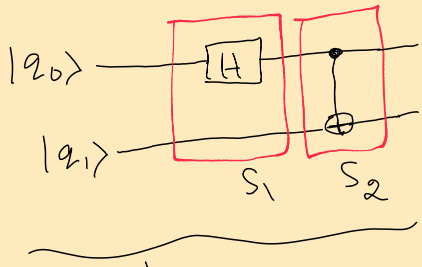
Bell States:-

* If we start working with the basis $(|0\rangle, |1\rangle)$ then we apply some gates and get new basis. So one of the most important basis state is Bell States.

So we use 2 qubits in $(|0\rangle, |1\rangle)$ basis and use Hadamard gate and CNOT gate to achive Bell states.

$\boxed{q_0, q_1 \in [0,1]}$



$S_1 = H \otimes I$

$S_2 = CNOT_{1\to2}$

So we know $S_2 = CNOT_{1\to 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$S_1 = H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(4×4)

Circuit equation:-

We can write the circuit as an unitary operator like:-

$U = CNOT_{1\to 2} \cdot (H \otimes I)$ (After simplification)

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ X & -X \end{pmatrix}$$

$$\boxed{U\begin{pmatrix} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{pmatrix} = \underline{\text{Result}}}$$

So now the combined state is $|q_0 q_1\rangle$ and all possibilities of this is $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.

So U ~~transforms~~

$$|00\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|01\rangle \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|10\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|11\rangle \to \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

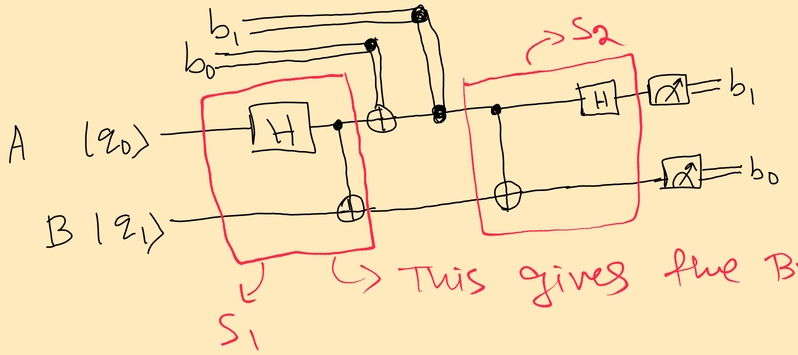
* You might wonder why we write 2-qubit basis in order $(|00\rangle, |01\rangle, |11\rangle, |10\rangle)$ instead $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.

  That's because we can only jump to next state by changing one bit at a time.

* This is important because we will use Bell states to understand Superdense coding and Quantum Teleportation.

learn about entanglement a bit before reading this)

Superdense Coding:- (we want to send 2 classical bits by sending only one qubit)



S_2 is inverse of S_1

This gives the Bell States.

The effective circuit equation is:-

$$(H\otimes I).(CNOT_{1\to 2}).(Z_1^{b_1}X_1^{b_0}).(CNOT_{1\to 2}).(H\otimes I)$$

* So let's say 2 person are there A and B. A want to send b_0 and b_1 to B. Initially they both have a qubit q_0 and q_1 respectively. Then they created Bell States out of their qubits. Note that their states are already entangled.

* Now if A and B are separated way long and A want to share 2 bits to B then we use this circuit.

let's say A want to send $(b_0 b_1 = 00)$ to B:-

Then The Bell State will be:- $|0_A 0_B\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$

as $b_0 = b_1 = 0$ then $(Z_1^{b_1}X_1^{b_0} = I)$, i.e. the Bell state will be unaffected and finally S_2 nullifies the effect of S_1 and after measurement B gets $b_1 = 0$ and $b_0 = 0$.

* Similarly if A want to send $b_1=0, b_0=1$ then $Z_1^{b_1=0}X_1^{b_0=1} = X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

if $\overset{b_1 b_0}{(1\,0)}$ then $Z_1^{b_1=1}X_1^{b_0=0} = Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

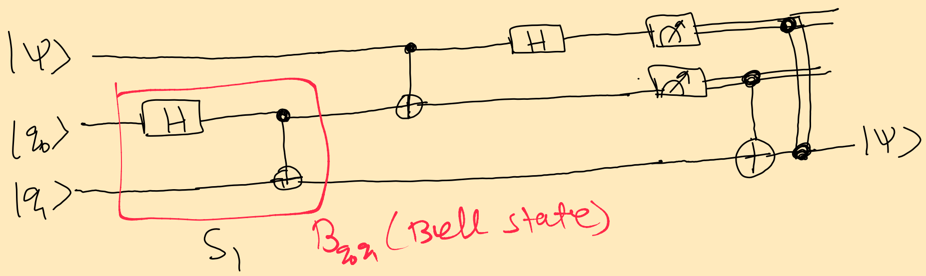
if (11) then $Z_1^{b_1=1}X_1^{b_0=1} = Z_1X_1 = iY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

These operations will modify the Bell States Before applying S_2 so that when B will measure he will get b_0 and b_1 which A sent him.

* Do the calculations yourself to see that the circuit works.

Quantum Teleportation:-

This is the opposite of SDC. Here we want to send the Qubit by using 2 classical bits.



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

So we can write combined qubit as $|\psi'\rangle = |\psi\rangle|q_0q_1\rangle$

let's set $(q_0q_1 = 00)$. $\Rightarrow |\psi'\rangle = |\psi\rangle|00\rangle$
(example)

① After S_1 $|\psi_1'\rangle = |\psi\rangle|B_{00}\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

② Then CNOT ($c = \psi$ and $t = q_0$) :- $|\psi_2'\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

③ Then applying Hadmard on ψ_2 :-

$$|\psi_3'\rangle = \frac{1}{2}\Big(\alpha\big(|000\rangle + |100\rangle + |011\rangle + |111\rangle\big) + \beta\big(|010\rangle - |110\rangle + |001\rangle - |101\rangle\big)\Big)$$

we can rearrange and can write:-

$$|\psi_3'\rangle = \frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big)$$

Now A will send 2 classical bits after the measurement on modified $|\psi\rangle$ and $|q_0\rangle$ and send to B. And based on the bits sent B applies $\underbrace{(Z^{\psi}_{q_1}, X^{q_0}_{q_1})}$ and get the $|\psi\rangle$ state.

This works exactly same way we have discussed earlier.