Game Theory and Applications (博弈论及其应用)

Chapter 7: Two-Player Zero-Sum Game

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Recap on the previous chapter

- Strategy game with incomplete information
- Bayes game $G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$
- Bayes Nash Equilibrium
- How to find Bayes Nash equilibrium

Two-Player zero-sum game

Definition A **two-player zero-sum game** is a strategy game $G = \{\{1,2\}, \{A_1, A_2\}, \{u_1, u_2\}\}$ such that

$$u_1(a_1, a_2) + u_2(a_1, a_2) = 0 \text{ for } a_1 \in A_1 \text{ and } a_2 \in A_2$$

One player wins while the other losses

Rock-Paper-Scissors

		Player 2		
		Rock	Paper	Scissors
	Rock	0 0	-1 1	1 -1
Player 1	Paper	1 -1	0 0	-1 1
	Scissors	-1 1	1 -1	0 0

Chess

War are seldom zero-sum game

Example

We consider a zero-sum game

Player 2

		L	M	R
	U	1 -1	1 -1	8 -8
Player 1	M	5 -5	2 -2	4 -4
	D	7 -7	0 0	0 0

It is not necessary to keep track of both payoffs. We keep the first player payoff only by convention.

The abbreviation is

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

Maxmin (最大化最小原则)

For this game, both player do not do too badly

Player 1 method

- Calculate minimization for each strategy, and maximize

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

Player 1 selects M

$$\mathbf{M \in \operatorname{argmax}_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)}$$

Maxmin

For this game, both player do not do too badly

Player 2 method:

- calculate minimization for each strategy and Maximize

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2)$$

Player 2 selects M

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

Maxmin (最小化最大原则)

Player 2 method:

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2)$$

From $u_2(a_1, a_2) = -u(a_1, a_2)$, we have

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2) = \max_{a_2 \in A_2} \min_{a_1 \in A_1} -u(a_1, a_2)$$

By $\max(-f(x)) = -\min(f(x))$ and $\max(-f(x)) = -\min(f(x))$

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2) = -\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Player 2 method:

$$\boldsymbol{\operatorname*{argmin}_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)}$$

Minmax

For this game, both player do not do too badly

Player 2 method:

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Player 2 selects M

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

Two-players zero-sum method

For this game, both player do not do too badly

Player 1 method

$$\boldsymbol{\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)}$$

Player 2 method

$$\boldsymbol{\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)}$$

		Player 2		
		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Another Example

Another example

		Player 2		
		L	M	R
	U	2	6	1
Player 1	M	3	1	4
	D	4	3	6

Player 1 method

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = 3$$

Player 2 method

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) = 4$$

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) > \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

MinMax ≥ MaxMin

Lemma For two-player zero-sum finite game G, we have

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) \geq \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Proof. See board.

Two-Players Zero-Sum Nash Equilibrium

Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, let player 1 select

$$a_1^* \in \underset{a_1 \in A_1}{\operatorname{argmax}} \min_{a_2 \in A_2} u(a_1, a_2),$$

and let player 2 select

$$a_2^* \in \underset{a_2 \in A_2}{\operatorname{argmin}} \max_{a_1 \in A_1} u(a_1, a_2).$$

The strategy outcome (a_1^*, a_2^*) is a Nash Equilibrium if and only if

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u_1(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u_1(a_1, a_2)$$

Proof. See board.

Find Nash Equilibrium

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

(M, M) is a NE

Player 2

Player 1

	L	M	R
U	2	6	1
M	3	1	4
D	4	3	6

(D, L) is not a NE

Mixed strategy

Strategic game

$N = \{1,2\}$

$A_1 = \{a_1, a_2, \dots, a_m\}, \quad A_2 = \{b_1, b_2, \dots b_n\}$

$u_1(a_i, b_j) = u(a_i, b_j) = u_{ij}$, $\mathrm{M} = (u_{ij})_{m\times n}$

Mixed strategy

$\mathrm{p} = (p_1, p_2, \dots, p_m) \in \Delta_1$ is a mixed strategy over A_1

$q = (q_1, q_2, \dots, q_n) \in \Delta_2$ is a mixed strategy over A_2

The expected payoff for player 1 on mixed outcome (p, q)

$$U(p,q) = \sum_{i,j} p_i q_j u(a_i, b_j) = \sum_{i,j} p_i q_j u_{ij} = pMq^\top$$

MinMax and MaxMin

Player 1's methods:

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} U(p,q) = \max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top$$

Player 2's methods:

$$\min_{q\in\Delta_2}\max_{p\in\Delta_1} U(p,q) = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top$$

Lemma We have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} U(p,q) \leq \min_{q\in\Delta_2}\max_{p\in\Delta_1} U(p,q)$$

Proof See board.

Nash Equilibrium

Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, let player 1 select

$$p^* \in \operatorname*{argmax}_{p \in \Delta_1} \min_{q \in \Delta_2} U(p, q)\,,$$

and let player 2 select

$$q^* \in \operatorname*{argmin}_{q \in \Delta_2} \max_{p \in \Delta_1} U(p, q)\,.$$

The mixed strategy outcome (p^*, q^*) is a MNE if and only if

$$\max_{p \in \Delta_1} \min_{q \in \Delta_2} U(p, q) = \min_{q \in \Delta_2} \max_{p \in \Delta_1} U(p, q)$$

John von Neumann’s Minimax Theorem (1928)

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\},\ \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2} \max_{p\in\Delta_1} pMq^\top .$$

Corollary: Two-person finite zero-sum games have at least one mixed-strategy Nash-equilibrium: any pair of optimal strategies is a Nash equilibrium.

How to Solve?

Theorem The optimization problem of $\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top$ is equivalent to

$$\max v$$

s.t.

$$e_iMq^\top \geq v \quad \text{for } i = 1 \dots n$$
$$q = (q_1, \dots, q_n) \in \Delta_2$$
$$e_i = (0, \dots, 0,1,0, \dots, 0)$$

Proof see board.

Linear programming: can be solved in polynomial time

How to Solve?

Theorem The optimization problem of $\min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top$ is equivalent to

$$\min v$$

s.t.

$$pMe_i{}^\top \leq v \quad \text{for } i = 1 \ldots n$$
$$p = (p_1, \ldots, p_m) \in \Delta_1$$
$$e_i = (0, \ldots, 0,1,0, \ldots, 0)$$

Proof see board.

Linear programming: can be solved in polynomial time

Symmetric Game (2-player zero-sum)

Symmetric strategic game

$N = \{1,2\}$

$A_1 = \{a_1, a_2, \dots, a_n\}, \; A_2 = \{b_1, b_2, \dots b_n\}$

$u_1(a_i, b_j) = u_{ij}$, $\mathrm{M} = (u_{ij})_{n\times n}$, $\mathbf{M} = -\mathbf{M}^\top$

Theorem For a symmetric game, we have

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2} \max_{p\in\Delta_1} pMq^\top = 0$$

Proof. See abroad.

NE for Symmetric Game (2-player zero-sum)

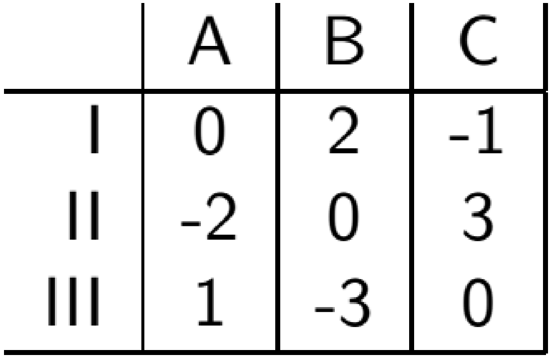
Symmetric strategic game

$N = \{1,2\}$

$A_1 = \{a_1, a_2, \ldots, a_n\}, \; A_2 = \{b_1, b_2, \ldots b_n\}$

$u_1(a_i, b_j) = u_{ij}$, $\mathrm{M} = (u_{ij})_{n\times n}$, $\mathbf{M} = -\mathbf{M}^\top$

Solve: $\mathbf{pM} = \mathbf{0}$ and $\mathbf{p} \in \Delta_1$ and q=p



	A	B	C
I	0	2	-1
II	-2	0	3
III	1	-3	0

How to find Nash Equilibria

1) Calculate directly

- I) find the best response functions
- II) calculate Nash equilibria

2) Eliminate all dominated strategy

3) For two-player zero-sum player, linear programming