CHAPTER 1

INFINITE SERIES

CHAPTER OBJECTIVE

At the end of the chapter, you should be able to:

1. Determine if a given sequence is convergent or divergent.
2. Determine if a given series is convergent or divergent.
3. Differentiate/integrate an infinite series.

CHAPTER OBJECTIVE

At the end of the chapter, you should be able to:

4. Find the interval and radius and convergence of a given series

5. Write the Maclaurin/Taylor series expansion of a function.

What's next in the sequence?

- $\frac{1}{2}, \frac{1\cdot 3}{2\cdot 4}, \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}, \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}, \boxed{\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}}$
- 0, 3, 8, 15, 24, 35, $\boxed{48}$
- 1, 1, 2, 3, 5, 8, 13, $\boxed{21}$

1.1 Sequences

A *sequence* of real numbers $a_1, a_2, \ldots, a_n, \ldots$ is a function that assigns to each positive integer n a number a_n.

DOMAIN: N

The numbers in the range are called the *elements* or *terms* of the sequence.

1.1 Sequences

NOTATIONS:

$$\left\{ a_n \right\}_{n=1}^{+\infty}$$

$$\left\{ a_n \right\}$$

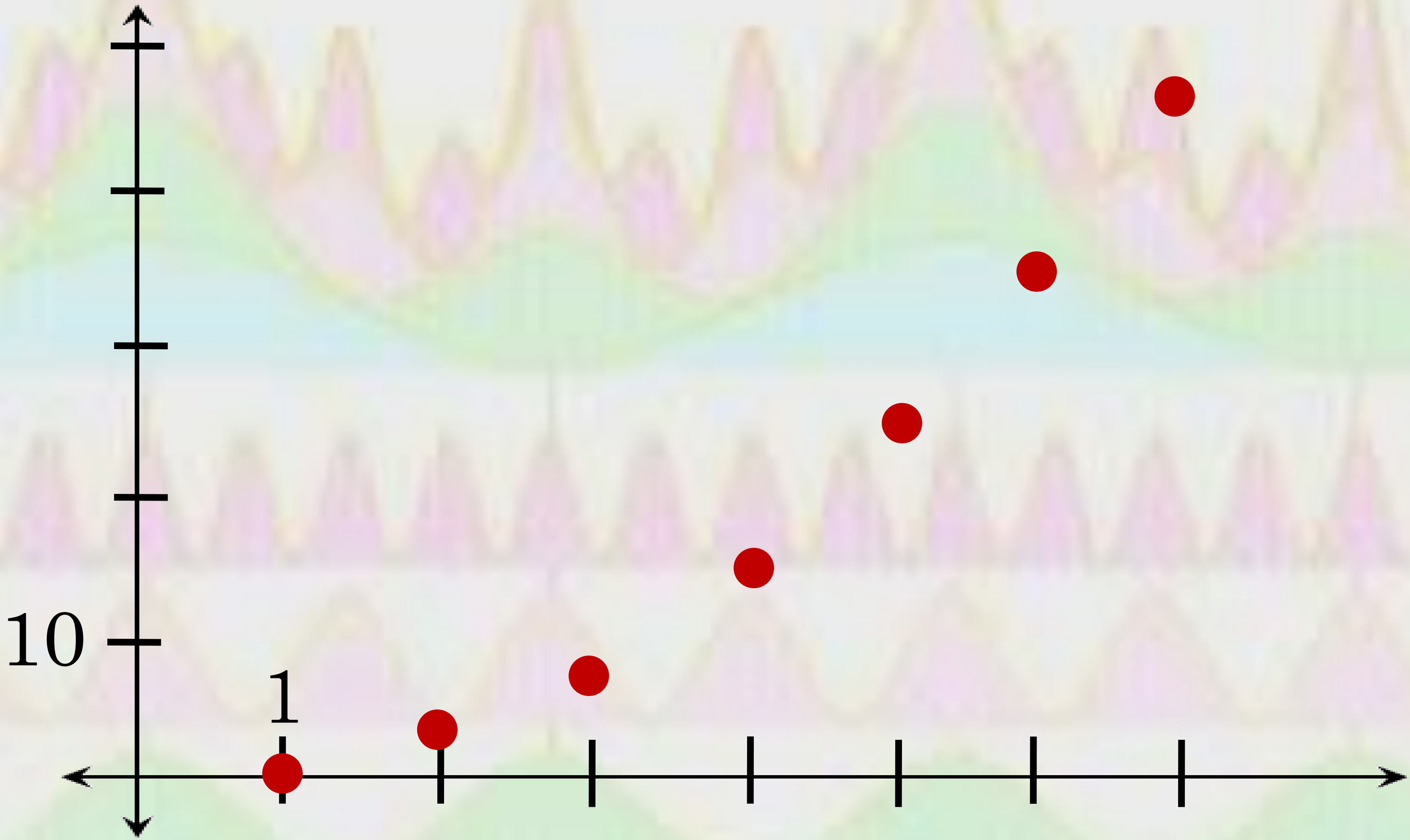
$$\left\{ f(n) \right\}$$

Example 1.

Let $f(n) = n^2 - 1$.

n	1	2	3	4	5	6	7
$f(n)$	0	3	8	15	24	35	48

n	1	2	3	4	5	6	7
$f(n)$	0	3	8	15	24	35	48

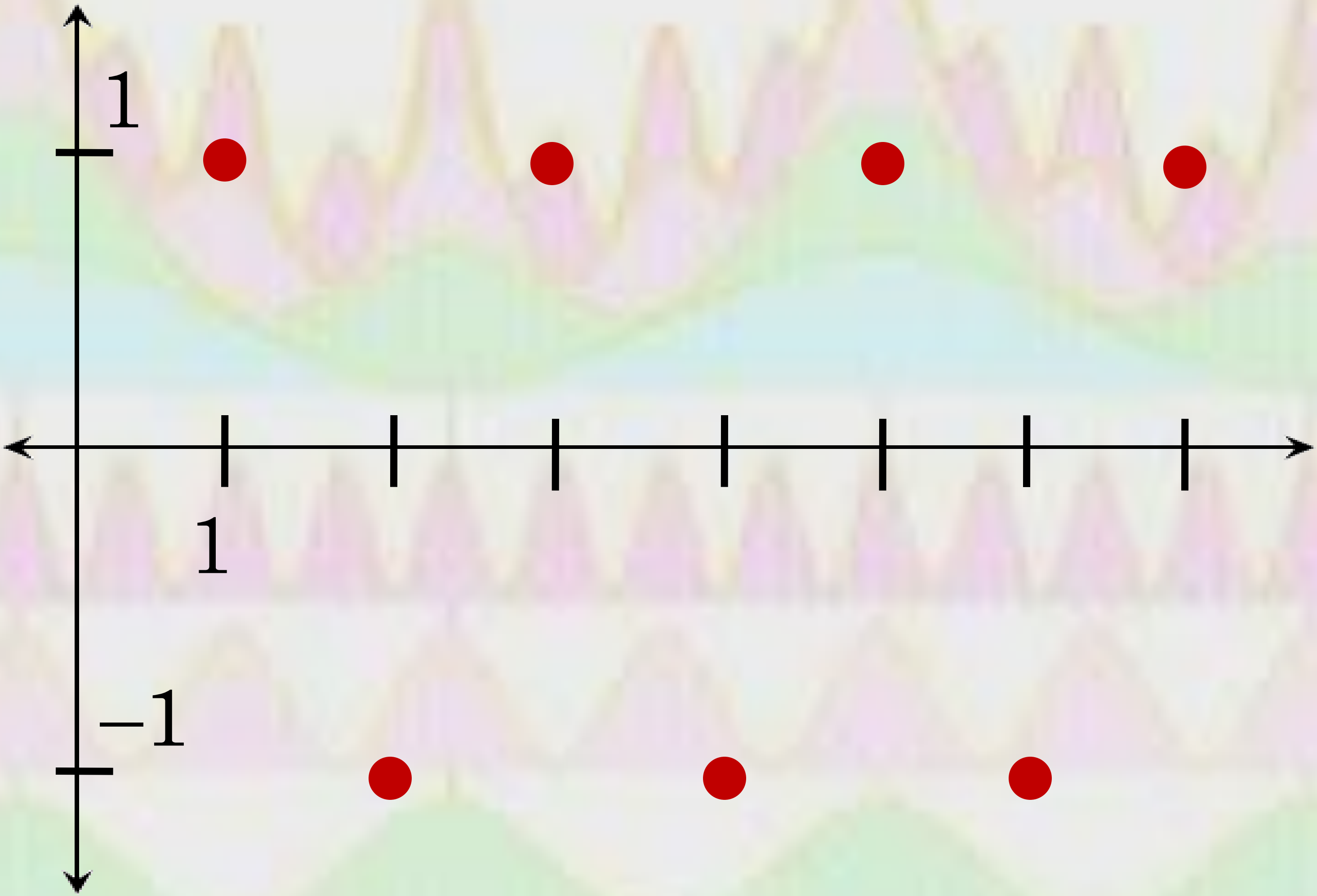


Example 2.

Let $g(n)=(-1)^{n+1}$.

n	1	2	3	4	5	6	7
$g(n)$	1	-1	1	-1	1	-1	1

n	1	2	3	4	5	6	7
$g(n)$	1	-1	1	-1	1	-1	1

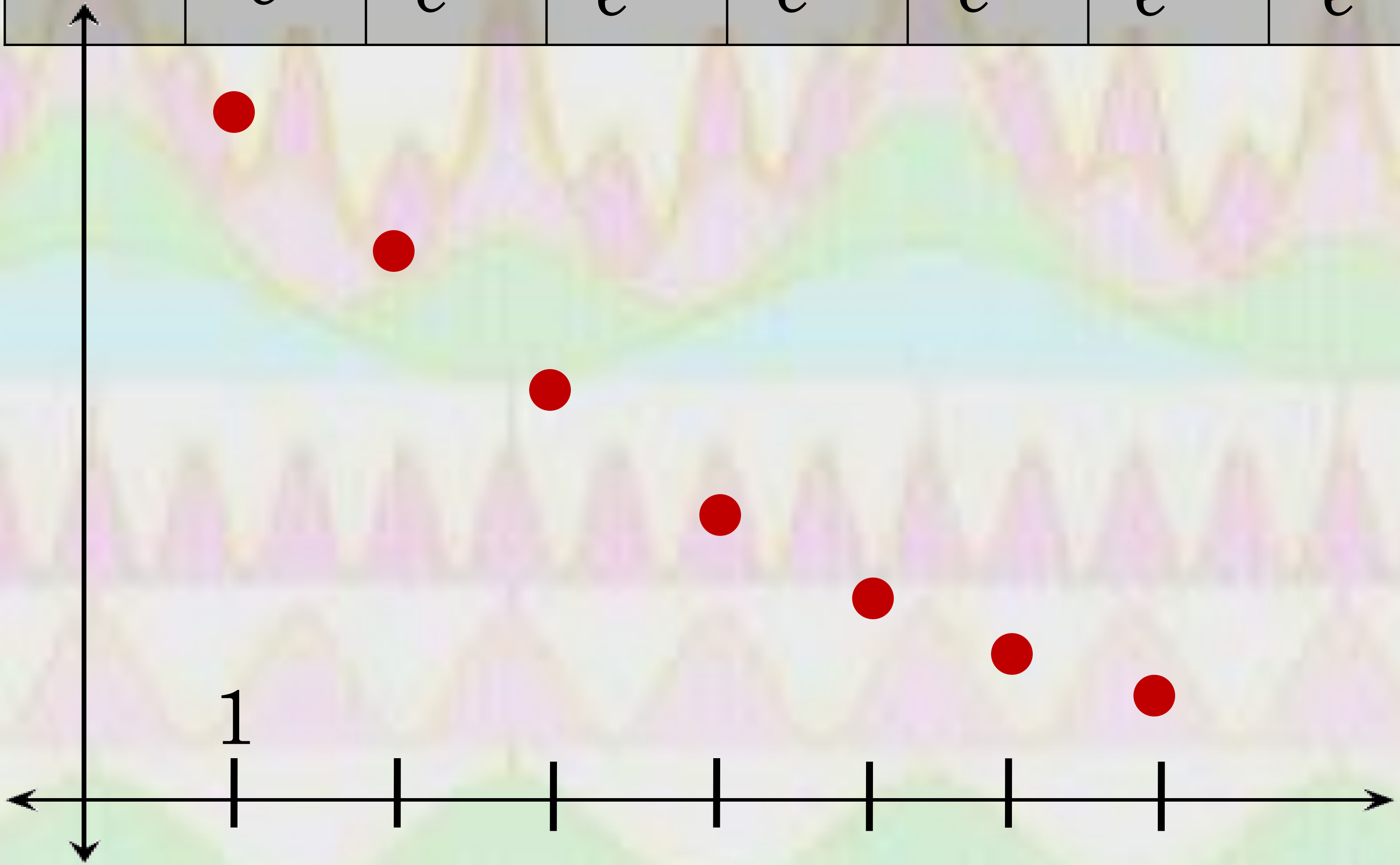


Example 3.

Let $h(n) = e^{-n}$.

n	1	2	3	4	5	6	7
$h(n)$	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$

n	1	2	3	4	5	6	7
$h(n)$	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$

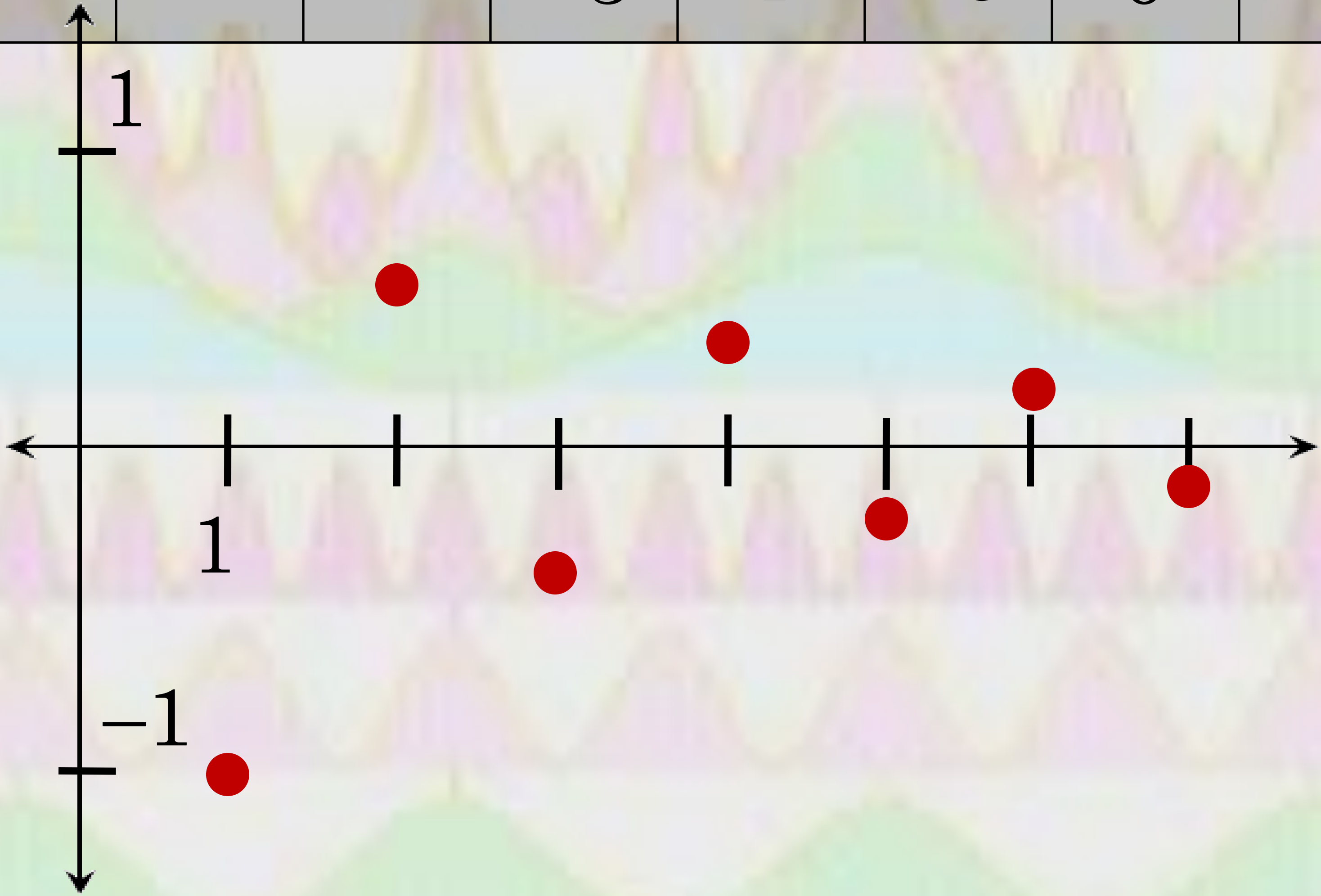


Example 4.

Let $j(n) = (-1)^n \frac{1}{n}$.

n	1	2	3	4	5	6	7
$j(n)$	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$	$\frac{1}{6}$	$-\frac{1}{7}$

n	1	2	3	4	5	6	7
$j(n)$	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$	$\frac{1}{6}$	$-\frac{1}{7}$



OUR INTEREST IN SEQUENCES:

Behavior of $f(n)$

as $n \to +\infty$

Let $\lim_{n \to +\infty} f(n) = L$.

The Limit of a Sequence

The *limit of a sequence* f is the real number *L* if for any $\varepsilon > 0$, however small, there exists a number $N > 0$ such that if n is a natural number and if $n > N$, then $\left| f(n) - L \right| < \varepsilon$.

We write: $\lim_{n \to +\infty} f(n) = L$

Theorem.

If $\lim_{x \to +\infty} f(x) = L$ and f is defined for every positive integer then $\lim_{n \to +\infty} f(n) = L$.

Recall: $\lim_{n \to +\infty} e^{-n} = 0$

Note that $h(n) = e^{-n}$ is defined for every positive integer and $\lim_{x \to +\infty} e^{-x} = 0$.

Definition.

If in $\lim_{n \to +\infty} f(n) = L$, L exists,

Then the sequence is said to be *convergent*.
Otherwise it is *divergent*.

Which of the ff sequences is/are convergent?

$$\left\{\frac{n}{3n+4}\right\}$$

$$\left\{\frac{3n}{n+2}+\cos\left(\frac{4}{n}\right)\right\}$$

$$\left\{\frac{2}{n}\left(Arc\tan n\right)\right\}$$

$$\left\{\left(1+\frac{7}{n}\right)^{n}\right\}$$

$$\left\{\left(-1\right)^{n+1}\right\}$$

$$\left\{\frac{n!}{10}\right\}$$

$$\left\{\frac{3^{n}}{(n+2)!}\right\}$$

1.2 Monotonic and Bounded Sequences

Theorem.

A bounded monotonic sequence is convergent.

When are sequences monotonic?

bounded?

1.2 Monotonic and Bounded Sequences

Definitions.

A sequence is *monotonic* if it is either increasing or decreasing.

A sequence $\left\{a_n\right\}$ is *increasing* if

$$a_n \le a_{n+1} \quad ,\forall n \in N$$

A sequence $\left\{a_n\right\}$ is *decreasing* if

$$a_n \ge a_{n+1} \quad ,\forall n \in N$$

How do we determine if a sequence is monotonic or not?

1. Observe a_n.
2. Obtain $\dfrac{a_n}{a_{n+1}}$. Then Compare result to 1(one).
2. Find $f'(x)$.

Definitions.

A sequence is *bounded* if it has both an upper bound and a lower bound.

A real number l is a *lower bound* of the sequence if $l \leq a_n \ ,\forall n \in N$

A lower bound g is the *greatest lower bound* of the sequence if $l \leq g$ for all lower bound l .

Definitions.

A real number u is an *upper bound* of the sequence if $u \geq a_n \ ,\forall n \in N$

An upper bound v is the *least upper bound* of the sequence if $u \geq v$ for all upper bound u .

Examples 1. $\left\{\dfrac{5n+1}{2n}\right\}$

Let $f(x)=\dfrac{5x+1}{2x} \quad \Rightarrow \quad f'(x)=\dfrac{-2}{4x^2}$

Since $f'(x)<0$, f is decreasing.

Now, $\dfrac{5n+1}{2n}>0$. f has 0 as a lower bound

and 3 as an upper bound.

Thus, the sequence is monotonic and bounded.

Examples 2. $\left\{\dfrac{n!}{10}\right\}$

Let $a_n = \dfrac{n!}{10} \Rightarrow a_{n+1} = \dfrac{(n+1)!}{10}$

Now, $\dfrac{a_n}{a_{n+1}} = \dfrac{n!}{10} \cdot \dfrac{10}{(n+1)!} = \dfrac{1}{(n+1)} < 1$

That is, $a_n < a_{n+1}$

Thus, the sequence is monotonic (increasing).

Examples 2. $\left\{\dfrac{n!}{10}\right\}$

Note that $\dfrac{n!}{10} > 0$.

$\left\{\dfrac{n!}{10}\right\}$ has 0 as a lower bound

but has no upper bound.

Thus, the sequence is unbounded.

Examples 3. $\left\{(-1)^{n+1}\right\}$

Recall:

n	1	2	3	4	5	6	7
a_n	1	-1	1	-1	1	-1	1

Thus, the sequence is bounded but is neither increasing nor decreasing.

Examples 4. $\left\{\dfrac{3^n}{(n+2)!}\right\}$

Let $a_n = \dfrac{3^n}{(n+2)!} \Rightarrow a_{n+1} = \dfrac{3^{n+1}}{(n+3)!}$

Now, $\dfrac{a_n}{a_{n+1}} = \dfrac{3^n}{(n+2)!} \cdot \dfrac{(n+3)!}{3^{n+1}} = \dfrac{(n+3)}{3} > 1$

That is, $a_n > a_{n+1}$

Thus, the sequence is monotonic (decreasing).

Examples 4. $\left\{\dfrac{3^n}{(n+2)!}\right\}$

Note that $\dfrac{3^n}{(n+2)!} > 0$.

$\left\{\dfrac{3^n}{(n+2)!}\right\}$ has 0 as a lower bound

and has ½ as an upper bound.

Thus, the sequence is bounded.

REMARKS:

A bounded decreasing sequence converges to its greatest lower bound.

Similarly, a bounded increasing sequence converges to its least upper bound.

END