

Game Theory and Applications (博弈论及其应用)

# Chapter 5.1 : Applications II

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## Recap on Previous Chapter

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Continuous game  $G = \{N, \{A_i\}, \{u_i\}\}$

Every continuous game has at least one mixed strategy NE

If  $u_i(a_i, a_{-i})$  is continuous and concave in  $a_i$  for a continuous game  $\{N, \{A_i\}, \{u_i\}\}$ , then there exists a pure strategy NE

## Applications

- ① Product Competition Model (Cournot and Bertrand)
- ② War of attribution

# Auction

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- Open bid auctions
  - Ascending-bid auction
    - Price is raised until only one bidder remains, who wins and pays the final prize
  - Descending-bid auction
    - Price is lowered until someone accepted, who wins the product at the current prize
- Sealed bid auctions
  - First/**second prize** auction
    - Highest bidder wins, pays the first/second highest bid

# First Price Auction (Two players)

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$N = \{1,2\}$ : players bid a building

$v_i \geq 0$ : the true value for player  $i$  ( $v_1 > v_2 > 0$ )

$b_i \geq 0$ : the bid price for player  $i$

Player 1 bids successfully if  $b_1 = b_2$

The payoff functions for player  $i$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1 \\ 0 & \text{otherwise} \end{cases}$$

# First Price Auction (Two players)

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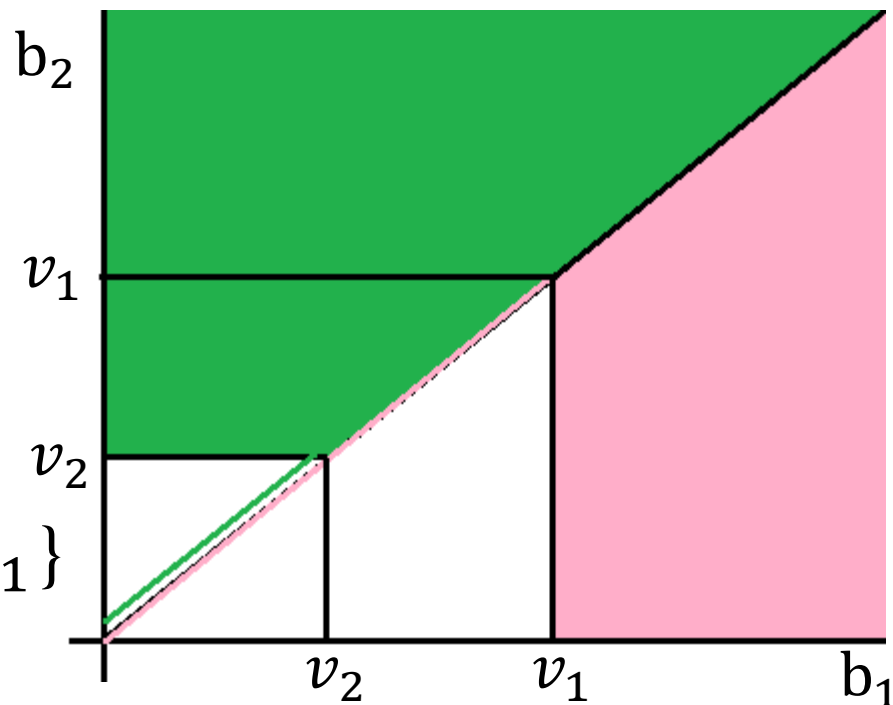
The best response functions

$$B_1(b_2) = \begin{cases} \{b_1: b_1 < b_2\} & \text{if } b_2 \geq v_1 \\ \{b_1: b_1 = b_2\} & \text{if } b_2 < v_1 \end{cases}$$

$$B_2(b_1) = \begin{cases} \{b_2: b_2 \leq b_1\} & \text{if } b_1 \geq v_2 \\ \{b_2: b_2 = b_1 + \epsilon\} & \text{if } b_1 < v_2 \end{cases}$$

The Nash Equilibrium

$$\{(b_1^*, b_2^*): v_2 \leq b_1^* = b_2^* \leq v_1\}$$



## First Price Auction ( $N$ players)

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$N = \{1, 2, \dots, N\}$ : players bid a building

$v_1 > v_2 > \dots > v_N > 0$ : the true value for player  $i$

$b_i \geq 0$ : the bid price for player  $i$

The payoff functions for player  $i$

$$u_1(b_1, \dots, b_N) = \begin{cases} v_1 - b_1 & \text{if } b_1 \geq \max \{b_j\}_{j \neq 1} \\ 0 & \text{otherwise} \end{cases}$$

$$u_i(b_1, \dots, b_N) = \begin{cases} v_i - b_i & \text{if } b_i > \max \{b_j\}_{j \neq i} \\ 0 & \text{otherwise} \end{cases}$$

# Necessary Condition

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**Theorem** If  $(b_1^*, \dots, b_N^*)$  is a NE, then  $b_1^* \geq b_i^*$  and  $b_1^* \geq v_2$

*Pf.* Assume  $b^* = (b_1^*, \dots, b_N^*)$  is a NE, and there is  $b_i^* > b_1^*$ .

If  $b_i^* > v_2$ , then  $u_i(b^*) < 0 < u_i(b_1^*, \dots, b_{i-1}^*, 0, b_{i+1}^*, \dots, b_N^*)$ , and  $b^*$  is not a NE.

If  $b_i^* \leq v_2$ , then  $u_1(b^*) = 0 < u_1(v_2^*, b_2^*, \dots, b_N^*)$  and  $b^*$  is not a NE.

If  $b_1^* < v_2$ , then  $u_2(b^*) = 0 < u_2(b_1^*, b_1^* + (v_2 - b_1^*)/2, b_3^* \dots b_N^*)$

# First Price Auction ( $N$ players)

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There are many Nash equilibria

$$\left\{ (b_1^*, \dots, b_N^*) : \begin{array}{l} \text{i) } v_1 \geq b_1^* \geq v_2; \quad \text{ii) } b_1^* \geq b_i^* \text{ for all } i; \\ \text{iii) } b_1^* = b_k^* \text{ for some } k \end{array} \right\}$$



# Meeting Problem

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- Persons A and B chat very well today, and they decide to meet again between 1:00 and 2:00 tomorrow
- However, they forget to decide the specific time and they do not have the contact information
- Rule: One person will wait at most 10 minutes, and then leave if he do not meet the other
- Problem: do the two persons will meet

# General Persons

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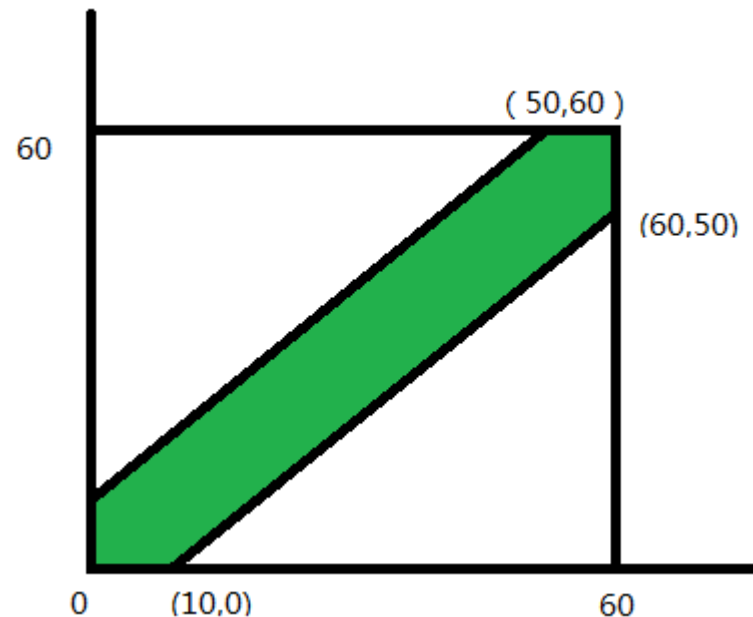
A arrives:  $x \in [0,60]$       B arrives:  $y \in [0,60]$

If A and B meet, then

$$|x - y| \leq 10$$

which implies  $x - y \leq 10$  and  $x - y \geq -10$

Probability is  $11/36 (< 1/3)$



# Smart Persons

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- If A arrives 1:00, then B meets [1:00, 1:10], prob. 10/60
- If A arrives 1:01, then B meets [1:00, 1:11], prob. 11/60
- ...
- If A arrives 2:10, then B meets [2:00, 2:20], prob. 20/60
- Both A and B are very smart, they will select [2:10-2:50]
- Repeat this process, they will select [2:20 2:40]
- Repeat this process, they will select [2:30 2:30]
- The NE is {2:30,2:30}

# Election

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- Several candidates vote for political office
- Each candidate chooses a policy position
- Each citizen, who has preferences over policy positions, votes for one of the candidates
- Candidate who obtains the most votes wins.

## Strategic game:

- Players: candidates
- Set of actions of each candidate: set of possible positions
- Payoff is 1 for winner; is 0.5 for ties; and is 0 for loser
- Note: Citizens are not players in this game

## Example

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- Two candidates  $N = \{1,2\}$
- Set of possible position:  $b_1, b_2 \in [0,1]$
- Citizens are continuous, and are distributed uniformly on  $[0,1]$ , and vote for the candidate with closet position.
- Payoff

$$u_i(b_1, b_2) = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0.5 & \text{if } i \text{ ties} \\ 0 & \text{if } i \text{ loses} \end{cases}$$

# Best Response

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The best response function  $B_i(b_j)$  is give as follows:

- If  $b_j < 1/2$ , then  $B_i(b_j) = \{b_i: b_j < b_i < 1 - b_j\}$
- If  $b_j = 1/2$ , then  $B_i(b_j) = \{b_i: b_i = 1/2\}$
- If  $b_j > 1/2$ , then  $B_i(b_j) = \{b_i: 1 - b_j < b_i < b_j\}$

The Nash Equilibrium  $(1/2, 1/2)$

