Game Theory and Applications (博弈论及其应用)

Chapter 2： Mixed Strategy Game and Nash Equilibrium

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Recap

- Strategy game
- Formal definition
- Nash equilibrium
- How to find Nash equilibria
  - Payoff matrix
  - Continuous and differentiable payoff function

An Example

- $G = \left\{ \{1,2\},\ \{\{U,L\},\{L,R\}\},\ \{u_1,u_2\} \right\}$

		Player 2	
		L	R
Player 1	U	1 2	0 4
	D	0 5	3 2

Mixed Strategy

- Player 1 is mixing over the pure strategies U and D
- Player 2 is mixing over the pure strategies L and R

Player 2

		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

- Mixed strategy keeps the guess of player’s strategies, keep unpredictable on pure strategies
- Pure strategy can be viewed as a special mixed strategy

Pure and Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: each strategy in A_i

Mixed strategy: a probability over the set A_i of strategies

Denote by $\boldsymbol{\Delta(A_i)}$ the set of all prob. distributions over A_i

An **mixed outcome** $p = (p_1, p_2, \dots, p_N)$, where $p_i \in \Delta(A_i)$

For any p_i, we define

$$p_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N)$$

$$p = (p_i, p_{-i})$$

Pure and Mixed Strategies (cont.)

	Player *i*	Outcome	Players	Outcome
Pure strategy	$a_i \in A_i$	$\mathrm{a} = (a_1, a_2 \dots a_N)$	a_{-i}	$\mathrm{a} = (a_i, a_{-i})$
Mixed strategy	$p_i \in \Delta(A_i)$	$p = (p_1, p_2 \dots p_N)$	p_{-i}	$p = (p_i, p_{-i})$

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{?, ?, \dots, ?\}\}$$

The Expected Payoff

Given $G = \{N, \{A_i\}, \{u_i\}\}$ and mixed $p = (p_1, p_2, \dots, p_N)$, the expected payoff of player i is given by

$$U_i(p) = \sum_{a \in A} p(a)\, u_i(a)$$

$$= \sum_{a=(a_1,\dots,a_N) \in A} p_1(a_1) \times \cdots \times p_N(a_N)\, u_i(a)$$

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{U_1, U_2, \dots, U_N\}\}$$

Example

Player 2

		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

$$p = (p_1, p_2) = \big((0.4{,}0.6), (0.5{,}0.5)\big)$$

$$U_1(p) = p_1(U)p_2(L)u_1(U, L) + p_1(U)p_2(R)u_1(U, R)$$

$$+p_1(D)p_2(L)u_1(D, L)+p_1(D)p_2(D)u_1(D, R) = 1.1$$

$$U_2(p) = \cdots = 3.3$$

Continuous Expected Payoff Function

Lemma $U_i\ (p)$ is a **continuous** function for each variable.

Let

$$U_i(p_{-i}, a_i) = \sum_{a_{-i} \in A_{-i}} p_{-i}(a_{-i})\, u_i(a_i, a_{-i})$$

Then

$$U_i(p) = \sum\nolimits_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i)$$

Multi-linear Payoff Function

Lemma: The expected payoff function U_i is **multi-linear**

For mixed outcome $p = (p_1, p_2, \ldots, p_N)$ and p_i', we have

$$\mathrm{U}_i(\lambda p_i + (1-\lambda)p_i', p_{-i}) = \lambda \mathrm{U}_i(p_i, p_{-i})) + (1-\lambda)\mathrm{U}_i(p_i', p_{-i})$$

$\lambda \in [0,1]$.

Proof. See board from the definition

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i).$$

Mixed Strategy Nash Equilibrium

An mixed strategy outcome $p = (p_1, p_2, \ldots, p_N)$ is a **Nash equilibrium (NE)** if for each i, we have

$$U_i(p_i, p_{-i}) \geq U_i(p_i', p_{-i}) \text{ for } p_i' \in \Delta(A_i)$$

Given $G = \{N, \{\Delta(A_i)\}, \{\mathrm{U}_i\}\}$ and $p = (p_1 \ldots p_N)$, the **best response correspondence** of player i is given by

$$B_i(p_{-i}) = \{p_i : U(p_i, p_{-i}) \geq U(p_i', p_{-i}) \text{ for all } p_i' \in \Delta(A_i)\}$$

Theorem A mixed outcome $p = (p_1 \ldots p_N)$is a NE if and only if $p_i \in B_i(p_{-i})$

Nash Theorem

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Here finite strategic game means

- **finite players**
- each player has **finite pure strategies**

Why is this important

- Difficult to understand properties (NE) without existence
- Find the NE if we know the existence of NE

An Property of MNE

Theorem If a mixed strategy is a best response, then each of the pure strategies (positive prob.) involved in the mixed strategy must be a best response. Particularly, each must yield the same expected payoff.

If a mixed strategy p_i is a best response to the strategies of the others p_{-i}, then each pure strategy a_i s.t. $p_i(a_i) > 0$ is itself a best response to p_{-i}.

Particularly, all $U_i(a_i, p_{-i})$ must be equal

An Property of MNE (cont.)

Theorem $G = \{N, \{A_i\}, \{u_i\}\}$, $p = (p_1, p_2, \dots, p_N)$ is a mixed Nash equilibrium if and only if every pure strategy of player i with positive probability is a best response to p_{-i}

Proof. See board by contradiction and from

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i)\,.$$

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Mixed Strategy Nash Equilibrium: Example 1

Nash Equilibrium

		Player 2			
		L,π_2		$R,1-\pi_2$	
Player 1	U,π_1	1	2	0	4
	$D,1-\pi_1$	0	5	3	2

Fixed Player 1, the expectation payoff of Player 2 on *L*

$$2\pi_1 + 5(1-\pi_1)$$

the expectation payoff of Player 2 on *R*

$$4\pi_1 + 5(1-\pi_1)$$

Nash Equilibrium implies

$$2\pi_1 + 5(1-\pi_1) = 4\pi_1 + 2(1-\pi_1) \rightarrow \pi_1 = 3/5$$

Mixed Strategy Nash Equilibrium: Example 1

Nash Equilibrium

		Player 2	
		L, π_2	$R, 1-\pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1-\pi_1$	0 5	3 2

Fixed Player 2, the expectation payoff of Player 1 on U

$$\pi_2$$

the expectation payoff of Player 2 on R

$$3(1-\pi_2)$$

Nash Equilibrium implies

$$\pi_2 = 3(1-\pi_2) \rightarrow \pi_2 = 3/4$$

Mixed Strategy Nash Equilibrium: Example 1

Player 2

Player 1

	$L,3/4$		$R,1/4$	
$U,3/5$	1	2	0	4
$D,2/5$	0	5	3	2

Nash Equilibrium:

Player 1 selects the mixed strategy $p_1 = (3/5,2/5)$ over $\{U,D\}$

Player 2 selects the mixed strategy $p_2 = (3/4,1/4)$ over $\{L,R\}$

The expected payoff of Player 1 on mixed strategy $p = (p_1,p_2)$

3/5*3/4*1+2/5*1/4*3=3/4

The expected payoff of Player 2 on mixed strategy $p = (p_1,p_2)$

3/5*3/4*2+3/5*1/4*4+2/5*3/4*5+2/5*1/4*2=16/5

Prisoners’ Dilemma: Mixed Strategy NE

Prisoner 2

Prisoner 1

		Confess(c)	Don’t confess(d)
Prisoner 1	Confess(c)	-6 -6	0 -12
	Don’t confess(d)	-12 0	-1 -1

PNE and MNE coexists

Mixed Strategy Nash Equilibrium: 2×2 games

$$G = \{\{1{,}2\},\{\{a_1, a_2\},\{b_1, b_2\}\},\{u_1, u_2\}\}$$

Player 2

Player 1

	b_1, π_2	$b_2, 1-\pi_2$
a_1, π_1	a c	e g
$a_2, 1-\pi_1$	b d	f h

Exercise: Primitive Hunting

Find all Nash Equilibria (pure and mix NE)

		Hunter 2			
		Rabbit (r)		Deer (d)	
Hunter 1	Rabbit (r)	3	3	3	0
	Deer (d)	0	3	9	9

Mixed Strategy Nash Equilibrium: Example 2

Rock-Paper-Scissors

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0 0	-1 1	1 -1
	Paper	1 -1	0 0	-1 1
	Scissors	-1 1	1 -1	0 0

Nash Equilibrium (Proof on board):

$$p = (p_1, p_2) = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$$

General Method for MNE

General Method for 2-player games with payoff matrix

	Player 2		
Player 1	0 0	…	1 -1
	…	…	…
	-1 1	…	0 0

Step 1: Conjecture some rows and columns (positive prob.)

Step 2: Calculate the mixed strategy

Step 3: Check Nash Equilibria

The running time is exponential with # of strategies

How Many Nash Equilibria?

- A game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.
- So if the game has no pure strategy Nash equilibrium, then it must have at least one mixed strategy Nash equilibrium.
- In the worst case, the running time for find MNE is exponential in the # of strategies

Nash Theorem

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Theorem A mixed outcome p is a NE iff $p \in B(p)$

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \cdots \times \Delta(A_N) \rightarrow \Delta(A_1) \times \cdots \times \Delta(A_N)$$

Proof Sketch

Kakutani Fixed Point Theorem Let $f: A \to A$ be a correspondence with $f(x) \subset A$ for $x \in A$. If

1) A is compact, convex and non-empty (finite space);

2) $f(x)$ is non-empty for all $x \in A$;

3) $f(x)$ is a convex set;

4) $f(x)$ has a closed graph: if $\{x_n, y_n\} \to \{x, y\}$ and $y_n \in f(x_n)$ then $y \in f(x)$,

then, there is a $x \in A$ such that $x \in f(x)$

Proof Sketch

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \cdots \times \Delta(A_N) \to \Delta(A_1) \times \cdots \times \Delta(A_N)$$

1) $\Delta(A_1) \times \cdots \times \Delta(A_N)$ is compact, convex and non-empty;

2) $B(p)$ is non-empty for all p;

3) $B(p)$is a convex set;

4) $B(p)$has a closed graph: if $\{p^n, \hat{p}^n\} \to \{p, \hat{p}\}$ and $\hat{p}^n \in B(p^n)$ then $\hat{p} \in B(p)$,

then, there is a $p \in \Delta(A_1) \times \cdots \times \Delta(A_N)$ s.t. $p \in \mathrm{B}(p)$

Proof of condition 1

$\mathit{\Delta}(A_1) \times \cdots \times \mathit{\Delta}(A_N)$ is compact, convex and non-empty

Pf. It suffices to prove $\Delta(A_i)$ is compact, convex and non-empty. Let $n = |A_i|$. Then

$$\Delta(A_i) = \{(x_1, \dots, x_n) : x_i \in [0,1], \sum x_i = 1\}$$

is a simplex of dimension $n - 1$.

Proof of condition 2

$B(p) = \{(p_1', p_2', \ldots, p_N') : p_i' \in B_i(p_{-i})\}$is non-empty

Pf. It suffices to prove $B_i(p_{-i})$ is non-empty.

$$B_i(p_{-i}) = \mathrm{argmax}_{\mathrm{p}_i' \in \Delta(A_i)} U_i(p_i', p_{-i})$$

Let $f(x) = U_i(x, p_{-i}) = \sum_k x_k U_i(p_{-i}, a_k)$ for $x \in \Delta(A_i)$.

$f(x)$ is continuous and $\Delta(A_i)$ is an nonempty compact set. By Weierstrass Theorem, $f(x)$ has maximum in $\Delta(A_i)$.

$$B_i(p_{-i}) = \mathrm{argmin}_{x \in \Delta(A_i)} f(x) \text{ is not-empty}$$

Proof of condition 3

$B(p) = \{(p_1', p_2', \dots, p_N') : p_i' \in B_i(p_{-i})\}$ is a convex set

Pf. It suffices to prove $B_i(p_{-i})$ is convex. For any $\lambda \in [0,1]$, if $p_i', p_i'' \in B_i(p_{-i})$ then we need to prove

$$\lambda p_i' + (1-\lambda) p_i'' \in B_i(p_{-i}).$$

From $p_i, p_i' \in B_i(p_{-i})$, we have

$$U_i(p_i, p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(p_i', p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(\lambda p_i + (1-\lambda) p_i', p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

Proof of condition 4

$B(p) = \{(p_1', p_2', \dots, p_N') : p_i' \in B_i(p_{-i})\}$ has a closed graph

Pf. Assume $(p^n, \hat{p}^n) \to (p, \hat{p})$, $\hat{p}^n \in B(p^n)$ but $\hat{p} \notin B(p)$. There exists $\hat{p}_i \notin B_i(p_{-i})$, i.e., there exist $\bar{p}_i$ and $\epsilon > 0$ s.t.

$$U_i(\bar{p}_i, p_{-i}) \geq U_i(\hat{p}_i, p_{-i}) + 3\epsilon$$

For continuous U_i, $p^n_{-i} \to p_{-i}$ and $(\hat{p}_i^n, p^n_{-i}) \to (\hat{p}_i, p_{-i})$

$$U_i(\bar{p}_i, p^n_{-i}) > U_i(\bar{p}_i, p_{-i}) - \epsilon$$

$$U_i(\hat{p}_i, p_{-i}) > U_i(\hat{p}_i^n, p^n_{-i}) - \epsilon$$

We have $U_i(\bar{p}_i, p^n_{-i}) > U_i(\hat{p}_i^n, p^n_{-i}) + \epsilon$. Thus $\hat{p}_i^n \notin B_i(p^n_{-i})$

Summary on Mixed Strategy

- Mixed strategy, mixed strategy game
- Nash Theorem
- How to find mixed strategy Nash Theorem

An Exercise 1

- Find all pure strategy Nash equilibria

P2

		h	i	j	k	l	m
P1	a	3 5	8 9	2 7	6 3	3 9	6 5
	b	6 21	13 6	5 8	9 4	8 9	7 8
	c	9 7	1 1	7 9	9 2	2 6	4 12
	d	2 14	10 12	6 5	6 8	7 2	9 19
	e	8 9	15 9	13 9	7 5	13 15	12 7

Exercise 2 田忌赛马

		田忌					
		上中下	上下中	中上下	中下上	下上中	下中上
齐威王	上中下	3, -3	1, -1	1, -1	1, -1	-1, 1	1, -1
	上下中	1, -1	3, -3	1, -1	1, -1	1, -1	-1, 1
	中上下	1, -1	-1, 1	3, -3	1, -1	1, -1	1, -1
	中下上	-1, 1	1, -1	1, -1	3, -3	1, -1	1, -1
	下上中	1, -1	1, -1	1, -1	-1, 1	3, -3	1, -1
	下中上	1, -1	1, -1	-1, 1	1, -1	1, -1	3, -3

Find a mixed Nash equilibrium