2.6

CYLINDERS and QUADRIC SURFACES

Planes in 3D

General equation of a plane:

$$ax + by + cz + d = 0$$

if a, b and c are not all zero, $\langle a, b, c \rangle$ is a normal vector to the plane

Spheres

Standard equation of a sphere:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Center: (h, k, l)

Radius: r

General equation

The graph in three-dimensional space of

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$$

is either a sphere, a point or the empty set.

Example. Identify the graph of the given equation.

1. $x^2 + 2x + y^2 - 2y + z^2 - 4z - 3 = 0$
2. $x^2 + 2x + y^2 + z^2 - 4z + 5 = 0$
3. $x^2 + y^2 - 2y + z^2 + 4z + 7 = 0$

1. $x^2 + 2x + y^2 - 2y + z^2 - 4z - 3 = 0$

Solution:

$$\left(x^2 + 2x\right) + \left(y^2 - 2y\right) + \left(z^2 - 4z\right) = 3$$

$$\Rightarrow \left(x^2 + 2x + 1\right) + \left(y^2 - 2y + 1\right) + \left(z^2 - 4z + 4\right) = 3 + 6$$

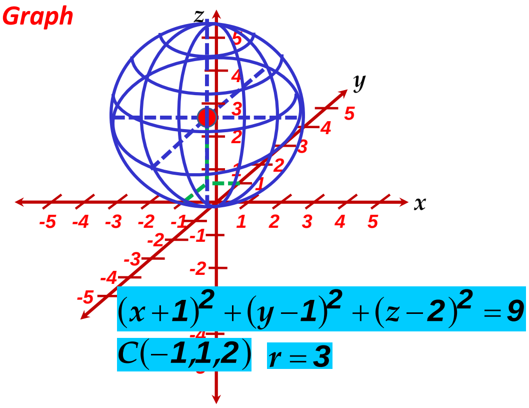
$$\Rightarrow (x+1)^2 + (y-1)^2 + (z-2)^2 = 9$$

Solution (continued)

$$(x+1)^2+(y-1)^2+(z-2)^2=9$$

is the sphere centered at $(-1,1,2)$ *of radius 3.*

Graph



2. $x^2+2x+y^2+z^2-4z+5=0$

Solution:

$$\left(x^2+2x\right)+y^2+\left(z^2-4z\right)=-5$$

$$\Rightarrow\left(x^2+2x+1\right)+y^2$$
$$+\left(z^2-4z+4\right)=-5+5$$

$$\Rightarrow(x+1)^2+y^2+(z-2)^2=0$$

Solution (continued)

$$(x+1)^2+y^2+(z-2)^2=0$$

is the point $(-1,0,2)$

3. $x^2+y^2-2y+z^2+4z+7=0$

Solution:

$$x^2+\left(y^2-2y\right)+\left(z^2+4z\right)=-7$$

$$\Rightarrow x^2+\left(y^2-2y+1\right)$$
$$+\left(z^2+4z+4\right)=-7+5$$

$$\Rightarrow x^2+(y-1)^2+(z+2)^2=-2$$

Solution (continued)

The graph of

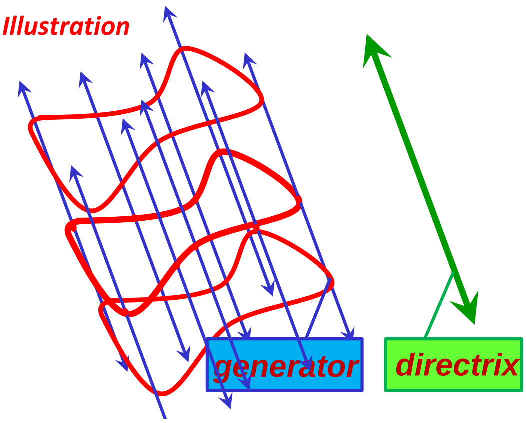
$$x^2+(y-1)^2+(z+2)^2=-2$$

is empty.

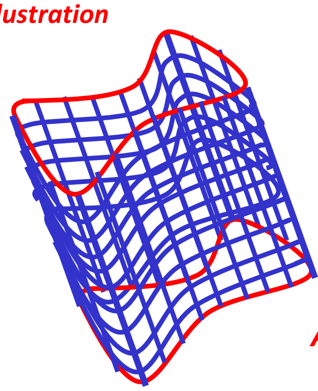
Cylinder

A cylinder is a surface generated by a line (generator) moving along a given plane curve in such a way that it is always parallel to a fixed line (directrix) not lying in the plane of the given curve.

Illustration



Illustration



A CYLINDER

Remark

In the three-dimensional space, the graph of an equation in two of the three variables x, y and z is a cylinder.

Example.

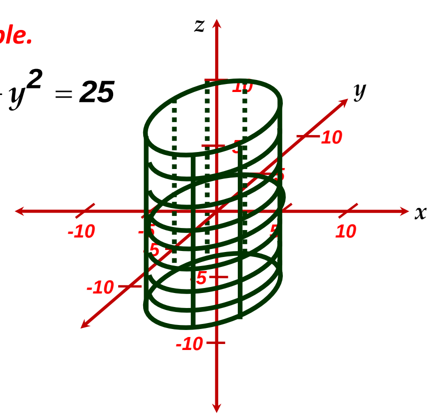
$x^2 + y^2 = 25$ is a cylinder in $\mathbb{R}^3$.

Plane curve: on the xy-plane

Directrix: z-axis

Example.

$x^2 + y^2 = 25$



Example.

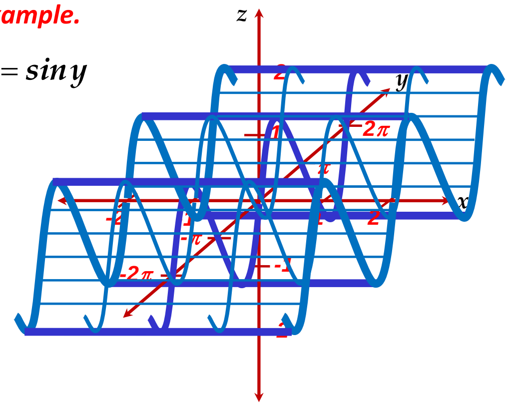
$z = \sin y$ ***is a cylinder in R³.***

Plane curve: ***on the*** yz***-plane***

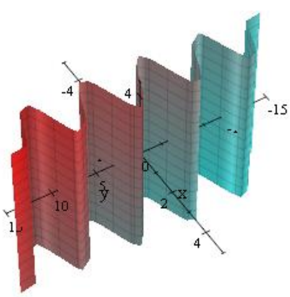
Directrix: x***-axis***

Example.

$z = \sin y$



$y = \sin x$



Example.

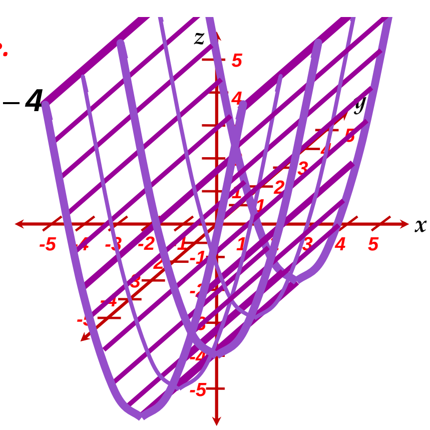
$z = x^2 - 4$ ***is a cylinder in R³.***

Plane curve: ***on the*** xz***-plane***

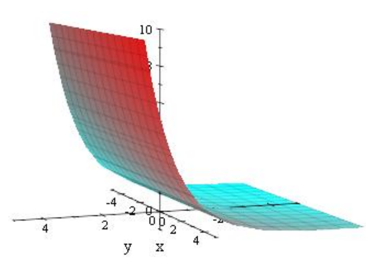
Directrix: y***-axis***

Example.

$z = x^2 - 4$



$z = e^x$



Quadric surfaces

The graph of the second-degree equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

is a quadric surface.

Restrictions

Equations that will be considered:

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + J = 0$$

These are expressed in standard forms.

Graphs

To graph quadric surfaces, obtain traces on the following:

xy-plane $z = 0$

yz-plane $x = 0$

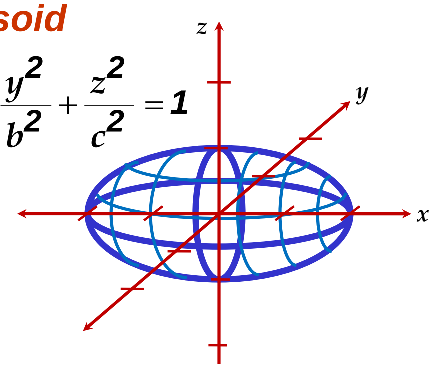
xz-plane $y = 0$

Level curves (cross-sections) on particular values of z can also be used.

Standard forms

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Surface # 1.

$$\frac{x^2}{9} + \frac{y^2}{25} + z^2 = 1$$

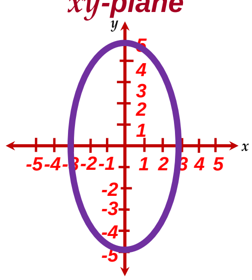
Traces

xy-plane: $z = 0$ $\quad \frac{x^2}{9} + \frac{y^2}{25} = 1$

yz-plane: $x = 0$ $\quad \frac{y^2}{25} + z^2 = 1$

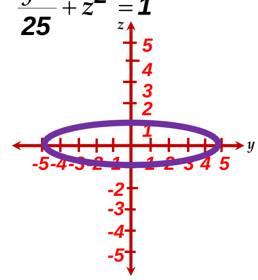
xz-plane: $y = 0$ $\quad \frac{x^2}{9} + z^2 = 1$

xy-plane

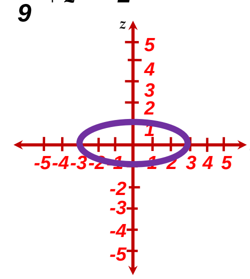


Surface # 1.

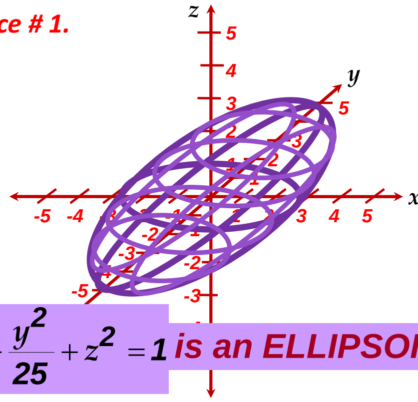
yz-plane: $\frac{y^2}{25} + z^2 = 1$



xz-plane: $\frac{x^2}{9} + z^2 = 1$



Surface # 1.



$\frac{x^2}{9}+\frac{y^2}{25}+z^2=1$ is an ELLIPSOID.

Standard forms

Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

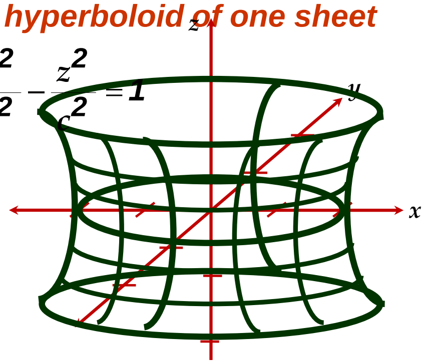
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Standard forms

Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Surface # 2.

$$-\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}=1$$

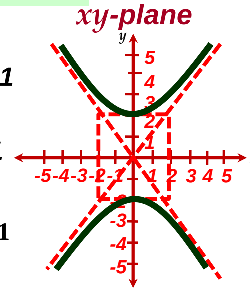
Traces

xy-plane: $z=0$ $\quad -\frac{x^2}{4}+\frac{y^2}{4}=1$

yz-plane: $x=0$ $\quad \frac{y^2}{4}+\frac{z^2}{16}=1$

xz-plane: $y=0$ $\quad -\frac{x^2}{4}+\frac{z^2}{16}=1$

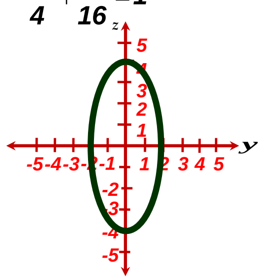
xy-plane



Surface # 2.

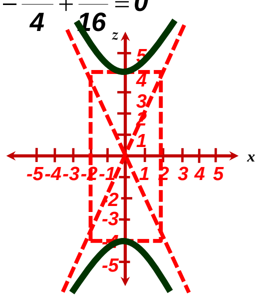
yz-plane:

$$\frac{y^2}{4}+\frac{z^2}{16}=1$$

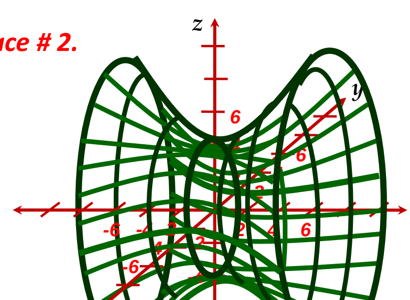


xz-plane:

$$-\frac{x^2}{4}+\frac{z^2}{16}=0$$



Surface # 2.



$-\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}=1$ is an ELLIPTIC HYPERBOLOID of one sheet.

Standard forms

Elliptic hyperboloid of two sheets

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

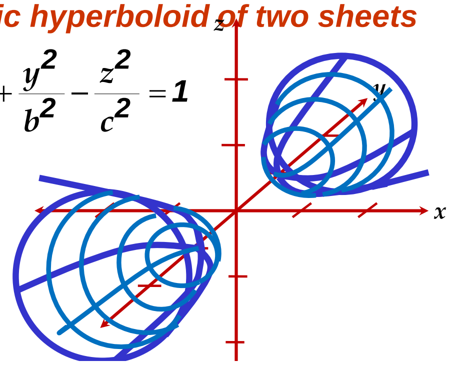
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Standard forms

Elliptic hyperboloid of two sheets

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Surface # 3.

$$x^2-\frac{y^2}{4}-\frac{z^2}{9}=1$$

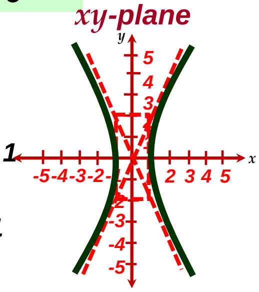
Traces

xy-plane: $z=0$ $\quad x^2-\frac{y^2}{4}=1$

yz-plane: $x=0$ $\quad -\frac{y^2}{4}-\frac{z^2}{9}=1$

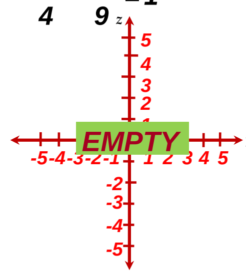
xz-plane: $y=0$ $\quad x^2-\frac{z^2}{9}=1$

xy-plane

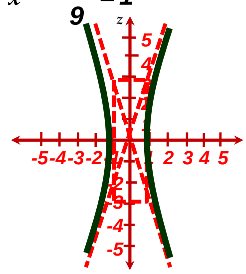


Surface # 3.

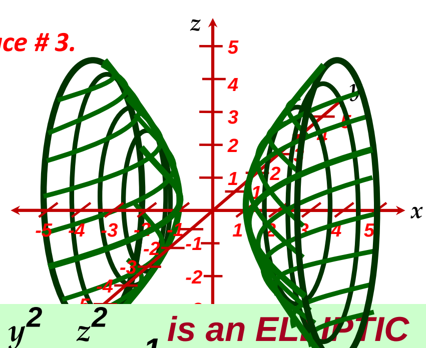
yz-plane: $-\frac{y^2}{4}-\frac{z^2}{9}=1$



xz-plane: $x^2-\frac{z^2}{9}=1$



Surface # 3.



$x^2-\frac{y^2}{4}-\frac{z^2}{9}=1$ is an ELLIPTIC HYPERBOLOID of two sheets.

Standard forms

Elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

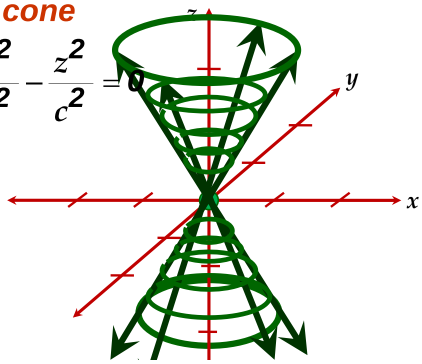
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Standard forms

Elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



Surface # 4.

$$-x^2+y^2+\frac{z^2}{9}=0$$

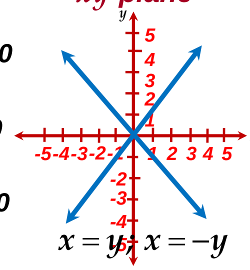
Traces

xy-plane: $-x^2+y^2=0$, $z=0$

yz-plane: $y^2+\frac{z^2}{9}=0$, $x=0$

xz-plane: $-x^2+\frac{z^2}{9}=0$, $y=0$

xy-plane

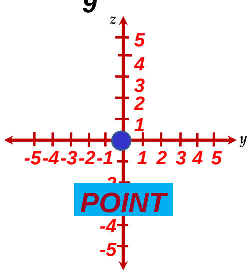


$x=y;\ x=-y$

Surface # 4.

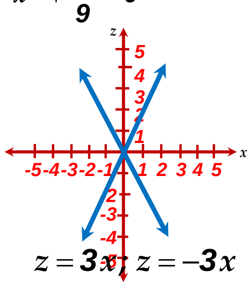
yz-plane:

$$y^2+\frac{z^2}{9}=0$$



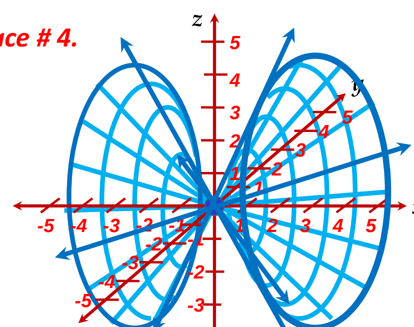
xz-plane:

$$-x^2+\frac{z^2}{9}=0$$



$z=3x;\ z=-3x$

Surface # 4.



$x^2-y^2+\frac{z^2}{9}=0$ is an ELLIPTIC CONE.

Standard forms

Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$$

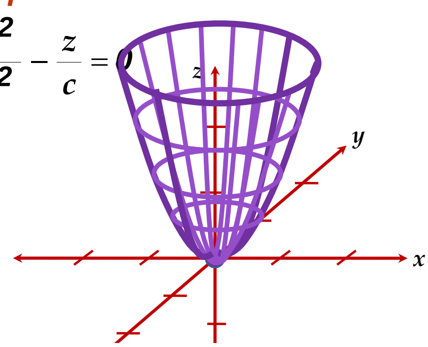
$$\frac{x^2}{a^2}-\frac{y}{b}+\frac{z^2}{c^2}=0$$

$$-\frac{x}{a}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Standard forms

Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$$



Example: Sketch the graph of

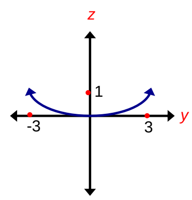
$$\cancel{\frac{x^2}{4}} + \frac{y^2}{9} = z.$$

solution:

By definition, the graph is an *elliptic paraboloid*.

If $x = 0$, we obtain the cross section of the graph in the *yz*-plane which is the parabola given by

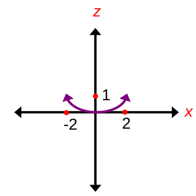
$$\frac{y^2}{9} = z$$



$$\frac{x^2}{4} + \cancel{\frac{y^2}{9}} = z$$

If $y = 0$, we obtain the cross section of the graph in the *xz*-plane which is the parabola given by

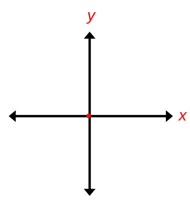
$$\frac{x^2}{4} = z \ .$$



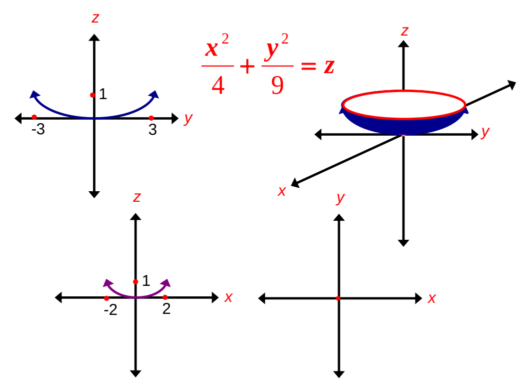
$$\frac{x^2}{4} + \frac{y^2}{9} = \cancel{z}$$

If $z = 0$, we obtain the cross section of the graph in the *xy*-plane which is the origin given by

$$\frac{x^2}{4} + \frac{y^2}{9} = 0 \cdot$$



$$\frac{x^2}{4} + \frac{y^2}{9} = z$$



Standard forms

Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 1$$

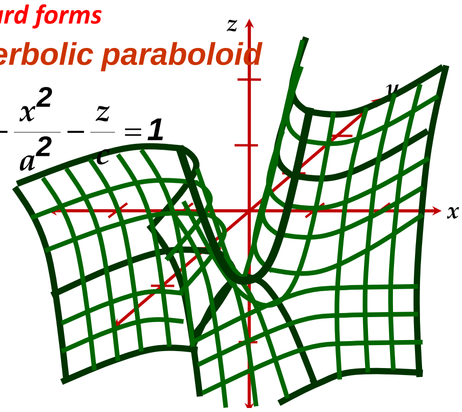
$$\frac{x^2}{a^2} - \frac{y}{b} - \frac{z^2}{c^2} = 1$$

$$-\frac{x}{a} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Standard forms

Hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} - \frac{z}{c} = 1$$



Example. Sketch the graph of

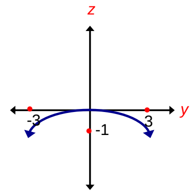
$$\frac{\cancel{x^2}}{4} - \frac{y^2}{9} = z\,.$$

solution:

By definition, the graph is a *hyperbolic paraboloid*.

If $x = 0$, we obtain the cross section of the graph in the *yz*-plane which is the parabola given by

$$\frac{-y^2}{9} = z\ .$$



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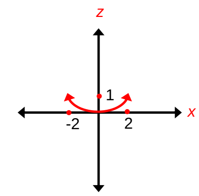
$$\frac{x^2}{4} - \frac{\cancel{y^2}}{9} = z$$

If $y = 0$, we obtain the cross section of the graph in the *xz*-plane which is the parabola given by

$$\frac{x^2}{4} = z\,.$$



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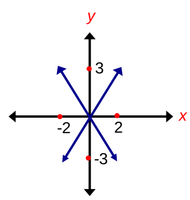
$$\frac{x^2}{4} - \frac{y^2}{9} = \cancel{z}$$

If $z = 0$, we obtain the cross section of the graph in the *xy*-plane which is the union of 2 lines given by

$$\frac{x^2}{4} - \frac{y^2}{9} = 0.$$



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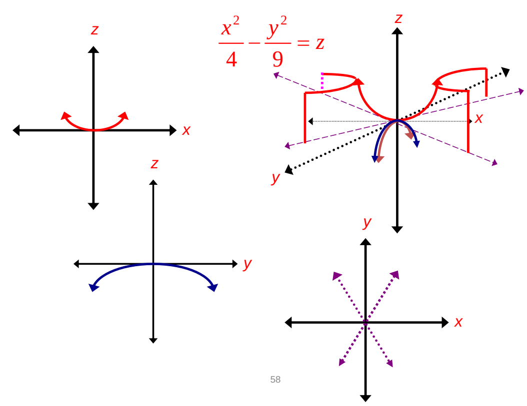

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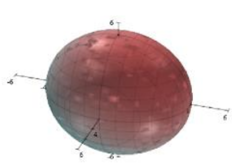
SUMMARY

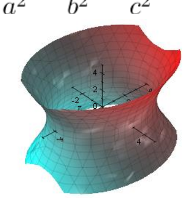
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



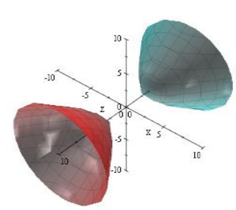
Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



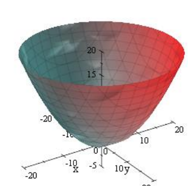
Elliptic hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



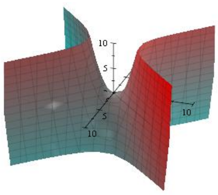
Elliptic paraboloid

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$



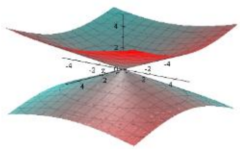
Hyperbolic paraboloid

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$



Elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



END