# 3.4 HIGHER-ORDER PARTIAL DERIVATIVES

## Second-order partial derivatives

**Given:** $z = f(x,y)$

**Partial derivative:** $\dfrac{\partial z}{\partial x}$

**Second-order partial derivatives of** $\dfrac{\partial z}{\partial x}$ **:**

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

## Second-order partial derivatives

**Given:** $z = f(x,y)$

**Partial derivative:** $\dfrac{\partial z}{\partial y}$

**Second-order partial derivatives of** $\dfrac{\partial z}{\partial y}$ **:**

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

| | | |
|---|---|---|
| $\dfrac{\partial^2 z}{\partial x^2}$ | $f_{xx}(x,y)$ | $D_{xx} f(x,y)$ |
| $\dfrac{\partial^2 z}{\partial y \partial x}$ | $f_{xy}(x,y)$ | $D_{xy} f(x,y)$ |
| $\dfrac{\partial^2 z}{\partial x \partial y}$ | $f_{yx}(x,y)$ | $D_{yx} f(x,y)$ |
| $\dfrac{\partial^2 z}{\partial y^2}$ | $f_{yy}(x,y)$ | $D_{yy} f(x,y)$ |

## Third-order partial derivatives

**Given:** $w = f(x,y,z)$

$$f_{xxy} = D_y(D_x(D_x f)) = \frac{\partial^3 f}{\partial y \partial x \partial x}$$

$$f_{yzx} = D_x(D_z(D_y f)) = \frac{\partial^3 f}{\partial x \partial z \partial y}$$

## Example. Consider

$$f(x,y,z) = x^2 \cos z + y^2 \sin z$$

Solve for $\dfrac{\partial^3 f}{\partial z^3}$

$$\frac{\partial f}{\partial z} = x^2 \cdot (-\sin z) + y^2 \cdot \cos z$$

$$\frac{\partial^2 f}{\partial z^2} = -x^2 \cdot \cos z + y^2 \cdot (-\sin z)$$

*Solution (continued)*

$$f(x,y,z)=x^2\cos z+y^2\sin z$$

$$\frac{\partial f}{\partial z}=x^2\cdot(-\sin z)+y^2\cdot\cos z$$

$$\frac{\partial^2 f}{\partial z^2}=-x^2\cdot\cos z+y^2\cdot(-\sin z)$$

$$\frac{\partial^3 f}{\partial z^3}=-x^2\cdot(-\sin z)-y^2\cdot\cos z$$

*Example.* Consider

$$f(x,y,z)=x^2\cos z+y^2\sin z$$

Solve for $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial z}$

$$\frac{\partial f}{\partial z}=x^2\cdot(-\sin z)+y^2\cdot\cos z$$

$$\frac{\partial^2 f}{\partial y\partial z}=2y\cdot\cos z \qquad \frac{\partial^3 f}{\partial x\,\partial y\,\partial z}=0$$

*Example.* Consider

$$f(x,y,z)=x^2\cos z+y^2\sin z$$

Solve for $\dfrac{\partial^3 f}{\partial x\,\partial z\,\partial x}$

$$\frac{\partial f}{\partial x}=2x\cos z \qquad \frac{\partial^2 f}{\partial z\partial x}=2x\cdot(-\sin z)$$

$$\frac{\partial^3 f}{\partial x\,\partial z\,\partial x}=-2\sin z$$

*MUST REMEMBER*

***Given:*** $z=f(x,y)$

***defined over*** $B((x_0,y_0);r)$

***If*** $f_x, f_y, f_{xy}$ ***and*** $f_{yx}$ ***are defined over*** $B$, ***and*** $f_{xy}$ ***and*** $f_{yx}$ ***are continuous over*** $B$, ***then***

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

*Example.* Verify that $f_{xy}=f_{yx}$

$$f(x,y)=y^2\sin x-e^y\cos x$$

*Solution:*

$$f_x(x,y)=y^2\cos x-e^y(-\sin x)$$

$$f_{xy}(x,y)=2y\cos x+e^y\sin x$$

$$f_y(x,y)=2y\sin x-e^y\cos x$$

$$f_{yx}(x,y)=2y\cos x-e^y(-\sin x)$$

*Solution (continued)*

$$f_{xy}(x,y)=2y\cos x+e^y\sin x$$

$$f_{yx}(x,y)=2y\cos x+e^y\sin x$$

***Hence,*** $f_{xy}(x,y)=f_{yx}(x,y)$.

# END