1.4

INFINITE SERIES of POSITIVE TERMS

Recall:

Let $\{s_n\}$ be the sequence of partial sums

defining the infinite series $\sum_{n=1}^{\infty} u_n$

Then for $n \geq 2$, $s_n = s_{n-1} + u_n$.

Remark:

If $u_n > 0 , \forall n$, then the sequence of partial sums is increasing.

Theorem.

An infinite series of positive terms is convergent if and only if the sequence of partial sums has an upper bound.

Show: *Use the theorem above to show that*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \quad \textit{is convergent.}$$

PROOF. In $\sum_{n=1}^{\infty} \frac{1}{n!}$, $s_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$

Now,

$$s_n = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots + \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$$

$$\leq \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 2} + \ldots + \frac{1}{1 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2} \right)^{n-1} = 2$$

PROOF. (cont.)

That is, $s_n \leq 2$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$ is convergent. ∎

Tests for Convergence for Infinite Series of Positive Terms:

1. Direct Comparison Test

2. Limit Comparison Test

3. Integral Test

Direct Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.

This series

- converges if there is a convergent series $\sum_{n=1}^{\infty} c_n$ with $a_n \leq c_n$ for all $n \in N$.

- diverges if there is a divergent series $\sum_{n=1}^{\infty} d_n$ with $a_n \geq d_n$ for all $n \in N$.

Examples. Determine if the series is convergent.

1. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

Recall: $-1 \le \sin n \le 1$

Thus, $\sin^2 n \le 1$

Define $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$.

By the remark given above, $a_n \le b_n$.

But, $\sum_{n=1}^{\infty} b_n$ is convergent.

So, $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ is also convergent. ▨

Examples. Determine if the series is convergent.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{7}{n+\sqrt{n}}$

Recall: $\sqrt{n}\le n$

whenever $n\ge 1$

Define $\displaystyle\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{7}{n+\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{7}{2n}$.

By the remark given above, $a_n\ge b_n$.

But, $\displaystyle\sum_{n=1}^{\infty}b_n$ is divergent.

So, $\displaystyle\sum_{n=1}^{\infty}\frac{7}{n+\sqrt{n}}$ is also divergent. ▨

Examples. Determine if the following series is/are convergent.

$$\sum_{n=1}^{\infty} \frac{2n+3}{n^3+1}$$

$$\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{3n}{n^2-\sin^2 n}$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{5n+1}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n+2n}$$

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of positive terms and $L = \lim_{n \to +\infty} \frac{a_n}{b_n}$.

- **If $L > 0$, then both series converge or both series diverge.**

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of positive terms and $L = \lim_{n \to +\infty} \frac{a_n}{b_n}$.

❖ **If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.**

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of positive terms and $L = \lim_{n \to +\infty} \dfrac{a_n}{b_n}$.

- **If $L = +\infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.**

Examples. Determine if the series is convergent.

1. $$\sum_{n=1}^{\infty} \frac{3n^2+5}{n^5+7}$$

Use: $$\frac{n^2}{n^5} = \frac{1}{n^3}$$

Define $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n^2+5}{n^5+7}$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

Now, $$L = \lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{3n^2+5}{n^5+7} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to+\infty} \frac{3n^5+5n^3}{n^5+7} = 3$$

Examples. Determine if the series is convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5}{n^5+7}$

Now, $L = 3 > 0$.

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5}{n^5+7}$ is also convergent.

Examples. Determine if the series is convergent.

2. $\displaystyle\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$

Use: $\dfrac{n}{n^2}=\dfrac{1}{n}$

Define $\displaystyle\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$ and $\displaystyle\sum_{n=2}^{\infty}b_n=\sum_{n=2}^{\infty}\frac{1}{n}$.

Now, $\displaystyle L=\lim_{n\to+\infty}\frac{a_n}{b_n}=\lim_{n\to+\infty}\frac{1+n\ln n}{n^2+5}\cdot\frac{n}{1}$

$$=\lim_{n\to+\infty}\frac{n+n^2\ln n}{n^2+5}=+\infty$$

Examples. Determine if the series is convergent.

2. $$\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$$

Now, $L=+\infty$.

and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$ is also divergent. ▨

Examples. Determine if the following series is/are convergent.

$$\sum_{n=1}^{\infty} \frac{4n+5}{n^2+2n+4}$$

$$\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt[3]{n^5+n^3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n-2n}$$

Integral Test

Let f be a function which is

1. Continuous and positive-valued for all $x \geq 1$
2. Decreasing to zero

Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if

the improper integral $\int_1^{+\infty} f(x)\,dx$.

Examples. Determine if the series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ Let $f(x) = x^p$.

   i. f is continuous and positive-valued for all $x \geq 1$

   ii. f is decreasing to zero since $\lim_{x \to +\infty} \frac{1}{n^p} = 0$

Examples. Determine if the series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ Let $f(x) = x^p$.

CASE 1: $p > 1$

$$\int_1^{+\infty} f(x)dx = \int_1^{+\infty} \frac{dx}{x^p} = \lim_{a \to +\infty} \int_1^a \frac{dx}{x^p} = \frac{-1}{1-p}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

Examples. Determine if the series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ Let $f(x) = x^{p}$.

CASE 2: $p = 1$

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{dx}{x} = \lim_{a \to +\infty} \int_1^{a} \frac{dx}{x} = +\infty$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Examples. Determine if the series is convergent.

1. $\sum_{n=1}^{\infty}\frac{1}{n^p}, p>0$ Let $f(x)=x^p$.

CASE 3: $p<1$

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty}\frac{dx}{x^p} = \lim_{a\to+\infty}\int_1^{a}\frac{dx}{x^p} = +\infty$$

Thus, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent if $p<1$.

Examples. Determine if the series is convergent.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{n}}{1+e^{2n}}$ Let $f(x)=\dfrac{e^{x}}{1+e^{2x}}$.

i. f is continuous and positive-valued for all $x \geq 1$

ii. f is decreasing to zero since $\displaystyle\lim_{x\to+\infty}\frac{e^{x}}{1+e^{2x}}=0$

Examples. Determine if the series is convergent.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}}$ Let $f(x)=\dfrac{e^{x}}{1+e^{2x}}$.

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty}\frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{a\to+\infty}\int_1^{a}\frac{e^x}{1+e^{2x}}\,dx = \frac{\pi}{2} - Arc\tan e$$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}}$ is convergent. ▨

Examples. Determine if the following series is/are convergent.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\left(1+\ln^2 n\right)}$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$

Remark: It is *NOT NECESSARY* that

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{+\infty} f(x)\,dx \quad \text{are equal.}$$

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{BUT} \qquad \int_1^{+\infty} \frac{dx}{x^2} = 1$$

END