Supplementary Exercises

1. Solve the following problems by the simplex method for bounded variables:

a)

$$\begin{aligned} \text{Minimize} \quad Z(\mathbf{x}) : \quad &= \quad x_1 + 2x_2 + 3x_3 - x_4 \\ \text{s.t.} \quad & \quad 2x_1 - x_2 + x_3 - 2x_4 \leq 6 \\ & \quad -x_1 + 2x_2 - x_3 + x_4 \leq 8 \\ & \quad 2x_1 + x_2 - x_3 \geq 2 \\ & \quad 0 \leq x_1 \leq 3, 1 \leq x_2 \leq 4, 0 \leq x_3 \leq 8, 2 \leq x_4 \leq 5 \end{aligned}$$

b)

$$\begin{aligned} \text{Maximize} \quad Z(\mathbf{x}) : \quad &= \quad 2x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & \quad 3x_1 + x_2 + x_3 \leq 12 \\ & \quad -x_1 + x_2 \leq 4 \\ & \quad x_2 + 2x_3 \leq 8 \\ & \quad 0 \leq x_1 \leq 3, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 4 \end{aligned}$$

c)

$$\begin{aligned} \text{Maximize} \quad Z(\mathbf{x}) : \quad &= \quad 2x_1 + 3x_2 - 2x_3 \\ \text{s.t.} \quad & \quad x_1 + 3x_2 + x_3 \leq 8 \\ & \quad 2x_1 + x_2 - x_3 \geq 3 \\ & \quad x_1 \leq 4, -2 \leq x_2 \leq 3, x_3 \geq 2 \end{aligned}$$

d)

$$\begin{aligned} \text{Minimize} \quad Z(\mathbf{x}) : \quad &= \quad 3x_1 + 7x_2 + 4x_3 + 5x_4 \\ \text{s.t.} \quad & \quad 2x_1 + x_2 + 3x_3 + 4x_4 \geq 6 \\ & \quad x_1 + 2x_2 + 4x_3 + 2x_4 \geq 6 \\ & \quad 3x_1 + 4x_2 + x_3 + x_4 \geq 6 \\ & \quad 0 \leq x_j \leq 1, \quad j = 1, \ldots, 4 \end{aligned}$$

e)

$$\begin{aligned} \text{Maximize} \quad Z(\mathbf{x}) : \quad &= \quad 4x_1 + 2x_2 + 6x_3 \\ \text{s.t.} \quad & \quad 4x_1 - x_2 \leq 9 \\ & \quad -x_1 + x_2 + 2x_3 \leq 8 \\ & \quad -3x_1 + x_2 + 4x_3 \leq 12 \\ & \quad 1 \leq x_1 \leq 3, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 2 \end{aligned}$$

2. Several warehouses are supplied for a set of factories. The supply available from each factory, the demand at each warehouse and the cost per unit of transportation goods from the factories to the warehouses are summarized in the following tables. Find the optimal solution of each problem, i.e., the solution that minimizes the transportation costs. (Use the Vogel's approximation method, the northwest corner rule and the minimal cost unit method to find an initial basic feasible solution, and compare the performance of the algorithm with each method).

a)

	warehouses 1	2	3	4	Supply
1	20	10	5	15	20
2	12	8	10	9	5
3	11	15	8	9	12
4	15	7	15	6	2
5	10	20	15	10	6
Demand	15	15	5	5	

b)

	warehouses 1	2	3	4	5	Supply
1	32	30	27	26	25	42
2	28	25	22	22	19	40
3	35	36	29	38	25	48
4	20	22	15	17	16	10
Demand	18	50	8	52	12	

c)

	warehouses 1	2	3	4	5	Supply
1	13	9	15	10	12	40
2	11	10	12	12	9	10
3	12	9	11	12	9	20
4	13	12	13	12	10	10
Demand	12	15	20	15	18	

d) (i)

	warehouses 1	2	3	4	Supply
1	4	3	4	2	4
2	8	6	7	5	4
3	6	4	5	3	6
4	7	5	6	4	4
Demand	2	7	3	6	

(ii)

	warehouses 1	2	3	4	Supply
1	5	2	6	5	25
2	7	12	5	6	30
3	8	9	7	8	50
Demand	17	38	20	30	

e) (i)

	warehouses 1	2	3	4	Supply
1	7	2	-1	0	10
2	4	3	2	3	30
3	2	1	3	4	25
Demand	10	15	25	5	

(ii)

	warehouses 1	2	3	4	Supply
1	14	87	48	27	71
2	52	35	21	81	47
3	99	20	71	63	95
Demand	71	35	47	60	

3. Consider the following cost minimization transportation problems in matrix format. Supply and destination points are numbered on the left and top of each table, respectively. For each problem, apply the Vogel's approximation, the northwest corner rule and the minimal cost unit method to construct a starting basic feasible solution, find an optimal solution and compare the performance of algorithm with each initial solution. Is the solution unique?

a)

	1	2	3	4	Supply
1	9	11	11	8	400
2	7	12	14	10	200
3	11	10	12	16	620
Demand	300	340	400	440	

b)

	1	2	3	4	5	Supply
1	80	40	60	30	25	30
2	50	20	40	35	28	30
3	65	50	30	22	26	30
Demand	10	10	20	20	30	

c)

	1	2	3	4	Supply
1	20	19	10	15	32
2	17	15	6	10	23
3	18	14	2	6	30
4	21	23	3	6	47
Demand	70	33	22	7	

d)

	1	2	3	4	Supply
1	15	23	20	25	30
2	14	17	11	17	12
3	14	7	6	10	5
4	8	9	10	5	10
Demand	20	4	10	31	

e)

	1	2	3	4	5	Supply
1	5	2	3	8	10	10
2	7	5	4	5	8	12
3	6	3	7	5	9	12
Demand	4	5	7	9	9	

f)

	1	2	3	4	5	Supply
1	15	4	9	16	11	4
2	10	15	8	14	11	6
3	13	10	13	15	–	9
Demand	3	4	7	4	6	

g)

	1	2	3	4	5	6	Supply
1	30	28	12	15	20	10	80
2	10	15	12	20	25	10	100
3	8	10	6	8	8	10	75
4	20	22	24	20	25	21	120
5	25	20	30	35	32	28	60
6	27	30	25	14	20	26	65
Demand	100	100	50	50	100	100	

h)

	1	2	3	4	Supply
1	10	5	6	10	15
2	8	2	7	6	26
3	12	3	4	8	50
Demand	16	10	30	36	

i)

	1	2	3	4	Supply
1	2	1	4	5	20
2	5	4	3	5	10
3	6	2	1	3	35
4	3	6	2	8	16
Demand	14	20	20	27	

j)

	1	2	3	4	Supply
1	2	3	4	5	10
2	5	4	3	1	15
3	1	3	3	2	21
Demand	6	11	17	12	

4. Solve the following assignment problems with the objective as stated.

a) (*Maximize*)

O_i \ D_j	1	2	3	4	5
1	4	2	0	1	5
2	1	3	5	2	7
3	7	4	2	8	9
4	10	0	3	4	5
5	2	8	9	10	1

b) (*Minimize*)

O_i \ D_j	1	2	3	4	5
1	28	25	35	33	34
2	20	30	23	25	26
3	36	32	36	32	40
4	36	33	37	33	42
5	28	30	33	35	35

c) (*Maximize*)

O_i \ D_j	1	2	3	4	5
1	8	3	7	6	2
2	5	1	4	9	3
3	6	0	1	7	4
4	8	3	8	2	8
5	4	1	5	0	1

d) (*Maximize*)

O_i \ D_j	1	2	3	4	5
1	4	3	9	6	2
2	3	8	6	6	5
3	9	1	7	4	4
4	8	6	7	5	3
5	4	9	5	8	2

e) (*Minimize*)

O_i \ D_j	1	2	3	4	5
1	1	6	4	2	8
2	6	3	4	7	2
3	4	5	6	3	7
4	6	2	8	6	3
5	2	8	5	2	6

5. Consider the following assignment costs tableau. Apply the Hungarian algorithm to find the optimal assignments among the origin points and the destination points.

a)

O_i \ D_j	1	2	3	4	5
1	3	6	4	5	3
2	4	5	4	-2	6
3	0	2	4	1	5
4	4	6	3	3	5
5	6	4	5	2	7

b)

O_i \ D_j	1	2	3	4
1	10	6	11	10
2	18	10	10	16
3	2	9	11	4
4	11	15	5	15

c)

$$\begin{array}{cc} & D_j \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ O_i \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{array}{|ccccc|} \hline 11 & 4 & 11 & 12 & 17 \\ 7 & 3 & 12 & 5 & 14 \\ 3 & 1 & 9 & 3 & 10 \\ 6 & 9 & 14 & 12 & 15 \\ 13 & 9 & 4 & 13 & 7 \\ \hline \end{array} \end{array}$$

d)

$$\begin{array}{cc} & D_j \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ O_i \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{array}{|ccccc|} \hline 23 & 17 & 2 & 27 & 9 \\ 29 & 8 & 3 & 25 & 19 \\ 24 & 34 & 22 & 38 & 5 \\ 13 & 11 & 32 & 15 & 30 \\ 36 & 26 & 4 & 39 & 37 \\ \hline \end{array} \end{array}$$

6. A company has four warehouses and supplies four customers. The distances involved are small, and the company charges its customers in terms of the charges per unit involved in loading at the warehouses and unloading at the destinations according to the following table. Formulate the problem, and use the transportation algorithm to find an optimal solution. Is it unique? (*Hint*: Construct an initial basic feasible solution by the three methods.)

Warehouse	Unit loading charge	Available suppliers	Customer	Unit unloading charge	Requirements
1	1	20	1	1	15
2	2	20	2	2	35
3	2	30	3	3	15
4	4	10	4	4	10

7. On Tuesday, the GT Railroad Company will have four locomotives at IE Junction, one locomotive at Centerville, and two locomotives at Wayover City. Student trains each requiring one locomotive will be al A-Station, Fine Place, Goodville, and Somewhere Street. The local map gives the following distances:

	A-Station	Fine Place	Goodville	Somewhere Street
IE Junction	15	38	45	11
Centerville	6	61	18	30
Wayover City	17	14	6	10

How should they assign power (locomotives) so that the total distance traveled is minimized? (*Hint*: Formulate the problem and use the transportation algorithm for solving it. Construct an initial basic feasible solution by the three methods.)

8. An automobile manufacturer has assembly plants located in the Northwest, Midwest, and Southeast. The cars are assembled and sent to major markets in the Southwest, West, East, and Northeast. The appropriate distance matrix, availabilities, and demands are given by the following chart.

	Southwest	East	West	Northeast	S_i
Northwest	1200	8500	1850	2250	2.500.000
Midwest	400	800	900	1400	1.800.000
Southeast	800	1200	1000	1100	1.600.000
D_j	2.000.000	1.500.000	1.200.000	1.200.000	

a) Assuming that cost is proportional to distance, find an optimal shipment pattern.

b) Assuming that cost is proportional to square of distance, find an optimal shipment pattern.

(*Hint*: For each case, formulate the problem and use the transportation algorithm for solving it. Construct an initial basic feasible solution by the three methods.)

9. A company has contracted for five jobs. These jobs can be performed in six of its manufacturing plants. Because of the size of the jobs, it is not feasible to assign more than one job to a particular manufacturing facility. Also, the second job cannot be assigned to the third manufacturing plant. The cost estimates, in thousands of dollars, of performing the jobs in the different manufacturing plants, are summarized bellow.

	PLANT					
JOB	1	2	3	4	5	6
1	60	55	42	57	20	52
2	66	73	–	68	75	63
3	81	78	72	80	85	78
4	30	42	38	50	46	72
5	50	55	40	60	56	70

a) Formulate the problem of assigning the jobs to the plants so that the total cost is minimized.

b) Solve the problem by the transportation algorithm.

c) Solve the problem by the Hungarian assignment algorithm.

10. An airline company can buy gasoline from three suppliers. The suppliers have available $2K$, $7K$ and $6K$ gallons, respectively. The company needs gasoline at four locations with each location requiring $5K$, $3K$, $2K$ and $4K$ gallons, respectively. The per/K gallon quoted price for gasoline delivered to each location is as follows:

		Location			
		1	2	3	4
Suppliers	1	3	1	1	4
	2	6	2	7	3
	3	1	9	12	8

How can the company buy the gasoline to minimize the total cost? Formulate the problem, and solve it by the transportation algorithm. (*Hint*: Construct an initial basic feasible solution by the three methods.)

11. A company manufactures a type of product in four different production plants $P_i,\ i = 1, \ldots, 4$. Each of these plants can produce up to 15 tons per month. The company supplies 30, 16 and 14 tons per month to three customers $C_j,\ j = 1, \ldots, 3$, respectively. The following table shows the distances (measured in Km) from each plant to each customer.

	C_j		
P_i	100	100	50
	650	110	100
	60	65	75
	150	90	70

The cost of transporting each ton of product is 0.5€ per Km. Formulate ans solve the transportation problem so as minimize the company's transportation cost. (*Hint*: Construct an initial basic feasible solution by the three methods.)

12. An enterprise manufactures a product in three production plants $P_i,\ i = 1, 2, 3$, with a production capacity of 130, 200 and 170 units of product, respectively. The demand of four customers $C_j,\ j = 1, \ldots, 4$, has to be satisfied as follows: C_1 demands 150 product units, C_2 demands 175, and C_3 demands at least 125. Both C_3 and C_4 are ready to buy any spare product units, and they both want to buy as many units of products as possible. The benefit obtained from the sale of units of product to the customers is given in the following table.

$$\begin{array}{c} \\ P_i \\ \\ \end{array} \begin{array}{c} C_j \\ \boxed{\begin{array}{cccc} 60 & 40 & 45 & 55 \\ 70 & 55 & 65 & 60 \\ 80 & 60 & 55 & 75 \end{array}} \end{array}$$

Formulate ans solve the transportation problem so as maximize the total benefit. (*Hint*: Construct an initial basic feasible solution by the three methods.)

13. A carpenter, plumber and engineer are available to perform certain tasks. Each person can perform only one task in the allotted time. There are four tasks available, three of which must be done. The inefficiency matrix for person i assigned to task j is as follows:

	Soldering	Framing	Drafting	Wiring
Carpenter	4	6	4	4
Plumber	3	4	2	3
Engineer	7	5	6	5

Which person should be assigned to which job? (*Hint*: Create a dummy person). Which job will go unfinished?

14. Sally, Susan and Sarah will go on a date with Bob, Bill and Ben. Sally likes Bill twice as much as Bob and three times as much as Ben. Susan likes Bill three times as much as Bob and seven times as much as Ben (Ben is a loser!). Sarah likes Bob about as much as Bill, but likes them both about six times as much as Ben. How should the couples pair up so that in the aggregate the girls are as happy as possible? If one girl is willing to stay home, which one should it be? (*Hint*: Formulate the problem, and solve it by the Hungarian algorithm.)

15. After qualifying, medical students must take two six-month jobs in hospital departments, but they cannot take both jobs in the same department. A hospital has four students and vacancies in four departments: Casualty, Maternity, medical and Surgical. The number of fatal mistakes each student will make in each department is given by the following table.

	Casualty	Maternity	Medical	Surgical
Student 1	3	0	2	6
Student 2	2	1	4	5
Student 3	4	2	5	7
Student 4	2	0	2	4

a) How should they be allotted to departments for the first job so as to minimize the total mistakes?

b) Given that allocation, how should they be allotted for the second six months so no one stays in the same job, and mistakes are minimized?

16. A council has three jobs to be done and receives tenders from three firms. The tenders in appropriate units are as follows, where i denotes the firm, and j denotes the job.

$$
\begin{array}{cc|ccc|}
 & & & j & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 20 & 35 & 8 \\
i & 2 & 15 & 40 & 7 \\
 & 3 & 12 & 33 & 6 \\
\hline
\end{array}
$$

Firm 1 is capable of doing all jobs at the same time, firm 2 is capable of doing jobs 2 and 3 at the same time, and firm 3 is capable of doing all jobs, but can only do one at a time. Set up the problem of assigning jobs to firms as an assignment problem, and find the solution. Is it unique?

17. An enterprise offers 4 new jobs T_j, $j = 1, \dots, 4$. With the aim of selecting the best candidate for each job, the personnel department has prepared a test, and 5 people $A_i, i = 1, \dots, 5$ have applied for it. The following table displays the number of errors made in the test.

$$
\begin{array}{c|cccc|}
 & & T_j & & \\
\hline
 & 16 & 4 & 17 & 3 \\
 & 13 & 14 & 8 & 11 \\
A_i & 2 & 19 & - & 9 \\
 & 21 & 12 & 13 & 16 \\
 & 22 & 16 & 25 & 12 \\
\hline
\end{array}
$$

A_3 is not able to perform job T_3, and therefore the assignment must be forbidden. Taking into account that the optimal assignment among applicants and jobs is the one that minimizes the number of errors, formulate the corresponding assignment problem, and use the Hungarian algorithm to find the optimal assignment. (*Hint*: Create a dummy job). Which of the applicants will remain unemployed?

18. A transportation company has 4 trucks T_i, $i = 1, \dots, 4$, in four different cities. Five production plants P_j, $j = 1, \dots, 5$, placed in five cities are demanding a truck. The trucks must be sent to the production plants so that their production can be delivered. The distances from the cities where the trucks are located to the production plants are shown in the following table.

$$
\begin{array}{c|ccccc|}
 & & & P_j & & \\
\hline
 & 13 & 1 & 8 & 7 & 10 \\
T_i & 12 & 6 & 4 & 4 & 7 \\
 & 18 & 10 & 14 & 21 & 20 \\
 & 14 & 13 & 7 & 12 & 11 \\
\hline
\end{array}
$$

Tacking into account that the optimal assignment among trucks and plants is the one that minimizes the total distance, formulate the corresponding assignment problem, and use the Hungarian algorithm to find the optimal assignment. (*Hint*: Create a dummy truck). Which is the production plant that does not receive any truck?

19. Consider the following linear programming problem, and its optimal final tableau shown below:

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + x_2 - x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Final tableau

	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	3	3	2	0	16
x_1	0	1	2	1	1	0	8
x_5	0	0	3	-1	1	1	12

a) Solve the foregoing problem by the simplex method, and write the optimal final tableau. Use this optimal tableau in the following questions.

b) Using sensitivity analysis, find a new optimal solution if the coefficient of x_2 in the objective function in changed from 1 to 5.

c) Suppose that the coefficient of x_2 in the first constraint is changed from 2 to $1/6$. Using sensitivity analysis, find a new optimal solution.

d) Suppose that the following constraint is added to the problem: $x_2 + 2x_3 = 3$. Using sensitivity analysis, find the new optimal solution.

e) Suppose a new activity x_6 is proposed with unit cost 6 and consumption vector $\mathbf{a}_6 = (2, 1)^t$. Find a new optimal solution.

20. Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + 3x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 8 \\ & x_1 - 2x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

a) Selecting the identity matrix as initial basis $\mathbf{B}$, solve the foregoing problem by the simplex method, and write the optimal final tableau. Use this final tableau in the following questions.

b) Using sensitivity analysis, find a new optimal solution if the coefficient of x_1 in the objective function in changed from 2 to 4.

c) Suppose that the coefficient of x_2 in the first constraint is changed from 2 to $1/6$. Using sensitivity analysis, find a new optimal solution.

d) Suppose that the following constraint is added to the problem: $x_2 - 2x_3 = 2$. Using sensitivity analysis, find the new optimal solution.

e) Suppose a new activity x_6 is proposed with unit cost 4 and consumption vector $\mathbf{a}_6 = (2, 1)^t$. Find a new optimal solution.

21. *A linear programming (LP) problem satisfies the non-degeneracy assumption when every basic feasible solution has all of its basic variables strictly positive. Does degeneracy cause problems? Not necessarily, in fact degeneracy arises frequently in practical LP problems and the simplex algorithm converges to the optimal solution.*

However, degeneracy can cause cycling in simplex-type algorithms, unless special rules are enforced (in fact, if there is no degeneracy, there would be no cycling). Because of this, degeneracy has been considered a "bad"phenomenon in the topic of linear programming. Indeed, while cycling can be avoided, the presence of degenerate solutions may temporarily suspend progress in the algorithm. (A related issue is the behaviour of the simplex algorithm in the presence of roundoff error. At a degenerate basic feasible solution there is a serious danger of selecting pivots which are small and have a high relative error).

Solve the following problems, using the lexicographic rule for noncycling. Repeat using the Bland's rule. Before, for each problem verify that the non-degeneracy assumption does not hold and the simplex algorithm presents cycling.

a)

$$\begin{aligned} \text{Maximize} \quad & Z(\mathbf{x}) = x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 + 3x_3 \le 4 \\ & -x_1 + x_2 + 4x_3 \le 1 \\ & x_1 + 3x_2 + x_3 \le 6 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

b)

$$\begin{aligned} \text{Maximize} \quad & Z(\mathbf{x}) = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{s.t.} \quad & 1/2\,x_1 - 11/2\,x_2 - 5/2\,x_3 + 9\,x_4 \le 0 \\ & 1/2\,x_1 - 3/2\,x_2 - 1/2\,x_3 + x_4 \quad \le 0 \\ & x_1 + x_2 + x_3 + x_4 \le 1 \\ & x_j \ge 0,\ j = 1, \ldots, 4 \end{aligned}$$

c)

$$\begin{aligned} \text{Maximize} \quad & Z(\mathbf{x}) = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{s.t.} \quad & 1/2\,x_1 - 11/2\,x_2 - 5/2\,x_3 + 9\,x_4 \le 0 \\ & 1/2\,x_1 - 3/2\,x_2 - 1/2\,x_3 + x_4 \quad \le 0 \\ & x_1 \qquad\qquad\qquad\qquad\qquad\quad \le 1 \\ & x_j \ge 0,\ j = 1, \ldots, 4 \end{aligned}$$

d)

$$\begin{aligned} \text{Maximize} \quad & Z(\mathbf{x}) = -x_1 + 7x_2 + x_3 - 2x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \le 1 \\ & 1/2\,x_1 - 11/2\,x_2 - 5/2\,x_3 + 9\,x_4 \le 0 \\ & 1/2\,x_1 - 3/2\,x_2 - 1/2\,x_3 + x_4 \quad \le 0 \\ & x_j \ge 0,\ j = 1, \ldots, 4 \end{aligned}$$

e)

$$\begin{aligned}
\text{Maximize} \quad & Z(\mathbf{x}) = 2.3\,x_1 + 2.15\,x_2 - 13.55\,x_3 - 0.4\,x_4 \\
\text{s.t.} \quad & 0.4\,x_1 + 0.2\,x_2 - 1.4\,x_3 - 0.2\,x_4 \le 0 \\
& -7.8\,x_1 - 1.4\,x_2 + 7.8\,x_3 + 0.4x_4 \le 0 \\
& x_j \ge 0,\ j = 1, \ldots, 4
\end{aligned}$$