Game Theory and Applications (博弈论及其应用)

# Chapter 5.2: Correlated Equilibrium

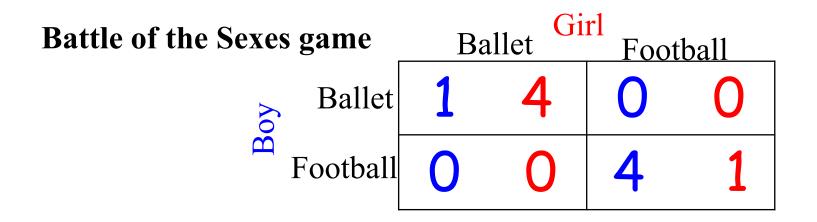
南京大学

高尉



# Correlated Strategies

- In a Nash equilibrium, players choose strategies (or randomize over strategies) independently.
- For games with multiple NE, one may want to allow for randomizations in NE by some form of communication prior.



For fairness, randomization between two pure strategy NE: flip a coin and go to the Ballet for heads coin; the Football otherwise.

Payoffs: (5/2, 5/2) that is not a Nash equilibrium payoff

Consider two cars arrive at an intersection simultaneously.

- Car A has the options U(up) or D(down)
- Car B has the options L(left) or R (right)

		Car B L R					
ГA	U	5	1	0	0		
Car	D	4	4	1	5		

Two pure strategy NE: (U, L) and (D, R)

To find a mixed strategy NE, assume car A plays U with probability p and player 2 plays L with probability q. We have

$$5q = 4q + (1 - q) \Rightarrow q = 1/2$$
  
 $5p = 4p + (1 - p) \Rightarrow p = 1/2$ 

There is a unique mixed strategy NE with expected payoff (5/2, 5/2)

- Assume there is a observable random variable (traffic light)
  - with probability 1/2 (Green): car A plays U and car B plays L
  - with probability 1/2 (Red): car A plays D and car B plays R.
- The expected payoff for this play of the game increases to (3,3)
- No player has an incentive to deviate from the "recommendation"
  - if car A sees a Head, he believes that car B will play L, and thus playing U is his best response (similar argument when he sees a Tail).
  - if car B sees a Head, he believes that car A will play U, and thus playing L is his best response (similar argument when he sees a Tail).
- When the recommendation of the traffic light is part of a Nash equilibrium, no player has an incentive to deviate

- With a observable random variable, we can get any payoff vector in the convex hull of Nash equilibrium payoffs
  - The convex hull of a finite number of vectors  $a_1, \ldots, a_k$  is given by  $\operatorname{Conv}(a_1, \ldots, a_k) = \{a : a = \sum a_i \lambda_i, \lambda_i \geq 0, \sum \lambda_i = 1\}$
- The traffic light is one way of communication prior to the play.
- A more elaborate signal scheme: suppose the players find a mediator who chooses  $\xi \in \{1, 2, 3\}$  with equal probability 1/3.
  - If  $\xi = 1$ , then car A plays U, car B plays L.
  - If  $\xi$  = 2, then car A plays D, car B plays L.
  - If  $\xi$  = 3, then car A plays D, car B plays R.

- We show that no player has an incentive to deviate from the "recommendation" of the mediator:
  - If car A gets the recommendation U, he believes car B will play L, so his best response is to play U.
  - If car A gets the recommendation D, he believes car B will play L, R with equal probability, so playing D is a best response.
  - If car B gets the recommendation L, he believes car A will play U, D with equal probability, so playing L is a best response.
  - If car B gets the recommendation R, he believes car A will play D, so his best response is to play R.
- The players will follow the mediator's recommendations.
- The expected payoffs are (10/3, 10/3), strictly higher than that of randomization in NEs

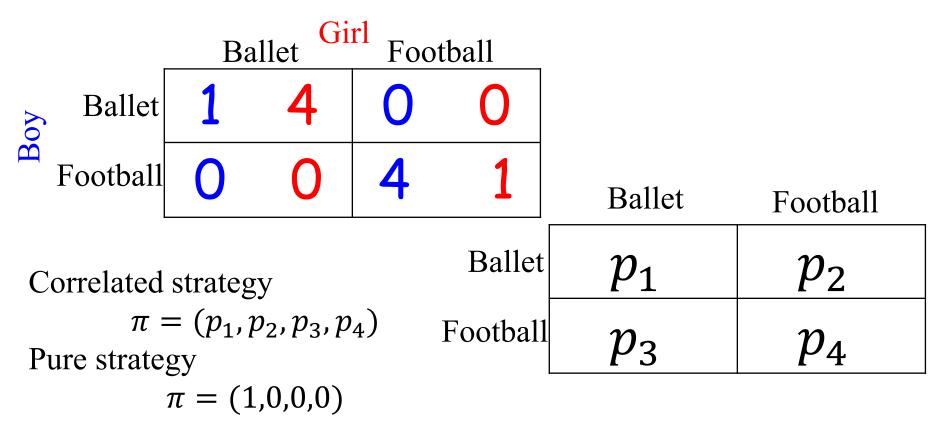
# Correlated Strategy

## Correlated strategies

- Let  $\Delta(A)$  denote set of probability over  $A = A_1 \times A_2 \times \cdots \times A_n$
- Let  $\pi$  be a distribution over A
- Let R be a random variable taking values in A according to  $\pi$ .

What's the difference among correlated strategy, pure strategy and mixed Strategy?

# Example



Mixed strategy

$$\pi = (pq, (1-p)q, p(1-q), (1-p)q, (1-p)(1-q))$$

The set of correlated strategy pairs is an extension of the set of mixed strategy pairs

# Correlated Equilibrium

**Definition**: A correlated equilibrium of game  $G = \{N, \{A_i\}, \{u_i\}\}$  is a joint probability distribution  $\pi \in \Delta(A)$  such that if R is a random variable distributed according to  $\pi$  then

$$\sum_{a_{-i}} \Pr(R = a | R_i = a_i) \left[ u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) \right] \ge 0$$

for all players i, all  $a_i, a_i' \in A_i$  such that  $Pr(R_i = a_i) > 0$ 

No player can ever expect to unilaterally gain by deviating from his recommendation, assuming the other players play according to their recommendations.

# Characterization of Correlated Equilibrium

**Proposition**: A joint distribution  $\pi \in \Delta(A)$  is a correlated equilibrium if and only if

$$\sum_{a_{-i}} \Pr(R = a) \left[ u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) \right] \ge 0$$

for all player i and all  $a_i, a'_i \in A_i$ 

Proof. See board

## Conclusions

**Theorem**: Every mixed NE is a correlated equilibrium

**Theorem**: The set of correlated equilibria is a convex set

**Corollary**: The set of correlated equilibria contains the convex hull of the set of Nash equilibria

# How to Compute?

			Gir B	1	F		
o j	В	2	5	0	0	$p_1$	$p_2$
$\mathbf{m}$	F	0	0	5	2	$p_3$	$p_4$

The condition for a correlated equilibrium are

$$2 \times p_{1} + 0 \times p_{2} \ge 0 \times p_{1} + 5 \times p_{2} \rightarrow p_{1} \ge 5p_{2}/2$$

$$0 \times p_{3} + 5 \times p_{4} \ge 2 \times p_{3} + 0 \times p_{4} \rightarrow p_{4} \ge 2p_{3}/5$$

$$5 \times p_{1} + 0 \times p_{3} \ge 0 \times p_{1} + 2 \times p_{3} \rightarrow p_{1} \ge 2p_{3}/5$$

$$0 \times p_{2} + 2 \times p_{4} \ge 5 \times p_{2} + 0 \times p_{4} \rightarrow p_{4} \ge 5p_{2}/2$$

$$p_{1} + p_{2} + p_{3} + p_{4} = 1$$

Find appropriate solutions by solving the LP problem

# Compute Correlated Equilibrium

For every player  $i, a_i, a'_i \in A$ 

$$\sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a_i, a_{-i}) \ge \sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a'_i, a_{-i})$$

subject to

$$p(a_i, a_{-i}) \ge 0$$
 and  $\sum p(a_i, a_{-i}) = 1$ 

variable  $p(a_i, a_{-i})$  constant  $u(a_i, a_{-i})$ 

# Comparisons

$$\sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a_i, a_{-i}) \ge \sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a'_i, a_{-i})$$
Subject to  $\sum p(a_i, a_{-i}) = 1$  and  $p(a_i, a_{-i}) \ge 0$ 

Correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities

$$p(a_i, a_{-i}) = \prod_j p_j(a_j)$$

which is not a linear programming