The method of Lagrange Multipliers

Chapter 3 Section 6

Method of Lagrange Multipliers

- Developed by Joseph Louis Lagrange in 1755
- maxima and minima of a function subject to constraints
- Important in economics, and engineering

Theorem.

Suppose f and g are functions of x and y with continuous first partial derivatives.

If f has a relative extremum value at the point $\left(x_0 y_0\right)$ subject to the constraint $g\left(x, y\right)=0$ and $\nabla g\left(x, y\right)\neq\vec{0}$ then there is a constant λ such that

$$\nabla f\left(x_0, y_0\right)-\lambda\nabla g\left(x_0, y_0\right)=0$$

HOW TO FIND THE RELATIVE EXTREMA OF A FUNCTION f OF x AND y SUBJECT TO THE CONSTRAINT $g(x,y)=0$:

1. Form the auxiliary function F of three variables x, y, and λ where

$$F(x,y,\lambda)=f(x,y)-\lambda g(x,y)$$

2. Set the first derivatives of F to zero and obtain the system of equations:

$$f_x(x,y)=\lambda g_x(x,y)$$

$$f_y(x,y)=\lambda g_y(x,y)$$

$$g(x,y)=0$$

3. Find the critical points of F by solving the system of equations in STEP2.

4. The first 2 coordinates of a critical point F will give the x and y values of the desired relative extrema.

NOTE: This method may be extended to a function of more than 2 variables.

Example.

Use Lagrange multipliers to find the greatest and smallest values that the function $f(x, y) = xy$ on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solution. $f(x, y) = xy \qquad g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$

$$f_x(x, y) = y \qquad\qquad g_x(x, y) = \frac{x}{4}$$

$$\Rightarrow \quad y = \lambda \frac{x}{4}$$

$$f(x,y)=xy \qquad g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1$$

$$\boxed{f_y(x,y)=\ x \qquad g_y(x,y)=\ y \quad \Rightarrow \quad x=\lambda y}$$

$$\boxed{g(x,y)=\ \frac{x^2}{8}+\frac{y^2}{2}-1=0}$$

$$\begin{cases} 4y=\lambda x \\ x=\lambda y \\ \dfrac{x^2}{8}+\dfrac{y^2}{2}=1 \end{cases}$$

$$\begin{cases} 4y = \lambda x & \textbf{(1)} \quad \Longrightarrow \quad \dfrac{4y}{x} = \lambda \quad \textbf{(4)} \\ x = \lambda y & \textbf{(2)} \\ \dfrac{x^2}{8} + \dfrac{y^2}{2} = 1 & \textbf{(3)} \end{cases}$$

(4) in **(2)**: $x = \dfrac{4y}{x} y \Longrightarrow x^2 = 4y^2$ **(5)**

(5) in **(3)**: $\dfrac{4y^2}{8} + \dfrac{y^2}{2} = 1 \Longrightarrow y^2 = 1 \Longrightarrow y = \pm 1$

Using **(5)**: $x = \pm 2$

Thus, the function f takes on its extreme values on the given ellipse at the four points:

$$(2,1) \qquad (-2,1)$$

$$(-2,-1) \quad (2,-1)$$

The extreme values are:

$$2 \qquad\qquad -2$$

Example.

Use Lagrange multipliers to find the point closest to the origin on the plane $2x + y - z - 5 = 0.$

Solution.

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$g(x,y,z) = 2x + y - z - 5$$

$$f_x(x,y,z) = 2x \qquad g_x(x,y,z) = 2$$

$$\Rightarrow \qquad 2x = 2\lambda$$

$$f(x,y,z)=x^2+y^2+z^2$$

$$g(x,y,z)=2x+y-z-5$$

$$\boxed{f_y(x,y,z)=2y \quad g_y(x,y,z)=1 \quad \Rightarrow \quad 2y=\lambda}$$

$$\boxed{f_z(x,y,z)=2z \quad g_z(x,y,z)=-1 \quad \Rightarrow \quad 2z=-\lambda}$$

$$\begin{cases} x=\lambda & z=\dfrac{-\lambda}{2} \\ y=\dfrac{\lambda}{2} & 2x+y-z-5=0 \end{cases}$$

$$\begin{cases} x = \lambda \quad \textbf{(1)} & z = \dfrac{-\lambda}{2} \quad \textbf{(3)} \\ y = \dfrac{\lambda}{2} \quad \textbf{(2)} & 2x + y - z - 5 = 0 \quad \textbf{(4)} \end{cases}$$

(1), (2), (3) in **(4)**: $\Rightarrow 2\lambda + \dfrac{\lambda}{2} + \dfrac{\lambda}{2} - 5 = 0$

$$\Rightarrow \lambda = \frac{5}{3}$$

$$x = \frac{5}{3} \qquad y = \frac{5}{6} \qquad z = \frac{-5}{6}$$

Thus, the point closest to the origin and is contained on the plane $2x + y - z - 5 = 0$ is the point

$$\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$$

The minimum distance from the origin is

$$\sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2} = \frac{5\sqrt{6}}{6}$$

Example.

Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi \ cm^3$.

Solution.

$$f(r,h) = 2\pi rh + 2\pi r^2 \qquad g(r,h) = \pi r^2 h - 16\pi$$

$$f_r(r,h) = 2\pi h + 4\pi r$$

$$g_r(r,h) = 2\pi rh$$

$$\Rightarrow \ h + 2r = \lambda rh$$

$$f(r,h) = 2\pi rh + 2\pi r^2 \qquad g(r,h) = \pi r^2 h - 16\pi$$

$$\boxed{f_h(r,h) = 2\pi r \qquad g_h(r,h) = \pi r^2 \Rightarrow 2 = r\lambda}$$

$$\boxed{g(r,h) = \pi r^2 h - 16\pi = 0 \Rightarrow r^2 h = 16}$$

$$\begin{cases} h + 2r = \lambda rh \\ 2 = r\lambda \\ r^2 h = 16 \end{cases}$$

$$\begin{cases} h+2r=\lambda rh & \textbf{(1)} \\ 2=r\lambda & \textbf{(2)} \\ r^2h=16 & \textbf{(3)} \end{cases}$$

$$\Rightarrow h=\frac{-2r}{1-\lambda r} \quad \textbf{(4)}$$

$$\Rightarrow \frac{2}{\lambda}=r \quad \textbf{(5)}$$

(5) in **(4)**: $h=\dfrac{-2\left(\dfrac{2}{\lambda}\right)}{1-\lambda\left(\dfrac{2}{\lambda}\right)}=\dfrac{4}{\lambda}$ **(6)**

(5), (6) in **(3)**: $\left(\dfrac{2}{\lambda}\right)^2\dfrac{4}{\lambda}=16 \Rightarrow 1=\lambda$

Thus, $r=2$ and $h=4$

are the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi \ cm^3$.

The minimum surface area is

$$f(2,4) = 2\pi(2)(4) + 2\pi(2)^2$$

$$= 24\pi$$

END