Extrema of Funtions of two or more variables

Chapter 3 Section 5

DEFINITION.

A function f of two variables x and y is said to have a ***relative maximum value*** at the point (x_0, y_0) if there exists an open disk $B((x_0, y_0); r)$ such that for all (x, y) in $\boldsymbol{B}$ $f(x_0, y_0) \geq f(x, y)$.

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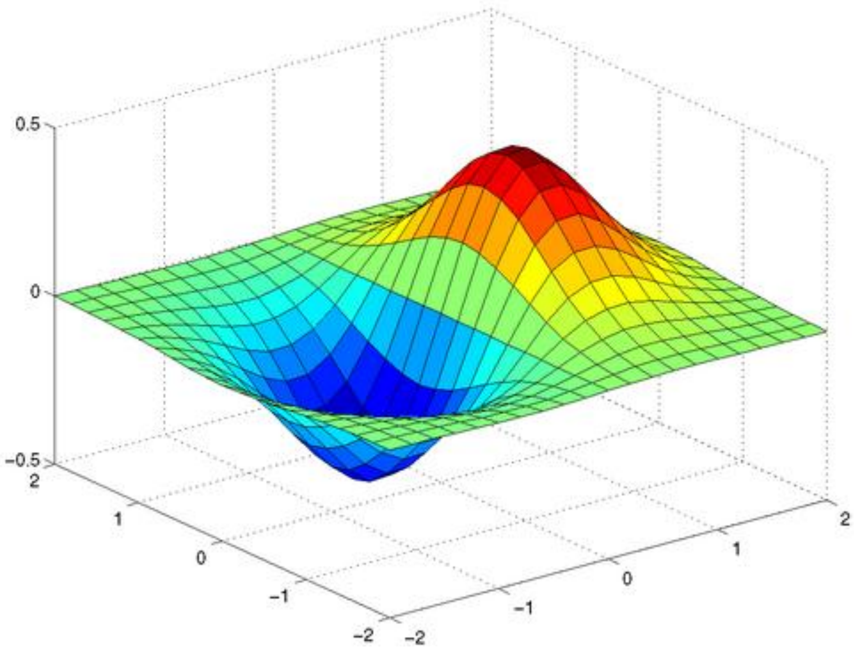
Definition.

A function f of two variables x and y is said to have an ***absolute maximum value*** on its domain $\boldsymbol{D}$ in the xy-plane if there is some point (x_0, y_0) in $\boldsymbol{D}$ such that for all (x, y) in $\boldsymbol{D}$, $f(x_0, y_0) \geq f(x, y)$.

A function f of two variables x and y is said to have an ***absolute minimum value*** on its domain $\boldsymbol{D}$ in the xy-plane if there is some point (x_0, y_0) in $\boldsymbol{D}$ such that for all (x, y) in $\boldsymbol{D}$, $f(x_0, y_0) \leq f(x, y)$.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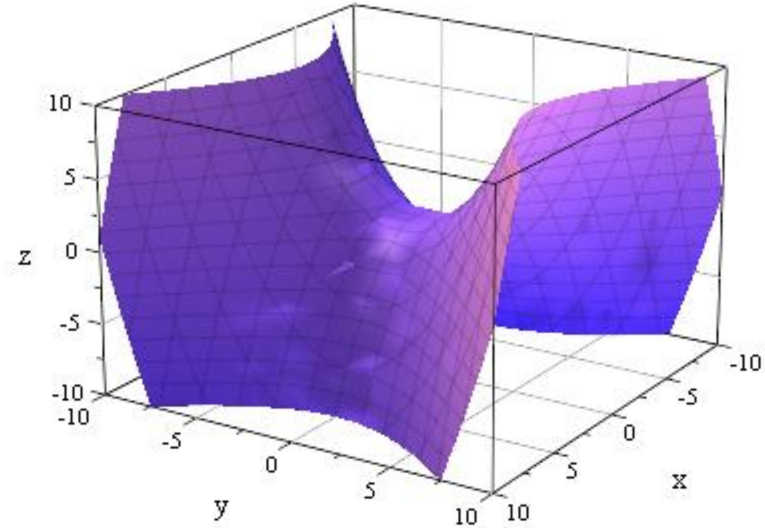
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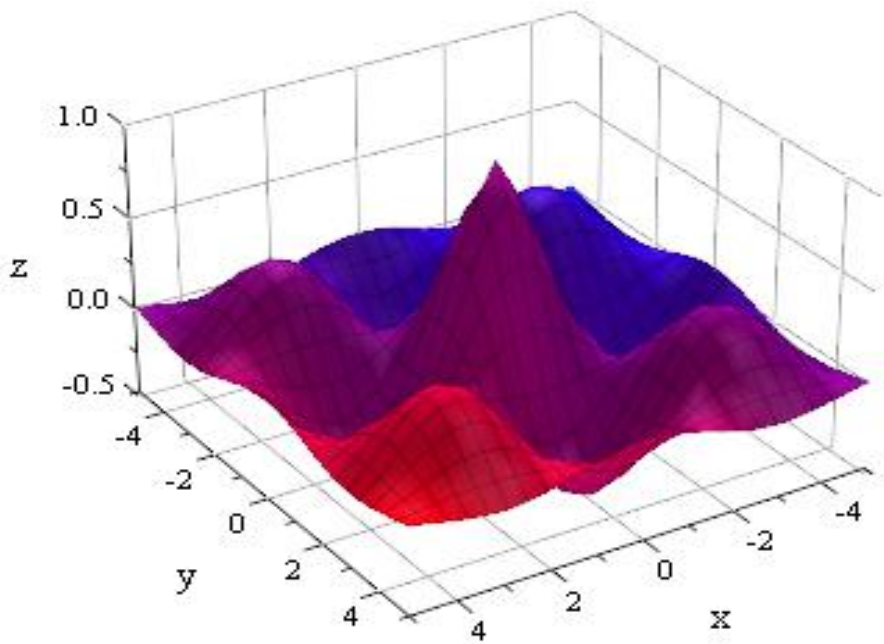
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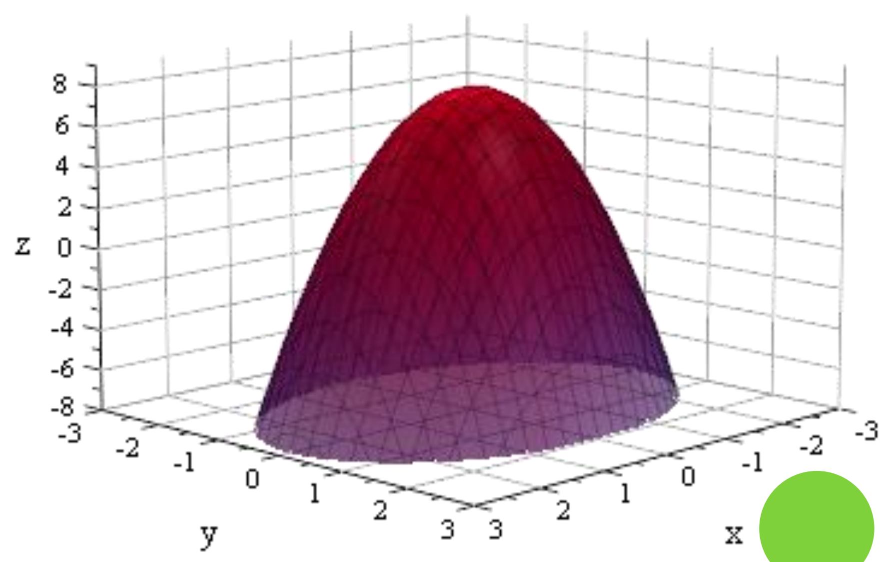
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REMARK:

A maximum or minimum value is also called an ***extremum***.

Finding Relative extrema

Theorem.

(First derivative test for relative extreme values)

If $f(x,y)$ exist at all points in some open disk $B((x_0,y_0);r)$ and if $\boldsymbol{f}$ has a relative extremum at (x_0,y_0) , then

If $f_x(x_0,y_0)$ *and* $f_y(x_0,y_0)$

exist, then

$$f_x(x_0,y_0)=0 \quad and \quad f_y(x_0,y_0)=0$$

DEFINITION.

If $f(x,y)$ exists at all points in some open disk $B((x_0,y_0);r)$ then (x_0,y_0) is a ***critical point*** of $\boldsymbol{f}$ if one of the following conditions holds:

1. $f_x(x_0,y_0)=0$ *and* $f_y(x_0,y_0)=0$

2. $f_x(x_0,y_0)$ *or* $f_y(x_0,y_0)$

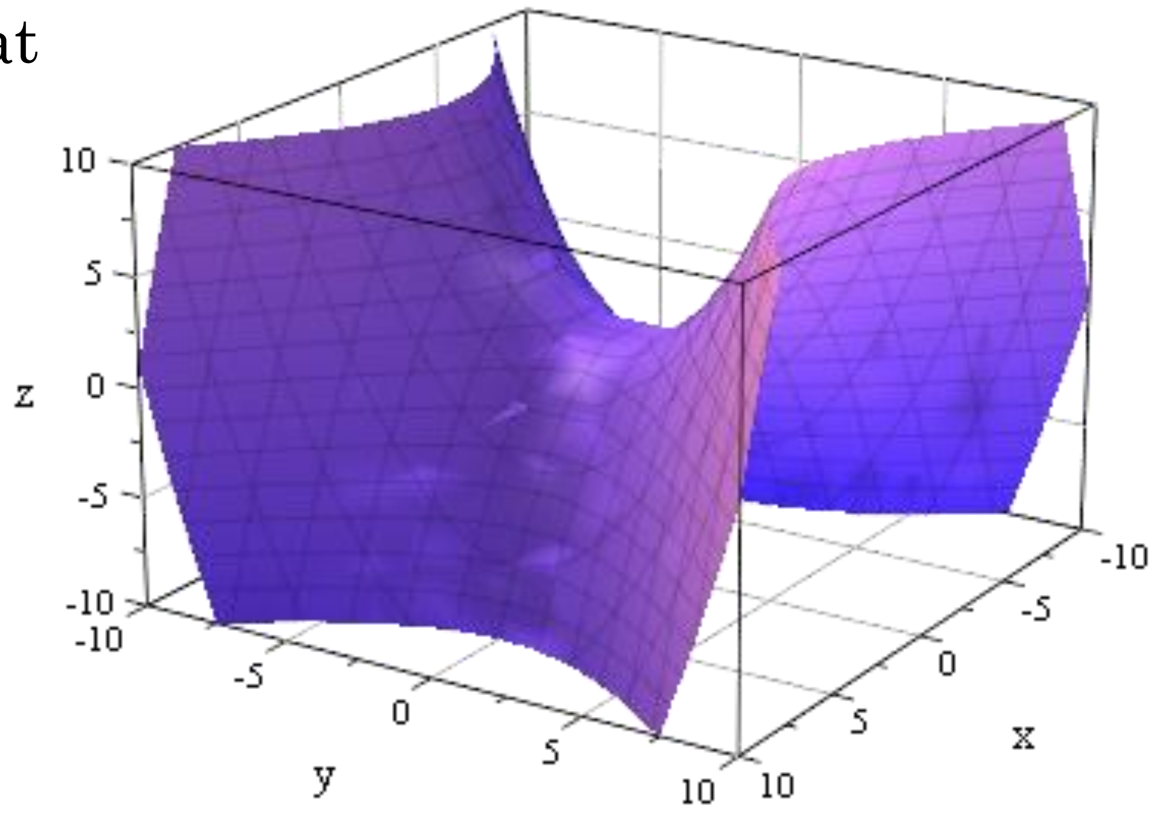
   does not exist.

CONSIDER THE FUNCTION $f(x,y)=\dfrac{x^2}{4}-\dfrac{y^2}{4}$.

Note that

$$\nabla f(x,y)=\left\langle \frac{x}{2},-\frac{y}{2}\right\rangle \Rightarrow \nabla f(0,0)=\langle 0,0\rangle$$

But recall that



Remarks:

1. The converse of the previous theorem is not always true, that is, if
$$f_x(x_0, y_0) = 0 \quad \textit{and} \quad f_y(x_0, y_0) = 0$$
then f does not necessarily have a relative extremum value at (x_0, y_0).

2. A critical point at which there is no relative extrema is called a ***saddle point*** of the graph of the function.

Example. Find the critical points:

1. $f(x,y) = x\left(e^{-y} - 1\right)$

Solution.

$$f_x(x,y) = \left(e^{-y} - 1\right)$$

Now,

$$f_x(x,y) = 0$$

$$\Rightarrow \quad y = 0$$

$$f_y(x,y) = -xe^{-y}$$

Now,

$$f_y(x,y) = 0$$

$$\Rightarrow \quad x = 0$$

CP: $(0,0)$

Example. Find the critical points:

2. $f(x,y) = 4xy - x^4 - y^4$

Solution.

$f_x(x,y) = 4y - 4x^3$

Now,

$f_x(x,y) = 0$

$\Rightarrow y = x^3$

$f_y(x,y) = 4x - 4y^3$

Now,

$f_y(x,y) = 0$

$\Rightarrow x = y^3$

CP: $(0,0)\quad(-1,-1)\quad(1,1)$

Example. Find the critical points:

3. $f(x,y) = x^3 - 3x^2 + y^2 - 6y - 4$

Solution.

$f_x(x,y) = 3x^2 - 6x$

Now,

$f_x(x,y) = 0$

$\Rightarrow 3x(x-2) = 0$

$f_y(x,y) = 2y - 6$

Now,

$f_y(x,y) = 0$

$\Rightarrow 2(y-3) = 0$

CP: $(0,3)$ $(2,3)$

THEOREM.

(2ND DERIVATIVE TEST FOR RELATIVE EXTREME VALUES)

If f be a function of two variables x and y such that f and its first and second-order partial derivatives are continuous on some open disk $B((x_0, y_0); r)$. Suppose further that $f_x(x_0, y_0) = 0$

and $, f_y(x_0, y_0) = 0$

Define $D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$

Then

a. f has a ***relative minimum value*** at (a,b)

if $D(a,b)>0$ and $f_{xx}(a,b)>0.$

b. f has a ***relative maximum value*** at (a,b)

if $D(a,b)>0$ and $f_{xx}(a,b)<0.$

c. $f(a,b)$ is ***not a relative extremum***,

instead f has a saddle point at $(a,b,f(a,b))$

if $D(a,b)<0.$

d. ***Test is inconclusive*** at (a,b) if $D(a,b)=0.$

Example. Determine the relative extrema of the function, if any and locate any saddle points of the function's graph.

1. $f(x,y) = x\left(e^{-y} - 1\right)$

Solution.

CP:	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$
$(0,0)$				

1. $f(x,y) = x\left(e^{-y} - 1\right)$

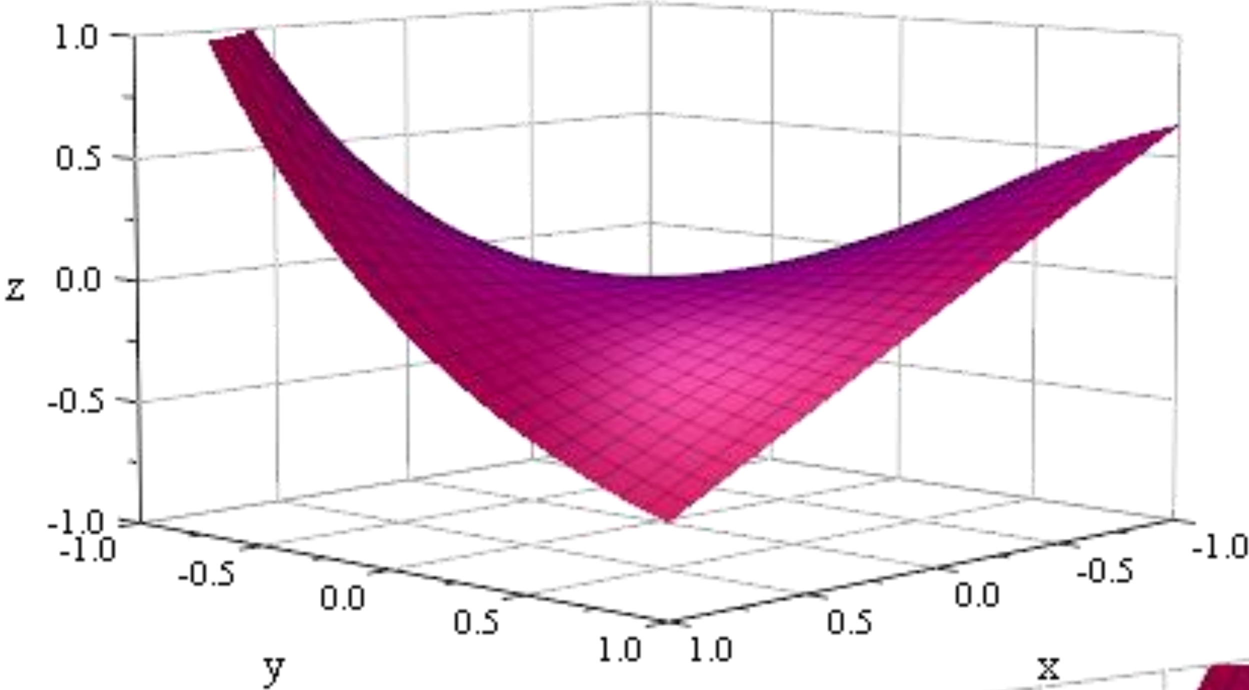
Solution.

$$f_x(x,y) = \left(e^{-y} - 1\right) \qquad f_y(x,y) = -xe^{-y}$$

$$f_{xx}(x,y) = 0 \qquad f_{yy}(x,y) = xe^{-y}$$

$$f_{xy}(x,y) = -e^{-y}$$

CP:	f_{xx}	f_{yy}	f_{xy}	D	*Conclusion:*
$(0,0)$	0	0	-1	-1	***f* has a saddle point at (0,0,0)**



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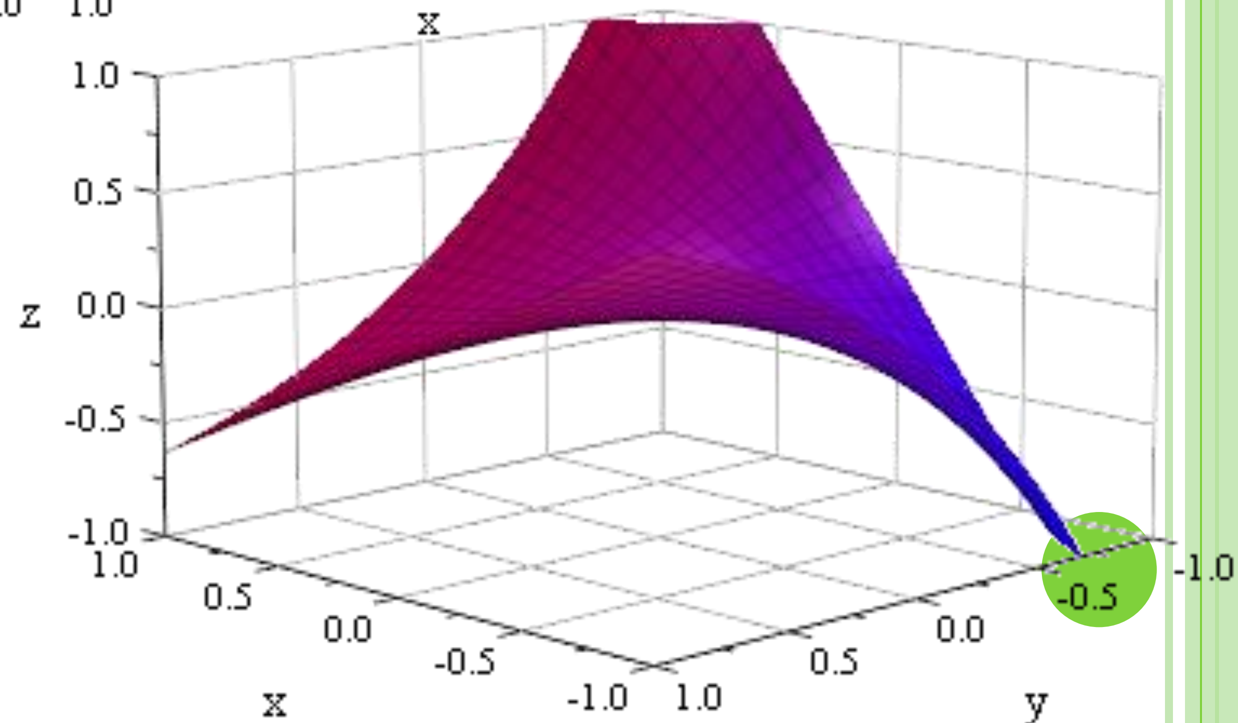

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x
1.0
0.5
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-1.0
y

2. $f(x,y)=4xy-x^4-y^4$

Solution. $f_x(x,y)=4y-4x^3$ $\quad$ $f_y(x,y)=4x-4y^3$

$f_{xx}(x,y)=-12x^2$ $\quad$ $f_{yy}(x,y)=-12y^2$

$f_{xy}(x,y)=4$

CP:	f_{xx}	f_{yy}	f_{xy}	D	*Conclusion:*
$(0,0)$	0	0	4	-16	***f has a saddle point at (0,0,0)***
$(-1,-1)$	-12	-12	4	128	***f has a relative maximum at (-1,-1)***
$(1,1)$	-12	-12	4	128	***f has a relative maximum at (1,1)***

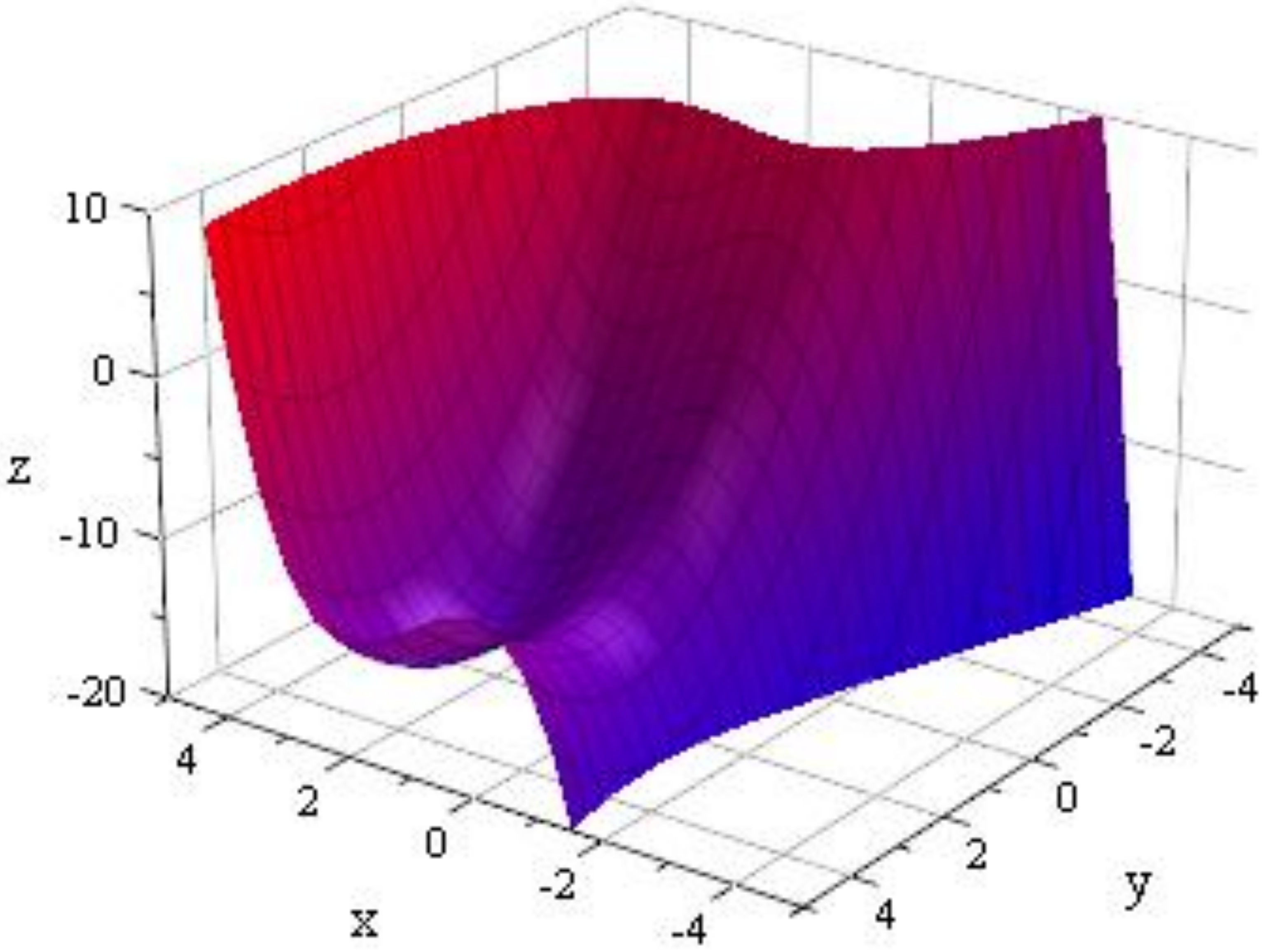
3. $f(x,y) = x^3 - 3x^2 + y^2 - 6y - 4$

Solution. $f_x(x,y) = 3x^2 - 6x$ $\quad f_y(x,y) = 2y - 6$

$f_{xx}(x,y) = 6x - 6$ $\quad f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 0$

CP:	f_{xx}	f_{yy}	f_{xy}	D	Conclusion:
$(0,3)$	-6	2	0	-12	*f has a saddle point at (0,3,-13)*
$(2,3)$	6	2	0	12	*f has a relative minimum at (2,3)*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Finding absolute extrema on a closed and bounded region

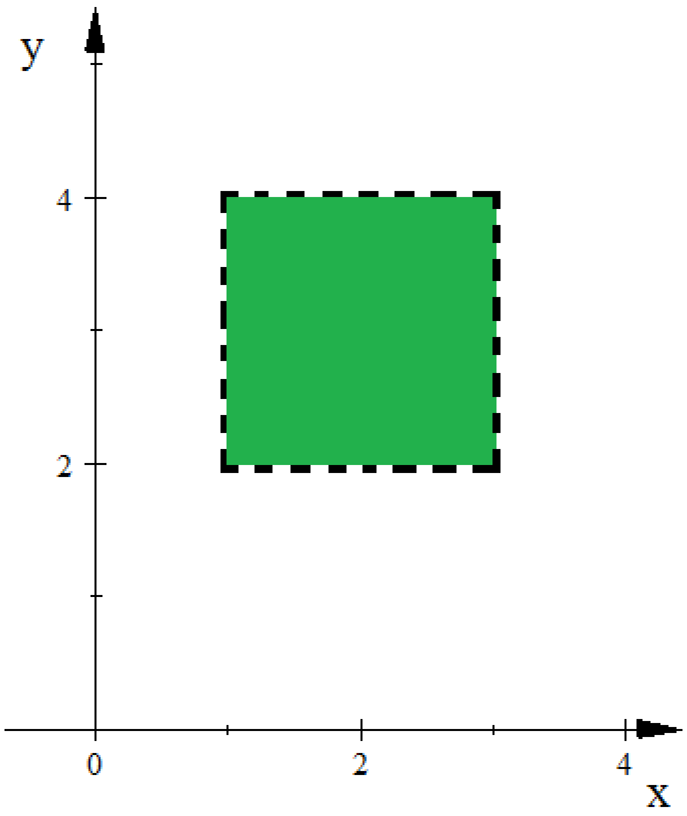
Definitions.

A region is ***bounded*** if it is a sub-region of a closed disk or closed ball.

The ***boundary*** of a region $\boldsymbol{R}$ is the set of all points $\boldsymbol{P}$ for which every open ball centered at $\boldsymbol{P}$ contains a point in $\boldsymbol{R}$ and a point not in $\boldsymbol{R}$.

A ***closed*** region is one that contains its boundary.

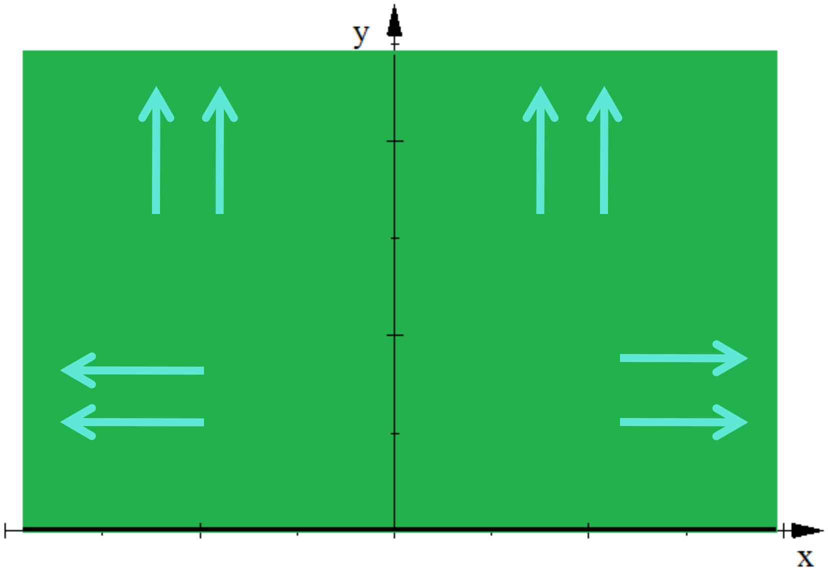
1. $S_1 = \{(x, y) | 1 < x < 3, 2 < y < 4\}$



BOUNDED

NOT CLOSED

2. $S_2 = \{(x, y) | y \geq 0\}$

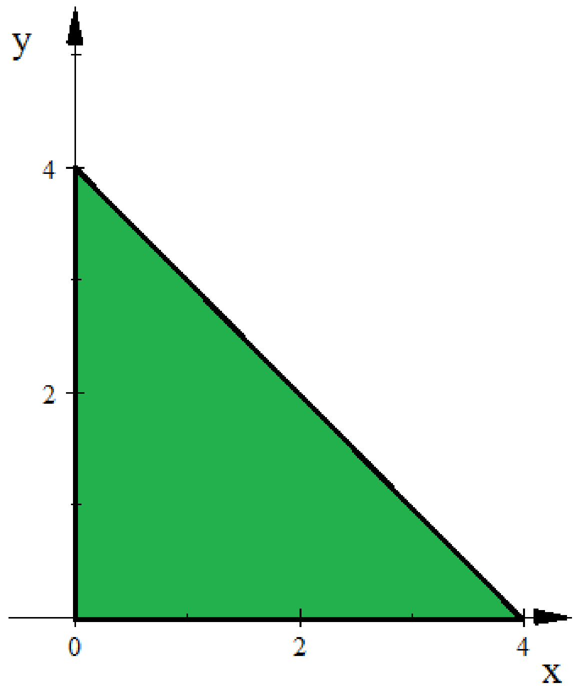


UNBOUNDED

CLOSED

3. Let S_3 be the collection of all points inside and on the triangle with vertices at

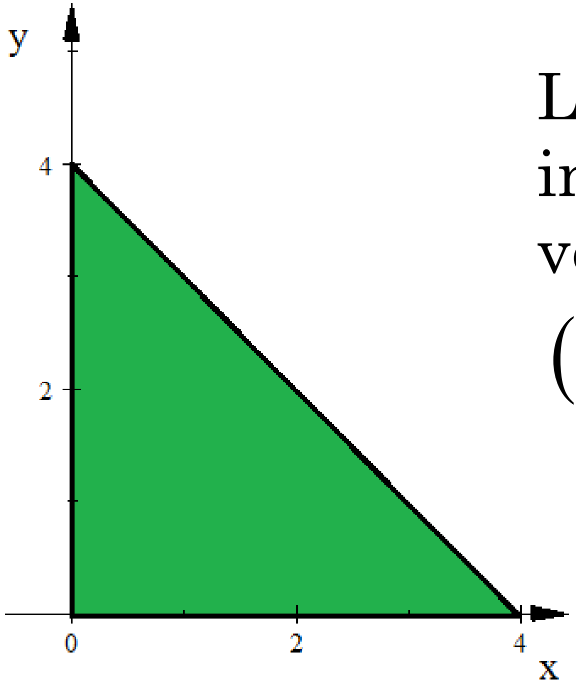
$$(0,0),(0,4) \textit{ and } (4,0).$$



BOUNDED

CLOSED

Of the three regions described earlier, one is a ***closed and bounded region***.



Let S_3 be the collection of all points inside and on the triangle with vertices at

$$(0,0),(0,4)\ \mathit{and}\ (4,0).$$

Theorem.

(Extreme Value Theorem or EVT)

Let $\boldsymbol{R}$ be a closed and bounded region in the xy-plane and let f be a continuous function on $\boldsymbol{R}$.

Then f has an absolute maximum and an absolute minimum value on $\boldsymbol{R}$.

REMARK:

Let a function f satisfies the EVT, then an absolute extremum occurs either at a critical point of f, in the interior of $\boldsymbol{R}$, or at a boundary point of $\boldsymbol{R}$.

Example. Find the absolute extreme values of $f(x,y)=2x^2-4x+y^2-4y+1$ on the triangular plate in the first quadrant bounded by the lines $y=2, x=0$ *and* $y=2x$.

Solution.

Interior: $f_x(x,y)= 4x-4$

Now,

$$f_x(x,y)=0$$

$$\Rightarrow \quad x=1$$

$f_y(x,y)= 2y-4$

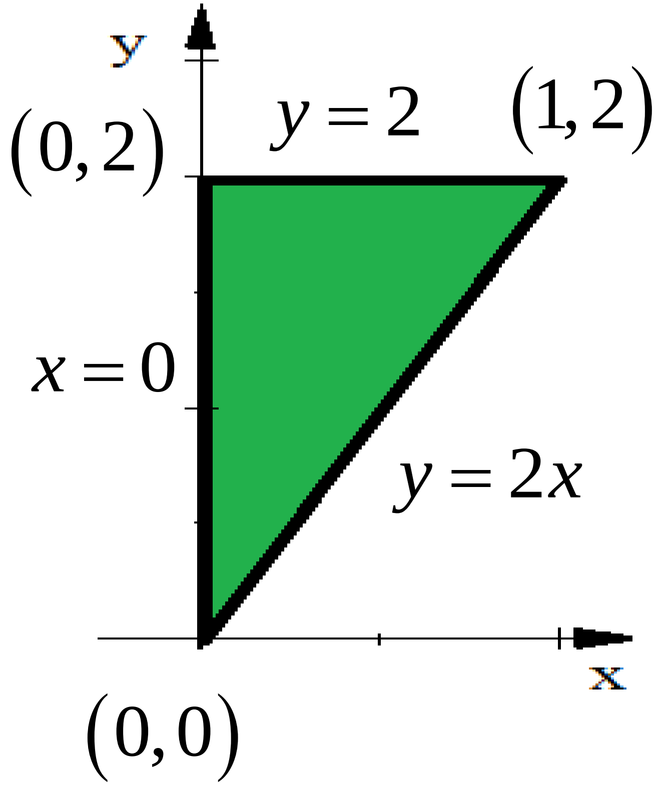
Now,

$$f_y(x,y)=0$$

$$\Rightarrow \quad y=2$$

CP: $(1,2)$

Along the boundary $x = 0:$



$$f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$$

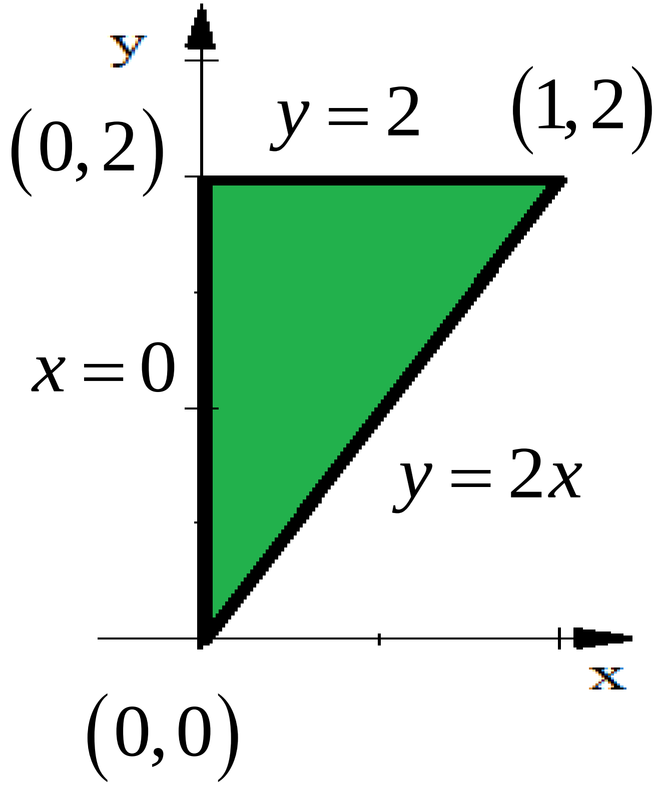
$$\Rightarrow \ f(y) = y^2 - 4y + 1$$

$$f'(y) = 2y - 4$$

$$f'(y) = 0 \ \Rightarrow \ y = 2$$

CP: $(1,2)$ $(0,2)$ $(0,0)$

Along the boundary $y=2$:



$$f(x,y)=2x^2-4x+y^2-4y+1$$

$$\Rightarrow f(x)=2x^2-4x-3$$

$$f'(x)= 4x-4$$

$$f'(x)=0 \Rightarrow x=1$$

CP: $(1,2)$ $(0,2)$ $(0,0)$

Along the boundary $y = 2x$:

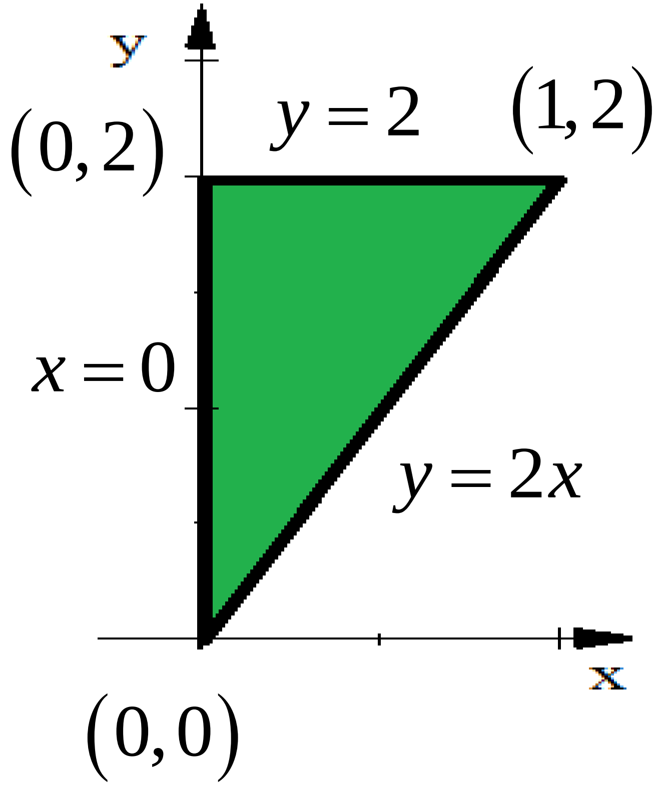
$$f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$$

$$\Rightarrow f(x) = 2x^2 - 4x + 4x^2 - 8x + 1$$

$$f(x) = 6x^2 - 12x + 1$$

$$f'(x) = 12x - 12$$

$$f'(x) = 0 \Rightarrow x = 1$$



CP: $(1,2)$ $(0,2)$ $(0,0)$

$$f(x,y)=2x^2-4x+y^2-4y+1$$

Now, we obtain $f(x,y)$ for every critical point (x,y).

CP: $(1,2)$ $(0,2)$ $(0,0)$

$f(1,2)= -5$

f has an absolute minimum of -5 at the point (1,2).

$f(0,2)= -3$

$f(0,0)= 1$

f has an absolute maximum of 1 at the point (0,0).

END