PreCalculus PreAP
Algebra Review

Name Fernando Trejano
Date ______________ Period ____

Remember the following factoring rules:

Cube of a binomial:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^3 - 3a^2b + 3ab^2 - b^3$$

Sum or difference of two cubes:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Factoring by grouping:

$$3x^2 + 6x - 4x - 8$$

$$(3x^2 + 6x) + (-4x - 8)$$

$$3x(x+2) - 4(x+2)$$

$$(x+2)(3x-4)$$

Factor the following. Show all work.

1. $12ab^2x + 6a^2bx^3 - 30ab^3$

   $6ab(2bx + ax^3 - 5b^2)$

2. $3a^4b^2m - 75a^3bm^4$

   $3a^3bm(ab - 25m^3)$

3. $x^2(x^2-25) - 9(x^2-25)$

   $x^2(x+5)(x-5) - 9(x+5)(x-5) = (x+3)(x-3)(x+5)(x-5)$

4. $2(a^2-1)b^2 - 8(a^2-1)$

   $2(a+1)(a-1)b^2 - 8(a+1)(a-1)$

5. $200x^2 - 50$

   $50(4x^2-1)$ $50(2x-1)(2x+1)$

6. $2x^3 + 6x^2 - 36x$

   $2x(x^2+3x-18)$ $2x(x+6)(x-3)$

7. $8x^2 + 10x + 3$

   $(4x+3)(2x+1)$

8. $t^3 + 1$

   $(t+1)(t^2 - 1t + 1)$

9. $x^3 - 27$

   $(x-3)(x^2+3x+9)$

10. $(a^2+2a)^2 - 2(a^2+2a) - 3$

    $b^2 - 2b - 3$ $(x-3)(x+1)$ $(a^2+2a-3)(a^2+2a+1)$ $(a+3)(a-1)(a+1)^2$

11. $(x+1)^3x - 2(x+1)^2x^2 + x^3(x+1)$

    $a = (x+1)$ $a^3x - 2a^2x^2 + ax^3$ $(a+x)^3$ $(\cancel{x+1+x})$ $(2x+1)^3$

12. $4x^{\frac{1}{2}}y^{\frac{3}{4}} - 8x^{\frac{3}{2}}y^{\frac{1}{4}}$

    $4x^{\frac{1}{2}}y^{\frac{1}{4}}(y^{\frac{2}{4}} - 2x^{\frac{2}{2}}) = 4x^{1/2}y^{1/4}(y^{1/2} - 2x)$

13. $5a^{\frac{3}{5}}b^{\frac{1}{3}} - 45a^{\frac{13}{5}}b^{\frac{7}{3}}$

    $5a^{\frac{3}{5}}b^{\frac{1}{3}}(1 - 9a^2b^2)$

14. $(3x+4)^{\frac{3}{4}} + 4(3x+4)^{\frac{7}{4}}$

    $(3x+4)^{3/4}(1 + 4(3x+4))$

15. $(3x+5)^{\frac{2}{3}}(4x-3)^{\frac{7}{2}} - (3x+5)^{\frac{5}{3}}(4x-3)^{\frac{1}{2}}$

    $(3x+5)^{2/3}(4x-3)^{1/2}((4x-3)^3 - (3x+5))$

Simplify.

16. $\dfrac{3y+15}{3y-15}$ $\quad \dfrac{\cancel{3}(y+5)}{\cancel{3}(y-5)}$

17. $\dfrac{x^2-25}{x^2-7x+10}$ $\quad \dfrac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}(x-2)}$

18. $\dfrac{x^2+4x-21}{x^2-6x+9}$ $\quad \dfrac{(x+7)\cancel{(x-3)}}{(x-3)\cancel{(x-3)}}$

19. $\dfrac{3x^3-21x^2+18x}{3x^2-3x}$ $\quad \dfrac{\cancel{3x}(x^2-7x+6)}{\cancel{3x}(x-1)}$ $\quad \dfrac{(x-6)\cancel{(x-1)}}{\cancel{(x-1)}}$

20. $\dfrac{x+\dfrac{1}{x}}{x-\dfrac{1}{x}}$ $\quad \dfrac{\frac{(x^2+1)}{\cancel{x}}}{\frac{(x^2-1)}{\cancel{x}}} = \dfrac{x^2+1}{x^2-1}$

21. $\dfrac{\dfrac{3}{x}+\dfrac{5}{y}}{\dfrac{5}{x}-\dfrac{1}{y}}$ $\quad \dfrac{\frac{3y+5x}{\cancel{xy}}}{\frac{5y-x}{\cancel{xy}}}$ $\quad \dfrac{3y+5x}{5y-x}$

22. $\dfrac{\dfrac{c^2-d^2}{cd}}{\dfrac{c-d}{c}}$ $\quad \dfrac{(c+d)\cancel{(c-d)}}{\cancel{c}d\cancel{(c-d)}}$ $\quad \dfrac{(c+d)}{d}$

Calculus BC
Ch 2 & 3 Review Notes day 2

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Chapter 2 Review

Explicit vs Implicit Form:

Explicit: $y = 3x - 5$
Explicitly defined in y form

Implicit: $xy = 1$
Cannot isolate y

Find $\frac{dy}{dx}$**:**

Ex 1: $x^2 + y^2 = 36$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y}$

↳ no need to symplify

Ex 2: $x^3y^3 - y = x$

$3x^2\frac{dx}{dx} \cdot y^3 + x^3 \cdot 3y^2\frac{dy}{dx} - (1)\frac{dy}{dx} = 1\frac{dx}{dx}$

$(3x^3y^2 - 1)\frac{dy}{dx} = 1 - 3x^2y^3$

$\frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$

Ex 3: $\sin x + 2\cos(2y) = 1$

$\cos x + -2\sin(2y) \cdot 2\frac{dy}{dx} = 0$

$-4\sin(2y)\frac{dy}{dx} = -\cos x$

$\frac{dy}{dx} = \frac{\cos x}{4\sin(2y)}$

Ex 4: $\tan(x+y) = x$

$\sec^2(x+y)\left(1 + \frac{dy}{dx}\right) = 1$

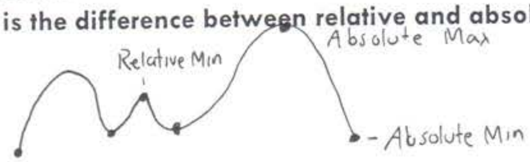
$1 + \frac{dy}{dx} = \frac{1}{\sec^2(x+y)}$

$\frac{dy}{dx} = \frac{1}{\sec^2(x+y)} - 1$

Chapter 3 Review

Extrema on an Interval:

What is the difference between relative and absolute extrema?



↳ derivative = 0 at points

How do we find extrema on a closed interval?

① Find critical #
derivative is 0 or undefined

② Find y values crit # and end points

Calculus BC
Ch 2 & 3 Review Notes day 1

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Chapter 2 Review

The slope at a point on a function is called the ____________ of the function.

In order for a function, f(x), to be differentiable, the following must be true:

1. continious (no asymtopes, no hiatus, no jumps)
2. no sharp turns
3. no vertical tangents

The definition of the derivative of a function:

$$\lim_{x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta X}$$

Ex 1: Find the derivative of the function $y = x^3 - 12x$ using the definition of a derivative.

Ex 2: Find the slope of the function $y = t^2 + 3$ using the definition of a derivative at the point $(-2, 7)$.

$$\lim_{t \to -2} \frac{[t^2+3]-7}{t-(-2)} = \lim_{t \to -2} \frac{t^2-4}{t+2} = \lim_{t \to -2} t-2 = -4$$

3.7 Optimization Problems ~ Day 1 Notes

Frequently, we want to optimize values in math and in life situations. For example, we often hear about the greatest profit, least time, optimum size, minimum cost, etc. In the first half of Chapter 3, we learned how to find the local or absolute extrema of a function. We apply these ideas in this section to optimize values in various situations.

Problem-Solving Strategy for Solving Applied Optimization (max/min) Problems:

1. Assign variables to all given quantities and quantities to be determined. When necessary, make a sketch.
2. Write a primary equation for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of a secondary equation, relating the variables that are in the primary equation.
4. Determine the domain of the primary equation – what are reasonable values that we could get for solutions?
5. Determine the desired maximum or minimum value by the calculus techniques discussed in sections 3.1-3.4

Example 1:
Find two positive numbers that satisfy the given requirements:
The product is 192 and the sum is minimum.

$xy = 192$

$y = \frac{192}{x}$

min

$S = x + y$

$S = x + \frac{192}{x}$

$\frac{dS}{dx} = 1 - \frac{192}{x^2}$

set equal to 0 → $1 - \frac{192}{x^2} = 0$

$1 = \frac{192}{x^2}$

$x^2 = 192$

$x = \pm\sqrt{192}$

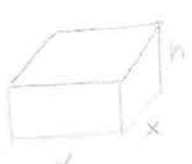
min b/c cc ↑

$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0$ when $x = \sqrt{192}$

$y = \frac{192}{\sqrt{192}} \cdot \frac{\sqrt{192}}{\sqrt{192}} = \sqrt{192}$

S is a min when $x = y = \sqrt{192}$

Example #2:
A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$SA = x^2 + 4xh$

$108 = x^2 + 4xh$

$108 - x^2 = 4xh$

$h = \frac{108 - x^2}{4x}$

max

$V = x^2 h$

$= x^2\left(\frac{108 - x^2}{4x}\right)$

$= \frac{108x^2 - x^4}{4x}$

$= 27x - \frac{x^3}{4}$

$\frac{dV}{dx} = 27 - \frac{3x^2}{4}$

set equal to 0 , $\frac{3x^2}{4} = 27$

$x^2 = 36$

$x = \pm 6$

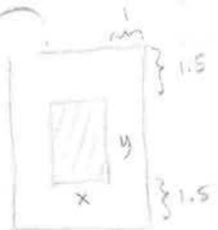
max b/c cc ↓

$\frac{d^2V}{dx^2} = \frac{-6x}{4} < 0$ when $x = 6$

V is a max when $x = 6$ in, $h = 3$ in

Example #3:

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are each 1.5 inches. The margins on each side are 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



$24 = xy$

$y = \frac{24}{x}$

min

$A = (x+2)(y+3)$

$= (x+2)\left(\frac{24}{x}+3\right)$

$= 24 + 3x + \frac{48}{x} + 6$

$= 3x + 30 + 48/x$

$\frac{dA}{dx} = 3 - 48/x^2$

set equal to 0

$\frac{48}{x^2} = 3$

$x^2 = 16$

$x = \pm 4$

$\frac{d^2A}{dx^2} = \frac{96}{x^3} > 0$ when $x = 4$

✓ min b/c cc ↑

A is a min when $x = 4$ in, $y = 6$ in

So page $= 6 \times 9$ in

Example 4:

A solid is formed by adjoining a hemisphere to each end of a right circular cylinder. The total volume of the figure is 12 cubic inches. Find the radius of the cylinder that produces the minimum surface area.

$12 = \pi r^2 h + 4/3\,\pi r^3$

$36 = 3\pi r^2 h + 4\pi r^3$

$h = \frac{36 - 4\pi r^3}{3\pi r^2}$

$h = \frac{12}{\pi r^2} - \frac{4r}{3}$

min

$S = 4\pi r^2 + 2\pi r h$

$= 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4r}{3}\right)$

$= 4\pi r^2 + \frac{24\pi r}{\pi r^2} - \frac{8\pi r^2}{3}$

$= \frac{4\pi r^2}{3} + \frac{24}{r}$

$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2}$

set equal to 0

$\frac{8\pi r}{3} = \frac{24}{r^2}$

$8\pi r^3 = 72$

$r^3 = 9/\pi$

$r = \sqrt[3]{9/\pi}$

min b/c cc ↑

$\frac{d^2S}{dr^2} = \frac{8\pi}{3} + \frac{48}{r^3} > 0$

when $r = \sqrt[3]{9/\pi}$

SA is a min when $r = \sqrt[3]{9/\pi}$ in, $h = 0$ in

→ sphere with $r \approx 1.42$ in

3.7

Optimization Problems

A rectangle has it's base on the x axis and it's upper two vertecies on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have and what are it's dimentions

$A = \text{base} \cdot \text{height}$

$A = 2xy$

$A = 2x(12 - x^2)$

$A = 24x - 2x^3$

$\frac{dA}{dx} = 24 - 6x^2$

$0 = 24 - 6x^2$

$x = 2$ $\cancel{x = -2}$

$\nearrow$ Area $= 4(8) = 32$

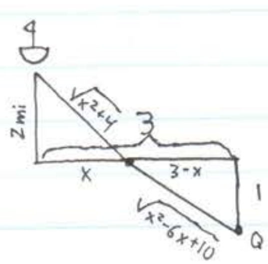
CHECK:

$\frac{d^2A}{dx^2} = -12x$

when $x = 2$, $\frac{d^2A}{dx^2} < 0$

so cc↓ ∴ max

A man is in a boat 2 miles from the nearest point on the coast. He is to go to point Q, located 3 miles down the coast and 1 mile inland. He can row 2 mph and walk 4. Toward what point on the coast should he row in order to reach point Q in the least time?



$v = d/t$

Row $\frac{2\,mi}{hR}$

WALK $\frac{4\,mi}{hR}$

min T

$T = d/v$

$T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

$\frac{dT}{dx} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(x^2+4)^{-1/2}(2x) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)(x^2-6x+10)^{-1/2}(2x-6)$

$0 = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$

$x = 1$ mile down the cost

CHECK

$\frac{d^2T}{dx^2}$ $x = 1$

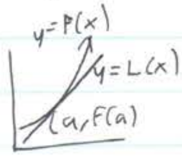
$= .2012 > 0$

↙

Math
↓
8 nDeriv
↓
Vars → YVars → Function → Y_1

Linear Approximations and Differentials

An equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is



$y - y_1 = m(x - x_1)$

$y - f(a) = f'(a)(x - a)$

* $L(x) = f'(a)(x-a) + f(a)$

Example

$f(x) = \sqrt{x+3}$ at $a = 1$

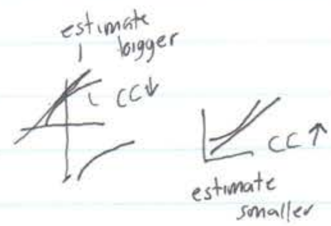
$f'(x) = \frac{1}{2}(x+3)^{-1/2}$

$L(x) = f'(a)(x-a) + f(a)$ $\quad$ $f'(1) = 1/4$

$L(x) = (1/4)(x-1) + 2$

↳ you can now get an ESTIMATE by plugging it in here

you can get an ACTUAL value by plugging in original



$f(x) = 1 + \sin x$ at $x = 0$

$L(x) = f'(a)(x-a) + f(a)$ $\quad$ $f'(x) = \cos x$

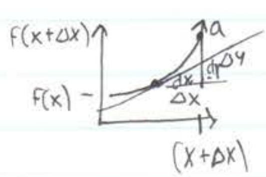
$L(x) = (1)(x-0) + 1$ $\quad$ $f'(0) = 1$

Differentials — change in y

dy = Estimated change in y - based on tangent line

$dx = \Delta x$ = Actual change in x

Δy Actual change in y



$\Delta y = \underbrace{f(x+\Delta x)}_{\text{new}} - \underbrace{f(x)}_{\text{old}}$

$\frac{dy}{dx} = f'(x) \rightarrow dy = f'(x)\,dx$

f(x)	Derivative	Differentials
$y = x\cos x$	$\frac{dy}{dx} = \cos x - x\sin x$	$dy = (\cos x - x\sin x)\,dx$

Antiderivatives and Indefinite Integration

$\frac{dy}{dx} = 2x$

$dy = 2x\,dx$

$\int dy = \int 2x\,dx$

$y = x^2 + c$

variable of integration

$y = \int f(x)\,dx = F(x) + C$ ← constant of integration

integrand antiderivative

p250 of Book

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

Example $\int 4x^2\,dx =$

$4\int x^2\,dx$ DO NOT FORGET

$= \frac{4 \cdot x^3}{3} + C$

CHECK BY taking derivative

2) $\int \frac{1}{x^2}\,dx$

$= \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C$

$= -1/x + C$

3) $\int \sqrt{x}\,dx =$

$\int x^{1/2}\,dx$

$\frac{x^{3/2}}{(3/2)} + C = \frac{2}{3}x^{3/2} + C$

4) $\int 2\sin x\,dx =$

$2\int \sin x\,dx = -2\cos x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int -\csc^2 x\,dx = \cot x + C$

5) $\int (2t^3 - 7t^2 + 4)\,dt$

$= \frac{2t^4}{4} - \frac{7t^3}{3} + 4t + C$

6) $\int \frac{x+1}{\sqrt{x}}\,dx$

$\int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx = \int (x^{1/2} + x^{-1/2})\,dx$

$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$

$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

7) $\int (2x+1)(x-1)\,dx$

$\int (2x^2 - 8x + x - 4)\,dx$

$\int (2x^2 - 7x - 4)\,dx$

$\frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + C$

Find $F(x)$ if $F'(x) = \frac{1}{x^2}$ $x>0$ when $F(1)=0$

know $F'(x) = 1/x^2$

$f(x) = \int x^{-1} dx = -1x^{-1} + C = -1/x + C$

general equation/solution

$0 = \frac{-1}{1} + C$ $\quad$ $C = 1$

$F(x) = -1/x + 1$ $\quad$ Particular solution

Find $f(x)$ if $f''(x) = x^2$ $\quad$ $f'(0) = 6$ $\quad$ $f(0) = 3$

$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C$

$6 = \frac{1}{3}x^3 + C$ $\quad$ $C = 6$

$f'(x) = \frac{1}{3}x^3 + 6$

$F(x) = \int(\frac{1}{3}x^3 + 6)dx$

$= \frac{1}{12}x^4 + 6x + C$ $\quad$ $c = 3$

$F(x) = \frac{1}{12}x^4 + 6x + 3$

A ball is thrown upwards with a $V_0 = 64$ ft/s $\quad$ $\Delta S = 80$ Initial height

a) Find the position function

know

$v(0) = 64$

$a(t) = -32$

$s(0) = 80$

$v(t) = \int -32\, dt = -32t + C$

$v(t) = -32t + 64$ $\quad$ $C = 64$

$s(t) = \int(-32t + 64)dt = -16t^2 + 64t + C$

$s(0) = 80$ so $C = 80$

$s(t) = -16t^2 + 64t + 80$

b) when does it hit the ground?

$s(t) = -16t^2 + 64t + 80$

$0 = -16t^2 + 64t + 80$

$-16(t^2 - 4t - 5)$

$(t-5)(t+1)$

$t = 5$ secs $\quad$ $t = -1$

A traffic light turns green, standing car accelerates at 6 ft/s^2

A moving truck at 30 ft/s passes the car

a) How far beyond the starting point will the automobile overtake the truck

<u>CAR</u>

$s(0)=0$

$v(0)=0$

$a(t)=6$

$v(t)=6t+\cancel{c}$ $c=0$

$s(t)=3t^2$

TRUCK

$s(0)=0$

$v(t)=30$

$s(t)=30t+\cancel{c}$ $c=0$

$S_c(t)=S_t(t)$ DO NOT DIVIDE BY T

$3t^2=30t$

$3t^2-30t=0$

$3t(t-10)=0$

$t=0$ $t=10$ secs

300 ft

AP Calculus BC

4.3 ~ Riemann Sums and Definite Integrals

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Last lesson: We learned to find the area under the curve by dividing a region into rectangles and adding the area of the rectangles. The widths of the rectangles were **equal.**

Riemann sum $\sum_{i=1}^{n} (f(c_i))\Delta x_i$

Today: We will look at dividing the area under the curve into rectangles with **unequal widths.**

To find the *exact* area under the curve last time...

equal width Δx

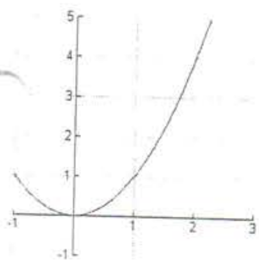
To find the *exact* area under the curve this time...

diff width $\|\Delta\|$ (norm)

How are these notations similar?

Connecting to the definite integral...

For example...



Area under the curve of $y = x^2$ on the interval [0,2]

$$\lim_{n\to\infty} \sum_{i=1}^{n} (c_i)^2 \Delta x = \int_0^2 x^2 dx$$

Definite Integral Notation: $\int_a^b f(x)dx$

If f is defined on [a, b] and f is integrable, then...

This is called the definite integral

a is the lower limit b is the upper limit

Theorem: Continuity implies integrability

If f is continuous on [a, b], then f is integrable of [a, b] and AREA = $\int_{a \leftarrow \text{start}}^{b \leftarrow \text{stop}} (f(x)) dx$ ← function, ← with respect to

So, the DEFINITE INTEGRAL gives the area under the curve of a bounded region.

4.4 Fundamental Theorem of Calculus

$$\int_0^4 f(x)\,dx$$

Area under the line / curve between 0,4

Rectangles touch dx

$$\int_0^2 (-2x+4)\,dx$$

$$= \left[-2x^2+4x\right]_0^2$$ - no +C b/c definite integral

$$4 - 0 = 4$$

FIRST FUNDAMENTAL THEOREM of CALCULUS

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_2^3 (2x)\,dx = \left[x^2\right]_{-2}^3$$ plugs in upper and lower

$$= 3^2 - (-2)^2 = 5$$

* Can also do in calculator

$$\int_1^4 (x^3-3)\,dx = \left[\frac{x^4}{4} - 3x\right]_1^4$$

$$= \left[\frac{4^4}{4} - 12\right] - \left[\frac{1}{4} - 3\right] = 52 - \frac{1}{4} + 3 = 54.75$$

$$\int_1^4 (3\sqrt{x})\,dx = \left[\frac{3x^{3/2}}{3/2}\right]_1^4 = \left[2x^{3/2}\right]_1^4$$

$$= 2(4)^{3/2} - 2(1)^{3/2} =$$

$$16 - 2 = 14$$

$$\int_0^{\pi/4} (\sec^2 x)\,dx = \left[\tan x\right]_0^{\pi/4}$$

$$1 - 0 = 1$$

$$\int_0^2 |2x-1|\,dx$$ SPLIT

$$2x - 1 = 0$$
$$= 1/2$$

$$\int_0^{1/2} (-2x+1)\,dx + \int_{1/2}^2 (2x-1)\,dx$$

$$= \left[-x^2+x\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^2$$

$$\left[\left(-\tfrac{1}{4}\right)+\tfrac{1}{2}-0\right] + \left[2-\left(\tfrac{1}{4}-\tfrac{1}{2}\right)\right] = \tfrac{1}{4} + 2 + \tfrac{1}{4} + \tfrac{1}{2} = 2.5$$

Mean Value Theorem for Integrals

If f(Continuous) on [a,b] then there is # in [a,b] such that

height = $\frac{\text{area}}{\text{base}}$

$$= \frac{\int_a^b F(x)\,dx}{b-a}$$

gives the average value of the function

avg height

Ex. Find the average value of

$f(x) = 3x^2 - 2x \quad [1,4]$

$h = \frac{A}{B}$

$A = \int_1^4 (3x^2 - 2x)\,dx = [x^3 - x^2]_1^4 = 48$

$b = 3$

height = value = $48/3 = 16$

Find c guaranteed by MVT

At what value does that height occur

$F(c)$ = average value

plug in original

$16 = 3x^2 - 2x$

$3x^2 - 2x - 16 = 0$

Fernando Trujano

4.4 Review Worksheet

Name Fernando Trujano

The Fundamental Theorems of Calculus

In-Class:

Given $\frac{dy}{dx} = 3x^2 + 4x - 5$ with the initial condition $y(2) = -1$. Find $y(3)$.

Method 1: Integrate $y = \int (3x^2 + 4x - 5)\,dx$, and use the initial condition to find C. Then write the particular solution, and use your particular solution to find $y(3)$.

$y = \frac{\cancel{3}x^3}{\cancel{3}} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} - 5x + C$

$-1 = 2^3 + 2(2)^2 - 5(2) + C$

$C = -7$

$y = x^3 + 2x^2 - 5x - 7$

$y(3) = 3^3 + 2(3)^2 - 5(3) - 7$

$y(3) = 23$

AP style

Method 2: Use the First Fundamental Theorem of Calculus: $\int_a^b f'(x)\,dx = f(b) - f(a)$

$\int_2^3 (3x^2 + 4x - 5)\,dx = y(3) - y(2)$

$24 = y(3) + 1$

$23 = y(3)$

Do the following problem by both methods.

1. $y' = 2 + \frac{1}{x^2}$ and $y(1) = 6$. Find $y(3)$.

Method 1

$y = 2x + \frac{x^{-1}}{-1}$

$y = 2x - \frac{1}{x} + C$

$6 = 2(1) - \frac{1}{1} + C$

$C = 5$

$y = 2x - \frac{1}{x} + 5$

$y(3) = 2(3) - \frac{1}{3} + 5$

$= 32/3$

Method 2

U - Substitution

Should only have u and du

$\int (x^2+1)^2 (2x)\,dx \quad = \int u^2\,du$

$u = x^2+1$

$du = 2x\,dx \qquad = \frac{u^3}{3} + C$

$= \frac{(x^2+1)^3}{3} + C$

$\int 5\cos 5x\,dx$

$u = 5x \qquad = \int \cos u\,du$

$du = 5\,dx \qquad = \sin u + C$

$= \sin(5x) + c$

$\int (3x^2+1)^4 (3x)\,dx = \int (3x^2+1)^4 (6x)\,dx \left(\frac{1}{2}\right)$

$u = 3x^2+1 \qquad = \frac{1}{2}\int u^4\,du$

$du = 6x\,d\cancel{x} \qquad = \frac{1}{2}\cdot\frac{u^5}{5} + C$

$\frac{(3x^2+1)^5}{10} + C$

$\frac{1}{2}\int \sqrt{2x-1}\,dx$ (2)

$u = 2x-1 \qquad = \frac{1}{2}\int u^{1/2}\,du$

$du = 2\,dx \qquad = \frac{1}{2}\cdot\frac{2u^{3/2}}{3} + C$

$= \frac{(2x-1)^{3/2}}{3} + C$

$\frac{1}{3}$ (3)

$\int \sin^2 3x \cos 3x\,dx$

$= (\sin 3x)^2 \cos 3x$

$u = \sin 3x \qquad = \frac{1}{3}\int u^2\,du$

$du = 3\cos(3x)\,dx \qquad \frac{1}{3}\cdot\frac{u^3}{3} + C$

$\frac{(\sin(3x))^3}{9} + C$

$\int x\sqrt{2x-1}\,dx$

$u = 2x-1$

$du = 2\,dx$

$x = \frac{u+1}{2}$

$\frac{1}{2}\int\left(\frac{u+1}{2}\right)u^{1/2}\,du$

$\frac{1}{4}\int(u+1)\,u^{1/2}\,du$

$\frac{1}{4}\int\left(u^{3/2}+u^{1/2}\right)du$

$\frac{1}{4}\left[\frac{2u^{5/2}}{5}+\frac{2u^{3/2}}{3}\right]+C$

$= \frac{(2x-1)^{5/2}}{10} + \frac{(2x-1)^{3/2}}{6} + C$

Definite Integrals

$\frac{1}{2}\int_0^1 x(x^2+1)^3\,dx$

when $x=1$ $\quad 1^2+1$ $\quad u=2$

$u = x^2+1$ $\quad = \frac{1}{2}\int_1^2 u^3\,du$

$du = 2x\,dx$

when $x=0$ $\quad 0^2+1=1$ $\quad u=1$

$\frac{1}{2}\cdot\frac{u^4}{4}\Big]_1^2 = \frac{u^4}{8}\Big]_1^2$

$\frac{16}{8}-\frac{1}{8}=\frac{15}{8}$

Calculus BC
Ch 5.1 notes

Name Fernando Trujano
Date ______ Period ____

5.1 Notes ~ The Derivative of $\ln x$

I. Introduction

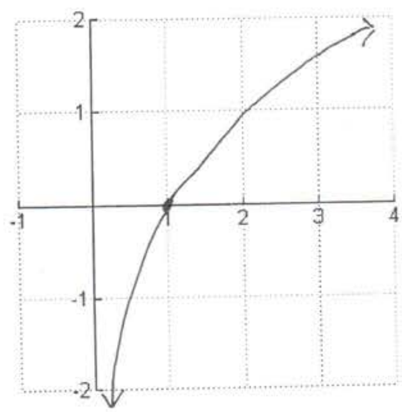
We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ … except when $n = -1$. Why?

New for today: $\int \frac{1}{x} dx = \ln x + C$

Since $\int \frac{1}{x} dx = \ln x + C$, we know that $\frac{d}{dx}[\ln x] = \frac{1}{x}$

II. Background about the natural logs

Graph:



Properties of Graph:
1) Domain: $(0, \infty)$
2) Range: $(-\infty, \infty)$
3) increasing f'(x)
4) cc↓

To evaluate derivatives involving logs, it's helpful to know some log properties:

1) $\ln(1) =$ 0
2) $\ln(ab) =$ ln(a) + ln(b)
3) $\ln(a^n) =$ n ln(a)
4) $\ln\left(\frac{a}{b}\right) =$ ln(a) - ln(b)
5) $\ln e =$ 1

5-2

Log Rule For Integration

$\int \frac{1}{x}dx = \ln|x| + C$

Example

$\int \frac{2}{x}dx = 2\int \frac{1}{x}dx$

$= 2\ln|x| + C$

$= \ln x^2 + C$

$\frac{1}{4}\int \frac{1}{4x-1} \cdot 4\, dx$

$u = 4x-1$ $\quad \frac{1}{4}\int \frac{1}{u}du$

$du = 4dx$ $\quad = \frac{1}{4}\ln|u| + C$

$= \frac{1}{4}\ln|4x-1| + C$

or $= \ln \sqrt[4]{4x-1} + C$

$\frac{1}{2}\int_0^3 \frac{x}{x^2+1} \cdot 2\, dx$

$u = x^2+1$ $\quad \frac{1}{2}\int_1^{10} \frac{1}{u}du$

$du = 2x\,dx$ $\quad = \frac{1}{2}\ln|u|\Big]_1^{10}$

$= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 1$

$= \ln\sqrt{10}$

Method 1 - rewrite

$\int \frac{x^2+x+1}{x^2+1}dx = \int \left(\frac{x^2}{x^2+1}\right) + \left(\frac{x}{x^2+1}\right) + \left(\frac{1}{x^2+1}\right) dx$

$u = x^2+1$

$du = 2x\,dx$ $\quad = \int \frac{x^2+1}{x^2+1} + \frac{x}{x^2+1}dx$

$= \int 1\,dx + \int \frac{x}{x^2+1}dx$

$x + \frac{1}{2}\ln|x^2+1| + C$

$x + \frac{1}{2}\ln(x^2+1) + C$

or $x + \ln\sqrt{x^2+1} + C$

$\int \frac{\sec^2 x}{\tan x}dx$ $\quad \int \frac{1}{u}du$

$u = \tan x$ $\quad = \ln|\tan x| + c$

$du = \sec^2 x\,dx$

Method 2 = division

$\int \frac{x^2-x+1}{x-2}$

Synthetic

~~-2~~	~~1~~	~~-1~~	~~1~~
	~~↓~~	~~-2~~	~~2~~
	~~1~~	~~1~~	~~-1~~

or

Long

$$\begin{array}{r} x+1 \\ x-2\,\overline{)\,x^2-x+1} \\ -(x^2-2x) \\ x+1 \\ -(x-2) \\ 3 \end{array}$$

$\int \left((x+1) + \frac{3}{x-2}\right)dx$

$\frac{x^2}{2} + x + 3\int \frac{1}{x-2}dx$

$u = x-2$

$du = dx$

$\frac{x^2}{2} + x + 3\ln|u| + C$

or $\frac{x^2}{2} + x + \ln|(x-2)^3| + C$

$\int \sin u\,du = -\cos u + C$

$\int \cos u\,du = \sin u + C$

$\int \tan u\,du = -\ln|\cos u| + C$

$\int \cot u\,du = \ln|\sin u| + C$

$\int \sec u\,du = \ln|\sec u + \tan u| + C$

$\int \csc u\,du = -\ln|\csc u + \cot u| + C$

$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx$

$\int_0^{\pi/4} \sec x\,dx$

$\ln|\sec x + \tan x|\Big]_0^{\pi/4}$

$\ln|\sqrt{2}+1| - \ln|1+0|$

$= \ln(\sqrt{2}+1)$

$\int \csc 2x \, dx$

$u = 2x$

$du = 2dx$

$\frac{1}{2} \int \csc u \, du$

$= -\frac{1}{2} \ln |\csc u + \cot u| + C$

$= -\frac{1}{2} \ln |\csc 2x + \cot 2x| + C$

Calculus BC
5.3 Inverse Functions Notes

Name ____________________
Date ____________________ Period____

Properties of Inverse Functions

- Coordinates Switch
  - Ex:

- $f(x)$ and $f^{-1}(x)$ "undo" each other
  - Ex: show that $f(x)=1-x^3$ and $g(x)=\sqrt[3]{1-x}$ are inverses.

- If f is the inverse of g, then g is the inverse of f.
- Not every function has an inverse. If it does, then the inverse is unique.
- $f(x)$ and $f^{-1}(x)$ are reflections of each other across the line y = x.

Determine Existence of Inverse

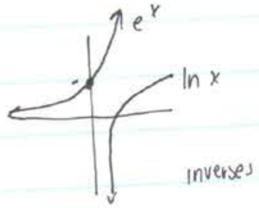
- Horizontal Line Test

- One-to-One

- Strictly Monotonic

- Ex: Which of the following has an inverse and why?
  - $f(x)=x^3-6x^2+12x$ $\qquad y=\cos\left(\dfrac{3x}{2}\right)$

The Natural Exponential Function



$y = e^x$
$\ln y = \ln e^x$
$\ln y = x$

$x = \ln_e y$
$e^x = y$

Example - Isolate x

$7 = e^{x+1}$
$\ln 7 = \ln e^{x+1}$
$\ln 7 = (x+1)\ln e$
$\ln 7 - 1 = x$

$\ln(2x+3) = 5$
$e^5 = 2x+3$
$x = \frac{e^5 - 3}{2}$

* $\frac{d}{dx}[e^u] = e^u \cdot u'$ (chain Rule!)

$y = e^{(2x+1)}$
$y' = e^{2x+1} \cdot 2$

$y = e^{-3/x}$
$y' = e^{-3/x} \cdot \frac{3}{x^2}$

Find relative extrema - derivative = 0 or undefined

$F(x) = xe^x$
$F'(x) = xe^x + e^x$
$F'(x) \neq$ und

$0 = e^x(1+x)$
$x = -1$

$\begin{array}{c|c} \multicolumn{2}{c}{-1} \\ - & + \end{array}$

minimum @ $(-1, -e^{-1})$ plugin original

* $\int e^u du = e^u + C$

$\int \sin x \, e^{\cos x} dx$
$u = \cos x$ $\quad -\int e^u du$
$du = -\sin x dx$ $\quad = -e^u + C = e^{\cos x} + C$

$\int e^{3x+1} dx =$
$u = 3x+1$ $\quad \frac{1}{3}\int e^u du$
$du = 3dx$ $\quad = \frac{1}{3}e^u + C$ $\quad \frac{1}{3}e^{3x+1} + C$

$\int_0^1 \frac{e^x}{1+e^x} dx$
$u = 1+e^x$
$du = e^x dx$
$\int_2^{1+e} \frac{1}{u} du = \ln|u| \Big]_2^{1+e}$

$\int 5xe^{-x^2} dx = -\frac{1}{2} \cdot -2 \cdot 5\int xe^{-x^2} dx$
$u = -x^2$
$du = -2x dx$
$= -\frac{5}{2}\int e^u du$
$= \frac{-5}{2}e^{-x^2} + C$

* $\int \frac{e^{1/x}}{x^2} dx$
$u = \frac{1}{x}$ $\quad -\int e^u du$
$du = \frac{-1}{x^2}$ $\quad -e^u + C = -e^{1/x} + C$

Bases other than e

$\log_b b^3 = x$

$b^x = b^3$

$x = 3$

$x^2 - x = \log_5 25$

$5^{x^2-x} = 5^2$

$x^2 - x = 2$ $\quad x = 2$, $x = -1$

* $\frac{d}{dx}[a^u] = (\ln a)a^u \cdot u'$

* $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \cdot u'$

Ex

$y = 2^{3x}$

$y' = \ln 2 \cdot 2^{3x} \cdot 3$

Log base, Original, $\frac{d}{dx}$ of exp

$y = 5^x$

$y' = \ln 5 \cdot 5^x \cdot 1$

$f(x) = \log_{10} \cos x$

$f'(x) = \frac{1}{\ln 10} \cdot \frac{-\sin x}{\cos x}$

$= -\tan x / \ln 10$

* $\int a^u \, du = \frac{1}{\ln a} \cdot a^u + C$

$\int 2^x dx$

$= \frac{1}{\ln 2} \cdot 2^x + C$

$\int (3-x)\, 7^{(3-x)^2} dx$

$u = (3-x)^2$

$du = 2(3-x)(-1)\,dx$

$= -2(3-x)\,dx$

$-\frac{1}{2}\int 7^u du$

$= -\frac{1}{2} \cdot \frac{1}{\ln 7} \cdot 7^u + C$

$= \frac{-7^{(3-x)^2}}{2\ln 7} + C$

* $y = e^e$

$y' = 0$

$y = e^x$

$y' = e^x$

$y = x^e$

$y' = ex^{(e-1)}$

$y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$y' = (x^x)(\ln x + 1)$

Chapter 5 Concept Review

Base: **variable**
Exponent: **constant**
How to take derivative: power rule

ex: $y = 2x^3$
$y' = 6x^2$

Base: **constant**
Exponent: **variable**
How to take derivative: $\frac{d}{dx}\left[a^u\right] = (\ln a)(a^u)(u')$

ex: $y = 2^{x^3}$
$y' = (\ln 2)(2^{x^3})(3x^2)$

Base: **variable**
Exponent: **variable**
How to take derivative: take ln of both sides, use power rule and other techniques

ex) $y = (2x)^{x^3}$

$\ln y = \ln(2x)^{x^3}$ — take ln of both sides

$\ln y = x^3 \cdot \ln(2x)$ — use log property to move exponent down

$\frac{y'}{y} = \left(3x^2\right)\left(\ln(2x)\right) + \left(\frac{2}{2x}\right)\left(x^3\right)$ — take derivative of both sides (product rule on right)

$y' = y\left[3x^2 \ln(2x) + x^2\right]$ — simplify and move y to the right

$y' = (2x)^{x^3}\left[3x^2 \ln(2x) + x^2\right]$ — replace y

<u>Other formulas</u>:

$$\frac{d}{dx}[\ln u] = \frac{u'}{u}$$

$$\int \frac{u'}{u}\,du = \ln|u| + C$$

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot u'$$

$$\int e^u\,du = e^u + C$$

$$\frac{d}{dx}\left[a^u\right] = \ln a \cdot a^u \cdot u'$$

$$\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)(u)} \cdot u'$$

$$\int a^u\,du = \frac{1}{\ln a} \cdot a^u + C$$

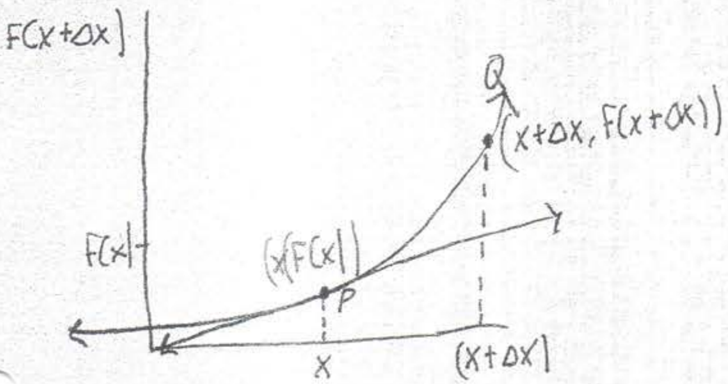
Name Fernando

2.1 Calculus Notes on Derivatives and the Tangent Line Problem

Calculus grew out of the major problems that European mathematicians were working on during the 17^{th} century. One of these is the tangent line problem that we discussed in Section 1.1. Isaac Newton and Gottfried Leibniz are given credit for the first general solution to this problem.

Goal: to find the slope of the tangent line at point P. This is also called the slope of the curve at point P.

To accomplish our goal we will take the limit of the slope of the secant lines:



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = m = \underline{derivative}$$

$(x+\Delta x) \to x$

Slope at a point

This equation gives the slope of the tangent line at point P. It also gives the slope of the graph of f at x = c.

Ex 1: A) Find the slope of the tangent line to the graph of $F(x) = x^2 + 1$ at any point.

$$\lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2}{\Delta x}$$

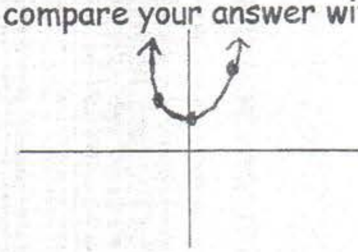
$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

B) So, the derivative of $F(x) = x^2 + 1$ is $f'(x) = $ <u>2x</u>. We can use this to find the slope at any point. Find the slope at the following points and compare your answer with the graph.

(0, 1) m = $2 \cdot 0 = 0$

(-1, 2) m = $2 \cdot -1 = -2$

(2, 5) m = $2 \cdot 2 = 4$



$\sin x = \cos x$
$\cos x = -\sin x$
$\tan x = \sec^2 x$
$\csc x = -\csc x \cot x$
$\cot x = -\csc^2 x$
$\sec x = \sec x \tan x$

The many representations of a DERIVATIVE =

Y'   f'(x)   slope at a pt. $\frac{dy}{dx}$ instanteous rate of change $\frac{d}{dx}[f(x)]$ $D_x[y]$

$y = \cos 3x^2 = \cos(3x^2)$

$y' = -\sin(3x^2) \cdot 6x = -6x\sin 3x^2$

$y = \sqrt{(\cos 3)x^2}$

$y_1 = 2(\cos 3)x$

$y = \cos(3x)^2$

$= \cos(9x^2)$ keep derivative

$y_1 = -\sin(9x^2) \cdot 18x$

$-18x\sin 9x^2$

$y = \cos^2 3x = (\cos 3x)^2$

$2\cos(3x) \cdot -\sin(3x) \cdot 3$

$-6\cos(3x)\sin(3x)$

$y = \sin^3 4t = (\sin 4t)^3$

$\cancel{3(\sin 4t)^2 \cdot 6(\sin 4t) \cdot \cos(4t) \cdot 4}$

$\cancel{\cos 4t}$

$3\sin(4t)^2 \cdot \cos(4t) \cdot 4$

$12\sin(4t)^2\cos 4t$

Implicit Differentiation

When you cannot isolate y

derivative of y

$\frac{dy}{dx}$ - Taking derivative of y with respect to x

when we get to a term involving y. Must use Chain Rule

$\frac{dy}{dx}[y^3] = 3y^2\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}[xy^2] = x \cdot 2y\frac{dy}{dx} + y^2(1)$
$= 2xy\frac{dy}{dx} + y^2$

$\frac{d}{dx}[x+3y] = 1 + 3\left(\frac{dy}{dx}\right)$

$\frac{dy}{dx}$ given $y^3 + y^2 - 5y - x^2 = -4$ der

$3y^2\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) - 5\left(\frac{dy}{dx}\right) - 2x = 0$

$\frac{dy}{dx}(3y^2+2y-5) = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$

$\frac{dy}{dx}$ still gives the slope of the tangent line

$\frac{dy}{dx}$

$x^3 - 3x^2y + 2xy^2 = 12$

$3x^2 - [3x^2\frac{dy}{dx} + y6x] + 2x \cdot 2y\left(\frac{dy}{dx}\right) + y^2 \cdot 2 = 0$

$3x^2 - 3x^2\frac{dy}{dx} - 6xy + 4xy\left(\frac{dy}{dx}\right) + 2y^2 = 0$

$\frac{dy}{dx}(-3x^2+4xy) = -3x^2 + 6xy - 2y^2$

$\frac{dy}{dx} = \frac{-3x^2+6xy-2y^2}{-3x^2+4xy}$

3-2

Rolle's Theorem

- must have 2 points in the same y-values
- $F(x)$ is continious and differentiable for all x
- * There must be one place where $F'(x)=0$
  - ↳ at least 1 horizontal tangent line

Examples — must show 3 requirements

$F(x) = x^4 - 2x^2 \quad [-2,2]$

① Continious b/c poly
② $F(-2)=8$
$F(2)=8$
③ differentiable ✓

$4x^3 - 4x$
$4x(x^2-1)$
$4x=0 \quad x^2-1=0$
$x=0 \quad x=\pm 1$

$$f(x) = \frac{x^2-2x-3}{x+2} \quad [-1,3]$$

$x=-2$ undefined - not on interval
① cont on $[-1,3]$
② $F(-1)=0$ ✓
$F(3)=0$
③ differentiable

$F'(x) = \dfrac{(x+2)(2x-2) - [x^2-2}{(x+2)^2}$

$\dfrac{2x^2-2x+4x-4-x^2+2x+3}{(x+2)^2}$

$0 = \dfrac{x^2+4x-1}{(x+2)^2}$

$x = \dfrac{-b \pm \sqrt{b^2}}{2a}$

$x^2+4x-1=0$

$x=.236$

or

$~~x=-4.2~~$

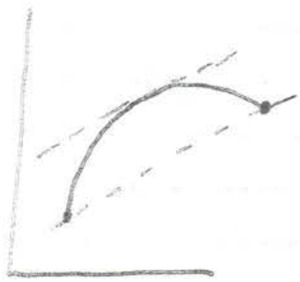
$x^2+4x+\underline{\ 4\ }-1-4$

$(x+2)^2-5=0$

$x+2=\pm\sqrt{5}$

Mean Value Theorem

- F(x) is continious and differentiable at on interval



-Somewhere in $x=c$
slope of tangent line will equal slope of secant line

↳ There must be a point in (a,b) where instantaneus rate of change = average rate of change

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$

tangent line (under $F'(c)$) — Secant line (under the fraction)

Examples

$F(x) = 5 - \frac{4}{x}$ $[1,4]$ ← Interval

cont ✓

differ ✓

$= 5-4x^{-1}$

$F'(x) = 4x^{-2}$

$= \frac{4}{x^2}$

$\frac{F(4)-F(1)}{4-1} = \frac{4-1}{3} = 1$ → Average slope

$\frac{4}{x^2} = 1$

$4 = x^2$

$x = \pm 2$

not on interval so

$\boxed{x = 2}$

$F(x) = x(x^2-x-2)$ $[-1,1]$

cont ✓ b/c poly

diff ✓

$F(x) = x^3-2x^2-2x$

$F'(x) = 3x^2-2x-2$

$\frac{F(1)-F(-1)}{1--1} = -1$

$3x^2-2x-2 = -1$?!

$3x^2-2x-1 =$

$(3x+1)(x-1)$

$x = -\frac{1}{3}$ ~~x=1~~

is an endpoint...

$F(x) = |x|$ $[-3,3]$

cont ✓

diff ✗ @ $x=0$

cannot apply MVT

Fernando Trujillo

AP Calculus BC
Calculator

14

Name Fernando Trujillo
Date ________________ Period____

Quiz ~ 3.7 Optimization

Show all work and check your answer with the second derivative. Please write neatly!

1. Find the positive number x such that the sum of x and its reciprocal is as small as possible.

$S = x + \frac{1}{x}$

$\frac{dS}{dx} = 1 - \frac{1}{x^2}$

$0 = 1 - \frac{1}{x^2}$

$1 = \frac{1}{x^2}$

$x = \pm 1$

$x = 1$

check

$\frac{d^2S}{dx^2} = \frac{2}{x^3}$

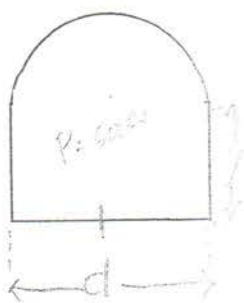
$\frac{2}{(1)^3} > 0$

ccu min

solution

$x = 1$

2. Suppose that 600 ft of fencing is used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle (see figure). Find the dimensions of the corral with maximum area.



$2r = d$

$600 = 2y + d + \frac{1}{2} d\pi$

$A = dy + \frac{\frac{d}{2}\pi^2}{2}$

$600 = 2y + 2r + r\pi$

$A = 2ry + \frac{r^2\pi}{2}$

$\frac{600 - 2r - r\pi}{2} = y$

$A = 2r\left(\frac{600 - 2r - r\pi}{2}\right) + \frac{r^2\pi}{2}$

$= 600r - 2r^2 - r^2\pi + \frac{1}{2}r^2\pi$

$\frac{dA}{dr} = 600 - 4r - r\pi$

$0 = 600 - 4r - r\pi$

$600 = 4r + r\pi$

$r = 84.015$ ft

$2r = d$

$d = 168.03$ ft

$y = 84.015$

label in your picture

solution

$d = 168.03$ ft

$y = 84.015$ ft

check

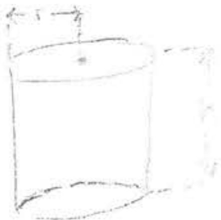
$\frac{d^2A}{dr^2} = -4 - \pi < 0$

ccd max!

C

surface area

3. Find the dimensions of a cylindrical can of volume 10 ft^3 so that it uses the least amount of metal.



$V = \pi r^2 h$

$10 = \pi r^2 h \qquad \frac{10}{\pi r^2} = h$

$SA = 2(\pi r^2) + (2\pi r)h$

$SA = 2\pi r^2 + (2\pi r)\left(\frac{10}{\pi r^2}\right)$

$= 2\pi r^2 + 20r^{-1}$

$\frac{dSA}{dr} = 4\pi r - \frac{20}{r^2}$

$0 = 4\pi r - \frac{20}{r^2}$

$4\pi r = \frac{20}{r^2} \qquad 4\pi r^3 = 20$

$r = 1.1675$

CHECK

$\frac{d^2SA}{dr^2} = 4\pi + \frac{40}{r^3} > 0$ ccup min

$h = \frac{10}{\pi r^2}$

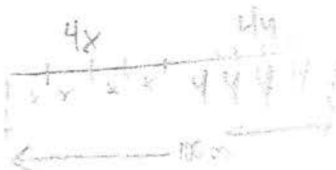
$h = 2.335$

solution

$r = 1.1675$ ft

$h = 2.335$ ft

4. A 100-in. piece of wire is divided into two pieces and each piece is bent into a square. What size should the two pieces be to minimize the sum of the areas of the two squares?



$P = 100$ in

$A = y^2 + x^2$

$P = 4x + 4y$

$100 = 4x + 4y$

$25 = x + y$

$25 - x = y$

$A = (25 - x)^2 + x^2$

$\frac{dA}{dx} = -2(25 - x) + 2x$

$0 = -2(25 - x) + 2x$

$50 - 2x = 2x$

$50 = 4x$

$x = 12.5$

CHECK

$\frac{dA}{dx} = -50 + 4x$

$\frac{d^2A}{dx^2} = 4$ ccup min

$25 - x = y$

$25 - 12.5 = y$

$y = 12.5$

Piece 1 = $4x = 4(12.5) = 50$ in

Piece 2 = $4y = 4(12.5) = 50$ in

solution

Piece 1 = 50 in

Piece 2 = 50 in

3.7

Optimization Problems

A rectangle has it's base on the x axis and it's upper two vertecies on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have and what are it's dimentions

$A = \text{base} \cdot \text{height}$

$A = 2xy$

$A = 2x(12 - x^2)$

$A = 24x - 2x^3$

$\frac{dA}{dx} = 24 - 6x^2$

$0 = 24 - 6x^2$

$x = 2 \quad \cancel{x = -2}$

Area $= 4(8) = 32$

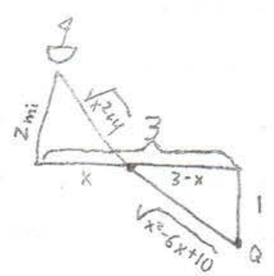
CHECK:

$\frac{d^2A}{dx^2} = -12x$

when $x = 2$, $\frac{d^2A}{dx^2} < 0$

so CC↓ ∴ max

A man is in a boat 2 miles from the nearest point on the coast. He is to go to point Q, located 3 miles down the coast and 1 mile inland. He can row 2 mph and walk 4. Toward what point on the coast should he row in order to reach point Q in the least time?



$v = d/t$

ROW $\frac{2\,mi}{hR}$

WALK $\frac{4\,mi}{hR}$

min T

$T = d/v$

$T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$

$\frac{dT}{dx} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(x^2+4)^{-1/2}(2x) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)(x^2-6x+10)^{-1/2}(2x-6)$

$0 = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$

$x = 1$ mile down the cost

CHECK

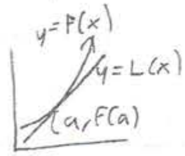
$\frac{d^2T}{dx^2} \quad x = 1$

$= .2012 > 0$

Math
↓
8 nDeriv
↓
Vars → YVARS → Function → Y_1

Linear Approximations and Differentials

An equation of the tangent line to $y = F(x)$ at $(a, F(a))$ is



$$y - y_1 = m(x - x_1)$$
$$y - f(a) = F'(a)(x-a)$$
* $$L(x) = F'(a)(x-a) + f(a)$$

Example

$F(x) = \sqrt{x+3}$ at $a = 1$

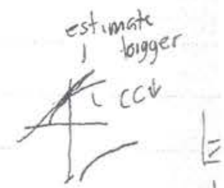
$F'(x) = \frac{1}{2}(x+3)^{-1/2}$

$L(x) = F'(a)(x-a) + F(a)$ $\quad F'(1) = 1/4$

$L(x) = (1/4)(x-1) + 2$

↳ you can now get an ESTIMATE by plugging it in here

you can get an ACTUAL value by plugging in original



$F(x) = 1 + \sin x$ at $x = 0$

$L(x) = f'(a)(x-a) + F(a)$ $\quad F'(x) = \cos x$

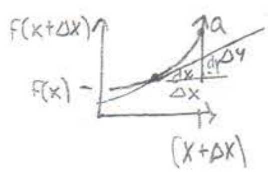
$L(x) = (1)(x-0) + 1$ $\quad F'(0) = 1$

Differentials — change in y

dy = Estimated change in y - based on tangent line

$dx = \Delta x$ = Actual change in x

Δy Actual change in y



$$\Delta Y = \underset{\text{new}}{F(x+\Delta x)} - \underset{\text{old}}{F(x)}$$

$\frac{dy}{dx} = f'(x) \to dy = f'(x)\,dx$

F(x)	Derivative	Differentials
$y = x\cos x$	$\frac{dy}{dx} = \cos x - x\sin x$	$dy = (\cos x - x\sin x)\,dx$

4.4 Review Worksheet

Name Key

The Fundamental Theorems of Calculus

In-Class:

Given $\frac{dy}{dx} = 3x^2 + 4x - 5$ with the initial condition $y(\overset{a}{2}) = -1$. Find $y(\overset{b}{3})$.

Method 1: Integrate $y = \int(\overset{y'}{3x^2 + 4x - 5})\,dx$, and use the initial condition to find C. Then write the particular solution, and use your particular solution to find $y(3)$.

$y = x^3 + 2x^2 - 5x + C$

$-1 = (2)^3 + 2(2)^2 - 5(2) + C$

$-1 = 6 + C$

$-7 = C$

$y = x^3 + 2x^2 - 5x - 7$

$y(3) = 3^3 + 2\cdot 3^2 - 5(3) - 7$

$y(3)\ \boxed{23}$

Method 2: Use the First Fundamental Theorem of Calculus: $\int_a^b f'(x)\,dx = f(b) - f(a)$

$\int_2^3 f'(x)\,dx = f(x)\Big]_2^3 = f(3) - f(2)$

$\int_2^3 (3x^2 + 4x - 5)\,dx = f(3) - f(2)$

$x^3 + 2x^2 - 5x\Big]_2^3 = f(3) + 1$

$(27 + 18 - 15) - (8 + 8 - 10) = f(3) + 1$

$24 = f(3) + 1$

$\boxed{23} = f(3)$

Do the following problem by both methods.

1. $y' = 2 + \frac{1}{x^2}$ and $y(1) = 6$. Find $y(3)$.

Method 1

$y = \int(2 + x^{-2})\,dx$

$y = 2x + \frac{x^{-1}}{-1} + C$

$y = 2x - \frac{1}{x} + C$

$6 = 2(1) - \frac{1}{1} + C$

$6 = 1 + C$

$5 = C$

$y = 2x - \frac{1}{x} + 5$

$y(3) = 2(3) - \frac{1}{3} + 5$

$y(3) = 11 - \frac{1}{3}$

$y(3) = \frac{33}{3} - \frac{1}{3}$

$y(3) = \boxed{\frac{32}{3}}$

Method 2

$\int_1^3 y'\,dx = y(3) - y(1)$

$\int_1^3 (2 + x^{-2})\,dx = y(3) - 6$

$\left[2x - \frac{1}{x}\right]_1^3 = y(3) - 6$

$\left(6 - \frac{1}{3}\right) - (2 - 1) = y(3) - 6$

$\frac{17}{3} - \frac{3}{3} = y(3) - 6$

$\frac{14}{3} + 6 = y(3)$

$y(3) = \boxed{\frac{32}{3}}$

Homework:

1. Define the 1st FTC: $\int_a^b f(x)\,dx = F(b) - F(a)$

2. Define the 2nd FTC: $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

Do the following problem by both methods. (See front of page)

3. $f'(x) = \cos(2x)$ and $f(0) = 3$. Find $f\left(\frac{\pi}{4}\right)$.

Method 1

$f(x) = \int \cos(2x)\,dx \quad u = 2x \quad du = 2dx$

$f(x) = \frac{1}{2}\int 2\cos(2x)\,dx = \frac{1}{2}\int \cos u\,du$

$f(x) = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$

$3 = \frac{1}{2}\sin 0 + C \qquad f(x) = \frac{1}{2}\sin 2x + 3$

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}\sin\frac{\pi}{2} + 3$

$= \frac{1}{2} + 3 = \boxed{3.5}$

Method 2

$\int_0^{\frac{\pi}{4}} \cos(2x)\,dx = f\left(\frac{\pi}{4}\right) - f(0)$

$\frac{1}{2}\sin 2x\Big]_0^{\frac{\pi}{4}} = f\left(\frac{\pi}{4}\right) - 3$

$\frac{1}{2} = f\left(\frac{\pi}{4}\right) - 3$

$\boxed{3.5} = f\left(\frac{\pi}{4}\right)$

Evaluate the following definite integrals.

4. $\int_{-1}^{1} 2x^2\,dx = \frac{2x^3}{3}\Big]_{-1}^{1}$

$\frac{2}{3} - \left(\frac{-2}{3}\right) = \boxed{\frac{4}{3}}$

5. $\int_{-3}^{-1} (2 - x^3)\,dx = \left[2x - \frac{x^4}{4}\right]_{-3}^{-1}$

$\left(-2 - \frac{1}{4}\right) - \left(-6 - \frac{81}{4}\right) = -2 - \frac{1}{4} + 6 + \frac{81}{4}$

$= 4 + \frac{80}{4} = 4 + 20 = \boxed{24}$

6. $\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi}$

$\sin\pi - \sin 0 = 0 - 0 = \boxed{0}$

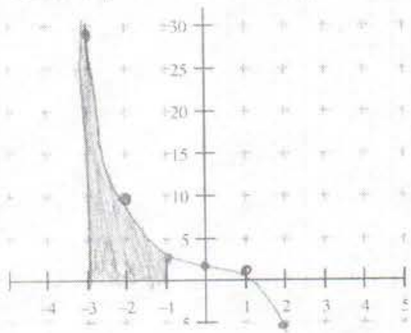
7. $\int_1^9 \frac{2t^2 + \sqrt{t^5} - 1}{t^2}\,dt = \int_1^9 \left(2 + t^{1/2} - t^{-2}\right)dt$

$= \left[2t + \frac{2t^{3/2}}{3} - \frac{t^{-1}}{-1}\right]_1^9$

$= \left[2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right]_1^9$

$= \frac{325}{9} - \frac{11}{3} = \boxed{\frac{292}{9}}$

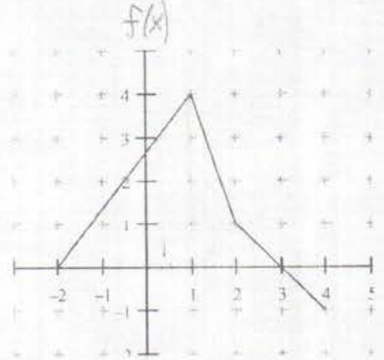
8. Draw a picture of the integral in #5.



9. Evaluate $\frac{d}{dx}\left[\int_0^x (t^2+3t-2)\,dt\right] = \boxed{x^2+3x-2}$

10. Evaluate $\frac{d}{dx}\left[\int_{23}^{5x^2} (\sqrt[4]{5t^2-1})\,dt\right] = \left(\sqrt[4]{5(5x^2)^2-1}\right)(10x) = \boxed{10x\sqrt[4]{125x^4-1}}$

11.



The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$

a) Compute $g(4)$ and $g(-2)$.

$g(4) = \int_1^4 f(t)\,dt = \boxed{2.5}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt = -\frac{1}{2}(3)(4) = \boxed{-6}$

b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

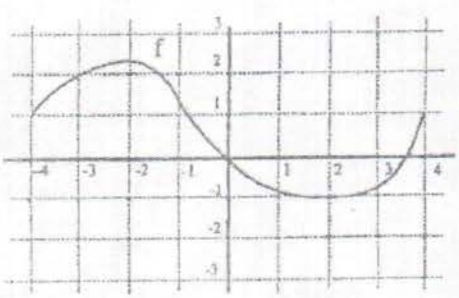
$g'(x) = f(x)$ $g'(1) = f(1) = \boxed{4}$

c) Find the absolute minimum value of g on the closed interval [-2,4]. Justify your answer.

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt = \boxed{-6}$

(b/c neg area)

12. The graph of f is shown in the figure. Estimate each of the following.



a) $f'(-1) = \boxed{-1}$

slope

b) $\int_{-4}^{1} f(x)\,dx$ $\boxed{6}$

area

c) $\frac{d}{dx}\int_{-4}^{x} f(t)\,dt$ at $x = -2$

$\boxed{2.2}$

13. What is the average value of $y = \cos x$ over the interval $\left[0, \frac{\pi}{2}\right]$?

$\int_0^{\pi/2} \cos x \, dx$

$= \sin x \Big]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$

avg. value $= \frac{1}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$

14. Find the value of c guaranteed by the Mean Value Theorem for Integrals for $f(x) = 3x^2 + 1$ over $[0,5]$?

$= \int_0^5 (3x^2+1)\, dx$

$= \left[x^3 + x\right]_0^5$

$= 5^3 + 5 = 130$

avg value $= \frac{130}{5} = 26$

$26 = 3x^2 + 1$

$0 = 3x^2 - 25$

$\boxed{x = 2.887}$

$25 = 3x^2$

$\frac{25}{3} = x^2$

$\pm\sqrt{\frac{25}{3}} = x$

$\boxed{+\frac{5}{\sqrt{3}} = x}$

15. The graph of the function $y = f(t)$ is given below.

a) Use this graph to sketch the graph of the function $F(x) = \int_{-2}^{x} f(t)\, dt$ on the xy-coordinate system given below on the right.

$F(-2) = 0$ $F(1) = 3.5$

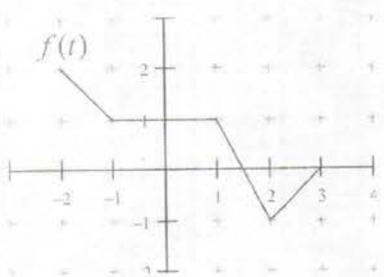
$F(-1) = 1.5$ $F(2) = 3.5$

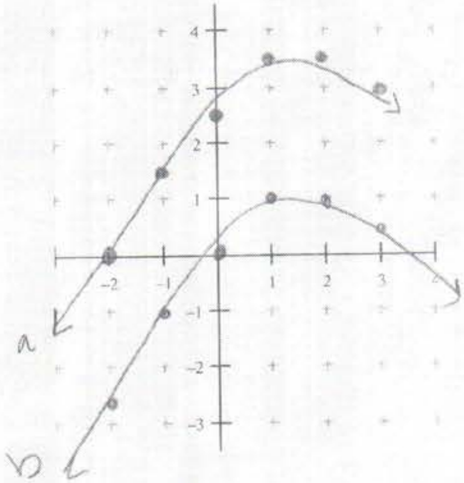
$F(0) = 2.5$ $F(3) = 3$

b) Use this graph to sketch the graph of the function $F(x) = \int_{0}^{x} f(t)\, dt$ on the xy-coordinate system given below on the right.

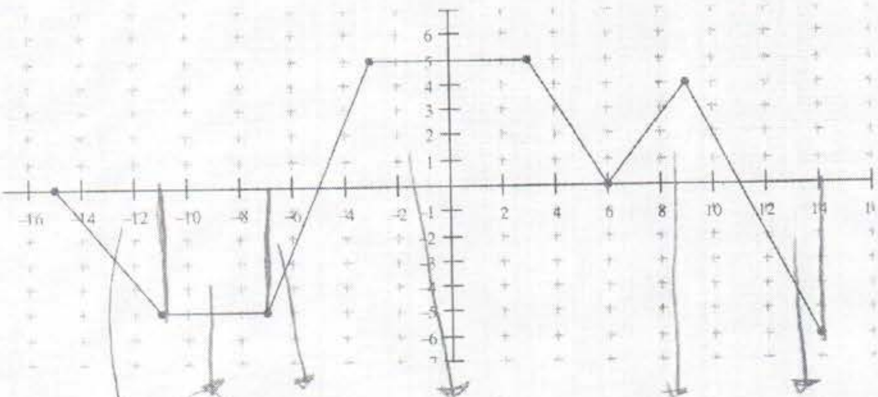
$F(-2) = -2.5$ $F(0) = 0$ $F(2) = 1$

$F(-1) = -1$ $F(1) = 1$ $F(3) = 0.5$





16. Find $\int_{-15}^{14} f(x)\,dx$, where the graph of f is given below.



$\frac{4(5)}{2}$ -20 $\frac{2(5)}{2}$ $\frac{5(11+6)}{2}$ $\frac{5(4)}{2}$ $\frac{3(6)}{2}$ $\int_{-16}^{14} f(x)\,dx = 8.5$

-10 -5 42.5 10 -9

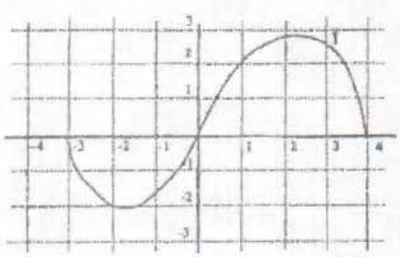
17. A car is decreasing in speed. The following table gives the speed of this car at quarter minute intervals. Use this table of data to find lower and upper estimates for the distance the car has traveled in this one minute.

Time (min)	0	0.25	0.5	0.75	1
Speed (ft/min)	94	64	42	30	25

left $S(4) = .5(94+64+42+30) = 57.5$ ft upper

Right $S(4) = .5(64+42+30+25) = 40.25$ ft lower

18. The figure shows the graph of f. Determine a so that $\int_{-3}^{a} f(t)\,dt$ will be as small as possible.

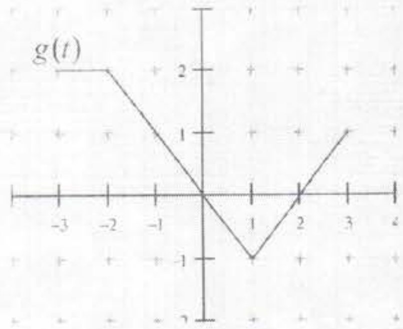


Will be as small as possible when we only have negative area.

$x = 0$

$\therefore \boxed{a = 0}$

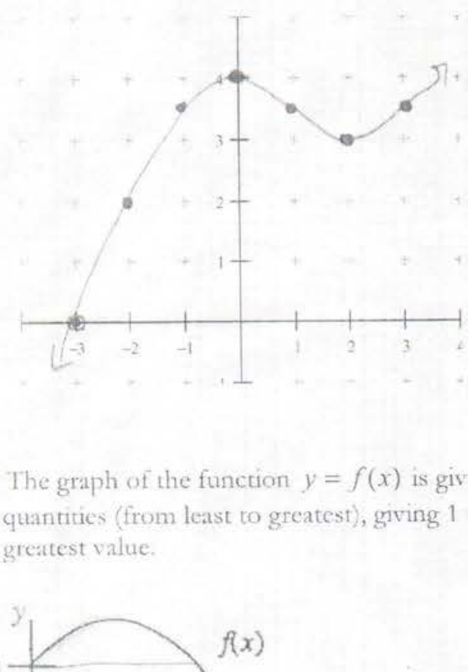
19. Let $G(x) = \int_{-3}^{x} g(t)\, dt$, $-3 \le x \le 3$, where g is the function graphed in the figure.



a) Complete the following table of values

x	-3	-2	-1	0	1	2	3
$G(x)$	0	2	3.5	4	3.5	3	3.5

b) Sketch a graph of G .



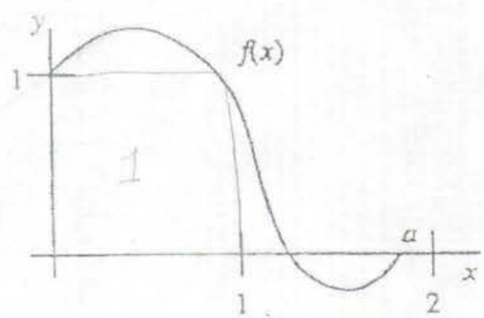
c) Which is larger: $G(0)$ or $G(1)$?
Justify your answer.

G(0) b/c from x = 0 ; x = 1
there is negative area
(area is removed)

d) Where is G increasing?

$-3 < x < 0$ $(-3, 0)$
$2 < x < 3$ $(2, 3)$ is when $g(t) > 0$

20. The graph of the function $y = f(x)$ is given below. Based on the graph, order the following quantities (from least to greatest), giving 1 to the least value, 2 to the next highest value, and 3 to the greatest value.



a) 3 $\int_0^a f(x)\, dx$

b) 2 The average value of $f(x)$ over the interval $[0,a]$

c) 1 $f'(1)$

Key

Worksheet 1. What You Need to Know About Motion Along the x-axis (Part 1)

In discussing motion, there are three closely related concepts that you need to keep straight. These are:

position : $x(t)$ or $s(t)$ – location of particle at time t

velocity : $v(t) = s'(t)$ – how fast position is changing / direct.

accel : $a(t) = v'(t) = s''(t)$ – how fast vel. is changing

If $x(t)$ represents the position of a particle along the x-axis at any time t, then the following statements are true.

1. "Initially" means when t = 0.
2. "At the origin" means $x(t)$ or $s(t)$ = 0.
3. "At rest" means $v(t)$ = 0.
4. If the velocity of the particle is positive, then the particle is moving to the right.
5. If the velocity of the particle is neg., then the particle is moving to the left.
6. To find average velocity over a time interval, divide the change in position by the change in time.
7. Instantaneous velocity is the velocity at a single moment (instant!) in time.
8. If the acceleration of the particle is positive, then the $v(t)$ is increasing.
9. If the acceleration of the particle is neg, then the velocity is decreasing.
10. In order for a particle to change direction, the $v(t)$ must change signs.
11. One way to determine total distance traveled over a time interval is to find the sum of the absolute values of the differences in position between all resting points. Here's an example: If the position of a particle is given by:

$$x(t) = \frac{1}{3}t^3 - t^2 - 3t + 4, \quad \text{so } v(t) = t^2 - 2t - 3$$

find the total distance traveled on the interval $0 \le t \le 6$.

- $x'(t) = v(t) = 0$ when $t = -1, 3$ (but -1 is not in interval)
- find position at endpoints and at $t = 3$ (where $v(t) = 0$)

x	$x(t)$	
0	4	left 9
3	−5	right 27
6	22	

the particle moves left 9 units and then right 27 units for a total of 36 units traveled

* on calculator : $\int_0^6 |v(t)|\,dt = \int_0^6 |t^2 - 2t - 3|\,dt = 36$

Worksheet 2. Sample Practice Problems for the Topic of Motion (Part 1)

Example 1 (numerical).

The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

At $t=0$, the particle is moving to the left because the velocity is negative.

2. Is there a time during the time interval $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer.

Yes; since the velocity function is differentiable, it is also continuous. By the IVT, since $V(t)$ goes from neg. to pos, it must go through 0, and $V(t)=0$ means the part. is at rest.

3. Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computations that lead to your answer and indicate units of measure.

$$V'(10) \approx \frac{V(12)-V(8)}{12-8} = \frac{5-7}{4} = -\frac{1}{2} \text{ m/min}^2$$

$V'(10) = a(10) =$ acceleration of particle at 10 min.

4. Let $a(t)$ denote the acceleration of the particle at time t. Is there guaranteed to be a time $t = c$ in the interval $0 \le t \le 12$ such that $a(c) = 0$? Justify your answer.

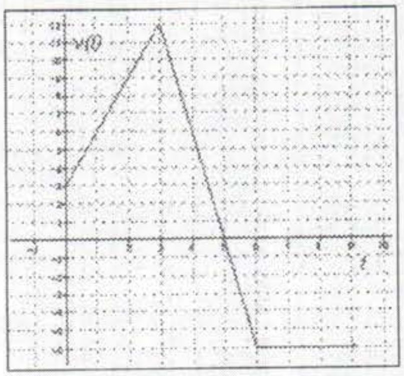
Since velocity is differentiable, it is also continuous over the interval $6<t<12$ and thus we can use MVT.

$$\text{avg rate of change} = \frac{V(12)-V(6)}{12-6} = \frac{5-5}{6} = 0 \text{ m/min}^2$$

thus, MVT implies there is a value where avg roc equals inst. roc, there is a place where $V'(t)=a(t)=0$ on $(6,12)$

Example 2 (graphical).

The graph below represents the velocity v, in feet per second, of a particle moving along the x-axis over the time interval from $t = 0$ to $t = 9$ seconds.



1. At $t = 4$ seconds, is the particle moving to the right or left? Explain your answer.

at $t=4$, the particle is moving to the right because the velocity is positive

2. Over what time interval is the particle moving to the left? Explain your answer.

the particle is moving to the left over the interval $5<t<19$ sec. b/c the velocity is negative.

3. At $t = 4$ seconds, is the acceleration of the particle positive or negative? Explain your answer.

$a(4) = v'(4)$ is negative because velocity is decreasing (has a neg slope)

4. What is the average acceleration of the particle over the interval $2 \le t \le 4$? Show the computations that lead to your answer and indicate units of measure.

avg velocity over $[2,4] = \frac{v(4)-v(2)}{4-2} = \frac{6-9}{2} = \frac{-3}{2}$ ft/sec^2

5. Is there guaranteed to be a time t in the interval $2 \le t \le 4$ such that $v'(t) = -3/2$ ft/sec^2? Justify your answer.

such a time cannot be guaranteed. MVT does not apply since the function is not differentiable at $t=3$ (sharp turn)

6. At what time t in the given interval is the particle farthest to the right? Explain your answer.

the particle is farthest right at $t=5$ seconds ;
$v(t)$ is positive on $0<t<5$ and negative on $5<t<9$,
thus moves right until $t=5$, then moves left.

Example 3 (analytic).

A particle moves along the x-axis so that at time t its position is given by:

$$x(t) = t^3 - 6t^2 + 9t + 11 \qquad \text{so } v(t) = 3t^2 - 12t + 9$$

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

particle is moving right because
$x'(0) = v(0) = 9$ (which is positive)

2. At $t = 1$, is the velocity of the particle increasing or decreasing? Explain your answer.

decreasing because $x''(1) = v'(1) = a(1) = -6$
and if acceleration is negative, $v(t)$ is decreasing

3. Find all values of t for which the particle is moving to the left.

particle is moving left whenever $v(t)<0$
$v(t) = 3t^2 - 12t + 9 < 0$ when $1<t<3$

4. Find the total distance traveled by the particle over the time interval $0 \le t \le 5$.

t	$x(t)$	
0	11	right 4
1	15	
2	13	left 4
3	11	
4	15	right 20
5	31	

the particle moves
- right 4 spaces
- left 4 spaces
- right 20 spaces

so particle has traveled a total of 28 units

* on calculator : $\int_0^5 |v(t)|dt = \int_0^5 |3t^2 - 12t + 9|\, dt = 28$

REMEMBER : total distance $\neq$ displacement !

U-Substitution Bingo

Name: Key

Solve each problem and write as a reduced fraction, if possible. Common denominators not necessary.

1) $\int \cot^4 x \csc^2 x\, dx$ $u = \cot x$, $du = -\csc^2 x\, dx$

$= -\int -\cot^4 x \csc^2 x\, dx$

$= -\int u^4\, du$

$= -\frac{u^5}{5} + C = \boxed{-\frac{\cot^5 x}{5} + C}$

2) $\int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + 1 + 1 + \frac{1}{x^2}\right) dx$

$= \int (x^2 + 2 + x^{-2})\, dx$

$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + C$

$= \boxed{\frac{1}{3}x^3 + 2x - \frac{1}{x} + C}$

3) $\int \frac{x^2}{\sqrt{x^3+3}}\, dx$ $u = x^3 + 3$, $du = 3x^2\, dx$

$= \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+3}}\, dx = \frac{1}{3}\int u^{-1/2}\, du$

$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\, du$ $= \frac{2}{3} u^{1/2} + C$

$= \boxed{\frac{2}{3}\sqrt{x^3+3} + C}$

4) $2\pi \int_{-1}^{0} x^2 \sqrt{x+1}\, dx$ $u = x+1$, $x = u - 1$, $du = 1\, dx$

$2\pi \int_0^1 (u-1)^2 \sqrt{u}\, du = 2\pi \int_0^1 (u^2 - 2u + 1)\, u^{1/2}\, du$

$= 2\pi \int_0^1 \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du = 2\pi \left(\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right)\Big]_0^1$

$= 2\pi \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) = \boxed{\frac{32\pi}{105}}$

5) $\int_0^{\pi} \cos \frac{x}{2}\, dx$ $u = \frac{x}{2}$, $du = \frac{1}{2}\, dx$

$= 2\int_0^{\pi} \frac{1}{2} \cos \frac{x}{2}\, dx$

$= 2\int_0^{\pi/2} \cos u\, du = 2\sin u \Big]_0^{\pi/2}$

$= 2(1) - 2(0) = \boxed{2}$

6) $\int \frac{x+3}{(x^2+6x-5)^2}\, dx$ $u = x^2 + 6x - 5$, $du = (2x + 6)\, dx$

$\frac{1}{2}\int \frac{2x+6}{(x^2+6x-5)^2}\, dx = \frac{1}{2}\int \frac{1}{u^2}\, du = \frac{1}{2}\int u^{-2}\, du$

$= \frac{1}{2}\frac{u^{-1}}{-1} + C = \frac{-1}{2u} + C = \boxed{\frac{-1}{2x^2 + 12x - 10} + C}$

7) $\frac{1}{\pi}\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\, \pi\, dx$

$u = 1 + \sec \pi x$ $du = (\sec \pi x \tan \pi x)(\pi)\, dx$

$= \frac{1}{\pi}\int u^2\, du = \frac{1}{\pi}\frac{u^3}{3} + C$

$= \boxed{\frac{(1 + \sec \pi x)^3}{3\pi} + C}$

8) $\int x \sin(3x^2)\, dx$ $u = 3x^2$, $du = 6x\, dx$

$\frac{1}{6}\int 6x \sin(3x^2)\, dx$

$\frac{1}{6}\int \sin u\, du = \frac{1}{6}(-\cos u) + C$

$\boxed{-\frac{1}{6}\cos(3x^2) + C}$

9) $\int \frac{\sin x}{\sqrt{1 - \cos x}}\, dx$ $u = 1 - \cos x$, $du = \sin x\, dx$

$= \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du$

$= 2u^{1/2} + C = \boxed{2\sqrt{1 - \cos x} + C}$

10) $\int_{-\pi/4}^{\pi/4} \sin 2x\, dx$ $u = 2x$ $du = 2\, dx$

$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 2 \sin 2x\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin u\, du$

$= \frac{1}{2}(-\cos u)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{-1}{2}\left[\cos \frac{\pi}{2} - \cos \frac{-\pi}{2}\right] = \frac{-1}{2}(0 - 0)$

$= \boxed{0}$

11) $\int \tan^n x \sec^2 x\,dx,\ n \neq -1$ $u = \tan x$, $du = \sec^2 x\,dx$

$\int u^n\,du$

$= \frac{u^{n+1}}{n+1} + C = \boxed{\frac{\tan^{n+1} x}{n+1} + C}$

12) $\int x(1-3x^2)^4\,dx$ $u = 1-3x^2$ $du = -6x\,dx$

$-\frac{1}{6}\int -6x(1-3x^2)^4\,dx$

$-\frac{1}{6}\int u^4\,du = -\frac{1}{6}\frac{u^5}{5} + C = \frac{-u^5}{30} + C$

$= \boxed{-\frac{(1-3x^2)^5}{30} + C}$

13) $\int \frac{\cos x}{\sqrt{\sin x}}\,dx$ $u = \sin x$ $du = \cos x\,dx$

$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du$

$= 2u^{1/2} + C = \boxed{2\sqrt{\sin x} + C}$

14) $\int_0^3 \frac{1}{\sqrt{1+x}}\,dx$ $u = 1+x$ $du = 1\,dx$

$= \int_1^4 \frac{1}{\sqrt{u}}\,du = \int_1^4 u^{-1/2}\,du = 2u^{1/2}\Big]_1^4$

$= 2\sqrt{u}\Big]_1^4 = 2(2) - 2(1) = \boxed{2}$

15) $\int_{-1}^{2} x(x^2-4)\,dx$ $u = x^2-4$ $du = 2x\,dx$

$\frac{1}{2}\int_{-1}^{2} 2x(x^2-4)\,dx = \frac{1}{2}\int_{-3}^{0} u\,du$

$= \frac{1}{2}\cdot\frac{u^2}{2}\Big]_{-3}^{0} = \frac{u^2}{4}\Big]_{-3}^{0} = 0 - \frac{9}{4} = -\frac{9}{4}$

16) $\int_0^1 x^2(x^3+1)^3\,dx$ $u = x^3+1$ $du = 3x^2\,dx$

$\frac{1}{3}\int_0^1 3x^2(x^3+1)^3\,dx = \frac{1}{3}\int_1^2 u^3\,du$

$= \frac{1}{3}\cdot\frac{u^4}{4}\Big]_1^2 = \frac{u^4}{12}\Big]_1^2 = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \boxed{\frac{5}{4}}$

17) $\int x^2\sqrt{x^3+3}\,dx$ $u = x^3+3$ $du = 3x^2\,dx$

$\frac{1}{3}\int 3x^2\sqrt{x^3+3}\,dx$

$\frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} = \frac{1}{3}\cdot\frac{2}{3}\cdot u^{3/2} + C$

$= \boxed{\frac{2}{9}(x^3+3)^{3/2} + C}$

18) $\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx$ $u = x^2-8$ $du = 2x\,dx$

$= \frac{1}{6}\int_3^6 \frac{2x}{\sqrt{x^2-8}}\,dx = \frac{1}{6}\int_1^{28}\frac{1}{\sqrt{u}}\,du = \frac{1}{6}\int_1^{28} u^{-1/2}\,du$

$= \frac{1}{6}\frac{u^{1/2}}{1/2}\Big]_1^{28}$ $\frac{u^{1/2}}{3}\Big]_1^{28} = \frac{1}{3}(\sqrt{28} - \sqrt{1})$

$= \boxed{\frac{1}{3}(2\sqrt{7} - 1)}$

19) $2\pi\int_0^1 (x+1)\sqrt{1-x}\,dx$ $u = 1-x$ $du = -1\,dx$, $x = 1-u$

$-2\pi\int_0^1 -(x+1)\sqrt{1-x}\,dx$

$-2\pi\int_1^0 (1-u+1)\sqrt{u}\,du = 2\pi\int_0^1 (2-u)u^{1/2}\,du$

$2\pi\int_0^1 (2u^{1/2} - u^{3/2})\,du = 2\pi\left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$2\pi\left(\frac{4}{3} - \frac{2}{5}\right) = \boxed{\frac{28\pi}{15}}$

20) $\int \sin^3 x\cos x\,dx$ $u = \sin x$ $du = \cos x\,dx$

$\int u^3\,du = \frac{u^4}{4} + C$

$\boxed{\frac{\sin^4 x}{4} + C}$

21) $\int (x^2+1)^3\,dx$

$\int (x^6 + 3x^4 + 3x^2 + 1)\,dx$

$\boxed{\frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + C}$

22) $\int \sec 2x\tan 2x\,dx$ $\int \frac{1}{\cos 2x}\cdot\frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\int \frac{-2\sin 2x}{\cos^2 2x}\,dx$

$u = \cos 2x$
$du = -2\sin 2x\,dx$ $= -\frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2}\int u^{-2}\,du$

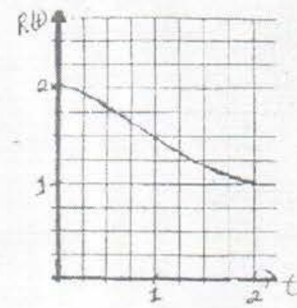
$= -\frac{1}{2}\frac{u^{-1}}{-1} + C = \frac{1}{2u} + C = \boxed{\frac{1}{2\cos 2x} + C} = \boxed{\frac{1}{2}\sec 2x + C}$

Calculus ~~AB~~ BC
4.2, 4.6 Integrals Wkst

Name ______________________
Date ______________________ Period___

INTEGRALS

1. Water is leaking out of a tank at a rate of R(t) gallons per hour as shown, where t is measured in hours. Give an estimate of the total amount of water that leaks out in the first two hours. Use left endpoint, right endpoint, and midpoint rules. Show sums used for each Riemann Sum. Let $\Delta t = \frac{1}{4}$.

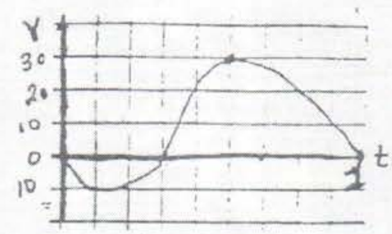


(L) $S(8) = \frac{1}{4}(2) + \frac{1}{4}(1.9) + \frac{1}{4}(1.75) + \frac{1}{4}(1.6) + \frac{1}{4}(1.5) + \frac{1}{4}(1.3) + \frac{1}{4}(1.2) + \frac{1}{4}(1.1)$
$= 3.0875$

(R) $S(8) = \frac{1}{4}(1.9 + 1.75 + 1.6 + 1.5 + 1.3 + 1.2 + 1.1 + 1) = 2.8375$

(M) $S(8) = \frac{1}{4}(1.95 + 1.8 + 1.65 + 1.55 + 1.3 + 1.25 + 1.1 + 1) = 2.9$

2. A bicyclist is pedaling along a straight road with velocity, v in miles per hour, as shown below, ,for t in hours. Suppose the cyclist starts 5 miles from a lake, and that positive velocity takes her away from the lake and negative velocity toward the lake. When is the cyclist farthest from the lake, and how far away is she then? Use left endpoint Riemann Sum to calculate. Let $\Delta t = \frac{1}{9}$.



(L) $S(9) = \frac{1}{9}[0 + (-10) + (-9) + 0 + 20 + 30 + 27 + 20 + 10]$
$= 9.778$ miles away

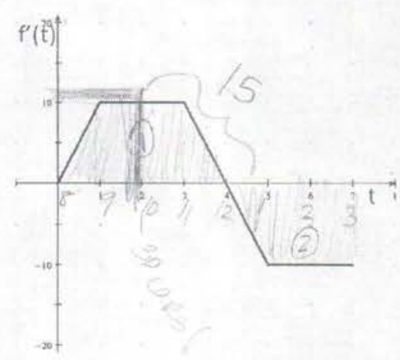
3. Evaluate a Riemann Sum for the following function using left-handed rule and right-handed rule with subintervals shown for x in the table:

x	0	1	4	6	9	10	12	16
y	.2	3	8	7	6	5	2	1

(L) $S(7) = 1(.2) + 3(3) + 2(8) + 3(7) + 1(6) + 2(5) + 4(2) = 70.2$

(R) $S(7) = 1(3) + 3(8) + 2(7) + 3(6) + 1(5) + 2(2) + 4(1) = 72$

4. The number of cars in a parking lot is given by f(t), where t is the hours from 8 am. The graph of f'(t) is shown below. If there were 30 cars in the lot at 10 am, how many cars are in the parking lot at 3 pm? You may use the geometry of triangles and rectangles to compute the Riemann Sum.



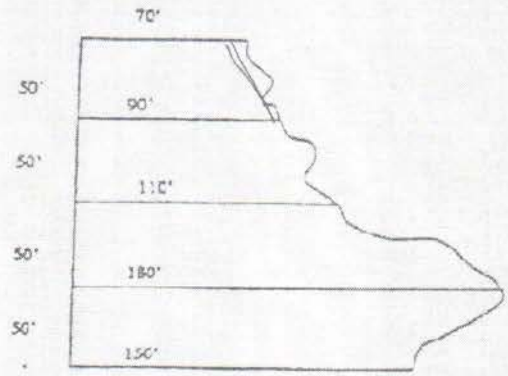
30 cars at 10 am

(2) $\frac{-10(3+2)}{2} = -25$

cars at 3pm $= 30 + 15 - 25 = 20$

to 8 to 9

5. A map of an ocean front property is shown. Estimate its area using a Riemann Sum.

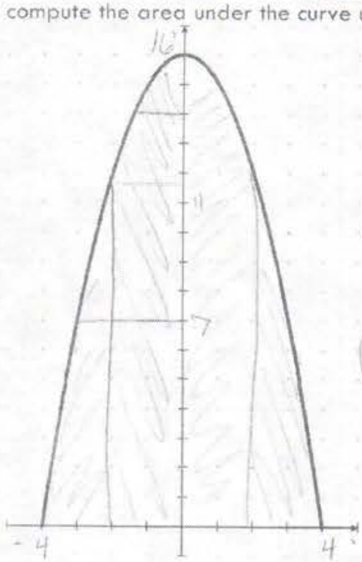


TRAP: $S(4) = \frac{50(70+90)}{2} + \frac{50(90+110)}{2} + \frac{50(110+180)}{2} + \frac{50(180+150)}{2}$

$= 22{,}025 \text{ ft}^2$

(M) $S(4) = 50(80) + 50(100) + 50(145) + 50(165) = 24{,}500 \text{ ft}^2$

6. See the graph for AREA II. Each space is one unit in length. Partition the interval [-4, 4] into 4 equal subintervals and compute the area under the curve using left endpoint, right endpoint, and midpoint rule. Show sums below.



(L) $S(4) = 2(0) + 2(11.5) + 2(16) + 2(11.5) = 78$

(R) $S(4) = 2(11.5) + 2(16) + 2(11.5) + 2(0) = 78$

(M) $S(4) = 2(7) + 2(14) + 2(14) + 2(7) = 84$

7. Estimate the area bounded by $y = x^2 + 1$ and the x-axis over the interval [0, 2] by using n = 4 intervals. Find left sum, right sum, and midpoint sum.

(L) $S(4) = .5(0^2+1) + .5(.5^2+1) + .5(1^2+1) + .5(1.5^2+1) = 3.75$

(R) $S(4) = .5(.5^2+1) + .5(1^2+1) + .5(1.5^2+1) + .5(2^2+1) = 5.75$

(M) $S(4) = .5(.25^2+1) + .5(.75^2+1) + .5(1.25^2+1) + .5(1.75^2+1) = 4.625$

Calculus Review
4.1-4.3, 4.6, Particle Motion

Name: Key 2011-2012

In addition to the following, you can also look at pg 316 #4, 6, 8, 9, 10, 12, 15, 16
***Calculus BC*: Also look over how to find area under the curve using the limit method!! (not in this review)**

Find the indefinite integral of each of the following:

1) $\int \frac{2-x}{\sqrt{x}}dx = \int (2-x)(x^{-1/2})dx$
$= \int (2x^{-1/2} - x^{1/2})dx = \frac{2x^{1/2}}{1/2} - \frac{x^{3/2}}{3/2} + C$
$= 4x^{1/2} - \frac{2}{3}x^{3/2} + C$

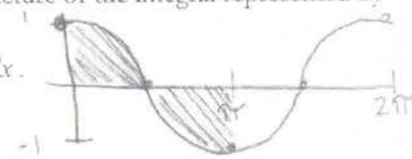
2) $\int (3x^3 - 2x^2 - 3)dx = \frac{3x^4}{4} - \frac{2x^3}{3} - 3x + C$

3) $\int -2\sin x\, dx = -2(-\cos x) + C$
$= 2\cos x + C$

4) $\int (2\tan^2 x + 2)dx = 2\int(\tan^2 x + 1)dx$
$= 2\int \sec^2 x\, dx = 2\tan x + C$

5) $\int \frac{2-x}{\sqrt[5]{x^3}}dx = \int (2-x)(x^{-3/5})dx$
$= \int (2x^{-3/5} - x^{2/5})dx = \frac{2x^{2/5}}{2/5} - \frac{x^{7/5}}{7/5} + C$
$= 5x^{2/5} - \frac{5}{7}x^{7/5} + C$

6) Draw a picture of the integral represented by $\int_0^{\pi} \cos x\, dx$.



$= 0$

Estimate the area under the curve with the given interval using the method(s) listed.

7) $y = 4 - x^2$ $[0, 2]$ - trapezoidal and left sum with $n = 4$.

Trap: $\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \quad + 2f(x_{n-1}) + f(x_n)\right]$
$\frac{2}{8}\left[(4-0^2) + 2(4-0.5^2) + 2(4-1^2) + 2(4-1.5^2) + (4-2^2)\right] = 5.25$

Left: $S(4) = \frac{1}{2}\left[(4-0^2) + (4-0.5^2) + (4-1^2) + (4-1.5^2)\right] = 6.25$

8) $y = x - x^3$ $[0, 3]$ - midpoint and right sum with $n = 3$.

Mid: $S(3) = 1(.5 - .5^3) + 1(1.5 - 1.5^3) + 1(2.5 - 2.5^3) = -14.625$

Right: $S(3) = 1(1 - 1^3) + 1(2 - 2^3) + 1(3 - 3^3) = -30$

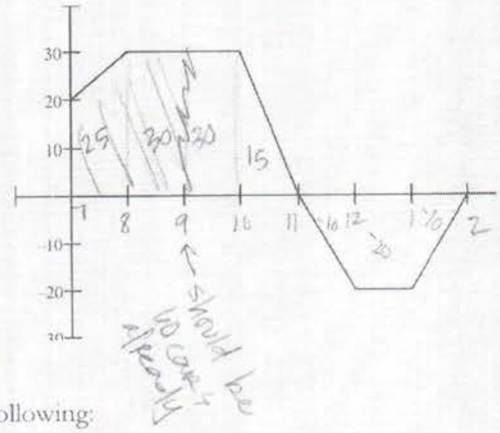
9) A car's speed is recorded in the chart below. Use a trapezoidal approximation to estimate the distance the car traveled over the trip. $A = \frac{h(b_1 + b_2)}{2} = \frac{1}{2}h(b_1 + b_2)$

t (in hours)	1	2	5	7	10	12
$v(t)$ (in km/hr)	40	60	65	70	45	55

$S(5) = \frac{1}{2}(1)(40+60) + \frac{1}{2}(3)(60+65) + \frac{1}{2}(2)(65+70) + \frac{1}{2}(3)(70+45) + \frac{1}{2}(2)(45+55) = 645$

* you must label x-axis

10) The rates at which cars are parked in the Stratford parking lot are recorded versus time in the graph below starting at 7 am. There are 60 cars in the lot at 9 am.



a) How many cars are in the lot at 2 pm?

$60+30+15-10-20-10 = 65$ cars

b) What is the maximum amount of cars in the lot during the day and when does it occur?

max at 11

$60+30+15 = 105$ cars

c) How many cars were in the lot at 7 am?

$60-30-25 = 5$ cars already at 7

11) If $\int_{-1}^{3} f(x)\,dx = 5$, $\int_{3}^{7} f(x)\,dx = 10$, $\int_{-1}^{2} f(x)\,dx = 2$, fin the following:

a) $\int_{2}^{3} f(x)\,dx = \int_{-1}^{3} f(x)\,dx - \int_{-1}^{2} f(x)\,dx = 5-2 = 3$

b) $\int_{7}^{3} f(x)\,dx = -\int_{3}^{7} f(x)\,dx = -10$

c) $\int_{2}^{7} f(x)\,dx = \int_{2}^{3} f(x)\,dx + \int_{3}^{7} f(x)\,dx = 3+10 = 13$

12) The following table gives the emissions, E, of nitrogen oxides in millions of metric tons per year in the United States. Let t be the number of years since 1940 and $E = f(t)$.

year	1940	1950	1960	1970	1980	1990
E	6.9	9.4	13.0	18.5	20.9	19.6

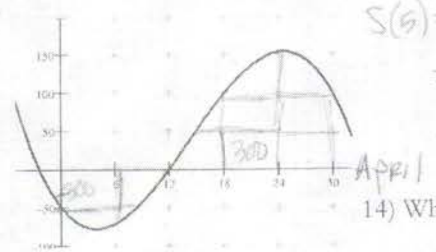
a) What are the units and meaning of $\int_0^{50} f(t)\,dt$?

total # of millions of metric tons of nitrogen oxides over 50 yrs

b) Estimate $\int_0^{50} f(t)\,dt$ by using a left Riemann sum.

$S(5) = 10(6.9) + 10(9.4) + 10(13) + 10(18.5) + 10(20.9) = 687$ million metric tons

13) The graph in the figure on the right shows the rate of change of the quantity of water in a water tower, in liters per day, during the month of April. If the tower had 12,000 liters of water in it on March 31, estimate the quantity of water in the tower at the end of the day on April 30.



$S(5) = -300(1.5) + -300(-1) + 300(1) + 300(2.5) + 300(2.5)$

$= 1{,}050$ liters water change

$= 12{,}000 + 1{,}050 = 13{,}050$ liters

14) What are the values of k for which $\int_{-3}^{k} x^2\,dx = 0$?

a) -3 ~~b) 0~~ ~~c) 3~~ ~~d) -3 and 3~~ ~~e) -3, 0, and 3~~

15) A table of the velocity, $v(t)$, in ft/sec, of a car traveling on a straight road for $0 \le t \le 50$ is shown.

$\int v(t) = s(t)$

a) Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using the midpoints of five subintervals of equal length.

$S(5) = 10(12) + 10(30) + 10(70) + 10(81) + 10(60) = 2530$ ft

t seconds	$v(t)$ feet per second
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

b) Using correct units, explain the meaning of this integral.

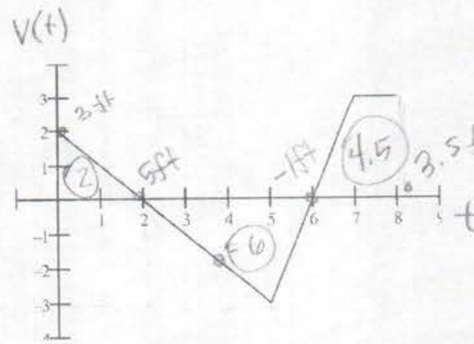
how far the car travelled in 50 sec

16) The table below gives the values for the rate (in gal/sec) at which water flows into a lake, with readings taken at specific times. Use a right endpoint approximation, with four subintervals indicated by the data in the table, to estimate the total amount of water that flows into the lake during the time period $0 \le t \le 60$.

time (sec)	0	18	28	48	60
rate (gal/sec)	300	200	150	100	50

$S(4) = 18(200) + 10(150) + 20(100) + 12(50) = 7700$ gallons

17) The velocity of a particle in ft/sec is shown to the right. The position of the particle at $t = 0$ is 3 ft.



a) When is the particle's position the greatest?

2 sec

b) What is its position at that point in time?

5 ft

c) When is the particle's speed the greatest?

speed $= |v(t)|$ $t = 5$ & $t = 7$ to $t = 8$

d) What is the acceleration of the particle at $t = 4$?

$a(t) = $ slope $v(t)$ $a(4) = -1$ ft/s²

e) What is the total distance that the particle traveled by $t = 8$? 3.5 ft

18) A table of values for a continuous function, f, is shown below. If four equal subintervals of [0,2] are used, which of the following is a trapezoidal approximation of $\int_0^2 f(x)\,dx$? $A = \frac{1}{2}h(b_1 + b_2)$

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13

$\frac{1}{2}\left(\frac{1}{2}\right)(3+3) + \frac{1}{2}\left(\frac{1}{2}\right)(3+5) + \frac{1}{2}\left(\frac{1}{2}\right)(5+8) + \frac{1}{2}\left(\frac{1}{2}\right)(8+13) = 12$

pg 316 # 4, 6, 8, 9, 10, 12, 15, 16

4. $\int \frac{2}{\sqrt[3]{3x}}\,dx$

$= \int 2 \cdot (3x)^{-1/3}\,dx$

$= \int 2 \cdot 3^{-1/3} \cdot x^{-1/3}\,dx$

$= 2 \cdot 3^{-1/3} \int x^{-1/3}\,dx$

$= 2 \cdot 3^{-1/3} \cdot \frac{x^{2/3}}{2/3} + C$

$= 3^{-1/3} \cdot 3x^{2/3} + C$

$= (3x)^{2/3} + C$

6. $\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx$

$= \int (x - 2 + 1x^{-2})\,dx$

$= \frac{x^2}{2} - 2x + \frac{x^{-1}}{-1} + C$

$= \frac{x^2}{2} - 2x - \frac{1}{x} + C$

8. $\int (5\cos x - 2\sec^2 x)\,dx$

$= 5\sin x - 2\tan x + C$

9. $f'(x) = -2x \quad (-1, 1)$

$f(x) = \int -2x\,dx = -\frac{2x^2}{2} + C$

$f(x) = -x^2 + C$

$f(-1) = -(-1)^2 + C = 1$

$-1 + C = 1$

$C = 2$

$f(x) = -x^2 + 2$

10. $f''(x) = 6(x-1) = 6x - 6 \quad (2, 1)$

tangent $3x - y - 5 = 0$

$3x - 5 = y$

$m = 3$

when $x = 2$

$f'(x) = \int (6x - 6)\,dx$

$= \frac{6x^2}{2} - 6x + C$

$f'(x) = 3x^2 - 6x + C$

$f'(2) = 3(2)^2 - 6(2) + C = 3$

$12 - 12 + C = 3$

$C = 3$

$f'(x) = 3x^2 - 6x + 3$

$f(x) = \int (3x^2 - 6x + 3)\,dx$

$= x^3 - 3x^2 + 3x + C$

$f(2) = (2)^3 - 3(2)^2 + 3(2) + C = 1$

$8 - 12 + 6 + C = 1$

$2 + C = 1$

$C = -1$

$f(x) = x^3 - 3x^2 + 3x - 1$

12. $\frac{dy}{dx} = \frac{1}{2}x^2 - 2x \quad (6, 2)$

$y = \frac{1}{2}\left(\frac{x^3}{3}\right) - \frac{2x^2}{2} + C$

$2 = \frac{(6)^3}{6} - (6)^2 + C$

$2 = 36 - 36 + C \qquad 2 = C$

$y = \frac{x^3}{6} - x^2 + 2$

15. $s(0) = 0$ ft
$v(0) = 96$ ft/sec
$a(0) = -32$ ft/sec^2

$v(t) = -32t + 96$
$s(t) = -16t^2 + 96t$

a) max height $\rightarrow v(t) = 0$

$-32t + 96 = 0$
$t = 3$ sec b) $h = 144$ ft

c) $v(t) = \frac{1}{2}$ initial $= 48$
$48 = -32t + 96$
$t = 1.5$ sec

d) $h = 108$ ft

16. $s(0) = 0$ m
$v(0) = 40$ m/s
$a(0) = -9.8$ m/s^2

$v(t) = -9.8t + 40$
$s(t) = -4.9t^2 + 40t$

a) max height $\rightarrow v(t) = 0$
$0 = -9.8t + 40$
$t = 4.08$ se

b) $h = 81.63$ m

c) $20 = -9.8t + 40$
$t = 2.04$ sec

d) $h = 61.2$ m

a) 8   b) 12   c) 16   d) 24   e) 30

Trabezoid

$\frac{b-a}{2n}$

or $\frac{b-a}{n}$

19) A particle's position is given by $s(t) = 3t^3 - 5t^2 + 9t - 11$

a) Where is the particle when $t = 2$? $s(2) = 11$

b) What is the velocity function of the particle? $v(t) = 9t^2 - 10t + 9$

c) What is the instantaneous velocity at $t = 1$? $v(1) = 8$

d) What is the acceleration function of the particle? $a(t) = 18t - 10$

e) Is the particle speeding up or slowing down at $t = 1$? $a(1) = 8$ $a(1) > 0$ $v(1) > 0$ $\therefore$ speeding up

f) What is the average velocity of the particle over the first 3 seconds?

$\frac{s(3) - s(0)}{3 - 0} = \frac{52 + 11}{3} = 21$

20) Given the graph of $f'(x)$, an even function, and that $\int_{\pi}^{0} f'(x)\,dx = \pi$, approximate the following:

a) $\int_{-4}^{-\pi} f'(x)\,dx = \frac{1}{2}bh = \frac{1}{2}(4-\pi)(-3.5) = -1.502$

b) $\int_{-4}^{0} f'(x)\,dx = -1.502 + \pi$

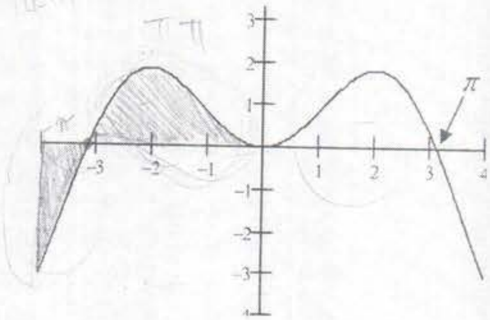
c) $\int_{-4}^{4} f'(x)\,dx = 2(-1.502 + \pi)$

d) $\int_{-4}^{4} |f'(x)|\,dx \; 2(1.502 + \pi)$

e) $\int_{-4}^{4} (|f'(x)| + 1)\,dx = \int_{-4}^{4} |f'(x)|\,dx + \int_{-4}^{4} 1\,dx$

$= 2(1.502 + \pi) + 8$

$= 11.004 + 2\pi$



AP Calculus – Review 5.1-5.5
Integrals and U-Substitution with e and ln

Name: Key

1) $\frac{1}{6} \int \frac{x\,dx}{\sqrt{3x^2+5}} =$

$u = 3x^2+5$
$du = 6x\,dx$

a. $\frac{1}{9}(3x^2+5)^{\frac{3}{2}}+C$
b. $\frac{1}{4}(3x^2+5)^{\frac{3}{2}}+C$
c. $\frac{1}{12}(3x^2+5)^{\frac{1}{2}}+C$
d. $\frac{1}{3}(3x^2+5)^{\frac{1}{2}}+C$
e. $\frac{3}{2}(3x^2+5)^{\frac{1}{2}}+C$

$\frac{1}{6}\int u^{-1/2}\,du$
$\frac{1}{6}\cdot 2u^{1/2}+C$
$\frac{1}{3}(3x^2+5)^{1/2}+C$

2) At what point does $(f^{-1})(x)$ have a instantaneous slope of ¼, if $f(x)=x^4-28x+3$?

$f'(x) = 4x^3-28$

a. $(2,-37)$
b. $(75,-2)$
c. $(-2,75)$
d. $(-37,2)$
e. $(4,147)$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{4}$
so $f'(f^{-1}(x)) = 4$
$4(f^{-1}(x))^3-28=4$
$4(f^{-1}(x))^3=32$
$(f^{-1}(x))^3=8$
$f^{-1}(x)=2$
which means $f(2)=x$
$f(2)=-37$
$f^{-1}(-37)=2$

3) $\int \sec^2 x\,dx =$

a. $\tan x + C$
b. $\csc^2 x + C$
c. $\cos^2 x + C$
d. $\frac{\sec^3 x}{3}+C$
e. $2\sec^2 x\ \tan x + C$

4) $\int x^2 e^{x^3}\,dx =$

$u = x^3$
$du = 3x^2$

a. $x^2e^{x^3}+C$
b. $x^2e^{x^3}-2xe^{x^3}+2e^{x^3}+C$
c. $\frac{1}{3}e^{x^3}+C$
d. $x^2e^{x^3}+2xe^{x^3}+2e^{x^3}+C$
e. $3e^{x^3}+C$

$\frac{1}{3}\int e^u\,du$
$\frac{1}{3}e^u+C$
$\frac{1}{3}e^{x^3}+C$

5) $\int_0^{\pi/4} \tan^2 x\,dx =$

$\int_0^{\pi/4}(\sec^2 x-1)\,dx$
$\tan x - x\Big]_0^{\pi/4}$
$\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)-(\tan 0-0)$
$1-\frac{\pi}{4}-0$

a. $\frac{\pi}{4}-1$
b. $1-\frac{\pi}{4}$
c. $1/3$
d. $\sqrt{2}-1$
e. $\frac{\pi}{4}+1$

6) $\int \frac{e^x}{e^x+1}\,dx =$

$u=e^x+1 \quad du=e^x\,dx$
$\int \frac{1}{u}\,du$
$\ln|u|+C$
$\ln|e^x+1|+C$

a. e^x+C
b. $\tan^{-1}(e^x+1)+C$
c. $\ln(e^x+1)+C$
d. $\frac{e^x}{e^x+1}+C$
e. $\frac{e^x+1}{e^x}+C$

7) $\frac{1}{3}\int_0^{\frac{\pi}{3}} \sin(3x)\,dx =$ (upper limit becomes 3)

$u=3x \quad du=3\,dx$

a. -2
b. -2/3
c. 0
d. 2/3
e. 2

$\frac{1}{3}\int_0^{\pi}\sin u\,du$
$-\frac{1}{3}\cos u\Big]_0^{\pi}$
$-\frac{1}{3}\cos\pi+\frac{1}{3}\cos 0$
$-\frac{1}{3}(-1)+\frac{1}{3}(1)$

8) $\int \frac{4\tan 3x}{5}\,dx =$

$u=3x \quad du=3\,dx$

a. $\frac{-4}{15}\ln|\cos 3x|+C$
b. $\frac{4}{15}\ln|\sec 3x|+C$
c. $\frac{4}{15}\ln|\sec 3x\tan 3x|+C$
d. $\frac{4}{15}\sec^2 3x+C$

$\frac{1}{3}\cdot\frac{4}{5}\int 3\tan(3x)\,dx$
$\frac{4}{15}\int \tan u\,du$
$-\frac{4}{15}\ln|\cos 3x|+C$

9) Which of the following are antiderivatives of $\frac{\ln^2 x}{x}$?

$u = \ln x \quad du = \frac{1}{x}dx$

$\int \frac{(\ln x)^2}{x}dx$

$\int u^2 du$

$\frac{1}{3}u^3 + C$

$\frac{1}{3}\ln^3 x + C$

I. $\frac{\ln^3 x}{3}$

II. $\frac{\ln^3 x}{3}+6$

III. $\frac{2\ln x-\ln^2 x}{x^2}$

a. I only
b. III only
c. I and II only
d. I and III only
e. II and III only

10) $\int \tan(2x)\, dx =$

$u = 2x \quad du = 2dx$

$\frac{1}{2}\int \tan u \, du$

$-\frac{1}{2}\ln|\cos 2x| + C$

a. $-2\ln|\cos(2x)|+C$
b. $-\frac{1}{2}\ln|\cos(2x)|+C$
c. $\frac{1}{2}\ln|\cos(2x)|+C$
d. $2\ln|\cos(2x)|+C$
e. $\frac{1}{2}\sec(2x)\tan(2x)+C$

11) $\int \cos^2 x \sin x \, dx =$

$u = \cos x \quad du = -\sin x \, dx$

$-\int u^2 du$

$-\frac{1}{3}u^3 + C$

a. $-\frac{\cos^3 x}{3}+C$
b. $-\frac{\cos^3 x\sin^2 x}{6}+C$
c. $\frac{\sin^2 x}{2}+C$
d. $\frac{\cos^3 x}{3}+C$
e. $\frac{\cos^3 x\sin^2 x}{6}+C$

12) If $f(x)=\sqrt{x}\ln x$ then $f'(2)=$

a. -.75
b. .95
c. 0
d. .25
e. .75

$f'(x) = \left(\frac{1}{2x^{1/2}}\right)(\ln x) + \left(\frac{1}{x}\right)(\sqrt{x})$

$= \frac{1}{2x^{1/2}}\ln x + \frac{1}{\sqrt{x}}$

$f'(2) = \frac{1}{2\sqrt{2}}\ln 2 + \frac{1}{\sqrt{2}}$

13) Given $f(x)=\begin{cases} x+1, & x<0 \\ \cos \pi x, & x\ge 0 \end{cases}$, $\int_{-1}^{1} f(x)dx =$

continuous ✓

$\int_{-1}^{0}(x+1)\,dx + \int_{0}^{1}\cos \pi x \, dx$

$\frac{1}{2}x^2 + x\Big]_{-1}^{0} + \frac{1}{\pi}\sin(\pi x)\Big]_{0}^{1}$

$(0 + 1/2) + (0-0)$

a. $\frac{1}{2}+\frac{1}{\pi}$
b. $-1/2$
c. $\frac{1}{2}-\frac{1}{\pi}$
d. ½
e. $-\frac{1}{2}+\pi$

14) $\frac{1}{2}\int 2\sin(2x+3)\,dx$

$u = 2x+3$
$du = 2dx$

$\frac{1}{2}\int \sin u \, du$

$-\frac{1}{2}\cos u + C$

$-\frac{1}{2}\cos(2x+3) + C$

a. $-2\cos(2x+3)+C$
b. $-\cos(2x+3)+C$
c. $-\frac{1}{2}\cos(2x+3)+C$
d. $\frac{1}{2}\cos(2x+3)+C$
e. $\cos(2x+3)+C$

15) If $f(x)=\ln x$, then $f\left(\frac{3}{2}\right)=$ $\ln\frac{3}{2}$

$= \ln 3 - \ln 2$

$= \ln x\Big]_2^3$

$= \int_2^3 \frac{1}{x}dx$

a. $\frac{\ln 3}{\ln 2}$
b. $\ln 2-\ln\frac{1}{2}$
c. $\int_{\ln 2}^{\ln 3} e^t dt$
d. $\int_2^3 \ln t \, dt$
e. $\int_2^3 \frac{1}{t}\, dt$

16) $\frac{1}{2}\int_0^1 2(x+1)e^{x^2+2x}dx =$

$u = x^2+2x$
$du = (2x+2)\,dx$

$\frac{1}{2}\int_0^3 e^u du$

$\frac{1}{2}e^u\Big]_0^3$

$\frac{1}{2}\left[e^3 - e^0\right]$

$\frac{1}{2}\left[e^3 - 1\right]$

a. $\frac{e^3}{2}$
b. $\frac{e^3-1}{2}$
c. $\frac{e^4-e}{2}$
d. e^3-1
e. e^4-e

17) $\int_0^1 x^3 e^{x^4}\,dx =$ $\frac{1}{4}$ 4

$u = x^4 \quad du = 4x^3\,dx$

a. $\frac{1}{4}(e-1)$
b. $\frac{1}{4}e$
c. $e-1$
d. e
e. $4(e-1)$

$\frac{1}{4}\int_0^1 e^u\,du$

$\frac{1}{4}\left[e^u\right]_0^1$

$\frac{1}{4}\left[e^1 - e^0\right]$

$\frac{1}{4}\left[e-1\right]$

18) $\int_0^3 (x+1)^{\frac{1}{2}}\,dx =$

$u = x+1 \quad du = dx$

a. $21/2$
b. 7
c. $16/3$
d. $14/3$
e. $-1/4$

$\int_1^4 u^{1/2}\,du$

$\frac{2}{3}u^{3/2}\Big]_1^4$

$\frac{2}{3}(2^3 - 1)$

19) $\int_1^2 \frac{x-4}{x^2}\,dx =$

$\int_1^2 \frac{x}{x^2}\,dx - \int_1^2 \frac{4}{x^2}\,dx$

$\int_1^2 \frac{1}{x}\,dx - \int_1^2 4x^{-2}\,dx$

$\left[\ln x\right]_1^2 + \left[\frac{4}{x}\right]_1^2$

$(\ln 2 - \ln 1) + (2-4)$

$\ln 2 - 2$

a. $-1/2$
b. $\ln 2 - 2$
c. $\ln 2$
d. 2
e. $\ln 2 + 2$

20) For all real b, $\int_0^b |2x|\,dx$ is

a. $-b|b|$
b. b^2
c. $-b^2$
d. $b|b|$
e. none of the above

$A = \frac{1}{2}(b)(|2b|)$
$= \frac{1}{2}\cdot 2\,(b)(|b|)$
$= b|b|$

21) $\frac{1}{2}\int e^{\frac{t}{2}}\,dt =$ 2 $\frac{1}{2}$

$u = t/2 \quad du = \frac{1}{2}\,dt$

a. $e^{-t} + C$
b. $e^{-\frac{t}{2}} + C$
c. $e^{\frac{t}{2}} + C$
d. $2e^{\frac{t}{2}} + C$
e. $e^t + C$

$\int e^u\,du$

$e^u - C$

$e^{t/2} + C$

22) $\int_1^2 \frac{1}{x^2}\,dx =$

$\int_1^2 x^{-2}\,dx$

a. $-1/2$
b. $7/24$
c. ½
d. 1
e. 2

$-\frac{1}{x}\Big]_1^2$

$-\frac{1}{2} + 1$

23) $\int_0^x \sin t\,dt =$

a. $\sin x$
b. $-\cos x$
c. $\cos x$
d. $\cos x - 1$
e. $1 - \cos x$

$-\cos t\Big]_0^x$

$-\cos x + \cos 0$

$-\cos x + 1$

24) $\int_1^2 (4x^3 - 6x)\,dx =$

$x^4 - 3x^2\Big]_1^2$

a. 2
b. 4
c. 6
d. 36
e. 42

$(16-12) - (1-3)$

$4 + 2$

25) Which of the following are the antiderivatives of $f(x) = \sin x \cos x$?

✓ I. $F(x) = \frac{\sin^2 x}{2}$ $F' = \sin x \cos x$

II. $F(x) = \frac{\cos^2 x}{2}$ $F' = -\cos x \sin x$

✓ III. $F(x) = \frac{-\cos(2x)}{4}$

$F' = \frac{\sin(2x)\cdot 2}{4}$

$= \frac{1}{2}\sin(2x)$

$= \frac{1}{2}(2\sin x\cos x)$

a. I only
b. II only
c. III only
d. I and III only
e. II and III only

26) $\int_1^e \left(\frac{x^2-1}{x}\right)dx$

$\int_1^e \frac{x^2}{x}\,dx - \int_1^e \frac{1}{x}\,dx$

$\int_1^e x\,dx - \int_1^e \frac{1}{x}\,dx$

$\left[\frac{1}{2}x^2\right]_1^e - \left[\ln x\right]_1^e$

$\left(\frac{e^2}{2} - \frac{1}{2}\right) - (\ln e - \ln 1)$ 0

$\frac{e^2}{2} - \frac{1}{2} - 1$

a. $e - \frac{1}{e}$
b. $e^2 - e$
c. $\frac{e^2}{2} - e + \frac{1}{2}$
d. $e^2 - 2$
e. $\frac{e^2}{2} - \frac{3}{2}$

27) Which of the following is NOT an antiderivative of secx?

a. $\ln|\sec x + \tan x| + C$

b. $-\ln|\sec x - \tan x| + C$

same (see next pg)

c. $\ln\left|\frac{1-\sin x}{\cos x}\right| + C$

d. $\ln\left|\frac{1+\sin x}{\cos x}\right| + C$

e. $\ln\left|\frac{\cos x}{1-\sin x}\right| + C$

$\boxed{\sec x + \tan x}$

$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$

$= \boxed{\frac{1+\sin x}{\cos x}} \cdot \frac{1-\sin x}{1-\sin x}$

$= \frac{1-\sin^2 x}{\cos x\,(1-\sin x)} = \frac{\cos^2 x}{\cos x\,(1-\sin x)}$

$= \boxed{\frac{\cos x}{1-\sin x}}$

28) $\frac{1}{5}\int_1^2 \sqrt{5x}\,dx =$

$u=5x$

$du=5dx$

a. 2.726

b. 2.981

c. 3.354

d. 13.628

e. 20.442

$\frac{1}{5}\int_5^{10} u^{1/2}\,du$

$\frac{1}{5}\cdot\frac{2}{3}u^{3/2}\Big]_5^{10}$

$\frac{2}{15}\left[10^{3/2} - 5^{3/2}\right]$

29) If f is a continuous function and if $F'(x) = f(x)$ for all real numbers x, then

$\frac{1}{2}\int_1^3 f(2x)dx =$

a. $2F(3) - 2F(1)$

b. $\frac{1}{2}F(3) - \frac{1}{2}F(1)$

c. $2F(6) - 2F(2)$

d. $F(6) - F(2)$

e. $\frac{1}{2}F(6) - \frac{1}{2}F(2)$

$u=2x \quad du=2dx$

$\frac{1}{2}\int_2^6 f(u)\,du$

$\frac{1}{2}\left[F(u)\right]_2^6$

$\frac{1}{2}\left[F(6) - F(2)\right]$

30) $\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\,dx =$

a. 0

b. 1

c. $e-1$

d. e

e. $e+1$

$u=\tan x \quad du=\sec^2 x\,dx$

$\int_0^1 e^u\,du$

$e^u\Big]_0^1 = e^1 - e^0$

31) $\int_1^2 x^{-3}dx =$

a. $-7/8$

b. $-3/4$

c. $15/64$

d. $3/8$

e. $15/16$

$\frac{x^{-2}}{-2}\Big]_1^2 = \frac{-1}{2x^2}\Big]_1^2$

$= \frac{-1}{2(4)} - \frac{-1}{2(1)}$

$-\frac{1}{8} + \frac{1}{2}$

32) If the substitution $u = \frac{x}{2}$ is made, the integral

$\therefore x = 2u$

$du = \frac{1}{2}dx$

$2\int_2^4 \frac{1-\left(\frac{x}{2}\right)^2}{x}\,dx =$

a. $\int_1^2 \frac{1-u^2}{u}\,du$

b. $\int_2^4 \frac{1-u^2}{u}\,du$

c. $\int_1^2 \frac{1-u^2}{2u}\,du$

d. $\int_1^2 \frac{1-u^2}{4u}\,du$

e. $\int_2^4 \frac{1-u^2}{2u}\,du$

$2\int_1^2 \frac{1-u^2}{2u}\,du$

$\int_1^2 \frac{1-u^2}{u}\,du$

33) If $\frac{dy}{dx} = \cos(2x)$, then $y = \frac{1}{2}\int 2\cos(2x)\,dx$

a. $-\frac{1}{2}\cos(2x) + C$

b. $-\frac{1}{2}\cos^2(2x) + C$

c. $\frac{1}{2}\sin(2x) + C$

d. $\frac{1}{2}\sin^2(2x) + C$

e. $-\frac{1}{2}\sin(2x) + C$

$u=2x \quad du=2dx$

$\frac{1}{2}\int \cos u\,du$

$\frac{1}{2}\sin u + C$

$\frac{1}{2}\sin(2x) + C$

34) $\frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx =$

a. $\frac{1}{2}\ln\frac{3}{2}$

b. $\frac{1}{2}\ln 2$

c. $\ln 2$

d. $2\ln 2$

e. $\frac{1}{2}\ln 5$

$u=x^2+1 \quad du=2x\,dx$

$\frac{1}{2}\int_5^{10} \frac{1}{u}\,du$

$\frac{1}{2}\left[\ln u\right]_5^{10}$

$\frac{1}{2}\left[\ln 10 - \ln 5\right]$

$\frac{1}{2}\left[\ln \frac{10}{5}\right] = \frac{1}{2}\ln 2$

35) $\frac{1}{3}\int_0^1 3(3x-2)^2\,dx =$

a. $-7/3$

b. $-7/9$

c. $1/9$

d. 1

e. 3

$u=3x-2 \quad du=3dx$

$\frac{1}{3}\int_{-2}^1 u^2\,du$

$\frac{1}{3}\cdot\frac{1}{3}u^3\Big]_{-2}^1$

$\frac{1}{9}\left[u^3\right]_{-2}^1 = \frac{1}{9}\left[1-(-8)\right] = 1$

from #27...

$$\ln|\sec x + \tan x| + C$$

$$\ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C$$

$$\ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$\ln\left|\frac{(1+\sin x)\cos x}{\cos^2 x}\right| + C$$

$$\ln\left|\frac{(1+\sin x)\cos x}{1-\sin^2 x}\right| + C$$

$$\ln\left|\frac{(1+\sin x)\cos x}{(1+\sin x)(1-\sin x)}\right| + C$$

$$\ln\left|\frac{\cos x}{1-\sin x}\right| + C$$

$$\ln\left|\frac{1}{\frac{1-\sin x}{\cos x}}\right| + C$$

$$\ln\left|\frac{1}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}\right| + C$$

$$\ln\left|\frac{1}{\sec x - \tan x}\right| + C$$

$$\ln|(\sec x - \tan x)^{-1}| + C$$

$$-\ln|\sec x - \tan x| + C$$

FIND THE BINGO

Calculus – e^x and ln(x) and log(x)

DIRECTIONS:
Work any problem below and locate your answer on your bingo card. Circle the answer.

Keep working problems in any order until you have five circled answers in a line -- horizontally, vertically, or diagonally.
WHEN YOU FIND THE BINGO, YOUR WORK IS FINISHED!

Your BINGO Card

12 $e^{-x}(1-x)$	$2xe^{-x^2}$	9 $\frac{1}{6}e^{6x}+C$	$4\tan(4x)$	-1
2 -3	13 1	5 4/(x+1)	$e^{-x}(x-1)$	1 x^4
$\frac{1}{4}\tan(4x)$	4 1 and 2	-2	15 $\frac{1}{2}\ln\lvert 2x+1\rvert+C$	3
8 $-2xe^{-x^2}$	14 $\frac{1}{2}\ln(x^2+1)+C$	3 2	11 $2xe^{x^2}$	7 $-4\tan(4x)$
$-2xe^{x^2}$	$-4\cot(4x)$	10 $-e^{\cos x}+C$	$\ln(x^2+1)+C$	6 $1/(x^2+x)$

Problems to solve:

1. Simplify $e^{\ln x^4}$

2. Evaluate $\log_3 \frac{1}{27}$

3. Solve $9^{\log_3 x}=4$ for x

4. Solve $3^{x^2-3x}=\frac{1}{9}$ for x $3^{x^2-3x}=\frac{1}{9}$

5. If $y=\ln(x+1)^4$, determine $\frac{dy}{dx}$

6. If $y=\ln\left(\frac{x}{1+x}\right)$, determine $\frac{dy}{dx}$

7. If $y=\ln(\cos 4x)$,, determine $\frac{dy}{dx}$

8. If $y=e^{-x^2}$, determine $\frac{dy}{dx}$

9. Evaluate $\int e^{6x}dx$

10. Evaluate $\int e^{\cos x}\sin x dx$

11. If $\ln y=x^2$, determine $\frac{dy}{dx}$

12. If $y=xe^{-x}$, determine $\frac{dy}{dx}$

13. If $y=\ln e^{x}$, determine $\frac{dy}{dx}$

14. Evaluate $\int\frac{xdx}{x^2+1}$

15. Evaluate $\int\frac{dx}{2x+1}$

e^x = ln x bingo

1. $e^{\ln x^4}$

$\boxed{x^4}$

$\ln e^{\ln x^4} = \ln y$
$\ln x^4 = \ln y$
$x^4 = y$

2. $\log_3 \frac{1}{27}$

$\boxed{-3}$

$\log_3 \frac{1}{27} = y$
$3^y = \frac{1}{27}$
$y = -3$

3. $9^{\log_3 x} = 4$

$\boxed{x=2}$

$\log_3 9^{\log_3 x} = \log_3 4$
$\log_3 x \cdot \log_3 3^2 = \log_3 4$
$\log_3 x \cdot 2 = \log_3 4$
$\log_3 x^2 = \log_3 4$
$x^2 = 4$

4. $3^{x^2-3x} = \frac{1}{9}$

$\boxed{x=1,\ x=2}$

$\log_3(3^{x^2-3x}) = \log_3 1/9$
$(x^2-3x)(\log_3 3) = \log_3 3^{-2}$
$x^2 - 3x = -2$
$x^2 - 3x + 2 = 0$
$(x-2)(x-1) = 0$

7. $y = \ln(\cos 4x)$

$\frac{dy}{dx} = \frac{-4\sin 4x}{\cos 4x}$

$\boxed{\frac{dy}{dx} = -4\tan 4x}$

6. $y = \ln\left(\frac{x}{1+x}\right)$

$y = \ln x - \ln(1+x)$
$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{1+x}$
$\frac{dy}{dx} = \frac{(1+x) - x}{x(1+x)}$

$\boxed{\frac{dy}{dx} = \frac{1}{x^2+x}}$

5. $y = \ln(x+1)^4$

$\boxed{\frac{dy}{dx} = \frac{4}{x+1}}$

$y = 4\ln(x+1)$
$\frac{dy}{dx} = 4 \cdot \frac{1}{x+1}$

8. $y = e^{-x^2}$

$\boxed{\frac{dy}{dx} = -2xe^{-x^2}}$

$\frac{dy}{dx} = e^{-x^2} \cdot -2x$

9. $\int e^{6x}\,dx$

$\boxed{\frac{1}{6}e^{6x} + C}$

$u = 6x \quad du = 6dx$
$= \frac{1}{6}\int e^{6x} \cdot 6dx$
$= \frac{1}{6}\int e^u du$
$= \frac{1}{6}e^u + C$

10. $\int e^{\cos x}\sin x\,dx$

$= \boxed{-e^{\cos x} + C}$

$u = \cos x \quad du = -\sin x dx$
$-\int e^u du$
$-e^u + C$

11. $\ln y = x^2$

$\boxed{\frac{dy}{dx} = 2xe^{x^2}}$

$y = e^{x^2}$
$\frac{dy}{dx} = e^{x^2} \cdot 2x$

12. $y = xe^{-x}$

$\boxed{\frac{dy}{dx} = e^{-x} - xe^{-x}}$ $= e^{-x}(1-x)$

$\frac{dy}{dx} = e^{-x} + xe^{-x} \cdot -1$

13. $y = \ln(e^x)$

$\boxed{\frac{dy}{dx} = 1}$

$y = x\ln e$
$y = x$

14. $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

$= \boxed{\frac{1}{2}\ln(x^2+1) + C}$

$u = x^2+1 \quad du = 2xdx$
$= \frac{1}{2}\int \frac{1}{u}du$
$= \frac{1}{2}\ln|u| + C$

15) $\frac{1}{2}\int \frac{2}{2x+1}\,dx$ $u = 2x+1$ $du = 2dx$

$\frac{1}{2}\int \frac{1}{u}\,du$

$\boxed{\frac{1}{2}\ln|2x+1| + c}$ $\frac{1}{2}\ln|u| + c$

WS: Integral Review 5.1-5.5

c 1) $\int -2e^{-2x}\,dx$ $\quad u=-2x \quad du=-2\,dx$

$\int e^u\,du$

$e^u + C = \boxed{e^{-2x}+C}$

b 2) $\int \frac{dx}{x+1}$ $\quad u=x+1 \quad du=dx$

$\int \frac{1}{u}\,du$

$\ln|u| + C = \boxed{\ln|x+1|+C}$

b 3) $\frac{1}{2}\int \frac{dx}{x \ln x^2} = \frac{1}{2}\int \frac{dx}{2x \ln x} = \frac{1}{4}\int \frac{dx}{x \ln x}$ $\quad u=\ln x \quad du = 1/x\,dx$

$= \frac{1}{4}\int \frac{1}{u}\,du$

$= \frac{1}{4}\ln|u| + C = \boxed{\frac{1}{4}\ln|\ln x|+C}$

a 4) $\int \frac{\ln x}{2x}\,dx = \frac{1}{2}\int \frac{\ln x}{x}\,dx$ $\quad u=\ln x \quad du = 1/x\,dx$

$= \frac{1}{2}\int u\,du$

$= \frac{1}{2}\cdot\frac{u^2}{2} = \boxed{\frac{(\ln x)^2}{4}+C}$

a 5) $\int \frac{1}{x\sqrt{\ln x}}\,dx$ $\quad u=\ln x \quad du = 1/x\,dx$

$\int u^{-1/2}\,du$

$= 2u^{1/2}+C = \boxed{2\sqrt{\ln x}+C}$

a 6) $\int \frac{2x}{\sqrt{x^2-1}}\,dx$ $\quad u=x^2-1 \quad du=2x\,dx$

$\int u^{-1/2}\,du$

$= 2u^{1/2}+C = \boxed{2\sqrt{x^2-1}+C}$

a 7) $\frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+3}}\,dx$ $\quad u=x^3+3 \quad du=3x^2+C$

$= \frac{1}{3}\int u^{-1/2}\,du$

$= \frac{1}{3}\cdot 2u^{1/2} = \boxed{\frac{2}{3}\sqrt{x^3+3}+C}$

8) $\int \frac{-2x+2}{e^{x^2-2x}} = \int (-2x+2)\,e^{-x^2+2x}$ $\quad u=-x^2+2x \quad du=-2x+2\,du$

c $= \int e^u\,du$

$= e^u + C = \boxed{e^{-x^2+2x}+C}$

a 9) $\int \tan x \sec^2 x \, dx$ $u = \tan x$ $du = \sec^2 x \, dx$

$\int u \, du$

$= u^2/2 + C = \boxed{\frac{\tan^2 x}{2} + C}$

b 10) $-\int \frac{-e^{-x}}{1+e^{-x}}$ $u = 1+e^{-x}$ $du = -e^{-x}$

$= -\int \frac{1}{u} \, du$

$= -\ln|u| + C = \boxed{-\ln|1+e^{-x}| + C}$

a 11) $\int \frac{1}{(x+1)^2} \, dx$ $u = x+1$ $du = dx$

$= \int \frac{1}{u^2} \, du = \int u^{-2} \, du$

$= \frac{-1}{u} + C = \boxed{\frac{-1}{x+1} + C}$

no u 12) $\int \frac{x^2+5}{x} \, dx = \int \left(\frac{x^2}{x} + \frac{5}{x}\right) dx = \int \left(x + \frac{5}{x}\right) dx$

$= \boxed{\frac{x^2}{2} + 5 \ln|x| + C}$

no u 13) $\int e^2 x^3 \, dx = e^2 \int x^3 \, dx$

$= e^2 \cdot x^4/4 + C = \boxed{e^2 x^4/4 + C}$

b 14) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx$ $u = e^x - e^{-x}$ $du = e^x + e^{-x} \, dx$

$= \int \frac{1}{u} \, du$

$= \ln|u| + C = \boxed{\ln|e^x - e^{-x}| + C}$

a 15) $\int \frac{1}{\sqrt{1+x}} \, dx$ $u = 1+x$ $du = dx$

$= \int u^{-1/2} \, du$

$= 2u^{1/2} + C = \boxed{2\sqrt{1+x} + C}$

c 16) $\int \cos x \left(e^{\sin x}\right) dx$ $u = \sin x$ $du = \cos x \, dx$

$= \int e^u \, du$

$= e^u + C = \boxed{e^{\sin x} + C}$

a 17) $\int \frac{2 \tan 2x}{\cos^2 2x}\, dx = \int 2 \tan 2x \sec^2 2x \, dx$

$u = \tan 2x \quad du = 2 \sec^2 2x$

$= \int u \, du$

$= u^2/2 + C = \boxed{\frac{\tan^2 2x}{2} + C}$

a 18) $-\int \sin x \cos^3 x \, dx \qquad u = \cos x \quad du = -\sin x \, dx$

$= \int u^3 \, du$

$= u^4/4 + C = \boxed{\frac{\cos^4 x}{4} + C}$

b 19) $-\int \frac{-\sin x}{1 + \cos x}\, dx \qquad u = 1 + \cos x \quad du = -\sin x \, dx$

$= -\int \frac{1}{u}\, du$

$= -\ln|u| + C = \boxed{-\ln|1+\cos x| + C}$

20) $\int ex \, dx = e \int x \, dx$

$= e \cdot \frac{x^2}{2} + C = \boxed{\frac{ex^2}{2} + C}$

alternate sol:

17) $\frac{2 \tan 2x}{\cos^2 2x} = 2 \frac{\sin 2x}{\cos 2x} \cdot \frac{1}{\cos^2 2x} = \int \frac{-2 \sin 2x}{\cos^3 2x} \qquad u = \cos 2x \quad du = -2 \sin 2x$

$-\int \frac{1}{u^3}\, dx = -\int u^{-3}\, du = -\frac{u^{-2}}{-2} = \frac{1}{2u^2} = \frac{1}{2\cos^2 2x} + C$

Name Fernando Trujano Date November 1st

Calculus BC Notes ~ Section 5.6
Inverse Trig Functions and Differentiation

I. <u>Intro and warm up</u>:

$\arcsin\frac{\sqrt{2}}{2} = \pi/4$

$\arctan(-1) = -\pi/4$ cannot go through white water

$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = 5\pi/6$

When doing these, remember your restrictions!

$\left.\begin{array}{l}\arcsin x\\ \arctan x\\ \operatorname{arccsc} x\end{array}\right\} \quad \left(\frac{-\pi}{2},\frac{\pi}{2}\right) \to QI, QIV$

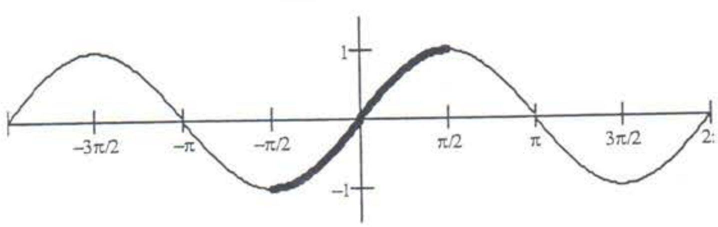
$\left.\begin{array}{l}\arccos x\\ \operatorname{arccot} x\\ \operatorname{arcsec} x\end{array}\right\} \quad (0,\pi) \to QI, QII$

<u>Why restrictions</u>?

→ None of the 6 basic trig functions has an inverse.

→ They are all periodic and **not one-to-one or strictly monotonic**.

ex) $f(x) = \sin x$ is always increasing on $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, so it is always monotonic <u>on that interval</u> and it is one-to-one. $f(x) = \sin x$ does not have an inverse across its entire domain.



→ We restrict all 6 periodic trig functions like this, so that we can find an inverse.

Inverse Trig Integrals

* $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$

$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$

a = constant

u = variable / expression

Examples

$\int \frac{dx}{\sqrt{4-x^2}}$

$a^2=4 \quad u^2=x^2 \quad = \arcsin \frac{x}{2} + C$

$a=2 \quad u=x$

$du=dx$

$\int \frac{dx}{2+9x^2}$

$a^2=2 \quad u^2=9x^2 \quad = \frac{1}{3}\int \frac{du}{2+u^2}$

$a=\sqrt{2} \quad u=3x$

$du=3dx \quad = \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$

$\int \frac{dx}{x^2-4x+7}$ ← u-sub will not work → inverse trig

$x^2-4x+\underline{4}+7\underline{-4} \qquad \int \frac{dx}{(x-2)^2+3}$

$(x-2)^2+3$

$a^2=3 \quad u^2=(x-2)^2 \quad = \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C$

$a=\sqrt{3} \quad u=x-2$

$du=dx$

$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$a^2=1 \qquad -\int \frac{du}{1+u^2}$

$a=1$

$u^2=\cos^2 x \qquad = -\arctan(\cos x)\Big]_{\pi/2}^{\pi}$

$u=\cos x$

$du=-\sin x\, dx \qquad -\arctan(-1)+\arctan(0)$

$= \frac{\pi}{4}$

$\int \frac{dx}{2x^2-8x+10} = \frac{1}{2}\int \frac{dx}{x^2-4x+5}$

$x^2-4x+4+5-4 \qquad \frac{1}{2}\int \frac{dx}{(x-2)^2+1}$

$(x-2)^2+1 \qquad a^2=1 \quad u^2=(x-2)^2$

$a=1 \quad u=x-2 \quad = \frac{1}{2} \arctan(x-2) + C$

$du=dx$

$\int \frac{3}{\sqrt{-x^2-10x}} dx = 3\int \frac{1}{\sqrt{25-(x+5)^2}} dx$

$-x^2-10x \qquad = 3\arcsin \frac{x+5}{5} + C$

$-[x^2+10x+25-25] \qquad a^2=25$

$a=5$

$-[(x+5)^2-25] \qquad u^2=(x+5)^2$

$u=x+5$

$25-(x+5)^2 \qquad du=dx$

$$\int \frac{x+2}{\sqrt{4-x^2}}\,dx$$

~~usub~~ ~~trig~~ → rewrite

$$\int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx$$

$a^2 = 4$
$a = 2$
$u^2 = x^2$
$u = x$
$du = dx$

$2\arcsin \frac{x}{2} + C$

$u = 4-x^2$
$du = -2x\,dx$ $\quad -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= -\frac{1}{2} \cdot \frac{2u^{-1/2}}{1}$ $\quad 1$ $\qquad -\frac{1}{2} \cdot \frac{2u^{-1/2}}{1} + 2\arcsin \frac{x}{2} + C$

$-\sqrt{4-x^2} + 2\arcsin \frac{x}{2} + C$

III. Euler's Method

Euler's Method is a numerical approach used to approximate the particular solution of a differential equation. Use the point (x_0, y_0) as a starting point, and use the slope (derivative) to determine where to go next. The formula is derived from the point-slope formula...

$y - y_1 = m(x - x_1)$	point-slope formula
$y - y_1 = m(\Delta x)$	$x - x_1$ gives change in x
$y = y_1 + m(\Delta x)$	move the initial y-value to other side

Euler's Method Formula: $\underbrace{y_n}_{\text{new}} = \underbrace{y_{n-1}}_{\text{old}} + \underbrace{(\Delta x)}_{\text{change}}\underbrace{(m_{n-1})}_{\text{old slope}}$

where $y_n =$ y-value

$y_{n-1} =$ previous y-value

$\Delta x = h =$ change in x-value from previous point

$m_{n-1} = f'(n-1) =$ slope at previous point

Ex) Use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use n steps of size h.

$y' = x - 2,\ y(0) = 5,\ n = 2,\ h = 0.4$

n	0	1	2
x_n	0	.4	.8
y_n	5	4.2	3.56

start

new y = old y + (Δx)(old slope)

$y = 5 + (.4)(\underset{f'(0)}{-2}) = 4.2$

$y = 4.2 + (.4)(-1.6) = 3.56$

Ex) (if time allows) Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2	1.92	1.78

new y = old y + (Δx)(old slope)

$y = 2 + (-.2)(.4) = 1.92$

$y = 1.92 + (-.2)(.7) = 1.78$

AP Calculus BC

Name Fernando Trujano

6.2 ~ Differential Equations: Growth and Decay Models

If $y' = ky$ (that is, the derivative of y is proportional to y), then the general solution to the differential equation is $y = Ce^{kt}$. This is the main equation for an exponential function.

Rate proportional

Derive: $y' = \underset{\text{constant}}{ky} \rightarrow y = Ce^{kt}$

$\frac{dy}{dt} = ky$

$dy = ky\,dt$

$\frac{1}{y}dy = k\,dt$

$\int \frac{1}{y}dy = \int k\,dt$

$\ln|y| = kt + C$

$|y| = e^{kt + C}$

$y = e^{kt}e^{C}$ constant

$y = Ce^{kt}$

Ex1) Solve the differential equation

$$y' = \frac{x}{2y}$$

$\frac{dy}{dx} = \frac{x}{2y}$

$\int 2y\,dy = \int x\,dx$

$y^2 = \frac{x^2}{2} + C$ →Done

Ex2) (#12) Write a differential equation:

The rate of change of P with respect to t is proportional to $10-t$.

$\frac{dP}{dt} = k(10-t)$

$\int dP = \int k(10-t)\,dt$

$P = k(10t - \frac{t^2}{t}) + C$

Ex3) (#26) Find the exponential function $y = Ce^{kt}$ that passes through the points (0, 4) and (5, $\frac{1}{2}$).

$4 = Ce^{k(0)}$

$C = 4$

$\frac{1}{2} = 4e^{5k}$

$\frac{1}{8} = e^{5k}$

$\ln(\frac{1}{8}) = \ln e^{5k}$

$\ln\frac{1}{8} = 5k$

$\frac{\ln(1/8)}{5} = k = -.416$

$y = 4e^{-.416t}$

Name Fernando Trujano

AP Calculus BC
6.1 ~ Slope Fields and Euler's Method

I. Intro to Differential Equations

From the text...

"A function $y = f(x)$ is called a solution of a differential equation (an equation that involves a derivative) if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives."

Ex) Determine whether $y = 4e^{-x}$ is a solution of the differential equation $y'' - y = 0$.

$y' = -4e^{-x}$ $\quad$ $4e^{-x} - 4e^{-x} = 0$

$y'' = 4e^{-x}$ $\quad$ $0 = 0$ ✓

YES

Ex) For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution, and find the particular solution determined by the initial condition $y = 2$ when $x = 3$.

SOLVE FOR C

$y' = 3Cx^2$ $\quad$ $2 = C(3^3)$

$x(3Cx^2) - 3(Cx^3) = 0$ $\quad$ $C = 2/27$

$0 = 0$ ✓

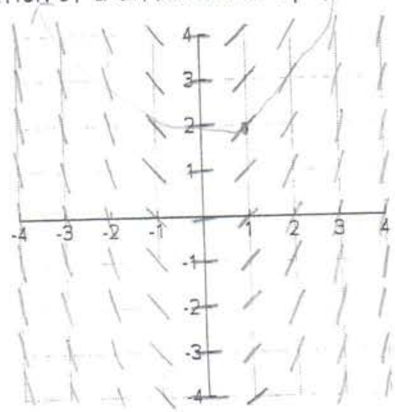
YES

$y = \frac{2}{27}x^3$

II. Slope Fields

Why do we use slope fields? We use slope fields when solving a differential equation can be difficult or even impossible. This is a graphical approach that can reveal the solution of a differential equation.

Ex) Sketch a slope field for the differential equation $y' = x$.



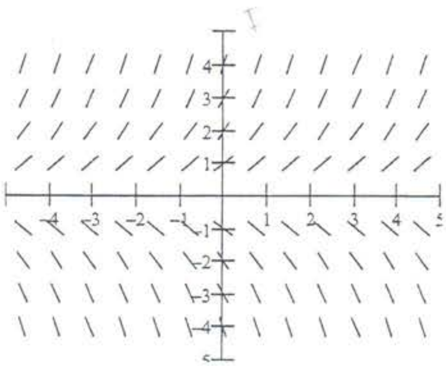
What do you notice?

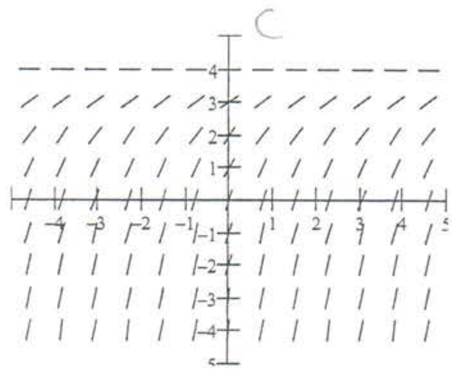
Draw a solution curve through the point (1, 2)

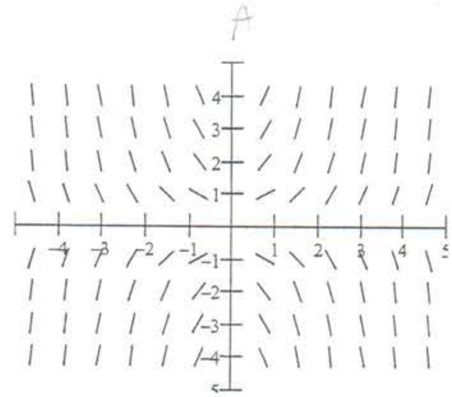
7. Match the slope fields with their differential equations.

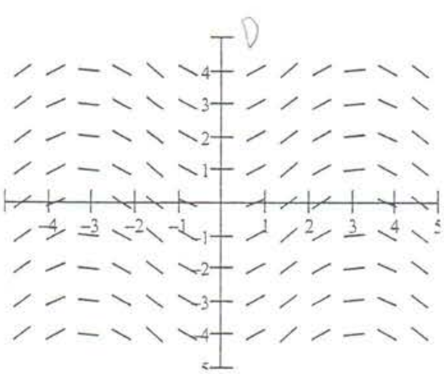
a) $\dfrac{dy}{dx} = xy$ b) $\dfrac{dy}{dx} = 1 + y^2$ c) $\dfrac{dy}{dx} = 4 - y$

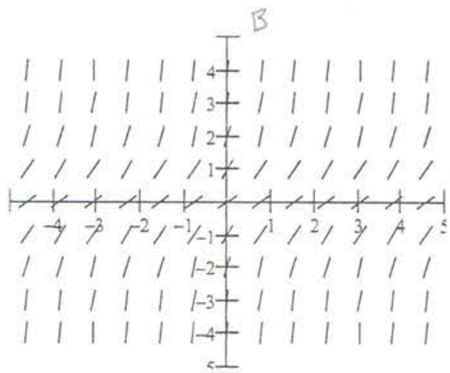
d) $\dfrac{dy}{dx} = \sin x$ e) $\dfrac{dy}{dx} = x$ f) $\dfrac{dy}{dx} = y$

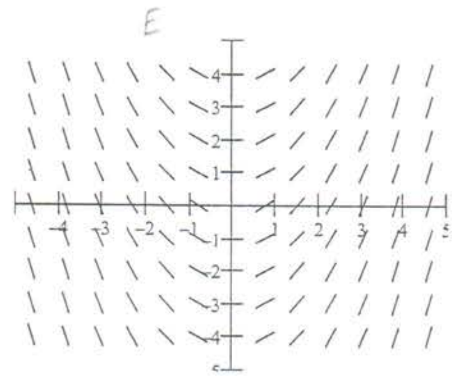












8. Shown below is the slope field for which of the following differential equations?

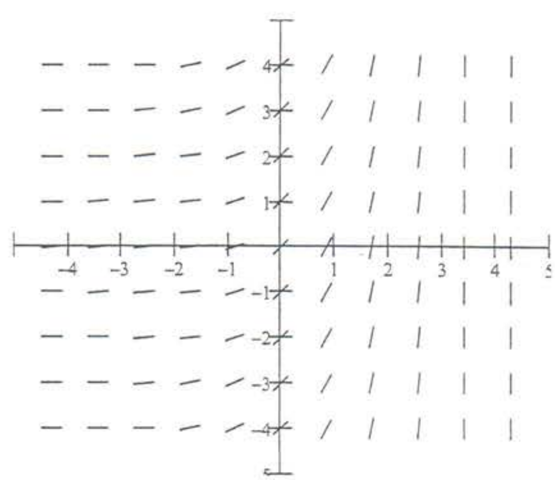
(A) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(B) $\dfrac{dy}{dx} = \ln y$

(C) $\dfrac{dy}{dx} = e^x$

(D) $\dfrac{dy}{dx} = x^2$

(E) $\dfrac{dy}{dx} = 1 + x$



AP Calculus BC

Name Fernando Trujano

6.3 ~ Separation of Variables and the Logistic Equation

I. Separation of Variables (continuation from 6.2)

Ex) Find the general solution of $(x^2+4)\frac{dy}{dx}=xy$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx \qquad u = x^2+4 \quad du = 2x\,dy$$

① Group Variables
② Integrate on each side

$$\ln|y| = \frac{1}{2}\ln|x^2+4| + C$$

$$y = e^{\frac{1}{2}\ln|x^2+4| + C}$$

$$y = Ce^{\ln\sqrt{x^2+4}}$$

$$y = C\sqrt{x^2+4}$$

#2) Same as 1 Solve the differential equation: $\frac{dy}{dx}=\frac{x^2+2}{3y^2}$

$$\int 3y^2\,dy = \int x^2+2\,dx$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

OK to leave it as y^2, y^3, y^8

cannot combine all T,t → capitalization matters

#22) Find the particular solution that satisfies the differential equation: $dT+k(T-70)dt=0,\ T(0)=140$

$$dT = -k(T-70)\,dt$$

$$\int \frac{1}{T-70}\,dT = \int -k\,dt$$

$$\ln|T-70| = -kt + C$$

$$e^{-kt+C} = T-70$$

$$T = Ce^{-kt} + 70$$

Particular solution

$$140 = Ce^{-k(0)} + 70$$

$$C = 70$$

$$T = 70e^{-kt} + 70$$

 sin tan sec  cos cot csc

5.6-5.7 Practice

Inverse Trig: Numerical Evaluation, Differentiation, and Integration

Find the following:

1. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

2. $\cos^{-1}(1) = 0$

3. $\tan^{-1}(-1) = -\frac{\pi}{4}$

4. $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

5. $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

6. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

7. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{-\pi}{6}$

9. $\tan^{-1}\left(-\sqrt{3}\right) = \frac{-\pi}{3}$

10. $\sin^{-1}(0) = 0$

11. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

12. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

13. $\tan^{-1}(0) = 0$

14. $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

15. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

16. $\sin^{-1}(-1) = -\frac{\pi}{2}$

17. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{-\pi}{4}$

18. $\cos^{-1}(0) = \frac{\pi}{2}$

19. $\sin^{-1}(1) = \frac{\pi}{2}$

20. $\tan^{-1}(1) = \frac{\pi}{4}$

21. $\cos^{-1}(-1) = \pi$

22. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Find the derivative of each.

1. $y = \cos^{-1} 2x$

$y' = \boxed{\dfrac{-2}{\sqrt{1-4x^2}}}$

2. $y = \sec^{-1} 5x$

$y' = \boxed{\dfrac{5}{|5x|\sqrt{25x^2-1}}}$

3. $y = \arcsin x^2$

$y' = \boxed{\dfrac{2x}{\sqrt{1-x^4}}}$

4. $y = \arctan\sqrt{x}$

$y' = \dfrac{\frac{1}{2}x^{-1/2}}{1+x} = \boxed{\dfrac{1}{2\sqrt{x}(1+x)}}$

5. $y = \tan^{-1}(\ln x)$

$y' = \dfrac{1/x}{1+(\ln x)^2} = \dfrac{1}{x(1+(\ln x)^2)} = \boxed{\dfrac{1}{x+x(\ln x)^2}}$

6. $y = x \arccos x$

$y' = \arccos x + x \cdot \dfrac{-1}{\sqrt{1-x^2}} = \boxed{\arccos x - \dfrac{x}{\sqrt{1-x^2}}}$

7. $y = \tan^{-1} e^x$

$y' = \boxed{\dfrac{e^x}{1+e^{2x}}}$

8. $y = \arccos e^{-x}$

$y' = \dfrac{-(-e^{-x})}{\sqrt{1-e^{-2x}}} = \boxed{\dfrac{e^{-x}}{\sqrt{1-e^{-2x}}}}$

Evaluate each integral.

1. $\int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^2}}dx$ $\arcsin x \Big]_0^{1/2}$

$\arcsin\frac{1}{2} - \arcsin 0 = \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$

2. $\frac{1}{2}\int \dfrac{2\ 1}{\sqrt{1-4x^2}}dx = \boxed{\frac{1}{2}\arcsin 2x + C}$

$a=1$
$u=2x$
$du=2$

3. $\int_{\sqrt{2}}^{2} \dfrac{1}{x\sqrt{x^2-1}}dx$ $\operatorname{arcsec}|x| \Big]_{\sqrt{2}}^{2}$

$u=x$
$a=1$

$\operatorname{arcsec} 2 - \operatorname{arcsec}\sqrt{2} = \arccos\frac{1}{2} - \arccos\frac{\sqrt{2}}{2}$
$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \boxed{\frac{\pi}{12}}$

4. $\int_{-1}^{1} \dfrac{1}{1+x^2}dx$

$\arctan x \Big]_{-1}^{1} = \arctan 1 - \arctan(-1)$
$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right) = \boxed{\frac{\pi}{2}}$

5. $\frac{1}{2}\int \dfrac{2x}{\sqrt{1-x^4}}dx = \boxed{\frac{1}{2}\arcsin x^2 + C}$

$u=x^2$
$du=2x$

6. $\int \dfrac{dx}{9+x^2} = \boxed{\frac{1}{3}\arctan\frac{x}{3} + C}$

7. $\int \dfrac{dx}{\sqrt{25-x^2}}$ $\boxed{\arcsin\frac{x}{5} + C}$

8. $\int \dfrac{e^{2x}}{\sqrt{1-e^{4x}}}dx = \boxed{\arcsin e^{2x} + C}$

$u=e^{2x}$
$du=e^{2x}dx$

9. $\frac{1}{2}\int \dfrac{2x}{4+x^4}dx$ $\frac{1}{2}\cdot\frac{1}{2}\arctan\frac{x^2}{2} + C$

$a=2$
$u=x^2$
$du=2x$

$= \boxed{\frac{1}{4}\arctan\frac{x^2}{2} + C}$

10. $-\int \dfrac{-\sin\theta}{1+\cos^2\theta}d\theta = \boxed{-\arctan(\cos\theta) + C}$

$u=\cos\theta$
$du=-\sin\theta$

Name: ______________________

6.2

Exponential Growth and Decay $y = Ce^{kt}$

1. A radioactive substance has a half-life of 100 years. If 6 grams of a substance were present initially, how much will be present in 50 years? $(0,6)$ $(100,3)$ $(50,?)$

$3 = 6e^{100k}$ $\ln\frac{1}{2} = 100k$

$\frac{1}{2} = e^{100k}$ $k = \frac{\ln\frac{1}{2}}{100} = -0.007$ $y = 6e^{-0.007(50)}$ 4.243 grams

2. A town's population is increasing at a rate proportional to the population. The population was initially 1000, and grew to 3000 in 20 years. $(0,1000)$ $(20,3000)$

   a. What would the population be in 50 years?

   $3000 = 1000e^{k(20)}$ $\ln 3 = 20k$ $y = 1000e^{0.0549(50)}$ 15,588.457 ppl

   $3 = e^{20k}$ $k = \frac{\ln 3}{20} = 0.0549$

   b. How long did it take the initial population to double?

   $2000 = 1000e^{0.0549(t)}$ $\ln 2 = 0.0549(t)$ $t = 12.619$ yrs

   $2 = e^{0.0549(t)}$

3. A radioactive element has a half-life of 1000 years. If 5 grams of the element were present initially, how long would it take to reduce the amount to 2 grams? $(0,5)$ $(1000, 2.5)$ $(?,2)$

$2.5 = 5e^{1000k}$ $\ln\frac{1}{2} = 1000k$

$.5 = e^{1000k}$ $k = \frac{\ln\frac{1}{2}}{1000}$ $2 = 5e^{kt}$ $\frac{2}{5} = e^{kt}$ $\ln\frac{2}{5} = kt$ $t = 1321.928$ yr

4. The number of bacteria present is increasing at a rate proportional to the number of bacteria present. There were 5000 bacteria present initially. The population tripled in 10 hours. How long would it take before the population was 9 times its original size? $(0,5000)$ $(10,15{,}000)$ $(?,45{,}000)$

$15{,}000 = 5{,}000e^{k(10)}$ $\ln 3 = 10k$

$3 = e^{10k}$ $k = \frac{\ln 3}{10} = 0.110$ $45{,}000 = 5{,}000e^{0.110t}$ $\ln 9 = 0.110t$

$9 = e^{0.110t}$ $t = 20$ hr

Integrals and Equations of Tangent Lines

5. Find the area bounded by: $y = e^{-x}$; $x = 0$; $y = 0$ and $x = 1$.

$u = -x$

$du = -dx$ $\int_0^1 e^{-x}dx = -e^{-x}\Big]_0^1 = -e^{-1} + e^0 = 1 - e^{-1}$

6. Write the equation of the line tangent to $y = e^{x^2}$ at $(2, e^4)$.

$y' = 2xe^{x^2}$ $y - e^4 = 4e^4(x-2)$

$y'|_{x=2} = 4e^4$

7. Find the area bounded by $y = \frac{1}{x}$; $y = 0$; $x = 1$ and $x = e$.

$\int_1^e \frac{1}{x}dx = \ln|x|\Big]_1^e = \ln e - \ln 1 = 1$

8. Write the equation of the line tangent to $y = \ln(2x)$ at $\left(\frac{1}{2}, 0\right)$.

$y' = \frac{2}{2x} = \frac{1}{x}$ $y'|_{x=\frac{1}{2}} = 2$ $y - 0 = 2(x - \frac{1}{2})$

9. Write the equation of the line tangent to $y = 1 + e^x$ at $(0,2)$.

$y' = e^x$ $y'|_{x=0} = 1$ $y - 2 = 1(x-0)$

10. Write the equation of the line tangent to $y = x\ln x$ at (e,e).

$y' = \ln x + x(\frac{1}{x})$ $y - e = 2(x-e)$

$y'|_{x=e} = 1 + 1 = 2$

Name: ______________________

6.3

Differential Equations and Separation of Variables

Solve the following differential equations. Remember not to leave "C" in the exponent!

1) $y' = \frac{y}{x}$ $\frac{dy}{dx} = \frac{y}{x}$

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\ln|y| = \ln|x| + C$

$y = e^{\ln|x| + C}$

$\boxed{y = Cx}$

2) $y' = 4x^3(1-y)$ $\frac{dy}{dx} = 4x^3(1-y)$

$\int \frac{1}{1-y}\,dy = \int 4x^3\,dx$

$-\ln|1-y| = x^4 + C$

$\ln|1-y| = -x^4 + C$

$1-y = e^{-x^4+C}$

$1-y = Ce^{-x^4}$

$\boxed{y = 1 - Ce^{-x^4}}$

3) $\frac{dy}{dx} = \frac{3x^2+1}{2y}$, given that $y(0) = 1$

$\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$

$1 = C$

$\boxed{y^2 = x^3 + x + 1}$

4) $2\frac{dy}{dx} = \frac{y}{x}(x+1)$

$\int \frac{2}{y}\,dy = \int \frac{x+1}{x}\,dx$

$2\int \frac{1}{y}\,dy = \int \left(1 + \frac{1}{x}\right)dx$

$2\ln|y| = x + \ln|x| + C$

$\ln|y| = \frac{x + \ln|x| + C}{2}$

$\boxed{y = e^{\frac{x+\ln|x|+C}{2}}}$

5) $t\,\frac{dx}{dt} - x = 3$

$t\frac{dx}{dt} = x + 3$

$\int \frac{1}{x+3}\,dx = \int \frac{1}{t}\,dt$

$\ln|x+3| = \ln|t| + C$

$x + 3 = e^{\ln|t| + C}$

$x = Ce^{\ln|t|} - 3$

6) $y' - y = 6$, given the point $(0,1)$

$\frac{dy}{dx} = y + 6$

$\int \frac{1}{y+6}\,dy = \int dx$

$\ln|y+6| = x + C$

$y + 6 = e^{x+C}$

$y = Ce^x - 6$

$1 = C(1) - 6$

$7 = C$

$\boxed{y = 7e^x - 6}$

AP Calculus
Review ~ 5.6-5.7, 6.1-6.3

Name: ____________________

E

1. Find the derivative of the function $f(x) = 8\arcsin(x-8)$.

(A) $f'(x) = \dfrac{8}{\sqrt{1-(x-8)^2}}$

$1-(x^2-16x+64)$

(B) $f'(x) = \dfrac{8}{\sqrt{16x-x^2-63}}$

(C) $f'(x) = \dfrac{8}{\sqrt{(x-8)^2-1}}$

(D) $f'(x) = \dfrac{8}{(x-8)^2-1}$

(E) both A and B

A

2. Find the derivative of the function $y = \arctan\left(\dfrac{x}{2}\right) + \dfrac{5x-1}{2(x^2+2)}$. $= \arctan\frac{x}{2} + \frac{1}{2}\left(\frac{5x-1}{x^2+2}\right)$

(A) $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{1}{1+(x/2)^2} + \dfrac{10+2x-5x^2}{(x^2+2)^2}\right)$

$y' = \dfrac{\frac{1}{2}}{1+(\frac{x}{2})^2} + \dfrac{1}{2}\left(\dfrac{(x^2+2)(5)-(5x-1)(2x)}{(x^2+2)^2}\right)$

(B) $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{1}{x+4} + \dfrac{5x}{(x^2+2)^2}\right)$

$y' = \dfrac{1}{2}\left(\dfrac{1}{1+(\frac{x}{2})^2} + \dfrac{5x^2+10-10x^2+2x}{(x^2+2)^2}\right)$

(C) $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{1}{(x^2+2)^2} + \dfrac{5x}{x^2+2^2}\right)$

$y' = \dfrac{1}{2}\left(\dfrac{1}{1+(\frac{x}{2})^2} + \dfrac{10+2x-5x^2}{(x^2+2)^2}\right)$

(E) both B and C

E

3. Find the indefinite integral. $\displaystyle\int \frac{1}{64+(x-2)^2}\,dx$

$a = 8$
$u = x-2$

(A) $\dfrac{1}{64}\arctan\left(\dfrac{x-2}{8}\right)+C$

(B) $8\arctan\left(\dfrac{x-2}{8}\right)+C$

(C) $\dfrac{1}{64}\arctan\left(\dfrac{x-2}{64}\right)+C$

(D) $64\arctan\left(\dfrac{x-2}{64}\right)+C$

(E) $\dfrac{1}{8}\arctan\left(\dfrac{x-2}{8}\right)+C$

$u = x^2 + 6x + 25$ $\quad (x^2+6x+9) + 25 - 9$

$du = (2x+6)\,dx$ $\quad (x+3)^2 + 16$

4. Find the indefinite integral. $\int \frac{2x-4}{x^2+6x+25}\,dx$

A

(A) $\ln\left|x^2+6x+25\right| - \frac{5}{2}\arctan\left(\frac{x+3}{4}\right) + C$

$\int \frac{2x+6}{x^2+6x+25}\,dx - 10\int \frac{1}{x^2+6x+25}\,dx$

(B) $\ln\left|x^2+6x+25\right| + C$

$\int \frac{1}{u}\,du - 10\int \frac{1}{(x+3)^2+4^2}\,dx$

$\ln|u| - 10\left(\frac{1}{4}\right)\arctan\frac{(x+3)}{4} + c$

(C) $\frac{5}{2}\arctan\left(\frac{x+3}{4}\right) + C$

(D) $\ln\left|x^2+6x+25\right| + \frac{5}{2}\arctan\left(\frac{x+3}{4}\right) + C$

$\ln|x^2+6x+9| - \frac{5}{2}\arctan\frac{x+3}{4} + c$

(E) $\frac{-5}{2}\arctan\left(\frac{x+3}{4}\right) + C$

$-1(x^2+12x+36-36)$

B

5. Find the indefinite integral. $\int \frac{dx}{\sqrt{-x^2-12x}}$

$-1\left[(x+6)^2 - 36\right]$

(A) $\arcsin\left(\frac{x+12}{12}\right) + C$

(D) $\arcsin\left(\frac{x-12}{12}\right) + C$

$\int \frac{1}{\sqrt{36-(x+6)^2}}\,dx$

(B) $\arcsin\left(\frac{x+6}{6}\right) + C$

(E) $\arcsin\left(\frac{x+6}{12}\right) + C$

$a = 6$

$u = x+6$

(C) $\arcsin\left(\frac{x-6}{6}\right) + C$

C

6. (1985 BC 33) If $\frac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

$\frac{1}{y}\,dy = -2\,dt$ $\quad y = e^{-2t}$ $\quad -\ln 2 = -2t$

(A) $\frac{-\ln 2}{2}$ (B) $\frac{-1}{4}$ (C) $\frac{\ln 2}{2}$

$\ln|y| = -2t + C$ $\quad \frac{1}{2} = e^{-2t}$ $\quad \frac{\ln 2}{2} = t$

$Ce^{-2t} = y$

(D) $\frac{\sqrt{2}}{2}$ (E) $\ln 2$

$\ln\frac{1}{2} = -2t$

$Ce^{0} = 1$ $\quad \ln 1 - \ln 2 = -2t$

A

7. (1985 BC 44) At each point (x, y) on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point (0,8), then its equation is

(A) $y = 8e^{x^3}$ (B) $y = x^3 + 8$ (C) $y = e^{x^3} + 7$

(D) $y = \ln(x+1) + 8$ (E) $y^2 = x^3 + 8$

$\frac{dy}{dx} = 3x^2y$ $\quad \ln|y| = x^3 + C$

$Ce^{x^3} = y$ $\quad y = 8e^{x^3}$

$\int \frac{1}{y}\,dy = 3\int x^2\,dx$ $\quad Ce^{0} = 8$

$\int \frac{1}{y}\,dy = \int \sec^2 x\,dx$

C 8. (1988 BC 39) If $\frac{dy}{dx} = y\sec^2 x$ and $y = 5$ when $x = 0$, then $y =$ $\ln|y| = \tan x + C$

(A) $e^{\tan x} + 4$ (B) $e^{\tan x} + 5$ (C) $5e^{\tan x}$

(D) $\tan x + 5$ (E) $\tan x + 5e^x$

$y = Ce^{\tan x}$ $y = 5e^{\tan x}$

$5 = Ce^{\tan 0}$

$5 = C$

A 9. (1988 BC 43) Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

(A) $\frac{3\ln 3}{\ln 2}$ (B) $\frac{2\ln 3}{\ln 2}$ (C) $\frac{\ln 3}{\ln 2}$

(D) $\ln\left(\frac{27}{2}\right)$ (E) $\ln\left(\frac{9}{2}\right)$

$y = Ce^{kt}$ $(0, C)$ $(3, 2C)$

$2C = Ce^{k(3)}$ $(?, 3C)$

$2 = e^{k(3)}$ $3C = Ce^{\frac{\ln 2}{3}(t)}$

$\ln 2 = 3k$ $3 = e^{\frac{\ln 2}{3}(t)}$

$k = \frac{\ln 2}{3}$ $\ln 3 = \frac{\ln 2}{3}(t)$

B 10. (1993 AB 33) If $\frac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$, then when $x = 2$, $y =$

(A) $\frac{-2}{3}$ (B) $\frac{-1}{3}$ (C) 0

(D) $\frac{1}{3}$ (E) $\frac{2}{3}$

$\int \frac{1}{y^2}\,dy = \int 2\,dx$ $\frac{-1}{y} = 2x + C$ $-\frac{1}{y} = 2x - 1$

$\int y^{-2}\,dy = 2\int dx$ $\frac{-1}{-1} = 2(1) + C$ $-\frac{1}{y} = 2(2) - 1$

$\frac{y^{-1}}{-1} = 2x + C$ $1 = 2 + C$ $-\frac{1}{y} = 3$

$-1 = C$ $-1 = 3y$

B 11. (1988 BC 8) If $\frac{dy}{dx} = \sin x\cos^2 x$ and if $y = 0$ when $x = \frac{\pi}{2}$, what is the value of y when $x = 0$?

(A) -1 (B) $\frac{-1}{3}$ (C) 0

(D) $\frac{1}{3}$ (E) 1

$\int dy = -\int \sin x\cos^2 x\,dx$ $\int dy = -\int u^2\,du$

$u = \cos x$ $y = -\frac{(\cos x)^3}{3} + C$

$du = -\sin x\,dx$

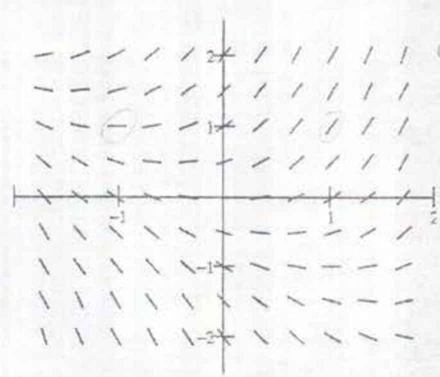
C 12. (1998 BC 24) Shown below is the slope field for which of the following differential equations?

look at y axis + x scale!

(A) $\frac{dy}{dx} = 1 + x$ (D) $\frac{dy}{dx} = \frac{x}{y}$

(B) $\frac{dy}{dx} = x^2$ (E) $\frac{dy}{dx} = \ln y$

(C) $\frac{dy}{dx} = x + y$



B 13. (1993 AB 42) A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

$(0,2)\ (3,?)$
$(2,3.5)$

(A) 4.2 pounds (B) 4.6 pounds (C) 4.8 pounds

(D) 5.6 pounds (E) 6.5 pounds

$3.5 = 2e^{2k}$ $y = 2e^{0.2798t}$

$\ln 1.75 = 2k$

$k = 0.2798$

C 14. (1993 BC 13) If $\dfrac{dy}{dx} = x^2 y$, then y could be

(A) $3\ln\left(\dfrac{x}{3}\right)$ (B) $e^{x^3/3}+7$ (C) $2e^{x^3/3}$

(D) $3e^{2x}$ (E) $\dfrac{x^3}{3}+1$

$\int \frac{1}{y}\,dy = \int x^2\,dx$ $Ce^{\frac{x^3}{3}} = y$

$\ln|y| = \frac{x^3}{3} + C$

15. (2000 AB 6 no calculator) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2}{e^{2y}}$

a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \dfrac{1}{2}$.

b) Find the domain and range of the function f found in part (a).

a)

$\frac{1}{2}\int 2e^{2y}\,dy = 3\int x^2\,dx$ $u = 2y$, $du = 2dy$

$\frac{1}{2}\int e^{u}\,du = 3\int x^2\,dx$

$\frac{1}{2}e^{2y} = x^3 + C$

$e^{2y} = 2x^3 + C$

$e^{2(\frac{1}{2})} = 2(0)^3 + C$

$e = C$

$e^{2y} = 2x^3 + e$

$2y = \ln(2x^3 + e)$

$\boxed{y = \ln\sqrt{2x^3+e}}$

b) D: $2x^3 + e \geq 0$ $x \geq \sqrt[3]{\frac{-e}{2}}$

$2x^3 \geq -e$

$x^3 \geq -\frac{e}{2}$

R: all real #

16. The rate at which the flu spreads through a community is modeled by the logistic differential equation
$$\frac{dP}{dt} = 0.001P(3000-P)\text{, where } t \text{ is measured in days.}$$

$0.001(3000)\left(1 - \frac{P}{3000}\right) = 3\left(1 - \frac{P}{3000}\right)$

a) If $P(0) = 50$, solve for P as a function of t.

b) Use your solution to (a) to find the size of the population when $t = 2$ days.

c) Use your solution to (a) to find the number of days that have occurred when the flu is spreading the fastest.

a) $k = 3$ $L = 3000$ $\boxed{y = \dfrac{3000}{1+59e^{-3t}}}$

$y = \dfrac{3000}{1+be^{-3t}}$

$50 = \dfrac{3000}{1+b}$

$1+b = \dfrac{3000}{50}$

$1+b = 60$

$b = 59$

b) $\boxed{y = 2617.238}$

c) $1500 = \dfrac{3000}{1+59e^{-3t}}$ $\boxed{t = 1.359 \text{ days}}$

$1+59e^{-3t} = 2$

$59e^{-3t} = 1$

$e^{-3t} = \frac{1}{59}$

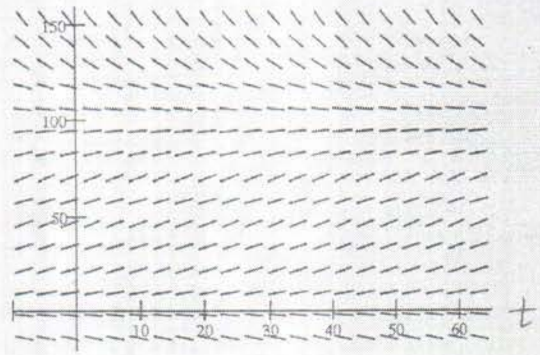
$\ln\frac{1}{59} = -3t$

17. Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks. $0.05P\left(1 - \frac{P}{100}\right)$

a) What is the carrying capacity?

100

b) A slope field for this equation is shown at the right.



Where are the slopes close to zero? $P = 100$

Where are they largest? $P = 50$

Which solutions are increasing? $P < 100$

Which solutions are decreasing? $P > 100$

c) Use the slope field to sketch solutions for initial populations of 20, 60, and 120.

What do these solutions have in common? all level out at P

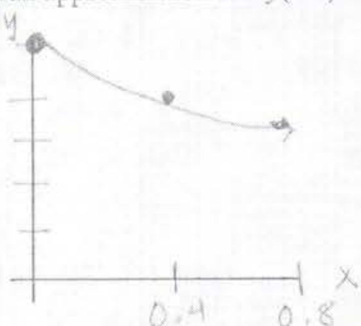
How do they differ? inc vs dec

Which solutions have inflection points? all $\rightarrow$ 6

At what population level do they occur? when $P = 50$

18. Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5\ldots$

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.



n	0	1	2
x	0	0.4	0.8
y	5	4.2	3.56

$y_1 = 5 + 0.4(0-2) = 4.2$

$y_2 = 4.2 + 0.4(0.4-2) = 3.56$

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

$y = \frac{(0.8)^2}{2} - 2(0.8) + 5 = \boxed{3.72}$

$\int dy = \int (x-2)\,dx$

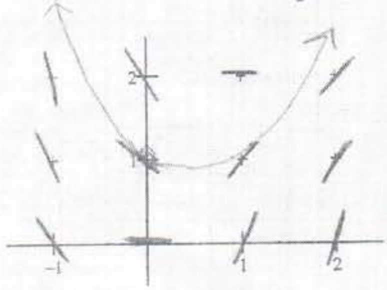
$y = \frac{x^2}{2} - 2x + C$

$C = 5$

$y = \frac{x^2}{2} - 2x + 5$

19. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point (0,1).



b) The solution curve that passes through the point (0,1) has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

$\frac{dy}{dx} = 0$ for minimum

$2x - y = 0$

$2x = y$

$\boxed{2(\ln(\tfrac{3}{2})) = y}$

c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

n	0	1	2
x	0	-0.2	-0.4
y	1	1.28	1.552

$f(-0.4) \approx 1.552$

$y_1 = 1 - 0.2(2(-0.2) - 1) = 1.28$

$y_2 = 1.28 - 0.2(2(-0.4) - 1.28) = 1.552$

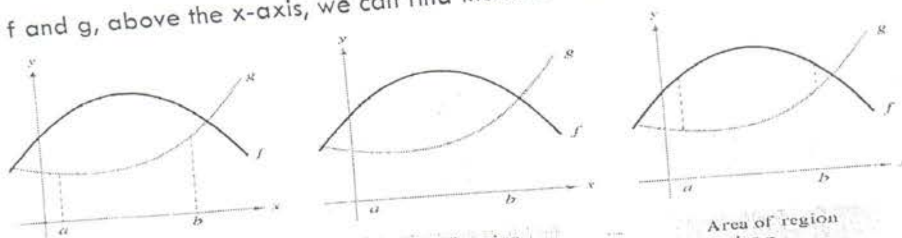
Also, look over matching slope fields to differential equations!

Name Fernando Trujano Period___
Date ______

Calculus BC
7.1 & 7.2 Day 1 Notes

7.1 Area of a Region Between Two Curves

Given two functions, f and g, above the x-axis, we can find the area between the two curves:



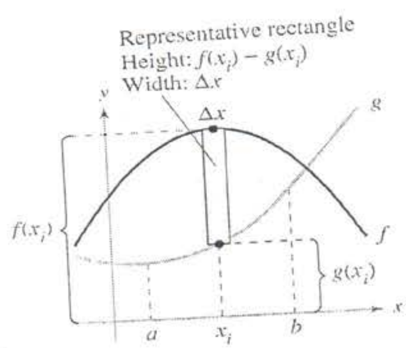
Area of region between f and g = Area of region under f − Area of region under g

$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

top   bottom

How do we get this formula mathematically?

Use n rectangles in [a, b] with width Δx:



$A_{rectangle}$ = height · base

=

$A_{region} = \int_a^b [f(x) - g(x)]\,dx$

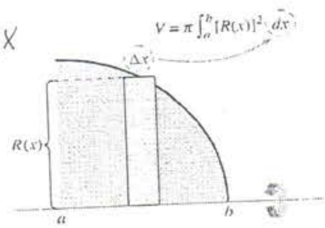
A=

Vertical Rectangle(Horizontal axis):

width = $\Delta x = dx$

height = $f(x) - g(x)$ top - bottom

$\int_a^b (\text{top} - \text{bottom})\,dx$



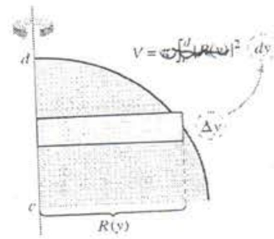
Horizontal axis of revolution

Horizontal Rectangle(Vertical axis):

width = $\Delta y = dy$

height = Right - left

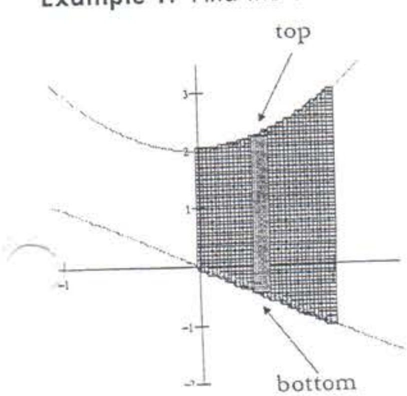
$\int_a^b (\text{Right} - \text{left})\,dy$



Vertical axis of revolution

Example 1: Find the area of the region bounded by $y = x^2 + 2$, $y = -x$, $x = 0$, $x = 1$

top   bottom



$A = \int_0^1 [(x^2+2) - (-x)]\,dx$

$= \int_0^1 (x^2 + x + 2)\,dx$

$= \frac{x^3}{3} + \frac{x^2}{2} + 2x \Big]_0^1$

$= \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}$

7.2 Volume: The Disk Method

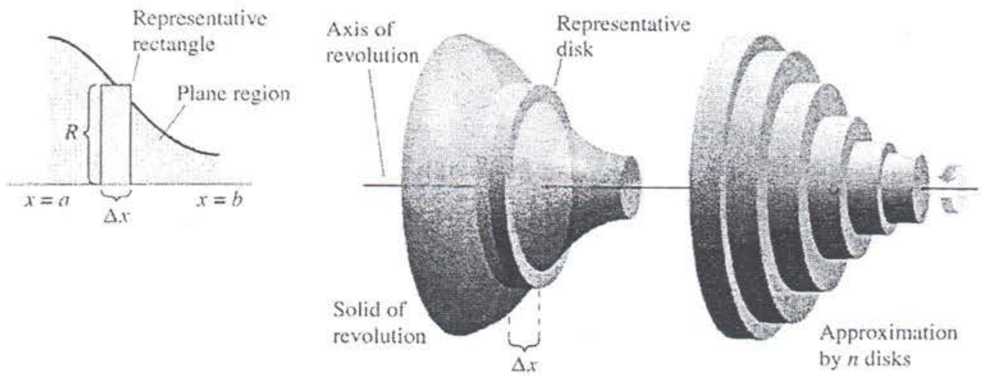
Day 1

We can use integration to find the volume of a solid!

-Spin a region around an axis to create a 3-D figure then find the volume.

-representative rectangles are perpendicular to axis of revolution

-rotating rectangles create disks

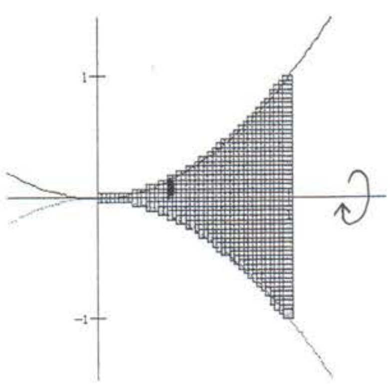


$V_{disk} = \pi r^2 \cdot \Delta X$

Vertical rectangles: $V_{region} = \int_{x_1}^{x_2} \pi r^2 \, dx$ ↳ Function

Horizontal rectangles: $V_{region} = \int_{y_1}^{y_2} \pi r^2 \, dy$ ↳ function

Example 1: Find the volume when $y = x^2$ is rotated around the x-axis from x=0 to x=1.



$$V = \int_0^1 \pi (x^2)^2 \, dx$$

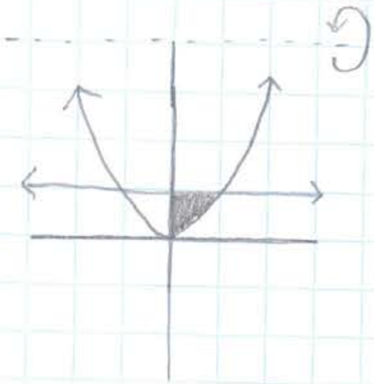
$$= \pi \int_0^1 x^4 \, dx$$

$$= \frac{\pi}{5} \approx 0.628$$

7-2 Day 2

Find the volume of solid formed by revolving the region bounded by $x=0,\ y=x^2,\ y=1$ around $y=4$

wk. 19



$$V=\pi\int_a^b\left[R^2-r^2\right]dx$$

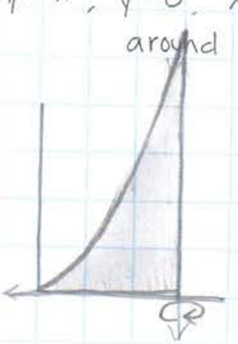
$R=4-x^2$

$r=4-1=3$

$$V=\pi\int_0^1\left[(4-x^2)^2-(3)^2\right]dx$$

$V=14.242$

$y=x^2,\ y=0,\ x=3$ around $x=3$



$y=x^2$

$x=\pm\sqrt{y}$

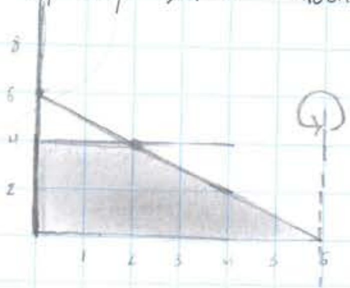
$R=3-\sqrt{y}$

$r=3-3=0$ ← no opening / hole

$$V=\pi\int_0^9\left[(3-\sqrt{y})^2-(0)^2\right]dy$$

$V=42.411$

$y=6-x,\ y=0,\ y=4,\ x=0$ around $x=6$



$R=6-0=6$

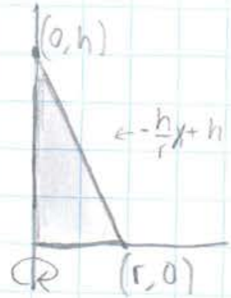
$r=6-(6-y)=y$

$$V=\pi\int_0^4\left[6^2-y^2\right]dy$$

$V=385.369$

Use the disc method to verify that the V of a right cylinder cone is $V=\frac{1}{3}\pi r^2h$

r and h are constants



$y=-\frac{h}{r}x+h$

$x=\frac{-r}{h}(y-h)$

$x=-\frac{r}{h}y+r$

$$V=\pi\int_0^h\left(\frac{-r}{h}y+r\right)^2dy$$

$$V=\pi\int_0^h\left(\frac{r^2}{h^2}y^2-\frac{2r^2}{h}y+r^2\right)dy$$

$$V=r^2\pi\int_0^h\left(\frac{y^2}{h^2}-\frac{2y}{h}+1\right)dy$$

$$=\pi r^2\left(\frac{y^3}{3h^2}-\frac{y^2}{h}+y\right]_0^h$$

$$=\pi r^2\left[\frac{h}{3}-h+h\right.$$

$$=\frac{1}{3}\pi r^2h$$

7-2
Day 3

Cross sections

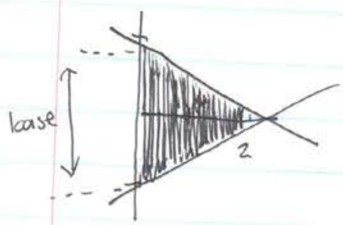
cross sections $\perp$ to x-axis $V = \int_{x_1}^{x_2} A(x)\,dx$

Area of shape

cross sections $\perp$ to y-axis $V = \int_{y_1}^{y_2} A(y)\,dy$

Ex

base bounded by $y = 1 - \frac{x}{2}$ $y = -1 + \frac{x}{2}$ $x = 0$



a) squares $A = b^2$

$\text{base} = (1 - \frac{x}{2}) - (-1 + \frac{x}{2}) = 2 - x$

top bottom

$V = \int_0^2 \left((2-x)^2 \right) dx$

b) semicircles

$A = \frac{\pi}{8} b^2$

base: $2 - x$

Find b and plug in

$V = \int_0^2 \left(\frac{\pi}{8} (2-x)^2 \right) dx$

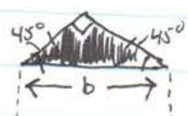
$= 1.047$

c) equalateral $\triangle$

$A = \frac{\sqrt{3}}{4} b^2$

$V = \int_0^2 \frac{\sqrt{3}}{4} (2-x)^2\, dx = 1.155$

d) isoceles right $\triangle$

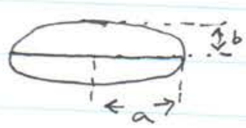


$\frac{1}{2}(b)(\frac{b}{2})$

$A = \frac{b^2}{4}$

$V = \int_0^2 \frac{1}{4} (2-x)^2\, dx = 2/3$

e) ellipse



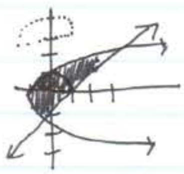
$A = \pi ab$

semi elipse $A = \frac{1}{2} \pi ab$

Find volume of figure

$x = y^2 - 1 \qquad y = x - 1$

Use cross sections of semi circles perpendicular to the y-axis



$y = x - 1$

$x = y + 1$

Base: $(y+1) - (y^2 - 1) = y - y^2 + 2$

Right - left

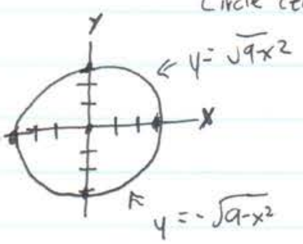
$A = \frac{\pi}{8} b^2$

$V = \int_0^2 \frac{\pi}{8} (y - y^2 + 2)^2 \, dy$

$= 3.181$

Find volume when $x^2 + y^2 = 9$ has square crossections perpendicular to the x-axis

Circle center $(0,0)$ $r = 3$



Base $= 2\sqrt{9 - x^2}$

$A = b^2$

$V = \int_{-3}^{3} \left(2\sqrt{9 - x^2}\right)^2 dx = 144$

AP Calculus BC

Notes ~ Section 7.4 ~ Arc Length

Name Fernando Trujano

I. Derivation of the Arc Length Formula

Question: How do you find the length of an arc?

Answer: Begin with the distance formula...

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Use this to find the distance between any two points on the curve that are close to each other...

$x_2-x_1=dx$

$y_2-y_1=dy$

$$d=\sqrt{dx^2+dy^2}$$

$$d=\sqrt{dx^2+dx^2\left(\frac{dy}{dx}\right)^2}$$

$$d=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

Sum up those infinitely small distances across the entire arc using calculus techniques.

Definition of Arc Length

Let the function given by $y=f(x)$ represent a smooth curve on the interval $[a,b]$. The arc length of f between a and b is

$$s=\int_a^b\sqrt{1+[f'(x)]^2}\ dx\text{, where } a \text{ and } b \text{ are } x\text{-values.}$$

Similarly, for a smooth curve given by $x=g(y)$, the arc length of g between c and d is

$$s=\int_a^b\sqrt{1+[g'(y)]^2}\ dy\text{, where c and d are } y\text{-values.}$$

$$s=\int_a^b\sqrt{1+(f'(x))^2}\ dx$$

II. Examples similar to those in the text

1. Find the distance between the points using (a) the Distance Formula and (b) integration: $(2,4)$ and $(5,8)$

a) $d=\sqrt{(5-2)^2+(8-4)^2}$

$d=5$

b) $f'(x)=\frac{4}{3}$

$m=\frac{8-4}{5-2}=\frac{4}{3}$

$$s=\int_2^5\sqrt{1+\left(\frac{4}{3}\right)^2}\ dx=\int_2^5\frac{5}{3}\,dx=\frac{5}{3}x\Big]_2^5$$

$$=5$$

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AP Calculus BC
Quiz 7.1, 7.2 (Day 1)
Calculator

Name Fernando Trujano

Date Nov 30 Period 4

1. Fill in the blanks of the general formulas, that are used when figuring the area between two curves.
Choose from: **bottom curve, x, left curve, y-values, right curve, x-values, y, top curve**

Horizontal Rectangles: $A=\int_a^b($ right curve $-$ left curve $)\,d$ y

a and b are y-values.

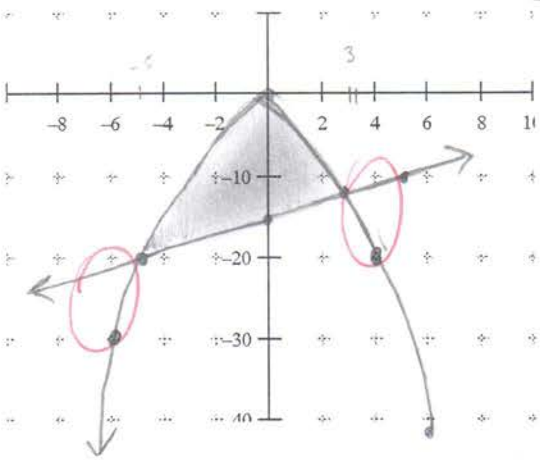
Vertical Rectangles: $A=\int_a^b($ top curve $-$ bottom curve $)\,d$ x

a and b are x values.

2. Sketch a graph and find the area of the regions bounded by the graphs of: -8

a. $f(x)=-x^2-x$ and $g(x)=x-15$ on [-6,5]. ?

A = ~~85.333~~

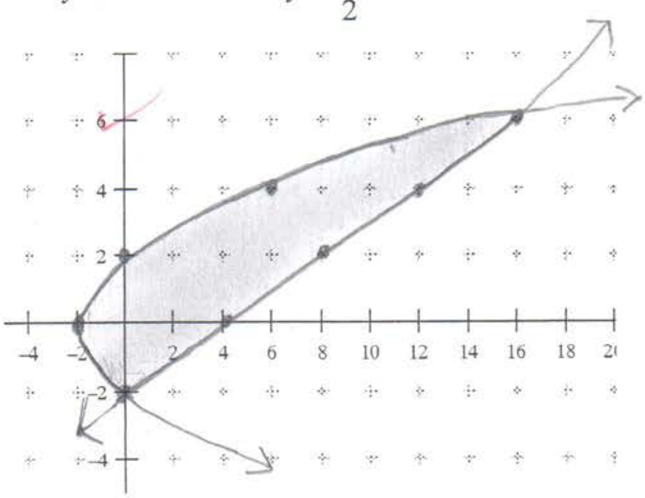


$$A=\int_{-5}^{3}\left[(-x^2-x)-(x-15)\right]dx$$

$$=85.333$$

b. $y^2=2x+4$ and $y=\frac{1}{2}x-2$

A = 42.6667



Horizontal

$$A=\int_{-2}^{6}\left[(2y+4)-\left(\frac{y^2-4}{2}\right)\right]dy$$

$$=42.6667$$

$2y+4=x \qquad x=\frac{y^2-4}{2}$

8

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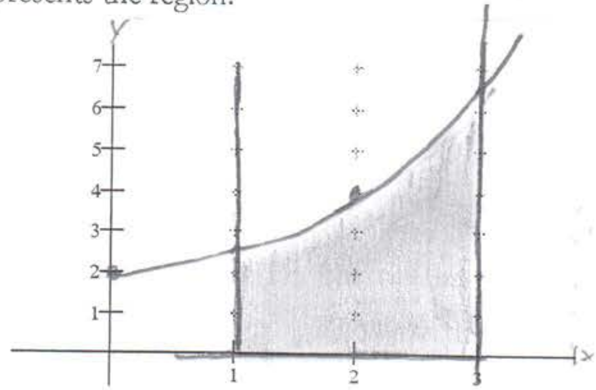
AP Calculus BC
Quiz 7.2 ~ Volume of Solids
Calculator

Name Fernando Trujano

1) Let B be the region bounded by $f(x)=\frac{1}{2}x^2+2, \quad x=1, \quad x=3,$ and the x-axis.

a) Draw a graph that represents the region.



b) Rotate B about the x-axis and calculate the volume. V = 37.433π

$V = \pi \int_1^3 \left(\frac{1}{2}x^2+2\right)^2 dx$

c) Rotate B about the line y=8 and calculate the volume. V = ~~131.6π~~

dx

$R = 8-\left(\frac{1}{2}x^2+2\right)$

$r = 1.5$

$V = \pi\int_1^3\left[\left(-\frac{1}{2}x^2+6\right)^2-(1.5)\right]^2 dx$

$R=8$

$r = 8-\left(\frac{1}{2}x^2+2\right) = 6-\frac{1}{2}x^2$

$= 131.6$

$V=\pi\int_1^3 \left(8^2-\left(6-\frac{1}{2}x^2\right)^2\right)dx$

8

d) Rotate B about the line x=4 and calculate the volume. V = ______

$y=\frac{1}{2}x^2+2$

$2y=x^2+4$

$x=\pm\sqrt{2y-4}$

$x^2=2y-4$

$R = 4-\sqrt{2y-4}$

$r = 4-3=1$

dy

$V=\pi\int_1^3\left[\left(4-\sqrt{2y-4}\right)^2-1^2\right]dy$

① $R = 4-1=3$

$r=4-3=1$

$V_1=\pi\int_0^{2.5}(3^2-1^2)dy$

② $R=4-\sqrt{2y-4}$

$r=4-3=1$

$V_2=\pi\int_{2.5}^{6.5}\left(\left(4-\sqrt{2y-4}\right)^2-1^2\right)dy$

$V=V_1+V_2=96.342$

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e) B is the base of a solid and cross sections of the solid are semicircles perpendicular to the x-axis. Calculate the volume of the solid.

$A=\frac{\pi}{8}b^2$

b = top - bottom

$= \frac{1}{2}x^2+2 - 0$

V = 14.700

$V=\int_1^3\left(\frac{\pi}{8}\left(\frac{1}{2}x^2+2\right)^2\right)dx$

$= 14.700$

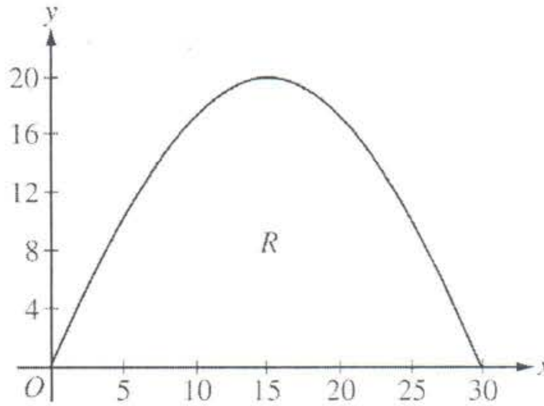
-2

Calculus BC
7.2 AP Problems

Name Fernando [illegible]
Date ______________ Period ___

2009B #1 (calc)

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure to the right. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.



(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

$A_{total} = 30 \times 20 = 800$

$A_{cut} = \int_0^{30} 20\sin\left(\frac{\pi x}{30}\right) dx$

$= 381.972$

$A_d = A_T - A_c = 218.028 \text{ cm}^2$

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 grams of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

$A = \frac{\pi}{8} B^2$

$\left[\int_0^{30} \frac{\pi}{8} f(x)^2 \, dx\right] \cdot .05 = 117.809$ grams

(c) Find the perimeter of the base of the cake.

$L = \int_0^{30} \sqrt{1 + f'(x)^2} = 51.804$

$+30$

81.8041 cm

7.4 p. 483, #1-3, 6-9, 11-23 odd

1) (0,0) (5,12)

a) $d = \sqrt{(5-0)^2+(12-0)^2} = 13$

b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$S = \int_0^5 \sqrt{1+(\frac{12}{5})^2}\, dx$

$S = \int_0^5 \sqrt{\frac{169}{25}}\, dx = \int_0^5 \frac{13}{5}\, dx$

$S = 13$

2) (1,2) (7,10)

a) $d = \sqrt{(7-1)^2+(10-2)^2}$

$= \sqrt{36+64}$

$= 10$

b) $y = \frac{4}{3}x + b$

$y' = \frac{4}{3}$

$S = \int_1^7 \sqrt{1+(\frac{4}{3})^2}\, dx$

$S = \int_1^7 \sqrt{\frac{25}{9}}\, dx = \int_1^7 \frac{5}{3}\, dx$

$S = 10$

3) $y = \frac{2}{3}x^{3/2} + 1 \quad [0,1]$

$y' = x^{1/2}$

$S = \int_0^1 \sqrt{1+(\sqrt{x})^2}\, dx$

$S = 1.219$

6) $y = \frac{x^4}{8} + \frac{1}{4x^2}$ $[1,2]$ $\qquad S = \int_1^2 \sqrt{1+\left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2}\,dx$

$y' = \frac{x^3}{2} - \frac{1}{2x^3}$ $\qquad S = 2.0625$

7) $y = \frac{x^5}{10} + \frac{1}{6x^3}$ $[1,2]$ $\qquad S = \int_1^2 \sqrt{1+\left(\frac{x^4}{2} - \frac{1}{2x^4}\right)^2}\,dx$

$y' = \frac{x^4}{2} - \frac{1}{2x^4}$ $\qquad S = 3.2458$

8) $y = \frac{3}{2}x^{2/3} + 4$ $[1,27]$ $\qquad S = \int_1^{27} \sqrt{1+\left(\frac{1}{x^{1/3}}\right)^2}\,dx$

$y' = \frac{1}{x^{1/3}}$ $\qquad S = 28.794$

9) $y = \ln(\sin x)$ $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ $\qquad S = \int_{\pi/4}^{3\pi/4} \sqrt{1+\cot^2 x}\,dx$

$y' = \frac{\cos x}{\sin x} = \cot x$ $\qquad S = \int_{\pi/4}^{3\pi/4} \csc x\,dx$

$S = 1.763$

11) $y = \frac{1}{2}(e^x + e^{-x})$ $[0,2]$ $\qquad S = \int_0^2 \sqrt{1+\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)^2}\,dx$

$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ $\qquad = 3.627$

$y' = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

10) $x = \frac{1}{3}(y^2+2)^{3/2}$ $0 \le y \le 4$ $\qquad S = \int_0^4 \sqrt{1+\left(y\sqrt{y^2+2}\right)^2}\,dy$

$x' = \frac{1}{2}(y^2+2)^{1/2} \cdot 2y$ $\qquad S = \int_0^4 \sqrt{1+y^2(y^2+2)}\,dy$

$= y\sqrt{y^2+2}$

$S = 25.333$

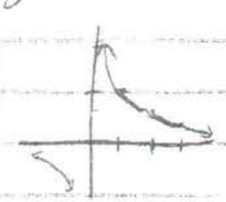
14 (con't)

15) $y = 4 - x^2 \quad 0 \le x \le 2$ b) $s = \int_0^2 \sqrt{1+(-2x)^2}\,dx$

a)  $y' = -2x$ $s = \int_0^2 \sqrt{1+4x^2}\,dx$

c) $s = 4.647$

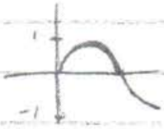
17) $y = \frac{1}{x} \quad 1 \le x \le 3$

a)  $y' = -\frac{1}{x^2}$ b) $s = \int_1^3 \sqrt{1+(-\frac{1}{x^2})^2}\,dx$

$s = \int_1^3 \sqrt{1+\frac{1}{x^4}}\,dx$

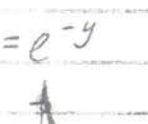
c) $s = 2.147$

19) $y = \sin x \quad 0 \le x \le \pi$ b) $s = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$

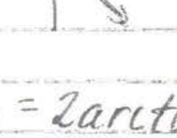
 $y' = \cos x$ $s = \int_0^\pi \sin x\,dx$

c) $s = 3.820$

21) $x = e^{-y} \quad 0 \le y \le 2$ b) $s = \int_0^2 \sqrt{1+(-e^{-y})^2}\,dy$

 $x' = -e^{-y}$ c) $s = 2.221$

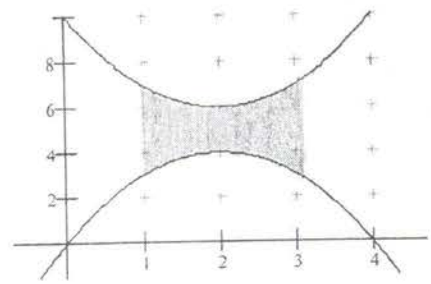
23) $y = 2\arctan x \quad 0 \le x \le 1$ b) $s = \int_0^1 \sqrt{1+(\frac{2}{1+x^2})^2}\,dx$

 $y' = \frac{2}{1+x^2}$ c) $s = 1.871$

AP Calculus AB
Review 7.1-7.2

Name: Key

1) (no calculator) Calculate the area of the region between the graphs of $f(x)=x^2-4x+10$ and $g(x)=4x-x^2$ over [1,3].

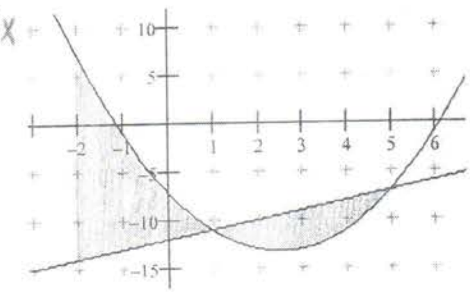


$$\int_1^3 \left[(x^2-4x+10)-(4x-x^2)\right]dx$$

$$\int_1^3 (2x^2-8x+10)\,dx$$

$$\frac{2}{3}x^3-4x^2+10x\Big]_1^3 = \boxed{\frac{16}{3}}$$

2) (no calculator) Find the area between the graphs of $f(x)=x^2-5x-7$ and $g(x)=x-12$ over [-2,5].



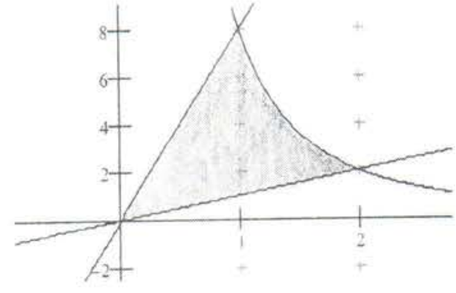
$$\int_{-2}^{1} \left[(x^2-5x-7)-(x-12)\right]dx + \int_1^5 \left[(x-12)-(x^2-5x-7)\right]dx$$

$$\int_{-2}^{1} (x^2-6x+5)\,dx + \int_1^5 (-x^2+6x-5)\,dx$$

$$\frac{1}{3}x^3-3x^2+5x\Big]_{-2}^{1} + -\frac{1}{3}x^3+3x^2-5x\Big]_1^5$$

$$= \boxed{\frac{113}{3}}$$

3) (no calculator) Find the area of the region bounded by the graphs of $y=\frac{8}{x^2}$, $y=8x$, and $y=x$.



$$\int_0^1 (8x-x)\,dx + \int_1^2 \left(\frac{8}{x^2}-x\right)dx$$

$$\int_0^1 7x\,dx + \int_1^2 \left(\frac{8}{x^2}-x\right)dx$$

$$\frac{7}{2}x^2\Big]_0^1 + -\frac{8}{x}-\frac{x^2}{2}\Big]_1^2$$

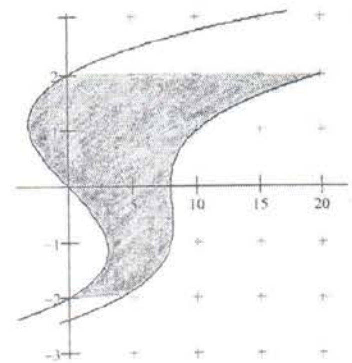
$$\frac{7}{2} + \frac{5}{2} = \boxed{6}$$

4) Calculate the area between the graphs of $x = y^3 - 4y$ and $x = y^3 + y^2 + 8$ for $-2 \le y \le 2$.

$$\int_{-2}^{2} \left[(y^3 + y^2 + 8) - (y^3 - 4y) \right] dy$$

$$\int_{-2}^{2} (y^2 + 4y + 8)\, dy$$

$$\frac{1}{3}y^3 + 2y^2 + 8y \Big]_{-2}^{2} = \boxed{\frac{112}{3}}$$



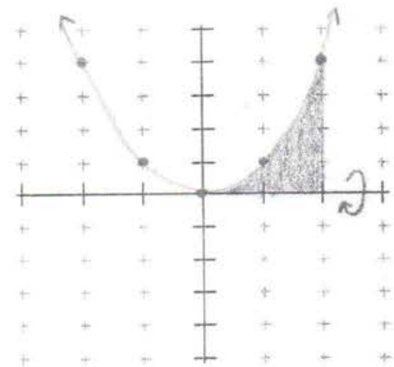
no calc 5) Calculate the volume V of the solid formed by rotating the region under $y = x^2$ about the x-axis for $0 \le x \le 2$.

$$V = \pi \int_0^2 (x^2)^2 \, dx$$

$$= \pi \int_0^2 x^4 \, dx$$

$$= \pi \cdot \frac{x^5}{5} \Big]_0^2$$

$$= \boxed{\frac{32\pi}{5}}$$



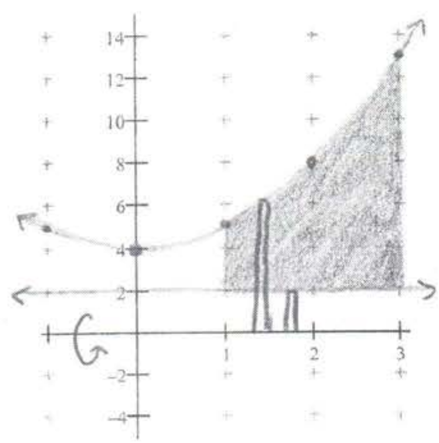
6) Calculate the volume V of the solid formed by rotating the region between $y = x^2 + 4$ and $y = 2$ about the x-axis for $1 \le x \le 3$.

$$V = \pi \int_1^3 \left[(x^2+4)^2 - (2)^2 \right] dx$$

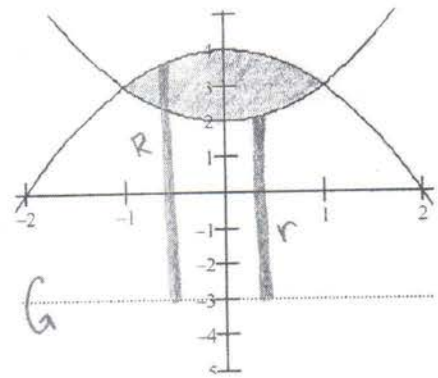
$$= \pi \int_1^3 (x^4 + 8x^2 + 12)\, dx$$

$$= \pi \cdot \frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x \Big]_1^3$$

$$= \boxed{\frac{2{,}126}{15}\pi}$$



7) Find the volume of the figure obtained by rotating the region between the graphs of $f(x) = x^2 + 2$ and $g(x) = -x^2 + 4$ about the horizontal line $y = -3$.



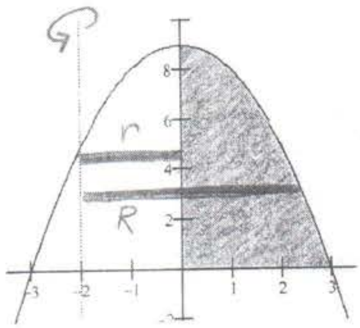
$R = -3 - (-x^2 + 4) = x^2 - 7$

$r = -3 - (x^2 + 2) = -x^2 - 5$

$V = \pi \int_{-1}^{1} \left[\overset{R}{(x^2 - 7)^2} - \overset{r}{(-x^2 - 5)^2} \right] dx$

$= \boxed{32\pi}$

8) Find the volume of the figure obtained by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the vertical line $x = -2$.



$x = \pm\sqrt{9-y}$

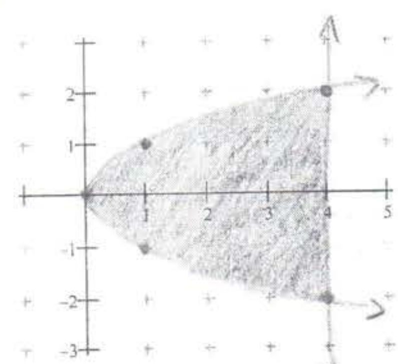
$R = -2 - (\sqrt{9-y})$

$r = -2 - (0) = -2$

$V = \pi \int_0^9 \left[\overset{R}{(-2 - \sqrt{9-y})^2} - \overset{r}{(-2)^2} \right] dy$

$= \pi \int_0^9 (9 - y + 4\sqrt{9-y})\, dy = \boxed{\frac{225}{2}\pi}$

9) Find the volume of the figure enclosed by the graphs of $x = y^2$ and $x = 4$. Use square cross-sections that are perpendicular to the y-axis.



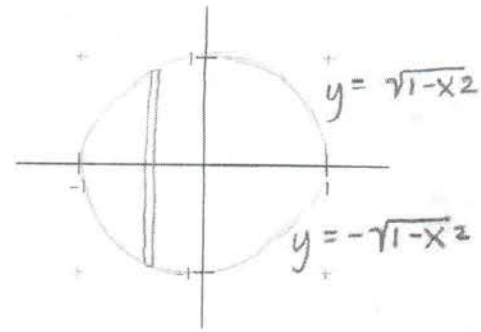
base $= 4 - y^2$

area of sqare $= (4 - y^2)^2$

$V = \int_{-2}^{2} (4 - y^2)^2\, dy$

$= \boxed{\frac{512}{15}}$

10) Find the volume of the figure whose base is given by $x^2 + y^2 = 1$.
Use semi-circle cross-sections that are perpendicular to the x-axis.



$y = \pm\sqrt{1-x^2}$

$$V = \int_{-1}^{1} \frac{\pi}{2}(1-x^2)\,dx$$

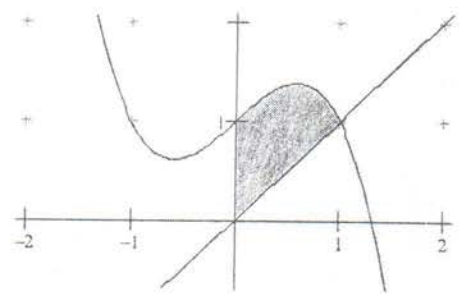
$$= \boxed{\frac{2}{3}\pi}$$

base $= 2\sqrt{1-x^2}$

area of semi-circles:

$$\frac{1}{2}\pi\left(\sqrt{1-x^2}\right)^2 = \frac{\pi}{2}(1-x^2)$$

11) Find the volume of the figure enclosed by the graphs of $y = -x^3 + x + 1$, $y = x$,and $x = 0$.
Use cross-sections of equilateral triangles that are perpendicular to the x-axis.



base $= (-x^3 + x + 1) - (x)$

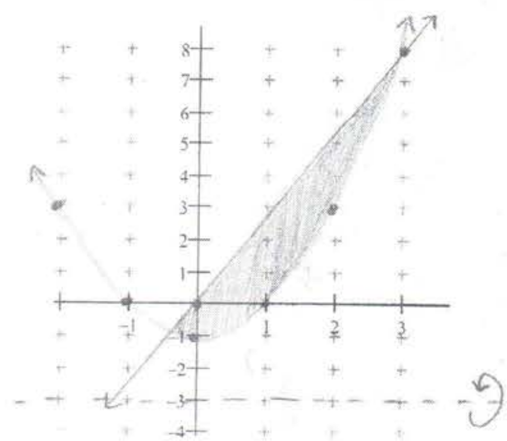
$= -x^3 + 1$

area $= \frac{\sqrt{3}}{4}(-x^3+1)^2$

$$V = \int_0^1 \frac{\sqrt{3}}{4}(-x^3+1)^2\,dx = \frac{9}{14}\cdot\frac{\sqrt{3}}{4} = \boxed{\frac{9\sqrt{3}}{56}}$$

12) Given $x = \frac{3}{8}y$ and $y = x^2 - 1\ldots$ $\qquad y = \frac{8x}{3}$

a) Sketch a graph and find the area of the bounded region.



$$\text{area} = \int_{-1/3}^{3}\left[\frac{8x}{3} - (x^2-1)\right]dx$$

$$= \boxed{\frac{500}{81}}$$

b) Find the volume when the bounded region is rotated around the line $y = -3$.

$R = -3 - \left(\frac{8x}{3}\right) \qquad r = -3-(x^2-1) = -2-x^2$

$$V = \pi\int_{-1/3}^{3}\left[\left(-3-\frac{8x}{3}\right)^2 - \left(-2-x^2\right)^2\right]dx$$

$$= \boxed{\frac{49000}{729}\pi}$$

AP Calculus BC
Section 8.2 Notes ~ Integration by Parts

Name Fernando Trujano

I. Introduction

Integration by parts is used to integrate a product, such as the product of an algebraic and a transcendental function: $\int xe^x\,dx$, $\int x\sin x\,dx$, $\int x\ln x\,dx$, etc

Product Rule: $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$

If you ingrate both sides, then $uv = \int u\,dv + \int v\,du$

Rearrange:

$$\int u\,dv = uv - \int v\,du$$

II. Examples

From text ~ #1) Match the antiderivative with the correct integral. Let $y = \sin x - x\cos x$.

(a) $\int \ln x\,dx$ (b) $\int x\sin x\,dx$ (c) $\int x^2e^x\,dx$ (d) $\int x^2\cos x\,dx$

$y' =$

CHECK:

$u = x$
$du = dx$
$dv = \sin x\,dx$
$v = -\cos x$
$uv - \int v\,du$
$(x)(-\cos x) + \int \cos x\,dx$
$-x\cos x + \sin x$

Tips for choosing u and dv:

u	dv
can take derivative of ↳ go to 0	can take the integral

Ex) $\int xe^{3x}\,dx$

$u = x$
$du = 1dx$
$dv = e^{3x}dx$
$\int e^{3x}dx$
$u = 3x$
$du = 3$
$\frac{1}{3}\int e^u du$
$v = \frac{1}{3}e^{3x}$

$uv - \int v\,du$
$= x(\frac{1}{3}e^{3x}) - \int \frac{1}{3}e^{3x}dx$
$= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx$
$= \frac{1}{3}xe^{3x} - \frac{1}{3}(\frac{1}{3})e^{3x} + C$
$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

AP Calculus BC
Section 8.3 Notes ~ Trigonometric Integrals

Name Fernando Trujano

$\int \sin^m x \cos^n x\, dx$

A. Power Rule immediately works ($m=1$ or $n=1$)

Ex) $\int \sin^5 x \cos x\, dx = \int u^5 du$

$u = \sin x$
$du = \cos x$

$\frac{u^6}{6} + C$

$= \frac{(\sin x)^6}{6} + C$

B. Simplify using identities ⟹ Power Rule just evens

If $m \neq 1$ and $n \neq 1$, use identities:

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = \frac{1 - \cos 2x}{2}$ $\qquad \cos^2 x = \frac{1 + \cos 2x}{2}$

Tips for evaluating integrals involving sine and cosine (from p. 534):

1) If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.
2) If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.
3) If the powers of both the sine and cosine are even and nonnegative, make repeated use of the last two identities listed above to convert the integrand to odd powers of the cosine. Then proceed as in Tip #2.

Ex) $\int \sin^5 x \cos^4 x\, dx$ (odd)

$= \int \sin x \cdot \sin^4 x \cdot \cos^4 x$ ① Break off odd power

$= \int \sin x (1 - \cos^2 x)^2 \cos^4 x\, dx$ ② Rewrite

$= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \cos^4 x\, dx$ ③ Expand

$= \int \sin x (\cos^4 x - 2\cos^6 x + \cos^8 x)\, dx$ ④ Usub

$u = \cos x$
$du = -\sin x\, dx$

$= -\int (u^4 - 2u^6 + u^8)\, du$

$= -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + C$

$$\lim_{x \to 1^+} \left(\frac{1}{(\ln x)} - \frac{1}{(x-1)} \right) = \lim_{x \to 1^+} \frac{(x-1) - \ln(x)}{(\ln(x))(x-1)} = \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{(\frac{1}{x})(x-1) + \ln x} \cdot \frac{x}{x} = \lim_{x \to 1^+} \frac{x-1}{(x-1) + x \ln x}$$

Can ignore coming from right or left

$\hookrightarrow 0/0$

Product Rule

$\hookrightarrow 0/0$

$$\lim_{x \to 1^+} \frac{1}{1 + \ln x + 1} = 1/2$$

$$\lim_{x \to \infty} \left(1 + \frac{2}{x} \right)^x = y$$

$$\ln y = \ln \left(\lim_{x \to \infty} \left(1 + \frac{2}{x} \right)^x \right)$$

$$\ln y = \lim_{x \to \infty} x \ln \left(1 + \frac{2}{x} \right)$$

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + \frac{2}{x})}{x^{-1}} \rightarrow \frac{v'}{v}$$ - Fraction Form

$$\ln y = \lim_{x \to \infty} \frac{\frac{-2/x^2}{1 + 2/x}}{-1/x^2}$$

$$\ln y = \lim_{x \to \infty} \frac{\frac{-2}{x^2}}{1 + \frac{2}{x}} \cdot \frac{-x^2}{1}$$ Multiply by Reciprocal

$$\ln y = \lim_{x \to \infty} \frac{2}{1 + \frac{2}{x}}$$

$$\ln y = 2$$

$$y = e^2$$

AP Calculus BC
Section 8.8 Notes~ Improper Integrals

Name Fernando Trujano

Improper Integrals with Infinite Limits of Integration

The definition of a definite integral $\int_a^b f(x)\,dx$ and the FTC require that the interval [a,b] be finite and continuous.

What if we have an integral that does not satisfy these requirements?

If one of the limits of integration is infinite or if f has a finite number of infinite discontinuities on the interval [a,b], the integral is an ***improper integral***.

Integrals such as $\int_0^{\infty} f(x)\,dx$, $\int_{-\infty}^{0} f(x)\,dx$, and $\int_{-\infty}^{\infty} f(x)\,dx$ are called improper integrals. They are evaluated by rewriting the integral as a proper integral and then using limits.

Ex 1) $\int_1^{\infty} \frac{1}{x}\,dx =$

$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} [\ln b - \ln 1] = \infty$

diverges

Ex 2) $\int_1^{\infty} e^{-x}\,dx =$

$\lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} [-e^{-x}]_1^b = \lim_{b\to\infty} [-e^{-b} + e^{-1}] = 1/e$

converges ↳ approaches a number

$u = -x$
$du = -dx$

Ex 3) $\int_{-\infty}^{0} e^{x/4}\,dx =$

$\lim_{a\to-\infty} \int_a^0 e^{x/4}\,dx = \lim_{a\to-\infty} [4e^{x/4}]_a^0 = \lim_{a\to-\infty} [4 - 4e^{a/4}] = 4$

converges

$u = x/4$
$du = 1/4\,dx$

converges

Ex 4) $\int_0^{27} \frac{1}{\sqrt[3]{27-x}}\,dx =$ $\lim_{b\to 27} \int_0^b (27-x)^{-1/3}\,dx = \lim_{b\to 27} \left[\frac{-3(27-x)^{2/3}}{2}\right]_0^b = \lim_{b\to 27} \left[-\frac{3}{2}(27-b)^{2/3} + \frac{27}{2}\right] = \frac{27}{2}$

vertical asymptote @ 27

$u = 27 - x$
$du = -dx$

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AP Calculus BC
Quiz 8.3, 8.5

$\sin^2 x = \frac{1-\cos 2x}{2}$

Name Fernando Trujano
Date 1/18/13 Period 4

Trig Integration, Partial Fractions

Evaluate the integrals. 15 points each.

1. $\int \sin^{20} x \cos x\,dx =$

$u = \sin x$
$du = \cos x\,dx$
$\int u^{20}\,du$
$= \frac{u^{21}}{21} + C$

1. $\frac{\sin^{21} x}{21} + C$

2. $\int \sin^3 x \cos^4 x\,dx =$

$\int \sin x \sin^2 x \cos^4 x\,dx$
$\int \sin x (1-\cos^2 x)\cos^4 x\,dx$
$\int \sin x(\cos^4 x - \cos^6 x)\,dx$
$u = \cos x$
$du = -\sin x$

$-\int u^4 - u^8\,du$
$= \frac{-u^5}{5} - \frac{u^9}{9} + C$
$= \frac{-\cos^5 x}{5} - \frac{\cos^9 x}{9} + C$

2. $\frac{-\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$

-3

3. $\int \sin^4(3x)\,dx$

$\int (\sin^2(3x))^2\,dx$
$\int \left(\frac{1-\cos(6x)}{2}\right)\left(\frac{1-\cos(6x)}{2}\right)dx$
$\int \frac{1-2\cos(6x)+\cos^2(6x)}{4}\,dx$

$\int \frac{1}{4}dx - \frac{1}{2}\int \cos(6x)\,dx + \frac{1}{4}\int \cos^2(6x)\,dx$
$u = 6x$
$du = 6dx$
$\frac{1}{4}x - \frac{1}{12}\sin(6x) + \frac{1}{4}\int \cos^2(6x)\,dx$
$\frac{1}{4}\int \frac{1+\cos(12x)}{2}\,dx$
$\frac{1}{4}\cdot\int \frac{1}{2}dx + \frac{1}{4}\cdot\frac{1}{2}\int \cos(12x)\,dx$
$u = 12x$
$du = 12dx$
$\frac{1}{8}x + \left(\frac{1}{96}\right)\sin(12x) + C$

3. $\frac{1}{4}x - \frac{1}{12}\sin(6x) + \frac{1}{8}x + \frac{1}{96}\sin(12x) + C$

4. $\int \sin 7x \sin 4x\,dx =$

$(m-n)x$

$\frac{1}{2}$

$\frac{1}{2}\int \cos(3x) + \cos(-3x)\,dx$
$u = 3x$
$du = 3$

$\frac{1}{6}\sin(3x) + \frac{1}{6}\sin(3x)$

4. $\frac{1}{6}\sin(3x) - \frac{1}{6}\sin(-3x) + C$

-5 Rule

-8

Calculus BC
Ch 8 Bingo Problems

Name Fernando Trujano
Date ______________ Period___

Integrate.

1. $\int_0^{\pi/2} \cos^8 x\,dx$
Wallis

2. $\int \sin^5 x \cos^2 x\,dx$
Trig-Sub

3. $\int x\sqrt{x^2-1}\,dx$
U-Sub

4. $\int x^2 \sin 2x\,dx$

5. $\int \cos^3(\pi x - 1)\,dx$

6. $\int_0^{\pi/6} \cos(3x)\cos x\,dx$

7. $\int \frac{x-28}{x^2-x-6}\,dx$

8. $\int \frac{2x^3-5x^2+4x-4}{x^2-x}\,dx$

9. $\int \frac{\sec^2\theta}{\tan\theta(\tan\theta-1)}\,d\theta$

10. $\int \cos x \ln(\sin x)\,dx$

11. $\int_4^8 x\sqrt{x-4}\,dx$
U-Sub
↳ Solve for x

Ch8 Bingo problems

1. $\int_0^{\pi/2} \cos^8 x\,dx$ Wallis' Rule n is even

$= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\frac{\pi}{2} = \boxed{\frac{35\pi}{256} \approx 0.430}$

2. $\int \sin^5 x \cos^2 x\,dx$ trig int

$\int \sin x \sin^4 x \cos^2 x\,dx$

$\int \sin x (1-\cos^2 x)^2 \cos^2 x\,dx$

$\int \sin x (1 - 2\cos^2 x + \cos^4 x)\cos^2 x\,dx$

$\int \sin x (\cos^2 x - 2\cos^4 x + \cos^6 x)\,dx$ $u = \cos x$ $du = -\sin x\,dx$

$-\int (u^2 - 2u^4 + u^6)\,du$

$-\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$

$= \boxed{\frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + C}$

3. $\int x\sqrt{x^2-1}\,dx$ u-sub $u = x^2 - 1$ $du = 2x\,dx$

$\frac{1}{2}\int u^{1/2}\,du$

$\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{\frac{1}{3}(x^2-1)^{3/2} + C}$

4. $\int x^2 \sin 2x\,dx$ integ. by parts $u = x^2$ $v = -\frac{1}{2}\cos 2x$ $du = 2x\,dx$ $dv = \sin 2x\,dx$

$-\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$

$-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$\boxed{-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C}$

$u = x$ $v = \frac{1}{2}\sin 2x$

$du = dx$ $dv = \cos 2x\,dx$

5. $\int \cos^3(\pi x - 1)\,dx$ trig $u = \sin(\pi x - 1)$ $du = \pi\cos(\pi x - 1)$

$\int \cos(\pi x - 1)\cos^2(\pi x - 1)\,dx$

$\int \cos(\pi x - 1)(1 - \sin^2(\pi x - 1))\,dx$

$\frac{1}{\pi}\int (1 - u^2)\,du$

$\frac{1}{\pi}\left[u - \frac{u^3}{3}\right] + C = \boxed{\frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{\sin^3(\pi x - 1)}{3}\right] + C}$

6. $\int_0^{\pi/6} \cos(3x)\cos x\,dx$ trig * remember: $\cos m \cos n = \frac{1}{2}[\cos(m-n)+\cos(m+n)]$

$\frac{1}{2}\int_0^{\pi/6}[\cos(2x)+\cos(4x)]\,dx$

$\frac{1}{2}\left[\frac{1}{2}\sin(2x)+\frac{1}{4}\sin(4x)\right]_0^{\pi/6}$

$\left[\frac{\sin 2x}{4}+\frac{\sin 4x}{8}\right]_0^{\pi/6} = \frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{16} = \boxed{\frac{3\sqrt{3}}{16}}$

7. $\int \frac{x-28}{x^2-x-6}\,dx$ partial fractions

$x-28 = A(x+2)+B(x-3)$

$x=-2$ $\quad -30 = B(-5)$ $\quad B=6$

$x=3$ $\quad -25 = A(5)$ $\quad A=-5$

$\int \frac{A}{x-3}\,dx + \int \frac{B}{x+2}\,dx$

$-5\int \frac{1}{x-3}\,dx + 6\int \frac{1}{x+2}\,dx$

$\boxed{-5\ln|x-3| + 6\ln|x+2| + C}$

8. $\int \frac{2x^3-5x^2+4x-4}{x^2-x}\,dx$ partial fractions

$$\begin{array}{r} 2x-3 \\ x^2-x+0\,\overline{)\,2x^3-5x^2+4x-4} \\ -(2x^3-2x^2+0x) \\ -3x^2+4x-4 \\ -(-3x^2+3x+0) \\ x-4 \end{array}$$

$\int(2x-3)\,dx + \int \frac{x-4}{x(x-1)}\,dx$

$x^2-3x + \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx$

$x^2-3x + 4\int \frac{1}{x}\,dx - 3\int \frac{1}{x-1}\,dx$

$x-4 = A(x-1)+B(x)$

$x=0$ $\quad -4 = A(-1)$ $\quad 4=A$

$x=1$ $\quad -3 = B$

$\boxed{x^2-3x+4\ln|x|-3\ln|x-1|+C}$

9. $\int \frac{\sec^2\theta}{\tan\theta(\tan\theta-1)}\,d\theta$ partial fractions $u=\tan\theta$ $\quad du=\sec^2\theta\,d\theta$

$\int \frac{1}{u(u-1)}\,du = \int \frac{A}{u}\,du + \int \frac{B}{u-1}\,du$

$1 = A(u-1)+Bu$

$u=1$ $\quad 1=B$

$u=0$ $\quad 1=A(-1)$ $\quad -1=A$

$-\int \frac{1}{u}\,du + \int \frac{1}{u-1}\,du$

$-\ln|u| + \ln|u-1| + C$

$\boxed{-\ln|\tan\theta| + \ln|\tan\theta-1| + C}$

10. $\int \cos x \ln(\sin x)\,dx$ integ by parts

$u=\ln(\sin x)$ $\quad v=\sin x$

$du=\frac{\cos x}{\sin x}\,dx$ $\quad dv=\cos x\,dx$

$\sin x\ln(\sin x) - \int \cos x\,dx$

$\boxed{\sin x\ln(\sin x) - \sin x + C}$

11. $\int_4^8 x\sqrt{x-4}\,dx$ U-sub $u=x-4$ $du=dx$

$x=u+4$

$\int_0^4 (u+4)\,u^{1/2}\,du$

$\int_0^4 (u^{3/2}+4u^{1/2})\,du$

$\left[\frac{2u^{5/2}}{5}+\frac{4\cdot 2u^{3/2}}{3}\right]_0^4 = \frac{2}{5}\cdot 32 + \frac{8}{3}\cdot 8 = \frac{64}{5}+\frac{64}{3} = \boxed{\frac{512}{15}}$

12. $\lim\limits_{x\to 1} \frac{(\ln x)^2}{x-1} = \lim\limits_{x\to 1} \frac{2(\ln x)(\frac{1}{x})}{1} = \lim\limits_{x\to 1} \frac{2\ln x}{x} = \boxed{0}$

13. $\lim\limits_{x\to\infty} (\ln x)^{2/x}$

$\ln y = \lim\limits_{x\to\infty} \ln(\ln x)^{2/x}$

$\ln y = \lim\limits_{x\to\infty} \frac{2}{x}\ln(\ln x)$

$\ln y = \lim\limits_{x\to\infty} \frac{2\ln(\ln x)}{x}$

$\ln y = \lim\limits_{x\to\infty} \frac{2\cdot\frac{(1/x)}{\ln x}}{1}$

$\ln y = \lim\limits_{x\to\infty} \frac{2}{x\ln x}$

$\ln y = 0$

$y = e^0 = \boxed{1}$

14. $\int_0^{16} x^{-1/4}\,dx$ improper @ $x=0$

$\lim\limits_{a\to 0} \int_a^{16} x^{-1/4}\,dx = \lim\limits_{a\to 0} \frac{4}{3}x^{3/4}\Big]_a^{16} = \lim\limits_{a\to 0}\left[\frac{32}{3}-\frac{4}{3}a^{3/4}\right] = \boxed{\frac{32}{3}}$

15. $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ improper $u=\ln x$ $v=-1/x$

$du = 1/x\,dx$ $dv = 1/x^2\,dx$

$\lim\limits_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim\limits_{b\to\infty}\left[-\frac{\ln x}{x}+\int\frac{1}{x^2}\,dx\right]_1^b$

$= \lim\limits_{b\to\infty}\left[\frac{-\ln x}{x}-\frac{1}{x}\right]_1^b = \lim\limits_{b\to\infty}\left[\left(\frac{-\ln b}{b}-\frac{1}{b}\right)-(-1)\right]$

$= \lim\limits_{b\to\infty}\left[\frac{-\ln b-1}{b}\right]+1 = \lim\limits_{b\to\infty}\left[\frac{-1/b}{1}\right]+1 = 0+1 = \boxed{1}$

16. $V = \pi \int_0^\infty (e^{-x})^2 dx = \pi \int_0^\infty e^{-2x} dx$ improper

$\lim_{b \to \infty} \pi \int_0^b e^{-2x} dx = \lim_{b \to \infty} \pi \left[-\frac{1}{2} e^{-2x} \right]_0^b$

$= \lim_{b \to \infty} \pi \left[-\frac{1}{2} e^{-2b} + \frac{1}{2} \right] = \lim_{b \to \infty} \left[-\frac{\pi}{2} e^{-2b} + \frac{\pi}{2} \right] = \boxed{\frac{\pi}{2}}$

Ch 8 Review problems

1. $\int x\sqrt{x^2-1}\,dx$ usub $u=x^2-1$ $du=2x\,dx$

$\frac{1}{2}\int u^{1/2}\,du$

$\frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C$

$\boxed{=\frac{1}{3}(x^2-1)^{3/2}+C}$

3. $\int \frac{x}{x^2-1}\,dx$ usub $u=x^2-1$ $du=2x\,dx$

$\frac{1}{2}\int\frac{1}{u}\,du$

$\frac{1}{2}\ln|u|+C$

$\boxed{\frac{1}{2}\ln|x^2-1|+C}$

5. $\int_1^e \frac{\ln(2x)}{x}\,dx$ usub $u=\ln(2x)$ $du=\frac{2}{2x}dx=\frac{1}{x}dx$

$\int_{\ln 2}^{\ln 2e} u\,du$

$\left.\frac{u^2}{2}\right]_{\ln 2}^{\ln 2e} = \boxed{\frac{(\ln 2e)^2}{2}-\frac{(\ln 2)^2}{2}\approx 1.193}$

7. $\int\frac{16}{\sqrt{16-x^2}}\,dx$ inverse trig

$\boxed{16\arcsin\left(\frac{x}{4}\right)+C}$

9. $\int e^{2x}\sin 3x\,dx$ integ by parts $u=e^{2x}$ $v=-\frac{1}{3}\cos 3x$

$du=2e^{2x}$ $dv=\sin 3x\,dx$

$\int e^{2x}\sin 3x\,dx=-\frac{1}{3}e^{2x}\cos 3x+\frac{2}{3}\int\cos 3x\,e^{2x}\,dx$

$u=e^{2x}$ $v=\frac{1}{3}\sin 3x$

$du=2e^{2x}$ $dv=\cos 3x\,dx$

$\int e^{2x}\sin 3x\,dx=-\frac{1}{3}e^{2x}\cos 3x+\frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x-\frac{2}{3}\int\sin 3x\,e^{2x}\,dx\right)$

$=-\frac{1}{3}e^{2x}\cos 3x+\frac{2}{9}e^{2x}\sin 3x-\frac{4}{9}\int\sin 3x\,e^{2x}\,dx$

$=\boxed{\frac{9}{13}\left(-\frac{1}{3}e^{2x}\cos 3x+\frac{2}{9}e^{2x}\sin 3x\right)}$

11. $\int x\sqrt{x-5}\,dx$ integ. by parts. $u = x$ $\quad du = dx$ $\quad v = \frac{2}{3}(x-5)^{3/2}$ $\quad dv = \sqrt{x-5}\,dx$

$= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2}dx$

$= \frac{2}{3}x(x-5)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x-5)^{5/2} + C$

$= \boxed{\frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C}$

13. $\int x^2 \sin 2x\,dx$ integ. by parts $u = x^2$ $\quad du = 2x\,dx$ $\quad v = -\frac{1}{2}\cos 2x$ $\quad dv = \sin 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$

$u = x$ $\quad du = dx$ $\quad v = \frac{1}{2}\sin 2x$ $\quad dv = \cos 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$= \boxed{-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C}$

17 $\int \cos^3(\pi x - 1)\,dx$ trig

$\int \cos(\pi x-1)\cos^2(\pi x-1)\,dx$

$\int \cos(\pi x-1)(1-\sin^2(\pi x-1))\,dx$ $\quad u = \sin(\pi x-1)$ $\quad du = \pi\cos(\pi x-1)\,dx$

$\frac{1}{\pi}\int(1-u^2)\,du$

$\frac{1}{\pi}\left(u - \frac{u^3}{3}\right) + C = \boxed{\frac{1}{\pi}\left(\sin(\pi x-1) - \frac{\sin^3(\pi x-1)}{3}\right) + C}$

18. $\int \sin^2 \frac{\pi x}{2}\,dx$ trig $\quad$ ** remember $\sin^2 x = \frac{1-\cos 2x}{2}$ **

$\int \frac{1-\cos\pi x}{2}\,dx$

$\int \frac{1}{2}\,dx - \frac{1}{2}\int \cos\pi x\,dx$

$\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{\pi}\sin\pi x + C$

$\boxed{\frac{1}{2}x + \frac{1}{2\pi}\sin\pi x + C}$

AP Calculus BC
Section 8.2 Notes ~ Integration by Parts

Name Fernando Trujano

I. Introduction

Integration by parts is used to integrate a product, such as the product of an algebraic and a transcendental function: $\int xe^x\,dx$, $\int x\sin x\,dx$, $\int x\ln x\,dx$, etc

Product Rule: $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$

If you ingrate both sides, then $uv = \int u\,dv + \int v\,du$

Rearrange:

$$\int u\,dv = uv - \int v\,du$$

II. Examples

From text ~ #1) Match the antiderivative with the correct integral. Let $y = \sin x - x\cos x$.

(a) $\int \ln x\,dx$ (b) $\int x\sin x\,dx$ (c) $\int x^2 e^x\,dx$ (d) $\int x^2\cos x\,dx$

$y' =$

$u = x$
$du = dx$
$dv = \sin x\,dx$
$v = -\cos x$

CHECK:

$uv - \int v\,du$
$(x)(-\cos x) + \int \cos x\,dx$
$-x\cos x + \sin x$

Tips for choosing u and dv:

u	dv
can take derivative of ↳ go to 0	can take the integral

Ex) $\int xe^{3x}\,dx$

$u = x$
$du = 1dx$
$dv = e^{3x}dx$

$\int e^{3x}dx$
$u = 3x$
$du = 3$
$\frac{1}{3}\int e^u du$
$v = \frac{1}{3}e^{3x}$

$uv - \int v\,du$

$= x(\frac{1}{3}e^{3x}) - \int \frac{1}{3}e^{3x}dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}(\frac{1}{3})e^{3x} + C$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

neither go to 0 !!

Ex) $\int e^x \sin x\, dx$

$u = \sin x$
$dv = e^x dx$
$= e^x$
$du = \cos x\, dx$

$= e^x \sin x - \int e^x \cos x\, dx$

$u = \cos x$
$du = -\sin x\, dx$
$v = e^x$
$dv = e^x dx$

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x\, dx \right]$

$\int e^x \sin x\, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x\, dx \right]$

$= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x$

From 1998 BC Multiple Choice

15. $\int x \cos x\, dx =$

$uv - \int v\, du$

$u = x$
$du = dx$
$dv = \cos x\, dx$
$v = \sin x$

$x \sin x - \int \sin x\, dx$

$x \sin x + \cos x + C$

(A) $x\sin x - \cos x + C$ (B) $x \sin x + \cos x + C$ (C) $-x \sin x + \cos x + C$

(D) $x \sin x + C$ (E) $\frac{1}{2} x^2 \sin x + C$

From 2003 BC Multiple Choice

23. $\int x \sin(6x)\, dx =$

$uv - \int v\, du$

$u = x$
$du = dx$
$dv = \sin(6x)\, dx$
$v = \frac{1}{6}\cos(6x)$

$(x)\frac{1}{6}\cos(6x) - \int \frac{1}{6}\cos(6x)\, dx$

(A) $-x\cos(6x) + \sin(6x) + C$ (B) $-\frac{x}{6}\cos(6x) + \frac{1}{36}\sin(6x) + C$ (C) $-\frac{x}{6}\cos(6x) + \frac{1}{6}\sin(6x) + C$

(D) $\frac{x}{6}\cos(6x) + \frac{1}{36}\sin(6x) + C$ (E) $6x\cos(6x) - \sin(6x) + C$

24. $\int_0^{\pi/6} \cos(3x)\cos x\,dx$ trig ** remember $\cos(m)\cos(n) = \frac{1}{2}(\cos(m-n)+\cos(m+n))$

$\frac{1}{2}\int_0^{\pi/6} (\cos 2x + \cos 4x)\,dx$

$\frac{1}{2}\left[\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x\right]_0^{\pi/6} = \left[\frac{\sin 2x}{4} + \frac{\sin 4x}{8}\right]_0^{\pi/6}$

$= \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16} = \boxed{\frac{3\sqrt{3}}{16}}$

33. $\int \frac{x-28}{x^2-x-6}\,dx$ partial fractions

$x-28 = A(x+2) + B(x-3)$

$x=-2$ $-30 = B(-5)$ $6=B$

$x=3$ $-25 = A(5)$ $-5=A$

$\int \frac{A}{x-3}\,dx + \int \frac{B}{x+2}\,dx$

$-5\int \frac{1}{x-3}\,dx + 6\int \frac{1}{x+2}\,dx$

$\boxed{-5\ln|x-3| + 6\ln|x+2| + C}$

35. $\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx$ partial fractions

$x^2+2x = A(x^2+1) + (Bx+C)(x-1)$

$x=1$ $3 = A(2)$ $A = 3/2$

$x^2+2x = Ax^2+A + Bx^2 - Bx + Cx - C$

$x^2+2x = (A+B)x^2 + (C-B)x + (A-C)$

$A+B=1$

$\frac{3}{2}+B=1$ $B=-\frac{1}{2}$ $A-C=0$ $C=\frac{3}{2}$

$\int \frac{x^2+2x}{(x-1)(x^2+1)}\,dx = \int \frac{A}{x-1}\,dx + \int \frac{Bx+C}{x^2+1}\,dx$

$\frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x-3}{x^2+1}\,dx$

$\frac{3}{2}\ln|x-1| - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+1}\,dx$

$\boxed{\frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C}$

37. $\int \frac{x^2}{x^2+2x-15}\,dx$ partial fractions

$$\begin{array}{r|l} & 1 \\ x^2+2x-15 & x^2+0x+0 \\ & -(x^2+2x-15) \\ & -2x+15 \end{array}$$

$\int dx + \int \frac{15-2x}{(x+5)(x-3)}\,dx = \int dx + \int \frac{A}{x+5}\,dx + \int \frac{B}{x-3}\,dx$

$15-2x = A(x-3) + B(x+5)$

$x=3$ $9 = 8B$ $9/8 = B$

$x=-5$ $25 = -8A$ $-25/8 = A$

$= x - \frac{25}{8}\int \frac{1}{x+5}\,dx + \frac{9}{8}\int \frac{1}{x-3}\,dx$

$= \boxed{x - \frac{25}{8}\ln|x+5| + \frac{9}{8}\ln|x-3| + C}$

54. $\int \frac{3x^3+4x}{(x^2+1)^2}\,dx$ partial fractions

$3x^3+4x = (Ax+B)(x^2+1) + Cx + D$

$3x^3+4x = Ax^3+Ax+Bx^2+B+Cx+D$

$3x^3+4x = Ax^3+Bx^2+(A+C)x+(B+D)$

$3=A \quad B=0 \quad 3+C=4 \quad B+D=0$

$C=1 \quad D=0$

$\int \frac{Ax+B}{x^2+1}\,dx + \int \frac{Cx+D}{(x^2+1)^2}\,dx$

$\int \frac{3x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \boxed{\frac{3}{2}\ln(x^2+1) - \frac{1}{2(x^2+1)} + C}$

55. $\int \cos x \ln(\sin x)\,dx$ integ. by parts

$u = \ln(\sin x) \quad V = \sin x$

$du = \frac{\cos x}{\sin x}\,dx \quad dV = \cos x\,dx$

$du = \cot dx$

$= \sin x \ln(\sin x) - \int \sin x \cdot \frac{\cos x}{\sin x}\,dx$

$= \sin x \ln(\sin x) - \int \cos x\,dx$

$= \boxed{\sin x \ln(\sin x) - \sin x + C}$

59. $y = \int \ln(x^2+x)\,dx$ integ. by parts

$u = \ln(x^2+x) \quad V = x$

$du = \frac{2x+1}{x^2+x}\,dx \quad dV = dx$

$= x\ln(x^2+x) - \int x \cdot \frac{2x+1}{x^2+x}\,dx$

$= x\ln(x^2+x) - \int \frac{2x+1}{x+1}\,dx = x\ln(x^2+x) - \int 2\,dx + \int \frac{1}{x+1}\,dx$

$x+1 \overline{)\,2x+1}$ quotient 2; $-(2x+2)$; remainder -1

$= \boxed{x\ln(x^2+x) - 2x + \ln|x+1| + C}$

61. $\int_2^{\sqrt{5}} x(x^2-4)^{3/2}\,dx$ u sub. $u = x^2-4 \quad du = 2x\,dx$

$\frac{1}{2}\int_0^1 u^{3/2}\,du = \frac{1}{2}\cdot\frac{2}{5}u^{5/2}\Big]_0^1 = \frac{1}{5}u^{5/2}\Big]_0^1 = \boxed{\frac{1}{5}}$

65. $\int_1^4 \frac{\ln x}{x}\,dx$ u sub $u = \ln x \quad du = \frac{1}{x}\,dx$

$\int_0^{\ln 4} u\,du = \frac{u^2}{2}\Big]_0^{\ln 4} = \frac{(\ln 4)^2}{2} = \boxed{0.961}$

65. $\int_0^{\pi} x \sin x\,dx$ integ by parts

$u = x \quad V = -\cos x$

$du = dx \quad dv = \sin x\,dx$

$= -x\cos x + \int \cos dx = \left[-x\cos x + \sin x\right]_0^{\pi}$

$= [-\pi(-1) + 0] - [0] = \boxed{\pi}$

67. $\int_0^4 x\sqrt{4-x}\,dx$ usub $u=4-x$ $du=-dx$

$x=4-u$

$-\int_4^0 (4-u)u^{1/2}\,du = \int_0^4 (4-u)u^{1/2}\,du = \int_0^4 (4u^{1/2}-u^{3/2})\,du$

$= \left[4\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4 = \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4 = \frac{64}{3} - \frac{64}{5} = \boxed{\frac{128}{15}}$

73. $\lim_{x\to 1} \frac{(\ln x)^2}{x-1} = \lim_{x\to 1} \frac{2(\ln x)(\frac{1}{x})}{1} = \lim_{x\to 1} \frac{2\ln x}{x} = \boxed{0}$

75. $\lim_{x\to\infty} \frac{e^{2x}}{x^2} = \lim_{x\to\infty} \frac{2e^{2x}}{2x} = \lim_{x\to\infty} \frac{e^{2x}}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1} = \boxed{\infty}$

77. $\lim_{x\to\infty} (\ln x)^{2/x}$

$\ln y = \lim_{x\to\infty} \ln(\ln x)^{2/x}$ $\quad \ln y = \lim_{x\to\infty} \frac{2\frac{1/x}{\ln x}}{1}$

$\ln y = \lim_{x\to\infty} \frac{2}{x}\ln(\ln x)$ $\quad \ln y = \lim_{x\to\infty} \frac{2}{x\ln x}$

$\ln y = \lim_{x\to\infty} \frac{2\ln(\ln x)}{x}$ $\quad \ln y = 0$

$y = e^0 = \boxed{1}$

81. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$ improper (vert. asym @ $x=0$)

$\lim_{a\to 0} \int_a^{16} x^{-1/4}\,dx = \lim_{a\to 0} \frac{4}{3}x^{3/4}\Big]_a^{16} = \lim_{a\to 0}\left[\frac{32}{3} - \frac{4a^{3/4}}{3}\right] = \boxed{\frac{32}{3}\text{ ; converge}}$

88. $\int_1^\infty x^2\ln x\,dx$ improper (unbounded/infinite)

$\lim_{b\to\infty} \int_1^b x^2\ln x\,dx$ $\quad u=\ln x \quad v = x^3/3\,dx$

$du = 1/x\,dx \quad dv = x^2\,dx$

$\lim_{b\to\infty} \left[(\ln x)(x^3/3) - \int (1/x)(x^3/3)\,dx\right]_1^b$

$\lim_{b\to\infty} \left[\frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\,dx\right]_1^b = \lim_{b\to\infty}\left[\frac{x^3\ln x}{3} - \frac{x^3}{9}\right]_1^b$

$\lim_{b\to\infty}\left[\left(\frac{b^3\ln b}{3} - \frac{b^3}{9}\right) + \left(\frac{1}{9}\right)\right] = \boxed{\infty\text{ ; diverges}}$

85. $\int_1^\infty \frac{\ln x}{x^2}\,dx$ improper (unbounded/infinite) $u = \ln x$ $v = -1/x$ $du = 1/x\,dx$ $dv = 1/x^2\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx\right]_1^b$

$= \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left[\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - (-1)\right]$

$= \lim_{b\to\infty} \left[\frac{-\ln b - 1}{b}\right] + 1 = \lim_{b\to\infty} \left[\frac{-1/b}{1}\right] + 1 = 0 + 1 = \boxed{1;\ \text{converges}}$

88. $\pi \int_0^\infty (xe^{-x})^2\,dx$ improper (unbounded/infinite)

$u = x^2$ $v = -\frac{1}{2}e^{-2x}$
$du = 2x\,dx$ $dv = e^{-2x}\,dx$

$\lim_{b\to\infty} \pi \int_0^b x^2 e^{-2x}\,dx$

$u = x$ $v = -\frac{1}{2}e^{-2x}$
$du = dx$ $dv = e^{-2x}\,dx$

$\lim_{b\to\infty} \pi \left[-\frac{x^2}{2}e^{-2x} + \int x e^{-2x}\,dx\right]_0^b$

$\lim_{b\to\infty} \pi \left[-\frac{x^2}{2}\cdot e^{-2x} - \frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx\right]_0^b$

$\lim_{b\to\infty} \pi \left[-\frac{x^2}{2}\cdot e^{-2x} - \frac{x}{2}\cdot e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$

$\lim_{b\to\infty} \pi \left[-\frac{e^{-2x}}{2}\left(x^2 - x - \frac{1}{2}\right)\right]_0^b = \lim_{b\to\infty} \left[\frac{-\pi e^{-2x}}{4}\left(2x^2 - 2x - 1\right)\right]_0^b$

$\lim_{b\to\infty} \left[\frac{-\pi e^{-2b}}{4}(2b^2 - 2b - 1)\right] + \frac{\pi}{4}$

$\lim_{b\to\infty} \left[\frac{-\pi(2b^2 - 2b - 1)}{4e^{2b}}\right] + \frac{\pi}{4} = \lim_{b\to\infty} \left[\frac{-\pi(4b - 2)}{8e^{2b}}\right] + \frac{\pi}{4}$

$= \lim_{b\to\infty} \left[\frac{-\pi(4)}{16e^{2b}}\right] + \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

CALCULUS BC
WORKSHEET 1 ON LOGISTIC GROWTH

Name: Key

Use your calculator on 4(b), 4(c), and 5(c) only.

1. Suppose the population of bears in a national park grows according to the logistic differential equation $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears at time t in years. $\frac{dP}{dt} = 5P\left(1 - \frac{1}{2500}\right)$

$\frac{1}{500}$

(a) If $P(0) = 100$, find $\lim_{t \to \infty} P(t)$. Is the solution curve increasing or decreasing? Justify your answer. Sketch the graph of $P(t)$.

$\lim_{t \to \infty} P(t) = 2500$

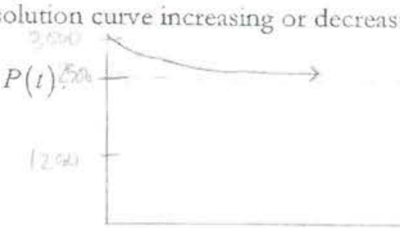
increasing b/c $\frac{dP}{dt} > 0$

(b) If $P(0) = 1500$, find $\lim_{t \to \infty} P(t)$. Is the solution curve increasing or decreasing? Justify your answer. Sketch the graph of $P(t)$.

$\lim_{t \to \infty} P(t) = 2500$

increasing b/c $\frac{dP}{dt} > 0$

(c) If $P(0) = 3000$, find $\lim_{t \to \infty} P(t)$. Is the solution curve increasing or decreasing? Justify your answer. Sketch the graph of $P(t)$.



$\lim_{t \to \infty} P(t) = 2500$

decreasing $\frac{dP}{dt} < 0$

(d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

$\frac{d^2P}{dt^2} = 5\frac{dP}{dt} - 0.004P\frac{dP}{dt} = 0.004\frac{dP}{dt}(1250 - P)$ P = 1250 bears

$0 = 0.004\frac{dP}{dt}$ $0 = 1250 - P$

2. **1998 BC MC** The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 2500 (B) 3000 (C) 4200 (D) 5000 (E) 10,000

$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10{,}000}\right)$

3. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t in years. Which of the following statements are true?

$\frac{1}{100}$

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{300}\right)$$

✓ I. $\lim_{t \to \infty} P(t) = 300$

✓ II. The growth rate of the wolf population is greatest at $P = 150$.

III. If $P > 300$, the population of wolves is increasing.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

4. A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where t is measured in years.

$$\frac{dP}{dt} = P\left(1 - \frac{P}{100}\right)$$

(a) If $P(0) = 20$, solve for P as a function of t.

$L = 100$

$k = 1$

$b = \frac{100-20}{20} = 4$

$$\boxed{P = \frac{100}{1+4e^{-t}}}$$

(b) Use your answer to (a) to find P when $t = 3$ years.

$P = \frac{100}{1+4e^{-3}} = 83.393 \approx 83$ animals

(c) Use your answer to (a) to find t when $P = 80$ animals.

$80 = \frac{100}{1+4e^{-t}}$

$1+4e^{-t} = \frac{5}{4}$

$4e^{-t} = \frac{1}{4}$

$e^{-t} = \frac{1}{16}$

$-t = \ln\frac{1}{16}$

$t = -\ln\frac{1}{16} \approx 2.7726$ yrs

5. The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hours after 9AM.

$\frac{dP}{dt} = 6P - 0.003P^2$

(a) How many students have heard the rumor when it is spreading the fastest? Justify your answer.

$P = 1000$ students

$\frac{d^2P}{dt^2} = 6\frac{dP}{dt} - 0.006P\frac{dP}{dt}$

$0 = 0.006\frac{dP}{dt}(1000 - P)$

(b) If $P(0)=5$, solve for P as a function of t.

$$\frac{dP}{dt} = 0.003P(2000-P)$$

$$\int \frac{1000}{3P(2000-P)}\,dP = dt$$

$$\int \frac{A}{3P} + \frac{B}{2000-P}\,dP = dt$$

$$1000 = A(2000-P) + B(3P)$$

$$A = \frac{1}{2} \qquad B = \frac{1}{6}$$

$$\frac{1}{6}\int \frac{1}{P}\,dP + \frac{1}{6}\int \frac{1}{2000-P}\,dP = dt$$

$$\frac{1}{6}\ln\left|\frac{P}{2000-P}\right| = t + c$$

$$\ln\left|\frac{P}{2000-P}\right| = 6t + c$$

$$\frac{P}{2000-P} = Ce^{6t}$$

$$\frac{P}{2000-P} = \frac{1}{399}e^{6t}$$

$$399P = 2000e^{6t} - Pe^{6t}$$

$$(399 - e^{6t})P = 2000e^{6t}$$

$$\boxed{P = \frac{2000e^{6t}}{399-e^{6t}} \qquad P = \frac{2000}{399e^{-6t}+1}}$$

(c) Use your answer to (b) to determine how many hours have passed when half the student Body has heard the rumor.

$$1500 = \frac{2000}{399e^{-6t}+1}$$

$$399e^{-6t}+1 = 2$$

$$399e^{-6t} = 1$$

$$e^{-6t} = \frac{1}{399}$$

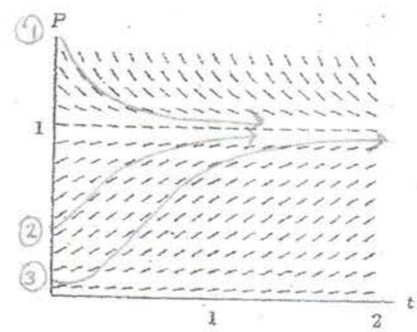
$$-6t = \ln\left|\frac{1}{399}\right|$$

$t = 0.998$ hR

(d) How many students have heard the rumor after 2 hours?

$$P = \frac{2000}{399e^{-6(2)}+1} = 1995.109 \approx 1995 \text{ students}$$

6. (a) On the slope field shown on the right for $\frac{dP}{dt} = 3P - 3P^2$, sketch three solution curves showing different types of behavior for the population P.



(b) Describe the meaning of the shape of the solution curves for the population.

Where is P increasing? when $P(0) < 1$

Decreasing? $P(0) > 1$

What happens in the long run? $\lim_{t\to\infty} P(t) = 1$

Are there any inflection points? yes

Where? $P(t) = 0.5$

What do they mean for the population? when the population is growing the fastest

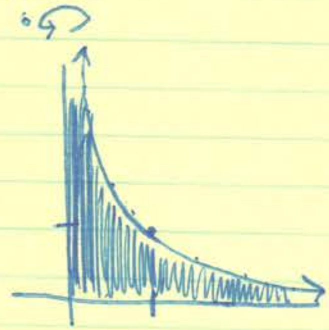
$\lim_{x\to 1} \frac{(\ln x)^2}{x-1} = y$ L Hôpital $= \lim_{x\to 1} \frac{2(\ln x)\frac{1}{x}}{1} = 0$

$\cancel{\ln y = \lim_{x\to 1} \ln\left|\frac{(\ln x)^2}{(x-1)}\right|}$

$f(x) = \frac{1}{x}$

$1 \leq x \leq \infty$

?



$A = \int_1^\infty \frac{1}{x}\,dx$

$V = \pi \int_1^\infty \frac{1}{x^2}\,dx$

$A = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln|x|\Big]_1^b$

$\lim_{b\to\infty} \ln|b| - \overset{0}{\ln|1|}$

$A = \infty$

$x \to \infty$

$\ln x = y$

$e^y = x$

$e^y = \infty$

$y = \infty$

$V = \pi \int_1^\infty \frac{1}{x^2}\,dx$

$(\pi) \lim_{b\to\infty} \int_1^b \frac{1}{x^2}$

x^{-2} $\frac{x^{-1}}{-1}$

$(\pi) \lim_{b\to\infty} -\frac{1}{x}\Big]_1^b$

$-\frac{1}{x^2}$ $+x^{-2}$

$(\pi)\left[\lim_{b\to\infty}\left[\frac{-1}{b^{-1}} - -\frac{-1}{1}\right]\right.$

$\frac{1}{x^2} = x^{-2}$

$-\frac{1}{x}$

$\pi \lim_{b\to\infty} \frac{1}{-b} + 1$

$\pi(0+1)$ $\cancel{=\infty}$

$= \pi$

$V = \pi$

$\boxed{\begin{matrix} A = \infty \\ V = \pi \end{matrix}}$?

$\frac{1}{2}$ $-$ $+$

sin sin   cos − cos
sin cos   sin + sin
cos cos   cos + cos

$\arcsin u \rightarrow \frac{u'}{\sqrt{1-u^2}}$ $\arccos u \rightarrow \frac{-u'}{\sqrt{1-u^2}}$

$\text{arccot}\, u \rightarrow \frac{-u'}{1+u^2}$ $\arctan u \rightarrow \frac{u'}{1+u^2}$

$\text{arcsec}\, u \rightarrow \frac{u'}{|u| \sqrt{u^2-1}}$ $\text{arccsc}\, u \rightarrow \frac{-u'}{|u| \sqrt{u^2-1}}$

$\sin^2 u = \frac{1-\cos 2u}{2}$ $\cos^2 u = \frac{1+\cos 2u}{2}$

u

$\int x e^{3x} dx$

$\int e^{3x} dx$

$u = 3x$
$du = 3dx$

$u = x \quad v = \frac{1}{3}e^{3x}$
$du = dx \quad dv = e^{3x} dx$

$uv - \int v\,du$

$\frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx$

$\boxed{1/3\, x e^{3x} - 1/9\, e^{3x} + C}$

$\int \arctan x \, dx$

$u = \arctan x \quad dv = dx$
$du = \frac{1}{1+x^2} dx \quad v = x$

$x \arctan x - \int x \frac{1}{1+x^2}$

$u = 1+x^2$
$du = 2x dx$

$x \arctan x - \frac{1}{2} \ln|1+x^2| + C$

$\int e^x \sin x \, dx$

$\sin x\, e^x - \int e^x \cos x$

$u = \sin x \quad v = e^x$
$du = \cos x \quad dv = e^x \, dx$

$u = \cos x \quad v = e^x$
$du = -\sin x \, dx \quad dv = e^x dx$

$\sin x\, e^x - \left[\cos x\, e^x - \int -e^x \sin x \, dx\right]$

$\int e^x \sin x dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$

$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x$

$\int x\cos x\,dx$

$u = x$ $\quad v = \sin x$

$du = dx$ $\quad dv = \cos x\,dx$

$x\sin x - \int \sin x\,dx$

$x\sin x + \cos x + C$

$\int x^3\cos x\,dx$

$u = x^3$ $\quad v = \sin x$

$du = 3x^2dx$ $\quad dv = \cos x\,dx$

$x^3\sin x - \int(\sin x)(3x^2)\,dx$

$u = 3x^2$ $\quad v = -\cos x$

$du = 6x$ $\quad dv = \sin x\,dx$

$x^3\sin x - \left[-3x^2\cos x - \int(-\cos x)(6x)\,dx\right]$

$u = 6x$ $\quad v = -\sin x$

$du = 6\,dx$ $\quad dv = -\cos x\,dx$

$x^3\sin x + 3x^2\cos x + \overset{-\sin x\,6x}{\cancel{\quad}} - \int -\cos x\,6$

$x^3\sin x + 3x^2\cos x - 6x\sin x - 6\sin x + C$

$x^3\sin x + 3x^2\cos x - 6x\sin x - 6\sin x + C$

$\int x\sin(6x)\,dx$

$u = x$ $\quad v = \frac{1}{6}\cos(6x)$

$du = dx$ $\quad dv = \sin(6x)\,dx$

$x\frac{1}{6}\cos(6x) - \int \frac{1}{6}\cos(6x)\,dx$

$x\frac{1}{6}\cos(6x) + \frac{1}{36}\sin(6x) + C$

$\frac{8}{3}$ $\int 3xe^{4x^2}dx$

$u = 4x^2$

$du = 8x\,dx$

$= \frac{8}{3}\int e^u = \frac{8}{3}e^{4x^2}$

$\int x\sqrt{x+4}\,dx$

$u = (x+4)^{1/2}$

$du = \frac{1}{2}(x+4)$

$\int x\sin^2 x \; dx$

$u = x \qquad v = \frac{1}{2}x - \frac{1}{4}\sin(2x)$

$du = dx \qquad dv = \sin^2 x \, dx$

$\int \sin^2 x \, dx$

$\int \frac{1-\cos 2x}{2} dx$

$= \int \frac{1}{2} dx - \int \frac{\cos 2x}{2}$

$\frac{1}{2}x - \frac{1}{4}\sin(2x)$

$uv - \int v\,du$

$x(\frac{1}{2}x - \frac{1}{4}\sin(2x)) - \int \frac{1}{2}x - \frac{1}{4}\sin(2x)\,dx$

$x(\frac{1}{2}x - \frac{1}{4}\sin(2x)) - [x^2 + \frac{1}{8}\cos(2x)] + C$

$x(\frac{1}{2}x - \frac{1}{4}\sin(2x)) - x^2 - \frac{1}{8}\cos(2x) + C$

$\int \frac{4x+41}{x^2+3x-10} dx = \int \frac{4x+41}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$

$4x+41 = A(x-2) + B(x+5)$

$x=2 \rightarrow 49 = 7B \qquad B = 7$

$x=-5 \qquad 21 = -7A \qquad A = -21/7 = -3$

$-3\int \frac{1}{x+5} + 7\int \frac{1}{x-2}$

$-3\ln|x+5| + 7\ln|x-2| + C$

$\int \frac{2x^3+x^2-7x+7}{x^2+x-2} dx = \int 2x-1 + \frac{-2x+5}{x^2+x-2} dx$

$\frac{A}{x+2} + \frac{B}{x-1}$

$$\begin{array}{r} 2x-1 \\ x^2+x-2 \,\overline{)\, 2x^3+x^2-7x+7} \\ -(2x^3+2x^2-4x) \\ -x^2-3x+7 \\ -(-x^2-x+2) \\ -2x+5 \end{array}$$

$= \int 2x-1\,dx + \int \frac{-2x+5}{(x+2)(x-1)} dx$

$x^2 - x$

$-2x+5 = A(x-1) + B(x+2)$

$x=1 \rightarrow 3 = B3 \qquad B=1$

$x=-2 \rightarrow 9 = -3A \qquad A=-3$

$x^2 - x + \ln|x-1| - 3\ln|x+2| + C$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx+D$$
$$8x^3+13x = Ax^3+2Ax+Bx^2+2B+Cx+D$$
$$8x^3+13x = Ax^3+Bx^2+x(2A+C)+(2B+D)$$ Combine Terms

$A=8 \quad B=0$

$2A+C=13 \qquad 2B+D=0$

$\cancel{8}+C=13 \qquad D=0$

$C=-3$

$$=\int \frac{8x}{(x^2+2)} + \frac{-3x}{(x^2+2)^2}\,dx$$

$U=x^2+2$

$du=2x\,dx$

$$\frac{1}{4}\ln|x^2+2| + \frac{3}{2}\cdot\frac{1}{(x^2+2)} + C$$

$$\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx = \int \frac{1}{u(u+1)}\,du = \frac{A}{u}+\frac{B}{u+1}$$

$u=\tan x$

$du=\sec^2 x\,dx$

$1=A(u+1)+Bu$

$u=0 \rightarrow 1=A$

$u=-1 \quad -1=B$

$$\int \frac{1}{u}+\frac{-1}{u+1}\,dx$$

$$= \ln|u| + \ln|u+1| + C$$

$$= \ln|\tan x| \mp \ln|\tan x+1| + C$$

$$= \ln\left|\frac{\tan x}{\tan x+1}\right| + C$$

$$\int x\sqrt{x+4}\,dx$$

$u=(x+4)^{1/2}$

$du=\frac{1}{2}(x+4)\,dx$

$=.5x+2$

$2\int u^{1/2}\,1-2\,du$

$2\left(\frac{2u^{3/2}}{3}-2u\right)du$

$2\left[\frac{2}{3}\right.$

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = y$$

$$\ln y = \lim_{x\to\infty}\left(\ln\left(1+\frac{2}{x}\right)^x\right)$$

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)$$

$$\ln y = \lim_{x\to\infty}\frac{\ln(1+2/x)}{x^{-1}}$$

$$\ln y = \lim_{x\to\infty}\frac{\frac{-2/x^2}{1+2/x}}{-1/x^2} = \lim_{x\to\infty}\frac{-2/x^2}{1+2/x}\cdot\frac{x^2}{1} = \frac{2}{1+2/x} = 2$$

$$\ln y = 2$$

$$e^2 = y$$

$$\int_1^\infty (1-x)e^{-x}\,dx$$

$$\lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx$$

Int Parts

$u = 1-x \qquad v = -e^{x}$

$du = -dx \qquad du = e^{-x}$

$$\lim_{b\to\infty}\left[(1-x)(-e^{x}) - \int +e^{-x}dx\right]_1^b$$

$$\lim_{b\to\infty}\left[(1-x)(-e^{x}) + e^{x}\right]_1^b$$

Distribute

$$\lim_{b\to\infty}\left[\cancel{-e^{x}} + xe^{x} \cancel{+e^{x}}\right]_1^b$$

$$\lim_{b\to\infty}\left[xe^{-x}\right]_1^b$$

$$\lim_{b\to\infty}\left[\left(be^{-b}\right) - e^{-1}\right]$$

Rearrange

$$\lim_{b\to\infty}\left(\frac{b}{e^b}\right) - e^{-1}$$

L Hôpital

$$\lim_{b\to\infty}\left[\frac{1}{e^b} - e^{-1}\right]$$

0

$$= -e^{-1} = -1/e$$

$$\int\frac{\ln x}{x}dx$$

$u = \ln x \qquad = \int u\,du$

$du = \frac{1}{x}dx \qquad = \frac{u^2}{2} \qquad \frac{(\ln x)^2}{2}$

$\int \sin^5 x \cos^2 x\, dx$ $\qquad (1-\cos^2 x)(1-\cos^2 x)$

$\int \sin x \sin^4 x \cos^2 x\, dx$

$\int \sin x (1-\cos^2 x)^2 \cos^2 x\, dx$

$\int \sin x (1 - 2\cos^2 x + \cos^4 x) \cos^2 x\, dx$

$\int \sin x (\cos^2 x - 2\cos^4 x + \cos^6 x)\, dx$

$u = \cos x$ $\qquad -\int u^2 - 2u^4 + u^6\, du$

$du = -\sin x\, dx$ $\qquad = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$

$$\frac{-\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

$\int x\sqrt{x^2-1}\, dx$

$u = x^2 - 1$

$du = 2x\, dx$ $\qquad \frac{1}{2}\int u^{1/2} = \frac{2u^{3/2}}{3}$ $\quad \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(x^2-1)^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C$

$\int_4^8 x\sqrt{x-4}\, dx$

$u = x - 4$ $\qquad \int_4^8 (u+4)(u)^{1/2}\, du$

$du = dx$

$x = u + 4$ $\qquad \int_4^8 u^{3/2} + 4u^{1/2}\, du$

$$\frac{2u^{5/2}}{5} + \frac{(4)(2)u^{3/2}}{3}\Bigg]_4^8 = \frac{2u^{5/2}}{5} + \frac{8u^{3/2}}{3}$$

$\int \cos^3(\pi x - 1)\, dx$

$\int \cos(\pi x - 1)(\cos^2(\pi x - 1))\, dx$

$\int \cos(\pi x - 1)(1 - \sin^2(\pi x - 1))\, dx$

$u = \sin(\pi x - 1)$

$du = \pi\cos(\pi x - 1)$ $\qquad \frac{1}{\pi}\int 1 - u^2\, du$

$$= \frac{1}{\pi}\left(u - \frac{u^3}{3}\right) = \frac{1}{\pi}\left(\sin(\pi x - 1) - \frac{\sin^3(\pi x - 1)}{3}\right)$$

$\int_0^{\pi/6} \cos(3x)\cos x\, dx$ $\qquad \cos\cos = \cos + \cos$

$\frac{1}{2}\int_0^{\pi/6}\left[\cos 2x + \cos 4x\right] dx$ $\qquad \frac{1}{2}\left[\frac{1}{2}\sin\frac{\pi}{3} + \frac{1}{4}\sin\frac{2\pi}{3}\right]$

$\frac{1}{2}\left[\frac{1}{2}\sin(2x) + \frac{1}{4}\sin(4x)\right]_0^{\pi/6}$ $\qquad \frac{1}{4}\frac{\sqrt{3}}{2} + \frac{1}{8}\frac{\sqrt{3}}{2}$

$\frac{(2)1}{(2)8}\sqrt{3} + \frac{1}{16}\sqrt{3}$ $\qquad \frac{3\sqrt{3}}{16}$

CALCULUS BC
WORKSHEET 1 ON VECTORS

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Work the following on **notebook paper.** Use your calculator on problems 10 and 13c only.

1. If $x = t^2 - 1$ and $y = e^{t^3}$, find $\dfrac{dy}{dx}$.

2. If a particle moves in the xy-plane so that at any time $t > 0$, its position vector is $\left\langle \ln\left(t^2 + 5t\right), 3t^2 \right\rangle$, find its velocity vector at time $t = 2$.

3. A particle moves in the xy-plane so that at any time t, its coordinates are given by $x = t^5 - 1$ and $y = 3t^4 - 2t^3$. Find its acceleration vector at $t = 1$.

4. If a particle moves in the xy-plane so that at time t its position vector is $\left\langle \sin\left(3t - \dfrac{\pi}{2}\right), 3t^2 \right\rangle$, find the velocity vector at time $t = \dfrac{\pi}{2}$.

5. A particle moves on the curve $y = \ln x$ so that its x-component has derivative $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(1, 0)$. Find the position of the particle at time $t = 1$.

6. A particle moves in the xy-plane in such a way that its velocity vector is $\left\langle 1 + t, t^3 \right\rangle$. If the position vector at $t = 0$ is $\left\langle 5, 0 \right\rangle$, find the position of the particle at $t = 2$.

7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\dfrac{dy}{dt} = 3$, what is the value of $\dfrac{dx}{dt}$?

8. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - \dfrac{3}{2}t^2 - 18t + 5$ and $y = t^3 - 6t^2 + 9t + 4$. For what value(s) of t is the particle at rest?

9. A curve C is defined by the parametric equations $x = t^3$ and $y = t^2 - 5t + 2$. Write the equation of the line tangent to the graph of C at the point $(8, -4)$.

Fernando Trojanu

ULUS BC - WORKSHEET 2 ON VECTORS

ork the following on **notebook paper**. Use your calculator on problems 7 – 11 only.

. If $x = e^{2t}$ and $y = \sin(3t)$, find $\dfrac{dy}{dx}$ in terms of t.

2. Write an integral expression to represent the length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^2 t$ for $0 \le t \le \dfrac{\pi}{2}$.

3. For what value(s) of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

4. For any time $t \ge 0$, if the position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t+3)$, find the acceleration vector.

5. Find the equation of the tangent line to the curve given by the parametric equations $x(t) = 3t^2 - 4t + 2$ and $y(t) = t^3 - 4t$ at the point on the curve where $t = 1$.

6. If $x(t) = e^t + 1$ and $y = 2e^{2t}$ are the equations of the path of a particle moving in the xy-plane, write an equation for the path of the particle in terms of x and y.

7. A particle moves in the xy-plane so that its position at any time t is given by $x = \cos(5t)$ and $y = t^3$. What is the peed of the particle when $t = 2$?

8. The position of a particle at time $t \ge 0$ is given by the parametric equations $x(t) = \dfrac{(t-2)^3}{3} + 4$ and $y(t) = t^2 - 4t + 4$.
   (a) Find the magnitude of the velocity vector at $t = 1$.
   (b) Find the total distance traveled by the particle from $t = 0$ to $t = 1$.
   (c) When is the particle at rest? What is its position at that time?

9. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $\dfrac{dx}{dt} = 1 + \tan(t^2)$ and $\dfrac{dy}{dt} = 3e^{\sqrt{t}}$. Find the acceleration vector and the speed of the object when $t = 5$.

10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = t + \cos t$ and $y(t) = 3t + 2\sin t$, $0 \le t \le \pi$. Find the velocity vector when the particle's vertical position is $y = 5$.

11. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\dfrac{dx}{dt} = 2\sin(t^3)$ and $\dfrac{dy}{dt} = \cos(t^2)$ for $0 \le t \le 4$. At time $t = 1$, the object is at the position (3, 4).
   (a) Write an equation for the line tangent to the curve at (3, 4).
   (b) Find the speed of the object at time $t = 2$.
   (c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.
   (d) Find the position of the object at time $t = 2$.

Fernando Trujano

CALCULUS BC
WORKSHEET 3 ON VECTORS

Work the following on **notebook paper**. Use your calculator only on problems 3 – 7.

1. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 2$, $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 2$.
(b) Set up an integral expression to find the total distance traveled by the particle from $t = 0$ to $t = 4$.
(c) Find $\frac{dy}{dx}$ as a function of x.
(d) At what time t is the particle on the y-axis? Find the acceleration vector at this time.

2. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with the velocity vector $v(t) = \left\langle \frac{1}{t+1}, 2t \right\rangle$. At time $t = 1$, the object is at (ln 2, 4).

(a) Find the position vector.
(b) Write an equation for the line tangent to the curve when $t = 1$.
(c) Find the magnitude of the velocity vector when $t = 1$.
(d) At what time $t > 0$ does the line tangent to the particle at $(x(t), y(t))$ have a slope of 12?

3. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$, with $x(t) = 2t + 3\sin t$ and $y(t) = t^2 + 2\cos t$, where $0 \leq t \leq 10$. Find the velocity vector at the time when the particle's vertical position is $y = 7$.

4. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = 1 + \sin(t^3)$. The derivative $\frac{dy}{dt}$ is not explicitly given. For any $t \geq 0$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $t + 3$. Find the acceleration vector of the object at time $t = 2$.

AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

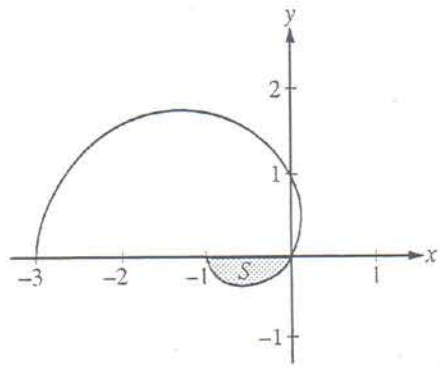
Question 4

The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.



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a) $A = \frac{1}{2}\int_0^{\pi/3}(1-2\cos\theta)^2\,d\theta$

$0 = 1 - 2\cos\theta$

$\theta = \pi/3$

b) $\frac{dx}{d\theta} = \frac{dR}{d\theta}\cos\theta - R\sin\theta = +2\sin\theta\cos\theta - (1-2\cos\theta)\sin\theta$ = [illegible]

$= +2\sin\theta\cos\theta - \sin\theta + 2\sin\theta\cos\theta$

$\frac{dy}{d\theta} = \frac{dR}{d\theta}\sin\theta + R\cos\theta = +2\sin^2\theta + (1-2\cos\theta)\cos\theta$ = [illegible]

$= -2\sin^2\theta + \cos\theta - 2\cos\theta$

c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dR}{d\theta}\sin\theta + R\cos\theta}{\frac{dR}{d\theta}\cos\theta - R\sin\theta}$

plug in $(0,1)$

$m = -2$

$$\frac{-2\sin^2\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)}$$

$y - 1 = -2(x - 0)$

AP Calculus BC
Parametric Equations and Vectors ~ Day 1
(Includes Sections 10.2, 10.3)

Name Fernando Trujano

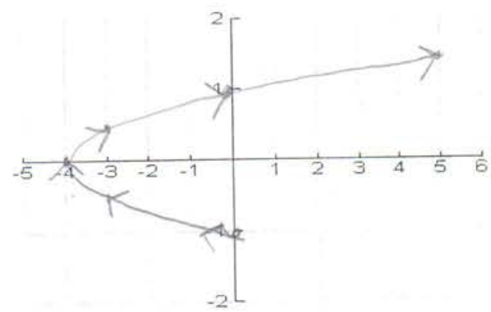
No Calculator

Review from Homework
In Algebra, equations are graphed in two variables, x and y. Now we will graph equations with x, y, and t, or with x, y, and θ, where x and y are expressed independently in terms of t or θ. The third variable, t or θ, is called the parameter, and the separate equations are called parametric equations.

Make a table and sketch the curve, indicating the direction of your graph. Then, eliminate the parameter.

Ex) $x = t^2 - 4$, and $y = \frac{t}{2}$, $-2 \le t \le 3$

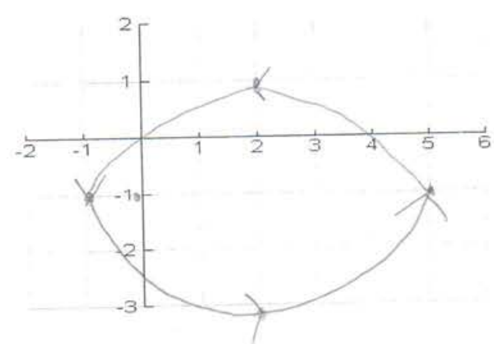
t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2



$y^2 = \frac{x-4}{4}$

Ex) $x = 2 + 3\cos t$, $y = -1 + 2\sin t$

t	0	π/2	π	3π/2	2π
x	5	2	-1	2	5
y	-1	1	-1	-3	-1



$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$

Practice with derivatives...

Ex) Given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate at $t = 1$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-2}{1/\sqrt{t}} = 6t^{3/2} - 2t^{1/2}$

$\left.\frac{dy}{dx}\right|_{t=1} = 4$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[6t^{3/2} - 2t^{1/2}\right] = \frac{18}{2}t^{1/2}\frac{dt}{dx} - t^{-1/2}\frac{dt}{dx}$

$= 9t - 1$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = 8$

AP Calculus BC
10.3 examples

Name: Fernando Trujano

#11) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $\theta = \frac{\pi}{6}$ for the parametric equations $x = 2 + \sec\theta$, $y = 1 + 2\tan\theta$. Determine the slope and concavity.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}\tan\theta} = \frac{2\sec\theta\cos\theta}{\sin\theta} = 2/\sin\theta = 2\csc\theta \qquad \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = 4$$

$\frac{dx}{d\theta}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[2\csc\theta\right] = -2\csc\theta\cot\theta\frac{d\theta}{dx}$$

$$= \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} = -2\cot^3\theta$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/6} = -6\sqrt{3}$$

#15) Find an equation of the tangent line at the point $\left(\frac{-2}{\sqrt{3}}, \frac{3}{2}\right)$ for $x = 2\cot\theta$, $y = 2\sin^2\theta$.

(do the other points for homework)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

$\frac{-2}{\sqrt{3}} = 2\cot\theta$ $\qquad$ $\frac{3}{2} = 2\sin^2\theta$

$-\frac{1}{\sqrt{3}} = \cot\theta$ $\qquad$ $\frac{3}{4} = \sin^2\theta$

$-\sqrt{3} = \tan\theta$ $\qquad$ $\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{2\pi}{3}, \cancel{\frac{5\pi}{3}}$ $\qquad$ $\theta = \cancel{\frac{\pi}{3}}, \frac{2\pi}{3}, \cancel{\frac{4\pi}{3}}, \cancel{\frac{5\pi}{3}}$

2nd Quadrant

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{2\pi}{3}} = \frac{3\sqrt{3}}{8}$$

$$y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$$

AP Calculus BC

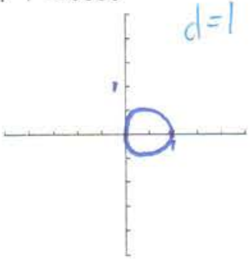
Name Fernando Trujano

Polar Coordinates and Polar Graphs ~ Day 1
(Includes Section 10.4)

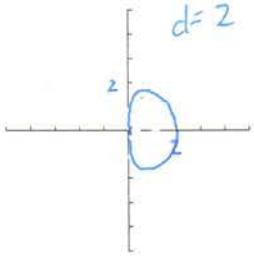
I. Polar Graphs

Put your graphing calculator in Polar mode and Radian mode. Graph the following equations on your calculator, sketch the graphs, and answer the questions.

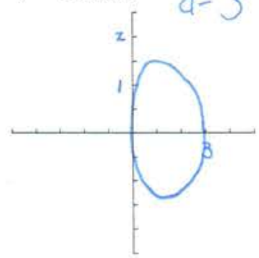
1. $r = \cos\theta$



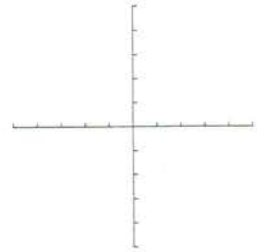
$r = 2\cos\theta$



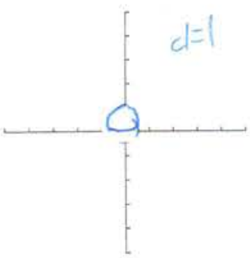
$r = 3\cos\theta$ d=3



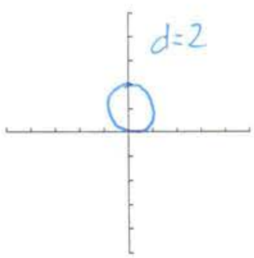
$r = 4\cos\theta$



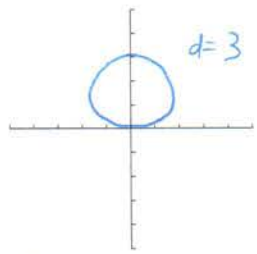
$r = \sin\theta$



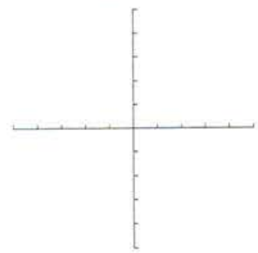
$r = 2\sin\theta$



$r = 3\sin\theta$



$r = 4\sin\theta$



What do you notice about these graphs? Circles

diameter = coefficient

sin - y axis
cos - x axis

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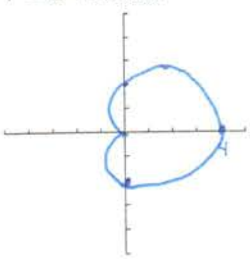
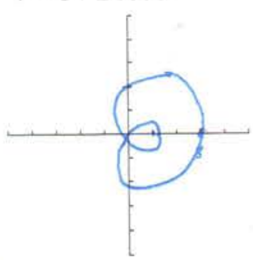
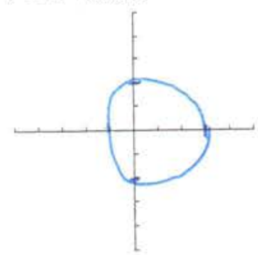
2. $r = 2 + 2\cos\theta$



$r = 1 + 2\cos\theta$



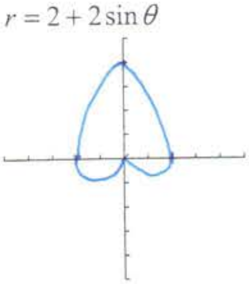
$r = 2 + \cos\theta$



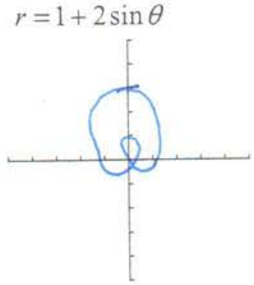
Cardioid

limacon

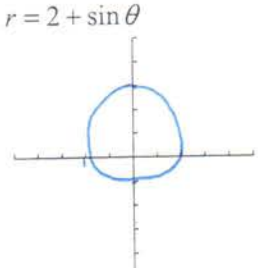
$r = 2 + 2\sin\theta$



$r = 1 + 2\sin\theta$



$r = 2 + \sin\theta$



Which graphs go through the pole?
Which ones do not go through the pole?
Which ones have an inner loop? - $a < b$
What causes these things to happen?

$a \pm b\cos n\theta$

10.4 - 10.5

Polar Coordinates and Graphs

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dR}{d\theta}\sin\theta + R\cos\theta}{\frac{dR}{d\theta}\cos\theta - R\sin\theta}$$

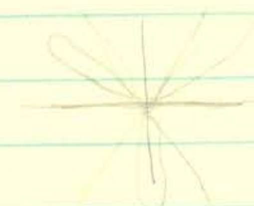
Horizontal Tangent lines · $dy/d\theta = 0$

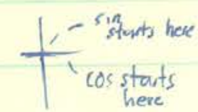
Vertical Tangent lines · $dx/d\theta = 0$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ then $\theta = \alpha$ is tangent to pole.

$r = |\sin 3\theta|$

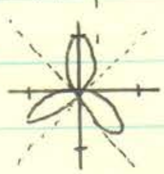
$u = \sin 3\theta$





a) $r = -\sin 3\theta$

① Graph



② Set $R = 0$

$0 = -\sin 3\theta$

$3\theta = 0, \pi, 2\pi$

$\theta = 0, \pi/3, 2\pi/3$

③ Find R'

$R' = -3\cos\theta$

$R' \neq 0$

④ Tangent lines at pole

$\theta = 0$

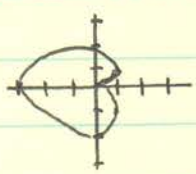
$\theta = \pi/3$

$\theta = 2\pi/3$

b) $r = 2(1-\cos\theta) = 2 - 2\cos\theta$

①

θ	r
0	0
$\pi/4$	1
$\pi/2$	2
$2\pi/3$	3
π	4



② $\cos 2 = 1$

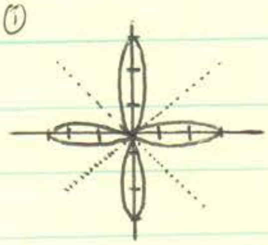
$\theta = 0$

③ $R' = 2\sin\theta$

$2\sin 0 = 0$ → No tangent lines at pole

c) $r = 3\cos 2\theta$

①



② $R = 0$

$0 = 3\cos 2\theta$

$\cos 2\theta = 0$

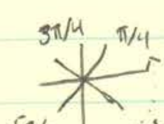
$2\theta = \pi/2, 3\pi/2, 5\pi/2$

$\theta = \pi/4, 3\pi/4, 5\pi/4$

③ $R' = -6\sin\theta$

$R' \neq 0$

④ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

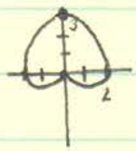


Same line not an answer

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

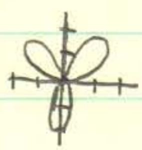
Area bounded by $r = 2 + 2\sin\theta$

θ	r
0	2
$\pi/2$	4
π	2
$3\pi/2$	0
2π	2



$$A = \frac{1}{2}\int_0^{2\pi} (2 + 2\sin\theta)^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (4 + 8\sin\theta + 4\sin^2\theta) \, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 8\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (4 + 8\sin\theta + 2 - 2\cos 2\theta) \, d\theta$$
$$= \frac{1}{2}\left[4\theta - 8\cos\theta + 2\theta - \sin 2\theta\right]_0^{2\pi}$$
$$= 2\theta - 4\cos\theta + \theta - \frac{\sin 2\theta}{2}\Big]_0^{2\pi}$$
$$= \left[2(2\pi) - 4\cos 2\pi + 2\pi - \frac{\sin 4\pi}{2}\right] - \left[2(0) - 4\cos 0 + 0 - \frac{\sin 0}{2}\right]$$
$$A = 4\pi - 4 + 2\pi - 0 - (0 - 4 + 0 - 0) = 6\pi$$

Area of one petal of $r = 2\sin 3\theta$



$R = 0$

$2\sin 3\theta = 0$

$3\theta = 0, \pi, 2\pi$

$\theta = 0, \pi/3, 2\pi/3$

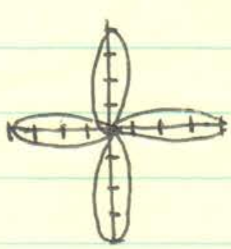
$R' = 6\cos 3\theta$

$R' \neq 0$

$$A = \frac{1}{2}\int (2\sin 3\theta)^2 d\theta$$

$$A = \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/3} (4\sin^2 3\theta) \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/3} 4\left(\frac{1-\cos\theta}{2}\right) d\theta = \frac{1}{2}\int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$
$$= \theta - \frac{1}{6}\sin 6\theta \Big]_0^{\pi/3} = \pi/3$$

Area of one petal of $r = 4\cos 2\theta$



$R = 0$

$4\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{\pi}{6}$ → $\theta = \frac{\pi}{6}, 0, \frac{3\pi}{6}$

$$A = \int_0^{\pi/4} (4\cos 2\theta)^2 \, d\theta$$
$$= \int_0^{\pi/4} (16\cos^2 2\theta) \, d\theta$$
$$= \int_0^{\pi/4} 16\left(\frac{1 + \cos 4\theta}{2}\right) d\theta$$
$$= \int_0^{\pi/4} (8 + 8\cos 4\theta) \, d\theta$$
$$= \left[8\theta + 2\sin 4\theta\right]_0^{\pi/4}$$
$$= 2\pi$$

Wk. 17

AP Calculus BC
Polar Coordinates and Polar Graphs ~ Day 3
(Includes Sections 10.5)

Name Fernando Trujano

Remember…

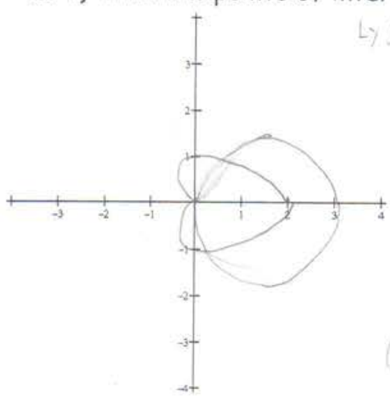
The area bounded by the polar curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is given by the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta .$$

Last time…

This time…

Ex 1) Find the points of intersection of $r = 1 + \cos\theta$ and $r = 3\cos\theta$. (Always graph… strange things often occur!)



↳ Set them equal

$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = 1/2$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ → plug into any equation

$(0,0)$ $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

$r = 1 + \cos\theta$

θ	r	θ	r
0	2	0	
$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	
π	0	π	
$\frac{3\pi}{2}$	1	$\frac{3\pi}{2}$	
2π	2	2π	

Ex 2) Find the area of the common interior of $r = 1 + \cos\theta$ and $r = 3\cos\theta$. (Calculator allowed)

Cardioid + Circle

Polar mode not decimal degrees

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\pi/3} (1+\cos\theta)^2\, d\theta + 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2\, d\theta\right] = 3.927$$

AP CALCULUS BC

Review worksheet on Polar

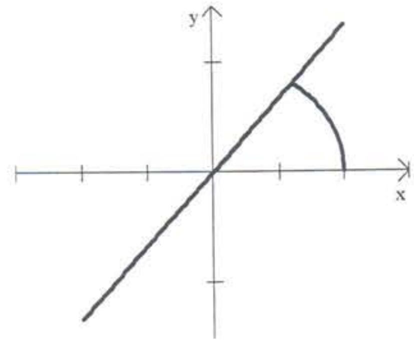
Name: Fernando Trujano

Work the following on notebook paper. You may use your calculator on problems 2 – 5.

On problems 1 – 2, sketch a graph, shade the region, and find the area.

1. inside $r = 2$ and outside $r = 2 - \sin\theta$
2. inside $r = 2 + 2\cos(2\theta)$ and outside $r = 2$

3. The figure shows the graphs of the line $y = \frac{2}{3}x$ and the curve C given by $y = \sqrt{1 - \frac{x^2}{4}}$. Let S be the region in the first quadrant bounded by the two graphs and the x-axis. The line and the curve intersect at point P.



(a) Find the coordinates of P.

(b) Set up and evaluate an integral expression with respect to x that gives the area of S.

(c) Find a polar equation to represent curve C.

(d) Use the polar equation found in (c) to set up and evaluate an integral expression with respect to the polar angle θ that gives the area of S.

4. A curve is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \cos(3\theta)$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, where r is measured in meters and θ is measured in radians.

(a) Find the area bounded by the curve and the y-axis.

(b) Find the angle θ that corresponds to the point on the curve with y-coordinate -1.

(c) For what values of θ, $\pi \le \theta \le \frac{3\pi}{2}$, is $\frac{dr}{d\theta}$ positive? What does this say about r? What does it say about the curve?

(d) Find the value of θ on the interval $\pi \le \theta \le \frac{3\pi}{2}$ that corresponds to the point on the curve with the greatest distance from the origin. What is the greatest distance? Justify your answer.

5. (From Calculus, 3rd edition, by Finney, Demana, Waits, Kennedy)
A region R in the xy-plane is bounded below by the x-axis and above by the polar curve defined by $r = \frac{4}{1 + \sin\theta}$ for $0 \le \theta \le \pi$.

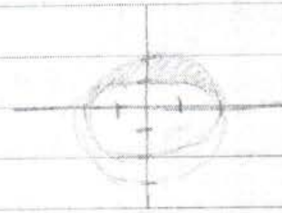
(a) Find the area of R by evaluating an integral in polar coordinates.

(b) The curve resembles an arch of the parabola $8y = 16 - x^2$. Convert the polar equation to rectangular coordinates, and prove that the curves are the same.

(c) Set up and evaluate an integral in rectangular coordinates that gives the area of R.

Review Worksheet on Polar

1) inside $r=2$, outside $r=2-\sin\theta$

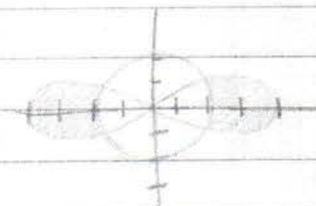


$$A = \frac{1}{2}\int_0^{\pi} (2)^2\, d\theta - \frac{1}{2}\int_0^{\pi} (2-\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(4-(2-\sin\theta)^2\right) d\theta$$

$$= \boxed{3.215}$$

2) inside $r = 2+2\cos 2\theta$, outside $r=2$



pts of intersection: $2+2\cos 2\theta = 2$

$$2\cos 2\theta = 0$$
$$\cos 2\theta = 0$$
$$2\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2$$
$$\theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$$

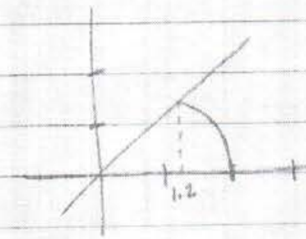
$$A = 4\left[\frac{1}{2}\int_0^{\pi/4} (2+2\cos(2\theta))^2\, d\theta - \frac{1}{2}\int_0^{\pi/4} (2)^2\, d\theta\right.$$

$$= 4\left[\frac{1}{2}\int_0^{\pi/4} \left((2+2\cos(2\theta))^2 - 4\right) d\theta\right.$$

$$= 2\int_0^{\pi/4} \left((2+2\cos(2\theta))^2 - 4\right) d\theta$$

$$= \boxed{11.142}$$

3) $y = \frac{2}{3}x \quad y = \sqrt{1 - x^2/4}$



ⓐ $\frac{2}{3}x = \sqrt{1 - x^2/4}$

$x = 1.2 \quad \boxed{P = (1.2, .8)}$

ⓑ $A = \int_0^{1.2} \left(\frac{2x}{3}\right) dx + \int_{1.2}^{2} \sqrt{1 - \frac{x^2}{4}}\, dx$

$= \boxed{.927}$

ⓒ $y = \sqrt{1 - x^2/4}$

$y^2 = 1 - \frac{x^2}{4}$

$4y^2 = 4 - x^2$

$4(r\sin\theta)^2 = 4 - (r\cos\theta)^2$

$4r^2\sin^2\theta = 4 - r^2\cos^2\theta$

$4r^2\sin^2\theta + r^2\cos^2\theta = 4$

$r^2(4\sin^2\theta + \cos^2\theta) = 4$

$\boxed{r^2 = \frac{4}{4\sin^2\theta + \cos^2\theta}}$

ⓓ $A = \frac{1}{2}\int_0^{\arctan(2/3)} \left(\frac{4}{4\sin^2\theta + \cos^2}\right) d\theta$

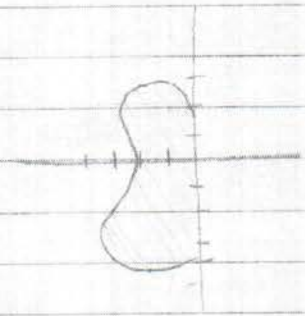
$= \boxed{.927}$

$y = 2/3\, x$

$r\sin\theta = 2/3\, r\cos\theta$

$\tan\theta = 2/3$

4) $r = \theta + \cos(3\theta) \qquad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$



ⓐ $A = \frac{1}{2}\int_{\pi/2}^{3\pi/2} (\theta + \cos 3\theta)^2 \, d\theta$

$= \underline{19.675}$

ⓑ $r = \theta + \cos(3\theta) \qquad y = r\sin\theta \rightarrow r = \frac{y}{\sin\theta}$

$-1 \rightarrow$

$\frac{y}{\sin\theta} = \theta + \cos(3\theta)$

$-1 = \sin\theta(\theta + \cos 3\theta) \qquad \underline{\theta = 3.485}$

ⓒ $\frac{dr}{d\theta} = 1 - 3\sin 3\theta$

$1 - 3\sin 3\theta = 0$

$\sin 3\theta = 1/3$

$3\theta = .340,\ \pi - .340,\ 2\pi + .340,\ 3\pi - .340,\ 4\pi + .340,\ 5\pi - .340$

$3\theta = .340,\ 2.802,\ 6.623,\ 9.085,\ 12.906,\ 15.368$

$\theta = .113,\ .934,\ 2.208,\ 3.028,\ 4.302,\ 5.123$

$\uparrow$ between π and $\frac{3\pi}{2}$

	π	(4)	4.302	(4.5)	$3\pi/2$
$\frac{dr}{d\theta}$		+		−	

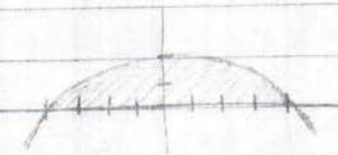
$\frac{dr}{d\theta} > 0$ for $\underline{(\pi, 4.302)}$. this means r is getting larger and the curve is getting further from origin

ⓓ r has a rel max at $\theta = 4.302$

θ	r
π	2.142
4.302	5.245
$3\pi/2$	4.712

the greatest distance is $\underline{5.245 \text{ when } \theta = 4.302}$

5) $r = \frac{4}{1+\sin\theta} \qquad 0 \le \theta \le \pi$



ⓐ $A = \frac{1}{2}\int_0^{\pi} \left(\frac{4}{1+\sin\theta}\right)^2 d\theta$

$= \underline{10.667}$

ⓑ $r = \frac{4}{1+\sin\theta}$

$r(1+\sin\theta) = 4$

$r + r\sin\theta = 4$

$\sqrt{x^2+y^2} + y = 4$

$\sqrt{x^2+y^2} = 4 - y$

$\left(\sqrt{x^2+y^2}\right)^2 = (4-y)^2$

$x^2 + \cancel{y^2} = 16 - 8y + \cancel{y^2}$

$8y = 16 - x^2$

$\underline{y = \frac{16-x^2}{8}}$

ⓒ $A = 2\int_0^4 \frac{16-x^2}{8}\, dx$

$= \underline{10.667}$ ✓

parametric and vector review

2) $x = 5t + 2$ $\qquad$ $y = 3t$

$\downarrow$ $\qquad$ $t = y/3$

$x = 5(y/3) + 2$

$x - 2 = \frac{5}{3}y$

$y = \frac{3}{5}x - \frac{6}{5}$ $\qquad$ $\boxed{m = \frac{3}{5}}$

10) $s(t) = \langle t^3 - t, (2t-1)^3 \rangle$

$v(t) = \langle 3t^2 - 1, 3(2t-1)^2(2) \rangle$

$= \langle 3t^2 - 1, 6(2t-1)^2 \rangle$

$a(t) = \langle 6t, 12(2t-1)(2) \rangle$

$= \langle 6t, 48t - 24 \rangle$

$\boxed{a(1) = \langle 6, 24 \rangle}$

21) arc length $= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^1 \sqrt{(t^2)^2 + (t)^2}\, dt$

$= \boxed{\int_0^1 \sqrt{t^4 + t^2}\, dt}$

77) $f(t) = (e^{-t}, \cos t)$

$f'(t) = (-e^{-t}, -\sin t)$

$\boxed{f''(t) = (e^{-t}, -\cos t)}$

4) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-3\sin t}$

m at $t=13$: $\frac{4\cos 13}{-3\sin 13} = -\frac{4}{3}\cot 13 = \boxed{\frac{-4}{3\tan 13}}$

7) at rest → velocity = 0

$s(t) = \langle t^3 - 3t^2,\ 2t^3 - 3t^2 - 12t \rangle$

$v(t) = \langle 3t^2 - 6t,\ 6t^2 - 6t - 12 \rangle$

$3t^2 - 6t = 0$ | $6t^2 - 6t - 12 = 0$

$3t(t-2) = 0$ | $6(t^2 - t - 2) = 0$

$t=0,\ t=2$ | $6(t-2)(t+1) = 0$

$t=-1,\ t=2$

at rest in both x- and y-direction at $\boxed{t=2}$

17) $x = t^2 - 4t + 1 \quad y = t^3$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t-4}\Big|_{t=2} = \frac{3(2)^2}{2(2)-4} =$ und.

(vertical line)

$-3 = t^2 - 4t + 1$ | $8 = t^3$

$0 = t^2 - 4t + 4$ | $t = \pm 2$ | $\boxed{x=-3}$

$0 = (t-2)(t-2)$ | $t = 2$

$t = 2$

84) speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(2t)^2 + (4\cos 4t)^2}$

$= \sqrt{4t^2 + 16\cos^2 4t}\Big|_{t=3} = \boxed{6.884}$

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AP Calculus BC
Quiz ~ Parametric Equations and Vectors
Calculator

Name Fernando Trujano
Date Feb 7th Per 4

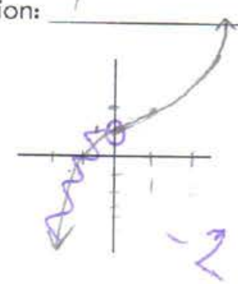
1. Eliminate the parameter: $x = e^t$ and $y = e^{3t} + 1$ and sketch a graph.

t	-2	-1	0	1	2
x	.135	.37	1	2.7	7.3
y	1.00	1.04	2	21.886	404.43

$x = e^t$

$x^3 = e^{3t}$ $y = x^3 + 1$

Equation: $y = x^3 + 1$



-2

2. Find the slope of the parametric equations $x = \sqrt{t}$ and $y = 3t - 1$ when t = 1.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{\frac{1}{2\sqrt{t}}} = 6\sqrt{t} =$

Slope = 6

3. A particle moves on a plane so that at any time $t > 0$ its x-coordinate is $3t^5 - t^4 + 8$ and its y-coordinate is $(-t+3)^3$. What is the acceleration vector of the particle at $t = 1$?

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3t^2 + 18t - 27}{15t^4 - 4t^3}$

$(-t+3)(-t+3)$

$(t^2 - 6t + 9)(-t+3)$

$-t^3 + 3t^2 + 6t^2 - 18t - 9t + 27$

$-t^3 + 9t^2 - 27t + 27$

$\frac{dy}{dx}\left[\quad \right]\Big|_{t=1} = 5.851$

$v(t) = \langle 15t^4 - 4t^3, 3(-t+3)^2(-1) \rangle$

$a(t) = \langle 60t^3 - 12t^2, -6(-t+3)(-1) \rangle$

$\langle 48, 12 \rangle$

$a(1) =$ ~~5.851~~ -10

4. A parametric curve is given by $x = 2t + 4\cos\theta$ and $y = -1 + \sin\theta$, $0 \le t < 2\pi$. At what t-values does the curve have a horizontal tangent line? Vertical tangent line? slope = undefined

slope = 0

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos\theta}{2 - 4\sin\theta}$

H: $\cos\theta = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

V: $2 - 4\sin\theta = 0$
$\sin\theta = \cancel{-\frac{1}{2}}$ $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Horizontal: $\frac{\pi}{2}, \frac{3\pi}{2}$

Vertical: ~~$\frac{7\pi}{6}, \frac{11\pi}{6}$~~ -3
$\frac{\pi}{6}, \frac{5\pi}{6}$

5. Given $x = t^2$, $y = \ln t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/t}{2t} = \frac{1}{2t^2}$

$\frac{d}{dx}\left[\frac{1}{2}t^{-2}\right] = \frac{1}{2}(-2)t^{-3} \cdot \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = t\,\frac{dt}{dx} = \frac{t}{2t} = 1/2$ $= -1/2t^4$

$\frac{dy}{dx} = 1/2t^2$

$\frac{d^2y}{dx^2} =$ ~~1/2~~ -6

-21

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AP Calculus BC
Quiz ~ 9.1, 9.2 ~ Sequences, Series and Convergence

Name Fernando Trejano

1. Find the general term of the sequence (a_n), starting with n=1: $2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \ldots$

$a_1 = 2$

$R = \frac{n+1}{n^2}$

$\cancel{a_n = 2+(n-1)\left(\frac{n+1}{n^2}\right)}$ $\quad a_n = 2\left(\frac{n+1}{n^2}\right)^{n-1}$

1 2 3 4 5

$a_n = \frac{n+1}{n^2}$

Determine if the sequence converges, and if so, find its limit. $\frac{n+1}{n^2}$

~~Geometric~~

0

L'Hopital

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

2. Find the general term of the sequence (a_n), starting with n=1: $7, \frac{-14}{3}, \frac{28}{9}, \frac{-56}{27}, \ldots$

$a_1 = 7$

$R = -\frac{2}{3}$

$a_n = 7\left(-\frac{2}{3}\right)^{(n-1)}$

$a_n = 7\left(-\frac{2}{3}\right)^{n-1}$

3. List the first five terms of the sequence given by $a_n = 2+(-1)^n$.

$a_1 = 1 \quad a_2 = 3 \quad a_3 = 1 \quad a_4 = 3 \quad a_5 = 1$

1, 3, 1, 3, 1

Determine if the sequence converges, and if so, find its limit.

diverges by oscillation

$\lim_{n\to\infty} 2+(-1)^n = DNE$ diverges by oscillation

4. List the first five terms of the sequence given by $a_n = \frac{1}{n}\sin n\pi$. $\frac{1}{2}\sin 2\pi$, $\frac{1}{3}\sin 3\pi$

$a_1 = 0 \quad a_2 = 0 \quad a_3 = 0 \quad a_4 = 0 \quad a_5 = 0$

0, 0, 0, 0, 0

Determine if the sequence converges, and if so, find its limit.

converges to 0

$\lim_{n\to\infty} \frac{1}{n}\sin n\pi = 0$ converges to 0

5. Determine whether $\sum_{k=1}^{\infty} \frac{e}{\pi^k}$ converges or diverges, and by which test. If it converges, find its sum.

Converges by Geometric Series Test

if converges, sum= $e/(1-\frac{1}{\pi})$

-3 -1 need to simplify

$a_1 r^n$

$\lim_{k\to\infty} \frac{e}{\pi^k} = 0$ nth term

$1-\frac{1}{\pi}$

$\sum_{k=1}^{\infty} e\left(\frac{1}{\pi}\right)^k$ $R < 1$ (converges)

$a_1 \quad R$

$S = \frac{a_1}{1-R} = \frac{e}{1-\left(\frac{1}{\pi}\right)}$

-4

AP Calculus BC

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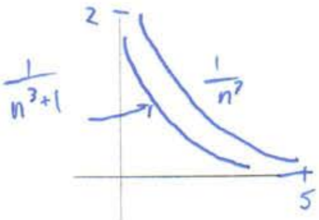
Section 9.4 ~ Comparison of Series

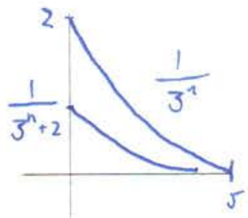
[use calc and just graph $\frac{1}{n^3}$ and $\frac{1}{n^3+1}$ - first quadrant]

Compare the graphs of...

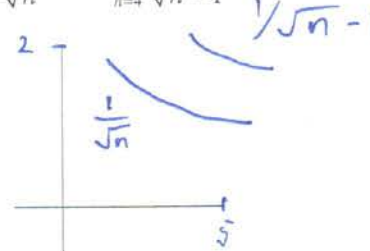
$\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$



$\sum_{n=1}^{\infty} \frac{1}{3^n}$ and $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$



$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$



For each of the series, there is another one for which all values are greater or all smaller

We can compare series to determine convergence or divergence.

- if the "bigger" series converges, then "smaller" converges.
- if the "smaller" series diverges, then "bigger" diverges.

Direct Comparison Test

If $a_n \geq 0$ and $b_n \geq 0$, and all terms are positive,

1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

Determine whether the following series converge or diverge.

Ex) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\quad \frac{1}{n^3}$ p series $\quad P=3 \quad P>1 \quad \therefore$ Converge by P-series

know: $0 \leq \frac{1}{n^3+1} < \frac{1}{n^3} \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges by direct comparison test

Ex) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ compare to $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ geom $\quad R=\frac{1}{3} \quad |R|<1 \quad \therefore$ Converge by Geometric Series

know: $0 < \frac{1}{3^n+2} < \frac{1}{3^n} \quad \therefore \sum_{n=1}^{\infty} \frac{1}{3^n+2}$ converges by DCT

Ex) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$ compare to $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}}$ $\quad p=1/2 \quad p \leq 1 \quad \therefore$ diverge by p-series

know: $0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1} \quad \therefore \sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges by DCT

* not every series has a simple comparison - if that is the case then you may have to use the integral test

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Section 9.6 ~ Ratio and Root Tests

↳ Factorial or variable in exponent

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Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

This test is especially good for series involving factorials and exponential functions.

Determine whether the series converge or diverge.

Ex) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{2^n/n!}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{\cancel{2^n}\cdot 2\cdot\cancel{n!}}{(n+1)\cancel{n!}\cdot\cancel{2^n}} = 0$

∴ converge by ratio test

Ex) $\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$ $\lim_{n\to\infty}\left|\frac{(n+1)^2 3^{n+2}}{2^{n+1}}\cdot\frac{2^n}{n^2 3^{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)^2\cdot 3}{2n^2} = 3/2$

∴ diverges by ratio test

Ex) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$ $\lim_{n\to !}\frac{(n+2)!}{3^{n+1}}\cdot\frac{3^n}{(n+1)!} = \lim_{n\to\infty}\frac{(n+2)\cancel{(n+1)!}\,\cancel{3^n}}{\cancel{3^n}\,3\,\cancel{(n+1)!}} = \infty$

∴ Diverges by ratio test

13) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ $\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\,2^{n+1}}{(n+1)!}\cdot\frac{n!}{\cancel{(-1)^n}\,2^n}\right| = \lim_{n\to\infty}\frac{2}{n+1} = 0$

∴ converge by ratio test

Name: ______________________

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$

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Review of Series ~ 9.1-9.6

Determine the convergence or divergence of each series. Tell which test you use, and justify your answer. You must use each of the ten tests at least once.

__________ 1. $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

__________ 2. $\sum_{n=1}^{\infty} \frac{5n+3}{2n-1}$

__________ 3. $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$

__________ 4. $\sum_{n=1}^{\infty} (-1.2)^n$

__________ 5. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

__________ 6. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

__________ 7. $\sum_{n=1}^{\infty} \frac{3n-2}{n^2-4n+7}$

__________ 8. $\sum_{n=3}^{\infty} \left(\frac{3n}{n-2} \right)^n$

__________ 9. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n} \right)^n$

__________ 10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^3+6}$

__________ 11. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$

__________ 12. $\sum_{n=1}^{\infty} \frac{5^n}{n^2 3^n}$

__________ 13. $\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n}$

__________ 14. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

Review of Series 9.1 - 9.6

1. $\sum_{n=1}^{\infty} \frac{1}{n^{.9}}$ $p = .9$ $.9 < 1$

diverges by P-series

2 $\sum_{n=1}^{\infty} \frac{5n+3}{2n-1}$

$\lim_{n\to\infty} \frac{5n+3}{2n-1} = \lim_{n\to\infty} \frac{5}{2} = \frac{5}{2} \neq 0$

diverges by n^{th} test

3 $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) \ldots = 1$

converges by telescoping series

4 $\sum_{n=1}^{\infty} (-1.2)^n = y$

~~$\ln y = \lim_{n\to\infty} n \ln$~~ $R = -1.2$

~~$R < 1$~~ $|R| > 1$ diverges by Geometric S.

5 $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ compare to $\frac{1}{\sqrt{n}}$ $p = 1/2$

$p < 1$

diverges by P-series

diverges by direct comparison

6 $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \frac{\cancel{2^n}(2)\cancel{n!}}{(n+1)\cancel{n!}\,\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{n+1} = \infty$

diverges by Ratio Test

7 $\sum_{n=1}^{\infty} \dfrac{3n-2}{n^2-4n+7}$

$$\cancel{\lim_{n\to\infty} \frac{3n-2}{n^2-4n+7} = \lim_{n\to\infty} \frac{3}{2n-4} = 0}$$

compare to $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{3n-2}{n^2-4n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^2-2n}{n^2-4n+7} = \lim_{n\to\infty} \frac{6n-2}{2n-4} = \lim_{n\to\infty} \frac{6}{2} = 6/2$$

Diverges ~~Converges~~ by limit comparison

8 $\sum_{n=3}^{\infty} \left(\dfrac{3n}{n-2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{n-2}\right)^n} = \lim_{n\to\infty} \frac{3n}{n-2} = 3$$

$3 > 1$

diverges ~~converges~~ by n^{th} root

9 $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ $\cancel{\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} 1+\frac{1}{n}}$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

diverges by n^{th} term

10 $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} n^3}{n^3+6}$ compare to $\dfrac{n^3}{n^3+6}$

$$\lim_{n\to\infty} \frac{n^3}{n^3+6} = \frac{6n}{6n} = 1$$

diverges by n^{th} term

11 $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ $(\ln x)^{-4}$

$\int (\ln x)^{-4}\,dx$

$\frac{}{-3\ln x^3}$

$\int_2^{\infty} \frac{1}{x} \cdot \frac{1}{(\ln x)^4}\,dx$ $\ln|x| \cdot \frac{1}{(\ln x)^3}$

$\cancel{\lim_{n\to\infty} \ln|x| \cdot \frac{3}{(\ln x)^3}\Big]_2^n}$

$\lim_{c\to\infty} \frac{-1}{3\ln x}$

12 $\sum_{n=1}^{\infty} \frac{5^n}{n^2 3^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{5^n}{n^2 3^n}} = \lim_{n\to\infty} \frac{5}{n^{2/n}\,3} = 5/3$

$5/3 > 1$

diverges by Root Test

13 $\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{e^{3n}}{n^n}} = \lim_{n\to\infty} \frac{e^3}{n} = 0$

$0 < 1$

converges by Root Test

14 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$, look at $\frac{1}{\ln n}$

$\lim_{n\to\infty} \frac{1}{\ln n} = 0$

Converges by alt. Series Test.

$\frac{1}{\ln(n+1)} < \frac{1}{\ln}$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$$

$u = \ln n$

$du = \frac{1}{n} dn$

$\int \frac{1}{n(\ln n)^4}$

$\int \frac{1}{u^4}\, du$

$\int u^{-4}$

$\frac{u^{-3}}{-3}$

$$\lim_{b \to \infty} \frac{-1}{3(\ln n)^3} \Big]_2^{\infty}$$

$$0 - \left[\frac{-1}{3(\ln 2)^3}\right] = \text{some value}$$

converge $\therefore$ Integral test

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Section 9.2 ~ Series and Convergence

Geometric Series Test, n^{th} Term Test, Telescoping Series Test

I. Review of Sequences (Section 9.1)

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:
1) If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity.
2) If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity.
3) If $\lim_{n\to\infty} a_n = c$, a finite number, then $\{a_n\}$ converges to c.
4) If $\lim_{n\to\infty} a_n$ oscillates between two numbers, then $\{a_n\}$ diverges by oscillation.

Determine whether the following sequences converge or diverge.

1. $a_n = \frac{n}{n+1}$ $\lim_{n\to\infty} \frac{n}{n+1} = 1$ Converge

2. $a_n = \frac{1}{2^n}$ $\lim_{n\to\infty} \frac{1}{2^n} = 0$ Converge

3. $a_n = 3 + (-1)^n$ $\lim_{n\to\infty} 3 + (-1)^n = DNE$ Diverges

4. $a_n = \frac{n}{1-2n}$ $\lim_{n\to\infty} \frac{n}{1-2n} = -\frac{1}{2}$ Converges

5. $a_n = \frac{\ln n}{n}$ $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ Converges

6. $a_n = \frac{n!}{(n+2)!}$ $\lim_{n\to\infty} \frac{n!}{(n+2)!} = \lim_{n\to\infty} \frac{n!}{(n+1)(n+2)n!} = 0$ Converges

7. $a_n = \frac{n!}{(n-1)!}$ $\lim_{n\to\infty} \frac{n!}{(n-1)!} = \lim_{n\to\infty} \frac{n(n-1)!}{(n-1)!} = DNE$ diverges

8. $a_n = \frac{n+(-1)^n}{n}$ $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1$ Converges

9. $a_n = \frac{(-1)^n(n-1)}{n}$ $\lim_{n\to\infty} \frac{(-1)^n(n-1)}{n} = \lim_{n\to\infty} (-1)^n\left(1-\frac{1}{n}\right) = DNE$ diverges

10. $a_n = \left(1+\frac{1}{n}\right)^n$ $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

Write an expression for the n^{th} term.

1. 2, 5, 10, 17, 26, ... (1 2 3 4 5)
$a_n = n^2+1$

2. 3, 8, 13, 18, ...

Type: Arithmetic
$a_1 = 3$
$d = 5$

Formula: $a_n = a_1 + (n-1)d$

$a_n = 5n-2$

3. 36, -18, 9, -4.5, ...

Type: Geometric
$a_1 = 36$
$R = -\frac{1}{2}$

Formula: $a_n = a_1 (R)^{n-1}$

$a_n = 36(-\frac{1}{2})^{n-1}$

Review of Series 9.1-9.6

1) $\sum_{n=1}^{\infty} \frac{1}{n^{.9}}$ $\quad p=.9<1$ $\quad \therefore$ series <u>diverges</u> by p-series

2) $\sum_{n=1}^{\infty} \frac{5n+3}{2n-1}$ $\quad \lim_{n\to\infty} \frac{5n+3}{2n-1} = \frac{5}{2} \neq 0$

$\therefore$ series <u>diverges</u> by n^{th} term test

3) $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = (1-1/3)+(1/3-1/5)+(1/5-1/7)+\ldots$

$\therefore$ series <u>conv. to 1</u> by tele. series test

4) $\sum_{n=1}^{\infty} (-1.2)^n$ $\quad |r|=1.2>1$ $\quad \therefore$ <u>diverges</u> by geom. series test

5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ $\quad$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent p-series

$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$ $\quad \therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ <u>diverges</u> by direct comparison

6) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

$\therefore$ <u>converges</u> by ratio test

7) $\sum_{n=1}^{\infty} \frac{3n-2}{n^2-4n+7}$ $\quad$ compare to $\sum \frac{1}{n}$, a divergent p-series

$\lim_{n\to\infty} \frac{\frac{3n-2}{n^2-4n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^2-2n}{n^2-4n+7} = 3$ (finite & positive)

$\therefore \sum_{n=1}^{\infty} \frac{3n-2}{n^2-4n+7}$ <u>diverges</u> by limit comparison

8) $\sum_{n=3}^{\infty} \left(\frac{3n}{n-2}\right)^n$ $\quad \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n}{n-2}\right)^n\right|} = \lim_{n\to\infty} \frac{3n}{n-2} = 3 > 1$

$\therefore$ <u>diverges</u> by root test

9) $\sum_{n=1}^{\infty} (1+1/n)^n$ $\quad \lim_{n\to\infty} (1+1/n)^n = e \neq 0$

<u>diverges</u> by n^{th} term test

10) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^3+6}$ $\lim_{n\to\infty} \frac{n^3}{n^3+6} = 1 \neq 0$ not an alt. series!

$\therefore$ diverges by n^{th} term test

11) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ let $f(x) = \frac{1}{x(\ln x)^4}$ → pos, cont, dec

$\int_2^{\infty} \frac{1}{x(\ln x)^4}\,dx = \lim_{c\to\infty} \int_2^c (\ln x)^{-4} \cdot \frac{1}{x}\,dx$

$= \lim_{c\to\infty} \left. \frac{-1}{3(\ln x)^3} \right]_2^c = \frac{-1}{3(\ln \infty)^3}^{\nearrow 0} - \frac{-1}{3(\ln 2)^3}$

converges by integral test

12) $\sum_{n=1}^{\infty} \frac{5^n}{n^2 3^n}$ $\lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{n^2 3^n}}{\frac{5^n}{n^2 3^n}} \right| = \lim_{n\to\infty} \frac{5^{n+1}}{n^2 3^{n+1}} \cdot \frac{n^2 3^n}{5^n}$

$= \lim_{n\to\infty} \frac{5n^2}{3(n+1)^2} = \frac{5}{3} > 1$ diverges by ratio test

13) $\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{e^3}{n}\right)^n$ $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{e^3}{n}\right)^n\right|} = \lim_{n\to\infty} \frac{e^3}{n} = 0 < 1$

converges by root test

14) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ $\left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{\ln n} = 0 \\ \frac{1}{\ln(n+1)} < \frac{1}{\ln n} \end{array}\right\}$ converges by alt. series test

also on test:

max error / remainder for integral test $= \int_N^{\infty} f(x)\,dx$

finite sum $1\to N$ $\leq$ infinite sum $1\to\infty$ $\leq$ finite sum $1\to N$ + max error integral $N\to\infty$

max error / remainder for alt series test $= a_{N+1}$

finite sum $1\to N$ − error a_{n+1} $\leq$ infinite sum $1\to\infty$ $\leq$ finite sum $1\to N$ + error a_{N+1}

absolute vs. conditional convergence for alt. series.

9.5 pg 636 # 11-27 odd, 37, 39, 41, 45-61 odd, 66

11) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$, $\frac{1}{n+1} < \frac{1}{n}$ } converges by alt series test

13) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ $\quad \lim_{n\to\infty} \frac{1}{2n-1} = 0$, $\frac{1}{2(n+1)-1} = \frac{1}{2n+1} < \frac{1}{2n-1}$ } conv by alt series test

15) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{n^2+1}$ $\quad \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

$\therefore$ diverges by n^{th} term test

17) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\quad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ } converges by alt series test

19) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$ $\quad \lim_{n\to\infty} \frac{n+1}{\ln(n+1)} \overset{LHOP}{=} \lim_{n\to\infty} \frac{1}{1/n+1} = \infty$

$\therefore$ diverges by n^{th} term test

21) $\sum_{n=1}^{\infty} \sin \frac{(2n-1)\pi}{2} = \sin\frac{\pi}{2} + \sin\frac{3\pi}{2} + \sin\frac{5\pi}{2} + \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1}$ $\quad \lim_{n\to\infty} 1 = 1$ $\therefore$ div by n^{th} term test

23) $\sum_{n=1}^{\infty} \cos n\pi = \cos\pi + \cos 2\pi + \cos 3\pi + \dots$

$\sum_{n=1}^{\infty} (-1)^n$ $\quad \lim_{n\to\infty} 1 = 1$ $\therefore$ div by n^{th} term test

25) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ $\quad \lim_{n\to\infty} \frac{1}{n!} = 0$, $\frac{1}{(n+1)!} < \frac{1}{n!}$ } converges by alt series test

27) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$ $\quad \lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$, $\frac{\sqrt{n+1}}{n+3} < \frac{\sqrt{n}}{n+2}$ } converges by alt series

true for $n \geq 2$

37) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$

$|R_N| \le a_{N+1} = \frac{1}{(N+1)!} < .001$

$\frac{1}{6!} = .001$ $\quad \frac{1}{7!} = .000198$ $\quad$ so $N=6$

let $n=6$ (to get 7 terms that are added b/c start w/ $n=0$)

$S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!} = 1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + 1/720 = .368 = \frac{1}{e}$ ✓

39) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin 1$

$|R_N| = a_{n+1} = \frac{1}{[2(N+1)+1]!} = \frac{1}{(2N+3)!} < .001$

when $N=1$ $\frac{1}{5!} = .0083$ $\quad$ when $N=2$ $\frac{1}{7!} = .00019$

when $n=2$, we actually add up 3 terms: $n=0$, $n=1$, $n=2$

$\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!} = \underset{n=0}{1} - \underset{n=1}{\frac{1}{3!}} + \underset{n=2}{\frac{1}{5!}} = 1 - \frac{1}{6} + \frac{1}{120} = .8416 \approx \sin 1$ ✓

41) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

$|R_N| < a_{N+1} = \frac{1}{N+1} < .001$ $\qquad$ valid when $N = 1000$

$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = \text{sum}(\text{seq}(\frac{(-1)^{x+1}}{x}, x, 1, 999)) = .693 = \ln 2$ ✓

limit on TI-83+

45) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

$$|R_N| < a_{n+1} = \frac{1}{2(N+1)^3-1} < .001$$

$$\frac{1}{2(N+1)^3-1} < \frac{1}{1000}$$

$$1000 < 2(N+1)^3-1$$

$$\frac{1001}{2} < (N+1)^3$$

$$500.5 < (N+1)^3 \qquad 8^3 = 512$$

$$N = 7$$

47) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$ $\left.\begin{matrix} \lim_{n\to\infty} \frac{1}{(n+1)^2} = 0 \\ \frac{1}{(n+2)^2} < \frac{1}{(n+1)^2} \end{matrix}\right\}$ converges by alt series

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{(n+1)^2}\right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ compare to $\frac{1}{n^2}$, $p=2$, converges

since $0 < \frac{1}{(n+1)^2} < \frac{1}{n^2}$, $\sum \frac{1}{(n+1)^2}$ conv by direct comp.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$ is absolutely convergent

49) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ $\left.\begin{matrix} \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \\ \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \end{matrix}\right\}$ converges by alt series

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $p = \frac{1}{2}$, diverges by p-series test

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent

51) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$ $\quad \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \neq 0$

$\therefore$ diverges by n^{th} term test

53) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ $\quad \left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{\ln n} = 0 \\ \frac{1}{\ln(n+1)} < \frac{1}{\ln n} \end{array}\right\}$ converges by alt series test

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ compare to $\frac{1}{n}$ $p=1$, diverges

since $\frac{1}{n} < \frac{1}{\ln n}$, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by direct comp

$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent

55) $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3-1}$ $\quad \left.\begin{array}{l} \lim_{n\to\infty} \frac{n}{n^3-1} = 0 \\ \frac{n+1}{(n+1)^3-1} < \frac{n}{n^3-1} \end{array}\right\}$ converges by alt series test

checked graphically

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n n}{n^3-1} \right| = \sum_{n=2}^{\infty} \frac{n}{n^3-1}$ compare to $\sum_{n=2} \frac{1}{n^2}$ $p=2$, converges

$\lim_{n\to\infty} \frac{\frac{n}{n^3-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1$ (finite & pos)

so $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ converges by limit comparison

$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3-1}$ is absolutely convergent

57) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ $\quad \left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{(2n+1)!} = 0 \\ \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} \end{array}\right\}$ converges by alt series test

$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(2n+1)!} \right| = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ compare to 2^n conv. geom series

since $0 < \frac{1}{(2n+1)!} < 2^n$, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ conv by dir. comp

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely

59) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

$\left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{n+1} = 0 \\ \frac{1}{n+2} < \frac{1}{n+1} \end{array}\right\}$ converges by alt series

$\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n+1}$ compare to $\sum_{n=0}^{\infty} \frac{1}{n}$ $p=1$, diverges

$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ (pos, finite) $\therefore \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges

$\therefore \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$ is conditionally convergent

61) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

$\left.\begin{array}{l} \lim_{n\to\infty} \frac{1}{n^2} = 0 \\ \frac{1}{(n+1)^2} < \frac{1}{n^2} \end{array}\right\}$ converges by alt series test

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $p=2$, converges

$\therefore \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ is absolutely convergent

66) (b) the partial sums alternate above and below the horizontal line representing the sum

AP Calculus BC

Name Key

Section 9.8 ~ Power Series (Radius and Interval of Convergence)

Power Series

If x is a variable, then an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

is called a power series.

have x and n. 10 tests are for series of constants (only n, no x)

$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$ is a power series centered at c, where c is a constant.

For a power series centered at c, precisely one of the following is true:

1) The series converges only at $x = c$
2) The series converges for all x.
3) There exists an R>0 such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$.

R is called the *radius of convergence* of the power series.

In part 1, the radius is 0. (b/c when $x=c$, then $c-c=0$)

In part 2, the radius is ∞. (R must be larger than anything you obtain from $x-c$.

The corresponding domain is called the *interval of convergence.*

Intervals of Convergence

$\sum_{n=0}^{\infty} a(r)^n$ converges when $-1 < r < 1$. → geometric series $|r| < 1$

Over what intervals do the following series converge:

1. $\sum_{n=0}^{\infty} 5(x)^n$ $\quad -1 < x < 1$

2. $\sum_{n=0}^{\infty} 5(3x)^n$ $\quad -1 < 3x < 1$
$-1/3 < x < 1/3$

will it converge when $x = 1/3$? $x = -1/3$? how do you know? geometric series test

3. $\sum_{n=0}^{\infty} 3(2x-1)^n$
$-1 < 2x-1 < 1$
$0 < 2x < 2$
$0 < x < 1$

What if the series is not a geometric series?

use the ratio test!

ratio test: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, series conv. $=1$ inconclusive

In the following examples, find the radius of convergence and the interval of convergence.

4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)(2)^{(n+1)}}}{\frac{(-1)^{n+1}(x-5)^n}{n2^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{x-5}{2}\cdot\frac{\overset{\nearrow 1}{\cancel{n}}}{\cancel{n+1}}\right|$$

$= \frac{x-5}{2}$ series conv. by ratio test if ... $\frac{x-5}{2} < 1$

$-1 < \frac{x-5}{2} < 1$

$-2 < x-5 < 2$

$3 < x < 7$

interval of convergence

radius of conv $= 2$

test endpoints:

when $x=3$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n(2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$

div by p-series

when $x=7$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

1) $\lim_{n\to\infty}\frac{1}{n}=0$

2) $\frac{1}{n+1} < \frac{1}{n}$ ✓

conv by alt series

$\boxed{\text{interval of convergence: } 3 < x \leq 7 \text{ or } (3,7]}$

5. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\frac{x^{2(n+1)+1}}{[2(n+1)+1]!}\cdot\frac{(2n+1)!}{x^{2n+1}} = \lim_{n\to\infty}\frac{x^{2n+3}(2n+1)!}{x^{2n+1}(2n+3)!} = \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)} = 0 < 1$$

no matter what x is, always conv.

radius of conv $= \infty$ $\boxed{\text{interval of conv: } -\infty < x < \infty}$

6. $\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$ geometric series

$\left|\frac{x}{3}\right| < 1$

$-1 < \frac{x}{3} < 1$

$-3 < x < 3$

can test endpts (but don't need to)

• when $x=-3$, $\sum_{n=0}^{\infty}(-1)^n$ $|r|=1\geq 1$ $\therefore$ div by geom series test

• when $x=3$, $\sum_{n=0}^{\infty}(1)^n$ $|r|=1\geq 1$ $\therefore$ "

radius of conv $= 3$ $\boxed{\text{interval of conv: } -3 < x < 3}$

7. $\sum_{n=0}^{\infty}n!(x-3)^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n}\right| = \lim_{n\to\infty}\left|(n+1)(x-3)\right| = |x-3|\lim_{n\to\infty}(n+1) = \infty > 1$$

diverges $\forall x$ except $x=3$

radius of conv $= 0$ $\boxed{\text{converges only if } x=3}$

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Calculus BC
Quiz 9.7-9.8
Calculator

Name Fernando Trejano
Date ______________ Period____

1) Find the third degree Maclaurin polynomial for the function $f(x)=e^{-3x}$

$P_3(x) = 1 + -3x + \frac{9x^2}{2!} + \frac{-27x^3}{3!}$ ✓

$f(x) = e^{-3x}$ $f(0) = 1$
$f'(x) = -3e^{-3x}$ $f'(0) = -3$
$f''(x) = 9e^{-3x}$ $f''(0) = 9$
$f'''(x) = -27e^{-3x}$ $f'''(0) = -27$

$1 + -3x + \frac{9x^2}{2!} + \frac{-27x^3}{3!}$

2) Find the third degree Taylor polynomial centered at $c=5$ for the function $f(x)=\sqrt{x}$.

$(x-5)$

$P_3(x) = \sqrt{5} + \frac{.224(x-5)}{1} + \frac{-.0224(x-5)^2}{2!} + \frac{.0067(x-5)^3}{3!}$

-5 no rounding

$f(x) = x^{1/2}$ $f(5) = \sqrt{5}$
$f'(x) = \frac{1}{2}x^{-1/2}$ $f'(5) = .224$
$f''(x) = -\frac{1}{4}x^{-3/2}$ $f''(5) = -.0224$
$f'''(x) = 3/8\, x^{-5/2}$ $f'''(5) = .0067$

$\sqrt{5} + .224(x-5) + \frac{-.0224(x-5)^2}{2!} + \frac{.0067(x-5)^3}{3!}$

$\frac{f(0)}{1}$ $\frac{f'(0)}{}$ $\frac{f''(0)}{2!}$

3) Given that $P_2(x)=-3+x+5x^2$ is a second degree Maclaurin polynomial, does f have a relative maximum, relative minimum, or neither at $x=0$? Justify your answer.

-7

$f(0) = -3$
$f'(0) = 1$
$f''(0) = 10$ CC↑ ~~Relative min b/c $f''(0) = 10$~~ slope = 0

neither because slope is not 0
↳ not critical #

-12

AP Calculus BC

Section 9.10 ~ Taylor and Maclaurin Series

Name Fernando Trujano

Taylor Series centered at $x=c$

└ infinite # of turns

start at $n=0$

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$, then the series is called a Maclaurin Series.

Recall: There are several techniques for finding a power series to represent a given function.

1) Recognizing it as a sum of a geometric power series (Section 9.9)
2) Adding or subtracting two power series.
3) Multiplying or dividing a power series by a variable and/or a constant.
4) Differentiating or integrating a power series.
5) Substituting into a known series.

} Focus for today

Sometimes to get a power series for a function it helps to manipulate a known series.

Ex) Find a Taylor series for $f(x)=e^{5x}$ centered at $c=2$. Give the first four nonzero terms and the general term.

$f(x)=e^{5x}$ $f(2)=e^{10}$

$f'(x)=5e^{5x}$ $f'(2)=5e^{10}$

$f''(x)=25e^{5x}$ $f''(2)=25e^{10}$

$f'''(x)=125e^{5x}$ $f'''(2)=125e^{10}$

$f^{n}(x)=5^ne^{5x}$ $f^{n}(2)=5^ne^{10}$

$$P_n(x)=e^{10}+5e^{10}(x-2)+\frac{25e^{10}(x-2)^2}{2!}+\frac{125e^{16}(x-2)^3}{3!}+\ldots$$

General $\displaystyle\sum_{n=0}^{\infty}\frac{5^ne^{10}(x-2)^n}{n!}$

There are three special Maclaurin series you must know. There are the series for e^x, $\sin x$, and $\cos x$.

$\to c=0$

Derive a series for e^x:

$$P_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!} \quad \text{or} \quad \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Derive a series for $\sin x$:

$f'(x)=\cos x$ $f(0)=1$

$f''(x)=-\sin x$ $f''(0)=0$

$f'''(x)=-\cos x$ $f'''(0)=-1$

$$P_n(x)=0+x+\frac{0x^2}{2!}+\frac{-x^3}{3!}+\ldots \quad \text{or} \quad \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}$$

$$P_n(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^nx^{2n+1}}{(2n+1)!}$$

Derive a series for $\cos x$:

$f(0)=1$

$f'(x)=-\sin x$ $f'(0)=0$

$f''(x)=-\cos x$ $f''(0)=-1$

$f'''(x)=\sin x$ $f'''(0)=0$

$$P_n(x)=1+0x+\frac{-1x^2}{2!}+\frac{0x^3}{3!}+\ldots+\frac{(-1)^nx^{2n}}{(2n)!} \quad \text{or} \quad \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!}$$

AP Calculus BC

Name Fernando Trujano

Section 9.7 ~ Lagrange Form of the Remainder

(also called Lagrange Error Bound or Taylor's Theorem Remainder)

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$f(x) = P_n(x) + R_n(x)$ so $R_n(x) = f(x) - P_n(x)$

Written in words:

Function = Polynomial + Remainder so Remainder = Function - Polynomial

<u>Taylor's Theorem:</u>

If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$\underbrace{f(x)}_{\text{Actual}} = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \overbrace{R_n(x)}^{\text{error}}$$

where the remainder $R_n(x)$ (or error) is given by

derivative

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$ **The Lagrange Remainder**

Historically the remainder was not due to Taylor but to a French mathematician, Joseph Louis Lagrange (1736-1813). For this reason, $R_n(x)$ is called the **Lagrange form** of the remainder.

When applying Taylor's Formula, we would not expect to be able to find the exact value of z. Rather, we would attempt to find a maximum bound for the $(n+1)^{th}$ derivative from which we will be able to tell how large the remainder or error, $R_n(x)$, is.

Ex) The 3rd Maclaurin polynomial for $\sin x$ is given by $P_3(x) = x - \frac{x^3}{3!}$. (Similar to HW and test)

$c = 0$

Use Taylor's Theorem to approximate sin(0.1) by $P_3(0.1)$ and determine the accuracy of the approximation.

① Approximate

$P_3(.1) = .0998$

② Remainder / error

$R_3(x) = \frac{f^4(z)}{4!} x^4 \le \frac{1 x^4}{4!}$

$R_3(.1) \le \frac{.1^4}{4!}$

$R_3(.1) \le .000004167$

Review

1) (a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$\cos(.8) \approx P_4(.8) = .697$

$|R_4(.8)| \le \frac{\cos^{(5)}(z)}{5!}(x-0)^5$ max value of cos x is 1

$\le \frac{1}{5!} \cdot x^5 = \frac{1}{120} \cdot (.8)^5$

$\le .003$

(b) $.697 - .003 \le \cos(.8) \le .697 + .003$
$.694 \le \cos(.8) \le .700$

(c) yes; .695 is in the interval

2) (a) $f(x) = x^{5/2}$ $f(4) = 32$
$f'(x) = \frac{5}{2}x^{3/2}$ $f'(4) = 20$
$f''(x) = \frac{15}{4}x^{1/2}$ $f''(4) = 15/2$

$P_2(x) = 32 + 20(x-4) + \frac{15/2}{2!}(x-4)^2$
$= 32 + 20(x-4) + \frac{15}{4}(x-4)^2$

$f(4.3) \approx P_2(4.3) = 38.338$ (store value!)

(b) $|R_2(x)| \le \frac{f^{(3)}(z)}{3!}(x-4)^3$

$= \frac{15/16}{3!}(x-4)^3$

$|R_2(4.3)| \le \frac{5}{32}(4.3-4)^3$

$\le .0042...$

$f'(x) = \frac{5}{2}x^{3/2}$ $f''(x) = \frac{15}{4}x^{1/2}$ $f'''(x) = \frac{15}{8x^{1/2}}$

→ graph $f'''(x)$ and look at [4, 4.3]
→ graph is decreasing
→ since $f'''(x)$ is dec on interval, the max value of the deriv. occurs at $f'''(4)$ and $f'''(4) = 15/16$

(c) $38.334 \leq f(4.3) \leq 38.342$

(d) no, 38.3 does not lie in the interval

(e) $|f(4.3) - P_3(4.3)|$ use stored values

$= |38.34168\ldots - 38.3375\ldots|$

$= .00418$

this is the actual error!

3) (a) $f(5) = 6$ $f'(5) = 8$ $f''(5) = 30$ $f'''(5) = 48$

$P_3(x) = 6 + 8(x-5) + \frac{30}{2!}(x-5)^2 + \frac{48}{3!}(x-5)^3$

$= 6 + 8(x-5) + 15(x-5)^2 + 8(x-5)^3$

(b) $f(5.2) \approx P_3(5.2) = 8.264$

$|R_3(x)| \leq \frac{75}{4!}(x-5)^4$

$|R_3(5.2)| \leq .005$

(c) $8.259 \leq f(5.2) \leq 8.269$

(d) no, 8.254 does not lie in the interval

4) $f(x) = \sin x$ $f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $f'\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $f''\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $f'''\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$ $f^{(4)}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin x \approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \frac{\sqrt{2}}{2 \cdot 3!}\left(x - \frac{3\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!}\left(x - \frac{3\pi}{4}\right)^4 \ldots$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$

$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \frac{x^{24}}{8!} \ldots$

$x\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \ldots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}$

6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ or $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

7) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$

ratio test: $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1}(n+1)^2} \cdot \frac{3^n n^2}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2) \cdot \cancel{n^2}}{3 \cdot \cancel{(n+1)^2}} \right| = \left| \frac{x-2}{3} \right|$

$-1 < \frac{x-2}{3} < 1$

$-3 < x-2 < 3$

$-1 < x < 5$

$\boxed{\text{interval: } -1 \le x \le 5 \quad \text{radius: } 3}$

test endpts:

$x = -1 : \sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n n^2} = \sum_{n=0}^{\infty} \frac{(-1)^{2n} 3^n}{3^n n^2}$

$= \sum_{n=0}^{\infty} \frac{1}{n^2}$ conv by p-series

$x = 5 : \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n n^2}$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$ conv by alt ser.

8) $\sum_{n=0}^{\infty} (2n)!\,(x-5)^n$

ratio test: $\lim_{n\to\infty} \left| \frac{(2n+2)!\,(x-5)^{n+1}}{(2n)!\,(x-5)^n} \right|$

$\lim_{n\to\infty} (2n+2)(2n+1)(x-5) = \infty$

diverges except when $x=5$!

converges only if $x=5$.

9) $f(x) = \frac{7}{4x+1}$ $\quad c = 3$

$= \frac{7}{4(x-3)+1+12} = \frac{7}{13+4(x-3)} = \frac{7/13}{1-\left[\frac{-4}{13}(x-3)\right]}$

$a_1 = 7/13 \qquad r = \frac{-4}{13}(x-3)$

interval: $-1 < \frac{-4}{13}(x-3) < 1$

$\frac{13}{4} > x-3 > \frac{-13}{4}$

$\frac{25}{4} > x > \frac{-1}{4} \Rightarrow \boxed{\frac{-1}{4} < x < \frac{25}{4}}$

power series: $\boxed{\sum_{n=0}^{\infty} \frac{7}{13}\left[\frac{-4}{13}(x-3)\right]^n}$

10) $f(x) = \sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{6n+3}}{(2n+1)!}$

so there can never be a term with x^6!

the coefficient is 0