CONTINUITY
of Functions of More Than One Variable

Continuity

Let $f : R^n \to R$ such that $w = f(X)$ where $X \in R^n$.

f is **continuous** at $P \in R^n$ if the following are satisfied:

i. $f(P)$ is defined.

ii. $\lim_{X \to P} f(X)$ exists.

iii. $\lim_{X \to P} f(X) = f(P)$

If a function f is discontinuous at a point A, the discontinuity is said to be ***removable*** if $\lim_{P \to A} f(P)$ exists.

If a function f is discontinuous at a point A, the discontinuity is said to be ***essential*** if $\lim_{P \to A} f(P)$ does not exist.

Theorems

1. Polynomial functions are continuous everywhere.

2. Rational functions are continuous over their respective domains.

Theorems

3. If $\lim_{(x,y) \to (x_0, y_0)} g(x,y) = b$ and f is continuous at b, then

$$\lim_{(x,y) \to (x_0, y_0)} (f \circ g)(x,y) = f(b)$$

Theorems

4. If f and g are functions of n variables and are continuous at a point P, then the following are continuous at P

$$f + g \qquad f \cdot g$$

$$f - g \qquad \frac{f}{g},\ g(P) \neq 0$$

Example. Examine continuity at the given point.

$$f(x,y)=\frac{x^4-y^4}{x^2+y^2} \text{ at } (0,0)$$

f is undefined at $(0,0)$.

Hence, f is discontinuous at $(0,0)$.

Example 1. Evaluate

$$\lim_{(x,y)\to(-1,-2)}(2x-3y+5)$$

$$=2(-1)-3(-2)+5$$

$$=-2+6+5$$

$$=9$$

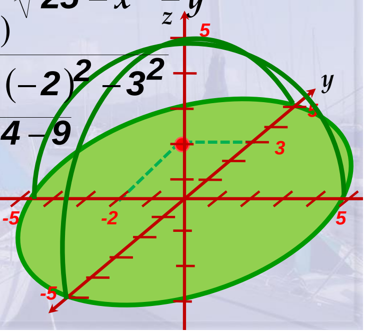
Example 2. Evaluate

$$\lim_{(x,y)\to(-2,3)}\sqrt{25-x^2-y^2}$$

$$=\sqrt{25-(-2)^2-3^2}$$

$$=\sqrt{25-4-9}$$

$$=\sqrt{22}$$

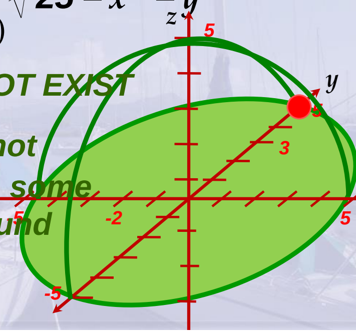


Example 2. Evaluate

$$\lim_{(x,y)\to(0,5)}\sqrt{25-x^2-y^2}$$

DOES NOT EXIST

since f is not defined on some points around (0,5)



Example 4. Evaluate

$$\lim_{(x,y)\to(5,1)}\left(\frac{x+y}{x-y}\right)$$

$$=\frac{5+1}{5-1}$$

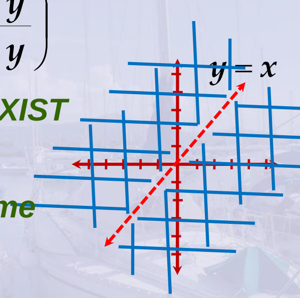
$$=\frac{6}{4}=\frac{3}{2}$$

Example 5. Evaluate

$$\lim_{(x,y)\to(1,1)}\left(\frac{x+y}{x-y}\right)$$

DOES NOT EXIST

since f is not defined on some points around (1,1)



Example 6. Evaluate

$$\lim_{(x,y,z)\to(-3,2,2)} \left(\frac{2x+y}{z}\right)$$

$$=\frac{2(-3)+2}{2}$$

$$=\frac{-4}{2} = -2$$

Example 7. Evaluate

$$\lim_{(x,y,z)\to\left(\frac{\pi}{3},e,4\right)} (\sin x + \ln y - z)$$

$$=\sin\frac{\pi}{3} + \ln e - 4$$

$$=\frac{\sqrt{3}}{2} + 1 - 4 = \frac{\sqrt{3}}{2} - 3$$

Example. Examine continuity at the given point.

$$f(x,y)=\begin{cases} \dfrac{x+y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

at $(0,0)$

i. f is defined at $(0,0)$.

Solution (continued)

Let $S_1 : \{(x,y) \mid x=0\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_1}} \frac{x+y}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0+y}{0+y^2}$$

$$= \lim_{y\to 0} \frac{y}{y^2} = \lim_{y\to 0} \frac{1}{y}$$

DOES NOT EXIST.

Solution (continued)

Since $\lim_{\substack{(x,y)\to(0,0)\\ P \text{ in } S_1}} f(x,y)$ does not exist,

$\lim_{(x,y)\to(0,0)} f(x,y)$ DOES NOT EXIST.

Hence, f is discontinuous at $(0,0)$.

Example. Determine where the given function is continuous

$$f(x,y)=\frac{x^2+y^2}{y-x^2}$$

Since f is a rational function, it is continuous over its domain,

$$R^2 - \left\{(x,y) \mid y = x^2\right\}$$

Example. Determine where the given function is continuous

$$g(x,y)=\frac{1}{\sqrt{1-x^2-y^2}}$$

g is continuous over all points where $1-x^2-y^2>0$ or

$$\{(x,y)\,|\,x^2+y^2<1\}$$

Example. Determine where the given function is continuous

$$h(x,y,z)=\frac{xyz}{x^2+y^2+z^2}$$

Since h is a rational function, it is continuous over its domain,

$$R^3-\{(0,0,0)\}$$

END