Game Theory and Applications (博弈论及其应用)

Chapter 14: Extensive Game with Imperfect Information-III

南京大学

高 尉

Recap on Previous Chapter

- Extensive game with imperfect information
- Formal definition $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \ldots I_N\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- SPNE and NE
- Perfect recall and imperfect recall

Definition of Mixed and Behavioral Strategies

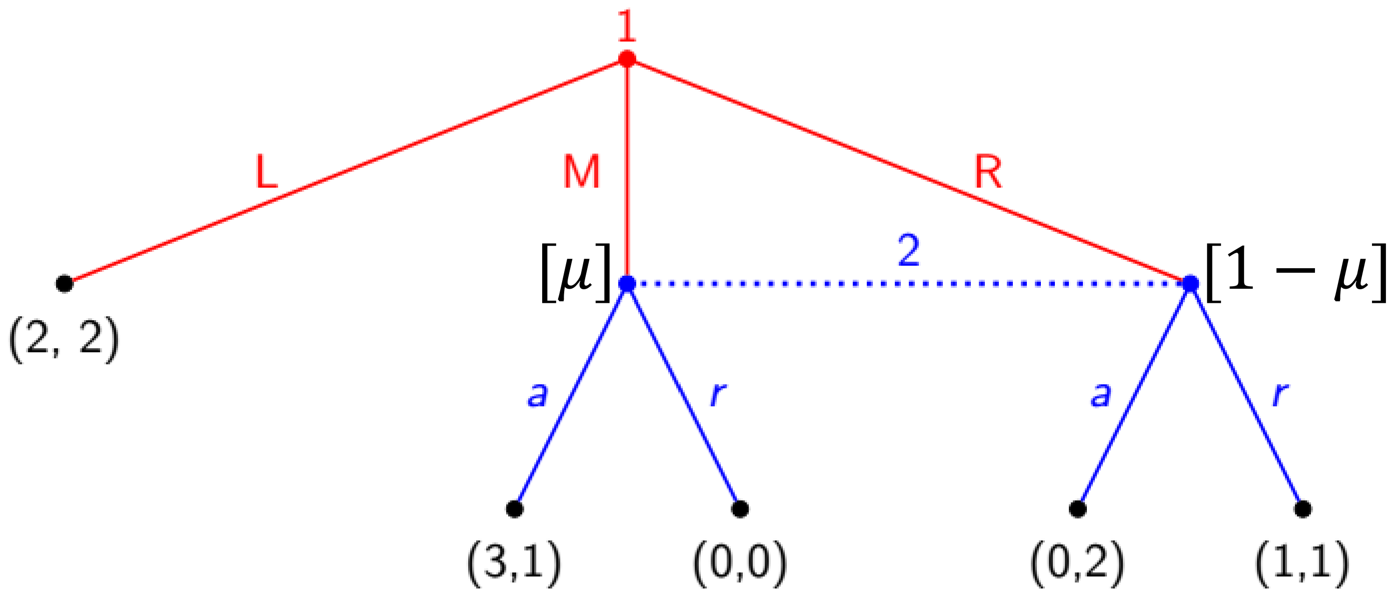
- Mixed Strategies: A mixed strategy of player i is a probability over the set of player i's pure strategy
- Behavioral strategies: A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik}\in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $\mathrm{A}(I_{ik})$

Theorem In an finite extensive game with perfect recall

- any mixed strategy of a player can be replaced by an equivalent behavioral strategy
- any behavioral strategy can be replaced by an equivalent mixed strategy
- Two strategies are equivalent

Beliefs

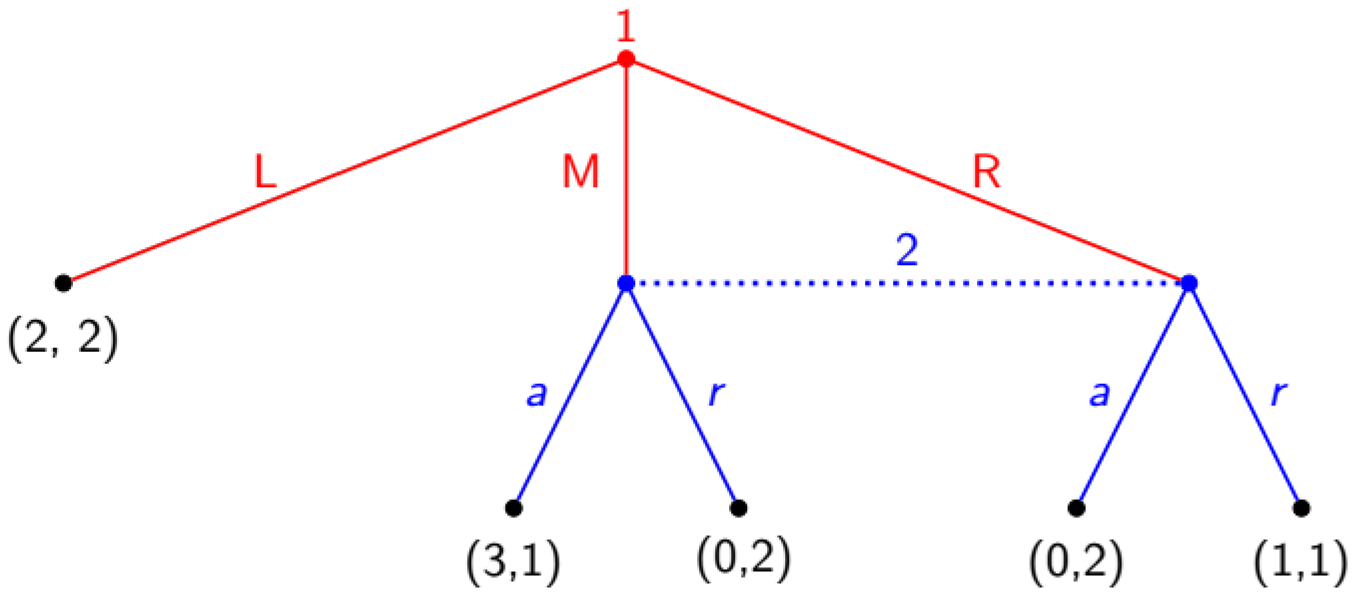
- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- A behavior strategy β a collection of independent probability measure over the actions after information set



Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

Assessment (评估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h},\mathrm{a}) = \beta_{h,a}(h,a) / \sum_a \beta_{h,a}(h,a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent $(\beta^n, \mu^n) \to (\beta, \mu)$ as $n \to \infty$
- (β, μ) is consistent $\rightarrow$ (β, μ) Bayesian consistent

Expected Payoffs in Information Sets

Fix assessment (β, μ) and information set I_{ij} of player i. We consider the expected payoff of player i on I_{ij} as

- Given I_{ij}, the belief μ assigns probability over I_{ij} with $\mu(h)$ for $h \in I_{ij}$
- For $h \in I_{ij}$, let $P(e|h, \beta)$ the probability from h to e under the behavioral strategy β, and the payoff is $u_i(e)$

The expected payoff for player i in the information I_{ij} w.r.t. (β, μ), is

$$u_i\left(\beta_i, \beta_{-i} \middle| I_{ij}, \mu\right) = \sum_{h \in I_{ij}} \mu(h) \left(\sum_e P(e|h, \beta) u_i(e)\right)$$

Sequential Rational

Assessment (β, μ) is **sequentially rational** if for each information set I_{ij}, player i makes a best response w.r.t. belief μ, that is,

$$u_i\left(\beta_i, \beta_{-i} \middle| I_{ij}, \mu\right) \geq u_i\left(\beta_i', \beta_{-i} \middle| I_{ij}, \mu\right)$$

for all other behavior strategies β_i' of player i

- Consistency: beliefs have to make sense w.r.t strategies, without requirements on strategies
- Sequential rationality: strategies have to make sense w.r.t. beliefs, without requirements on beliefs

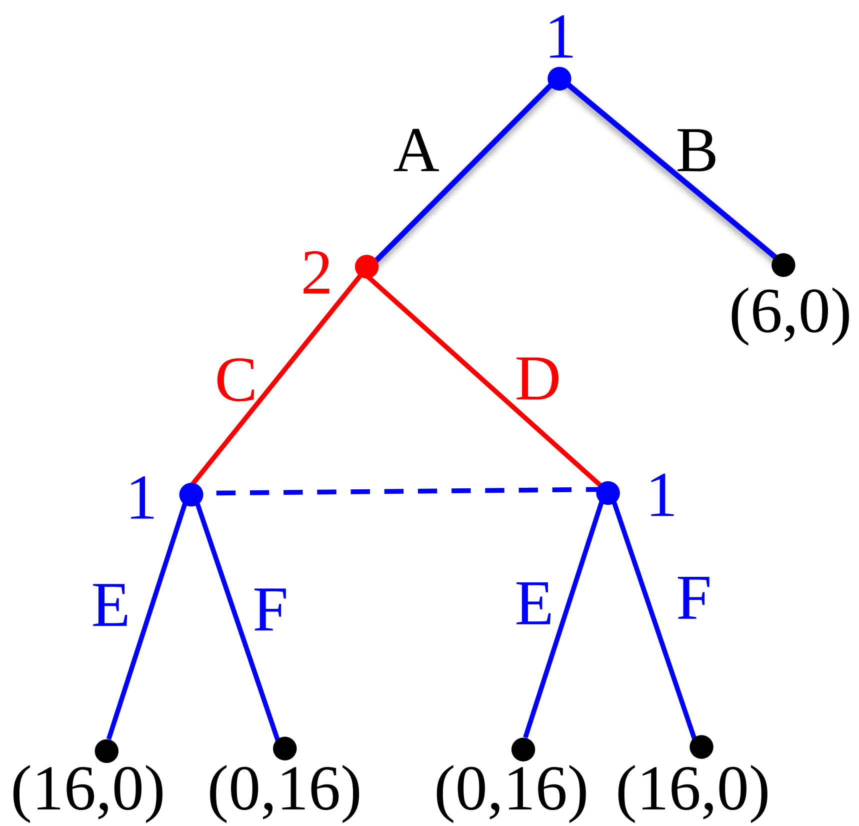
Sequential Equilibrium

An assessment (β, μ) is a **sequential equilibrium** if it is both consistent and sequentially rational.

Theorem

- a) Each finite extensive form game with perfect recall has a sequential equilibrium.
- b) If assessment (β, μ) is a sequential equilibrium, then β is a subgame perfect equilibrium.

Example



How to calculate the sequential equilibrium?

Example (Consistency)

Behavioral strategies $\beta = (\beta_1, \beta_2) = (p, r; q)$, where

- p: probability that 1 chooses A;
- q: probability that 2 chooses C;
- r: probability that 1 chooses E;

Belief μ can be summarized by one probability α

- α: probability assigns to history AC in inform. set {AC,AD}

- If $p, q, r \in (0,1)$, then Bayes' law gives

$$\alpha = \frac{pq}{pq + p(1-q)} = q$$

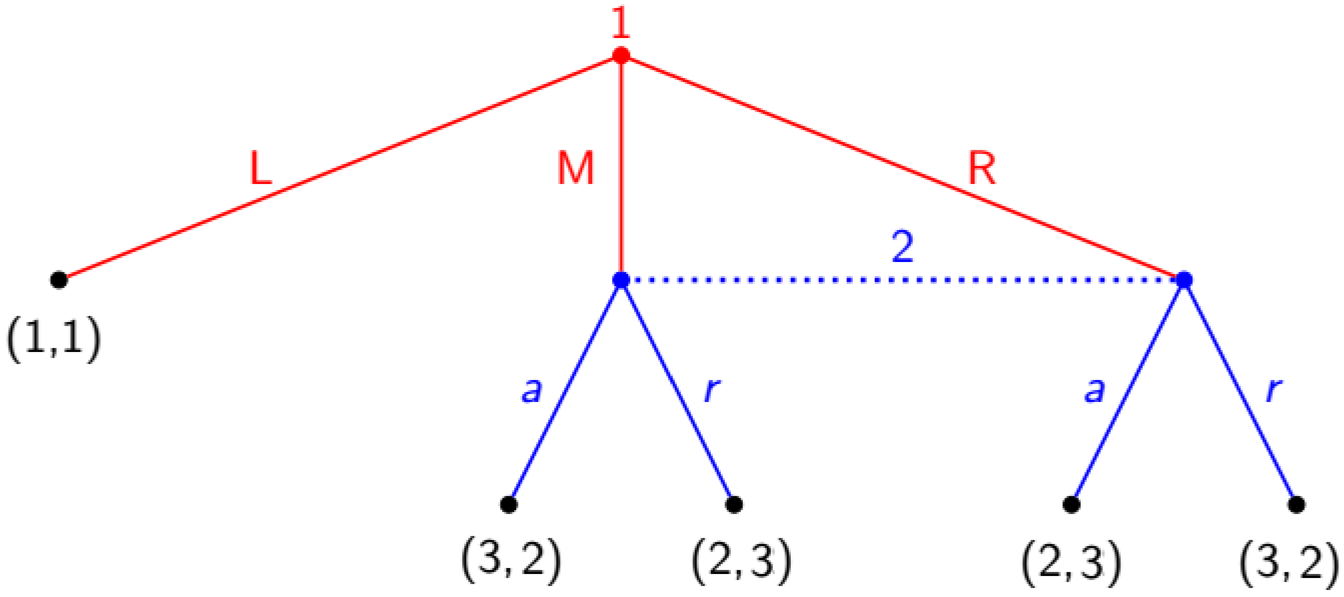
For each consistent (β, μ), we have $\alpha = q$

Example (Rationality)

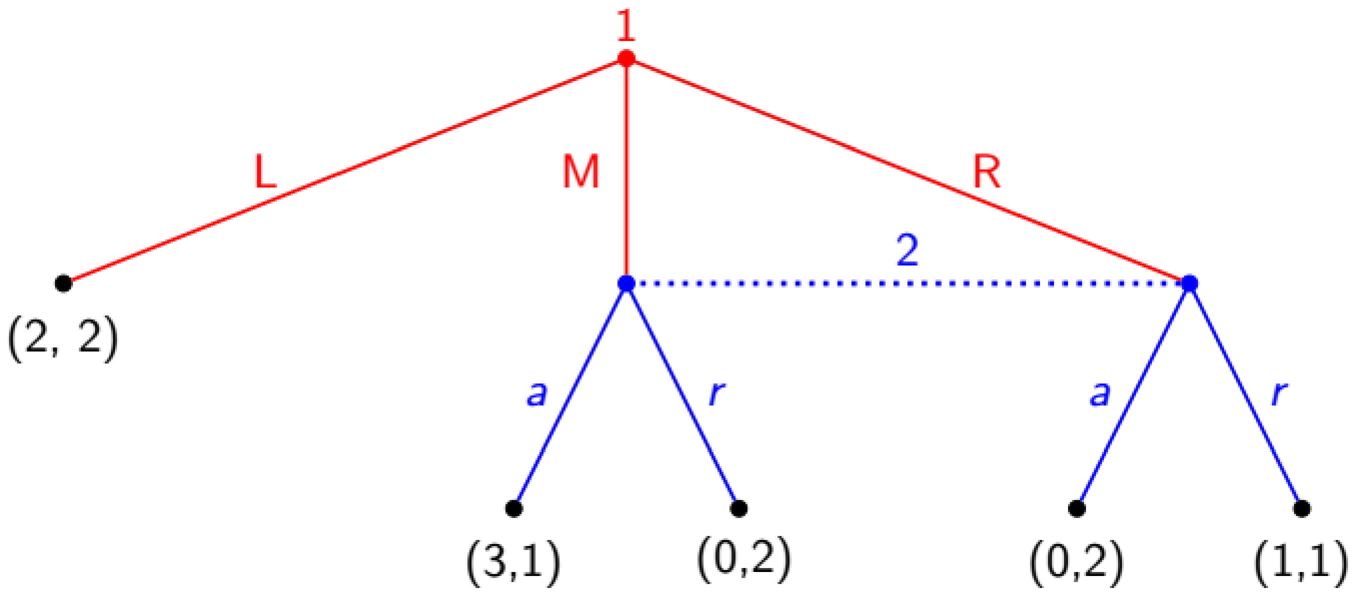
- If $q = 0$, then $\alpha = 0$ and $r = 0$ is player 1's unique best reply in the final info set. But if $r = 0$, then $q = 0$ is not a best reply in 2's info set. Contradiction.
- If $q = 1$, then $\alpha = 1$ and $r = 1$ is player 1's unique best reply in the final info set. But if $r = 1$, then $q = 1$ is not a best reply in 2's info set. Contradiction.
- If $q \in (0,1)$
  - rationality of 2 dictates that both C and D must be optimal and equal, i.e., $16(1 - r) = 16r$, this gives $r = 1/2$
  - In info set (AC,AD), the expected payoff of player 1 is $\alpha 16r + (1 - \alpha)16(1 - r) = 16 - 16q + 16r(1 - 2q)$
    - $r = 0$ if $q > 1/2$; $r = 1$ if $q < 1/2$; and $r \in [0,1]$ if $q = 1/2$

$r = 1/2$ if and only if $q = 1/2$. Finally $p = 1$

Exercise



Exercise



Signaling games (信 号 传 递 博 弈)

The most interesting class of games that are solved used the sequential Equilibrium concept are signaling games

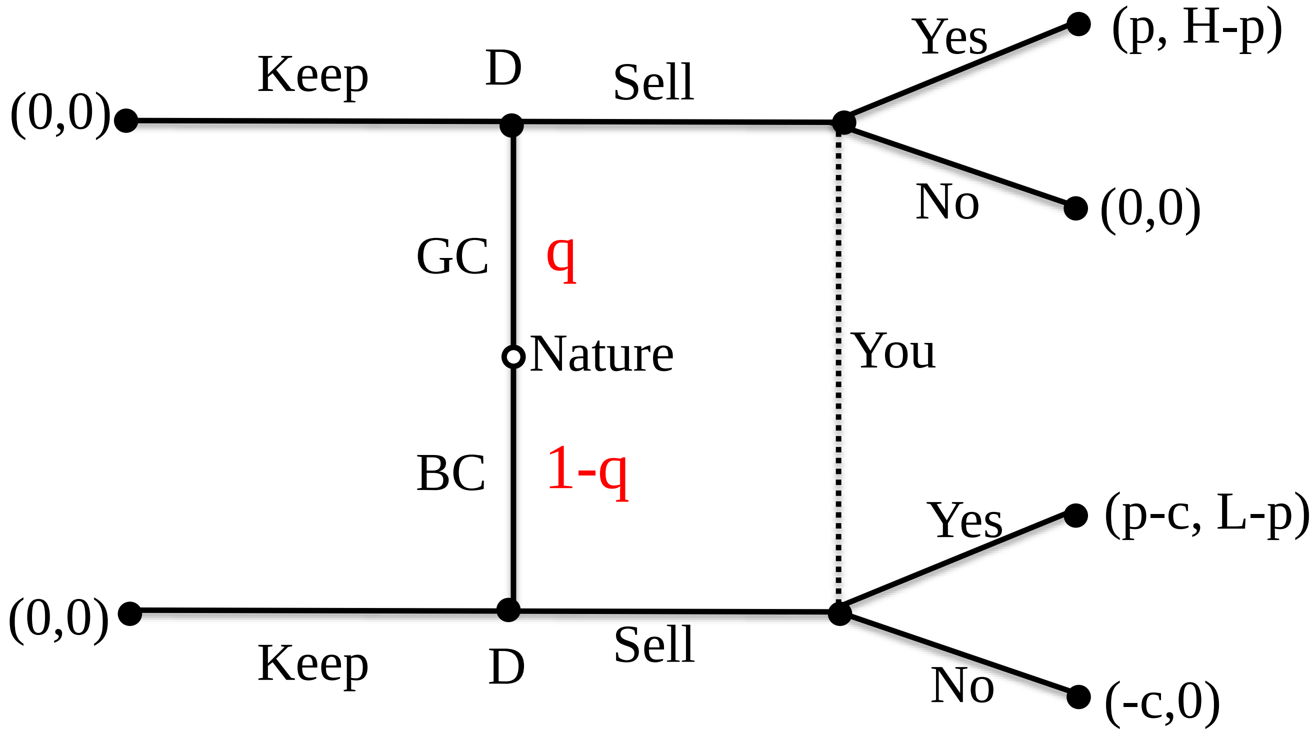
Michael Spence, 2001 Nobel Memorial Prize in economics: job-market signaling model

- A prospective employer can hire an applicant.
- The applicant has high or low ability, but the employer doesn't know which
- Applicant can give a signal about ability, e.g., education

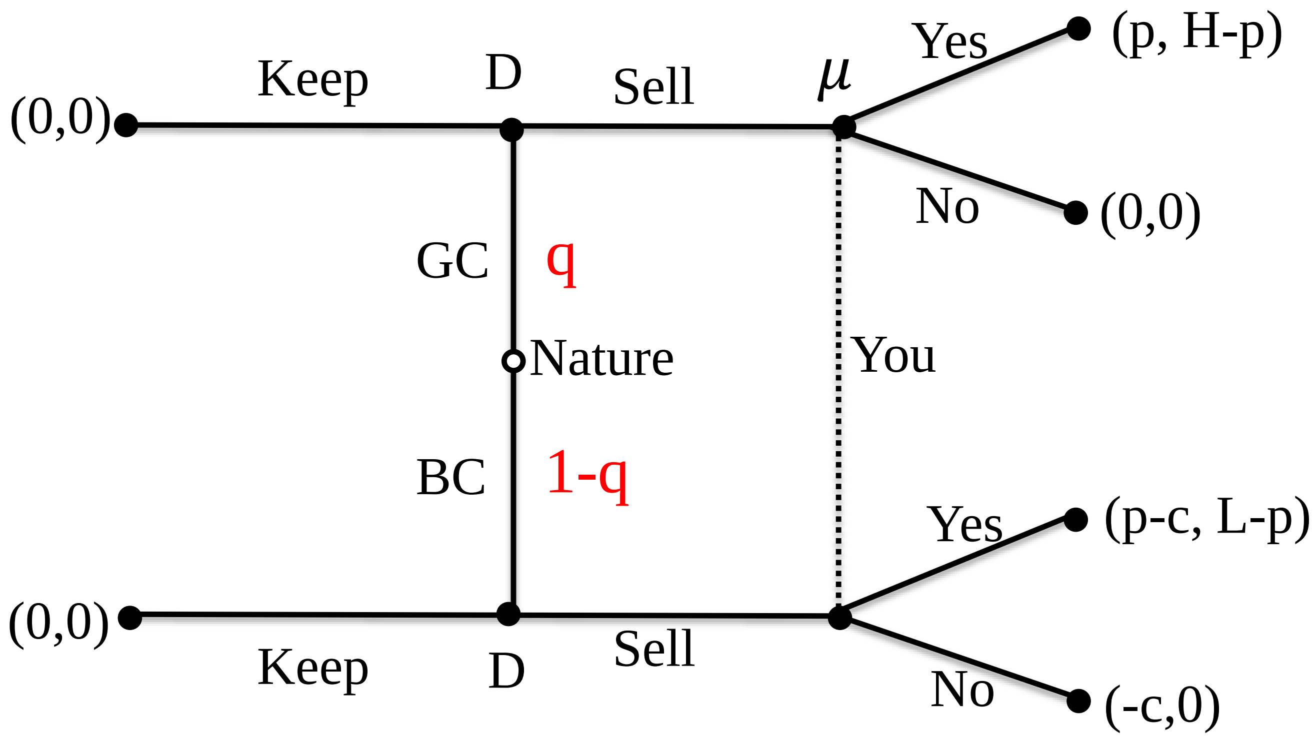
Signaling Games: Used-Car Market

- You want to buy a used-car which may be either good or bad
- A good car is worth H and a bad one L dollars
- You cannot tell a good car from a bad one but believe a proportion q of cars are good
- The car you are interested in has a price p
- The dealer knows quality but you don't
- The bad car needs additional costs c to make it look like good
- The dealer decides whether to put a given car on sale or keep
- You decide whether to buy or not
- Assume $H > p > L$

Signaling Games: Used-Car Market



Belief



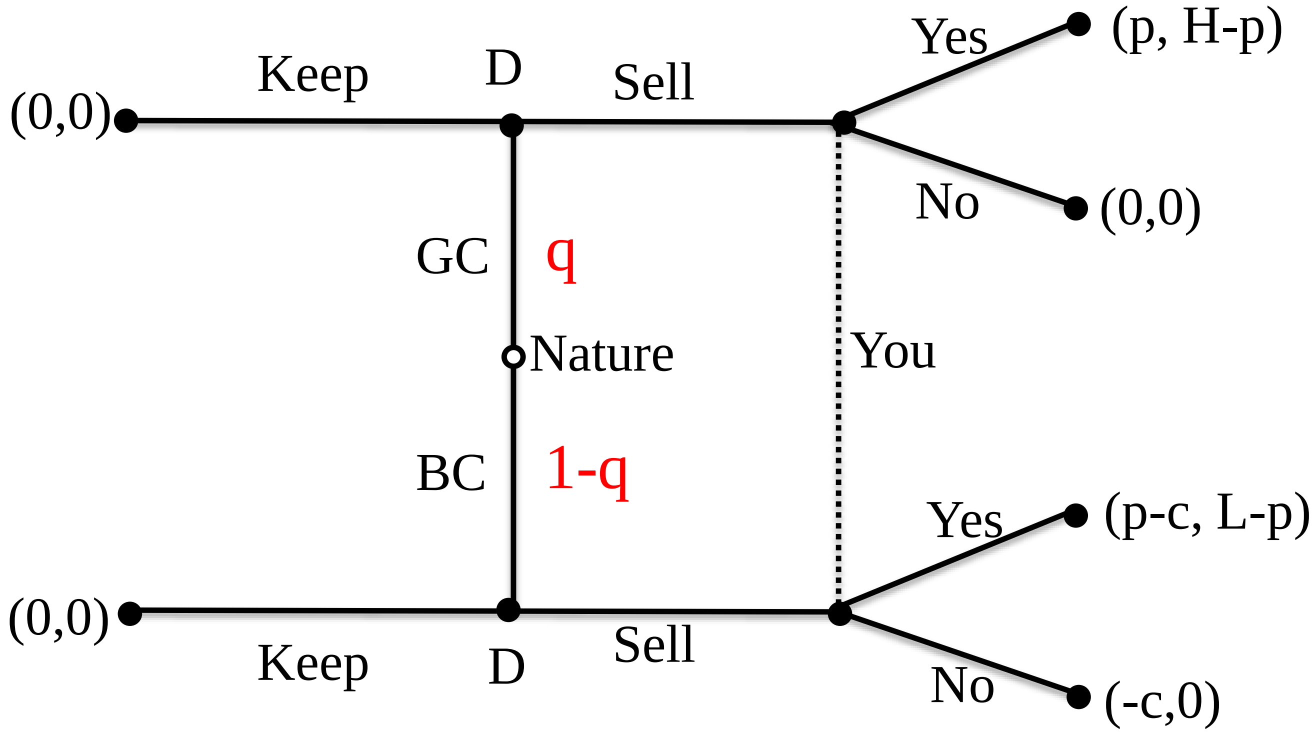
Dealer strategy: Offer if good; Hold if bad

每一种情况都客观存在
这种是保证信誉

What is your consistent belief if you observe the dealer sell a car?

$$\mu = \frac{P(\text{GC and sell})}{P(\text{sell})} = \frac{q \times 1}{q \times 1 + 0 \times (1-q)} = 1$$

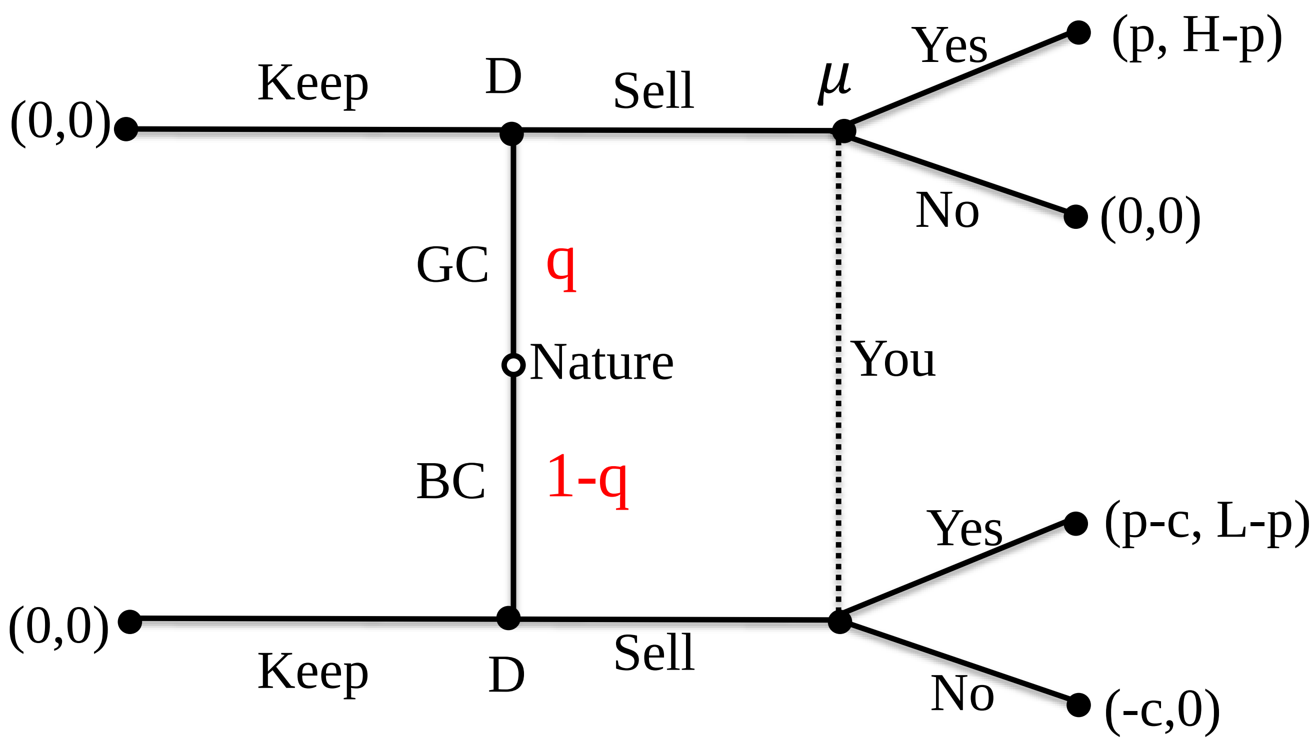
Signaling Games: Used-Car Market



We look for two types of equilibria

1) Pooling Equilibria: GC and BC dealer paly the same strategy
2) Separating Equilibria: GC and BC dealer paly different strategy

Pooling Strategy: Both Sell



Both strategies: Sell
Belief:

$$\mu = \frac{q}{1 \times q + 1 \times \ (1 - q)} = q$$

Pooling Strategy: Both Sell

- If Y buys a car with your prior beliefs q your expected payoff is
  $$V = q \times (H - p) + (1 - q) \times (L - p) \geq 0$$
- What does sequential rationality of seller imply?
- You must be buying and it must be the case that $p \geq c$

Pooling Equilibrium I

If $p \geq c$ and $V \geq 0$ the following is a PBE

Behavioral Strategy Profile: (GC: Sell, BC: Sell), (Y: Yes)

Belief System: $\mu = q$

Pooling Equilibria: Both Keep

You must be saying No

– Otherwise Good car dealer would offer

Under what conditions would Ysay No?

$$\mu \times (H - p) + (1 - \mu) \times (L - p) \leq 0$$

So we can set $\mu = 0$

The following is a PBE

Behavioral Strategy Profile: (Good: Hold, Bad: Hold),(You: No)

Belief System: $\mu = 0$

Market failure: a few bad car can ruin a market

Separating Equilibria - Good: Offer and Bad: Hold

- What about your beliefs?

$$\mu = 1$$

- What does you sequential rationality imply?
  - You say Yes
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - Yes if $p \leq c$
- If $p \leq c$ the following is a PBE
  Behavioral Strategy Profile: (Good: Offer, Bad: Hold), (You: Yes)
  Belief System: $\mu = 1$

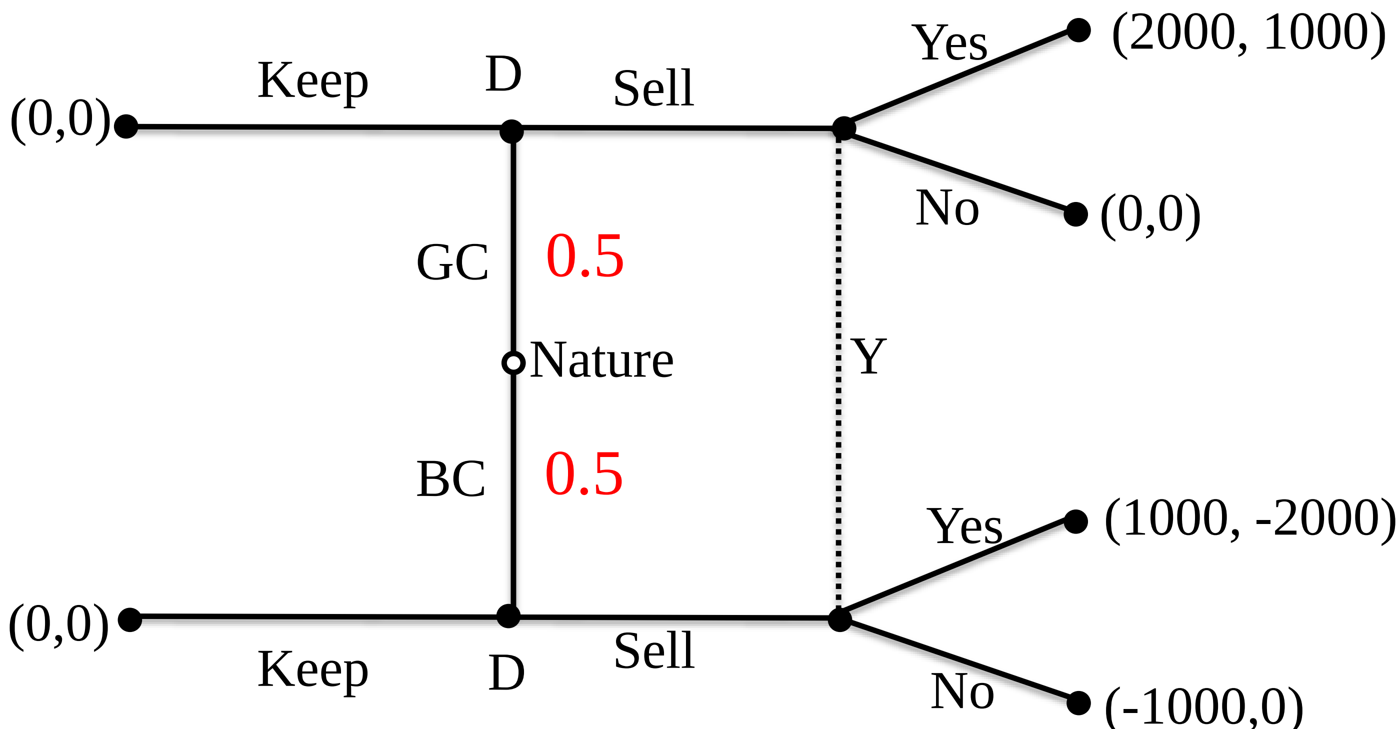
Separating Equilibria - Good: Keep and Bad: Sell

- What does Bayes Law imply about your beliefs?

$$\mu = 0$$

- What does you sequential rationality imply?
  - You say No
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - No
- There is no PBE in which Good dealer Holds and Bad dealer Offers

Behavior Strategy



Behavior strategy: Yes Prob. $x \in (0,1)$

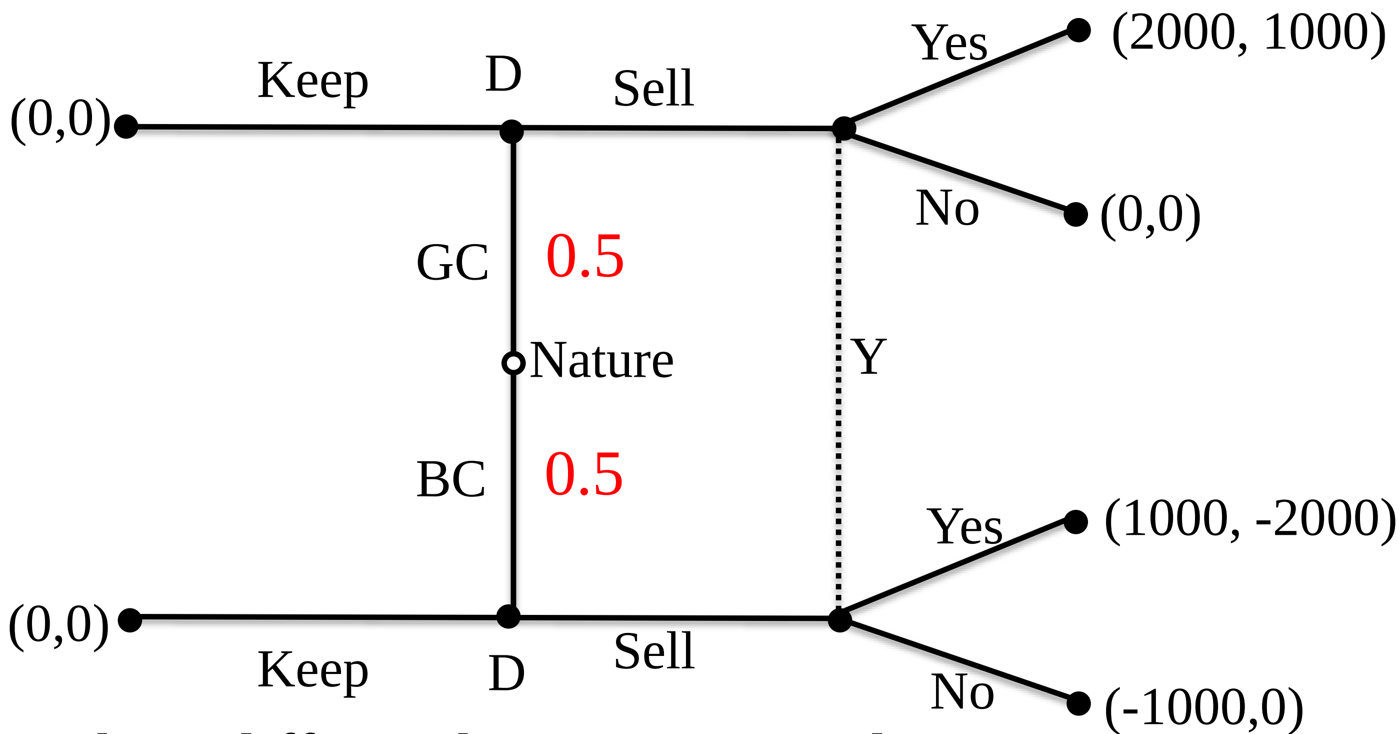
Behavior strategy: BC – sell Prob. y

Belief: GC – sell Prob. μ

You must be indifferent between Yes and No

$1000\mu - (1-\mu)2000 = 0$ implies μ=2/3

Behavior Strategy



You must be indifferent between Yes and No

$$1000\mu - (1-\mu)2000 = 0 \text{ implies } \mu = 2/3$$

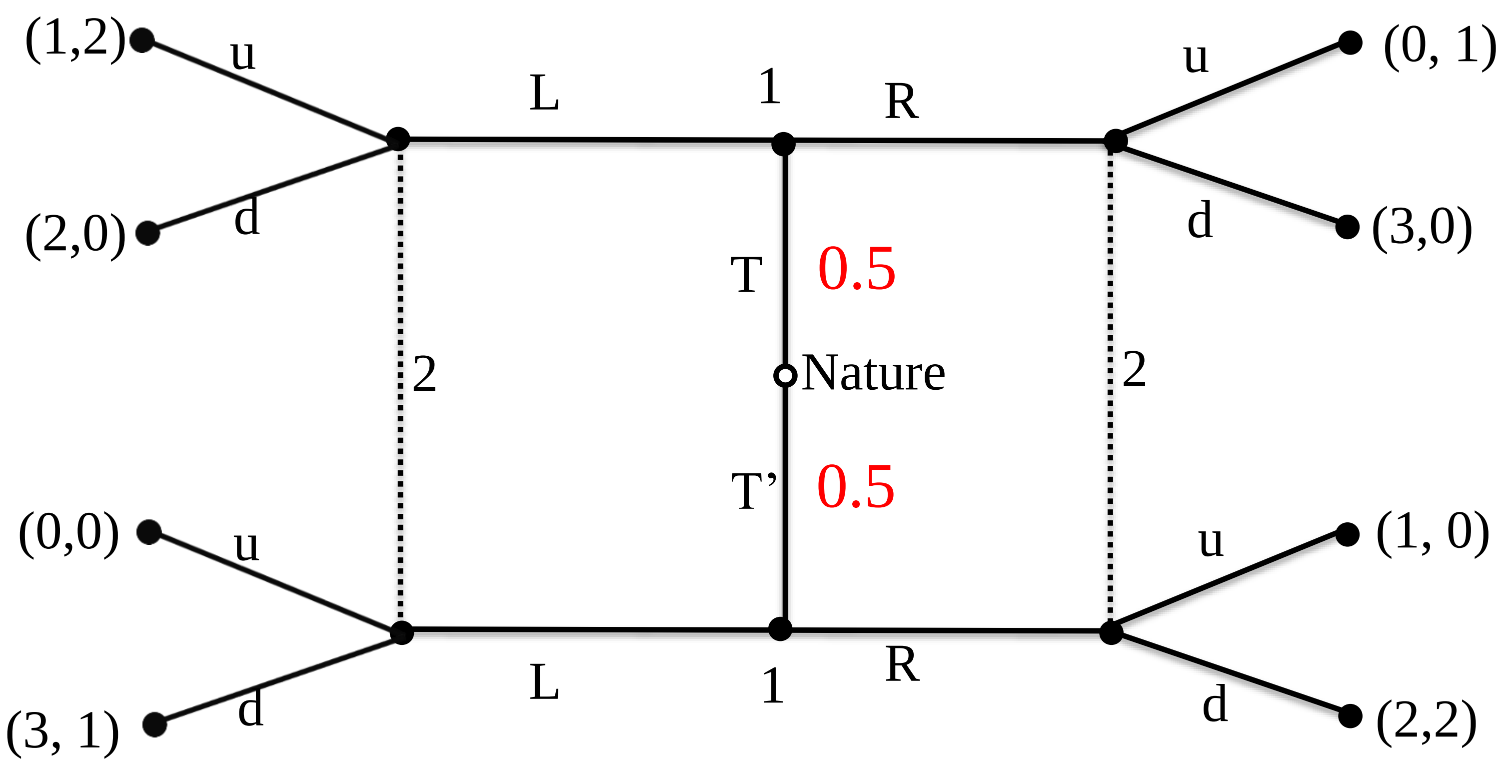
$$\frac{0.5}{0.5+0.5y} = \frac{2}{3} \text{ implies } y = 0.5$$

Bad car dealers must be indifferent between Keep and Sell

$$0 = 1000x - 1000(1-x) \text{ implies } x = 0.5$$

Signaling Game: Another Example

所有的序列Equ都是SPNE



(a) Find the corresponding strategic form game and its pure-strategy Nash equilibria.
(b) Determine (if any) the game's separating equilibria.
(c) Determine (if any) the game's pooling equilibria.

Signaling Game: Another Example

a) P1's pure strategies are pairs in LL, LR, RL, RR

P2's pure strategies are pairs in uu, ud, du, dd

还要带上belief

	uu		ud		du		dd	
LL 上下	0.5	1	0.5	1	2.5	0.5	2.5	0.5
LR	1	1	1.5	2	1.5	0	2	1
RL	0	0.5	1.5	0	1.5	1	3	0.5
RR	0.5	0.5	2.5	1	0.5	0.5	2.5	1

Nash equilibrium ((R; R); (u; d)).

(b): Separating equilibria must be Nash equilibria:

((R; R); (u; d))

Pooling equilibria, no separating equilibria.

Signaling Game: Another Example

The candidate strategy ((R; R); (u; d))

But what should the belief system be? Let $\alpha_1, \alpha_2 \in [0,1]$ denote the prob. assigned to the top

Bayesian consistency: requires that $\alpha_2 = 1/2$, $\alpha_1 \in [0,1]$

Sequential rationality:

- ((R; R); (u; d)) is a NE
- P2's payoff from u is $2\alpha_1 + 0(1-\alpha_1)$ and from d is $0\alpha_1 + 1(1 - \alpha_1)$, so requires $\alpha_1 \geq \frac{1}{3}$

Conclude: Assessments (s1; s2; β) with strategies

- (s1; s2) = ((R; R); (u; d)) and belief system
- $\beta = (\alpha_1, \alpha_2), \alpha_1 \in [1/3,1]$ $\alpha_2 = 1/2$ are pooling equilibria