Derivation of Equations used in ArduPilots Autonomous Autorotation Flight Mode

1   Introduction

This short report gives an overview of the equations used within the autonomous autorotation flight mode implimented in ArduPilot, for use with traditional helicopters.

The purpose of this document is to provide a guide to developers, giving the derivation of the equations used, making the code implementation easier to follow, and preserving the intent and rational as to why the controller is written as it is.

In an attempt to keep this document brief, where possible, equations from other resources have been used as a starting point, to avoid duplication of already available and well documented methods. All references and links are provided.

2   Nomenclature

A	(m^2) Rotor disc area.
C_{l_α}	(rad^{-1}) lift-curve slope.
C_T	Coefficient of thrust.
k_*	Grouping variable. Numerous are used in this derivation. They do not have physical significa
R	(m) Rotor Radius.
T	(N) Thrust.
θ	(rad) Blade pitch angle.
λ	Rotor inflow ratio (normalised by tip speed).
λ_i	Induced inflow ratio.
λ_h	Induced inflow ratio in the hover.
λ_c	Climb inflow ratio.
ρ	(kg m^{-3})Far-field air density.
σ	Rotor solidity ratio.
Ω	(rad s^{-1}) Rotor speed.

TODO: Add a few diagrams of rotor geometry to help introduce the nomenclature.

3   Note on Thrust Coefficient

Within the world of helicopter aerodynamics and performance analysis there are two slightly different ways of non-dimensionalising Thrust. These are often refereed to the US and European customary definitions, and the only difference between them is factor of $^1/_2$. Therefore to avoid confusion, the thrust coefficient used in this work is formally defined as:

$$C_T = \frac{T}{\rho A(\Omega R)^2}, \tag{1}$$

which is the US customary definition.

4   Flare Hight Estimate

In order to approximate what height the aircraft will begin to flare, Blade Element Momentum Theory (BEMT) is used to construct an approximation of the deceleration that can be expected from the helicopter in the flare, hence at what height the flare must begin to achieve a successful landing.
TODO: Give a little flow diagram of the calculation order to provide context

4.1   Approximation of Hover Inflow Ratio

To be able to determine an approximation for rotor drag, the inflow ratio required to maintain the rotor head speed in autorotation is first required. In this work it is assumed that the flow induced by the main rotor in the hover case, is sufficient to drive the rotor head at the same speed when the flow is reversed for the autorotation case.
To avoid duplication of already published work the BEMT equations used herein will not be derived from first principles. Instead, the derivation will start from equation (3.22) of [1] (Page 120), which gives a relationship for the coefficient of thrust (C_T) with respect to the inflow ratio (λ) for a rotor head with **zero blade twist** and by assuming a **uniform inflow**:

$$C_T = \frac{1}{2}\sigma C_{l_\alpha}\left(\frac{\theta}{3} - \frac{\lambda}{2}\right), \tag{2}$$

in which C_{l_α} is the lift curve slope of the rotor blade, σ is the solidity ratio, and θ is the blade collective angle. For cleanliness in the equations, σC_{l_α} will be grouped into one variable k_1:

$$C_T = \frac{1}{2}k_1\left(\frac{\theta}{3} - \frac{\lambda}{2}\right). \tag{3}$$

As the hover case is of specific interest in this application the following simplification can be made:

$$\lambda = \lambda_i + \lambda_c = \lambda_i, \tag{4}$$

as the inflow ratio due to climbing flight (λ_c) is zero in this case leaving only the induced inflow ratio (λ_i). From simple momentum theory, the following relationship between λ_i and C_T is known (equation (2.32) in [1] (Page 67)) for the hovering case:

$$\lambda_i = \sqrt{\frac{C_T}{2}}. \tag{5}$$

Substitution of equation (5) into (3), and rearranging, gives a quadratic equation for the inflow ratio:

$$2\lambda_i^2 = \frac{1}{2}k_1\left(\frac{\theta}{3} - \frac{\lambda_i}{2}\right),$$

$$\lambda_i^2 = \frac{1}{4}k_1\left(\frac{\theta}{3} - \frac{\lambda_i}{2}\right),$$

$$\lambda_i^2 = \frac{k_1}{12}\theta - \frac{k_1}{8}\lambda_i,$$

$$\lambda_i^2 + \frac{k_1}{8}\lambda_i - \frac{k_1}{12}\theta = 0. \tag{6}$$

$$\tag{7}$$

Resolving the roots of the quadratic equation:

$$\lambda_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$\lambda_i = \frac{-\dfrac{k_1}{8} \pm \sqrt{\left(\dfrac{k_1}{8}\right)^2 - 4 \times 1 \times \left(\dfrac{k_1\,\theta}{12}\right)}}{2 \times 1},$$

$$\lambda_i = -\frac{k_1}{16} \pm \sqrt{\frac{k_1^2}{256} - \frac{k_1\,\theta}{12}}, \tag{8}$$
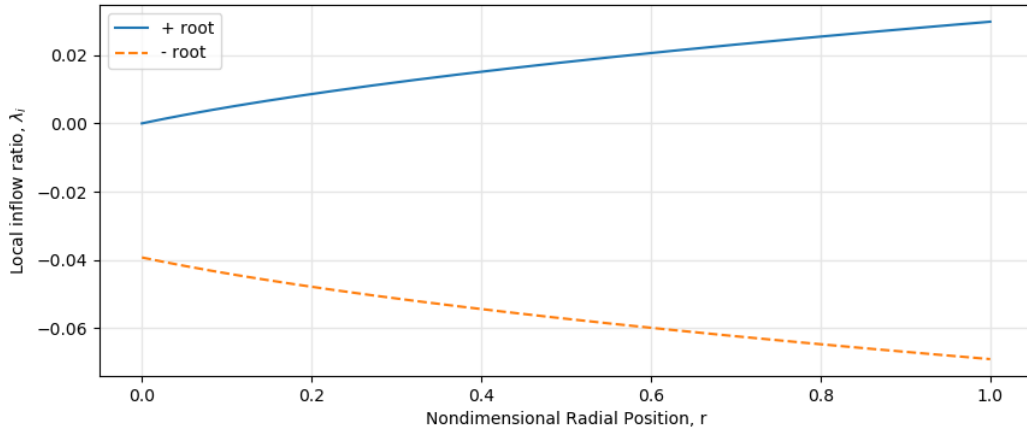

Figure 1: Local inflow distribution obtained by the roots of equation **??**.

4.2 Flare Time Calculation

From conservation of momentum (Newtons 2nd Law):

$$m\frac{dv}{dt} = C_R v^2 - mg. \tag{9}$$

Re-arranging to obtain an expression for the change in time with respect to change in velocity:

$$\frac{dv}{dt} = \frac{C_R v^2 - mg}{m}, \tag{10}$$

$$dt = \frac{m}{C_R v^2 - mg}\, dv. \tag{11}$$

Integrating to allow computation of the time required for the aircraft to move from the initial flare velocity (v_i) to the exit flare velocity (v_e):

$$\int_0^{t_f} dt = \int_{v_i}^{v_e} \frac{m}{C_R v^2 - mg}\, dv, \tag{12}$$

$$t_f = \frac{m}{C_R} \int_{v_i}^{v_e} \frac{1}{v^2 - \dfrac{mg}{C_R}}\, dv. \tag{13}$$

To maintain simplicity in the equations, defining a new grouping variable (k_1):

$$k_1 = \sqrt{\frac{mg}{C_R}}, \tag{14}$$

hence equation (13) becomes:

$$t_f = \frac{m}{C_R} \int_{v_i}^{v_e} \frac{m}{v^2 - k_1} \, dv \tag{15}$$

To evaluate the quadratic integral, recall [2, 3]:

$$\int \frac{1}{ax^2 + bx + c} \, dx = \frac{1}{\sqrt{b^2 - 4ac}} ln \left(\left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| \right) + C, \tag{16}$$

if the discriminant $(b^2 - 4ac)$ is positive. Hence, using equation (16) for the case in presented in equation (15) the following is obtained:

$$t_f = \frac{m}{C_R} \left[\frac{1}{\sqrt{0^2 - 4 \times 1 \times (-k_1^2)}} ln \left(\left| \frac{2 \times 1 \times v + 0 - \sqrt{0^2 - 4 \times 1 \times (-k_1^2)}}{2 \times 1 \times v + 0 + \sqrt{0^2 - 4 \times 1 \times (-k_1^2)}} \right| \right) \right]_{v_i}^{v_e},$$

$$t_f = \frac{m}{C_R} \left[\frac{1}{\sqrt{4k_1^2}} ln \left(\left| \frac{2v - \sqrt{4k_1^2}}{2v + \sqrt{4k_1^2}} \right| \right) \right]_{v_i}^{v_e},$$

$$t_f = \frac{m}{C_R} \left[\frac{1}{2k_1} ln \left(\left| \frac{v - k_1}{v + k_1} \right| \right) \right]_{v_i}^{v_e}, \tag{17}$$

The final expression for time required to flare the aircraft is therefore:

$$t_f = \frac{m}{C_R} \left(\frac{1}{2k_1} ln \left(\left| \frac{v_e - k_1}{v_e + k_1} \right| \right) - \frac{1}{2k_1} ln \left(\left| \frac{v_i - k_1}{v_i + k_1} \right| \right) \right) \tag{18}$$

Note, in this problem, the math is being constructed to develop a means to determine when to initiate the flare phase of the auto rotation. Therefore, when computing the time required to flare initial velocity (v_i) is known

References

[1] Leishman. *Principles of Helicopter Aerodynamics.* Cambridge Aerospace Series, 2nd edition, 2006.

[2] Wikipedia contributors. Quadratic integral — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Quadratic_integral&oldid=1069728369`, 2022. [Online; accessed 1-April-2024].

[3] Math Industries SRL. Fundemental integrals. `https://www.youmath.it/lezioni/analisi-matematica/integrali/596-integrali-notevoli.html`, 2023. [Online; accessed 1-April-2024].