Game Theory and Applications (博弈论及其应用)

Chapter 9: One Deviation, Back Induction

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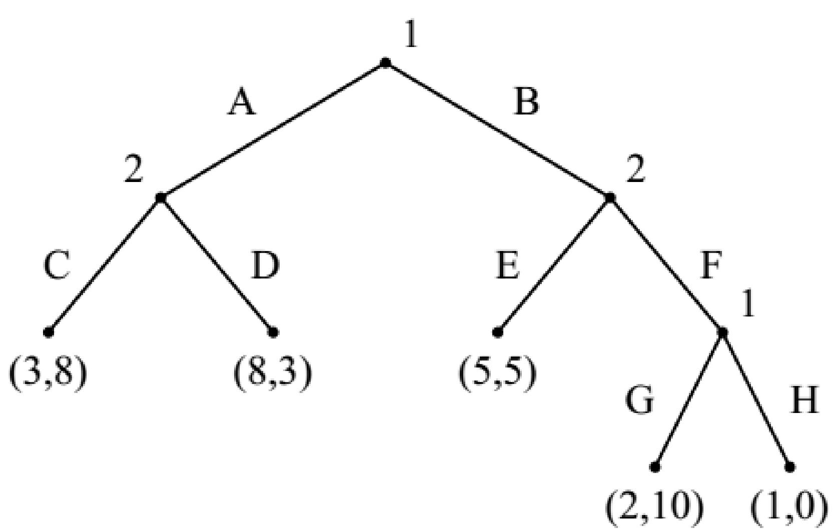
Recap on Previous Chapter

- The strategy game does not incorporate any information of time, or sequence of strategies of players
- The extensive game is an alternative representation that makes the temporal structure explicit
- Perfect information: game tree



Formalize $G = \{N, H, P, \{u_i\}\}$

Pure strategy (Mixed)

Nash Equilibrium

Subgame

Subgame Perfect

Motivation

- Existence:
  - Does every extensive game with perfect information have an SPE
  - If not, which extensive games with perfect information do have an SPE

- Computation:
  - If an SPE exists, how to compute it

Back Induction (后向归纳)

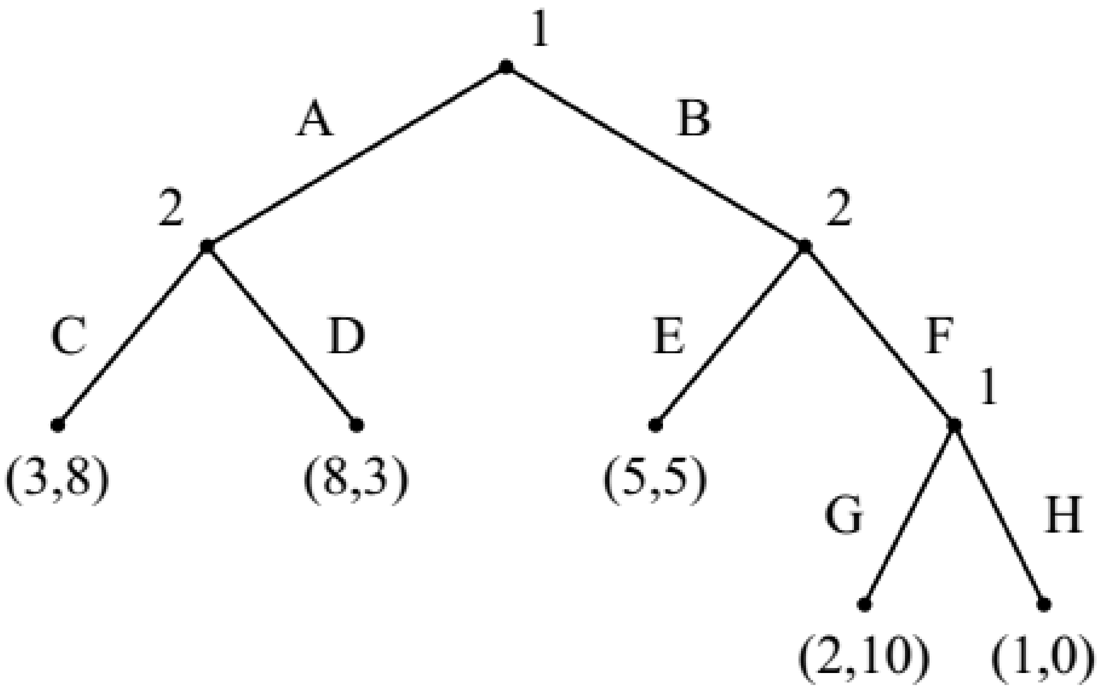
How to find subgame perfect Equilibria (SPE)

Back induction is the process of “pruning the game tree” described as follows:

- Step 1: start at each of the final subgame in the game, and solve for the player’s equilibrium. Remove that subgame and replace it with payoff of the player’s choice
- Step 2: Repeat step 1 until we arrive at the first node in the extensive game

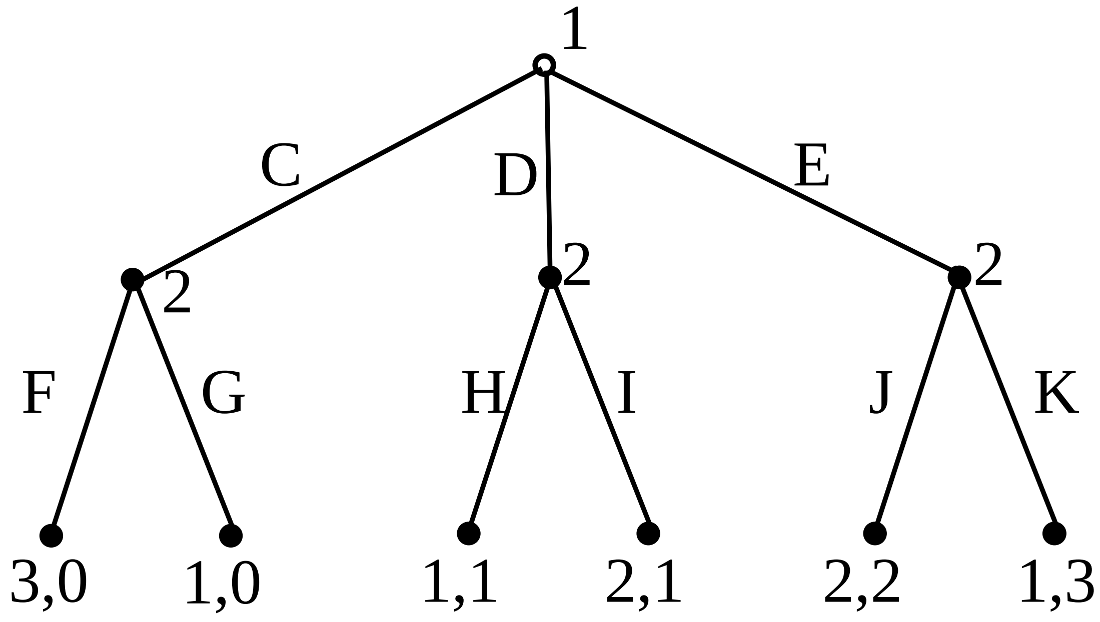
Theorem The set of strategy game constructed by backwards induction is equivalent to the set of SPE

Example



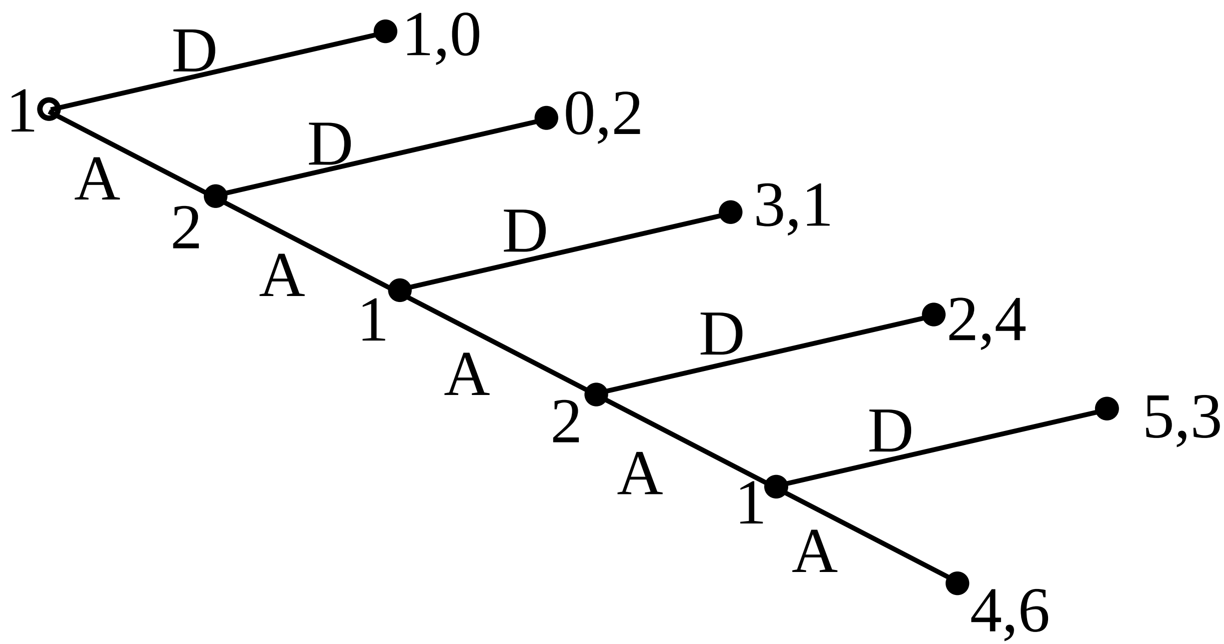
- Find a Sub-game perfect Equilibrium

Multiplicity of Subgame Perfect Equilibria



What happens for multiple optimal strategies?

Centipede Game（蜈蚣游戏）



What happens for centipede game?

Notations

Given game $G = \{N, H, P, \{u_i\}\}$

- define the initial history of $h \in H$ as

$$A(h) = \{a\colon\ (h, a) \in\ H\}$$

- define the length of G as

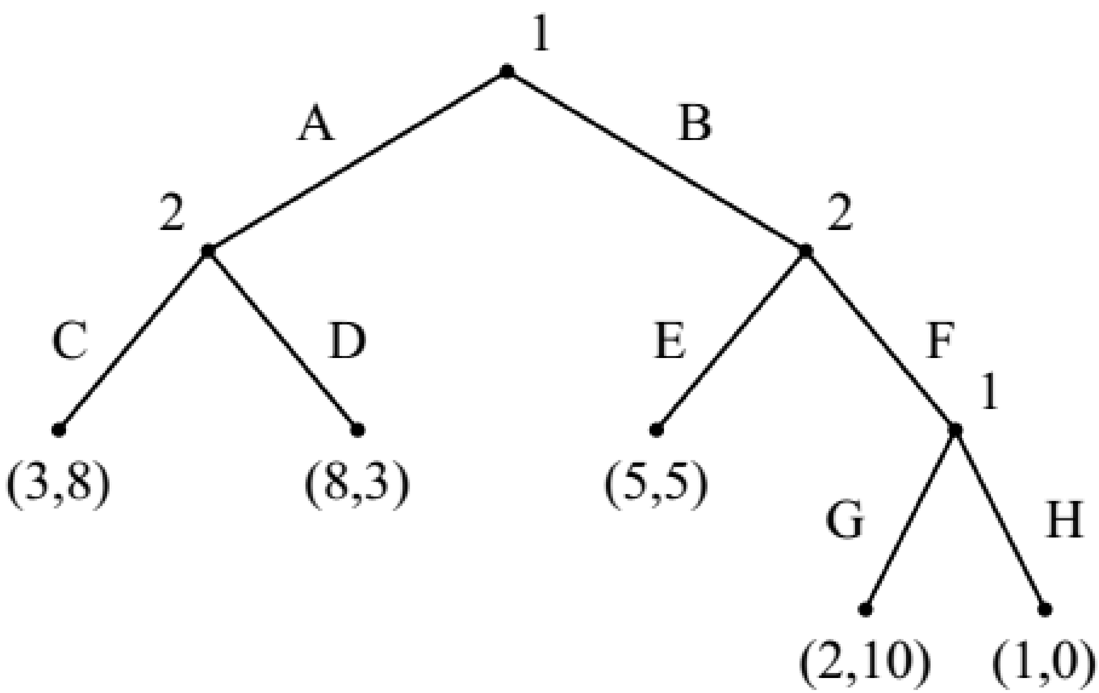
$$\ell(G) = \max_{h \in H}\{|h|\}$$

the length of the longest history in H

Given pure strategy s_i, and h such that $P(h) = i$, then

$$s_i(h) = a \text{ s.t. } a \in A(h) \text{ and } a \in\ s_i$$

Example



ℓ(G)=?

A(BF)=? A(G)=?

Given pure strategy $s_1 = (AG)$, s_1(BF)=?

Formal Definition of Subgame

Given $G = \{N, H, P, \{u_i\}\}$, the subgame of extensive game after the history h is

$$\boldsymbol{G(h)} = \{N, H|_h, P|_h, \{u_i|_h\}\}$$

- $H|_h$ is the set of sequence h' s.t. $(h, h') \in H$;
- $P|_h(h') = P(h, h')$ for every non-terminal his. $h' \in H|_h$;
- $u_i|_h(h') = u_i(h, h')$ for every terminal his. $h' \in H|_h$.

Given pure strategy s_i and history h

- $s_i|_h$ the strategy that s_i induces in subgame $G(h)$.
- $s_i|_h(h') = s_i(h, h')$ for every $h' \in H|_h$

Subgame Perfect Equilibrium

Theorem For **finite** game $G = \{N, H, P, \{u_i\}\}$, $s^* = (s_1^*, s_2^*, \dots, s_N^*)$ is a subgame perfect equilibrium (SPE) iff

$$\forall\, i \in N, \forall h \in H \setminus \mathrm{Z} \ \ \text{s.t.} \ \ \mathrm{P}(h) = i$$

$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i, s_{-i}^*|_h)$$

for every s_i in $G(h)$.

In words: $s^*|_h$ is a NE in every $G(h)$

One Deviation Principle (单步偏离原则)

Theorem For finite game $G = \{N, H, P, \{u_i\}\}$, $s^* = (s_1^*, s_2^*, \dots, s_N^*)$ is a subgame perfect equilibrium (SPE) iff

$$\forall\, i \in N, \forall h \in H \setminus \mathrm{Z} \quad \text{s.t.} \quad \mathrm{P}(h) = i$$

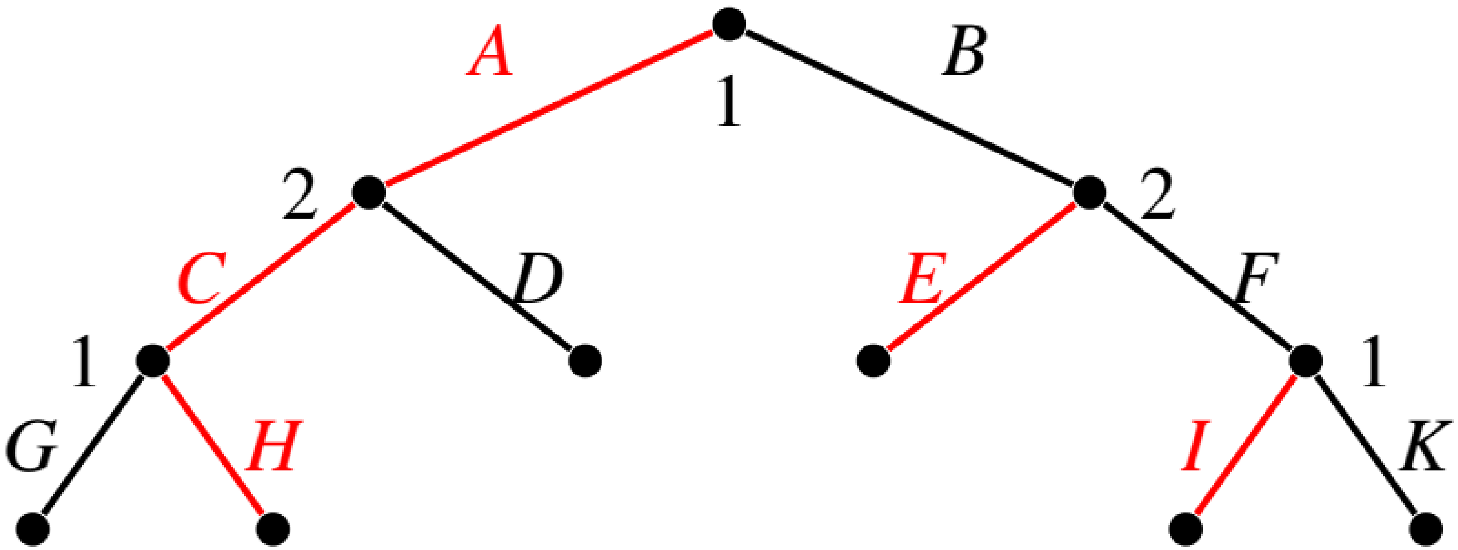
$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i, s_{-i}^*|_h)$$

for every s_i in $G(h)$ that differs from $s_i^*|_h$ only in $A(h)$.

- $s_i(\emptyset) \neq s_i^*|_h(\emptyset)$
- $s_i(h, h') = s_i^*|_h(h, h')$ for $(h, h') \in H$ and $h' \neq \emptyset$

One Deviation

Example: One Deviation Principle



Check whether (AHI, CE) is an SPE, it suffices to check

Player 1: Player 2

G in the subgame $G(AC)$ D in $G(A)$

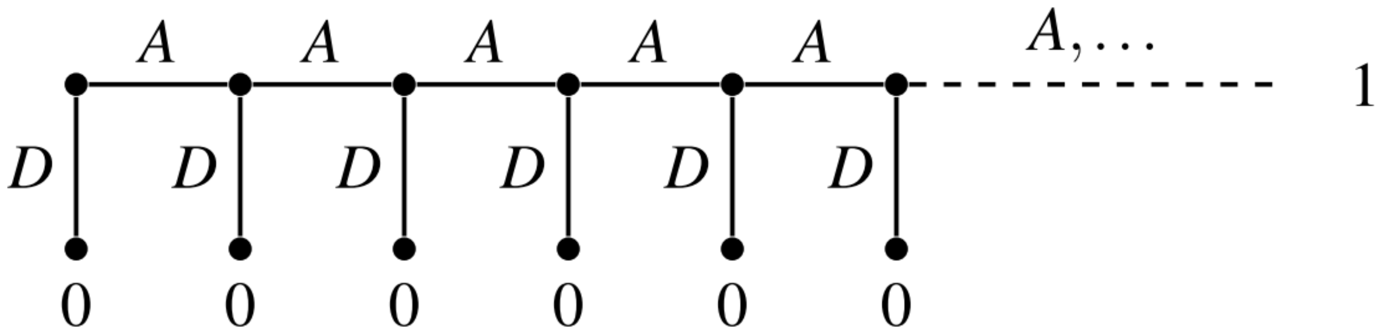
K in the subgame $G(BF)$ F in $G(B)$

BHI in G, and it is not necessary to check BGK, AHK, BHK ...

Infinite Games for One Deviation Property

One deviation does NOT hold for infinite-length game

For example



Strategy DDD… satisfies one-stage deviation property

AAA…is an SPE

Kuhn's Theorem

Theorem Every **finite** extensive game with perfect information has a subgame perfect equilibrium.

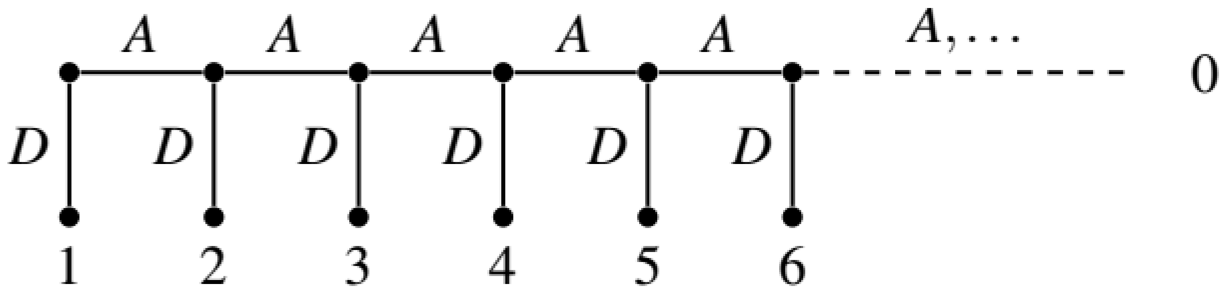
- The SPE consists of pure strategies (no mixing);
- If all payoffs for each player are different, then SPE is unique;
- Proof is constructive and builds an SPE bottom-up (backward induction).
- Finite means 'finite length'

Proof See board.

Infinite games

Kuhn's theorem does not holds for infinite-length games

Counter example (for one player)



$u_1(AAA\ldots) = 0$

$u_1(DDD\ldots) = 1$

$u_1(AAA\ldots D) = n + 1$ no SPE

Cournot Competition (Strategy game)

Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

- Price $p(q_1+q_2) = \max(0, a - b(q_1+q_2))$
- Costs $c_i(q_i) = cq_i$
- Payoffs $u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$
- Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

The Nash equilibria is give by $\left\{\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)\right\}$

Stackleberg Competition(主从博弈)

Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

- Price $p(q_1+q_2) = \max(0, a - b(q_1+q_2))$
- Costs $c_i(q_i) = cq_i$
- Payoffs $u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$
- Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Difference: player 1 choose q_1 first, then player 2 choose q_2 after observe q_1

Stackleberg Competition (Continued)

- This is an extensive game, and we look for SPE.
- Back Induction - Not a finite game but with finite length
- Look at a subgame by player 1 with q_1. Then, player 2's maximization problem is to

$$\max_{q_2 \geq 0} \quad u_2(q_1, q_2) = (a - b(q_1 + q_2) - c)q_2$$

- This gives the best response for player 2

$$q_2 = (a - c - bq_1)/2b$$

No difference

Stackleberg Competition (Continued)

The difference: player 1 will choose q_1 after the recognition of player 2's best response.

Player 1 is the leader; player 2 is the follower

The problem of player 1 is

$$\max_{q_1 \geq 0} \quad u_1(q_1, q_2) = (a - b(q_1 + q_2) - c)q_1$$

$$\text{subject to} \quad q_2 = (a - c - bq_1)/2b$$

This implies that

$$\max_{q_1 \geq 0} \quad (a - b(q_1 + (a - c - bq_1)/2b) - c)q_1$$

Stackleberg Competition (Continued)

We get the best response for player 1

$$q_1 = (a - c)/2b$$

This gives the best response for player 2

$$q_2 = (a - c)/4b$$

SPE: The player 1 has advantages

Ultimatum Game

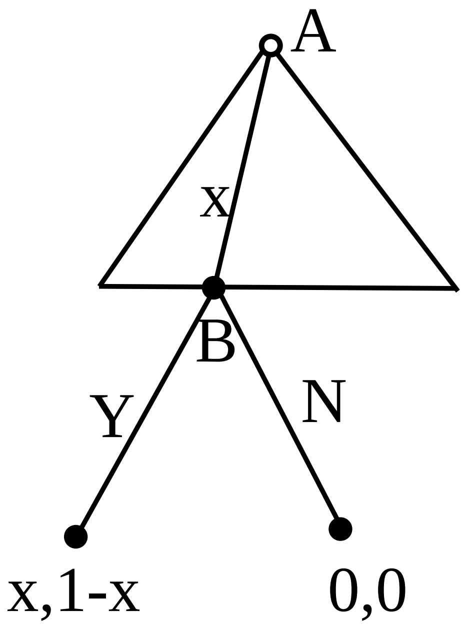
The ultimatum game

- Two players bargain over 1 unit:
  - Player A offers player B some amount $1 - x \leq 1$
  - If player B accepts the outcome is: $(x, 1 - x)$
  - If player B rejects the outcome is: (0, 0)
- Each person cares about the amount of money received. Assume that x can be any scalar, not necessarily integral.
- Question: What is an SPE for this game?

Ultimatum Game

Back induction to find the SPE

- Player B's optimal strategy
  - If $x < 1$, then accept
  - If $x = 1$, then accept or reject
- If player B accept for any $x \in [0,1]$
  - What is the optimal offer by A? $x = 1$
  - The SPE is (1,Y)
- If player B accept if and only if $x \in [0,1)$
  - What is the optimal offer by A? No solution



Unique SPE (1,Y)