DERRANGEMENTS

DERRANGEMENTS

PERMUTATIONS

wherein n-k objects are

NOT IN THEIR ORIGINAL POSITIONS

and k are

IN THEIR ORIGINAL POSITIONS

DERRANGEMENTS

$$D(n,k) =$$

$$\frac{n!}{k!}\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots\pm\frac{-1^{(n-k)}}{(n-k)!}\right)$$

A professor is returning exam papers at random to his three students (because the students didn't put their names on their exam papers).

How many ways can the professor give the wrong exam paper to all the three students?

POSSIBLE WAYS THE EXAMS CAN BE RETURNED

STUDENTS		
A	B	C
A	B	C
A	C	B
B	A	C
B	C	A
C	B	A
C	A	B

How many ways can the professor give the wrong exam paper to all the three students?

$$n-k = 3$$
$$k = 0$$

POSSIBLE WAYS THE EXAMS CAN BE RETURNED

STUDENTS			
A	B	C	
A	B	C	
A	C	B	
B	A	C	
B	C	A	k=0
C	B	A	
C	A	B	k=0

POSSIBLE WAYS THE EXAMS CAN BE RETURNED

STUDENTS			
A	B	C	
A	B	C	k=3
A	C	B	k=1
B	A	C	k=1
B	C	A	k=0
C	B	A	k=1
C	A	B	k=0

How many ways can the professor give the wrong exam paper to all the three students?

$D(3,0) = 2$

PIGEONHOLE PRINCIPLE

PIGEONHOLE PRINCIPLE

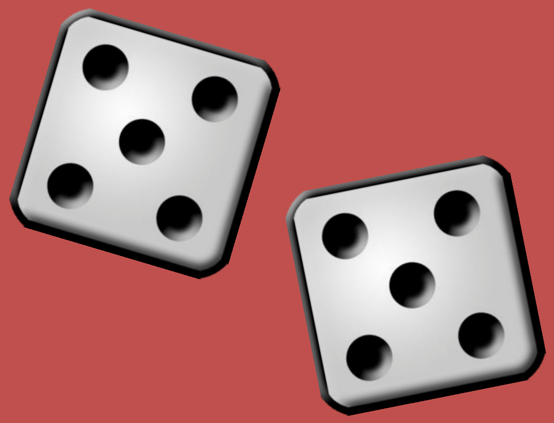
If k pigeons fly to n pigeonholes and $k > n$, then one of the pigeonholes will contain at least two pigeons

EXTENDED

PIGEONHOLE PRINCIPLE

If k pigeons are assigned to n pigeonholes, then one of the pigeonholes will contain at least $\lfloor (k-1) / n \rfloor + 1$ pigeons

Two identical dice are rolled a total of 25 times. Are the results of all the rolls unique?



# of rolls

Pigeons

possible outcomes

Pigeonholes

$25 > M(6,2) = 21$

Two identical dice are rolled a total of 25 times. Are the results of all the rolls unique?

NO

In a STAT1 quiz with five multiple choice items, each of 63 students answered correctly four out of five.

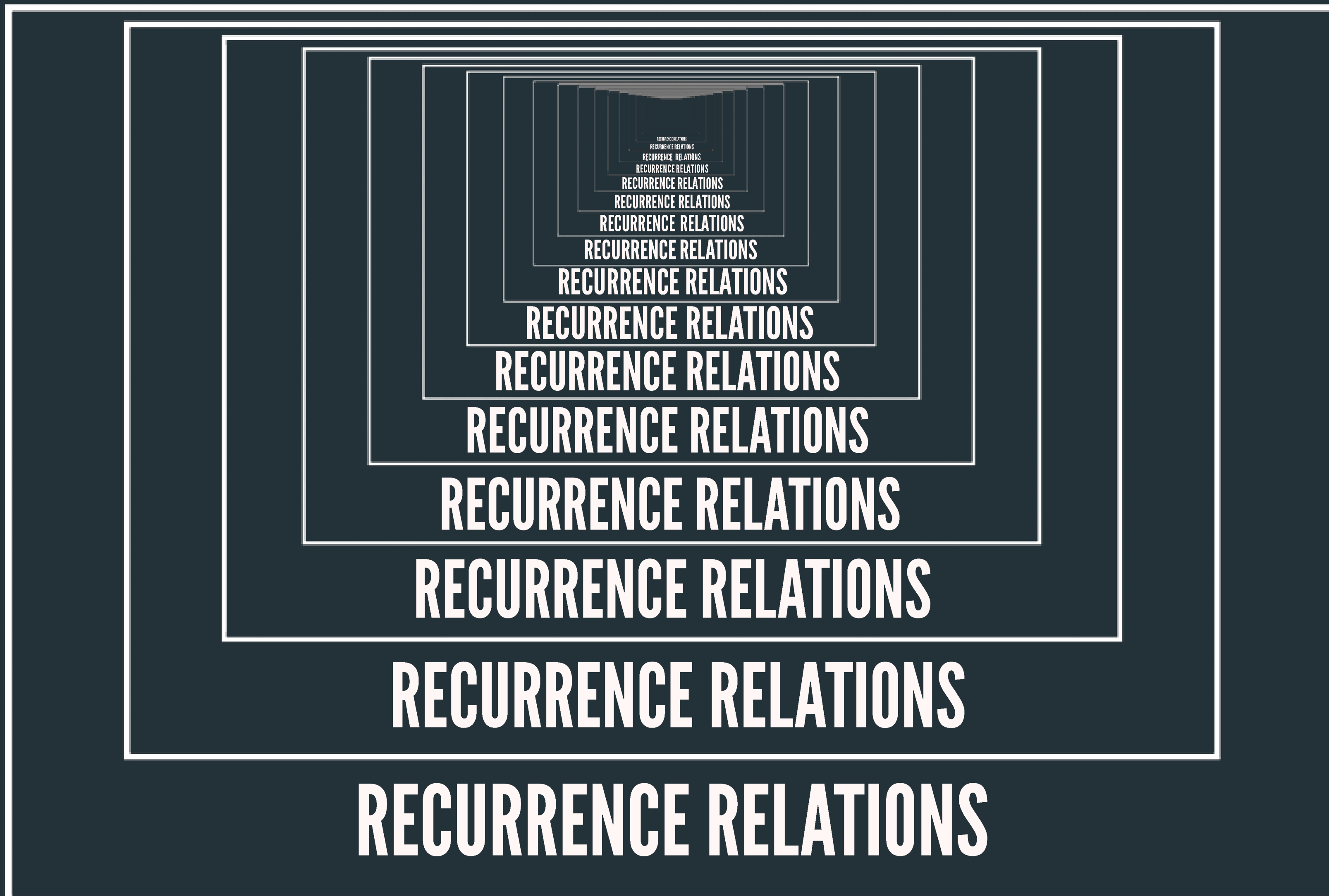
If there were four choices per item, at least how many students had exactly the same set of answers for the quiz?

# of students

Pigeons

# of possible answers with 4 correct answers

Pigeonholes



RECURRENCE RELATIONS

Given a sequence $a_0, a_1, a_2, \ldots$

A **RECURRENCE RELATION** is an equation that relates a_n to one or more of its predecessors $a_0, a_1, a_2, \ldots a_{n-1}$ for $n \geq 0$

a_n

EQUATION

a_4 a_1 a_0 a_{n-1} a_2 a_3

INITIAL CONDITIONS

$a_0 =$
$a_1 =$
$a_2 =$
$\ldots$
$a_k =$

EXPLICITLY
GIVEN
VALUES

TYPICAL FORM

$$
a_n = \begin{cases} a_0 & \text{when } n=0 \\ a_1 & \text{when } n=1 \\ a_2 & \text{when } n=2 \\ \ldots & \\ a_t & \text{when } n=t \\ f(a_0, a_1, \ldots, a_{n-1}) & \text{otherwise} \end{cases}
$$

dealing with RECURRENCE RELATIONS

DEFINE the recurrence relation

SOLVE the recurrence relation

DEFINE the recurrence relation

INITIAL CONDITIONS

$a_0, a_1, a_2, \ldots, a_k$

RECURRENCE RELATION

$a_n = f(a_0, a_1, \ldots, a_{n-1})$

DEFINE the recurrence relation

$$a_n = \begin{cases} ? & \text{when } n=0 \\ ? & \text{when } n=1 \\ ? & \text{when } n=2 \\ \ldots & \\ ? & \text{when } n=t \\ ? & \text{otherwise} \end{cases}$$

Try to **DEFINE** the recurrence relation of the sequence

3, 10, 17, 24, 31, ...

EXAMPLE

Try to **DEFINE** the recurrence relation of the sequence

S_n = The sum of the first n positive integers

EXAMPLE

Try to **DEFINE** the recurrence relation of the sequence

2, 5, 7, 12, 19, 31, ...

EXAMPLE

How many ways can a person climb a **STAIRS WITH n STEPS** if he/she can do **1 OR 2 STEPS AT A TIME**?

EXAMPLE

Try to **DEFINE** the recurrence relation of the sequence

C_n = the number of ways a person can climb a stairs with n steps if he/she can do 1 or 2 steps at a time.

EXAMPLE

Find the number of **BINARY STRINGS** of length n with **NO REPEATED ZEROES.**

EXAMPLE

Try to **DEFINE** the recurrence relation of the sequence

B_n = the number of binary strings of length n with no repeated zeroes.

EXAMPLE

9149103490

A computer program considers a **STRING OF DECIMAL DIGITS** a **VALID CODEWORD** if it contains an **EVEN NUMBER OF 0 DIGITS**.

0513656104 0

EXAMPLE

Try to **DEFINE** the recurrence relation of the sequence

W_n = number of valid n-digit codewords

EXAMPLE

SOLVE the recurrence relation

ITERATION method

method by **CHARACTERISTICS ROOTS**

S_n = The sum of the first n positive integers

Which one would you prefer?

$S_n = S_{n-1} + n$	$S_n = n(n+1) / 2$

SOLVE the recurrence relation

$$a_n = f(a_0, a_1, \ldots, a_{n-1}) \rightarrow a_n = f(n)$$

ITERATION

method

1

Write
$a_n = f(a_0, a_1, \ldots, a_{n-1})$
Using the defined
Recurrence relation.

2

Use the relation to replace each of a_{n-1}, a_{n-2}, …
by their respective predecessors.

3

Repeat 2 until an explicit formula depending only on n and the initial conditions.

ITERATION
method

Solve

$$a_n = a_{n-1} + 6$$

subject to initial condition

$$a_0 = 2$$

EXAMPLE

ITERATION
method

Solve

$$b_n = 2nb_{n-1}$$

subject to initial condition

$$b_1 = 3$$

EXAMPLE

ITERATION
method

Solve

$$c_n = (1/2)c_{n-1}$$

subject to initial condition

$$c_1 = 3$$

QUIZ

method by

CHARACTERISTIC ROOTS