Game Theory and Applications (博弈论及其应用)

Chapter 10: Bargaining Game and Nash Bargaining Solutions

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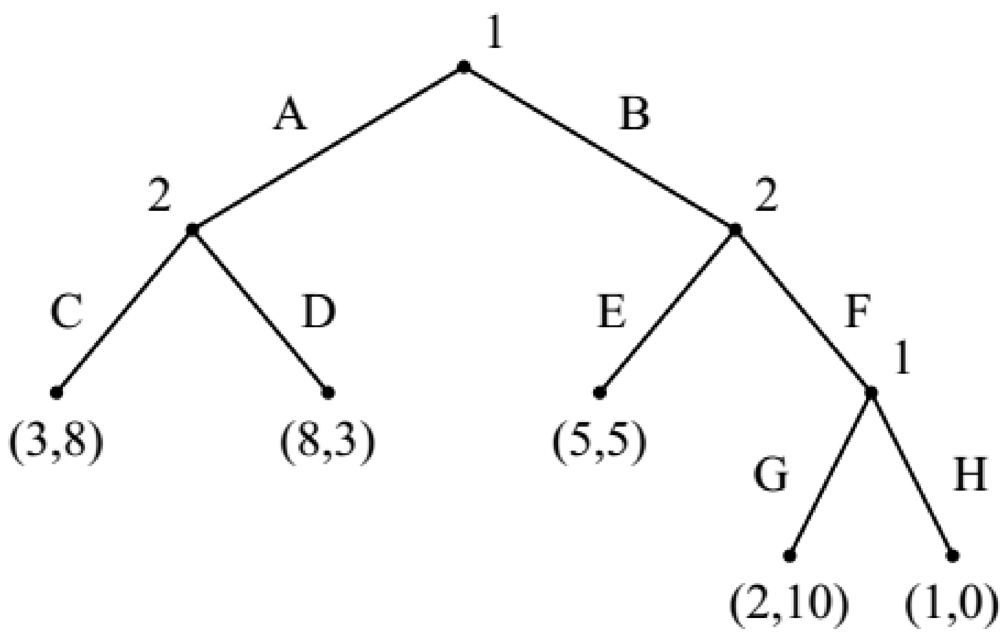
Recap on Previous Chapter

- Subgame perfect equilibrium (SPE): an outcome is SPE if it is Nash Equilibrium in every subgame

- The existence of SPE – back induction



Ultimatum Game

Back induction to find the SPE

For player 2, the optimal action:

- If x<1 → yes
- If x=1 → yes or no

The optimal strategies for player 2:

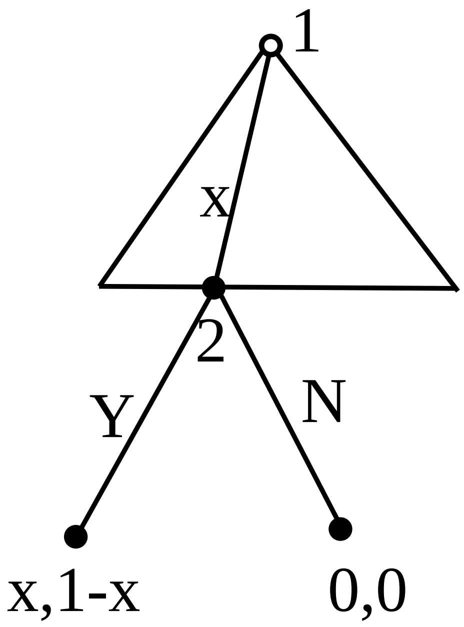
a) Yes for all x≤1

b) Yes for x<1 and No for x=1

The optimal strategies for player 1:

For case a), the optimal is x=1

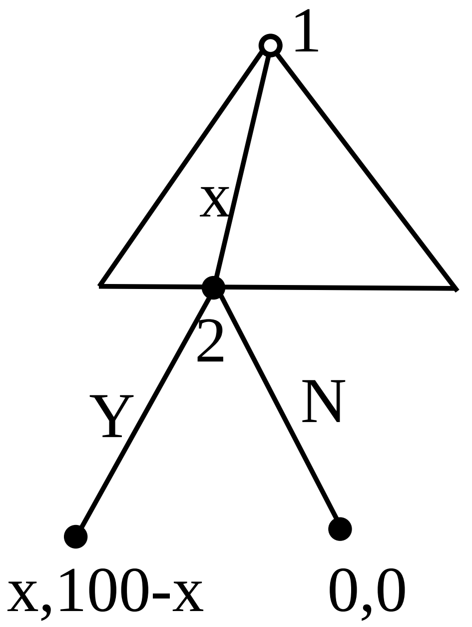
For case b), $\max_{x<1} x$ no solution



Ultimatum Game

Back induction to find the SPE

$$x \in \{1,2,\ldots,100\}$$



For player 2, the optimal action:

- If x<100 → yes
- If x=100 → yes or no

The optimal strategies for player 2:

a) Yes for all x≤100

b) Yes for x<100 and No for x=100

The optimal strategies for player 1:

For case a), the optimal is x=100

For case b), $\max_{x<100} x = 99$

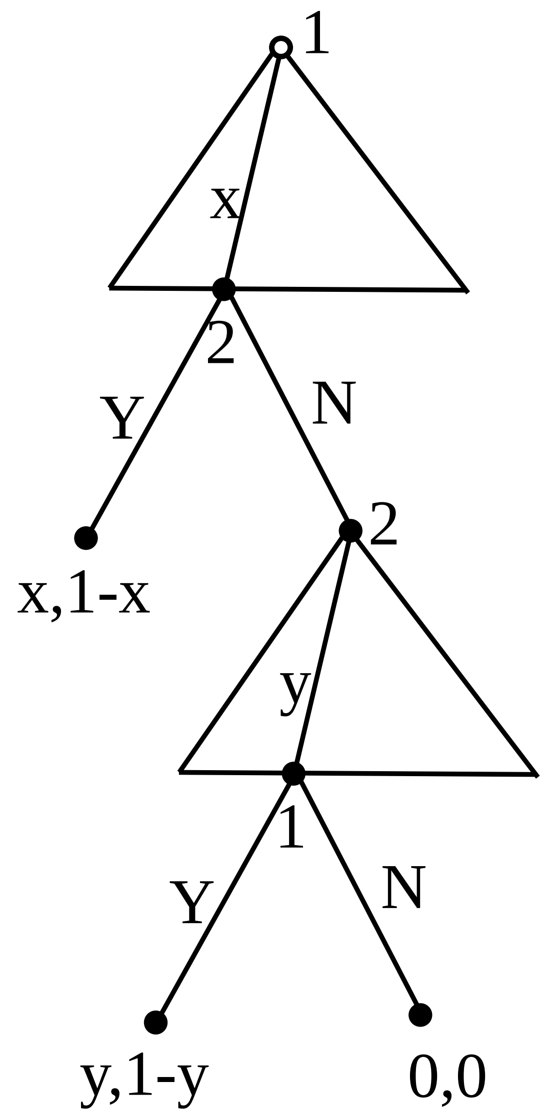
Bargaining Game (讨价还价博弈)

- In the ultimatum game, player 2 is powerless. His only strategy is to accept or reject

- Let us extend the model to give player 2 more power

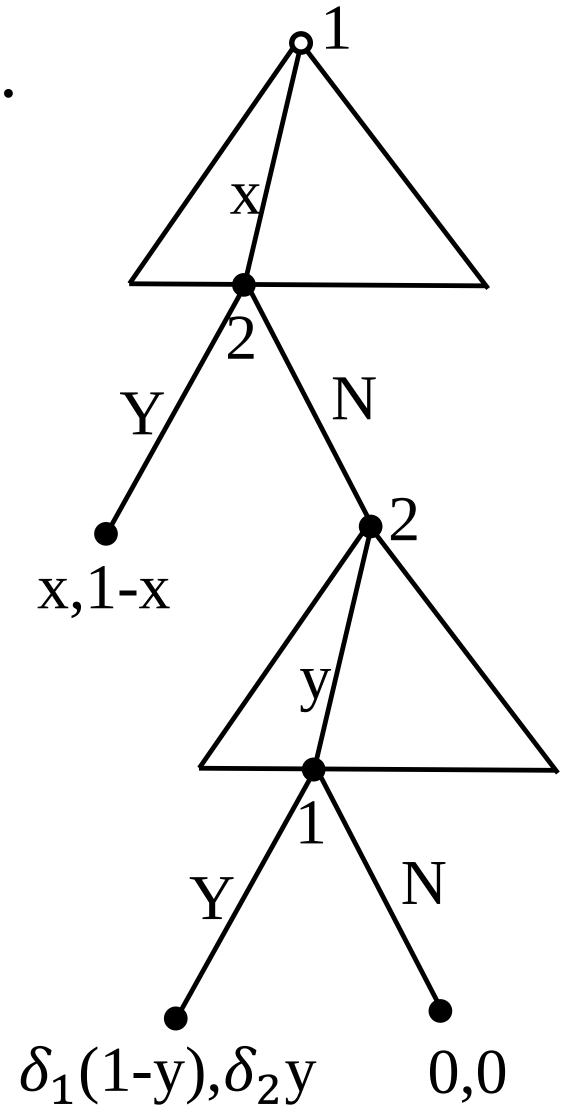
- If player 2 reject, then he can offer 1-y to player 1

- How to find the SPE?



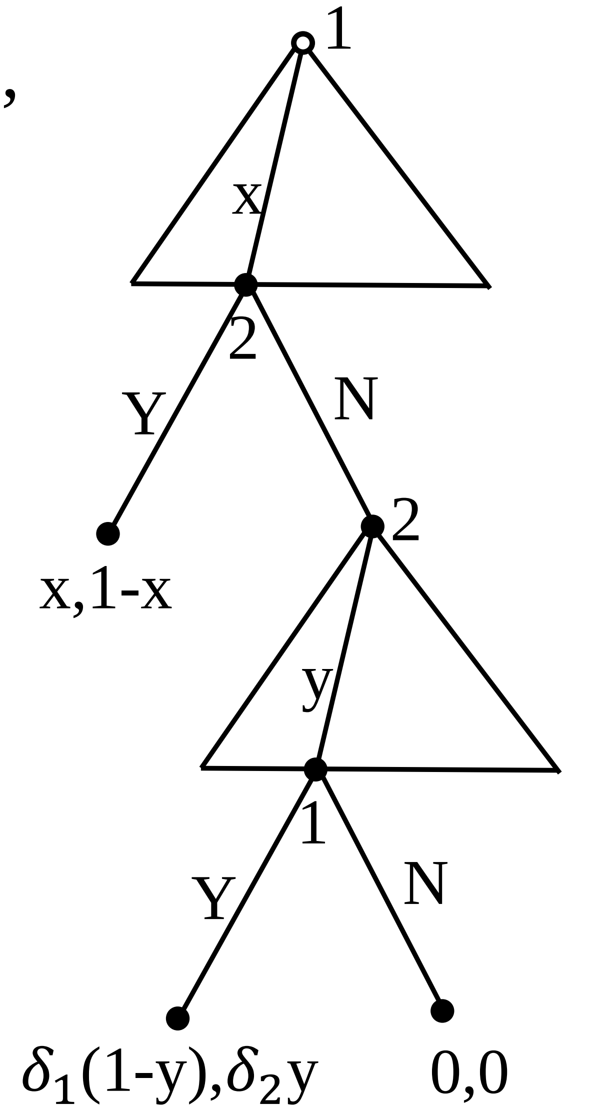
Bargaining Game

- Previous model does not consider time. In real life, bargaining takes time and time is valuable.
- Players alternate proposal, there is a discounted δ_1 and δ_2, respectively.
- How to find the SPE.

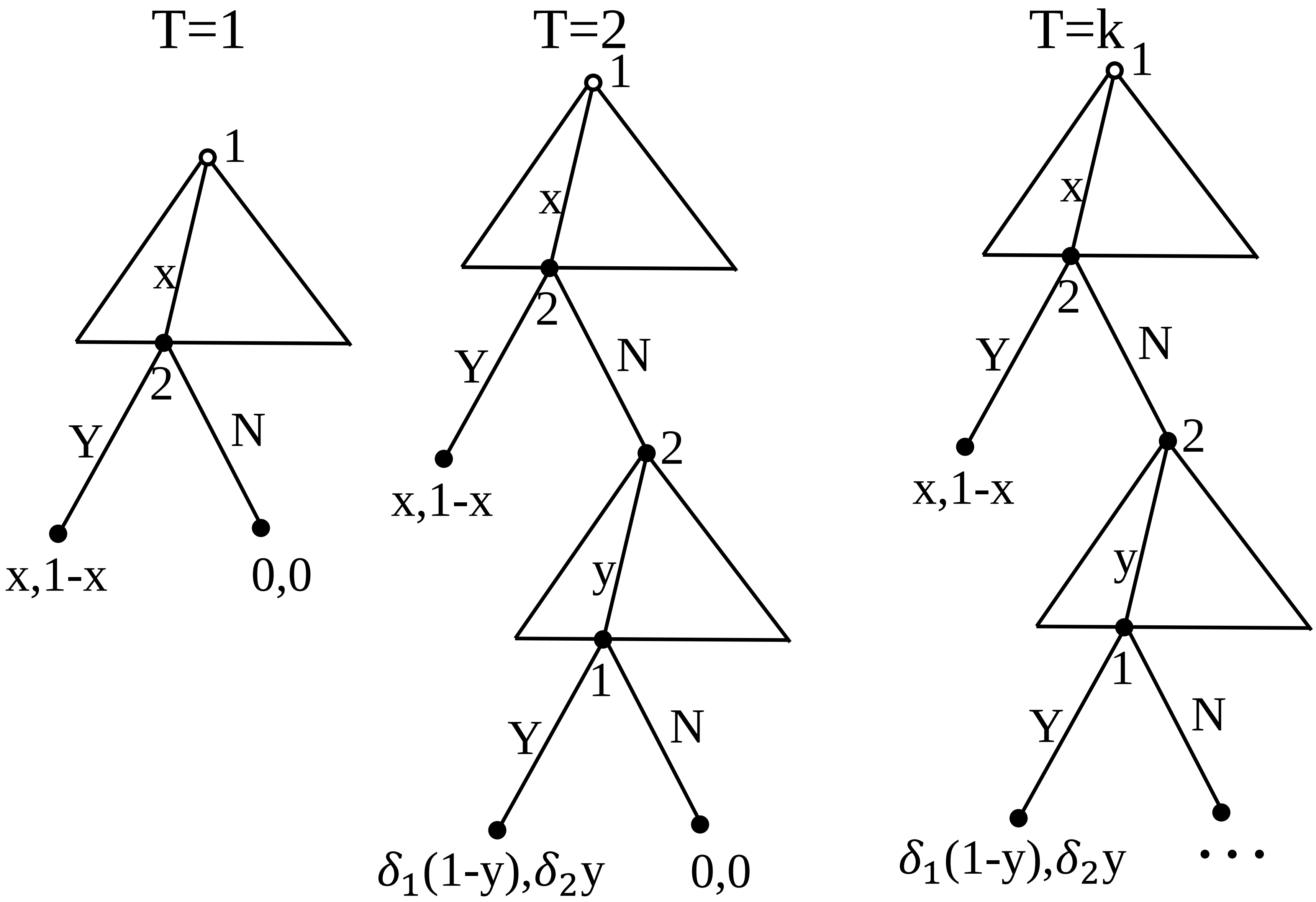


Bargaining Game

- Previous model does not consider time, and player 2 is powerless. Bargaining takes time and time is valuable.
- Players alternate proposal, there is a discounted δ_1 and δ_2, respectively.
- How to find the SPE.
- Rubinstein bargaining model with alternating offers



Rubinstein Bargaining Model with Alternating Offers



Rubinstein Bargaining Model with Alternating Offers

- $T = 2$ Player 1 get $x_1^* = 1 - \delta_2$
- $T = 4$ Player 1 get $x_1^* = 1 - \delta_2\big(1 - \delta_1(1 - \delta_2)\big)$
- $T = 6$ Player 1 gets

$$x_1^* = 1 - \delta_2\left(1 - \delta_1\left(1 - \delta_2\big(1 - \delta_1(1 - \delta_2)\big)\right)\right)$$

- …
- $T = 2k$ Player 1 gets

$$x_1^* = \frac{(1 - \delta_2)\big(1 - (\delta_1\delta_2)^k\big)}{1 - \delta_1\delta_2}$$

- Player 2 gets $1 - x_1^*$

Rubinstein Bargaining Model with Alternating Offers

- $T = 1$ Player 1 gets $x_1^* = 1$
- $T = 3$ Player 1 gets $x_1^* = 1 - \delta_2(1 - \delta_1)$
- $T = 5$ Player 1 gets

$$x_1^* = 1 - \delta_2 \left(1 - \delta_1\big(1 - \delta_2(1 - \delta_1)\big)\right)$$

- …
- $T = 2k - 1$ Player 1 gets

$$x_1^* = \frac{(1 - \delta_2)\big(1 - (\delta_1\delta_2)^k\big)}{1 - \delta_1\delta_2} + (\delta_1\delta_2)^k$$

- Player 2 gets $1 - x_1^*$

Rubinstein Bargaining Model with Finite Length

- $T = 2k$ player 1 proposes the offer
$$\left(\frac{(1-\delta_2)\left(1-(\delta_1\delta_2)^k\right)}{1-\delta_1\delta_2}, 1-\frac{(1-\delta_2)\left(1-(\delta_1\delta_2)^k\right)}{1-\delta_1\delta_2}\right)$$

- $T = 2k+1$ player 1 proposes the offer
$$\left(\frac{(1-\delta_2)\left(1-(\delta_1\delta_2)^k\right)}{1-\delta_1\delta_2} + (\delta_1\delta_2)^k,\right.$$

The Rubinstein Model with Infinite-Length

For $T = +\infty$, the game is much harder since we cannot use back induction

Rubinstein (1982) shows that the solution is simple

Theorem The Rubinstein bargaining game with infinite length has a unique SNE: in any period in which player i make a decision that he gets

$$x_i^* = (1 - \delta_j)/(1 - \delta_i \delta_j)$$

and the player j gets $1 - x_i^*$. player j accept if player i gets x $\leq x_i^*$ and rejects if player i gets x $> x_i^*$.

Proof. See board.

Properties of the SPE of Rubinstein Model

- Efficiency: Player j will accept player i's first decision, resulting in immediate agreement without delay (which is costly due to discounting)
- The higher δ_i reduces x_i^* and vice versa
- If $\delta_i = \delta_j = \delta$, then the first decision player makes a higher payoff in SPE
  - When $\delta \to 0$, the first decision person approaches to 1
  - When $\delta \to 1$, the first decision person approaches to 1/2

Bargaining: Nash’s Axiomatic Model

- Bargaining problems represent situations in which:
  - There is a conflict of interest about agreements.
  - Individuals have the possibility of a beneficial agreement.
  - No agreement on any individual without his approval.
- We will next adopt an axiomatic approach, which involves abstracting away the details of the process of bargaining and considers only the set of outcomes or agreements that satisfy ”reasonable” properties.
- This approach was proposed by Nash

Nash's Axiomatic Model

Example Suppose 2 players must split one unit of good. If no agreement is reached, then players do not receive anything. Two players are identical. We expect:

- Players to agree (Efficiency)
- Each obtains half (Symmetry)

We consider a more general setting:

- X denote the set of possible agreement and
- D denotes the disagreement outcome
- An example

$$X = \{(x_1, x_2): x_1 + x_2 = 1, x_i \geq 0\} \text{ and } D = (0,0)$$

Nash's Axiomatic Model

- Each player i has payoff function u_i on $X \cup \{D\}$. U is the set of possible agreement in terms of payoffs, that is

$$U = \{(v_1, v_2) : v_1 = u_1(x) \text{ and } v_2 = u_2(x) \text{ for } x \in U\}$$

- The disagreement point is

$$d = \left(u_1(D), u_2(D)\right)$$

- A bargaining problem is a pair (U, d) where $U \subset \mathbb{R}^2$ and d $\in U$. Assume that
  - U is a convex and compact set.
  - There is a $v \in U$ such that $v > d$ $(v_i > d_i)$

Nash's Axiomatic Model

- $\mathcal{B}$: the set of all possible bargaining problems
- A bargaining solution is a function $f\colon \mathcal{B} \to U$.

$$u = (u_1, u_2) = f(U, d) \text{ for } (U, d) \in \mathcal{B}$$

- The intuitive interpretation: the solution tells that the agreement u will be reached.
- The main goal is to study a list of reasonable axioms, can we define

$$u = (u_1, u_2) = f(U, d) \text{ for } (U, d) \in \mathcal{B}$$

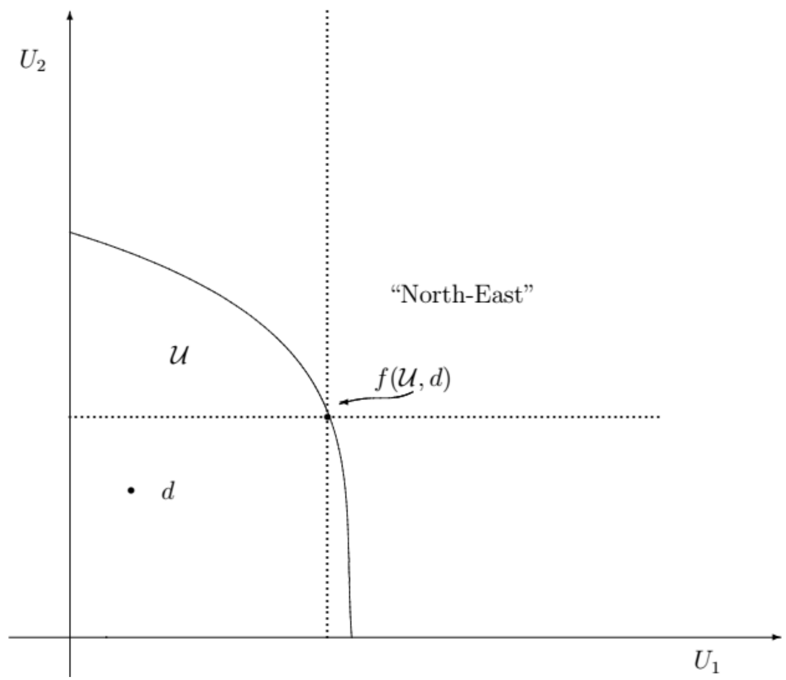
Nash's Axiomatic Model

Pareto Efficiency

- A bargaining solution $f(U,d)$ is Pareto efficient if there does not exist a

$$v = (v_1, v_2) \in U \text{ s.t. } v \geq f(U,d)$$

- There does not exist optimal point



Nash's Axiomatic Model

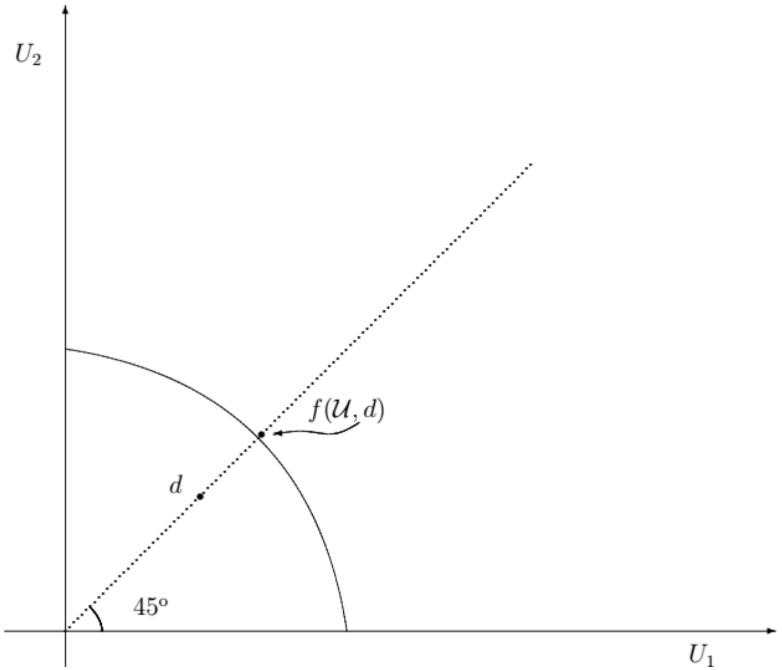
Symmetry:

$(v_1, v_2) \in U$ if and only if $(v_2, v_1) \in U$;

$d = (d_1, d_2)$ satisfies $d_1 = d_2$;

Then, $(v_1, v_2) = f(U, d)$ satisfies $v_1 = v_2$

For symmetric U and d, solution $f(U, d)$ is symmetric.



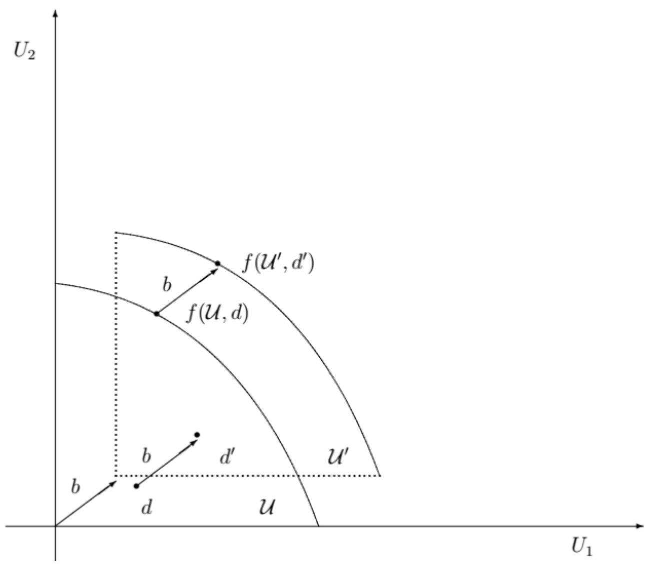
Nash's Axiomatic Model

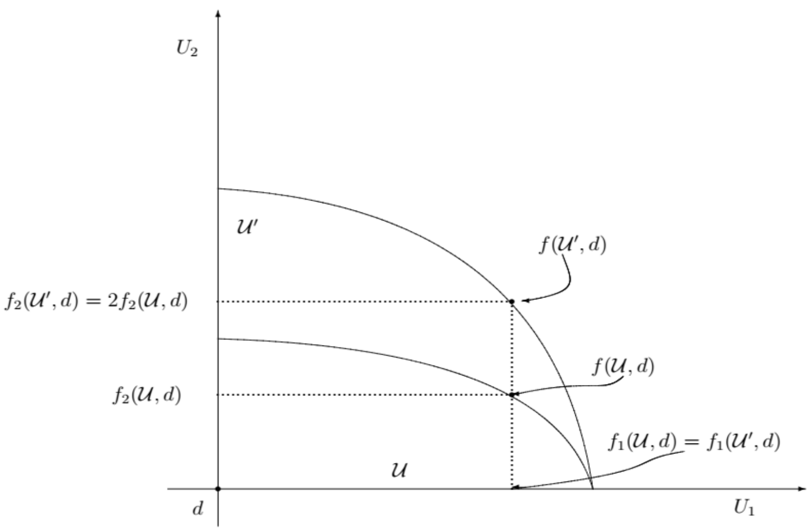
Invariance to Equivalent Payoff Representations

- Given a bargaining problem (U, d), consider a different bargaining problem (U', d') for some $\alpha > 0, \beta$:

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) | (v_1, v_2) \in U\}$$
$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$$

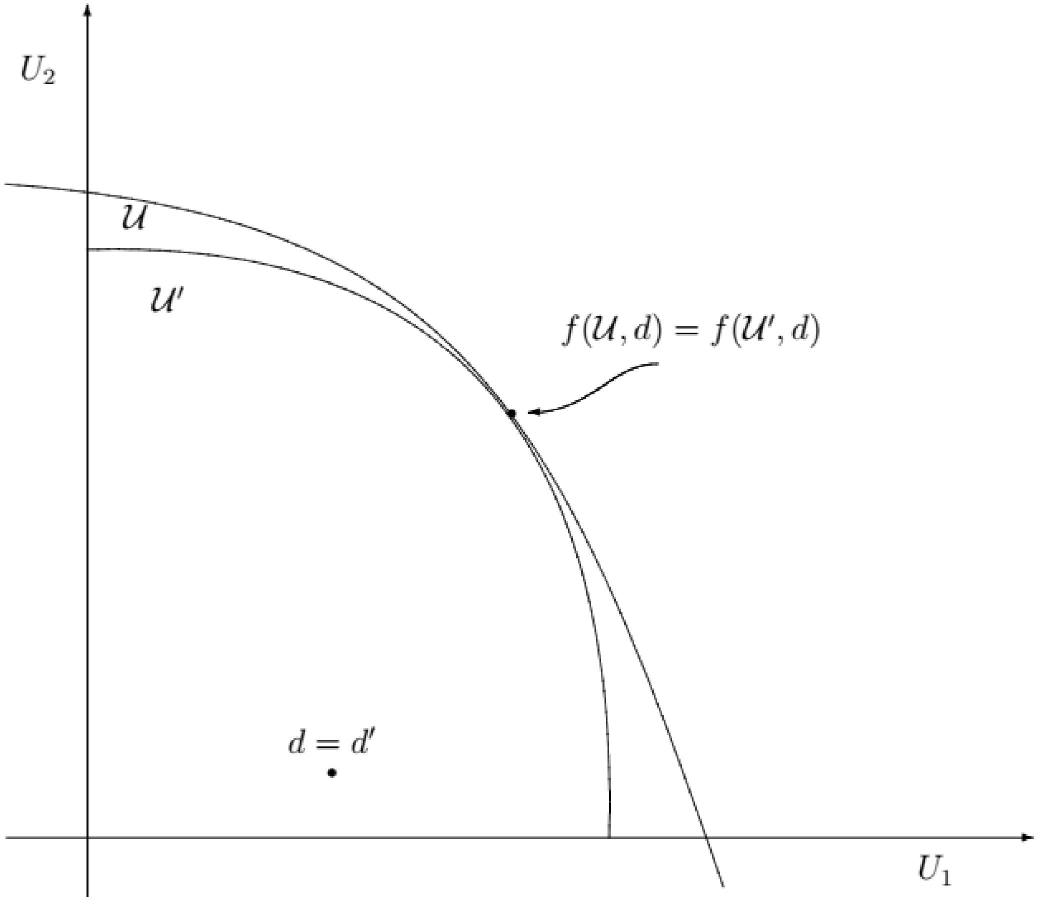
- Then $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$
- This is similar to linear transformation.





Independence of Irrelevant Alternatives

- Let (U, d) and (U', d) be two bargaining problems s.t. $U' \subseteq U$. If $f\ (U, d) \in U'$, then $f(U', d) = f(U, d)$.



Nash's Axiomatic Model Summaries

- Pareto Efficiency
- Symmetry
- Invariance to Equivalent Payoff Representations
- Independence of Irrelevant Alternatives

Based on those axiom, can we define $\boldsymbol{f(U,d)}$

Nash Bargaining Solution

Definition. The payoffs (v_1^*, v_2^*) is a Nash bargaining solution if it solves the following optimization problem:

$$\max_{v_1, v_2} (v_1 - d_1)(v_2 - d_2)$$

$$\text{subject to } (v1, v2) \in U \text{ and } (v_1, v_2) \geq (d_1, d_2).$$

- $f^N(U, d)$ denotes the Nash bargaining solution.

- Existence of $f^N(U, d)$ is from the continuous objective and compact U;
- Uniqueness of $f^N(U, d)$ is from the strictly quasi-concave objective function.

Nash Bargaining Solution

Theorem Nash bargaining solution $f^N(U, d)$ is the unique bargaining solution that satisfies the 4 axioms.

Proof. See draft.