Chapter 2

DIFFERENTIAL CALCULUS OF FUNCTIONS OF MORE THAN ONE VARIABLE

Objectives:

At the end of this chapter, you should be able to

- Find the domain and range,
- Perform operations,
- Sketch the graph,
- Find the limit,
- Obtain the partial derivative,

of a function of two or more variables.

RECALL:

- **Cartesian Coordinate System**

$$R^2 = \left\{ \left(x, y \right) \middle| x, y \in R \right\}$$

- **Distance between two points**

$$J\left(x_1, y_1\right), P\left(x_2, y_2\right) \in R^2$$

$$d\left(J, P\right) = \left|\overline{JP}\right|$$

$$= \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}$$

Definition.

The set of all ordered *n*-tuples of real numbers is called the ***n-dimensional number space*** .

NOTATION: R^n

Point: $\left(x_1, x_2, x_3, \ldots, x_n\right)$

- ***n*-dimensional number space**

$$R^n = \left\{ \left(x_1, x_2, \ldots, x_n \right) \middle| x_i \in R \right\}$$

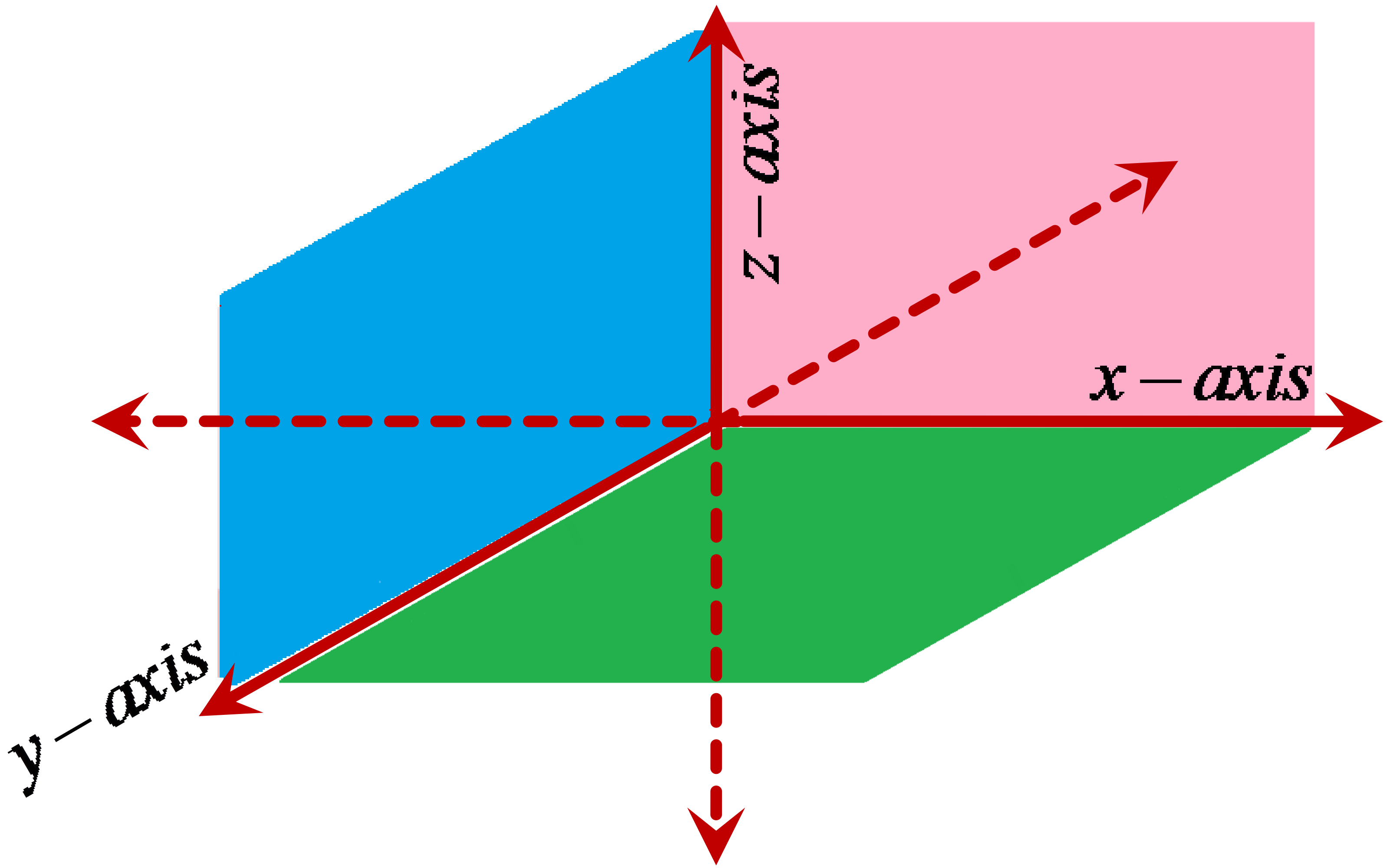
- **Distance between two points**

$$J\left(x_1, x_2, \ldots, x_n\right),\ P\left(y_1, y_2, \ldots, y_n\right) \in R^n$$

$$d\left(J, P\right) = \left|\overline{JP}\right| =$$

$$\sqrt{\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2 + \ldots + \left(x_n - y_n\right)^2}$$

In particular, when $n = 3$



RECALL:

A ***function*** is a set of ordered pairs, such that no two distinct ordered pairs have the same first element.

Definition

A ***function of n variables*** is a set of ordered pairs of the form (P, w), such that no two distinct ordered pairs have the same first element.

REMEMBER: $\begin{matrix} P \in R^n \\ w \in R \end{matrix}, f(P) = w$

Definition

If $f\left(P\right) = w$ **, then**

DOMAIN:

the set of all admissible points P

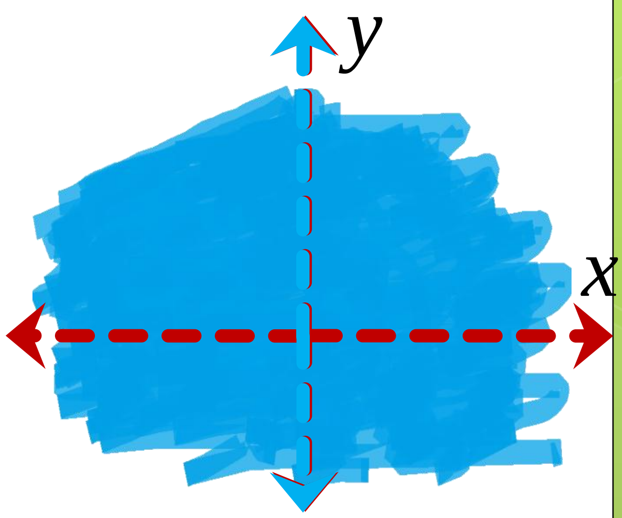
RANGE:

the set of all resulting values of w

Exercise

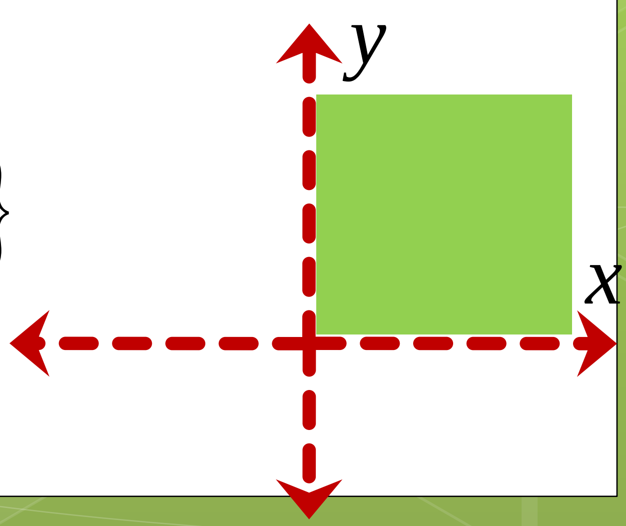
1. $f(x,y) = \dfrac{x}{y}$

DOMAIN: $\{(x,y) \mid y \neq 0\}$



2. $f(x,y) = \ln x + \ln y$

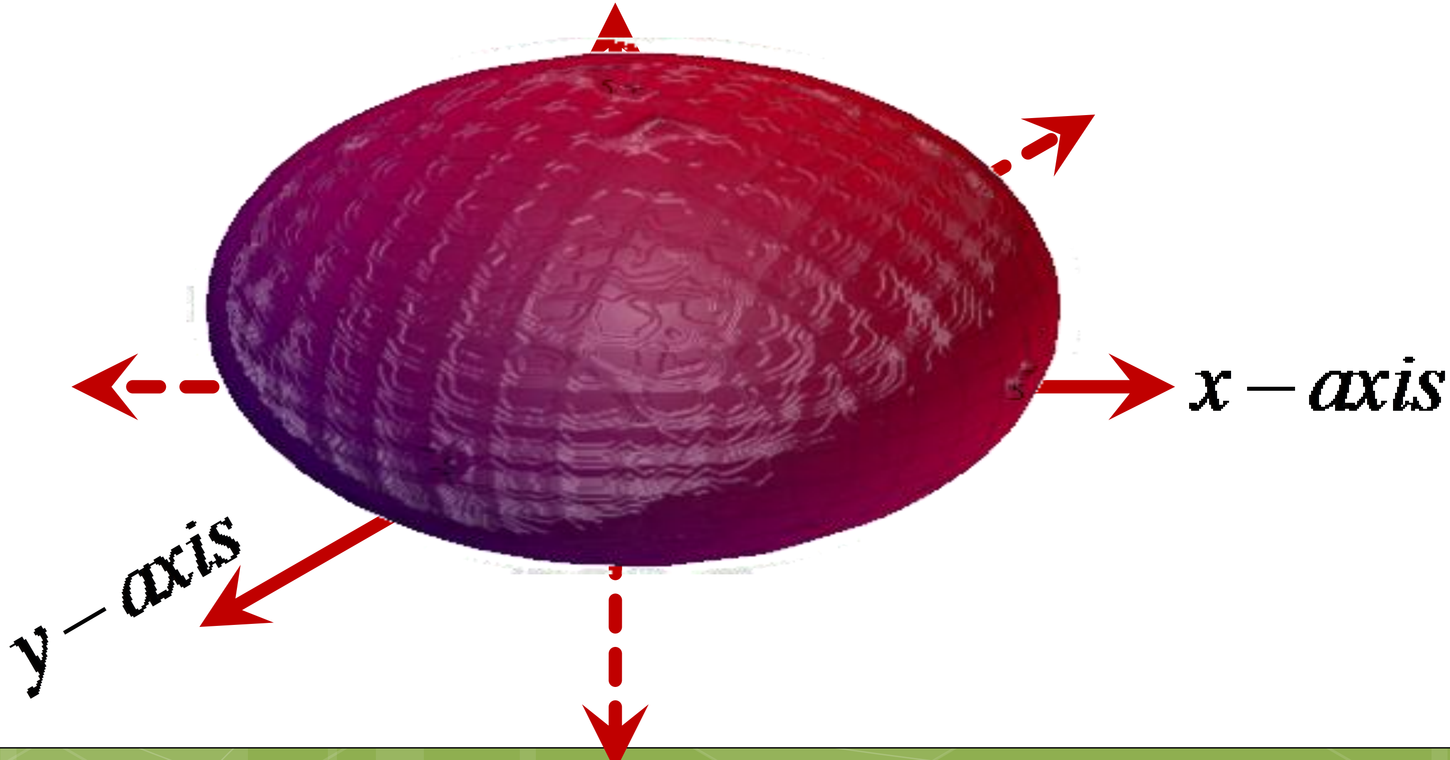
DOMAIN: $\{(x,y) \mid x, y > 0\}$



Exercise

3. $f(x,y,z)=\sqrt{25-x^2-y^2-z^2}$

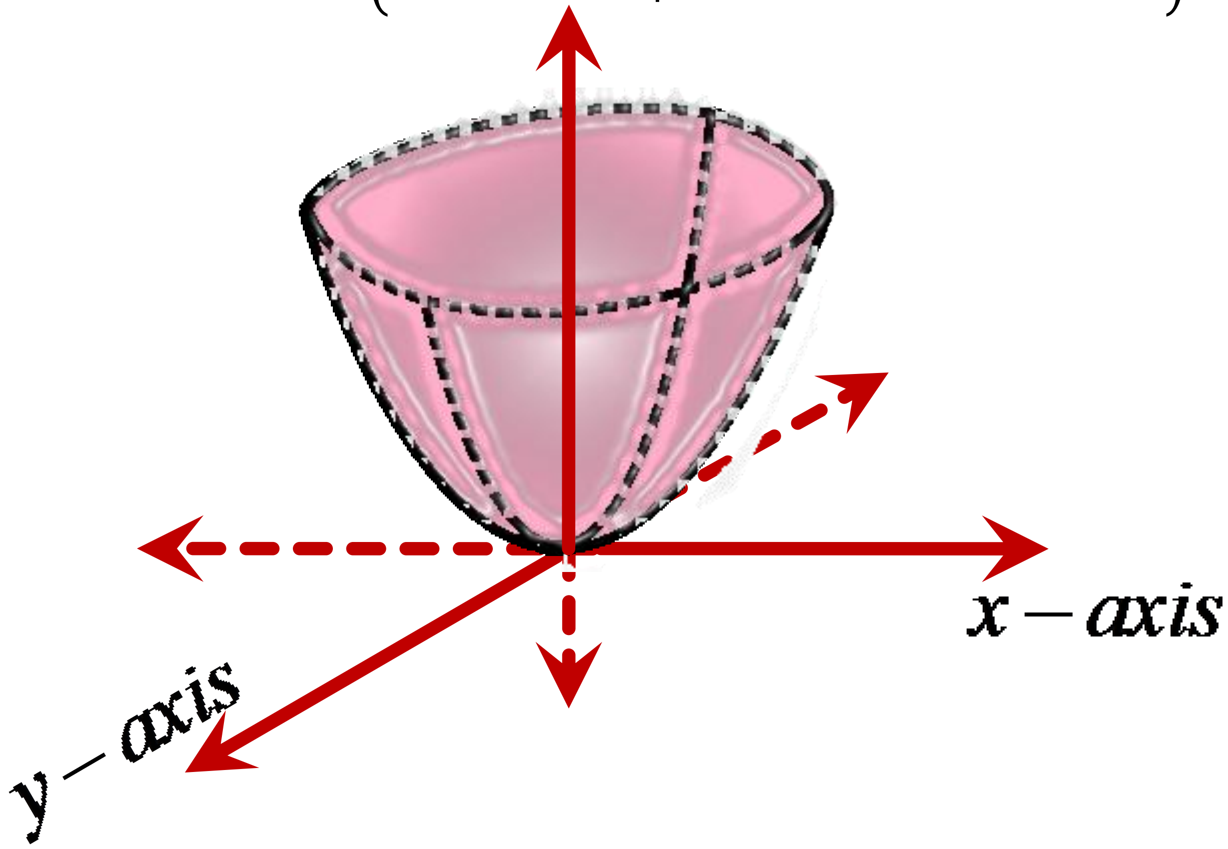
DOMAIN: $\left\{(x,y,z)\,\middle|\,x^2+y^2+z^2\le 25\right\}$



Exercise

4. $f(x,y,z)=\ln\left(z-x^2-y^2\right)$

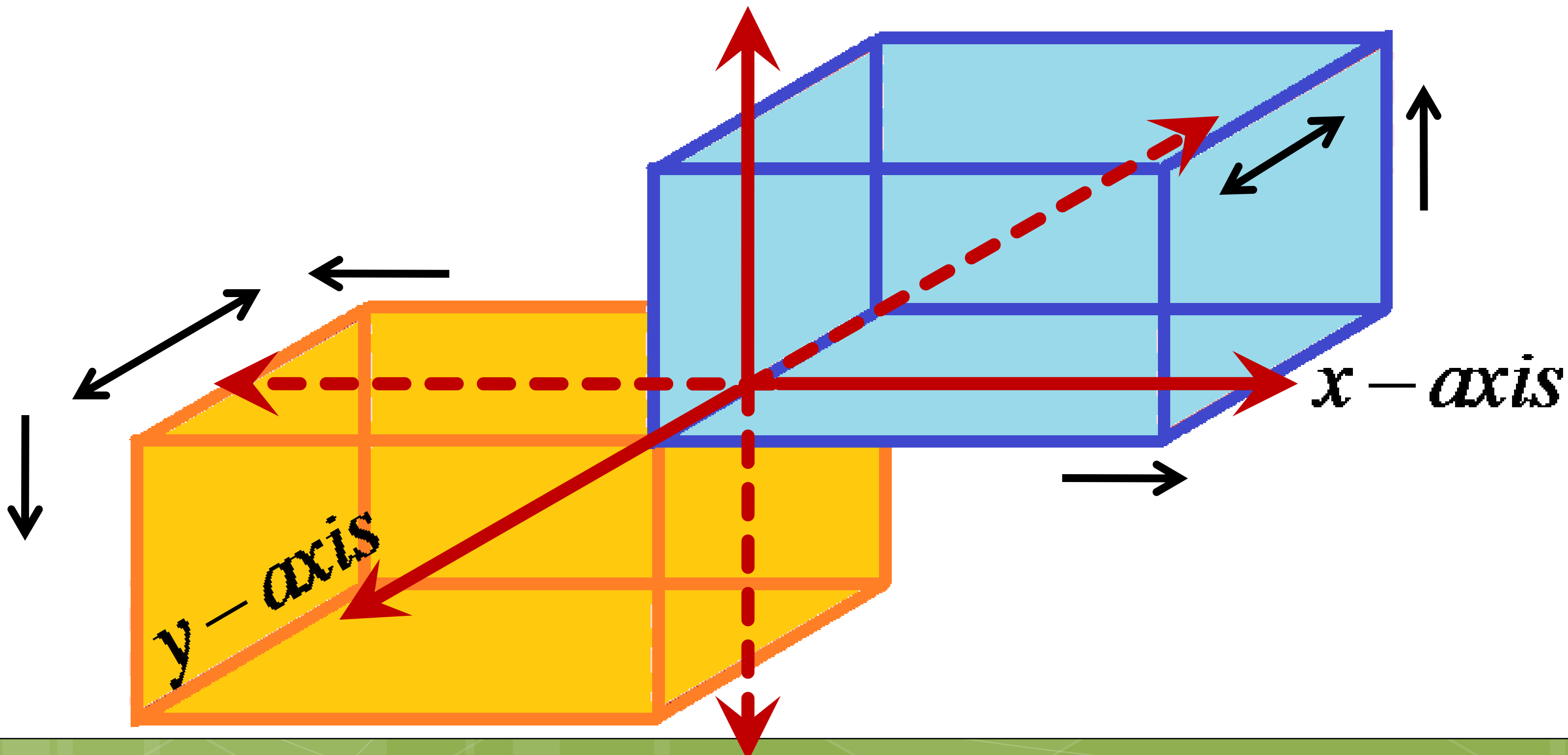
DOMAIN: $\left\{(x,y,z)\middle| z-x^2-y^2>0\right\}$



Exercise

5. $f\left(x,y,z\right)=y\sqrt{xz}$

DOMAIN: $\left\{\left(x,y,z\right)\middle| xz\ge 0\right\}$



Exercise

Let $f\left(x,y\right)=e^{y}\cos x$

$g\left(x,y,z\right)=\ln\left(xy-z^{2}\right)$

1. $f\left(\frac{\pi}{3},0\right)=e^{0}\cos\frac{\pi}{3}=1\cdot\frac{1}{2}=\frac{1}{2}$

2. $g\left(1,1,\frac{1}{4}\right)=\ln\left(1-\frac{1}{4^{2}}\right)=\ln\frac{15}{16}$

$$=\ln 15-\ln 16$$

OPERATIONS on FUNCTIONS

Recall

- The ***composite function*** $(f \circ g)(x)$

$$(f \circ g)(x) = f\left(g(x)\right)$$

DOMAIN:

$$\left\{ x \middle| x \in D_g \cap g(x) \in D_f \right\}$$

Exercise

- Let $f(t) = \sin t$

$$g(x) = \sqrt{x-1}$$

$$(f \circ g)(x) = f(g(x)) = \sin\sqrt{x-1}$$

DOMAIN: $\left\{x \middle| x \in D_g \cap g(x) \in D_f\right\}$

$$= \left\{x \middle| x \in D_g \cap g(x) \in R\right\}$$

$$= \left\{x \middle| x \in D_g\right\} = D_g = [1, +\infty)$$

Definition

If f is a function of a single variable and g is a function of n variables, the ***composite function*** $(f \circ g)$ is the function of n variables defined by

$$(f \circ g)(x_1, x_2, \ldots, x_n) = f\left(g\left(x_1, x_2, \ldots, x_n\right)\right)$$

DOMAIN:

$$\left\{P\left(x_1, x_2, \ldots, x_n\right) \middle| P \in D_g \cap g(P) \in D_f\right\}$$

Exercise

- **Let** $f(t) = e^t$

$$g(x, y, z) = \ln\left(xy - z^2\right)$$

$$f \circ g$$

DOMAIN: $\left\{P(x, y, z) \middle| P \in D_g \cap g(P) \in D_f\right\}$

$$= \left\{P(x, y, z) \middle| P \in D_g \cap g(P) \in R\right\}$$

$$= \left\{P(x, y, z) \middle| P \in D_g\right\} = D_g$$

$$= \left\{(x, y, z) \middle| xy - z^2 > 0\right\}$$

Exercise

- **Let** $f(t) = e^{t}$
  $g(x,y,z) = \ln\left(xy - z^{2}\right)$

$f \circ g$

$$(f \circ g)(x,y,z) = f\left(g(x,y,z)\right)$$

$$= e^{\ln\left(xy - z^{2}\right)}$$

$$= xy - z^{2}$$

Exercise

Determine the domain and the range of each function.

1. $f(x,y)=\sin\left(ye^{-x}\right)$

2. $g(x,y)=\dfrac{1}{\sqrt{x^2+y^2-1}}$

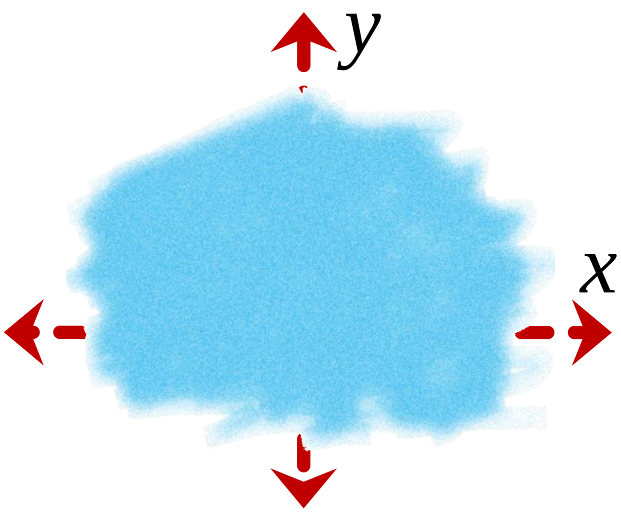
3. $h(x,y,z)=\sqrt{9-x^2-y^2}$

Exercise

1. $f(x,y)=\sin\left(ye^{-x}\right)$

Domain: R^2

Range: $[-1,1]$

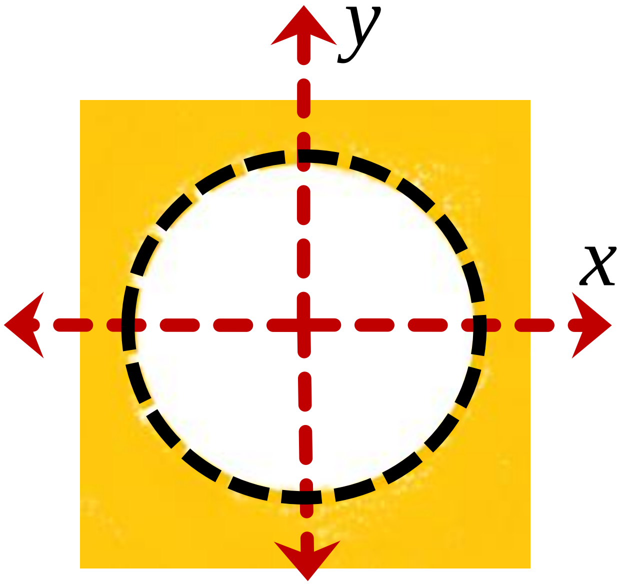


Exercise

2. $g(x,y)=\dfrac{1}{\sqrt{x^2+y^2-1}}$

Domain: $\left\{(x,y)\,\middle|\,x^2+y^2>1\right\}$

Range: $(0,+\infty)$

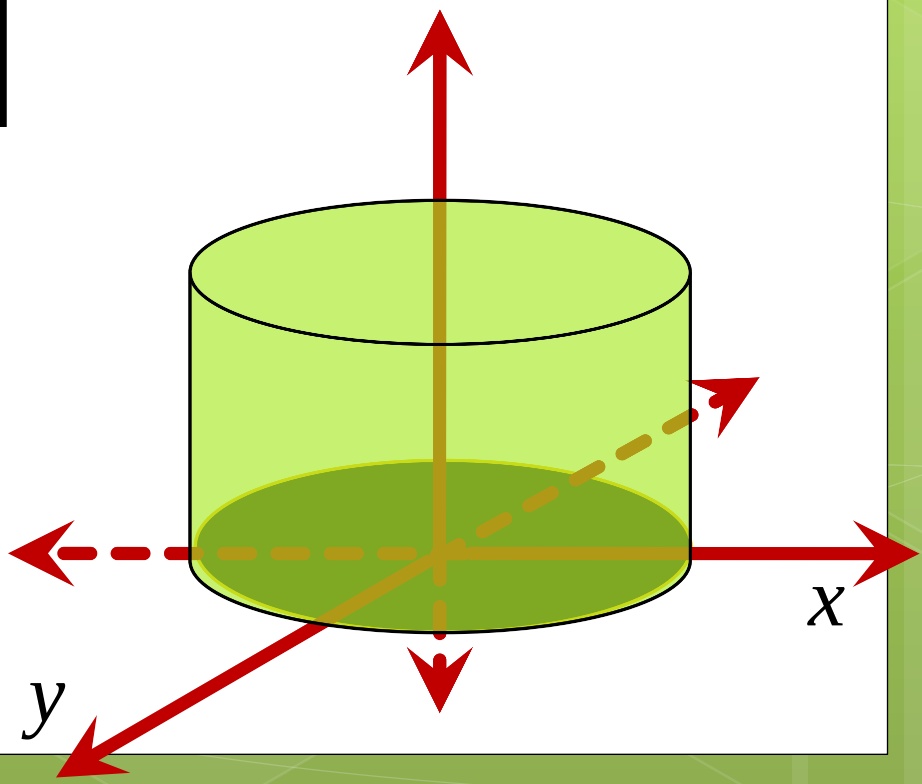


Exercise

3. $h(x,y,z)=\sqrt{9-x^2-y^2}$

Domain: $\left\{(x,y,z)\,\middle|\,x^2+y^2\le 9\right\}$

Range: $[0,3]$



GRAPHS of FUNCTIONS

Definition

If f is a function of n variables, then the **graph** of f is the set of all points

$$\left(x_1, x_2, \ldots, x_n, w\right)$$

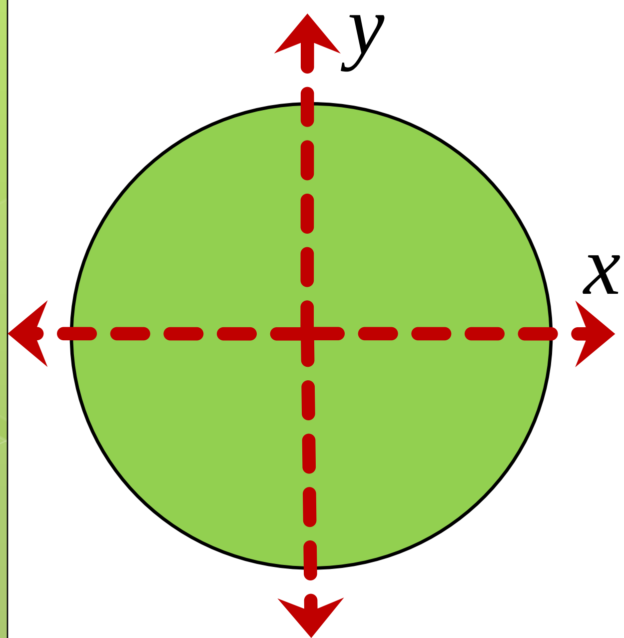
in R^{n+1} for which $\left(x_1, x_2, \ldots, x_n\right) \in D_f$

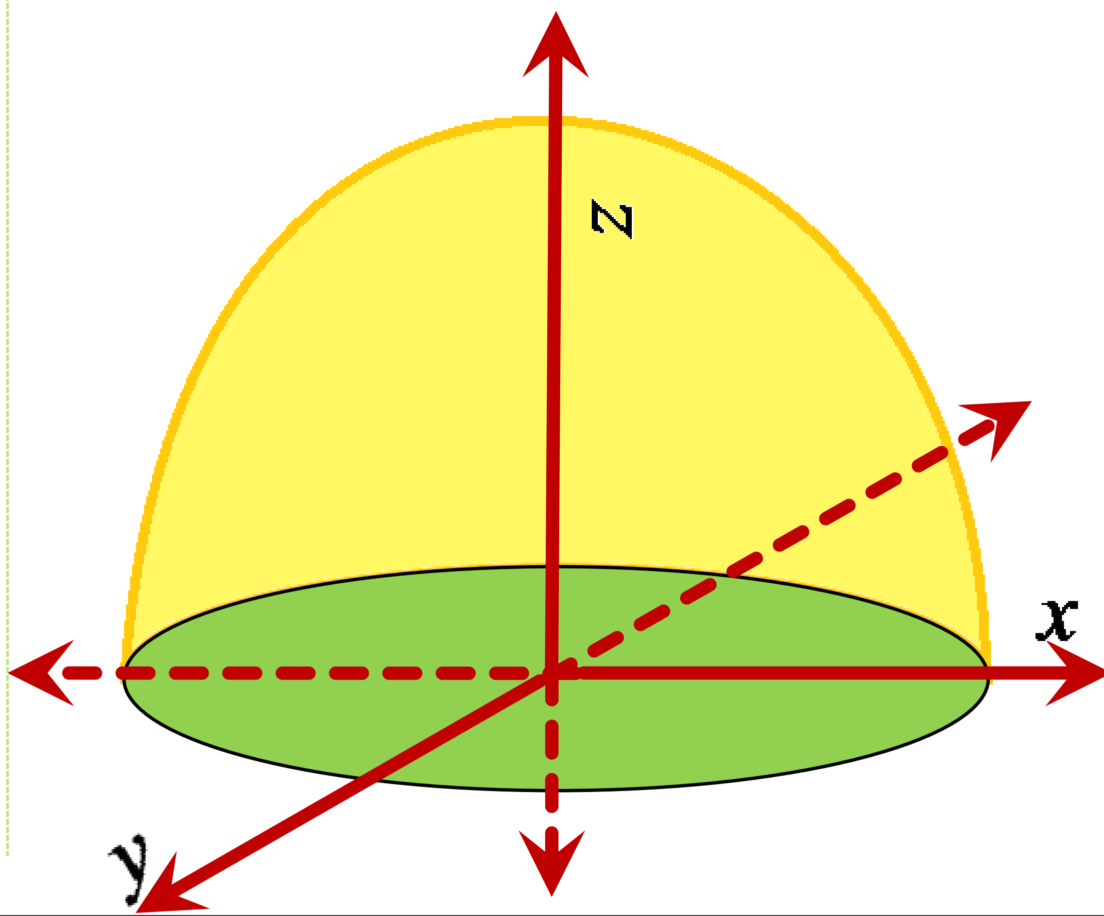
and $w = f\left(x_1, x_2, \ldots, x_n\right)$.

Exercise

$$f(x,y)=\sqrt{25-x^2-y^2}$$

$Domain:$





Definition

The set of all points in R^2 where $f(x, y)$ has a constant value k is called a ***level curve of f***.

The set of all points in R^3 where $f(x, y, z)$ has a constant value c is called a ***level surface of f***.

Defintion

A set of level curves or level surfaces obtained by considering different values of k is called a ***contour map***.

Exercise

Identify and sketch a contour map of the given function.

1. $f(x,y)=\dfrac{y}{x^2}$

Range: R

$$f(x,y)=k \Rightarrow \frac{y}{x^2}=k \Rightarrow y=k\,x^2$$

That is, the level curve of *f* is a

parabola if $k \neq 0$,

line if $k=0.$

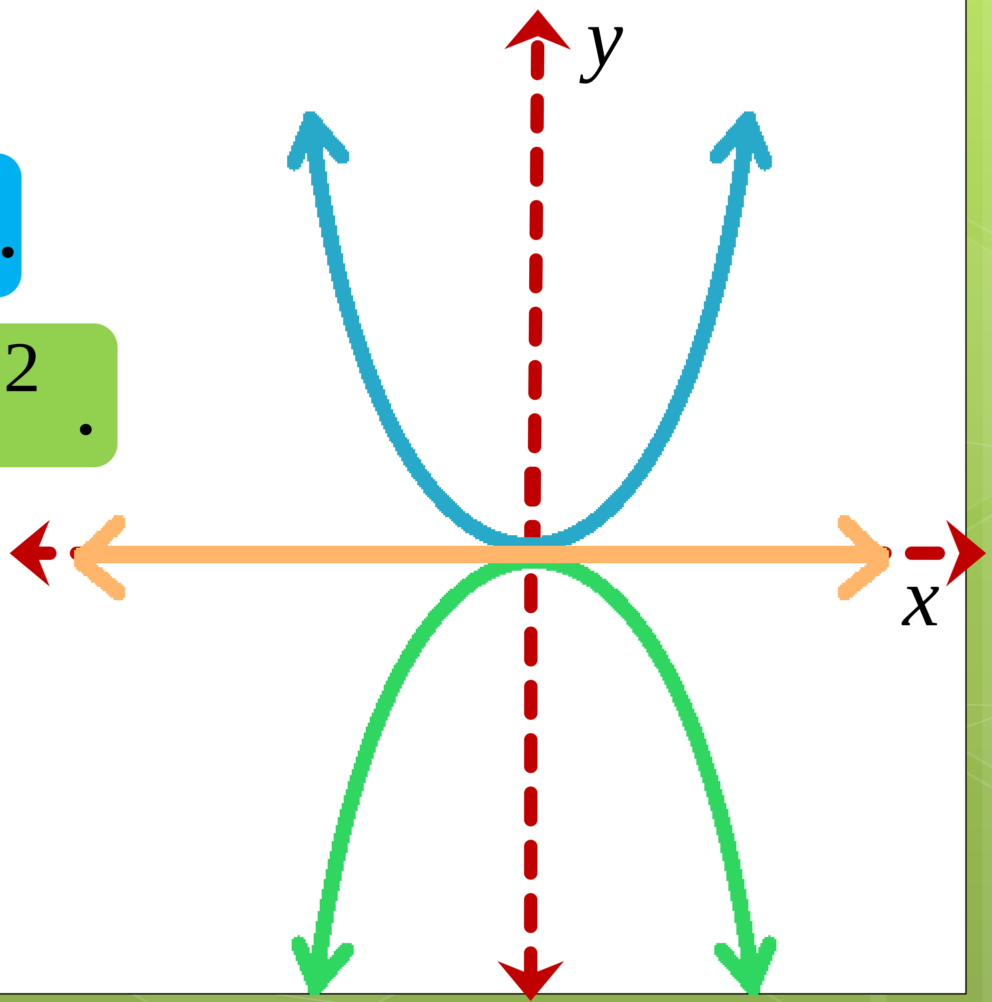
Exercise

$$f(x,y)=\frac{y}{x^2}=k \quad \Rightarrow y=k\,x^2$$

$$k=0 \Rightarrow y=0\,.$$

$$k=1 \Rightarrow y=x^2\,.$$

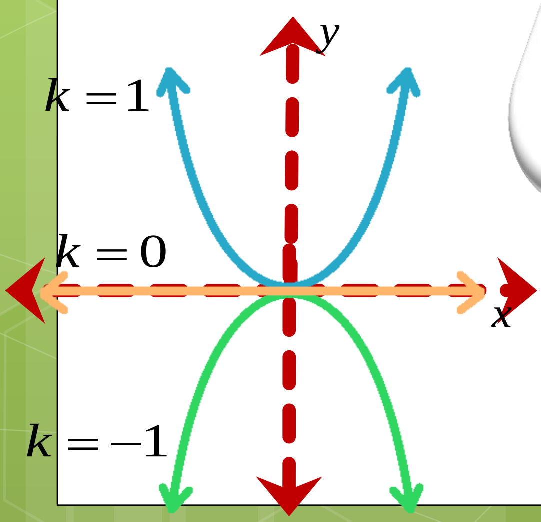
$$k=-1 \Rightarrow y=-x^2\,.$$

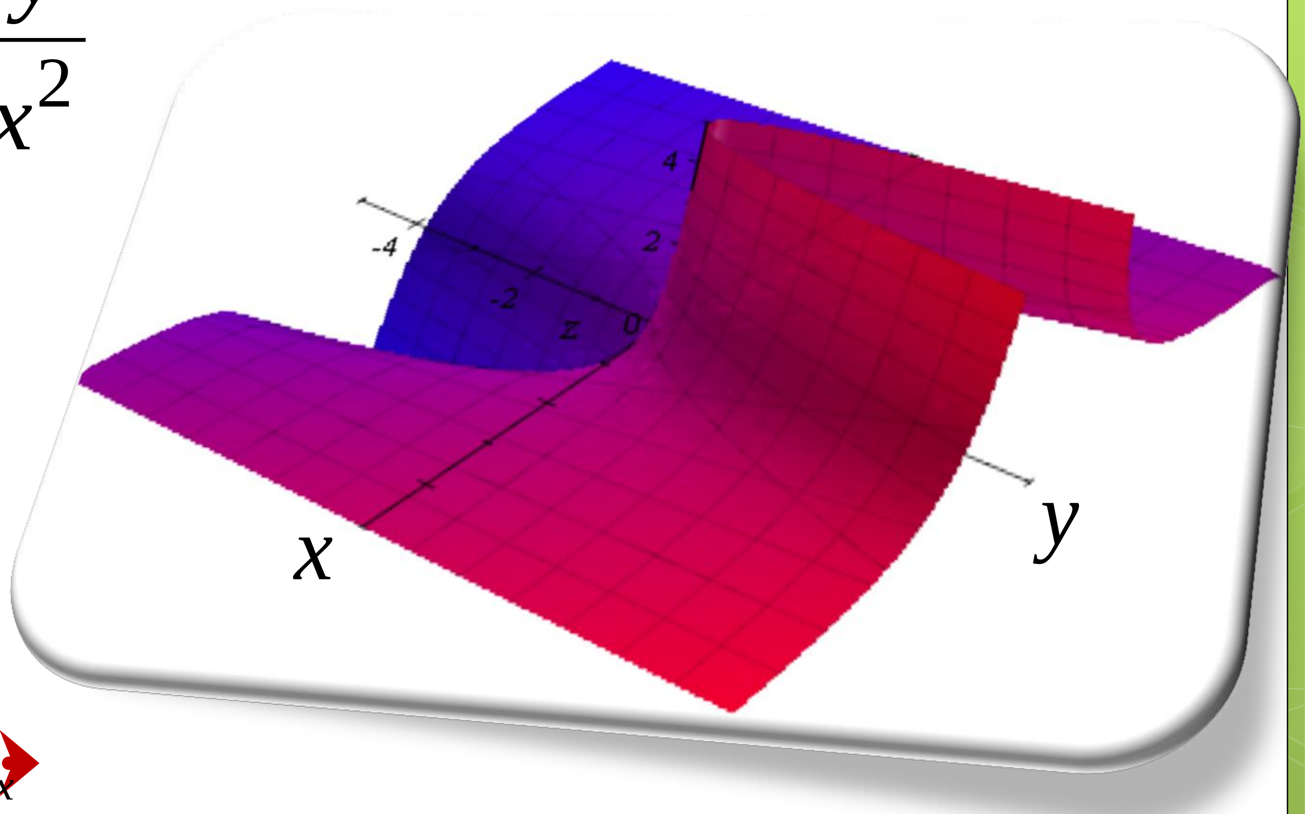


Exercise

A sketch of the graph of *f* is shown below:

$$f(x,y)=\frac{y}{x^2}$$





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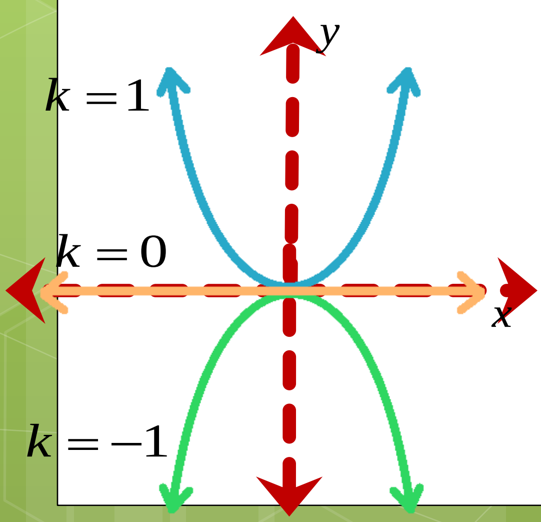
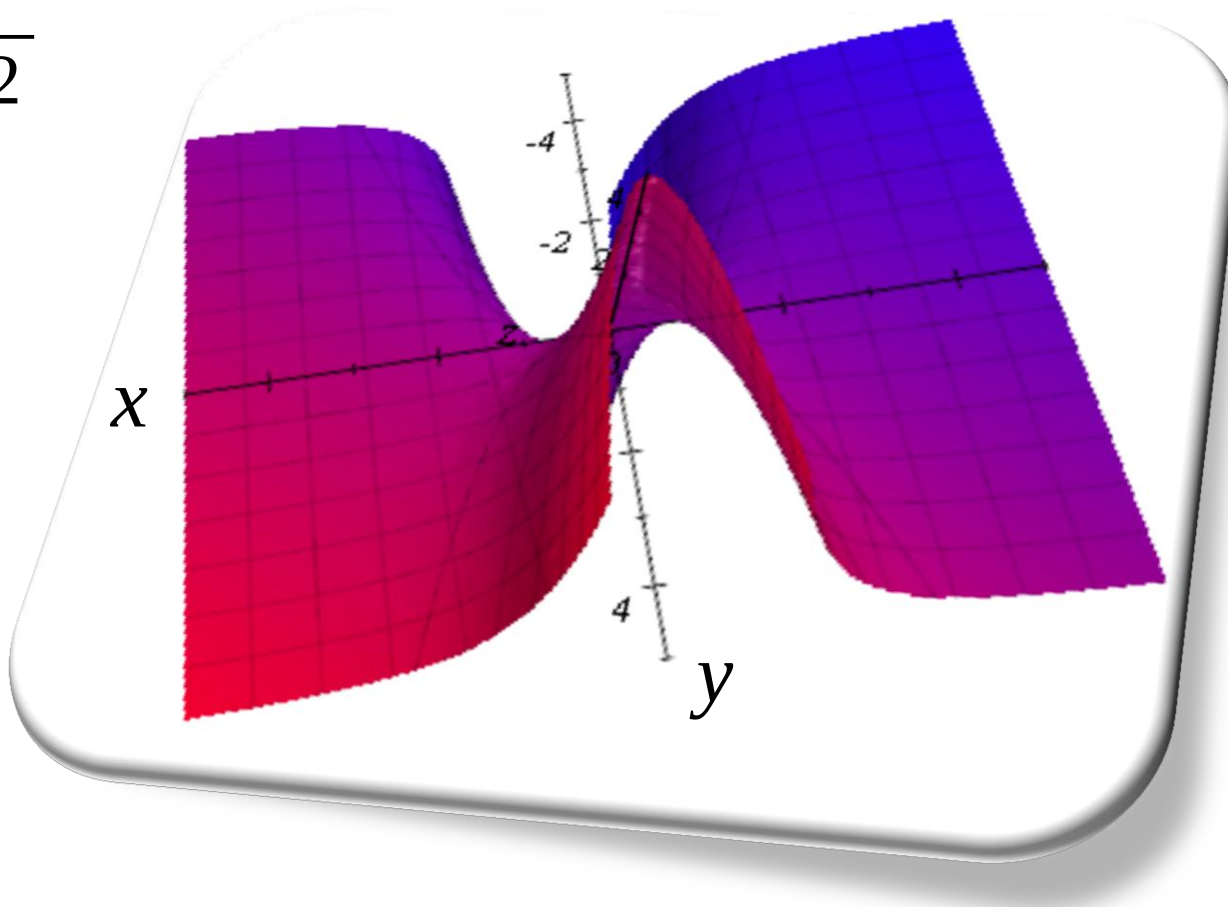




Exercise

Range: R

2. $g(x,y) = y^2 - x^2$

$$g(x,y) = k \;\Rightarrow y^2 - x^2 = k$$

That is, the level curve of *f*

is a *vertical hyperbola* **if** $k > 0$,

is a *horizontal hyperbola* **if** $k < 0$,

are *2 intersecting lines* **if** $k = 0.$

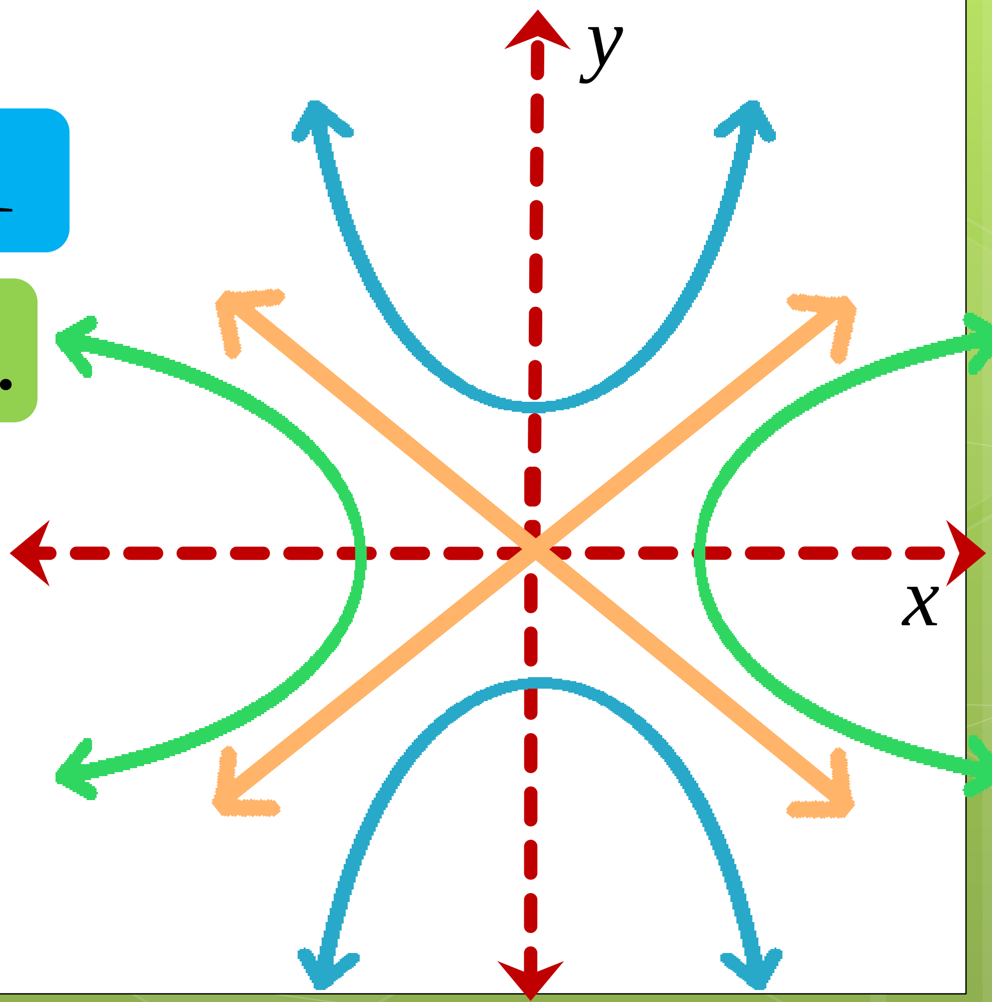
Exercise

$$g(x,y)=k \Rightarrow y^2-x^2=k$$

$$k=0 \Rightarrow y=\pm x\,.$$

$$k=1 \Rightarrow y^2-x^2=1$$

$$k=-1 \Rightarrow y=-x^2\,.$$



Exercise

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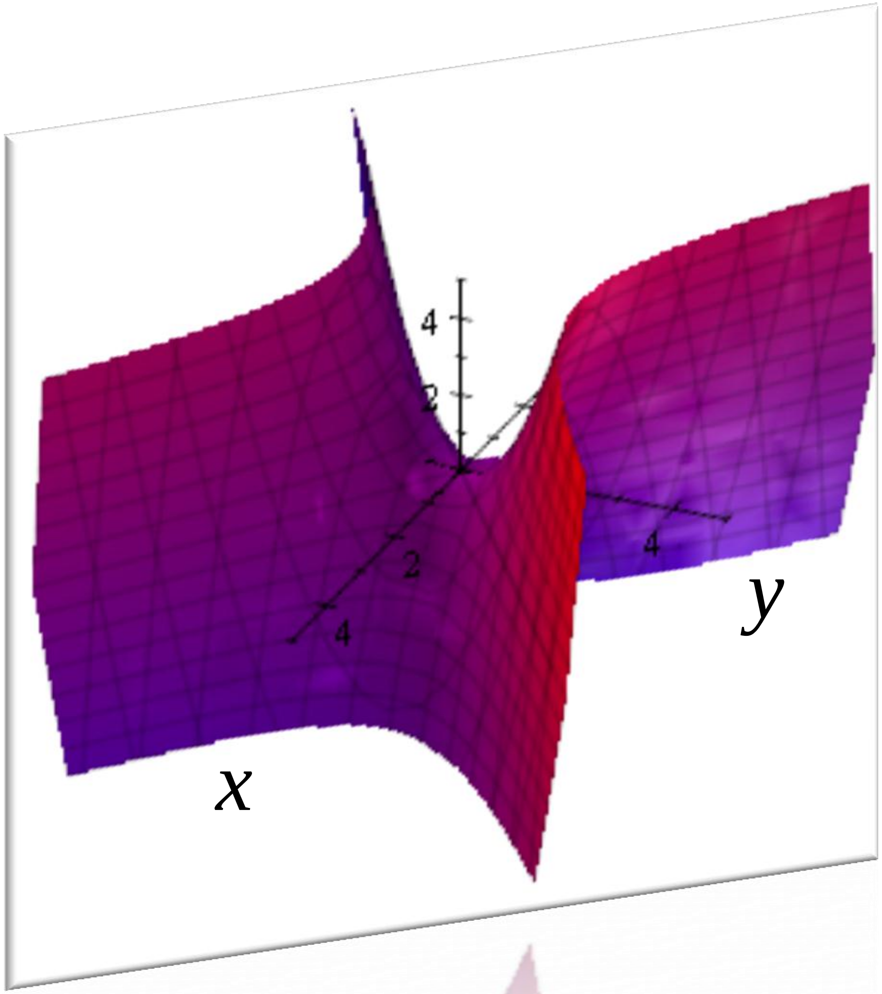
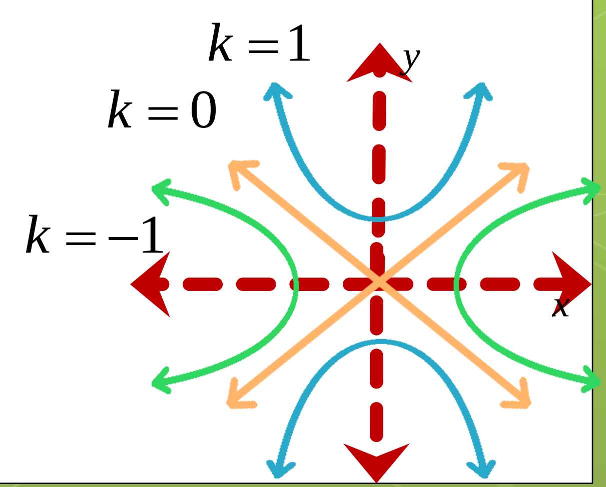
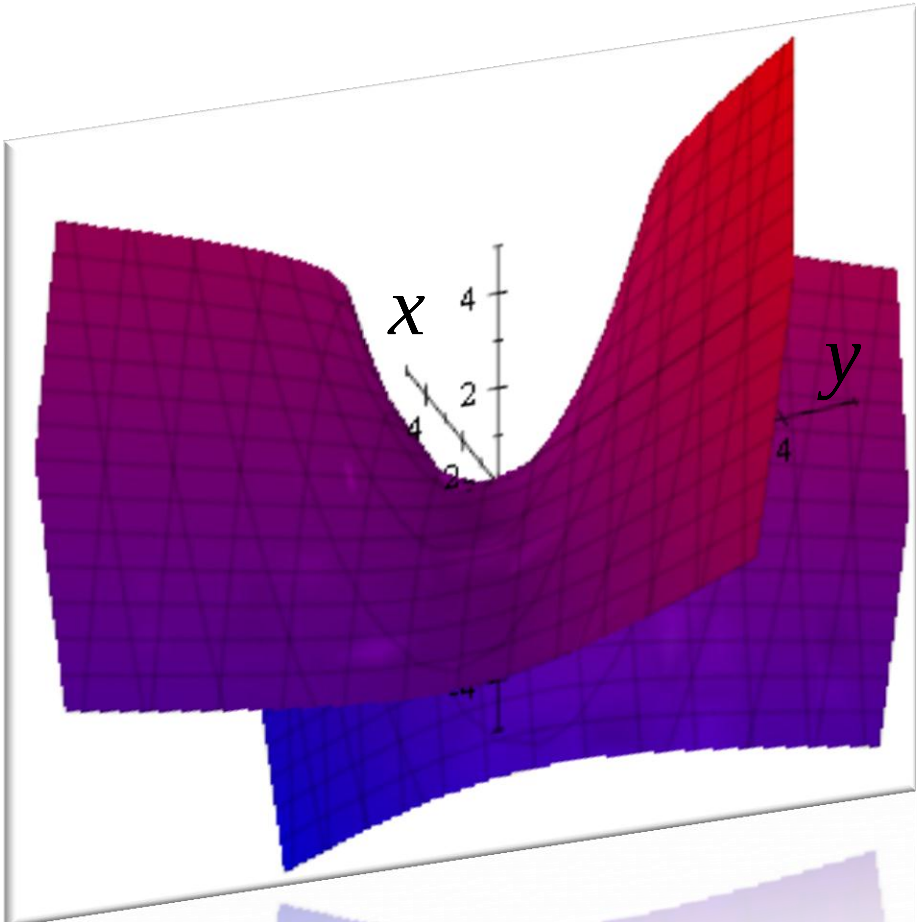


$$g(x,y) = y^2 - x^2$$

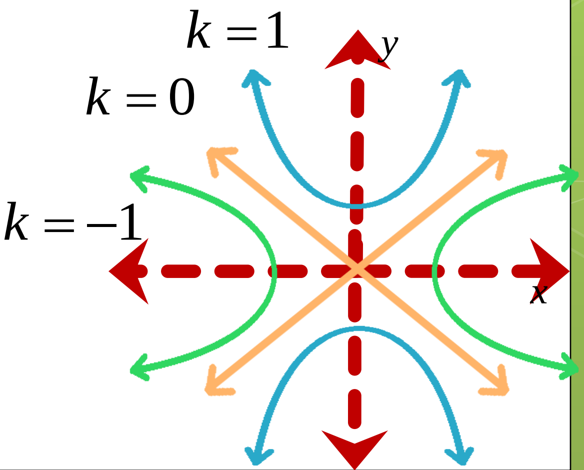


Exercise

A sketch of the graph of f is shown below:



$$g(x, y) = y^2 - x^2$$



Exercise

Range: R

$$f\left(x,y,z\right)=3x-2y+z$$

$$f\left(x,y,z\right)=c\Rightarrow 3x-2y+z=c$$

That is, the level surface of f **is a *plane*.**

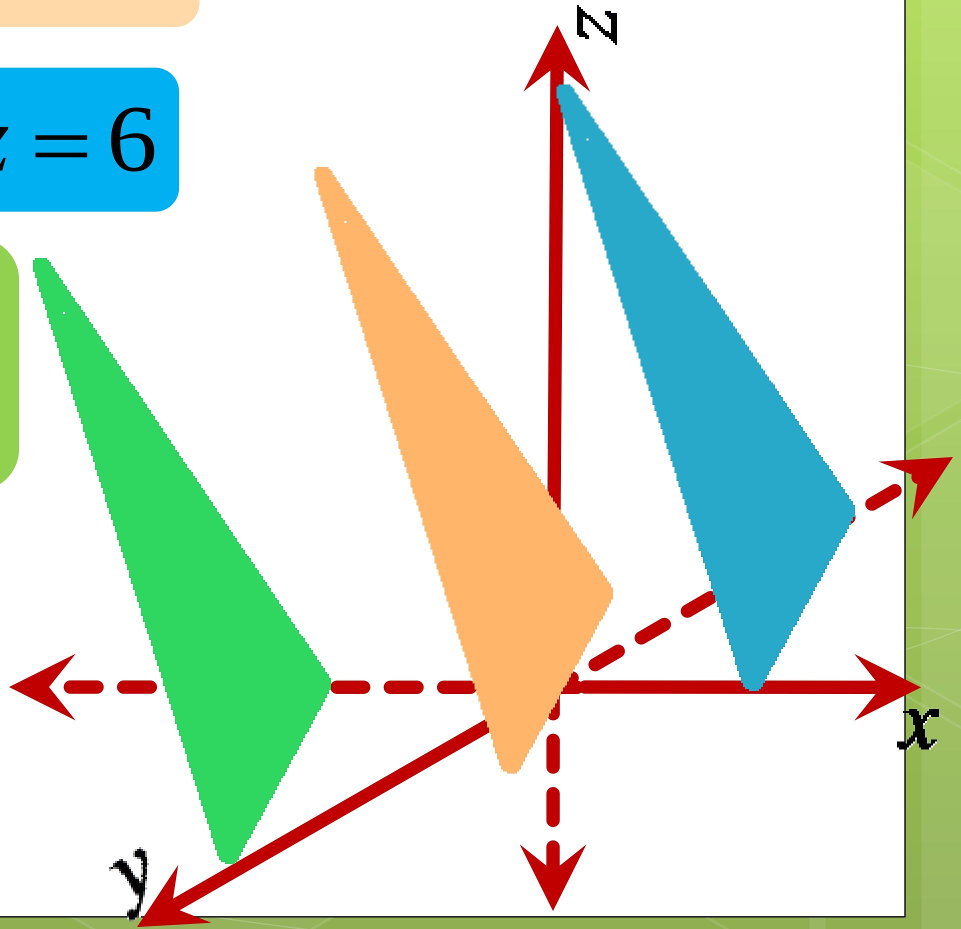
Exercise

$$f(x,y,z)=c \Rightarrow 3x-2y+z=c$$

$$c=0 \Rightarrow 3x-2y+z=0$$

$$c=6 \Rightarrow 3x-2y+z=6$$

$$c=-6$$
$$\Rightarrow 3x-2y+z=-6$$



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