Lets see what have you got correct here,

Global phase is rotation of the state about unit hypersphere?

(Note: Unit hypersphere is the surface satisfying norm 1 condition)

Yes, any Unitary transformation is rotation about unit hypersphere and global phase can be written as:

$$e^{i\theta}|\psi\rangle = U|\psi\rangle$$

where, $U = e^{i\theta} I$

But global phase represents only a small subset of the general rotations possible, where each component of the basis chages by the same factor.

Don't get it wrong that this global phase rotation doesn't change the vector $|\psi\rangle$. It does change mathematically but it still represents the same physical system. So, this invariance under global phase is a physical constraint put by us (just like the unit norm condition). Lets see why,

Prob. of getting eigenvalue λ as our measurement result $= P(\lambda)$

Now, $P(\lambda) = |\langle\lambda|\psi\rangle|^2$

Now, both $|\psi\rangle$ & $e^{i\theta}|\psi\rangle$ gives the same result hence both represents same quantum system.

But, if we take

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$$

$$\& \ |\psi'\rangle = |\psi_1\rangle + e^{i\theta}|\psi_2\rangle$$

then, as you can see $P(\lambda)$ is different for $|\psi\rangle$ & $|\psi'\rangle$ so adding relative phase changes the system.

But then why do we use unitary matrices U as time propagator? Because U doesn't any represent global phase rotation, it can also change the state vector such that it adds a relative phase or even change the amplitudes of the component vector all together. Such type of rotation changes the quantum system. So, since U can represent any general rotation about unit hyper sphere, we can use it as our propagator.

Think of all like this, suppose we have,

$$\vec{v} = \frac{\hat{i} + \hat{j}}{\sqrt{2}} \quad \text{(On the real 2-D space)}$$

If we write,

$$e^{i\pi}\vec{v} = \frac{-\hat{i} - \hat{j}}{\sqrt{2}}$$

then this is a different vector on the unit circle obtained using a special subset of rotation (reflection about origin here). But in QM we put our physical constraint on it through which both vectors represent the same system. But if we perform any rotation outside this small subset like

$$\vec{v}_1 = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

then both mathematically & physically $\vec{v}$ & $\vec{v}_1$ are different.

Something for you to think & find on your own - What is the rotation subset on the vector space C^n leaving state invariable?