Game Theory and Applications (博弈论及其应用)

Chapter 15: Extensive Game with Imperfect Information-III

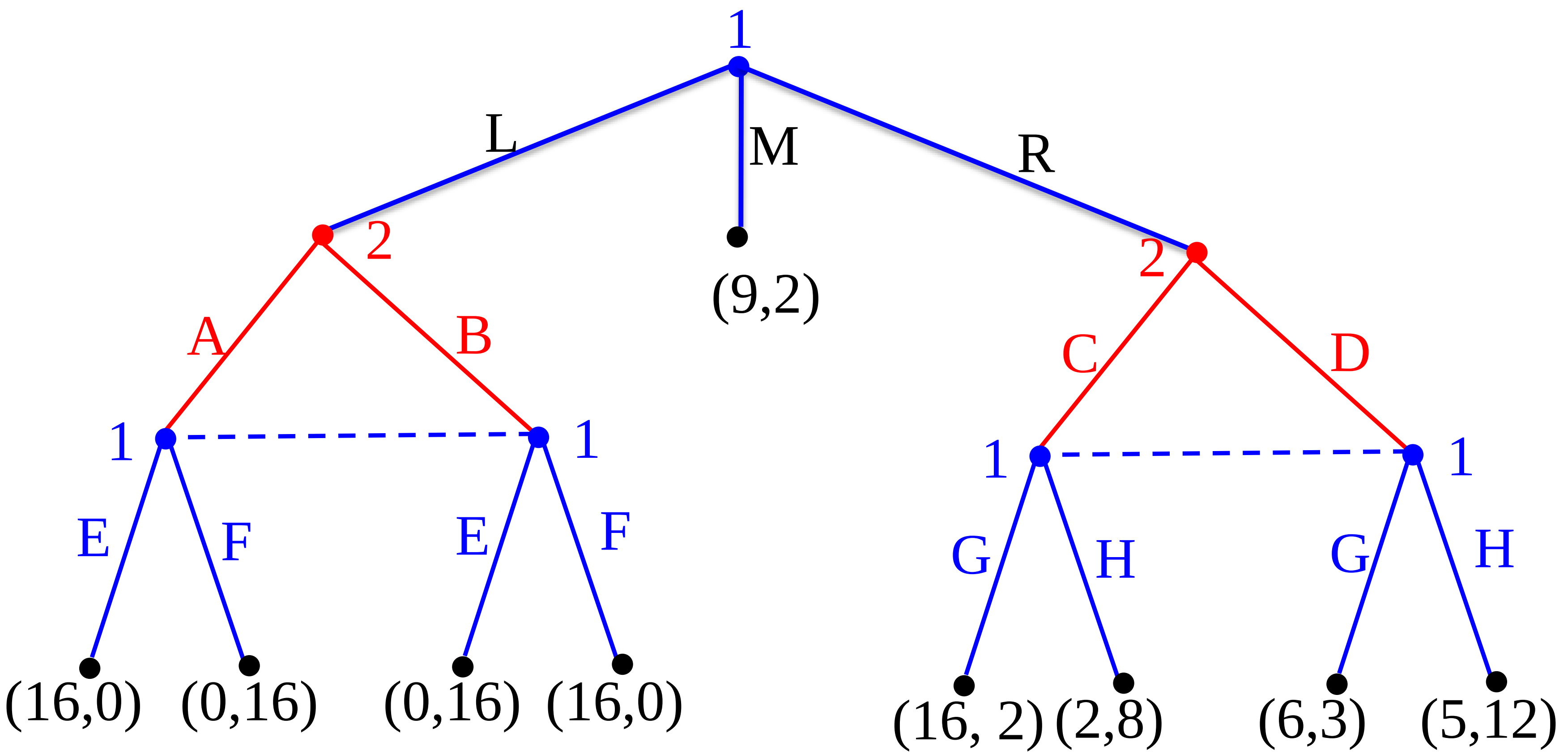
南京大学

高 尉

Recap on Previous chapter

- Extensive game with imperfect information $G = \{N, H, P, I, \{u_i\}\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- Mixed strategies
- Behavior strategies
- Subgame Perfect Nash Equilibrium

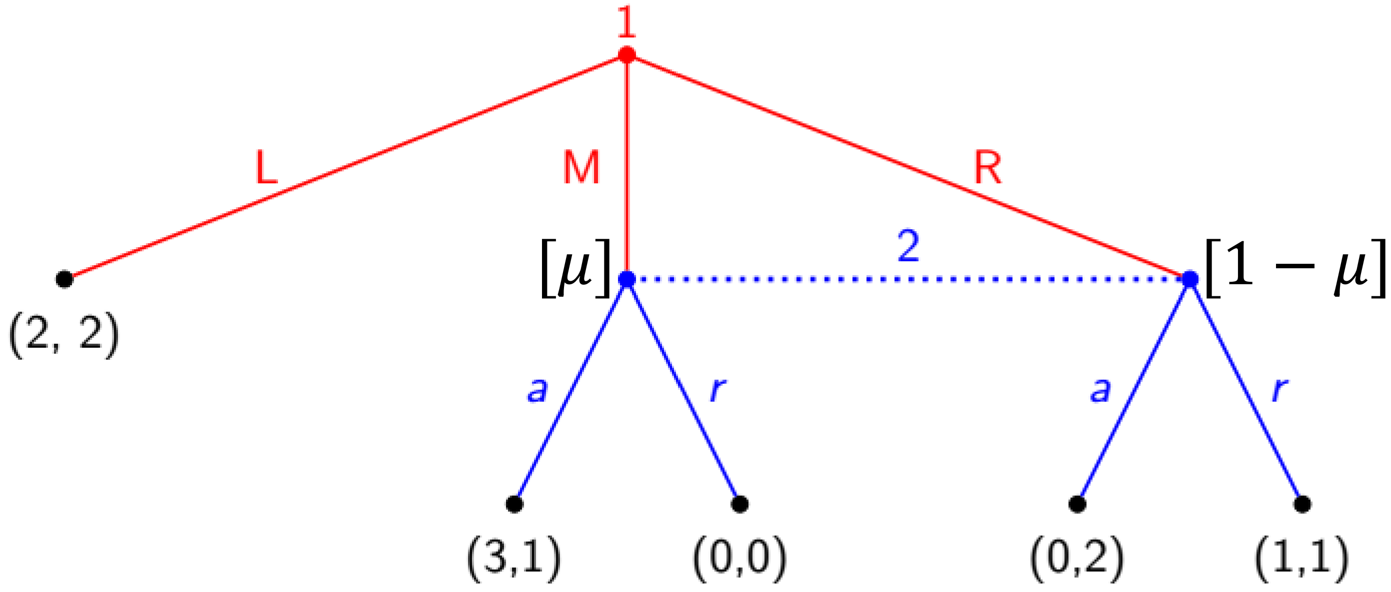
Example



How to solve SPNE?

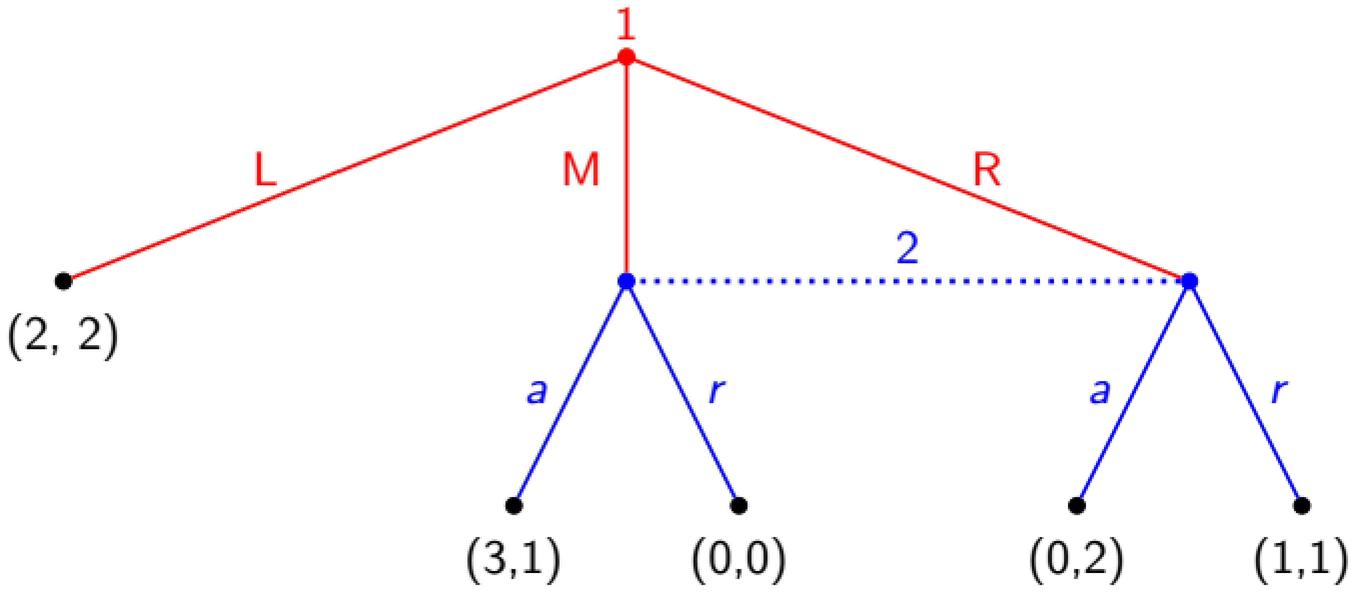
Beliefs

- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- The probability is 1 for the information set of size 1

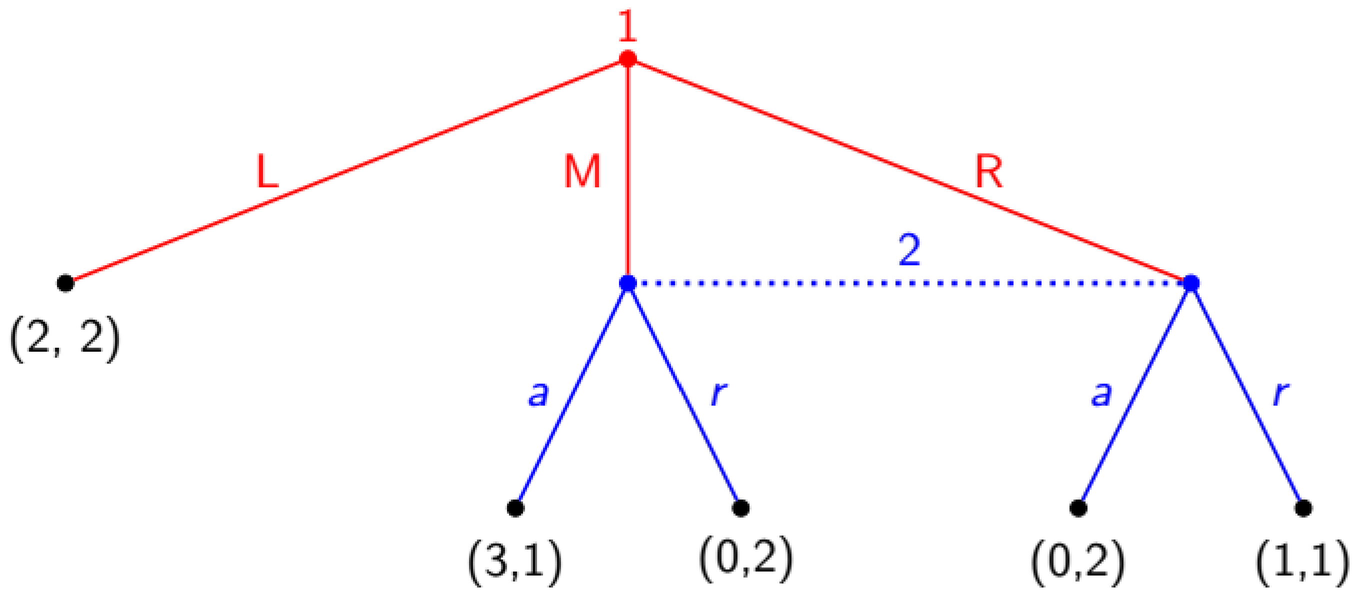


Behavioral Strategies

- A behavior strategy β a collection of independent probability measure over the actions after information set



Beliefs and Optimal Behavior Strategies

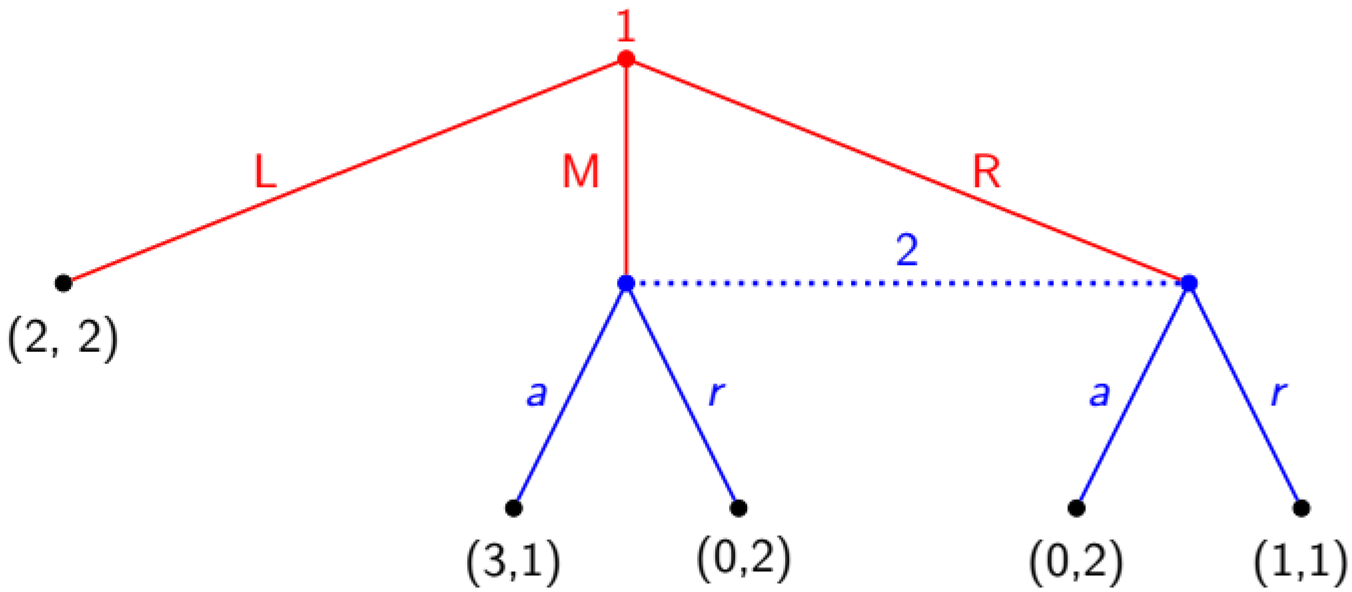


- Beliefs affect optimal strategies: For 2, a is the best strategies iff 2 assigns a belief $\mu(M) \leq 1/2$
- Strategies affect reasonable beliefs: If 1 assigns to action (L,M,R) prob. (0.1,0.3,0.6), then Bayes rule requires the belief (1/3,2/3) of 2
- What are reasonable beliefs if 1 select L with prob. 1

Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

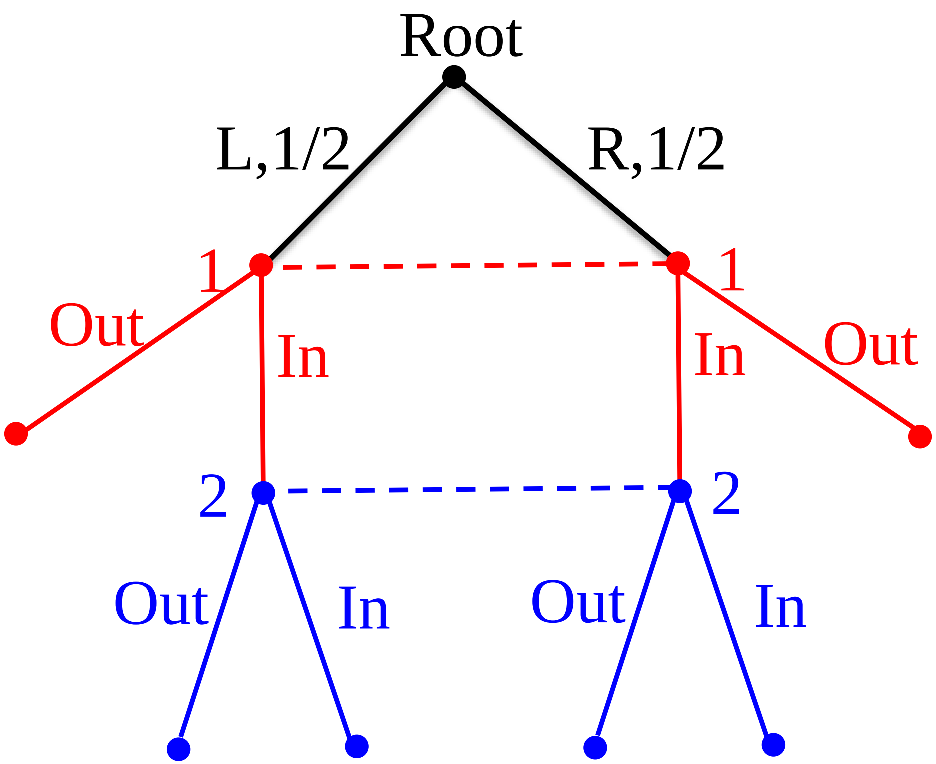
Assessment (评估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h}, \mathrm{a}) = \beta_{h,a}(h, a) / \sum_a \beta_{h,a}(h, a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent $(\beta^n, \mu^n) \rightarrow (\beta, \mu)$ as $n \rightarrow \infty$
- (β, μ) is consistent $\rightarrow$ (β, μ) Bayesian consistent

Example



- The payoffs are omitted since they are irrelevant
- Find all Bayesian consistent assessments
- Find all consistent assessments

Bayesian consistency

An assessment (β, μ) by a 4-tuple $(\beta_1, \beta_2, \mu_1, \mu_2) \in [0,1]^4$

- β_1 is the probability that 1 chooses In
- β_2 is the probability that 2 chooses In
- μ_1 is the belief assigns to the left node in 1's info set
- μ_2 is the belief assigns to the left node in 2's info set

Two cases:

– i) If $\beta_1 \in (0,1]$, 2's information set is reached with positive probability. Bayes' Law dictates that $\mu_1 = \mu_2 = 1/2$.

$$(\beta_1, \beta_2, \mu_1, \mu_2) = (0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

are Bayesian consistent

– ii) If $\beta_1 = 0$, then 2's information set is reached with zero probability and $\mu_2 \in [0,1]$

$$(\beta_1, \beta_2, \mu_1, \mu_2) = \{0\} \times [0,1] \times \{1/2\} \times [0,1]$$

are Bayesian consistent

Consistency

- Every complete outcome of behavioral strategies leads to $\mu_1 = \mu_2 = 1/2$.
- 2's information set, both nodes are reached with equal probability.
- Conclusion:

$$(\beta_1, \beta_2, \mu_1, \mu_2) = [0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

are consistent

Expected Payoffs in Information Sets

Fix assessment (β, μ) and information set I_{ij} of player i. We consider the expected payoff of player i on I_{ij} as

- Given I_{ij}, the belief μ assigns probability over I_{ij} with $\mu(h)$ for $h \in I_{ij}$
- For $h \in I_{ij}$, let $P(e|h, \beta)$ the probability from h to e under the behavioral strategy β, and the payoff is $u_i(e)$

The expected payoff for player i in the information I_{ij} w.r.t. (β, μ), is

$$u_i(\beta_i, \beta_{-i} | I_{ij}, \mu) = \sum_{h \in I_{ij}} \mu(h) (\sum_e P(e|h, \beta) u_i(e))$$

Sequential Rational

Assessment (β, μ) is **sequentially rational** if for each information set I_{ij}, player i makes a best response w.r.t. belief μ, that is,

$$u_i\left(\beta_i, \beta_{-i} \middle| I_{ij}, \mu\right) \geq u_i\left(\beta_i', \beta_{-i} \middle| I_{ij}, \mu\right)$$

for all other behavior strategies β_i' of player i

- Consistency: beliefs have to make sense w.r.t strategies, without requirements on strategies
- Sequential rationality: strategies have to make sense w.r.t. beliefs, without requirements on beliefs

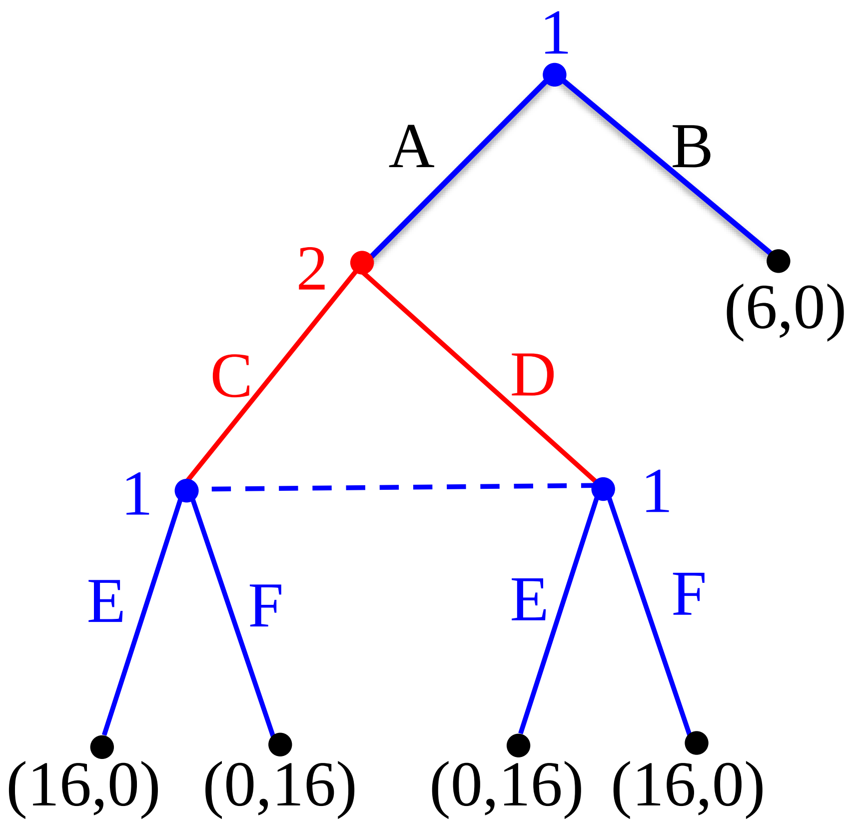
Sequential Equilibrium

An assessment (β, μ) is a **sequential equilibrium** if it is both consistent and sequentially rational.

Theorem

- a) Each finite extensive form game with perfect recall has a sequential equilibrium.
- b) If assessment (β, μ) is a sequential equilibrium, then β is a subgame perfect equilibrium.

Example



How to calculate the sequential equilibrium?

Example (Consistency)

Behavioral strategies $\beta = (\beta_1, \beta_2) = (p, r; q)$, where

- p: probability that 1 chooses A;
- q: probability that 2 chooses C;
- r: probability that 1 chooses E;

Belief μ can be summarized by one probability α

- α: probability assigns to history AC in inform. set {AC,AD}

- If $p, q, r \in (0,1)$, then Bayes' law gives

$$\alpha = \frac{pq}{pq + p(1-q)} = q$$

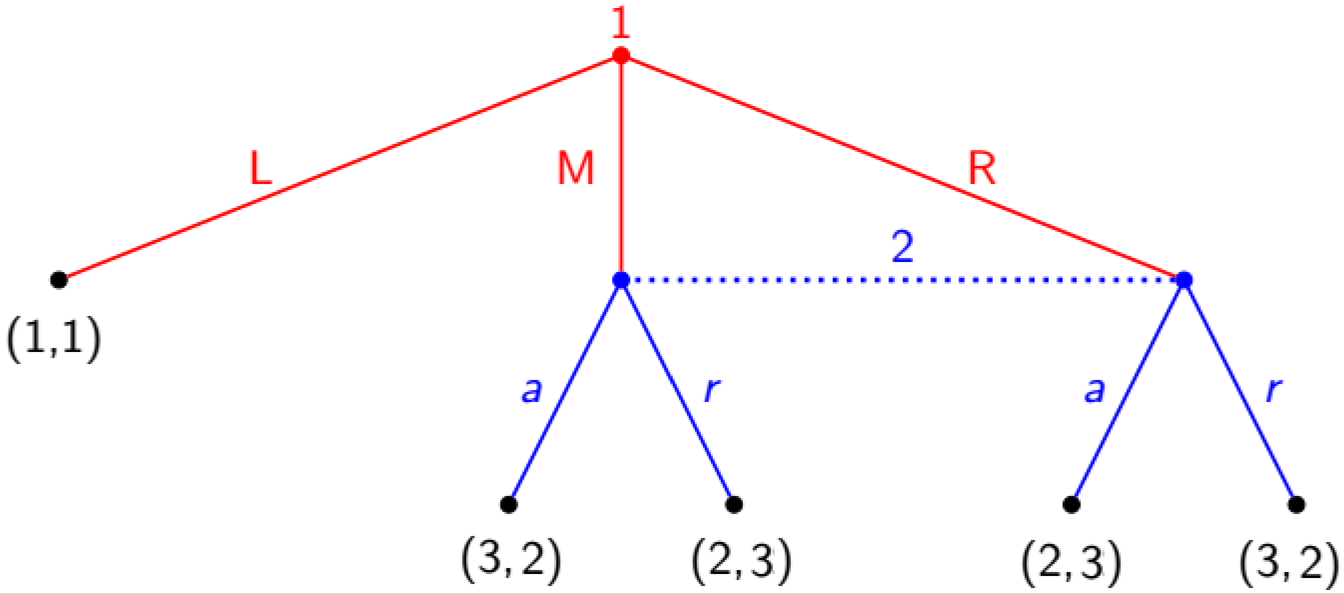
For each consistent (β, μ), we have $\alpha = q$

Example (Rationality)

- If $q = 0$, then $\alpha = 0$ and $r = 0$ is player 1's unique best reply in the final info set. But if $r = 0$, then $q = 0$ is not a best reply in 2's info set. Contradiction.
- If $q = 1$, then $\alpha = 1$ and $r = 1$ is player 1's unique best reply in the final info set. But if $r = 1$, then $q = 1$ is not a best reply in 2's info set. Contradiction.
- If $q \in (0,1)$
  - rationality of 2 dictates that both C and D must be optimal and equal, i.e., $16(1 - r) = 16r$, this gives $r = 1/2$
  - In info set (AC,AD), the expected payoff of player 1 is $\alpha 16r + (1 - \alpha)16(1 - r) = 16 - 16q + 16r(1 - 2q)$
    - $r = 0$ if $q > 1/2$; $r = 1$ if $q < 1/2$; and $r \in [0,1]$ if $q = 1/2$

$r = 1/2$ if and only if $q = 1/2$. Finally $p = 1$

Exercise



Exercise