Game Theory and Applications (博弈论及其应用)

Chapter 2: Mixed Strategy Game and Nash Equilibrium

南京大学

高尉

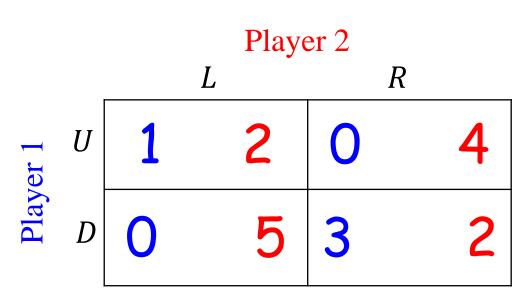


Recap

- Strategy game
- Formal definition
- Nash equilibrium
- How to find Nash equilibria
 - Payoff matrix
 - Continuous and differentiable payoff function

An Example

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$$G = \{\{1,2\}, \{\{U, \frac{L}{L}\}, \{L,R\}\}, \{u_1, u_2\}\}$$



Mixed Strategy

- Player 1 is mixing over the pure strategies *U* and *D*
- Player 2 is mixing over the pure strategies L and R

		Player 2							
		L,	π_2	R , $1-\pi_2$					
er 1	U , π_1	1	2	0	4				
Player	D , $1-\pi_1$	0	5	3	2				

- Mixed strategy keeps the guess of player's strategies, keep unpredictable on pure strategies
- Pure strategy can be viewed as a special mixed strategy

Pure and Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: each strategy in A_i

Mixed strategy: a probability over the set A_i of strategies

Denote by $\Delta(A_i)$ the set of all prob. distributions over A_i

An mixed outcome
$$p = (p_1, p_2, ..., p_N)$$
, where $p_i \in \Delta(A_i)$

For any p_i , we define

$$p_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_N)$$

 $p = (p_i, p_{-i})$

Pure and Mixed Strategies (cont.)

	Player i	Outcome	Players	Outcome		
Pure strategy	$a_i \in A_i$	$a = (a_1, a_2 a_N)$	a_{-i}	$\mathbf{a} = (a_i, a_{-i})$		
Mixed strategy	$p_i \in \Delta(A_i)$	$p = (p_1, p_2 \dots p_N)$	p_{-i}	$p = (p_i, p_{-i})$		

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{?, ?, \dots, ?\}\}$$

Given $G = \{N, \{A_i\}, \{u_i\}\}$ and mixed $p = (p_1, p_2, ..., p_N)$, the expected payoff of player i is given by

$$U_{i}(p) = \sum_{a \in A} p(a) u_{i}(a)$$

$$= \sum_{a=(a_{1},...,a_{N}) \in A} p_{1}(a_{1}) \times \cdots \times p_{N}(a_{N}) u_{i}(a)$$

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{U_1, U_2, \dots, U_N\}\}$$

Example

Player 2
$$L, \pi_{2} \qquad R, 1 - \pi_{2}$$

$$U, \pi_{1} \qquad 1 \qquad 2 \qquad 0 \qquad 4$$

$$D, 1 - \pi_{1} \qquad 0 \qquad 5 \qquad 3 \qquad 2$$

$$p = (p_1, p_2) = ((0.4, 0.6), (0.5, 0.5))$$

$$U_1(p) = p_1(U)p_2(L)u_1(U, L) + p_1(U)p_2(R)u_1(U, R)$$

$$+p_1(D)p_2(L)u_1(D, L) + p_1(D)p_2(D)u_1(D, R) = 1.1$$

$$U_2(p) = \dots = 3.3$$

Continuous Expected Payoff Function

Lemma U_i (p) is a continuous function for each variable.

Let

$$U_i(p_{-i}, a_i) = \sum_{a_{-i} \in A_{-i}} p_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

Then

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i)$$

Multi-linear Payoff Function

Lemma: The expected payoff function U_i is multi-linear

For mixed outcome
$$p = (p_1, p_2, ..., p_N)$$
 and p'_i , we have $U_i(\lambda p_i + (1 - \lambda)p'_i, p_{-i}) = \lambda U_i(p_i, p_{-i}) + (1 - \lambda)U_i(p'_i, p_{-i})$ $\lambda \in [0,1].$

Proof. See board from the definition

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i).$$

An mixed strategy outcome $p = (p_1, p_2, ..., p_N)$ is a Nash equilibrium (NE) if for each i, we have

$$U_i(p_i, p_{-i}) \ge U_i(p'_i, p_{-i}) \text{ for } p'_i \in \Delta(A_i)$$

Given $G = \{N, \{\Delta(A_i)\}, \{U_i\}\}$ and $p = (p_1 ... p_N)$, the **best** response correspondence of player i is given by

$$B_i(p_{-i}) = \{p_i : U(p_i, p_{-i}) \ge U(p_i', p_{-i}) \text{ for all } p_i' \in \Delta(A_i)\}$$

Theorem A mixed outcome $p = (p_1 ... p_N)$ is a NE if and only if $p_i \in B_i(p_{-i})$

Nash Theorem

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Here finite strategic game means

- > finite players
- > each player has **finite pure strategies**

Why is this important

- ➤ Difficult to understand properties (NE) without existence
- Find the NE if we know the existence of NE

Theorem If a mixed strategy is a best response, then each of the pure strategies (positive prob.) involved in the mixed strategy must be a best response. Particularly, each must yield the same expected payoff.

If a mixed strategy p_i is a best response to the strategies of the others p_{-i} , then each pure strategy a_i s.t. $p_i(a_i) > 0$ is itself a best response to p_{-i} .

Particularly, all $U_i(a_i, p_{-i})$ must be equal

An Property of MNE (cont.)

Theorem $G = \{N, \{A_i\}, \{u_i\}\}, p = (p_1, p_2, ..., p_N)$ is a mixed Nash equilibrium if and only if every pure strategy of player i with positive probability is a best response to p_{-i}

Proof. See board by contradiction and from

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i).$$

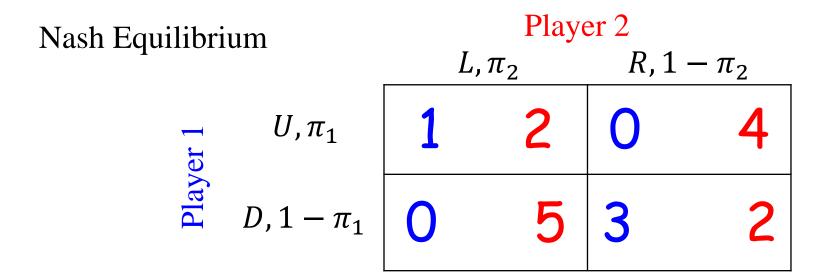
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Fixed Player 1, the expectation payoff of Player 2 on L $2\pi_1 + 5(1 - \pi_1)$

the expectation payoff of Player 2 on R $4\pi_1 + 2(1 - \pi_1)$

Nash Equilibrium implies

$$2\pi_1 + 5(1 - \pi_1) = 4\pi_1 + 2(1 - \pi_1) \rightarrow \pi_1 = 3/5$$



Fixed Player 2, the expectation payoff of Player 1 on U π_2

the expectation payoff of Player 2 on R $3(1-\pi_2)$

Nash Equilibrium implies

$$\pi_2 = 3(1 - \pi_2) \rightarrow \pi_2 = 3/4$$

		Player 2 L, 3/4 R, 1/4						
er 1	<i>U</i> ,3/5	1	2	0	4			
Player	D, 2/5	0	5	3	2			

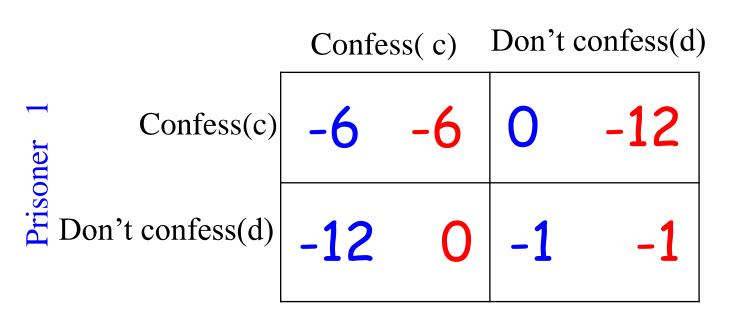
Nash Equilibrium:

Player 1 selects the mixed strategy $p_1 = (3/5,2/5)$ over $\{U,D\}$ Player 2 selects the mixed strategy $p_2 = (3/4,1/4)$ over $\{L,R\}$ The expected payoff of Player 1 on mixed strategy $p = (p_1,p_2)$ 3/5*3/4*1+2/5*1/4*3=3/4The expected payoff of Player 2 on mixed strategy $p = (p_1,p_2)$

3/5*3/4*2+3/5*1/4*4+2/5*3/4*5+2/5*1/4*2=16/5

Prisoners' Dilemma: Mixed Strategy NE

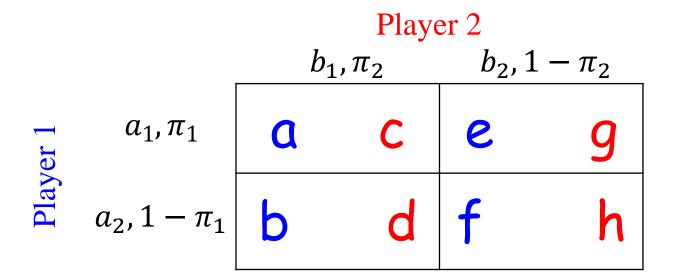
Prisoner 2



PNE and MNE coexists

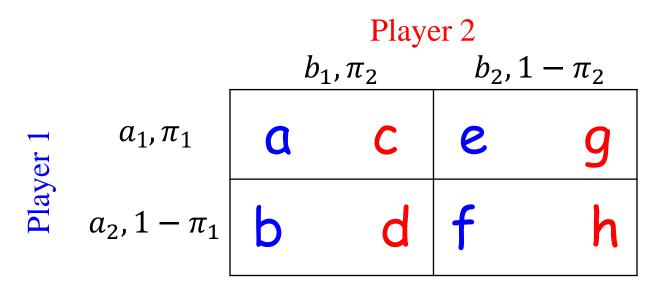
Mixed Strategy Nash Equilibrium: 2×2 games

$$G = \{\{1,2\}, \{\{a_1, a_2\}, \{b_1, b_2\}\}, \{u_1, u_2\}\}$$



Mixed Strategy Nash Equilibrium: 3 Players

$$G = \{\{T, B, J\}, \{\{a_1, a_2\}, \{b_1, b_2\}\}, \{u_1, u_2\}\}$$



Mixed Strategy Nash Equilibrium: 3 Players

$$G = \{\{1,2,3\}, \{\{a,b\}, \{x,y\}, \{L,R\}\}, \{u_i\}_{i=1}^3\}$$

Exercise: Primitive Hunting

Find all Nash Equilibria (pure and mix NE)

Hunter 2 Rabbit (r) Deer (d) Rabbit (r) 3 3 3 0 Deer (d) 0 3 9 9

Rock-Paper-Scissors

					Plag	yer 2		
			Ro	ock	Pa	per	Scissors	
		Rock	0	0	-1	1	1	-1
Player	1	Paper	1	-1	0	0	-1	1
		Scissors	-1	1	1	-1	0	0

Nash Equilibrium (Proof on board):

$$p = (p_1, p_2) = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right)$$

General Method for MNE

General Method for 2-player games with payoff matrix

Player 2

Player 1

0	0	•••	1	-1
•	••	•••	••	• •
-1	1	•••	0	0

Step 1: Conjecture some rows and columns (positive prob.)

Step 2: Calculate the mixed strategy

Step 3: Check Nash Equilibria

The running time is exponential with # of strategies

How Many Nash Equilibria?

- A game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.
- So if the game has no pure strategy Nash equilibrium, then it must have at least one mixed strategy Nash equilibrium.
- In the worst case, the running time for find MNE is exponential in the # of strategies

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Theorem A mixed outcome p is a NE iff $p \in B(p)$

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \cdots \times \Delta(A_N) \to \Delta(A_1) \times \cdots \times \Delta(A_N)$$

Kakutani Fixed Point Theorem Let $f: A \rightarrow A$ be a correspondence with $f(x) \subset A$ for $x \in A$. If

- 1) A is compact, convex and non-empty (finite space);
- 2) f(x) is non-empty for all $x \in A$;
- 3) f(x) is a convex set;
- 4) f(x) has a closed graph: if $\{x_n, y_n\} \to \{x, y\}$ and $y_n \in f(x_n)$ then $y \in f(x)$,

then, there is a $x \in A$ such that $x \in f(x)$

Proof Sketch

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \dots \times \Delta(A_N) \to \Delta(A_1) \times \dots \times \Delta(A_N)$$

- 1) $\Delta(A_1) \times \cdots \times \Delta(A_N)$ is compact, convex and non-empty;
- 2) B(p) is non-empty for all p;
- 3) B(p) is a convex set;
- 4) B(p) has a closed graph: if $\{p^n, \hat{p}^n\} \to \{p, \hat{p}\}$ and $\hat{p}^n \in B(p^n)$ then $\hat{p} \in B(p)$,

then, there is a $p \in \Delta(A_1) \times \cdots \times \Delta(A_N)$ s.t. $p \in B(p)$

 $\Delta(A_1) \times \cdots \times \Delta(A_N)$ is compact, convex and non-empty

Pf. It suffices to prove $\Delta(A_i)$ is compact, convex and non-empty. Let $n = |A_i|$. Then

$$\Delta(A_i) = \{(x_1, ..., x_n) : x_i \in [0,1], \sum x_i = 1\}$$

is a simplex of dimension n-1.

$$B(p) = \{(p'_1, p'_2, ..., p'_N): p'_i \in B_i(p_{-i})\}$$
 is non-empty

Pf. It suffices to prove $B_i(p_{-i})$ is non-empty.

$$B_i(p_{-i}) = \operatorname{argmax}_{p_i' \in \Delta(A_i)} U_i(p_i', p_{-i})$$

Let
$$f(x) = U_i(x, p_{-i}) = \sum_k x_k U_i(p_{-i}, a_k)$$
 for $x \in \Delta(A_i)$.

f(x) is continuous and $\Delta(A_i)$ is an nonempty compact set. By Weierstrass Theorem, f(x) has maximum in $\Delta(A_i)$.

$$B_i(p_{-i}) = \operatorname{argmin}_{x \in \Delta(A_i)} f(x)$$
 is not-empty

$$B(p) = \{(p'_1, p'_2, ..., p'_N): p'_i \in B_i(p_{-i})\}$$
 is a convex set

Pf. It suffices to prove $B_i(p_{-i})$ is convex. For any $\lambda \in [0,1]$, if $p_i', p_i'' \in B_i(p_{-i})$ then we need to prove $\lambda p_i' + (1-\lambda)p_i'' \in B_i(p_{-i})$.

From $p_i, p_i' \in B_i(p_{-i})$, we have

$$U_i(p_i, p_{-i}) \ge U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(p_i', p_{-i}) \ge U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(\lambda p_i + (1 - \lambda)p_i', p_{-i}) \ge U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$B(p) = \{(p'_1, p'_2, ..., p'_N): p'_i \in B_i(p_{-i})\}$$
 has a closed graph

Pf. Assume $(p^n, \hat{p}^n) \to (p, \hat{p}), \hat{p}^n \in B(p^n)$ but $\hat{p} \notin B(p)$. There exists $\hat{p}_i \notin B_i(p_{-i})$, i.e., there exist \bar{p}_i and $\epsilon > 0$ s.t. $U_i(\bar{p}_i, p_{-i}) \ge U_i(\hat{p}_i, p_{-i}) + 3\epsilon$

For continuous $U_{i}, p_{-i}^{n} \to p_{-i}$ and $(\hat{p}_{i}^{n}, p_{-i}^{n}) \to (\hat{p}_{i}, p_{-i})$ $U_{i}(\bar{p}_{i}, p_{-i}^{n}) > U_{i}(\bar{p}_{i}, p_{-i}) - \epsilon$ $U_{i}(\hat{p}_{i}, p_{-i}) > U_{i}(\hat{p}_{i}^{n}, p_{-i}^{n}) - \epsilon$

We have $U_i(\bar{p}_i, p_{-i}^n) > U_i(\hat{p}_i^n, p_{-i}^n) + \epsilon$. Thus $\hat{p}_i^n \notin B_i(p_{-i}^n)$

Summary on Mixed Strategy

- Mixed strategy, mixed strategy game
- Nash Theorem
- How to find mixed strategy Nash Theorem

An Exercise 1

• Find all pure strategy Nash equilibria

			h		i		j		k		1		m
	a	3	5	8	9	2	7	6	3	3	9	6	5
	b	6	21	13	6	5	8	9	4	8	9	7	8
P1	c	9	7	1	1	7	9	9	2	2	6	4	12
	d	2	14	10	12	6	5	6	8	7	2	9	19
	e	8	9	15	9	13	9	7	5	13	15	12	7

Exercise 2 田忌赛马

田忌 上中下 上下中 中上下 中下上 下上中 下中上 上中下 3, -3 | 1, -1 | 1, -1 | 1, -1 | -1, 1 | 1, -1 上下中 1, -1 | 3, -3 | 1, -1 | 1, -1 | 1, -1 | -1, 1 中上下 1, -1 | -1, 1 | 3, -3 | 1, -1 | 1, -1 | 1, -1 齐威王 中下上 | -1, 1 | 1, -1 | 1, -1 | 3, -3 | 1, -1 | 1, -1 下上中 | 1, -1 | 1, -1 | 1, -1 | -1, 1 | 3, -3 | 下中上 1, -1 1, -1 -1, 1 1, -1 1, -1 3, -3

Find a mixed Nash equilibrium