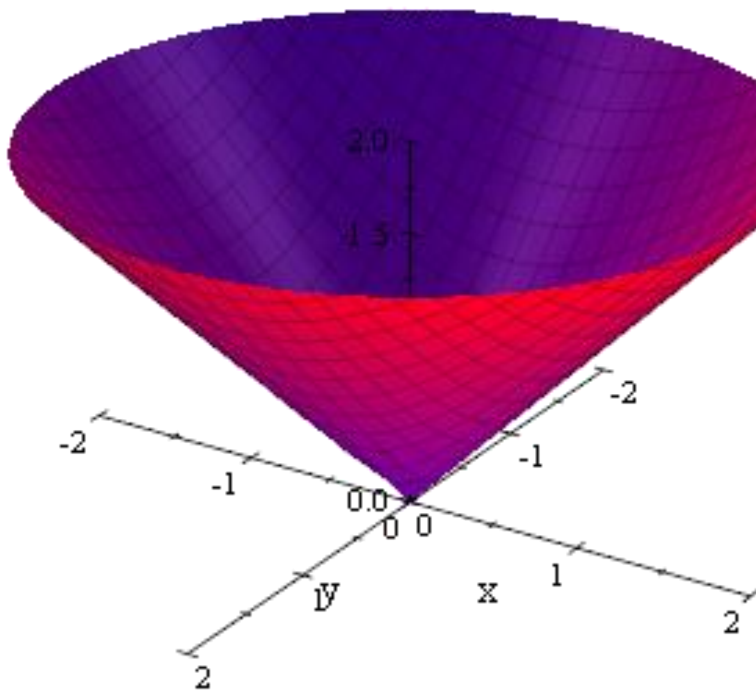


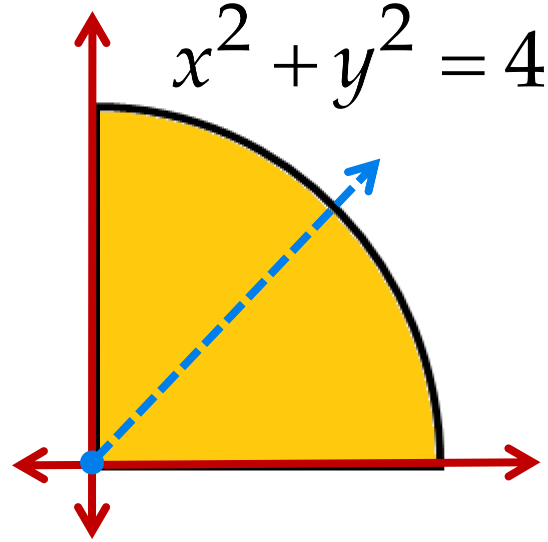
$$V = 8\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{9 - 9r^2}\, r\,dr\,d\theta$$



Exercise. **SET-UP** the double integral which gives the volume of the solid described.

b. Enclosed by $z=\sqrt{x^2+y^2}$ and $z=2$



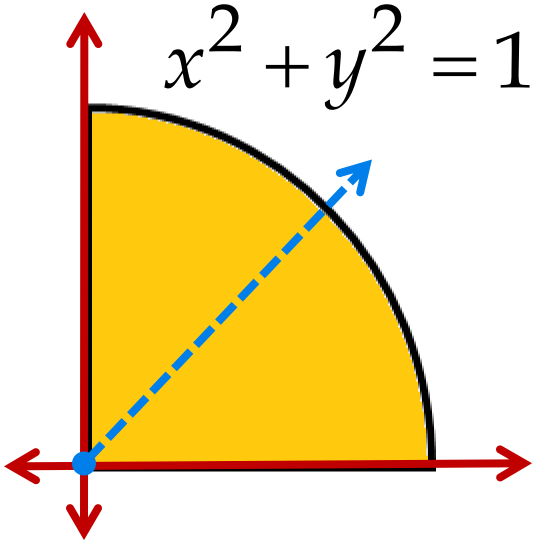


$$V=4\int_0^{\frac{\pi}{2}}\int_0^2 (2-r)\,r\,dr\,d\theta$$

Evaluating Integrals Using Polar Coordinates

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \left[\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}\right] dx\, dy$$

$$=\int_0^{\frac{\pi}{2}} \int_0^1 \left[\sqrt{2-r^2}-r\right] r\,dr\, d\theta$$

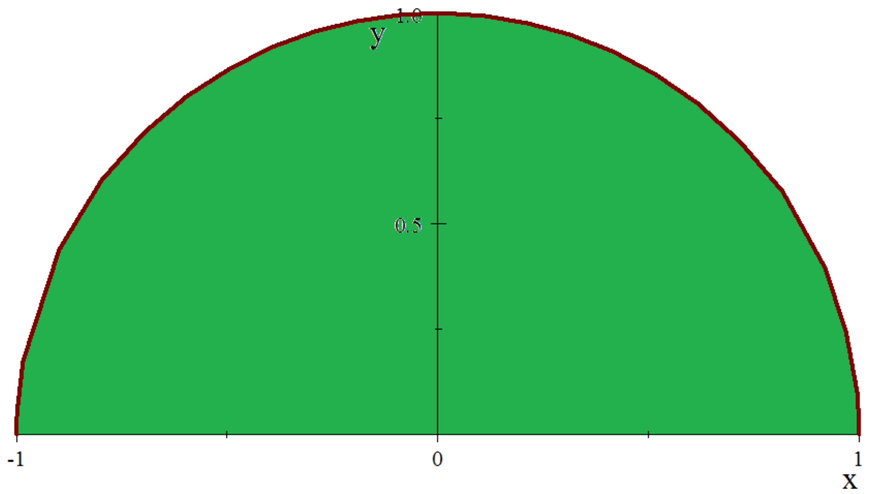


$$=\frac{\pi}{3}\left(\sqrt{2}-1\right)$$

Evaluating Integrals Using Polar Coordinates

Evaluate: $$\iint_R e^{x^2+y^2}\, dA$$

where ***R*** is the semicircle region bounded by the *x*-axis and the curve $y=\sqrt{1-x^2}$



$$\int_0^{\pi}\int_0^{1} e^{r^2}\, r\, dr\, d\theta$$

Evaluating Integrals Using Polar Coordinates

$$\iint_R e^{x^2+y^2}\,dA = \int_0^{\pi}\int_0^{1} e^{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\left(\int_0^1 e^{r^2}\, r dr\right)d\theta = \int_0^{\pi}\left(\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\left(\frac{1}{2}e - \frac{1}{2}\right)d\theta = \left(\frac{1}{2}e - \frac{1}{2}\right)\theta\Big|_0^{\pi} = \frac{\pi}{2}(e-1)$$

END