Game Theory and Applications (博弈论及其应用)

Chapter 3: Dominant Strategy Equilibrium and Rationality

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Recap on Previous Chapter

- Mixed strategy game
- Mixed strategy Nash equilibrium
- Nash Theorem
- How to find mixed strategy Nash equilibria

Prisoner 2

Prisoner 1	L	R
U	6 0	0 6
D	0 3	1 1

Dominant Strategy

- In most strategy games, one player's optimal choice depends on others' choice
- For some special cases, however, there is a optimal strategy independent of others' choice, e.g., dominant strategy

		Prisoner 2	
		Confess(c)	Don't confess(d)
Prisoner 1	Confess(c)	-6 -6	0 -12
	Don't confess(d)	-12 0	-1 -1

Prisoner 2 will select c whatever Prisoner 1 how to choose

Formal Definition

- $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$ outcome of strategy taken by all player other than i
- A_{-i} denotes the set of all such outcomes

A pure strategy a_i **strictly dominates** a_i' if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

A pure strategy a_i **weakly dominates** a_i' if

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

$$\text{while } u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \in A_{-i}$$

a_i is **strictly dominant** if it strictly dominates all other strategies in A_i, and it is called **weak dominant** if it weakly dominates all other strategy in A_i

Dominant Strategy Equilibrium

If every player has a (strictly and weakly) dominant strategy, then the corresponding outcome is a **(strictly and weakly) dominant strategy equilibrium**.

Dominant strategy equilibrium is belong to NE.

		Prisoner 2	
		c	d
Prisoner 1	c	-6 -6	0 -12
	d	-12 0	-1 -1

		Player 2	
		l	r
Player 1	u	3 3	3 0
	d	0 3	3 3

It is very simple

It may not exist in many games

Second Price Auction

N: players bid a building

$v_i \geq 0$: the true value for player i

$b_i \geq 0$: the bid price for player i

$v_i - b_i$: the payoff for player i

The rule of **second-price auction** is given as follow:

- Players make bids $b = (b_1, b_2, \ldots, b_N)$ simultaneously
- The higher player wins the building, yet pays the second highest bid price
- If there are more than one highest players, then randomly select one player and pay his own bid price

Second Price Auction

Theorem In second price auction, the strategy $b_i = v_i$ is a weakly dominant strategy for each player i.

$(v_1, v_2, \dots, v_N)$ is a weakly dominant strategy equilibrium.

Pf. It suffice to show $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all b_i, b_{-i}

- If someone's bid $b_k \geq v_i$, then player i has to pay $b_i > b_k \geq v_i$ by winning. Payoff is $v_i - \max_{k \neq i} b_k \leq 0$. It is optimal to select $b_i = v_i$.
- If each bid prize $b_k < v_i$, then payoff is $v_i - \max_{k \neq i} b_k > 0$, since the payoff is always the same when winning. It is optimal to select $b_i = v_i$.

Second Price Auction

- Honesty strategy is the best strategy

- Many internet auctions can be regarded as variants of second price auction.

- We can also consider first and third price auction. Is it a dominant strategy to bid your true value?

Dominated Strategies

A pure strategy a_i **strictly dominates** a_i' $\Longrightarrow$ a_i' is **strictly dominated** by a_i if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i};$$

a_i **weakly dominates** a_i' $\Longrightarrow$ a_i' is **weakly dominated** by a_i if

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

$$\text{while } u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \in A_{-i}$$

Iterated Elimination of Dominated Strategies

If every strategy eliminated is a strictly dominated strategy

- Iterated elimination of strictly dominated strategy

If at least one strategy eliminated is a weakly DS

- Iterated elimination of weakly dominated strategy

Iterated Elimination and Pure DS

		Player 2				
	l		m		r	
Player 1 u	10	10	2	15	10	10
m	15	2	5	5	5	5
d	10	10	5	5	10	10

- For player 1, the strategy ‘u’ is weakly dominated by ‘d’
- For player 2, the strategy ‘l’ is weakly dominated by ‘r’

Iterated Elimination and Pure DS (cont.)

Therefore, we have the game

Player 2

Player 1		m	r
	m	5 5	5 5
	d	5 5	10 10

- For player 1, the strategy 'm' is weakly dominated by 'd'
- For player 2, the strategy 'm' is weakly dominated by 'r'

By iterated elimination of weakly dominated strategy

(d,r) is a weakly dominant strategy Equilibrium

Mixed Strategy and Dominant Strategy

- A strategy may be not dominated by other strategies, yet can be dominated by a mixed strategy

Player 2

Player 1	l	r
u	1, 1	1, 0
m	3, 0	0, 3
d	0, 1	4, 0

- For player 1, no strategy dominates 'u'
- The mixed strategy $p_1 = (0, 0.5, 0.5)$ dominates 'u'

Mixed Strategy and Dominant Strategy

Figure out the dominated strategy for expected payoff

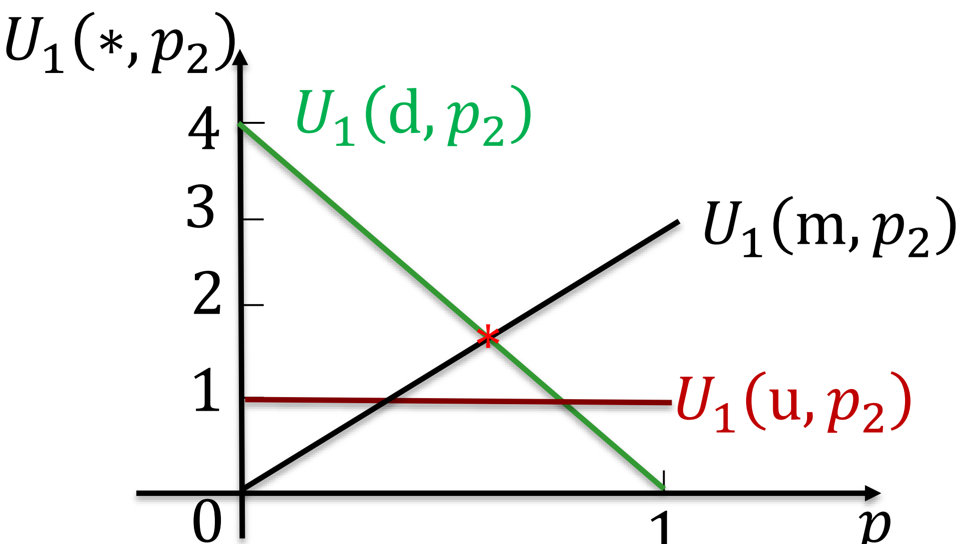
Let $p_2 = (p, 1 - p)$ be the mixed strategy for player 2

		Player 2	
		l	r
Player 1	u	1, 1	1, 0
	m	3, 0	0, 3
	d	0, 1	4, 0

$U_1(\mathrm{u}, p_2) = 1$

$U_1(\mathrm{m}, p_2) = 3p$

$U_1(\mathrm{d}, p_2) = 4(1 - p)$



- The mixed strategy $p_1 = (0, 0.5, 0.5)$ dominates ‘u’
- ‘u’ is a never best strategy

Mixed Strategy and Dominant Strategy

Theorem A **strictly** dominated strategy is never used with positive probability in a mixed strategy Nash equilibrium

Let $p = (p_1, \dots, p_N)$ be a mixed strategy NE.

For player i, $a_i, a_i' \in A_i$ s.t. a_i is strictly dominated by a_i',

$$p_i(a_i) = 0$$

Proof. See board.

Find Mixed Strategy Nash Equilibria

Step 1: eliminate all strictly dominated strategies

Step 2: use our previous methods introduced in Chapter 2

Player 2

Player 1	l	m	r
u	0, 5	2, 3	2, 3
m	2, 3	1, 5	3, 2
d	5, 0	3, 2	2, 3

Exercise on Class

Find all pure and mixed strategy NE

Player 2

Player 1

	x		y		z		r	
a	0	5	2	3	2	0	1	3
b	2	3	5	5	2	4	3	2
c	5	0	6	2	2	1	2	3
d	4	1	3	1	2	0	1	1

Belief

Given a strategy game $G = \{N, \{A_i\}, \{u_i\}\}$

- A mixed strategy outcome $p = (p_1, p_2, \dots p_N)$
- $p = (p_i, p_{-i})$
- p_{-i} is called **a belief**

A belief $\boldsymbol{p_{-i}}$ of player $\boldsymbol{i}$ is a probability over $\boldsymbol{A_{-i}}$

A strategy $a_i \in A_i$ is a best response to belief p_{-i} if

$$U_i(a_i, p_{-i}) \geq U_i(a, p_{-i}) \text{ for all } a \in A_i$$

Rationality and NE

A pure strategy $\boldsymbol{a_i \in A_i}$ is rational if there is a belief p_{-i} s.t. a_i is a best response to belief p_{-i}

The relationship between NE and rationality:

Theorem Every pure strategy with positive probability in a mixed strategy Nash equilibrium is rational.

Pf. Assume $p = (p_1, p_2, \dots, p_N)$ is a mixed strategy NE. Then, p_{i} is a best response to $p_{-\mathrm{i}}$, and every strategy with positive probability in p_{i} is also a best response to $p_{-\mathrm{i}}$.

Rationality and Strictly Dominant Strategy

$a_i \in A_i$ is rational if a_i is a best response to some belief p_{-i}

$$U_i(a_i, p_{-i}) \geq U_i(a, p_{-i}) \text{ for all } a \in A_i$$

A mixed strategy $p_i \in \Delta(A_i)$ strictly dominates $a_i \in A_i$

$$U_i(p_i, p_{-i}) > U_i(a_i, p_{-i}) \text{ for all } p_{-i} \in \Delta(A_{-i})$$

The relationship between rationality and strict domination

Theorem A strategy $a_i \in A_i$ is rational if and only if a_i is not strictly dominated.

Pf. See the board from the definition.

An Example

Player 2

Player 1	l	r
u	2, 0	-1, 1
m	0, 10	0, 0
d	-1, -6	2, 0

Player 1 is rational

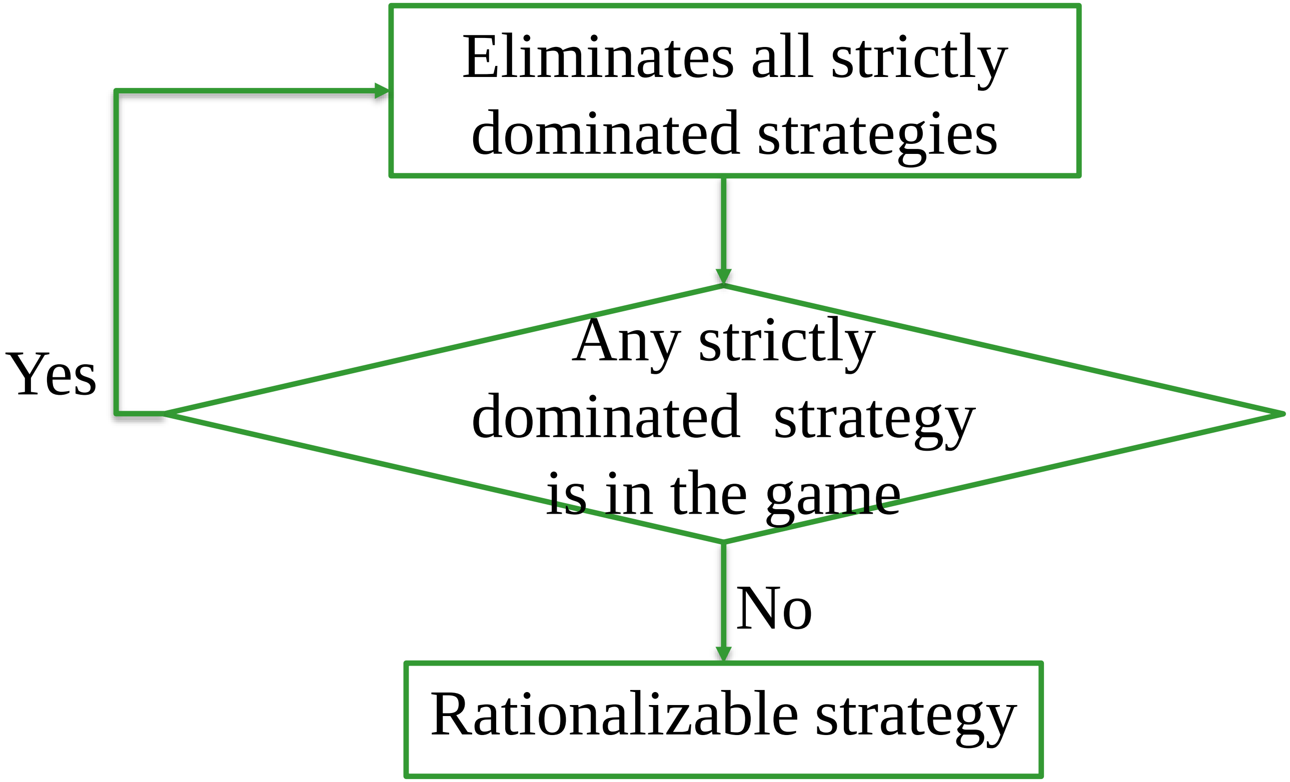
Player 2 is rational and

knows that Player 1 is rational

Player 1 is rational, and knows that player 2 is rational

and knows that 2 knows that 1 is rational

Rationalizability



Notice

1) Eliminate all strictly DS and keep weakly DS

2) Eliminate all strictly DS by pure and mixed strategy

Beauty Contest (选 美 竞 赛 游 戏)

- There are n players (two method)
- Each player selects a number $a_i \in [0,50]$
- The payoff for each student is $50 - \left(a_i - \frac{2}{3}\frac{\sum_i a_i}{n}\right)^2$

Given a_{-i}, the best strategy for player i is

$$a_i^* = \frac{2}{3}\frac{\sum_{j,j\neq i} a_i}{n - 2/3}$$

$$a_i^* \in \left[0, \frac{2}{3}\frac{n-1}{n-2/3}50\right]$$

Beauty Contest (cont)

- After round 1: $\left[0, \frac{2}{3}\frac{n-1}{n-2/3}50\right]$
- After round 2: $\left[0, \left(\frac{2}{3}\frac{n-1}{n-2/3}\right)^2 50\right]$

…

- After round k: $\left[0, \left(\frac{2}{3}\frac{n-1}{n-2/3}\right)^k 50\right]$

…

- Rational ={0}

Summary

- Strictly/weakly Dominant Strategy
- Dominant Strategy Equilibrium
- Dominated strategy and Nash Equilibrium
- How to find NE
- Rational
- Rationalizability