CHAPTER 5
VECTORS AND PARAMETRIC EQUATIONS

1

Chapter objectives:

At the end of the chapter, you must be able to

1. enumerate and apply properties of vectors in the plane and in space,
2. perform and interpret vector operations,
3. find the equations of a line and equation of a plane in space and
4. identify and sketch cylinders and quadric surfaces.

2

5.1 VECTORS IN THE PLANE

A *vector in the plane* is an ordered pair

$$\langle x, y \rangle$$

of real numbers.

The numbers x and y are called the *components* of the vector .

3

Equality of vectors

Two vectors

$\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$

are said to be *equal*, written ,

$$\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$$

if and only if

$x_1 = x_2$ and $y_1 = y_2$

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Example 5.1.1 Let $\vec{A} = \langle 2a+b, a-2b \rangle$ and $\vec{B} = \langle 1,3 \rangle$. If $\vec{A} = \vec{B}$, find the values of a and b.

5

Let and

Solution:

$2a + b = 1$

$a - 2b = 3$

Ans. $a = 1$, $b = -1$

If , find the values of a and b.

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Position representation of a vector

Consider a plane, let $\vec{A} = \langle a, b \rangle$ and O be the origin in the plane.

If A is the point (a, b), then vector $\vec{A}$ may be represented geometrically by the directed line segment $\overrightarrow{OA}$.

Such a directed line segment is called the *position representation* of the vector.

Example 5.1.2

Draw the position representation of $\vec{A} = \langle 2, 3 \rangle$.

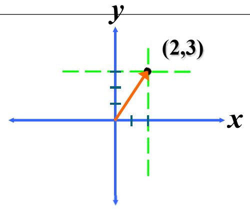

Figure 5.1.1

Direction angle of a non-zero vector

The *direction angle* of any non-zero vector is the smallest angle θ measured from the positive side of the x-axis counterclockwise to the position representation of the vector.

Example 5.1.3 Let $\vec{A} = \langle -4, 4 \rangle$.

Find the direction angle of $\vec{A}$.

Solution:

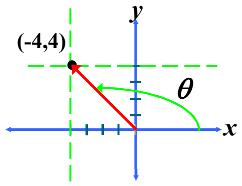

Figure 5.1.2

If θ is the direction angle of $\vec{A}$, then

$$\tan\theta = \frac{4}{-4} = -1$$

$$\Rightarrow \theta = \frac{3\pi}{4}.$$

Magnitude of a vector

The *magnitude* of a vector $\vec{A}$, denoted by $\|\vec{A}\|$ is the length of any of its representations.

Theorem 5.1.1 If $\vec{A} = \langle a, b \rangle$, then

$$\|\vec{A}\| = \sqrt{a^2 + b^2}.$$

Example 5.1.4 Find the magnitude of the given vector.

a. $\vec{A} = \langle 3, -4 \rangle$ b. $\vec{B} = \langle -2, 5 \rangle$

solution:

a. $\|\vec{A}\| = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-4)^2}$
$= \sqrt{9 + 16} = \sqrt{25} = 5.$

b. $\|\vec{B}\| = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + 5^2}$
$= \sqrt{4 + 25} = \sqrt{29}.$

Any directed line segment formed by taking a copy of the position representation of a vector and pasting this copy anywhere on a plane without changing its direction is *another representation* of the vector.

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Example 5.1.5 Let $\vec{A} = \langle 2,3 \rangle$.

Shown in Figure 1.1.3 is the position representation of $\vec{A}$. We show in Figure 1.1.4, 3 other representations of $\vec{A}$.

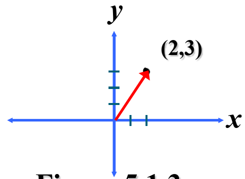

Figure 5.1.3

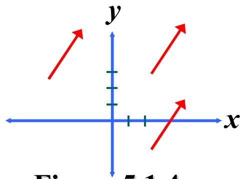

Figure 5.1.4

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In Figure 5.1.5, the direction of the line segment shown is opposite and therefore different from that of $\vec{A}$.

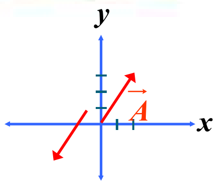

Figure 5.1.5

Thus, the vector is <u>not</u> a representation of $\vec{A}$.

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In Figure 5.1.6, the vector shown is in the same direction as $\vec{A}$.

But it is <u>not</u> a representation of $\vec{A}$ since its magnitude differs from that of $\vec{A}$.

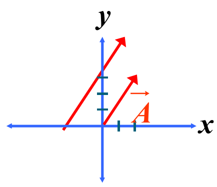

Figure 5.1.6

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Let C and D be points on the same plane. If the directed line segment $\overrightarrow{CD}$ is a representation of a vector $\vec{A}$, then C and D are called the *initial point* and *terminal point*, respectively, of this representation of $\vec{A}$.

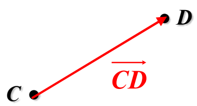

Figure 5.1.7

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Example 5.1.6 Consider the points $C(1,3)$ and $D(4,7)$. If $\overrightarrow{CD}$ is a representation of $\vec{A}$, find $\vec{A}$.

solution:

$$x_{\vec{A}} = x_D - x_C = 4 - 1 = 3.$$

$$y_{\vec{A}} = y_D - y_C = 7 - 3 = 4.$$

$$\vec{A} = \langle 3,4 \rangle$$

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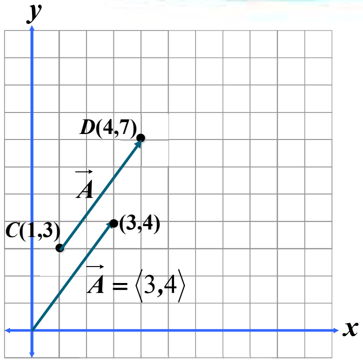

Figure 5.1.8

If $\vec{A} = \langle a, b \rangle$ and θ is the direction angle θ of $\vec{A}$, then

$$\cos\theta = \frac{a}{\|\vec{A}\|}$$

and

$$\sin\theta = \frac{b}{\|\vec{A}\|}.$$

Thus,

$$\vec{A} = \langle a, b \rangle = \left\langle \|\vec{A}\| \cos\theta, \|\vec{A}\| \sin\theta \right\rangle$$

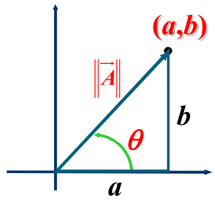

Figure 5.1.9

Example 5.1.7 If the direction angle of vector $\vec{A}$ is $\frac{2\pi}{3}$ and its magnitude is 4, then

$$\begin{aligned}\vec{A} &= \left\langle \|\vec{A}\| \cos\theta, \|\vec{A}\| \sin\theta \right\rangle \\ &= \left\langle 4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3} \right\rangle \\ &= \left\langle 4\left(\frac{-1}{2}\right), 4\cdot\frac{\sqrt{3}}{2} \right\rangle = \left\langle -2, 2\sqrt{3} \right\rangle.\end{aligned}$$

Vector sum

The *sum* of two vectors

$$\vec{A} = \langle a_1, a_2 \rangle \text{ and } \vec{B} = \langle b_1, b_2 \rangle$$

is the vector $\vec{A} + \vec{B}$ given by

$$\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2 \rangle.$$

Example 5.1.8 Let $\vec{A} = \langle 2,3 \rangle$ and $\vec{B} = \langle -3,4 \rangle$. Then

$$\vec{A} + \vec{B} = \langle 2 + (-3), 3 + 4 \rangle = \langle -1, 7 \rangle$$

Negative of a vector

If $\vec{A} = \langle a_1, a_2 \rangle$, the *negative* of $\vec{A}$ is the vector

$$-\vec{A} = \langle -a_1, -a_2 \rangle.$$

Example 5.1.9 Let $\vec{A} = \langle -3,4 \rangle$. Then

$$\begin{aligned}-\vec{A} &= \langle -(-3), -4 \rangle \\ &= \langle 3, -4 \rangle\end{aligned}$$

Vector difference

The *difference* of any two vectors $\vec{A}$ and $\vec{B}$, denoted by

$$\vec{A} - \vec{B}$$

is the vector obtained by adding $\vec{A}$ to the negative of $\vec{B}$, that is,

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right).$$

Example 5.1.10 Let

$$\vec{A} = \langle 2,3 \rangle \text{ and } \vec{B} = \langle -3,4 \rangle \ .$$

Then

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$$
$$= \langle 2,3 \rangle + \langle 3,-4 \rangle$$
$$= \langle 2+3, 3-4 \rangle = \langle 5,-1 \rangle .$$

Geometric interpretation of vector sum, negative and difference

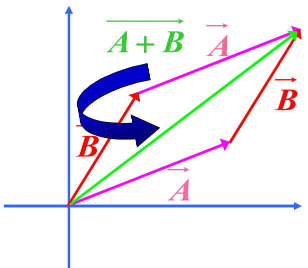

Figure 5.1.10

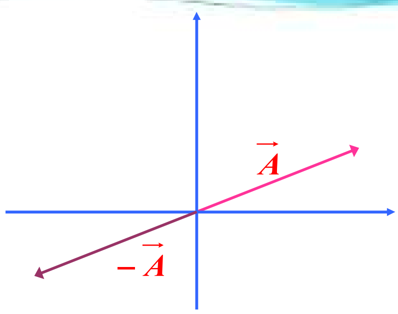

Figure 5.1.11

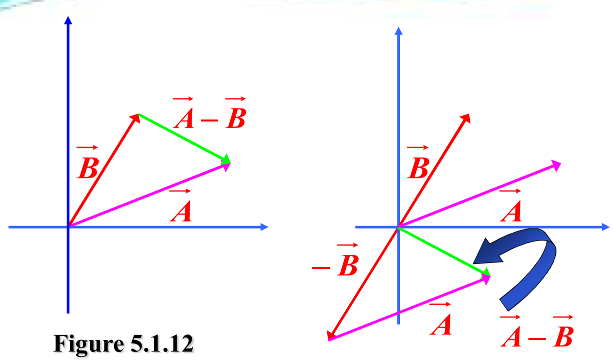

Figure 5.1.12

Scalar multiplication

If α is a scalar and

$$\vec{A} = \langle a, b \rangle,$$

the product of α and $\vec{A}$, denoted by

$$\alpha \vec{A}$$

is a vector and is given by

$$\alpha \vec{A} = \langle \alpha a, \alpha b \rangle .$$

Example 5.1.11 If

$$\vec{A} = \langle 2,-4 \rangle,$$

then

$$-3\vec{A} = \langle -3(2), -3(-4) \rangle = \langle -6, 12 \rangle$$

$$2\vec{A} = \langle 2(2), 2(-4) \rangle = \langle 4, -8 \rangle$$

$$0 \cdot \vec{A} = \langle 0(2), 0(-4) \rangle = \langle 0, 0 \rangle$$

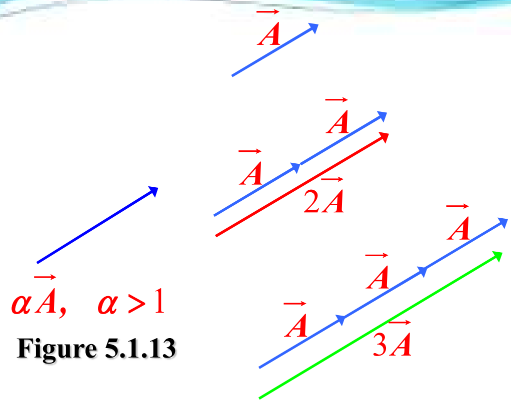

Figure 5.1.13

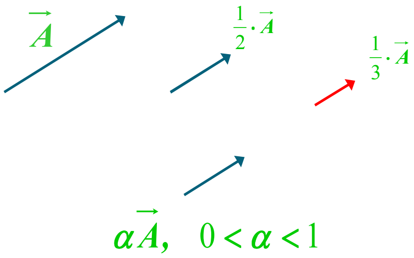

Figure 5.1.14

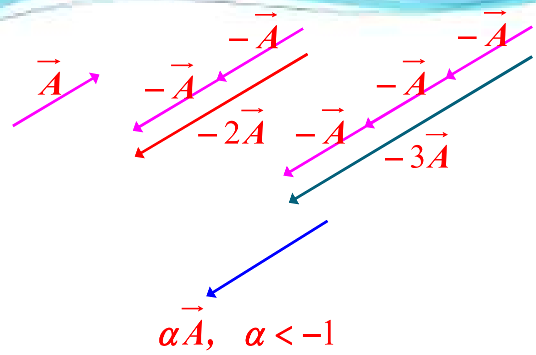

Figure 5.1.15

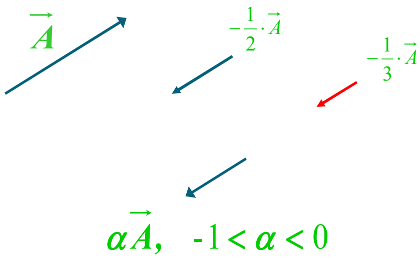

Figure 5.1.16

Theorem 5.1.2

Let V_2 be the set of all vectors in the plane.

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are any vectors in V_2 and α and β are any scalars, then vector addition and scalar multiplication satisfy the following properties:

$i.\ \vec{A}+\vec{B}=\vec{B}+\vec{A}$ (commutative law)

$ii.\ (\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$ (associative law)

$iii.$ There is a vector $\vec{O}$ in V_2 such that

$$\vec{A}+\vec{O}=\vec{A}.$$

(existence of additive identity)

$iv.$ There is a vector $(-\vec{A})$ in V_2 such that

$$\vec{A}+-\vec{A}=\vec{O}.$$

(existence of additive inverse)

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$v.\ (\alpha\beta)\vec{A}=\alpha(\beta\vec{A}).$

(associative law)

$vi.\ \alpha(\vec{A}+\vec{B})=\alpha\vec{A}+\alpha\vec{B}.$

(distributive law)

$vii.\ (\alpha+\beta)\vec{A}=\alpha\vec{A}+\beta\vec{A}.$

(distributive law)

$viii.\ 1\cdot\vec{A}=\vec{A}.$

(existence of scalar identity)

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A *real vector space* V is a set of elements, called *vectors*, together with the set of real numbers, called *scalars*, with two operations called *vector addition*, and *scalar multiplication* such that for every pair of vectors $\vec{A}$ and $\vec{B}$, and for each scalar α, $\vec{A}+\vec{B}$ and $\alpha\vec{A}$ are defined so that properties (i) - (viii) of Theorem 1.1.2 are satisfied.

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Any vector whose magnitude is 1 is called a *unit vector*.

Remarks:

1. **Let $i=\langle 1,0\rangle$ and $j=\langle 0,1\rangle$ so that i and j are unit vectors.**
2. **If $\vec{A}=\langle a,b\rangle$, then since**

$$\langle a,b\rangle=\langle a,0\rangle+\langle 0,b\rangle$$
$$=a\langle 1,0\rangle+b\langle 0,1\rangle$$

it follows that $\vec{A}=ai+bj$.

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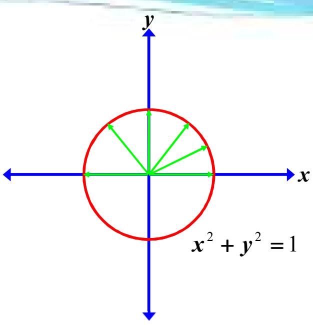

Figure 5.1.17

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3. If θ is the direction angle of $\vec{A}=\langle a,b\rangle$, then since

$$a=\|\vec{A}\|\cos\theta, b=\|\vec{A}\|\sin\theta$$

and

$$\vec{A}=ai+bj$$

$$\vec{A}=\|\vec{A}\|\cos\theta i+\|\vec{A}\|\sin\theta j$$
$$=\|\vec{A}\|(\cos\theta i+\sin\theta j).$$

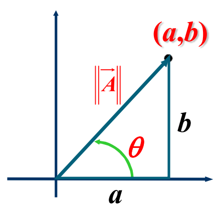

Figure 5.1.18

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Example 5.1.12 If the direction of a vector $\vec{A}$ is $\dfrac{5\pi}{6}$ and its magnitude is 4, find $\vec{A}$.

solution:

$$\vec{A} = \left\|\vec{A}\right\|(\cos\theta i + \sin\theta j)$$
$$= 4\left(\cos\left(\frac{5\pi}{6}\right)i + \sin\left(\frac{5\pi}{6}\right)j\right)$$
$$= 4\left(\frac{-\sqrt{3}}{2}i + \frac{1}{2}j\right)$$
$$= -2\sqrt{3}i + 2j.$$

Theorem 5.1.3 If $\vec{A} = a_1 i + a_2 j$ is a non-zero vector, the unit vector in the same direction as $\vec{A}$ is given by

$$\vec{U}_A = \frac{a_1}{\left\|\vec{A}\right\|}i + \frac{a_2}{\left\|\vec{A}\right\|}j.$$

Example 5.1.13 Find the unit vector in the same direction as the given vector.

a. $\vec{A} = 3i + 4j$ b. $\vec{B} = -2i + 3j$

solution:

a. $\left\|\vec{A}\right\| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5.$

$$\vec{U}_A = \frac{a_1}{\left\|\vec{A}\right\|}i + \frac{a_2}{\left\|\vec{A}\right\|}j \; = \frac{3}{5}i + \frac{4}{5}j$$

b. $\left\|\vec{B}\right\| = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}.$

$$\vec{U}_B = \frac{b_1}{\left\|\vec{B}\right\|}i + \frac{b_2}{\left\|\vec{B}\right\|}j \; = \frac{-2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$$

a. $\vec{A} = 3i + 4j$

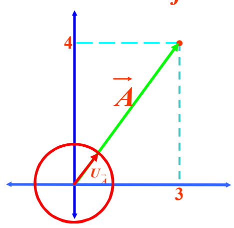


Figure 5.1.19

b. $\vec{B} = -2i + 3j$

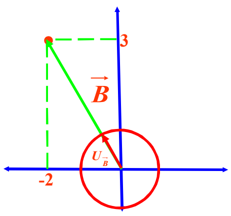


Figure 5.1.20

5.2 THE THREE-DIMENSIONAL NUMBER SPACE

The set of all ordered triples of real numbers is called the *three-dimensional number space*, denoted by R^3.

Each ordered triple

$$(x, y, z)$$

of real numbers is called a *point* in R^3 .

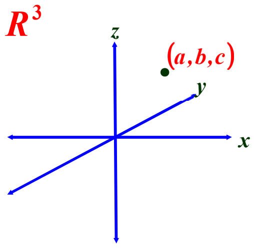
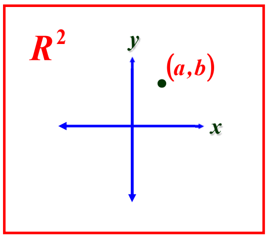


Figure 5.2.1



Example 5.2.1 Plot the following points:

a. $A(1,2,3)$ b. $B(-2,3,-1)$

To plot $A(1,2,3)$:

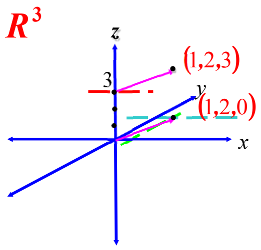
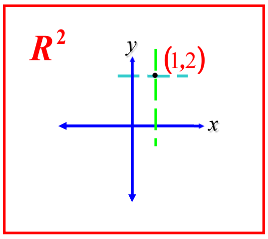


Figure 5.2.2a



To plot $B(-2,3,-1)$:

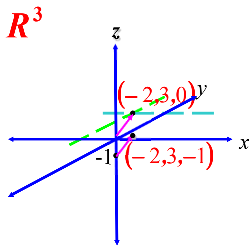
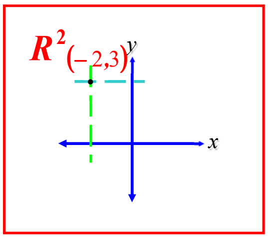


Figure 5.2.2a



Theorem 5.2.1

1. A line is parallel to the *yz* plane if and only if all points on the line have equal *x-coordinates*.
2. A line is parallel to the *xz* plane if and only if all points on the line have equal *y-coordinates*.
3. A line is parallel to the *xy* plane if and only if all points on the line have equal *z-coordinates*.

Example of a line parallel to the *yz* plane

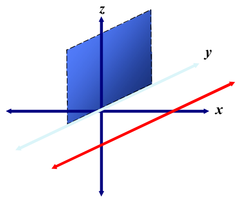


Figure 5.2.3a

Example of a line parallel to the *xz* plane

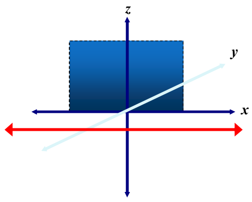


Figure 5.2.3b

Example of a line parallel to the *xy* plane

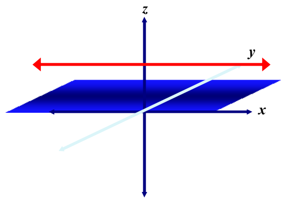


Figure 5.2.3c

Theorem 5.2.2

1. A line is parallel to the *x-axis* if and only if all points on the line have equal *y coordinates* and equal *z* coordinates.
2. A line is parallel to the *y*-axis if and only if all points on the line have equal *x* coordinates and equal *z* coordinates.
3. A line is parallel to the *z*-axis if and only if all points on the line have equal *x* coordinates and equal *y* coordinates.

Theorem 5.2.3

1. If $A(x_1, y, z)$ and $B(x_2, y, z)$ are two points on a line parallel to the *x-axis*, the distance between *A* and *B*, denoted by

$$\left|\overline{AB}\right|$$

is given by

$$\left|\overline{AB}\right| = \left|x_2 - x_1\right|.$$

2. If $C(x, y_1, z)$ and $D(x, y_2, z)$ are two points on a line parallel to the *y-axis*, the distance between *C* and *D*, denoted by

$$\left|\overline{CD}\right|$$

is given by

$$\left|\overline{CD}\right| = \left|y_2 - y_1\right|.$$

3. If $E(x, y, z_1)$ and $F(x, y, z_2)$ are two points on a line parallel to the *y-axis*, the distance between *E* and *F*, denoted by

$$\left|\overline{EF}\right|$$

is given by

$$\left|\overline{EF}\right| = \left|z_2 - z_1\right|.$$

Theorem 5.2.4 The distance between points

$$P_1(x_1, y_1, z_1) \text{ and } P_2(x_2, y_2, z_2)$$

is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 5.2.2 Find the distance between $P_1(2,3,-1)$ and $P_2(4,1,9)$.

solution:

With $(x_1, y_1, z_1) = (2,3,-1)$ and $(x_2, y_2, z_2) = (4,1,9)$,

$$\begin{aligned}|P_1P_2| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(4-2)^2 + (1-3)^2 + (9-(-1))^2} \\ &= \sqrt{2^2 + (-2)^2 + 10^2} \\ &= \sqrt{4+4+100} = \sqrt{108} = \sqrt{36 \cdot 3} = 6\sqrt{3}.\end{aligned}$$

Theorem 5.2.5 The midpoint of the line segment having

$$P_1(x_1, y_1, z_1) \text{ and } P_2(x_2, y_2, z_2)$$

as endpoints is

$$M(\bar{x}, \bar{y}, \bar{z})$$

where,

$$\bar{x} = \frac{x_1 + x_2}{2}, \quad \bar{y} = \frac{y_1 + y_2}{2} \text{ and } \bar{z} = \frac{z_1 + z_2}{2}.$$

Example 5.2.3 The midpoint of the line segment having

$$P_1(2,3,-1) \text{ and } P_2(4,1,9)$$

as endpoints is $M(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{2+4}{2} = 3, \quad \bar{y} = \frac{3+1}{2} = 2, \quad \bar{z} = \frac{-1+9}{2} = 4.$$

The *graph of an equation in* R^3

is the set of all points

$$(x, y, z)$$

whose coordinates are real numbers satisfying the equation.

Example 5.2.4 Determine if each given point lies on the graph of $x^2 + 4y^2 - z = 10$.

a. $A(2,1,-2)$ b. $B(-4,0,1)$

solution:

a. Since

$$(2)^2 + 4(1)^2 - (-2) = 4 + 4 + 2 = 10,$$

$A(2,1,-2)$ <u>lies</u> on the graph of

$$x^2 + 4y^2 - z = 10.$$

b. Since

$$(-4)^2 + 4(0)^2 - (1) = 16 + 0 - 1 = 15,$$

$B(-4,0,1)$ <u>does not lie</u> on the graph of

$$x^2 + 4y^2 - z = 10.$$

A *sphere* is the set of all points in R^3 *equidistant* from a fixed point.

The fixed point is called the *center* of the sphere while the measure of the constant distance is called the *radius* of the sphere.

Theorem 5.2.6 An equation of the sphere of radius r and centered at (h,k,l) is given by

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2.$$

The equation given in Theorem 2.2.6 is called the *standard equation of a sphere*.

Example 5.2.5 Write the standard equation of the sphere of radius 3 and centered at $(2,-1,4)$. Sketch the sphere.

solution:

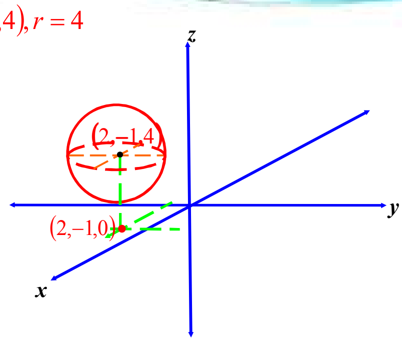
From Theorem 5.2.6, the *standard equation* of the sphere is

$$(x-2)^2+(y-(-1))^2+(z-4)^2=3^2.$$

The same sphere is given by the equation

$$(x-2)^2+(y+1)^2+(z-4)^2=9.$$

$C(2,-1,4), r=4$



Example 5.2.6 Write the standard equation of the sphere having $P_1(2,3,-1)$ and $P_2(4,1,9)$ as endpoints of a diameter.

solution:

In Example 5.2.2, the distance between the given points is $6\sqrt{3}$. Thus, the radius of the sphere is

$$\frac{1}{2}\cdot 6\sqrt{3}=3\sqrt{3}.$$

In Example 5.2.3, the midpoint of the segment whose endpoints are the given points is (3,2,4). The standard equation of the sphere is

$$(x-3)^2+(y-2)^2+(z-4)^2=\left(3\sqrt{3}\right)^2.$$

Theorem 5.2.7 The graph in R^3 of any second-degree equation in x, y, and z, of the form

$$x^2+y^2+z^2+Gx+Hy+Iz+J=0$$

where G, H, I and J are constants is either a *sphere*, a *point*, or *empty*.

Example 5.2.7 Identify the graph in R^3 of each of the following equations.

a. $x^2 + y^2 + z^2 + 2x - 4y - 4z = 0$

b. $x^2 + y^2 + z^2 - 2x - 4z + 5 = 0$

c. $x^2 + y^2 + z^2 - 2y + 4z + 7 = 0$

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solution:

a. $x^2 + y^2 + z^2 + 2x - 4y - 4z = 0$

$$(x^2 + 2x) + (y^2 - 4y) + (z^2 - 4z) = 0$$

$$\underline{(x^2 + 2x + 1 - 1)} + \underline{(y^2 - 4y + 4 - 4)} + \underline{(z^2 - 4z + 4 - 4)} = 0$$

$$(x+1)^2 + (y-2)^2 + (z-2)^2 - 9 = 0$$

$$(x+1)^2 + (y-2)^2 + (z-2)^2 = 9.$$

$$(x-(-1))^2 + (y-2)^2 + (z-2)^2 = 3^2.$$

The graph is the sphere centered at (-1,2,2) with radius 3.

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b. $x^2 + y^2 + z^2 - 2x - 4z + 5 = 0$

$$(x^2 - 2x) + y^2 + (z^2 - 4z) + 5 = 0$$

$$\underline{(x^2 - 2x + 1 - 1)} + y^2 + \underline{(z^2 - 4z + 4 - 4)} + 5 = 0$$

$$(x-1)^2 + (y-0)^2 + (z-2)^2 + 5 - 5 = 0$$

$$(x-1)^2 + (y-0)^2 + (z-2)^2 = 0$$

The graph is the point (1,0,2).

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c. $x^2 + y^2 + z^2 - 2y + 4z + 7 = 0$

$$x^2 + (y^2 - 2y) + (z^2 - 4z) + 7 = 0$$

$$x^2 + \underline{(y^2 - 2y + 1 - 1)} + \underline{(z^2 - 4z + 4 - 4)} + 7 = 0$$

$$x^2 + (y-1)^2 + (z-2)^2 - 5 + 7 = 0$$

$$x^2 + (y-1)^2 + (z-2)^2 = -2$$

The graph is empty.

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5.3 VECTORS IN SPACE

A *vector in* R^3 is an ordered triple $\langle x, y, z \rangle$ of real numbers.

The numbers x, y and z are called the *components* of the vector $\langle x, y, z \rangle$.

The set of all vectors in space will be denoted by V_3.

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Two vectors

$$\langle a, b, c \rangle \text{ and } \langle x, y, z \rangle$$

are said to be *equal*, written ,

$$\langle a, b, c \rangle = \langle x, y, z \rangle$$

if and only if

$$a = x, \quad b = y \quad \text{and} \quad c = z.$$

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Example 5.3.1 Let

$\vec{A} = \langle 8,-3,-9 \rangle$

and

$\vec{B} = \langle 2a+b-c, a-2b+c, 3b+4c \rangle$.

If $\vec{A} = \vec{B}$, find the values of a, b and c.

solution:

If

$\langle 8,-3,-9 \rangle = \langle 2a+b-c, a-2b+c, 3b+4c \rangle$,

then

$$2a + b - c = 8$$
$$a - 2b + c = -3$$
$$3b + 4c = -9$$

Solving this system , we get

$a = 2,\ \ b = 1$ and $c = -3$.

Let $\vec{A} = \langle a,b,c \rangle$ and O be the origin in R^3.

If A is the point (a,b,c), then vector $\vec{A}$ may be represented geometrically by the directed line segment $\overrightarrow{OA}$.

Such a directed line segment is called the *position representation* of the vector .

Example 5.3.2 Draw the position representation of $\vec{A} = \langle 2,3,4 \rangle$.

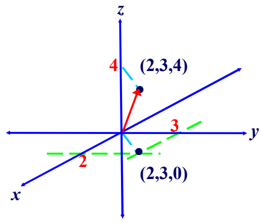


Figure 5.3.1

The *direction angles* of any nonzero vector in R^3 are the three angles which have the smallest non-negative radian measures α, β, γ measured from the *positive side* of the x, y and z axes, respectively, to the *position representation* of the vector.

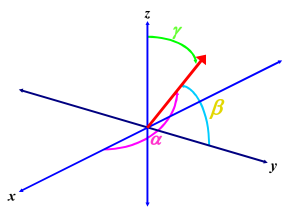
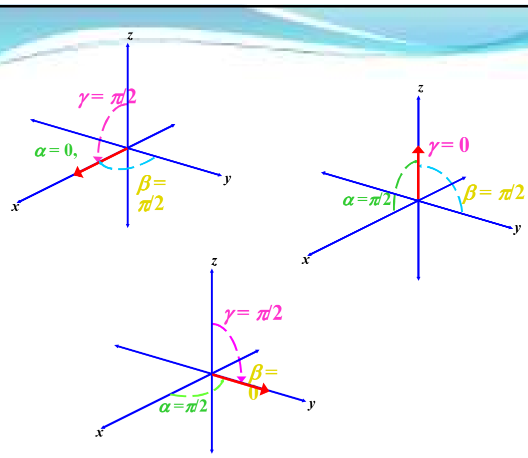


Figure 5.3.2



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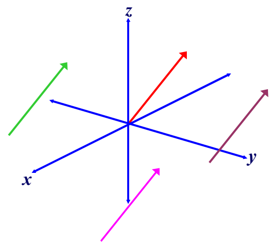
Any directed line segment formed by taking a copy of the position representation of a vector and then pasting this copy anywhere in space without changing its direction angles is *another representation* of the vector.



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Theorem 5.3.1 If $\vec{A} = \langle a, b, c \rangle$, then

$$\left\|\vec{A}\right\| = \sqrt{a^2 + b^2 + c^2}.$$

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Example 5.3.3 Find the magnitude of the given vector.

a. $\vec{A} = \langle 2, -6, 3 \rangle$ b. $\vec{B} = \langle -3, 0, 4 \rangle$

solution:

a. $\left\|\vec{A}\right\| = \sqrt{2^2 + (-6)^2 + 3^2}$

$= \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

b. $\left\|\vec{B}\right\| = \sqrt{(-3)^2 + 0^2 + 4^2}$

$= \sqrt{9 + 16} = \sqrt{25} = 5$

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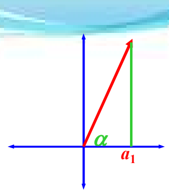
If $\vec{A} = \langle a_1, a_2, a_3 \rangle$

and the direction angles of $\vec{A}$

measure α, β, and γ, then

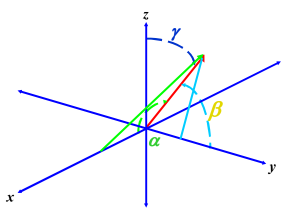


$$\cos\alpha = \frac{a_1}{\left\|\vec{A}\right\|},$$

$$\cos\beta = \frac{a_2}{\left\|\vec{A}\right\|},$$

and

$$\cos\gamma = \frac{a_3}{\left\|\vec{A}\right\|}.$$



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Since

$$\cos\alpha = \frac{a_1}{\left\|\vec{A}\right\|}, \cos\beta = \frac{a_2}{\left\|\vec{A}\right\|}, \text{ and } \cos\gamma = \frac{a_3}{\left\|\vec{A}\right\|},$$

$$a_1 = \left\|\vec{A}\right\|\cos\alpha,\ a_2 = \left\|\vec{A}\right\|\cos\beta \text{ and } a_3 = \left\|\vec{A}\right\|\cos\gamma.$$

Thus,

$$\vec{A} = \langle a_1, a_2, a_3 \rangle$$

$$= \left\langle \left\|\vec{A}\right\|\cos\alpha, \left\|\vec{A}\right\|\cos\beta, \left\|\vec{A}\right\|\cos\gamma \right\rangle$$

$$= \left\|\vec{A}\right\| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle.$$

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Example 5.3.4 If $\vec{A} = \langle 2,-1,-2 \rangle$, then since

$$\left\|\vec{A}\right\| = \sqrt{2^2 + (-1)^2 + (-2)^2}$$
$$= \sqrt{4+1+4} = \sqrt{9} = 3$$

it follows that

$$\cos\alpha = \frac{a_1}{\left\|\vec{A}\right\|} = \frac{2}{3},$$
$$\cos\beta = \frac{a_2}{\left\|\vec{A}\right\|} = \frac{-1}{3},$$
$$\cos\gamma = \frac{a_3}{\left\|\vec{A}\right\|} = \frac{-2}{3}.$$

Example 5.3.5 If the magnitude of vector $\vec{A}$ is 4, and the direction angles of $\vec{A}$ are $\frac{\pi}{3}, \frac{3\pi}{4}$ and $\frac{\pi}{6}$, find $\vec{A}$.

solution:

Let $\vec{A} = \langle a_1, a_2, a_3 \rangle$. Then

$$a_1 = \left\|\vec{A}\right\|\cos\alpha = 4\cos\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} = 2,$$
$$a_2 = \left\|\vec{A}\right\|\cos\beta = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(\frac{-\sqrt{2}}{2}\right) = -2\sqrt{2},$$
$$a_3 = \left\|\vec{A}\right\|\cos\gamma = 4\cos\left(\frac{\pi}{6}\right) = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

Thus, $\vec{A} = \langle 2,-2\sqrt{2},2\sqrt{3} \rangle$.

Theorem 5.3.2 If

$$\cos\alpha,\ \cos\beta,\ \text{and}\ \cos\gamma$$

are the direction cosines of a vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 5.3.6 If $\vec{A} = \langle 2,-1,2 \rangle$ and the direction angles of $\vec{A}$ measure α, β, and γ, verify that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

solution:

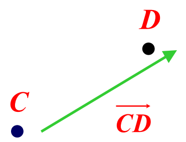
In Example 1.3.4, we found that

$$\cos\alpha = \frac{2}{3},\ \cos\beta = \frac{-1}{3}\ \text{and}\ \cos\gamma = \frac{2}{3}.$$

Thus,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2$$
$$= \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = \frac{9}{9} = 1.$$

Let C and D be points in R^3. If the directed line segment $\overrightarrow{CD}$ is a representation of a vector $\vec{A}$, then C and D are called the *initial point* and *terminal point*, respectively, of this representation of $\vec{A}$.



Example 5.3.7 Consider the points $C(1,2,3)$ and $D(5,3,6)$. If $\overrightarrow{CD}$ is a representation of $\vec{A}$, find $\vec{A}$.

solution:

Let $\vec{A} = \langle a,b,c \rangle$. Then

$$a = x_D - x_C = 5 - 1 = 4$$
$$b = y_D - y_C = 3 - 2 = 1$$
$$c = z_D - z_C = 6 - 3 = 3$$

Thus, $\vec{A} = \langle 4,1,3 \rangle$.

The *sum* of two vectors

$\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$

is the vector $\vec{A} + \vec{B}$ given by

$$\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

Example 5.3.8

Let

$\vec{A} = \langle 5,2,3 \rangle$ and $\vec{B} = \langle 6,-3,4 \rangle$.

Then

$$\vec{A} + \vec{B} = \langle 5+6, 2+(-3), 3+4 \rangle = \langle 11,-1,7 \rangle.$$

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If $\vec{A} = \langle a_1, a_2, a_3 \rangle$, the *negative* of $\vec{A}$

is the vector

$$-\vec{A} = \langle -a_1, -a_2, -a_3 \rangle.$$

Example 5.3.9 Let $\vec{A} = \langle 6,-3,4 \rangle$. Then

$$-\vec{A} = \langle -6, -(-3), -4 \rangle = \langle -6,3,-4 \rangle.$$

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The *difference* of any two vectors

$\vec{A}$ and $\vec{B}$,

denoted by

$$\vec{A} - \vec{B}$$

is the vector obtained by adding $\vec{A}$ to the negative of $\vec{B}$, that is,

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right).$$

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Example 5.3.10

Let

$\vec{A} = \langle 5,2,3 \rangle$ and $\vec{B} = \langle 6,-3,4 \rangle$.

Then

$$\begin{aligned}\vec{A} - \vec{B} &= \vec{A} + \left(-\vec{B}\right)\\ &= \langle 5,2,3 \rangle + \langle -6,3,-4 \rangle\\ &= \langle 5-6, 2+3, 3-4 \rangle = \langle -1,5,-1 \rangle.\end{aligned}$$

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If α is a scalar and $\vec{A} = \langle a,b,c \rangle$,

the product of α and $\vec{A}$, denoted by

$$\alpha\vec{A}$$

is a vector and is given by

$$\alpha\vec{A} = \langle \alpha a, \alpha b, \alpha c \rangle.$$

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Example 5.3.11 If $\vec{A} = \langle 2,-3,4 \rangle$, then

$$\begin{aligned}-5\vec{A} &= \langle -5(2), -5(-3), -5(4) \rangle\\ &= \langle -10,15,-20 \rangle\end{aligned}$$

$$\begin{aligned}2\vec{A} &= \langle 2(2), 2(-3), 2(4) \rangle\\ &= \langle 4,-6,8 \rangle\end{aligned}$$

$$\begin{aligned}0\cdot\vec{A} &= \langle 0(2), 0(-3), 0(4) \rangle\\ &= \langle 0,0,0 \rangle\end{aligned}$$

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Let V_3 be the set of all vectors in space.

If $\vec{A}, \vec{B}$ and $\vec{C}$ are any vectors in V_3, and α and β are scalars, then properties given in Theorem 5.1.2 still hold.

Remarks:

Let $i = \langle 1,0,0 \rangle, j = \langle 0,1,0 \rangle$ and $k = \langle 0,0,1 \rangle$ so that i, j and k are unit vectors in space.

If $\vec{A} = \langle a,b,c \rangle$, then since

$$\langle a,b,c \rangle = \langle a,0,0 \rangle + \langle 0,b,0 \rangle + \langle 0,0,c \rangle$$
$$= a\langle 1,0,0 \rangle + b\langle 0,1,0 \rangle + c\langle 0,0,1 \rangle,$$

it follows that

$$\vec{A} = ai + bj + ck.$$
$$\langle 1,2,3 \rangle = i + 2j + 3k.$$
$$\langle 2,-4,3 \rangle = 2i - 4j + 3k.$$

Theorem 5.3.3 If $\vec{A} = a_1 i + a_2 j + a_3 k$ is a non-zero vector, the unit vector in the same direction as $\vec{A}$ is given by

$$\vec{U}_A = \frac{1}{\|\vec{A}\|} \cdot \vec{A} = \frac{a_1}{\|\vec{A}\|} i + \frac{a_2}{\|\vec{A}\|} j + \frac{a_3}{\|\vec{A}\|} k .$$

Example 5.3.12 Find the unit vector in the same direction as the given vector.

a. $\vec{A} = 2i - j - 2k$ b. $\vec{B} = -2i + 4j + 3k$

solution:

a. $\vec{A} = 2i - j - 2k$

$$\|\vec{A}\| = 3$$

$$\vec{U}_A = \frac{2}{3} i - \frac{1}{3} j - \frac{2}{3} k = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle$$

b. $\vec{B} = -2i + 4j + 3k$

$$\|\vec{B}\| = \sqrt{(-2)^2 + 4^2 + 3^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\vec{U}_B = \frac{-2}{\sqrt{29}} i + \frac{4}{\sqrt{29}} j + \frac{3}{\sqrt{29}} k$$
$$= \left\langle \frac{-2}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}} \right\rangle$$
$$= \left\langle \frac{-2\sqrt{29}}{29}, \frac{4\sqrt{29}}{29}, \frac{3\sqrt{29}}{29} \right\rangle.$$

5.4 DOT PRODUCT

If $\vec{A} = \langle a_1, a_2 \rangle$ and $\vec{B} = \langle b_1, b_2 \rangle$ are two vectors in V_2, the *dot product* or *scalar product* or *inner product* of $\vec{A}$ and $\vec{B}$, denoted by $\vec{A} \cdot \vec{B}$ is given by

$$\vec{A} \cdot \vec{B} = a_1 \cdot b_1 + a_2 \cdot b_2 .$$

Example 5.4.1 Let

$\vec{A} = \langle 2,-3 \rangle$ and $\vec{B} = \langle -2,4 \rangle$.

Then

$$\vec{A}\cdot\vec{B} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle$$
$$= a_1 \cdot b_1 + a_2 \cdot b_2$$
$$= 2(-2) + (-3)4$$
$$= -4 - 12 = -16.$$

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If $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$ are two vectors in V_3, the *dot product* or *scalar product* or *inner product* of $\vec{A}$ and $\vec{B}$, denoted by

$$\vec{A}\cdot\vec{B}$$

is given by

$$\vec{A}\cdot\vec{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.$$

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Example 5.4.2 Let

$\vec{A} = \langle 4,2,-3 \rangle$ and $\vec{B} = \langle 3,-2,4 \rangle$.

Then

$$\vec{A}\cdot\vec{B} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$
$$= a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$
$$= 4(3) + 2(-2) + (-3)4$$
$$= 12 - 4 - 12 = -4.$$

Remark: The dot product of 2 vectors is not a vector but a real number.

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Theorem 5.4.1 If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors in V_2 (or V_3), then

a. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (commutative law)

b. $\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ (distributive law)

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Example 5.4.3 If $\vec{A} = \langle 4,2,-3 \rangle$, $\vec{B} = \langle 3,-2,4 \rangle$ and $\vec{C} = \langle 1,2,5 \rangle$, verify that $\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$.

solution:

$$\vec{B}+\vec{C} = \langle 3+1, -2+2, 4+5 \rangle = \langle 4,0,9 \rangle,$$
$$\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \langle 4,2,-3 \rangle \cdot \langle 4,0,9 \rangle$$
$$= 4\cdot 4 + 2\cdot 0 + (-3)9$$
$$= 16 + 0 - 27 = -11.$$

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$$\vec{A}\cdot\vec{B} = \langle 4,2,-3 \rangle \cdot \langle 3,-2,4 \rangle$$
$$= 4\cdot 3 + 2(-2) + (-3)4$$
$$= 12 - 4 - 12 = -4$$
$$\vec{A}\cdot\vec{C} = \langle 4,2,-3 \rangle \cdot \langle 1,2,5 \rangle$$
$$= 4\cdot 1 + 2(2) + (-3)5$$
$$= 4 + 4 - 15 = 8 - 15 = -7$$
$$\vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} = -4 + (-7) = -11$$

Thus, $\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$.

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Theorem 5.4.2 If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are any vectors in V_2 (or V_3) and α is any scalar, then

a. $\alpha(\vec{A}\cdot\vec{B})=(\alpha\vec{A})\cdot\vec{B}$

b. $\vec{O}\cdot\vec{A}=0$

c. $\vec{A}\cdot\vec{A}=\|\vec{A}\|^2$

Example 5.4.4 If $\vec{A}=\langle 4,2,-3\rangle$ and $\vec{B}=\langle 3,-2,4\rangle$, verify that $2(\vec{A}\cdot\vec{B})=(2\vec{A})\cdot\vec{B}$.

solution:

$$\begin{aligned}\vec{A}\cdot\vec{B}&=\langle 4,2,-3\rangle\cdot\langle 3,-2,4\rangle\\&=4\cdot 3+2(-2)+(-3)4\\&=12-4-12=-4\end{aligned}$$

$$2(\vec{A}\cdot\vec{B})=2(-4)=-8.$$

$$2\vec{A}=2\langle 4,2,-3\rangle=\langle 8,4,-6\rangle$$

$$\begin{aligned}(2\vec{A})\cdot\vec{B}&=\langle 8,4,-6\rangle\cdot\langle 3,-2,4\rangle\\&=8\cdot 3+4(-2)+(-6)4\\&=24-8-24=-8\end{aligned}$$

Thus, $2(\vec{A}\cdot\vec{B})=(2\vec{A})\cdot\vec{B}$.

Example 5.4.5 If $\vec{A}=\langle 4,2,-3\rangle$, verify that $\vec{O}\cdot\vec{A}=0$ and $\vec{A}\cdot\vec{A}=\|\vec{A}\|^2$.

solution:

$$\begin{aligned}\vec{O}\cdot\vec{A}&=\langle 0,0,0\rangle\cdot\langle 4,2,-3\rangle\\&=0\cdot 4+0\cdot 2+0\cdot 3=0.\end{aligned}$$

$$\begin{aligned}\vec{A}\cdot\vec{A}&=\langle 4,2,-3\rangle\cdot\langle 4,2,-3\rangle\\&=4\cdot 4+2\cdot 2+(-3)(-3)\\&=4^2+2^2+(-3)^2\\&=\left(\sqrt{4^2+2^2+(-3)^2}\right)^2=\|\vec{A}\|^2.\end{aligned}$$

Let $\vec{A}$ and $\vec{B}$ be two nonzero vectors.
If $\vec{A}$ is not a scalar multiple of $\vec{B}$,

$\overrightarrow{OP}$ is the position representation of $\vec{A}$,

$\overrightarrow{OQ}$ is the position representation of $\vec{B}$,

the *angle between vectors* $\vec{A}$ and $\vec{B}$ is defined to be the angle (of positive measure) interior to the triangle determined by the points O, P, and Q.

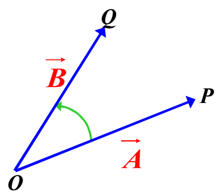


Figure 5.4.1

Remarks:

If $\vec{A} = \alpha \vec{B}$ where α is a scalar, then

a. if $\alpha > 0$, the angle between the vectors has radian measure 0;

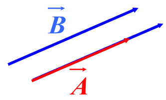


Figure 5.4.2a

b. if $\alpha < 0$, the angle between the vectors has radian measure π.

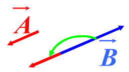


Figure 5.4.2b

Theorem 5.4.3

If α is the angle between the two non-zero vectors $\vec{A}$ and $\vec{B}$ in V_2 or V_3, then

$$\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\| \cos \alpha .$$

Equivalently,

$$\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|\|\vec{B}\|}.$$

Example 5.4.6 Let $\vec{A} = \langle 2,-2 \rangle$ and $\vec{B} = \langle -3,3 \rangle$.

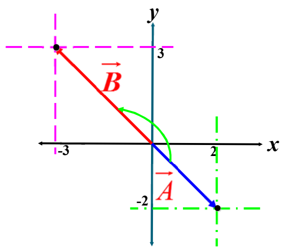
$$\vec{A} \cdot \vec{B} = 2(-3) + (-2)3 = -6 - 6 = -12$$

$$\|\vec{A}\| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{23}$$

$$\|\vec{B}\| = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{-12}{2\sqrt{2} \cdot 3\sqrt{2}} = \frac{-12}{12} = -1$$

$$\alpha = Arc\cos(-1) = \pi$$



Example 5.4.7 Let $\vec{A} = \langle 1,2,-2 \rangle$ and $\vec{B} = \langle -3,0,3 \rangle$.

$$A \cdot B = 1(-3) + 2 \cdot 0 + -2(3) = -9$$

$$\|\vec{A}\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$$

$$\|\vec{B}\| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{-9}{3 \cdot 3\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\alpha = Arc\cos\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

Two non-zero vectors are said to be *parallel* if and only if one is a scalar multiple of the other.

Example 5.4.8.a

Let $\vec{A} = \langle 3,4 \rangle$ and $\vec{B} = \langle 6,8 \rangle$.

Since

$2\vec{A} = \vec{B}$ or equivalently, $\frac{1}{2}\vec{B} = \vec{A}$,

vectors $\vec{A}$ and $\vec{B}$ are parallel.

Example 5.4.8.b

Let $\vec{A} = \langle 3,0,-3 \rangle$ and $\vec{B} = \langle -1,0,1 \rangle$.

Since

$\frac{-1}{3} \cdot \vec{A} = \vec{B}$ or equivalently, $\vec{A} = -3\vec{B}$,

vectors $\vec{A}$ and $\vec{B}$ are parallel.

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Example 5.4.8.c

Let $\vec{A} = \langle 1,2 \rangle$ and $\vec{B} = \langle 3,4 \rangle$.

Since we can not find any scalar α such that

$$\vec{A} = \alpha \vec{B}$$

vectors $\vec{A}$ and $\vec{B}$ are not parallel.

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Two non-zero vectors $\vec{A}$ and $\vec{B}$ are said to be *orthogonal* (*perpendicular*) if and only if

$$\vec{A} \cdot \vec{B} = 0.$$

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$$\vec{A} \cdot \vec{B} = 0$$

Then

$$\cos\alpha = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{0}{\|\vec{A}\|\|\vec{B}\|} = 0$$

$\rightarrow$ $\cos\alpha = 0$

$\rightarrow$ $\alpha = \frac{\pi}{2}$

$\rightarrow$ The vectors are perpendicular.

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Example 5.4.9

a. Let $\vec{A} = \langle 4,2 \rangle$ and $\vec{B} = \langle 2,-4 \rangle$.

$$\vec{A} \cdot \vec{B} = 4 \cdot 2 + 2(-4) = 8 - 8 = 0$$

$\vec{A}$ and $\vec{B}$ are orthogonal.

b. Let $\vec{A} = \langle -4,-6 \rangle$ and $\vec{B} = \langle 2,3 \rangle$.

$$\vec{A} \cdot \vec{B} = (-4) \cdot 2 + (-6)3$$
$$= -8 - 18 = -26 \neq 0$$

$\vec{A}$ and $\vec{B}$ are not orthogonal.

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c. Let $\vec{A} = \langle -4,-6,-2 \rangle$ and $\vec{B} = \langle 2,3,1 \rangle$.

$$\vec{A} \cdot \vec{B} = (-4) \cdot 2 + (-6) \cdot 3 + (-2) \cdot 1$$
$$= -8 - 18 - 2 = -28 \neq 0$$

$\vec{A}$ and $\vec{B}$ are not orthogonal.

d. Let $\vec{A} = \langle 4,2,0 \rangle$ and $\vec{B} = \langle 2,-4,3 \rangle$.

$$\vec{A} \cdot \vec{B} = 4(2) + 2(-4) + 0 \cdot 3$$
$$= 8 - 8 + 0 = 0$$

$\vec{A}$ and $\vec{B}$ are orthogonal.

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If $\vec{A}$ and $\vec{B}$ are non-zero vectors and α is the angle between them, the *scalar projection of* $\vec{B}$ *onto* $\vec{A}$ is defined to be $\|\vec{B}\|\cos\alpha$.

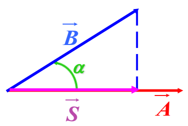


Figure 5.4.3

$$\cos\alpha = \frac{\|\vec{S}\|}{\|\vec{B}\|}$$

$$\|\vec{B}\|\cos\alpha = \|\vec{S}\|$$

Theorem 5.4.4 The scalar projection of a vector $\vec{B}$ onto the vector $\vec{A}$ is $\dfrac{\vec{A}.\vec{B}}{\|\vec{A}\|}$.

$$\boxed{\cos\alpha = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\|\vec{B}\|} \rightarrow \|\vec{B}\|\cos\alpha = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|}.}$$

Example 5.4.9 Let $\vec{A} = \langle 3,4\rangle$ and $\vec{B} = \langle 1,2\rangle$.

Find the scalar projection of

a. $\vec{B}$ onto $\vec{A}$ b. $\vec{A}$ onto $\vec{B}$

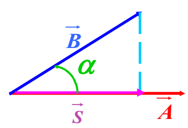
solution:

a. $\dfrac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|} = \dfrac{\langle 3,4\rangle\cdot\langle 1,2\rangle}{\sqrt{3^2+4^2}} = \dfrac{3\cdot 1+4\cdot 2}{\sqrt{25}} = \dfrac{11}{5}.$

b. $\dfrac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|} = \dfrac{\langle 3,4\rangle\cdot\langle 1,2\rangle}{\sqrt{1^2+2^2}} = \dfrac{3\cdot 1+4\cdot 2}{\sqrt{5}} = \dfrac{11}{\sqrt{5}} = \dfrac{11\sqrt{5}}{5}.$

Theorem 5.4.5 The vector projection of a vector $\vec{B}$ onto a non-zero vector $\vec{A}$ is

$$\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2}\left(\vec{A}\right).$$



$$u_{\vec{A}} = \frac{1}{\|\vec{A}\|}\cdot\vec{A}$$

$$\vec{S} = \left(\frac{\vec{A}.\vec{B}}{\|\vec{A}\|}\right)u_{\vec{A}}$$

$$\|\vec{S}\| = \|\vec{B}\|\cos\alpha = \|\vec{B}\|\cos\alpha\cdot\vec{U}_A$$

Since $\|\vec{B}\|\cos\alpha = \dfrac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|}$ and $\vec{U}_A = \dfrac{1}{\|\vec{A}\|}\cdot\vec{A}$,

$$\vec{S} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|}\right)\left(\frac{1}{\|\vec{A}\|}\left(\vec{A}\right)\right) = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2}\left(\vec{A}\right).$$

Example 5.4.9 Let $\vec{A} = \langle 3,4\rangle$ and $\vec{B} = \langle 1,2\rangle$.

Find the vector projection of $\vec{B}$ onto $\vec{A}$.

Draw the position representations of $\vec{A}$ and $\vec{B}$ and the vector projection of $\vec{B}$ onto $\vec{A}$.

solution:

$$\vec{A}\cdot\vec{B} = \langle 3,4\rangle\cdot\langle 1,2\rangle$$
$$= 3+8=11$$
$$\|\vec{A}\| = \sqrt{3^2+4^2}$$
$$=\sqrt{25}=5$$
$$\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2}\left(\vec{A}\right) = \frac{11}{5^2}\langle 3,4\rangle$$
$$=\left\langle \frac{33}{25},\frac{44}{25}\right\rangle$$

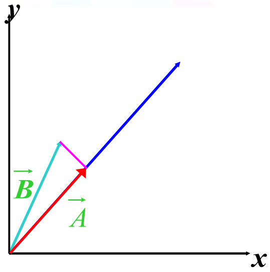


Figure 5.4.4

5.5 PLANES AND LINES IN SPACE

If $\vec{N}$ is a given non-zero vector and P_0 is a given point, then the set of all points P for which $\vec{N}$ and $\overrightarrow{P_0P}$ are orthogonal is defined to be the *plane through P_0 and having $\vec{N}$ as a normal vector*.

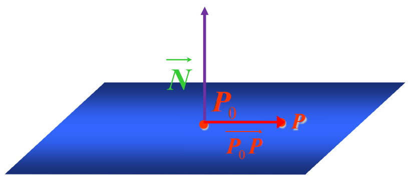


Figure 5.5.1

Theorem 5.5.1 If $P_0(x_0,y_0,z_0)$ is a point in a plane and $\langle a,b,c\rangle$ is a normal vector to the plane, an equation of the plane is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

Example 5.5.1 Find an equation of the plane which passes through $(1,2,3)$ and has $\langle 4,-1,5\rangle$ as a normal vector.

solution:

With $(x_0,y_0,z_0)=(1,2,3)$ and $\langle a,b,c\rangle = \langle 4,-1,5\rangle$, an equation of the plane is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
$$4(x-1)+(-1)(y-2)+5(z-3)=0$$
$$4(x-1)-(y-2)+5(z-3)=0$$
$$4x-y+5z-17=0$$

Example 5.5.2 Find an equation of the plane which passes through $(1,2,3)$ and has a normal vector parallel to the line passing through $B(2,-1,0)$ and $C(5,6,-2)$.

solution:

$$\overrightarrow{BC} = \langle 5-2, 6-(-1), -2-0\rangle = \langle 3,7,-2\rangle$$

With $(x_0,y_0,z_0)=(1,2,3)$ and $\langle a,b,c\rangle = \langle 3,7,-2\rangle$, an equation of the plane is

$$3(x-1)+7(y-2)+(-2)(z-3)=0$$

Theorem 5.5.2 If a, b, c and d are constants such that a, b, and c are not all zero, the graph of an equation of the form

$$ax + by + cz + d = 0$$

is a plane and

$$\langle a, b, c \rangle$$

is a normal vector to the plane.

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Example 5.5.3 Find two vectors which are normal to the plane given by

$$3x - 2z + 4y + 6 = 0.$$

solution:

By Theorem 1.5.2, $\langle 3,4,-2 \rangle$ is a normal vector to the plane. Thus,

$$-\langle 3,4,-2 \rangle = \langle -3,-4,2 \rangle$$

is another normal vector to the plane.

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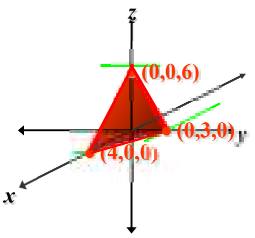
Example 5.5.4 Sketch the plane given by

$$3x + 4y + 2z - 12 = 0.$$

solution:



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Two planes are *parallel* if and only if their *normal vectors* are parallel.

Example 5.5.5 Determine if the indicated pair of planes are parallel.

a. $2x - 3y + 4z + 3 = 0$ and $4x - 6y + 8z + 1 = 0$

b. $2x - 3y + 5z - 2 = 0$ and $x - 2y + 3z + 7 = 0$

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a. $2x - 3y + 4z + 3 = 0$ and $4x - 6y + 8z + 1 = 0$

From Theorem 1.5.2, a normal vector to the plane given by $2x - 3y + 4z + 3 = 0$ is

$$\langle 2,-3,4 \rangle$$

while a normal vector to the plane given by $4x - 6y + 8z + 1 = 0$ is

$$\langle 4,-6,8 \rangle.$$

Since

$$2\langle 2,-3,4 \rangle = \langle 4,-6,8 \rangle$$

the normal vectors are parallel and so are the planes.

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b. $2x - 3y + 5z - 2 = 0$ and $x - 2y + 3z + 7 = 0$

From Theorem 1.5.2, a normal vector to the plane given by $2x - 3y + 5z - 2 = 0$ is

$$\langle 2,-3,5 \rangle$$

while a normal vector to the plane given by $x - 2y + 3z + 7 = 0$ is

$$\langle 1,-2,3 \rangle.$$

Since $\langle 2,-3,5 \rangle$ and $\langle 1,-2,3 \rangle$ are not parallel the planes are not parallel.

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Two planes are *perpendicular* if and only if their normal vectors are perpendicular.

Theorem 5.5.3

Two planes are perpendicular if and only if the dot product of their normal vectors is zero.

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Example 5.5.6 Determine if the indicated pair of planes are perpendicular.

a. $4x+2y=3=0$ and $2x-4y+3z-4=0$

b. $2x-3y+5z-2=0$ and $x-2y+3z+7=0$

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a. $4x+2y-3=0$ and $2x-4y+3z-4=0$

From Theorem 5.5.2, a normal vector to the plane given by $4x+2y-3=0$ is

$$\langle 4,2,0\rangle$$

while a normal vector to the plane given by $2x-4y+3z-4=0$ is

$$\langle 2,-4,3\rangle.$$

Since $\langle 4,2,0\rangle\cdot\langle 2,-4,3\rangle=8-8-0=0$ the normal vectors are perpendicular and so are the plane.

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b. $2x-3y+5z-2=0$ and $x-2y+3z+7=0$

From Theorem 1.5.2, a normal vector to the plane given by $2x-3y+5z-2=0$ is

$$\langle 2,-3,5\rangle$$

while a normal vector to the plane given by $x-2y+3z+7=0$ is $\langle 1,-2,3\rangle$.

Since

$$\langle 2,-3,5\rangle\cdot\langle 1,-2,3\rangle=2+6+15=23\neq 0,$$

the normal vectors are not perpendicular and so are the plane.

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Theorem 5.5.4 The perpendicular distance between the parallel planes given by

$$ax+by+cz+d_1=0$$

and

$$ax+by+cz+d_2=0$$

is

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

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Example 5.5.7 Find the distance between the parallel planes given by

$2x+4y-4z+5=0$ and $2x+4y-4z-7=0$.

solution:

With $a=2,\ b=4,\ c=-4,\ d_1=5$ and $d_2=-7$,

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{|5-(-7)|}{\sqrt{2^2+4^2+(-4)^2}}$$

$$=\frac{12}{\sqrt{4+16+16}}=\frac{12}{\sqrt{36}}=\frac{12}{6}=2.$$

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Theorem 5.5.5 The undirected distance from the plane given by

$$ax+by+cz+d=0$$

to the point

$$(x_0, y_0, z_0)$$

is

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

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Example 5.5.8 Find the undirected distance from the plane given by

$$2x+4y-4z+5=0$$

to the point $(3,-2,1)$.

solution:

With $a=2,\ b=4,\ c=-4,\ d=5$

and $(x_0, y_0, z_0)=(3,-2,1)$

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|2(3)+4(-2)-4(1)+5|}{\sqrt{2^2+4^2+(-4)^2}}$$

$$=\frac{|6-8-4+5|}{6}=\frac{1}{6}.$$

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Let L be a line containing the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $R=\langle a,b,c\rangle$.

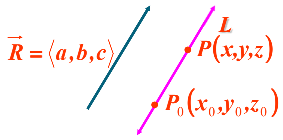

Figure 5.5.3.

$P(x,y,z)\in L$ if and only if $\overrightarrow{P_0P}$ is parallel to $\vec{R}$

iff $\langle x-x_0, y-y_0, z-z_0\rangle$ is parallel to $\vec{R}$.

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$\leftrightarrow$ $\langle x-x_0, y-y_0, z-z_0\rangle$ is a scalar multiple of $\vec{R}$.

$\leftrightarrow$ $\langle x-x_0, y-y_0, z-z_0\rangle = t\langle a,b,c\rangle$

$\leftrightarrow$ $\langle x-x_0, y-y_0, z-z_0\rangle = \langle ta,tb,tc\rangle$

$\leftrightarrow$ $\langle x-x_0, y-y_0, z-z_0\rangle = \langle at,bt,ct\rangle$

$\leftrightarrow$ $\boxed{\begin{aligned} x-x_0&=at\\ y-y_0&=bt\\ z-z_0&=ct\end{aligned}}$ These are called the *parametric equations of L.*

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$\boxed{\begin{aligned} x-x_0&=at\\ y-y_0&=bt\\ z-z_0&=ct\end{aligned}}$ Again, these are called the *parametric equations* of L.

If none of the numbers a, b, c is zero, t can be eliminated from the parametric equations to obtain

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

which are called *symmetric equations* of L.

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Example 5.5.9 Find the parametric equations and symmetric equations of the line passing through $(1,-2,3)$ and which is parallel to the vector $\langle -2,4,5\rangle$.

solution:

With $(x_0,y_0,z_0)=(1,-2,3)$

and $\langle a,b,c\rangle=\langle -2,4,5\rangle$,

the parametric equations of the line are

$$\begin{aligned} x-1&=-2t\\ y--2&=4t\\ z-3&=5t\end{aligned} \quad\text{or}\quad \begin{aligned} x-1&=-2t\\ y+2&=4t\\ z-3&=5t\end{aligned}$$

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From the parametric equations of the line, namely,

$$x-1=-2t$$
$$y+2=4t$$
$$z-3=5t$$

we get the following symmetric equations of the line

$$\frac{x-1}{-2}=\frac{y+2}{4}=\frac{z-3}{5}.$$

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5.6 CROSS PRODUCT

If $\vec{A}=\langle a_1,a_2,a_3\rangle$ and $\vec{B}=\langle b_1,b_2,b_3\rangle$, the *cross product* of $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\times\vec{B}$ is given by

$$\vec{A}\times\vec{B}=\langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$$

or equivalently,

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

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Example 5.6.1 Let $\vec{A}=\langle 2,3,1\rangle$ and $\vec{B}=\langle -1,4,2\rangle$.

Then with

$\langle a_1,a_2,a_3\rangle=\langle 2,3,1\rangle$ and $\langle b_1,b_2,b_3\rangle=\langle -1,4,2\rangle$,

$$\vec{A}\times\vec{B}=\langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$$
$$=\langle 3\cdot 2-1\cdot 4, 1(-1)-2\cdot 2, 2\cdot 4-3(-1)\rangle$$
$$=\langle 6-4, 1-1-4, 8+3\rangle$$
$$=\langle 2,-5,11\rangle.$$

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Using the second formula, with

$\langle a_1,a_2,a_3\rangle=\langle 2,3,1\rangle$ and $\langle b_1,b_2,b_3\rangle=\langle -1,4,2\rangle$,

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$=\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ -1 & 4 & 2 \end{vmatrix}=\begin{vmatrix} i & j & k & i & j \\ 2 & 3 & 1 & 2 & 3 \\ -1 & 4 & 2 & -1 & 4 \end{vmatrix}$$
$$=6i-j+8k+3k-4i-4j$$
$$=2i-5j+11k.$$

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Theorem 5.6.1 If $\vec{A}$ is any vector in V_3,

a. $\vec{A}\times\vec{A}=\vec{O}$

b. $\vec{O}\times\vec{A}=\vec{O}$

c. $\vec{A}\times\vec{O}=\vec{O}$

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Example 5.6.2 Verify Theorem 5.6.1 (a) and (b) for $\vec{A}=\langle 2,3,1\rangle$.

solution:

a. $\vec{A}\times\vec{A}=\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix}=3i+2j+6k-6k-3i-2j$

$=0i+0j+0k=\vec{O}.$

b. $\vec{O}\times\vec{A}=\begin{vmatrix} i & j & k \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix}=0i+0j+0k-0k-0i-0j$

$=0i+0j+0k=\vec{O}.$

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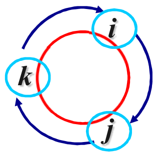
Theorem 5.6.2 If i, j and k are the unit vectors in space, then

$$i \times i = \vec{O}, \quad j \times j = \vec{O}, \quad k \times k = \vec{O},$$
$$\underline{i \times j = k}, \quad \underline{j \times k = i}, \quad \underline{k \times i = j},$$
$$j \times i = -k, \quad k \times j = -i, \quad i \times k = -j.$$



Theorem 5.6.3 If $\vec{A}$ and $\vec{B}$ are any vectors in V_3, then

$$\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right).$$

Example 5.6.3 Verify Theorem 1.6.3 for $\vec{A} = \langle 2,3,1 \rangle$ and $\vec{B} = \langle 4,5,6 \rangle$.

solution:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 18i + 4j + 10k - 12k - 12j - 5i$$

$$= 13i - 8j - 2k = \langle 13,\text{-}8,\text{-}2 \rangle.$$

$$\vec{B} \times \vec{A} = \begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= 5i + 12j + 12k - 10k - 4j - 18i$$

$$= -13i + 8j + 2k = \langle -13,8,2 \rangle.$$

Thus, $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$.

Theorem 5.6.4 If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are any vectors in V_3 then

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \left(\vec{A} \times \vec{B}\right) + \left(\vec{A} \times \vec{C}\right).$$

Example 5.6.4 Verify Theorem 5.6.4 for $\vec{A} = \langle 2,3,1 \rangle, \vec{B} = \langle 4,5,6 \rangle$ and $\vec{C} = \langle 3,1,2 \rangle$.

solution:

$$\vec{B} + \vec{C} = \langle 4,5,6 \rangle + \langle 3,1,2 \rangle = \langle 7,6,8 \rangle$$

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 7 & 6 & 8 \end{vmatrix}$$

$$= 24i + 7j + 12k - 21k - 16j - 6i$$

$$= 18i - 9j - 9k = \langle 18,-9,-9 \rangle.$$

From Example 5.6.3,

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 4 & 5 & 6\end{vmatrix}=13i-8j-2k=\langle 13,-8,-2\rangle.$$

$$\vec{A}\times\vec{C}=\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & 1 & 2\end{vmatrix}=6i+3j+2k-9k-4j-i$$

$$=5i-j-7k=\langle 5,-1,-7\rangle.$$

Thus,

$$\left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)=\langle 13,-8,-2\rangle+\langle 5,-1,-7\rangle$$

$$=\langle 18,-9,-9\rangle,$$

so that

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right).$$

Theorem 5.6.5 If $\vec{A}$ and $\vec{B}$ are any vectors in V_3 and α is any scalar, then

a. $\left(\alpha\vec{A}\right)\times\vec{B}=\vec{A}\times\left(\alpha\vec{B}\right)$

b. $\left(\alpha\vec{A}\right)\times\vec{B}=\alpha\left(\vec{A}\times\vec{B}\right)$

Example 5.6.5 Verify Theorem 5.6.5 (a) and (b) for $\vec{A}=\langle 2,3,1\rangle$, $\vec{B}=\langle 4,5,6\rangle$ and $\alpha=4$.

solution:

$\left(\text{to show that } \left(\alpha\vec{A}\right)\times\vec{B}=\vec{A}\times\left(\alpha\vec{B}\right)\right)$

$$4\vec{A}=4\langle 2,3,1\rangle=\langle 8,12,4\rangle$$

$$\left(4\vec{A}\right)\times\vec{B}=\begin{vmatrix} i & j & k \\ 8 & 12 & 4 \\ 4 & 5 & 6\end{vmatrix}$$

$$=72i+16j+40k-48k-48j-20i$$

$$=52i-32j-8k=\langle 52,-32,-8\rangle$$

$$4\vec{B}=4\langle 4,5,6\rangle=\langle 16,20,24\rangle$$

$$\vec{A}\times\left(4\vec{B}\right)=\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 16 & 20 & 24\end{vmatrix}$$

$$=72i+16j+40k-48k-48j-20i$$

$$=52i-32j-8k=\langle 52,-32,-8\rangle$$

Thus,

$$\left(\alpha\vec{A}\right)\times\vec{B}=\vec{A}\times\left(\alpha\vec{B}\right).$$

$\left(\text{to show that } \left(\alpha\vec{A}\right)\times\vec{B}=\alpha\left(\vec{A}\times\vec{B}\right)\right)$

From (a),

$$\left(4\vec{A}\right)\times\vec{B}=\begin{vmatrix} i & j & k \\ 8 & 12 & 4 \\ 4 & 5 & 6\end{vmatrix}=\langle 52,-32,-8\rangle$$

From Example 5.6.3,

$$\vec{A}\times\vec{B}=\langle 13,-8,-2\rangle.$$

Thus,

$$4\left(\vec{A}\times\vec{B}\right)=4\langle 13,-8,-2\rangle=\langle 52,-32,-8\rangle$$

and $\left(\alpha\vec{A}\right)\times\vec{B}=\alpha\left(\vec{A}\times\vec{B}\right)$.

Theorem 5.6.6 If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are any vectors in V_3 then

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}.$$

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Example 5.6.6 Verify Theorem 5.6.6 for $\vec{A}=\langle 2,3,1\rangle, \vec{B}=\langle 4,5,6\rangle$ and $\vec{C}=\langle 3,1,2\rangle$.

solution : (to show that $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$)

$$\vec{B}\times\vec{C} = \begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 3 & 1 & 2 \end{vmatrix} = 10i + 18j + 4k - 15k - 8j - 6i$$

$$= 4i + 10j - 11k = \langle 4,10,-11\rangle$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \langle 2,3,1\rangle\cdot\langle 4,10,-11\rangle$$

$$= 8 + 30 - 11 = 27$$

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From Example 5.6.4,

$$\vec{A}\times\vec{B} = \langle 13,-8,-2\rangle$$

so that

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \langle 13,-8,-2\rangle\cdot\langle 3,1,2\rangle$$

$$= 39 - 8 - 4 = 27.$$

Thus,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\vec{C}.$$

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Theorem 5.6.7 If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are any vectors in V_3 then

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}.$$

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Example 5.6.7 Verify Theorem 5.6.7 for $\vec{A}=\langle 2,3,1\rangle, \vec{B}=\langle 4,5,6\rangle$ and $\vec{C}=\langle 3,1,2\rangle$.

solution : (to show that $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$)

From Example 5.6.6, $\vec{B}\times\vec{C} = \langle 4,10,-11\rangle$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 4 & 10 & -11 \end{vmatrix}$$

$$= -33i + 4j + 20k - 12k + 22j - 10i$$

$$= -43i + 26j + 8k = \langle -43,26,8\rangle$$

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$$(\vec{A}\cdot\vec{C})\vec{B} = (\langle 2,3,1\rangle\cdot\langle 3,1,2\rangle)\vec{B}$$

$$= (2\cdot 3 + 3\cdot 1 + 1\cdot 2)\vec{B}$$

$$= 11\vec{B} = 11\langle 4,5,6\rangle = \langle 44,55,66\rangle$$

$$(\vec{A}\cdot\vec{B})\vec{C} = (\langle 2,3,1\rangle\cdot\langle 4,5,6\rangle)\vec{C}$$

$$= (2\cdot 4 + 3\cdot 5 + 1\cdot 6)\vec{C}$$

$$= 29\vec{C} = 29\langle 3,1,2\rangle = \langle 87,29,58\rangle$$

$$(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \langle 44,55,66\rangle - \langle 87,29,58\rangle$$

$$= \langle -43,26,8\rangle$$

Thus,

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}.$$

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Theorem 5.6.8 If $\vec{A}$ and $\vec{B}$ are any vectors in V_3 and θ is the measure of the angle between them, then

$$\left\|\vec{A}\times\vec{B}\right\| = \left\|\vec{A}\right\|\left\|\vec{B}\right\| \sin\theta$$

or equivalently,

$$\sin\theta = \frac{\left\|\vec{A}\times\vec{B}\right\|}{\left\|\vec{A}\right\|\left\|\vec{B}\right\|}.$$

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Example 5.6.8 Let

$\vec{A} = \langle 2,3,1\rangle$ and $\vec{B} = \langle 3,-2,0\rangle$.

Use Theorem 5.6.8 to find the measure of the angle θ between them.

solution:

By Theorem 5.6.8, $\theta = Arc\sin\dfrac{\left\|\vec{A}\times\vec{B}\right\|}{\left\|\vec{A}\right\|\left\|\vec{B}\right\|}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -2 & 0 \end{vmatrix} = 3j - 4k - 9k + 2i$$

$$= 2i + 3j - 13k = \langle 2,3,-13\rangle$$

$$\left\|\vec{A}\times\vec{B}\right\| = \sqrt{2^2+3^2+(-13)^2} = \sqrt{4+9+169} = \sqrt{182}$$

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$$\left\|\vec{A}\right\| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$\left\|\vec{B}\right\| = \sqrt{3^2+(-2)^2+0^2} = \sqrt{9+4} = \sqrt{13}$$

$$\theta = Arc\sin\frac{\left\|\vec{A}\times\vec{B}\right\|}{\left\|\vec{A}\right\|\left\|\vec{B}\right\|} = Arc\sin\frac{\sqrt{182}}{\sqrt{14}\sqrt{13}}$$

$$= Arc\sin\frac{\sqrt{182}}{\sqrt{182}}$$

$$= Arc\sin 1 = \frac{\pi}{2}.$$

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Theorem 5.6.9 If $\vec{A}$ and $\vec{B}$ are any non-zero vectors in V_3 then and $\vec{A}$ and $\vec{B}$ are parallel iff

$$\vec{A}\times\vec{B} = \vec{O}.$$

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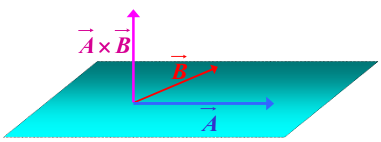
Theorem 5.6.10 If $\vec{A}$ and $\vec{B}$ are any nonzero vectors in V_3, then $\vec{A}\times\vec{B}$ is orthogonal to both $\vec{A}$ and $\vec{B}$.



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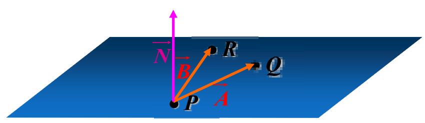
Example 5.6.10 Find an equation of the plane containing the points $P(2,3,1)$, $Q(3,-2,0)$ and $R(1,0,-2)$.

solution:



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Let

$A = \overrightarrow{PQ}$ and $B = \overrightarrow{PR}$.

Then

$$\vec{A} = \langle 3,-2,0\rangle - \langle 2,3,1\rangle = \langle 1,-5,-1\rangle$$

and

$$\vec{B} = \langle 1,0,-2\rangle - \langle 2,3,1\rangle = \langle -1,-3,-3\rangle$$

so that

$$\vec{N} = \vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 1 & -5 & -1 \\ -1 & -3 & -3 \end{vmatrix}$$

$$= 15\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} - 5\boldsymbol{k} + 3\boldsymbol{j} - 3\boldsymbol{i}$$

$$= 12\boldsymbol{i} + 4\boldsymbol{j} - 8\boldsymbol{k} = \langle 12,4,-8\rangle$$

Since $\langle 12,4,-8\rangle$ is a vector normal to the plane, and the point $(2,3,1)$ lies on the plane, an equation of the plane is

$$12(x-2)+4(y-3)-8(z-1)=0$$

or equivalently,

$$3(x-2)+(y-3)-2(z-1)=0$$

or equivalently,

$$3x+y-2z-7=0.$$

5.7 SURFACES

A *cylinder* is a surface generated by a line moving along a plane curve in such a way that it always remains parallel to a fixed line not lying on the plane of the given curve.

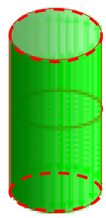
The moving line is called a *generator of the cylinder* and the given curve is called the *directrix* of the cylinder.

Any position of a generator is called a *ruling* of the cylinder.

Consider a circle on the *xy-plane*.

Let a line move along the circle so that it remains parallel to the *z-axis*.

Then by definition, the surface formed is a *cylinder*.



Example 5.7.1

1. In Figure 5.7.1, we show a cylinder whose directrix is the unit circle given by

$$x^2+y^2=1$$

and whose rulings are parallel to the *z*-axis.

A cylinder whose directrix is a circle is called a *right-circular cylinder*.

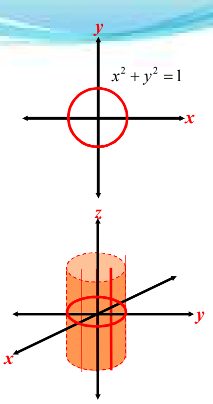


Figure 5.7.1

2. In Figure 5.7.2, we show a cylinder whose directrix is the parabola given by

$$z^2=8x$$

and whose rulings are parallel to the *y*-axis.

A cylinder whose directrix is a parabola is called a *parabolic cylinder*.

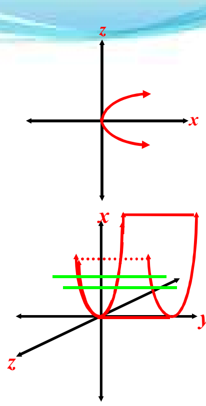


Figure 5.7.2

3. In Figure 5.7.3, we show a cylinder whose directrix is the ellipse given by

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

and whose rulings are parallel to the *z-axis*.

A cylinder whose directrix is an ellipse is called an *elliptic cylinder*.

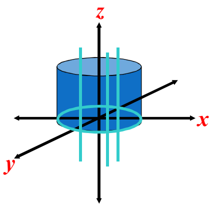


Figure 5.7.3

4. In Figure 5.7.4, we show a cylinder whose directrix is the hyperbola given by

$$x^2-y^2=4$$

and whose rulings are parallel to the *z-axis*.

A cylinder whose directrix is a hyperbola is called a *hyperbolic cylinder*.

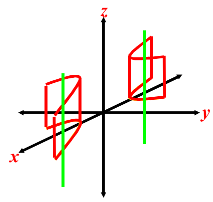


Figure 5.7.4

Theorem 5.7.1

In three-dimensional space, the graph of an equation in two of the 3 variables *x*, *y* and *z* is a cylinder whose rulings are parallel to the axis associated with the missing variable, and whose directrix is a plane curve in the plane associated with the two variables appearing in the equation.

Example 5.7.2 Sketch the graph of each of the following in R^3.

a. $z=x^2$ b. $z=e^y$ c. $y=x^2+1$

solution:

By Theorem 5.7.1, the graph of

$z=x^2$

is a cylinder whose ruling is parallel to the *y*-axis and whose directrix is the parabola given by $z=x^2$ in the *xz*-plane.

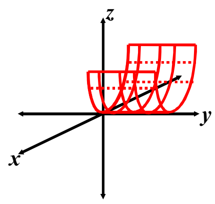


Figure 5.7.5

b. $z=e^y$

solution:

By Theorem 5.7.1, the graph of

$z=e^y$

is a cylinder whose ruling is parallel to the *x-axis* and whose directrix is the graph of in $z=e^y$ the *yz*-plane.

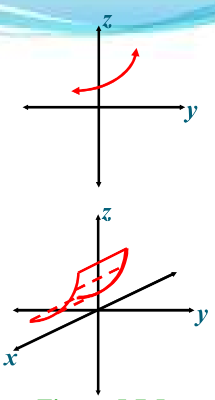


Figure 5.7.5

c. $y=x^2+1$

solution:

By Theorem 5.7.1, the graph of

$y=x^2+1$

is a cylinder whose ruling is parallel to the *z-axis* and whose directrix is the graph of in the *xy*-plane.

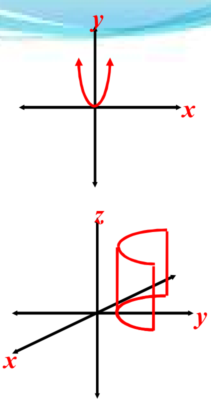


Figure 5.7.6

The graph of

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A, B, C, D, E, F, G, H, I and J are constants such that the first eight constants are not all zero is called a *quadric surface*.

Some examples of quadric surfaces are right-circular, parabolic, elliptic and hyperbolic cylinders.

We study 6 other types of quadric surfaces whose equations are of the form

$$Ax^2 + By^2 + Cz^2 + J = 0$$

Quadric surfaces whose equations are of the form

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + J = 0$$

can be drawn by translation of axes.

The Ellipsoid

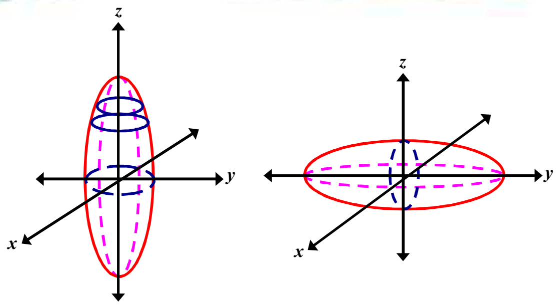
The graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b and c are positive constants is called an *ellipsoid*.

If exactly two of a, b and c in the equation are equal, the surface is called a *spheroid*.

If a, b and c are all equal, the surface is called a *sphere*.

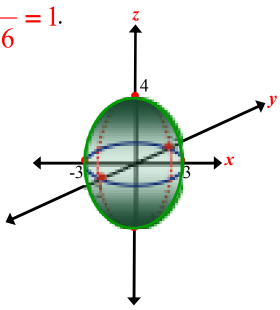


Example 5.7.3 Sketch the graph of

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 1.$$

solution:

By definition, the graph is an *ellipsoid*.



$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 1$$

$$z = 0 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

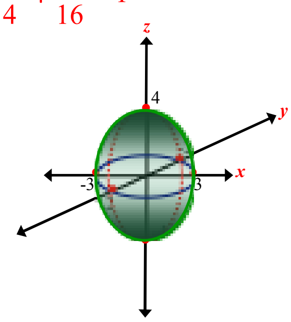
$$a = 3, \quad b = 2$$

$$y = 0 \Rightarrow \frac{x^2}{9} + \frac{z^2}{16} = 1$$

$$a = 4, \quad b = 3$$

$$x = 0 \Rightarrow \frac{y^2}{4} + \frac{z^2}{16} = 1$$

$$a = 4, \quad b = 2$$



The Elliptic Hyperboloid of one sheet

The graph of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

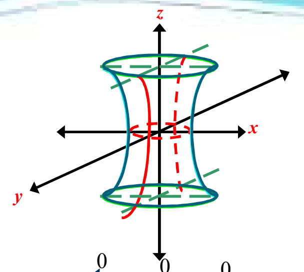
or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

or

$$\frac{-x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a, b and c are positive constants is called an *elliptic hyperboloid of one sheet*.



$$\cancelto{0}{\frac{x^2}{a^2}}+\cancelto{0}{\frac{y^2}{b^2}}-\cancelto{0}{\frac{z^2}{c^2}}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$$

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Example 5.7.4 **Sketch the graph of**

$$\frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{9}=1.$$

solution:

By definition, the graph is an ***elliptic hyperboloid of one sheet*.**

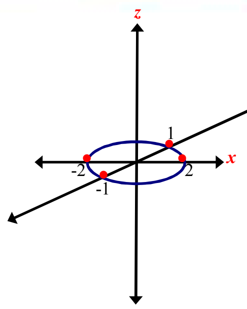
$$z=0 \Rightarrow \frac{x^2}{4}+\frac{y^2}{1}=1 \quad \Rightarrow a=2, \quad b=1$$

$$y=0 \Rightarrow \frac{x^2}{4}-\frac{z^2}{9}=1 \quad \Rightarrow a=2, \quad b=3$$

$$x=0 \Rightarrow \frac{y^2}{1}-\frac{z^2}{9}=1 \quad \Rightarrow a=1, \quad b=3$$

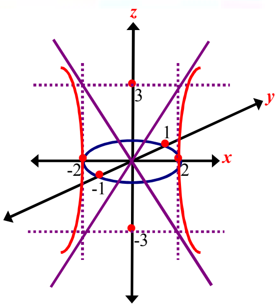
$$z=0 \Rightarrow \frac{x^2}{4}+\frac{y^2}{1}=1$$

$$\Rightarrow a=2, \quad b=1$$



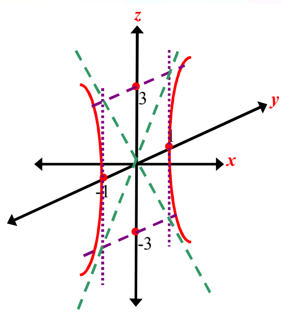
$$y=0 \Rightarrow \frac{x^2}{4}-\frac{z^2}{9}=1$$

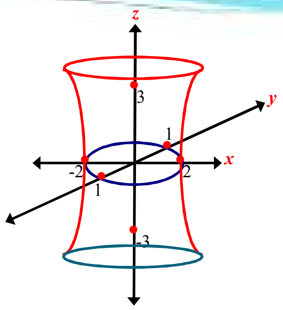
$$\Rightarrow a=2, \quad b=3$$



$$x=0 \Rightarrow \frac{y^2}{1}-\frac{z^2}{9}=1$$

$$\Rightarrow a=1, \quad b=3$$





$$\frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{9}=1$$

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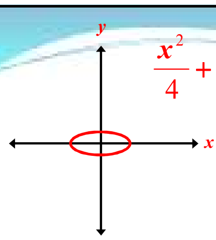
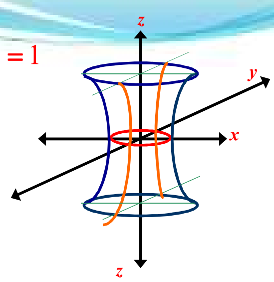
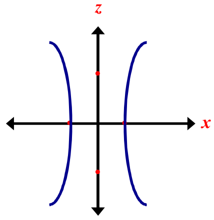
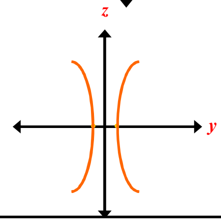
$$\frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{9}=1$$









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The Elliptic Hyperboloid of two sheets

The graph of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

or

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

or

$$\frac{-x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a, b **and** c **are positive constants is called an** ***elliptic hyperboloid of two sheets*****.**

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Example 5.7.5 **Sketch the graph of**

$$\cancel{\frac{x^2}{4}}-\frac{y^2}{16}-\frac{z^2}{9}=1$$

solution:

By definition, the graph is an ***elliptic hyperboloid of two sheets*****.**

If $x=0$**, we obtain**

$$\frac{-y^2}{16}-\frac{z^2}{9}=1$$

so that the graph of the given equation will not intersect the yz**-plane.**

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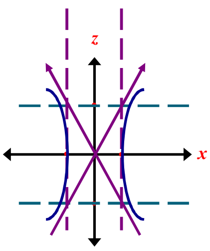
$$\frac{x^2}{4}-\cancel{\frac{y^2}{16}}-\frac{z^2}{9}=1$$

If $y=0$ **, we obtain the cross section of the graph in the** xz**-plane which is the hyperbola given by**

$$\frac{x^2}{4}-\frac{z^2}{9}=1.$$



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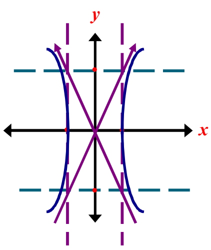
$$\frac{x^2}{4}-\frac{y^2}{16}-\cancel{\frac{z^2}{9}}=1$$

If $z=0$, we obtain the cross section of the graph in the xy-plane which is the hyperbola given by

$$\frac{x^2}{4}-\frac{y^2}{16}=1.$$



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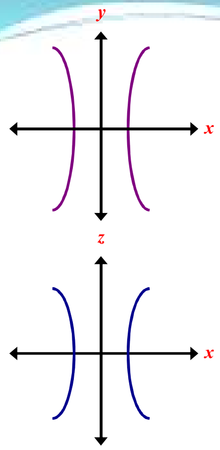
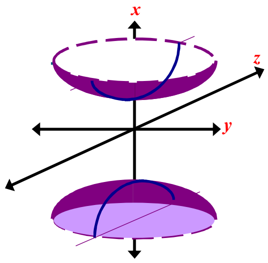
$$\frac{x^2}{4} - \frac{y^2}{16} - \frac{z^2}{9} = 1$$





The Elliptic Paraboloid

The graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

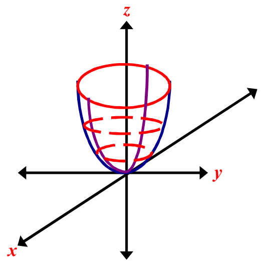
or

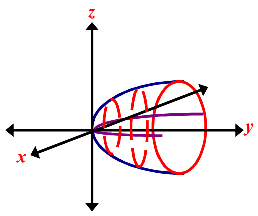
$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = \frac{y}{c}$$

or

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = \frac{x}{c}$$

where a, b and c are positive constants is called an *elliptic paraboloid*.





Example 5.7.6 Sketch the graph of

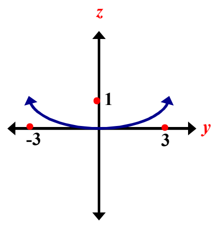
$$\cancel{\frac{x^2}{4}} + \frac{y^2}{9} = z.$$

solution:

By definition, the graph is an *elliptic paraboloid*.

If $x = 0$, we obtain the cross section of the graph in the yz-plane which is the parabola given by

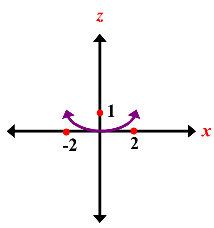
$$\frac{y^2}{9} = z$$



$$\frac{x^2}{4} + \cancel{\frac{y^2}{9}} = z$$

If $y = 0$, we obtain the cross section of the graph in the xz-plane which is the parabola given by

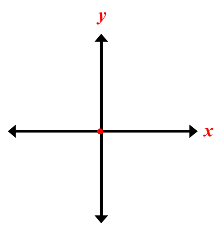
$$\frac{x^2}{4} = z.$$



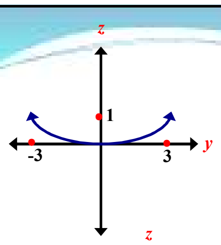
$$\frac{x^2}{4} + \frac{y^2}{9} = \cancel{z}$$

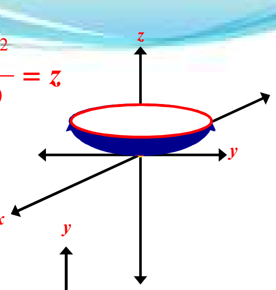
If $z = 0$, we obtain the cross section of the graph in the xy-plane which is the origin given by

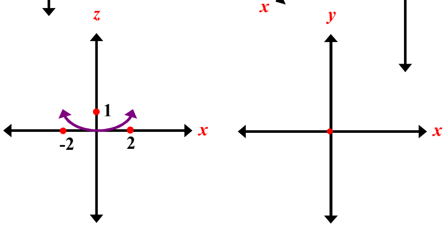
$$\frac{x^2}{4} + \frac{y^2}{9} = 0.$$



$$\frac{x^2}{4} + \frac{y^2}{9} = z$$







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The Hyperbolic Paraboloid

The graph of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

or

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = \frac{y}{c}$$

or

$$\frac{y^2}{a^2} - \frac{z^2}{b^2} = \frac{x}{c}$$

where a, b and c are positive constants is called a *hyperbolic paraboloid*.

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Example 5.7.7 Sketch the graph of

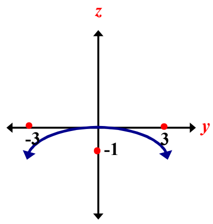
$$\frac{\cancel{x^2}}{4} - \frac{y^2}{9} = z.$$

solution:

By definition, the graph is a *hyperbolic paraboloid*.

If $x = 0$, we obtain the cross section of the graph in the *yz*-plane which is the parabola given by

$$\frac{-y^2}{9} = z.$$



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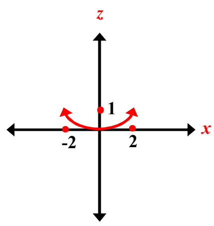
$$\frac{x^2}{4} - \frac{\cancel{y^2}}{9} = z$$

If $y = 0$, we obtain the cross section of the graph in the *xz*-plane which is the parabola given by

$$\frac{x^2}{4} = z$$



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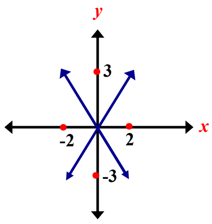
$$\frac{x^2}{4} - \frac{y^2}{9} = \cancel{z}$$

If $z = 0$, we obtain the cross section of the graph in the *xy*-plane which is the union of 2 lines given by

$$\frac{x^2}{4} - \frac{y^2}{9} = 0.$$



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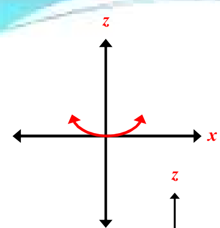
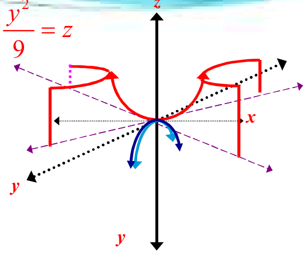
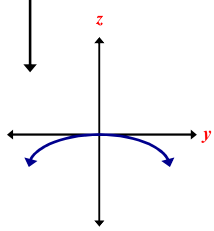
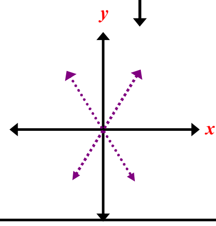
$$\frac{x^2}{4} - \frac{y^2}{9} = z$$









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The Elliptic Cone

The graph of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

or

$$\frac{-x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

where a, b and c are positive constants is called an *elliptic cone*.

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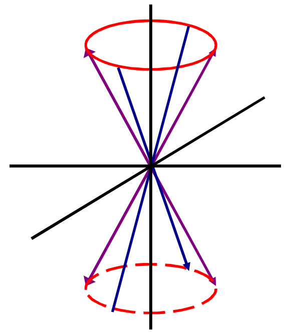
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



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Example 5.7.8 Sketch the graph of

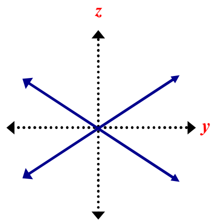
$$\cancel{\frac{x^2}{4}}+\frac{y^2}{9}-\frac{z^2}{4}=0.$$

solution:

By definition, the graph is an *elliptic cone*.

If $x=0$, we obtain the cross section of the graph in the yz-plane which is the hyperbola given by

$$\frac{y^2}{9}-\frac{z^2}{4}=0\cdot$$



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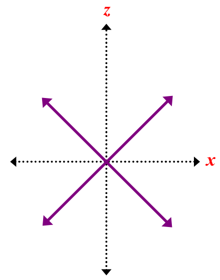
$$\frac{x^2}{4}+\cancel{\frac{y^2}{9}}-\frac{z^2}{4}=0$$

If $y=0$, we obtain the cross section of the ellipsoid in the xz-plane which is the union of 2 lines given by

$$\frac{x^2}{4}-\frac{z^2}{4}=0\cdot$$



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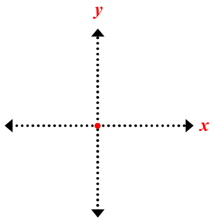
$$\frac{x^2}{4}+\frac{y^2}{9}-\cancel{\frac{z^2}{4}}=0$$

If $z=0$, we obtain the cross section of the graph in the xy-plane which is the origin given by

$$\frac{x^2}{4}+\frac{y^2}{9}=0\cdot$$



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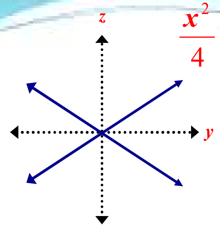
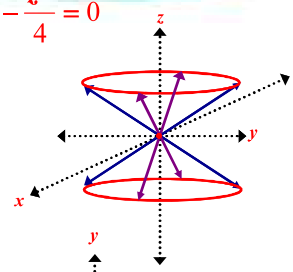
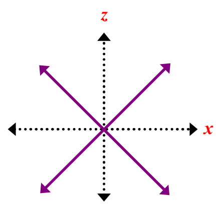
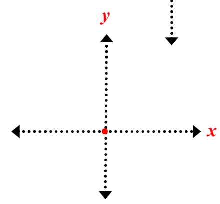
$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{4}=0$$









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