# 1.2

# *INFINITE SERIES*

***ADDITION to infinity and beyond . . .***

NOTION

## SERIES

- A SUM OF INFINITELY MANY NUMBER OF TERMS
- THE SUM OF ALL TERMS OF A SEQUENCE

### Infinite Series

Let $\{u_n\}$ be a sequence of real numbers.

Define $s_n = u_1 + u_2 + \ldots + u_n$

The sequence $\{s_n\}$ is an infinite series denoted by

$$\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + \ldots + u_n + \ldots$$

### Infinite Series

$$\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + \ldots + u_n + \ldots$$

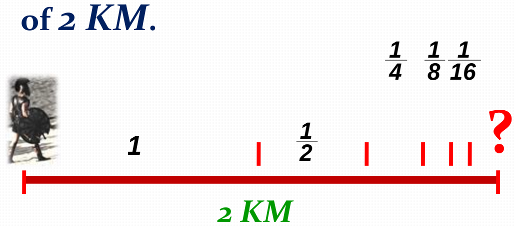
$u_i$'s : terms of the series

$$s_n = u_1 + u_2 + \ldots + u_n$$

$s_i$'s : partial sums

### Example. ZENO's PARADOX

ACHILLES wants to run a distance of 2 *KM*.




WILL ACHILLES REACH THE 2 *KM* MARK?

### Example. ZENO's PARADOX

| LEG | Distance (in km) |
|---|---|
| 1 | $1$ |
| 2 | $\frac{1}{2}$ |
| 3 | $\frac{1}{4}$ |
| 4 | $\frac{1}{8}$ |
| 5 | $\frac{1}{16}$ |
| $n$ | $\left(\frac{1}{2}\right)^{n-1}$ |

Sequence: $\left\{\left(\frac{1}{2}\right)^{n-1}\right\}$

**Example. ZENO's PARADOX**

Distance covered in leg $n$ = $\left(\frac{1}{2}\right)^{n-1}$

Distance covered for the first $n$ legs **?**

$$= 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{n-1}$$

$$= s_n$$

Total distance = INFINITE SERIES

*(up to infinity)*

**How to write infinite series**

1. Using partial sums: $\{ s_n \}$.

   PARTIAL SUM:

$$s_n = u_1 + u_2 + \ . \ . \ . \ + u_n$$

2. Using summation notation.

$$\sum_{n=1}^{+\infty} u_n$$

HOW TO GET THE SUM $\sum_{n=1}^{+\infty} u_n$?

$$\sum_{n=1}^{+\infty} u_n = \lim_{n \to +\infty} s_n$$

where $s_n = u_1 + u_2 + \ldots + u_n$

**Example.**

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{n-1} + \ldots$$

$\lim_{n \to +\infty} s_n$ BUT $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{n-1}$

NOT A GOOD FORM!

CONVENIENT FORM OF $s_n$!!!

**Consider**

$$s_n = 1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{4}} + \ldots + \cancel{\left(\frac{1}{2}\right)^{n-1}}$$

$$-\frac{1}{2} s_n = -\cancel{\frac{1}{2}} - \cancel{\frac{1}{4}} - \ldots - \cancel{\left(\frac{1}{2}\right)^{n-1}} - \left(\frac{1}{2}\right)^{n}$$

$$\frac{1}{2} s_n = 1 - \left(\frac{1}{2}\right)^n$$

$$s_n = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$

**Example.**

$$s_n = 2 \cdot \left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} 2 \cdot \left(1 - \left(\frac{1}{2}\right)^n\right) = 2$$

0

THUS, $\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^{n-1} = 2$

## Detour

WILL ACHILLES REACH THE 2 *KM* MARK ? **YES!**

up to $\infty$

2 *KM*

## Convergence / Divergence

$\sum_{n=1}^{+\infty} u_n$ is convergent if $\lim_{n \to +\infty} s_n$ exists.

If $\lim_{n \to +\infty} s_n = L$, then $\sum_{n=1}^{+\infty} u_n = L$ .

If $\lim_{n \to +\infty} s_n$ does not exist, then the series is divergent.

## Theorem

If $\sum_{n=1}^{+\infty} u_n$ is convergent, then $\lim_{n \to +\infty} u_n = 0.$

The CONTRAPOSITIVE

If $\lim_{n \to +\infty} u_n \neq 0,$ then $\sum_{n=1}^{+\infty} u_n$ is divergent.

## Clarification

If $\lim_{n \to +\infty} u_n = 0,$

NO CONCLUSION on convergence/divergence of $\sum_{n=1}^{+\infty} u_n.$

## Example 1. Show divergence.

$$\sum_{n=1}^{+\infty} \frac{e^n}{n^2}$$

## Example 2. Show divergence.

$$\sum_{n=1}^{+\infty} \frac{2n^2+3n}{1-3n^2}$$

## Illustration.

HARMONIC SERIES: $\sum_{n=1}^{+\infty} \frac{1}{n}$

$\sum_{n=1}^{+\infty} \frac{1}{n}$ is DIVERGENT.

BUT, $\lim_{n \to +\infty} \frac{1}{n} = 0$ .

## Supplement

**ADJUSTING THE TERMS OF A SERIES**

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=k}^{+\infty} u_{n-(k-1)}$$

$$\sum_{n=k}^{+\infty} u_n = \sum_{n=1}^{+\infty} u_{n+(k-1)}$$

## Observe . . .

$$\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + u_3 \;...\; + u_n + \;...$$

$$\sum_{n=k}^{+\infty} u_{n-(k-1)} = u_{k-(k-1)} + u_{(k+1)-(k-1)} + u_{(k+2)-(k-1)} + u_{(k+3)-(k-1)} + \;...$$

$$\sum_{n=k}^{+\infty} u_{n-(k-1)} = u_1 + u_2 + u_3 \;...\; + u_n + \;...$$

## Illustration

$$\sum_{n=1}^{+\infty} 2^{n-1} \cdot n = \sum_{n=3}^{+\infty} 2^{n-3} \cdot (n-2)$$

$$\sum_{n=3}^{+\infty} \frac{2^n}{n} = \sum_{n=1}^{+\infty} \frac{2^{n+2}}{n+2}$$

END