Game Theory and Applications (博弈论及其应用)

Chapter 13: Extensive Game with Imperfect Information

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Recap on Previous Chapter

- Repeated game: many real interactions have an ongoing structure; players consider short- and long-term payoffs.
- A repeated game $G^{\mathrm{T}}(\delta)$ consists of stage game G, terminal date T and discount factor δ
- Nash Folk Theorem (Nash equilibrium)
- Folk Theorem (SPNE)

Recap on Previous Chapter

Nash Folk Theorem

If $(u_1, u_2, \dots, u_N) \in U$ is strictly individually rational, then there exists some $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, there is Nash equilibrium of $G^{\infty}(\delta)$ with payoff $(u_1, u_2, \dots, u_N)$

Payoff vector $(u_1, u_2, \dots, u_N) \in R^N$ is strictly individually rational if $u_i > \min\limits_{a_{-i}}[\max\limits_{a_i} u_i(a_i, a_{-i})]$for all i

Recap on Previous Chapter

- Folk Theorem
  - An infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.
  - Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, $\hat{u}_i > u_i^*$ for every player i
  - There is a Subgame Perfect Nash Equilibrium for some discount factor δ

Solving for Equilibria in Repeated Games

1. Solve all equilibria of the stage game (Competition)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (Cooperation)
3. Design trigger strategies that support cooperation and punish with competition
4. Compute the maximum discount factor so that cooperation is an equilibrium
5. The trigger strategies are an SPEN of the infinitely repeated game for some larger discount factor

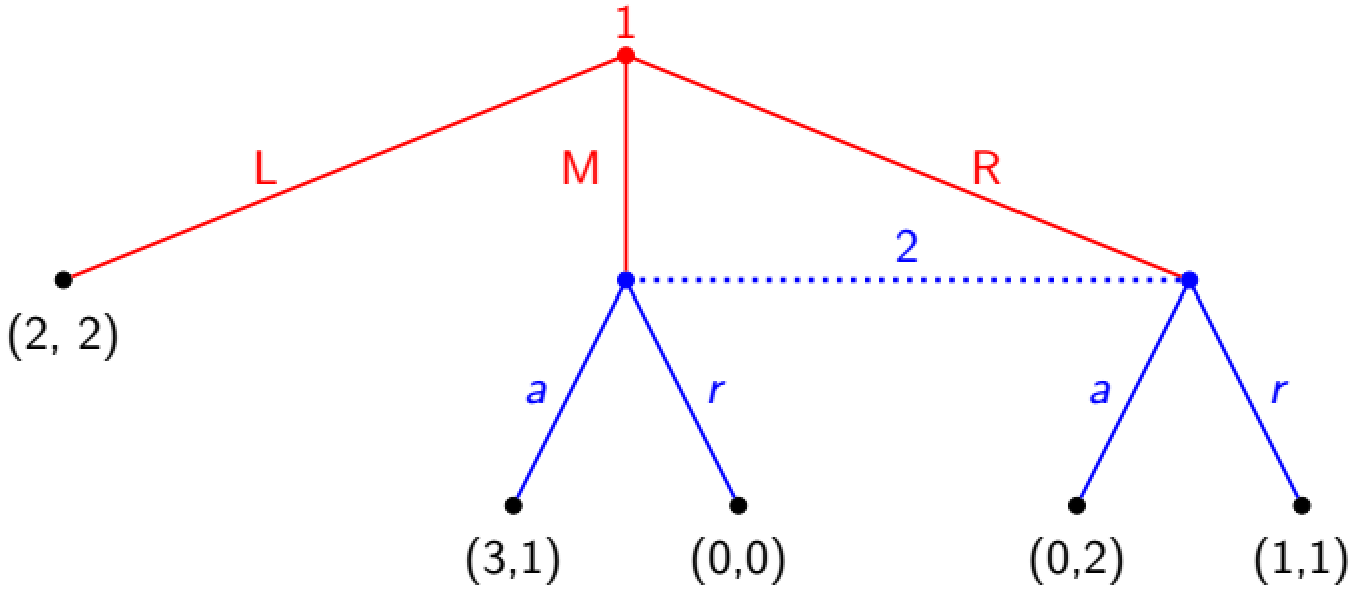
Recap on Extensive Game

- The extensive game is an alternative representation that makes the temporal structure explicit
- Nash equilibrium
- Subgame perfect equilibrium (SPE): an outcome is SPE if it is Nash Equilibrium in every subgame
- How to find SPE – back induction and one deviation
- Two variants
  - Perfect information: game tree
  - Imperfect information

Motivation

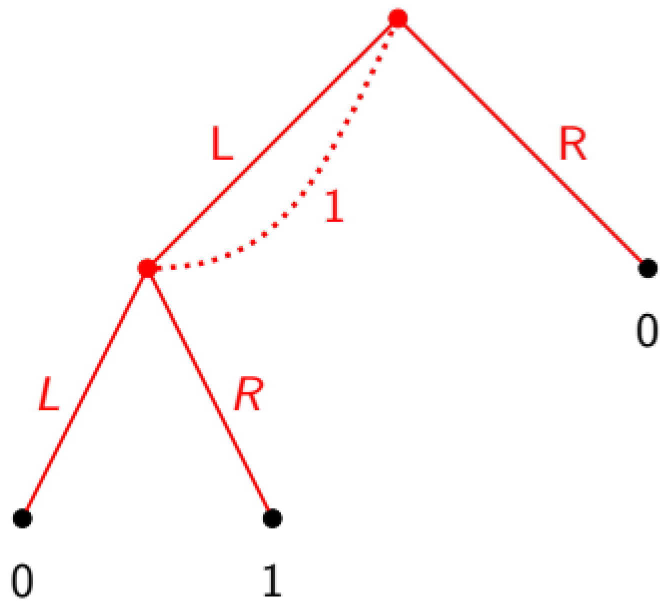
- Extensive game with perfect information
  - Know all prior strategies for all players
- Sometimes, players
  - Don’t know all the strategies the other take or
  - Don’t recall all their past actions
- Extensive game captures some of this ignorance
  - An earlier choice is made without knowledge of a later choice
- How to represent the case two players make choices at the same time, in mutual ignorance of each other

Example



Player 2 does not know the choice of player 1 over M or R

Example



Player 1 does not know if he has made a choice or not

Definition of extensive game with Perfect Information

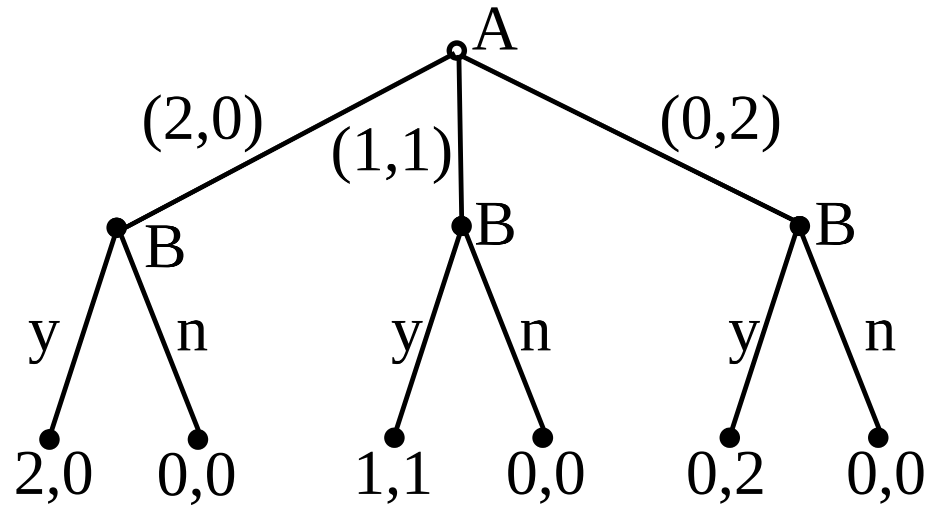
An extensive game with perfect information is defined by
$G = \{N, H, P, \{u_i\}\}$

- Players N is the set of N players
- Histories H is a set of sequence $a^1 \dots a^k$, where each component a^i is a strategy
- Player function $P(h): H \to N$ is the player who takes action after the history h
- Payoff function u_i
- Action set $A(h) = \{\text{a}:\ (h, a) \in H\}$

Ultimatum Game

$G = \{N, H, P, \{u_i\}\}$

N={A,B}



H={∅,(2,0),(1,1),(0,2), ((2,0),y)}

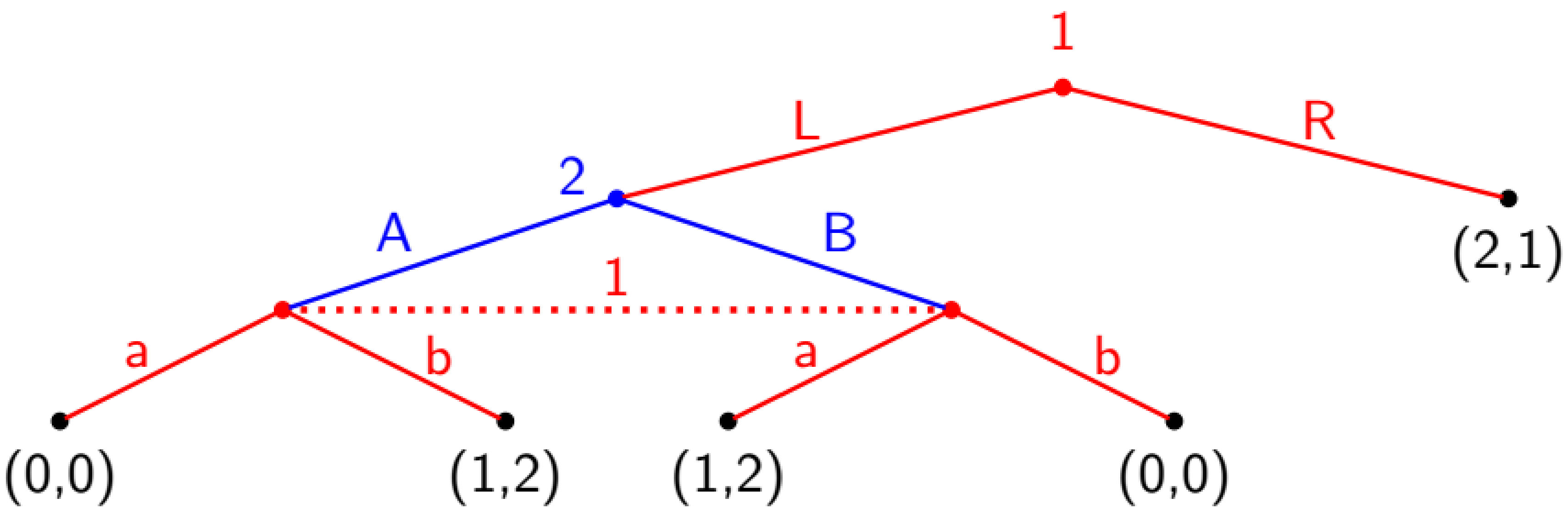
∪{((2,0),n), ((1,1),y), ((1,1),n)}

∪{((0,2),y), ((0,2),n)}

P: P(∅)=A; P((2,0))=B; P((1,1))=B; P((0,2))=B

A: A(∅)={(2,0),(0,2),(1,1)}; A((2,0))=A((0,2))=A((1,1))={y,n}

Extensive Game with Imperfect Information



Player 1 does not know the choice of player 2 over LA or LB

Nonterminal histories: $\{\emptyset, \text{L}, \text{LA}, \text{LB}\}$

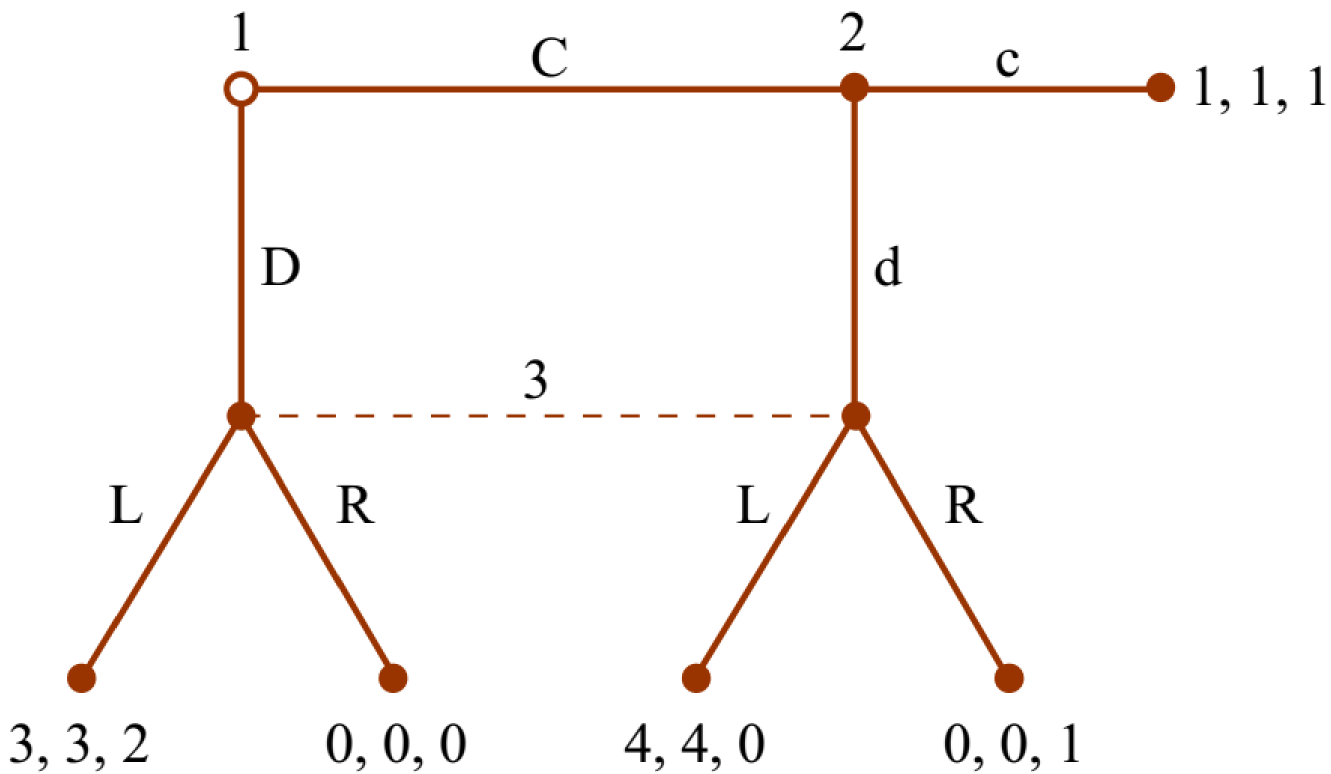
- Player 1 has information set $I_1 = \{\emptyset, \{\text{LA}, \text{LB}\}\}$,
- Player 2 has information set $I_2 = \{\text{L}\}$

Definition of Extensive Game with Imperfect Information

An extensive game with imperfect information is defined by $G = \{N, H, P, I, \{u_i\}\}$

- Information set $I = \{I_1, I_2, \dots I_N\}$ is the set of information partition of all players' strategy nodes, where the nodes in an information set are indistinguishable to player
  - $I_i = \{I_{i1}, \dots, I_{ik_i}\}$ is the information partition of player i
  - $I_{i1} \cup \cdots \cup I_{ik_i}$ ={all nodes of player i}
  - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq$ k
  - Action set $A(h) = A(h')$ for $h, h' \in I_{ij}$, denote by $A(I_{ij})$
  - $P(I_{ij})$ be the player who plays at information set I_{ij}
- An extensive game with perfect information is a special case where each I_{ij} contains only one node

Example



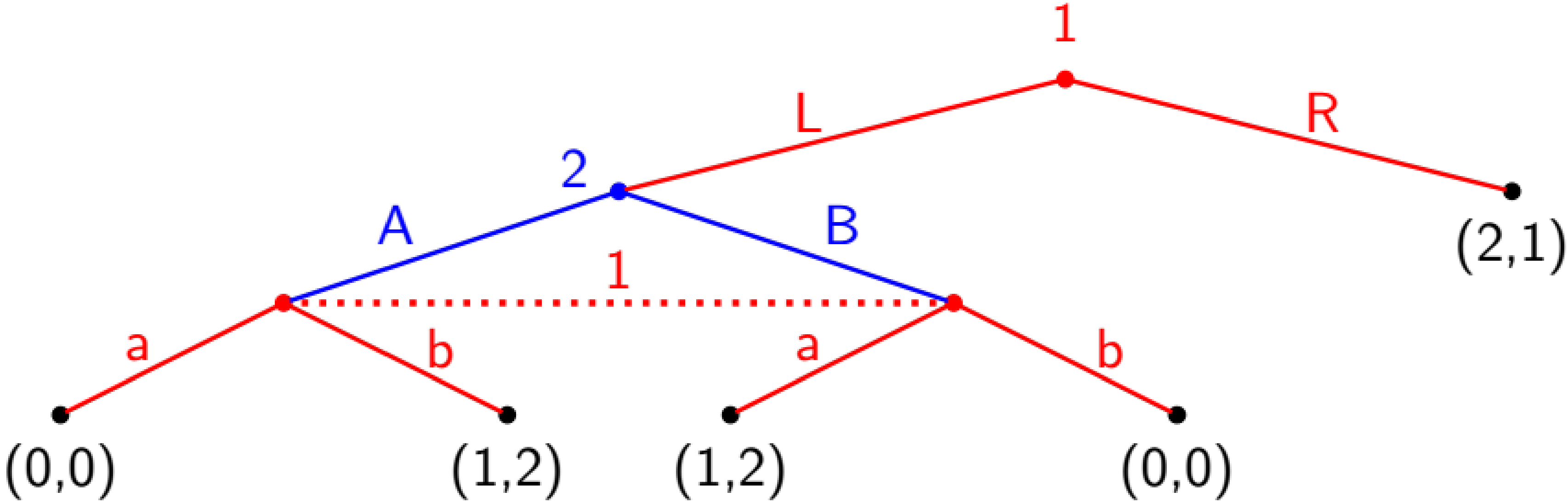
- Player 1 has information set $I_{11} = \{\emptyset\}$
- Player 2 has information set $I_{21} = \{\mathrm{C}\}$
- Player 3 has the information set $I_{31} = \{\{\mathrm{D}, \mathrm{Cd}\}\}$

Pure Strategies

- A pure strategy for player i selects an available action at each of i's information sets $I_{i1}, \dots, I_{im}$
- All pure strategies for player i is

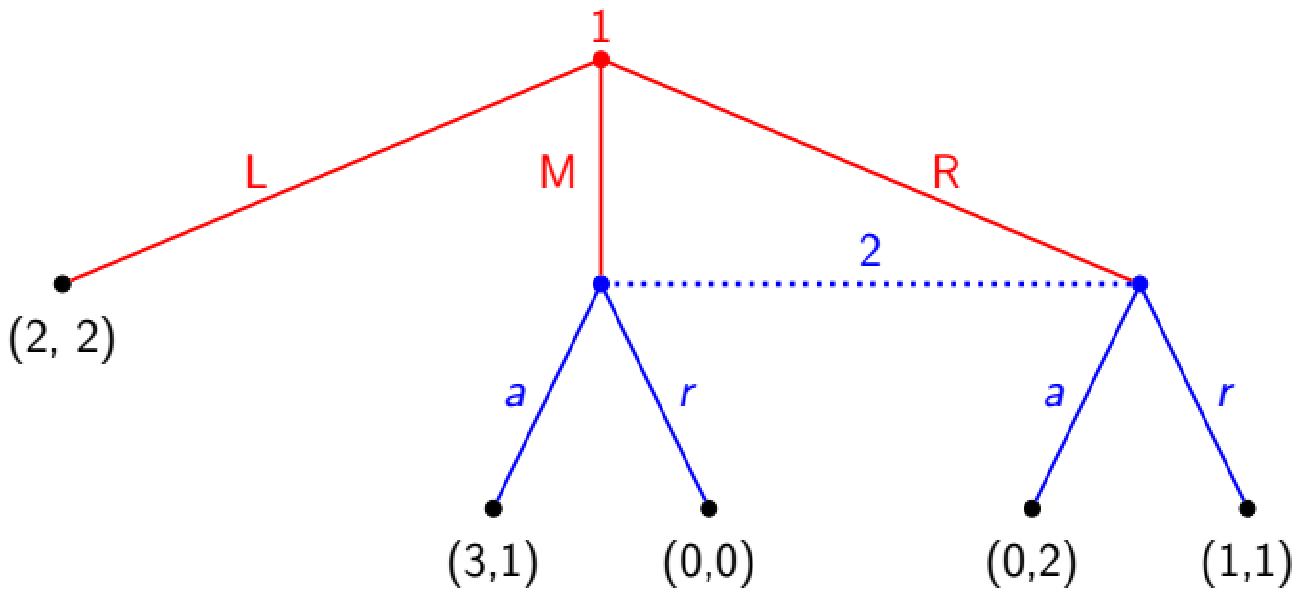
$$A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$$

where $A(I_{ij})$ denotes the strategies available in I_{ij}



What's the pure strategies for players 1 and 2?

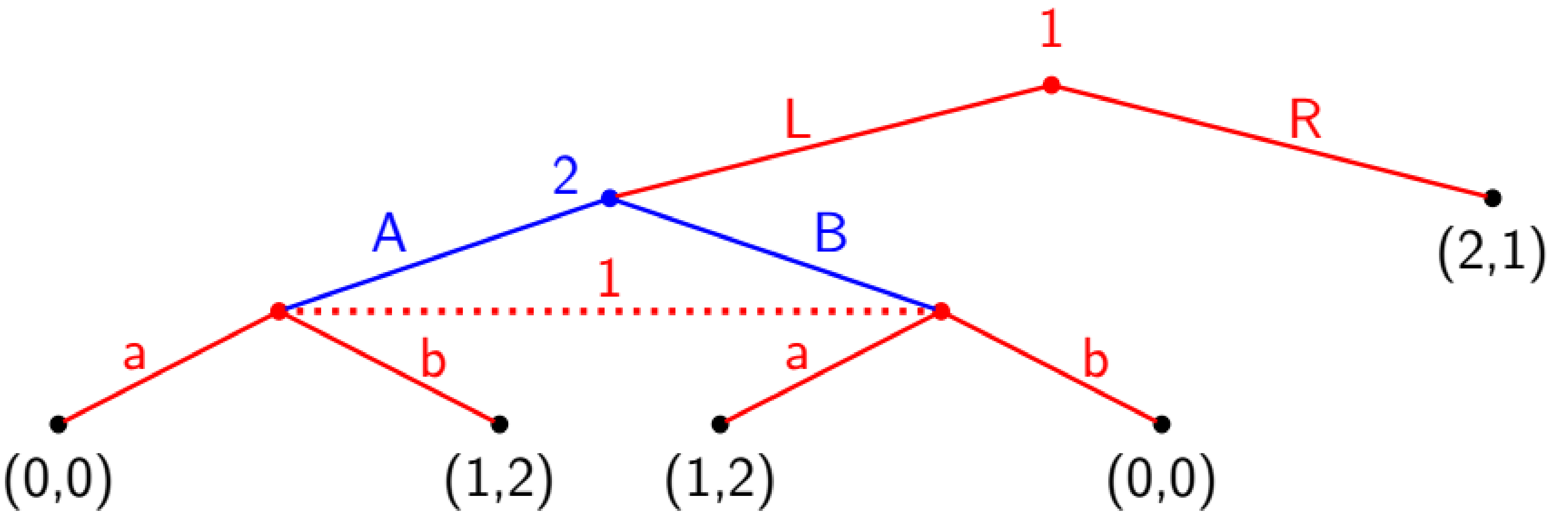
Normal-Form Representation of Extensive Imperf. Game



	a	r
L	2,2	2,2
M	3,1	0,0
R	0,2	1,1

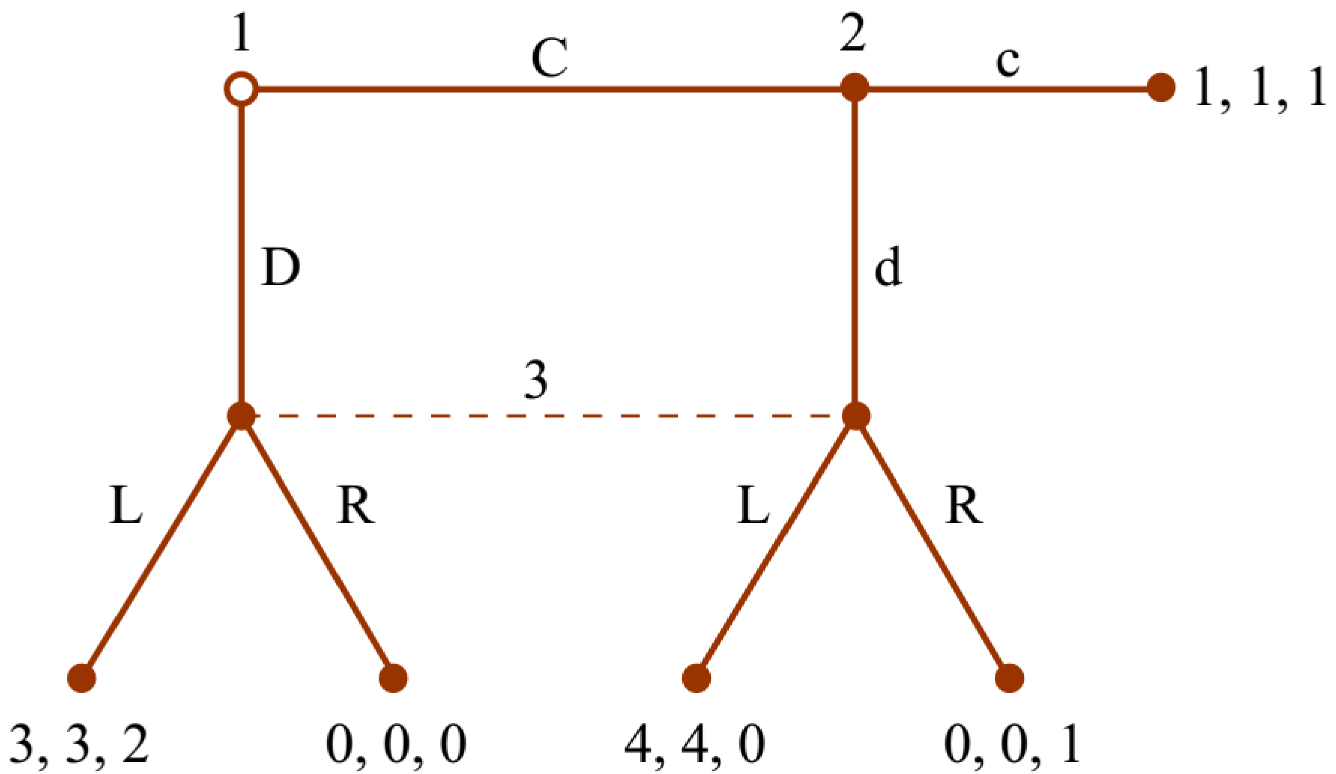
- The pure and mixed strategy Nash Equilibrium remains?
- What's the difference from the extensive game with perfect information game?

Exercise



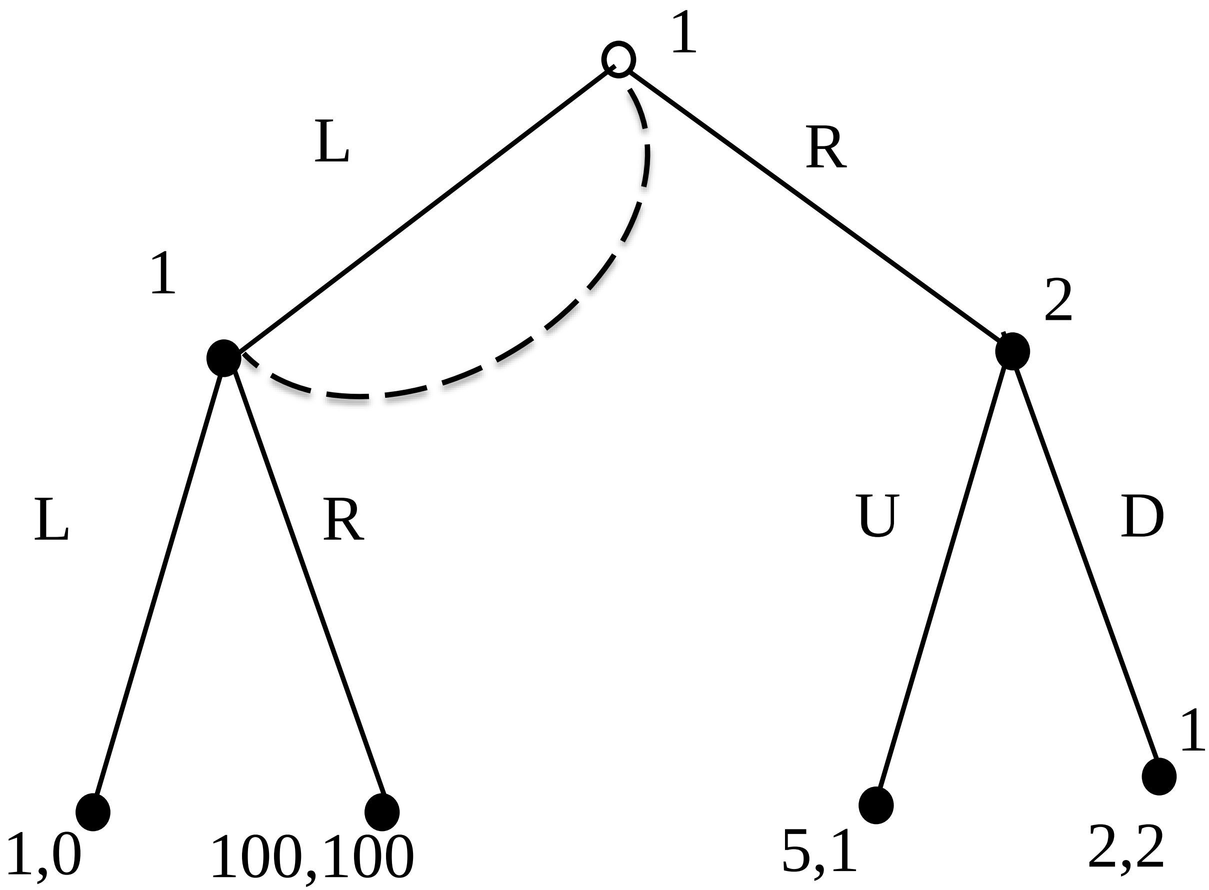
What are Nash Equilibria

Exercise



What are Nash Equilibria

Exercise



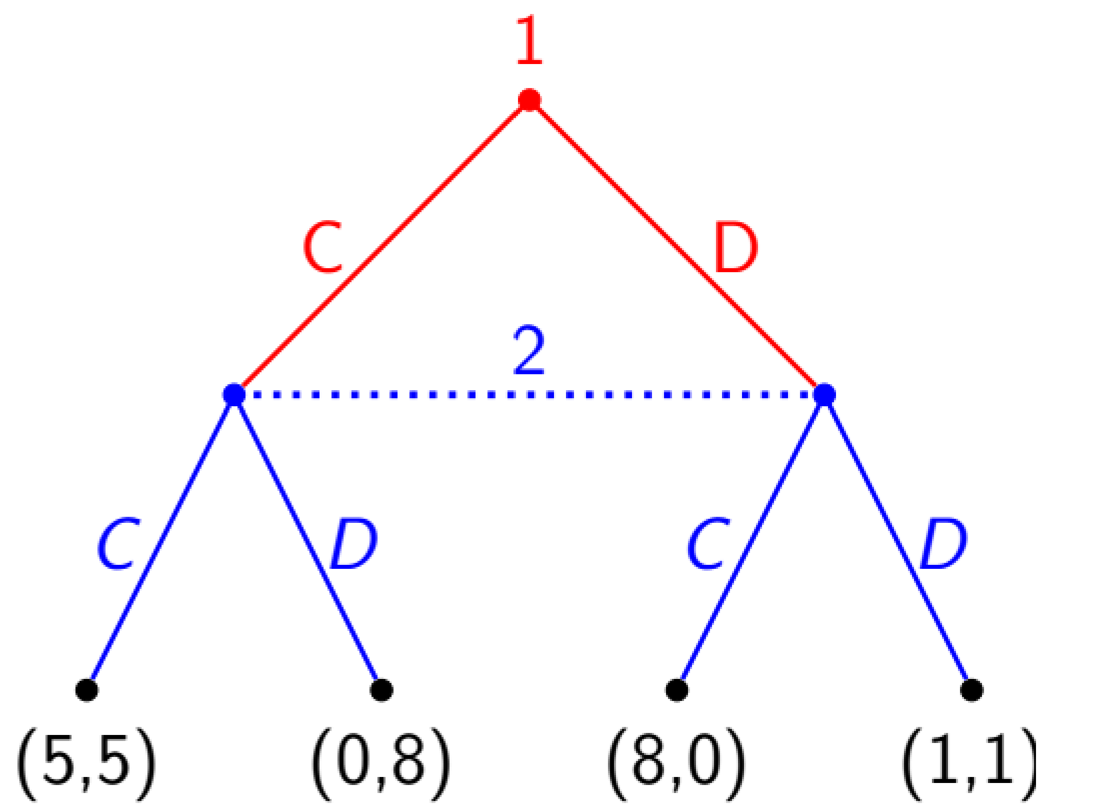
What are Nash Equilibria

Extensive Representation of Normal-Form Game

A strategy game $\Longrightarrow$ An extensive game with imp.

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Prisoner's Dilemma



Exercise: 3-Players Game

$$G=\{\{1,2,3\},\{\{a,b,c\},\{x,y,z\},\{L,R\}\},\{u_i\}_{i=1}^{3}\}$$

P3 chooses *L*

P2

P1		x			y			z		
	a	8	7	4	2	9	1	4	1	8
	b	4	6	5	7	2	6	1	3	7
	c	6	2	2	5	1	7	4	4	2

P3 chooses *R*

P2

P1		x			y			z		
	a	5	3	2	6	5	4	1	2	4
	b	8	6	2	2	8	10	5	2	6
	c	6	9	4	1	1	3	9	7	8

Perfect Recall (完美回忆) and Imperfect Recall

- An extensive game has perfect information if each information set consist of only one nodes

- An extensive game has perfect recall if each player recalls exactly what he did in the past

  – otherwise, this game has imperfect recall

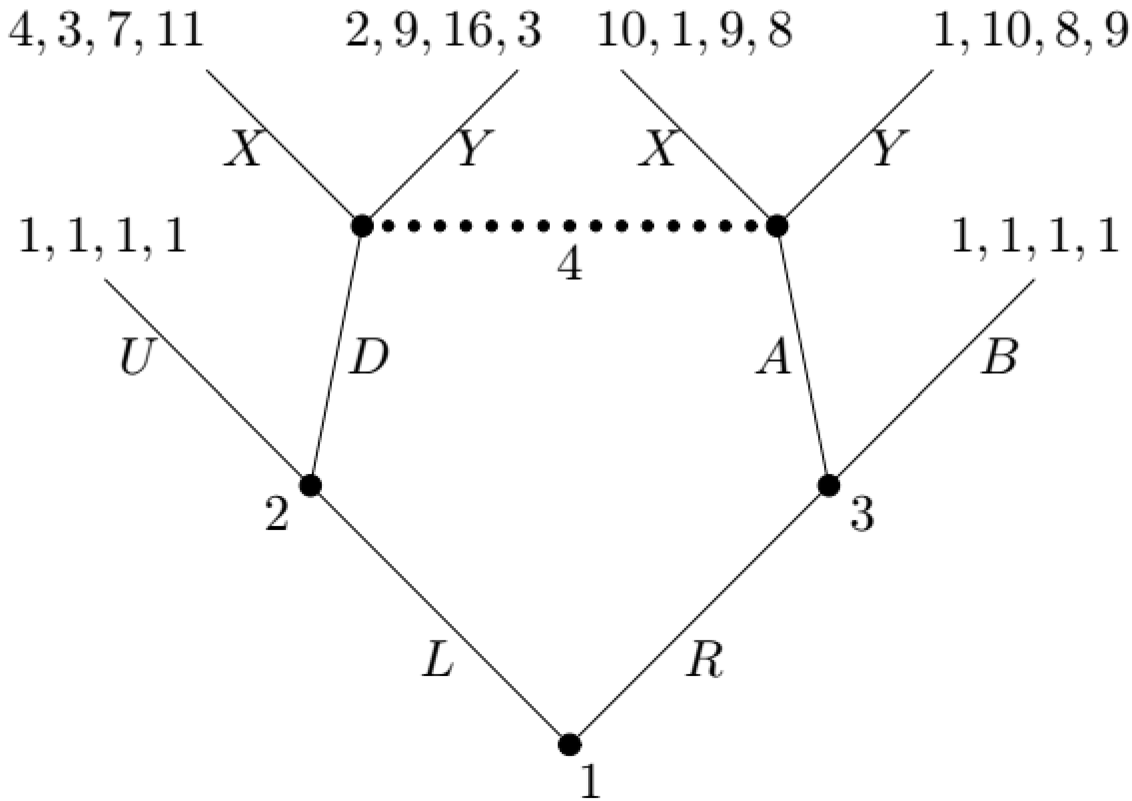
Formal Definition of Perfect Recall

Player i has **perfect recall** in game G if for any two history h and h' that are in the same information set for player i, for any path $h_0, h_1, \dots h_n, h$ and $h'_0, h'_1, \dots h'_m, h'$ from the root to h and $h'$with $P(h_k) = P(h'_k) = i$, we have

- $n = m$
- $h_i = h'_i$ for $1 \leq i \leq n$

G is **a game of perfect recall** if every player has perfect recall in it.

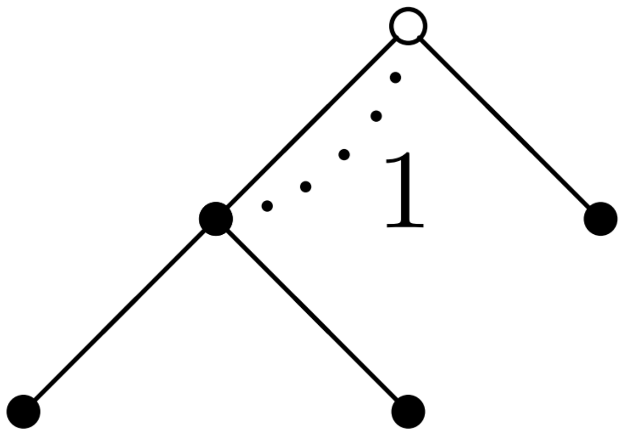
Example

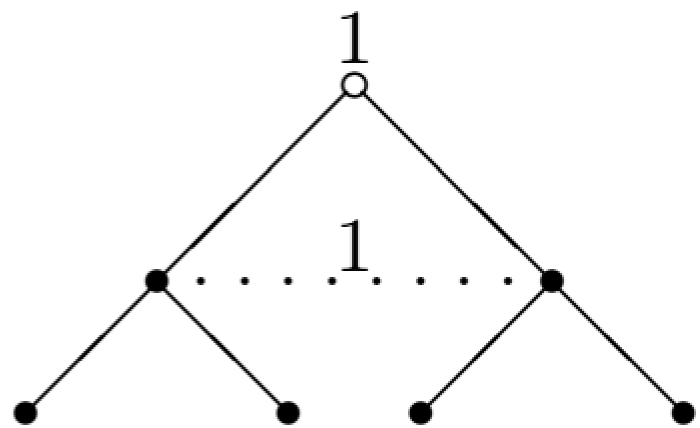


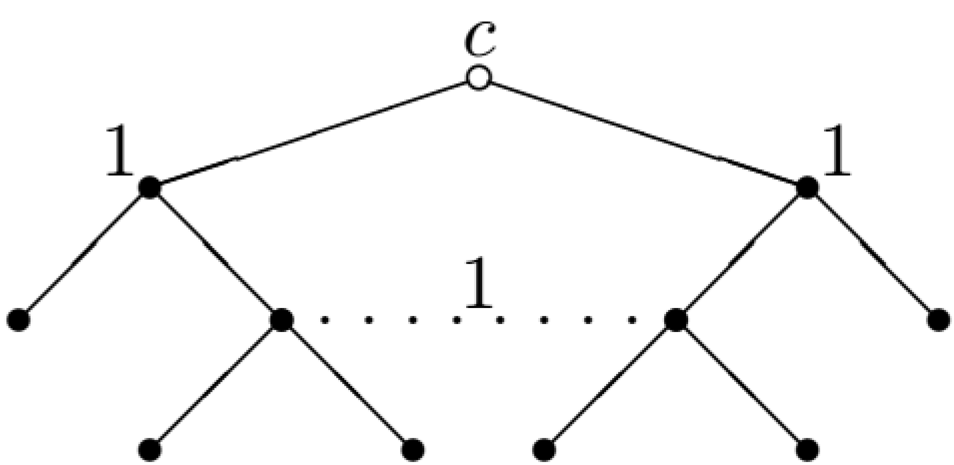
Perfect recall

If we change player 4 by player 1, is it a perfect recall

Example of Imperfect Recall







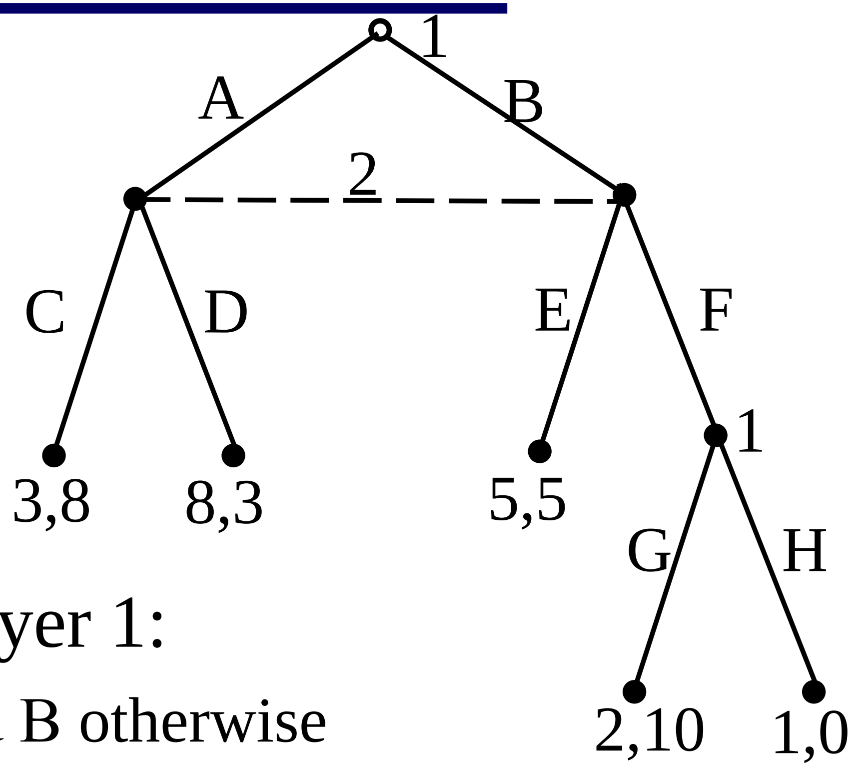
Definition of Mixed and Behavioral Strategies

- Mixed Strategies: A mixed strategy of player i in an extensive game is a probability over the set of player i's pure strategy

- Behavioral strategies: A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik} \in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $\mathrm{A}(I_{ik})$

Behavioral vs. Mixed Strategies

Behavioral strategies distinguish from mixed strategies



A behavioral strategy for player 1:

- Selects A with prob. 0.5, and B otherwise
- choose G with prob. 0.3, and H otherwise

Here's a mixed strategy that isn't a behavioral strategy

- Pure Strategy AG with probability 0.6, pure strategy BH 0.4
- The choices at the two nodes are not independent

Behavioral vs. Mixed Strategies

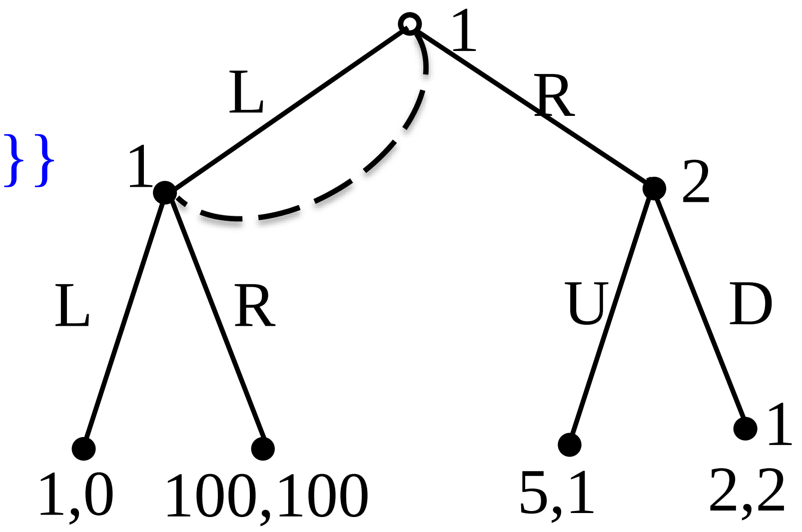
In imperfect-information games, mixed and behavioral strategies produce different sets of equilibria

- In some games, mixed strategies can achieve equilibria that aren’t achievable by any behavioral strategy
- In some games, behavioral strategies can achieve equilibria that aren’t achievable by any mixed strategy

Behavioral vs. Mixed Strategies

Consider game

Player 1 inform. set: {{∅, L}}

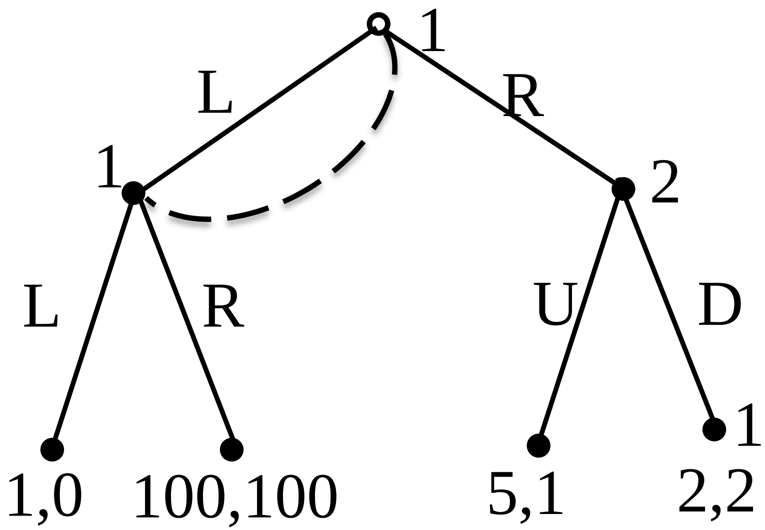


- Player 1: R is a strictly dominant strategy
- Player 2: D is a strictly dominant strategy
  - (R, D) is the unique Nash equilibrium for mixed strategy

Behavioral vs. Mixed Strategies

1: the information set is {(Ø,L)}

2: D is a strictly dominant strategy



Player 1's best response to D:

- Player 1's the behavioral strategy $[L, p; R, 1 - p]$ i.e., choose L with probability p
- The expected payoff of player 1 is
- $U_1 = p^2 + 100p(1 - p) + 2(1-p) = -99p^2 + 98p + 2$
- To find the maximum, we have $p = 49/99$

(R,D) is not an equilibrium for behavioral strategy