

Game Theory and Applications (博弈论及其应用)

Chapter 12: Extensive Game with Imperfect Information

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Recap on Previous Chapter

Repeated game: many real interactions have an ongoing structure; players consider short- and long-term payoffs.

A repeated game $G^T(\delta)$ consists of stage game G , terminal date T and discount factor δ

Folk Theorem (SPNE) If $(u_1, u_2, \dots, u_N) \in U$ is strictly **individually rational**, then there exists some $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, there is Nash equilibrium of $G^\infty(\delta)$ with payoff (u_1, u_2, \dots, u_N)

Payoff vector $(u_1, u_2, \dots, u_N) \in R^N$ is **strictly individually rational** if $u_i > \min_{a_{-i}} [\max_{a_i} u_i(a_i, a_{-i})]$ for all i

Recap on Previous Chapter

- Folk Theorem

- An infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.
- Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, $\hat{u}_i > u_i^*$ for every player i
- There is a Subgame Perfect Nash Equilibrium for some discount factor δ

Solving for Equilibria in Repeated Games

1. Solve all equilibria of the stage game (**Competition**)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (**Cooperation**)
3. Design **trigger strategies** that support cooperation and punish with competition
4. Compute **the maximum discount factor** so that cooperation is an equilibrium
5. The trigger strategies are an **SPEN** of the infinitely repeated game for some larger discount factor

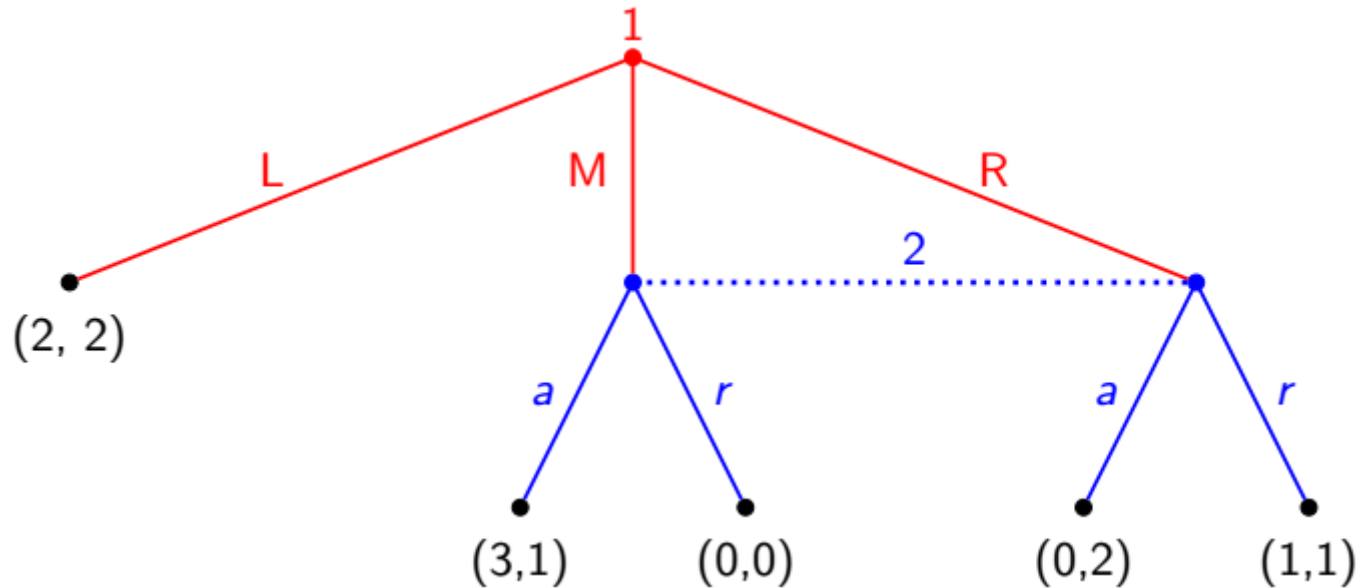
Recap on Extensive Game

- The **extensive game** is an alternative representation that makes the temporal structure explicit
- **Nash equilibrium**
- **Subgame perfect equilibrium** (SPE): an outcome is SPE if it is Nash Equilibrium in every subgame
- How to find SPE – **back induction and one deviation**
- Two variants
 - Perfect information: game tree
 - Imperfect information

Motivation

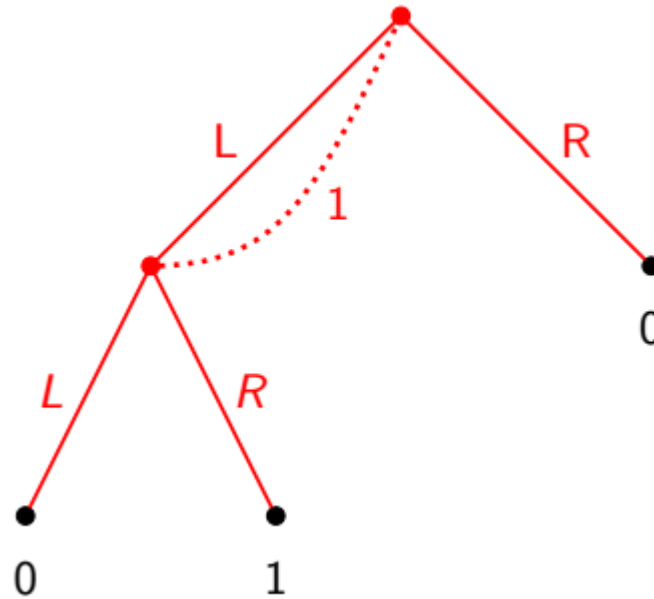
- Extensive game with perfect information
 - Know all prior strategies for all players
- Sometimes, players
 - Don't know all the strategies the other take or
 - Don't recall all their past actions
- Extensive game captures some of this ignorance
 - An later choice is made without knowledge of a earlier choice
- How to represent the case two players make choices at the same time, in mutual ignorance of each other

Example



Player 2 does not know the choice of player 1 over M or R

Example



Player 1 does not know if he has made a choice or not

Definition of extensive game with Perfect Information

An **extensive game** with **perfect information** is defined by

$$G = \{N, H, P, \{u_i\}\}$$

- **Players** N is the set of N players
- **Histories** H is a set of sequence $a^1 \dots a^k$, where each component a^i is a strategy
- **Player function** $P(h): H \rightarrow N$ is the player who takes action after the history h
- **Payoff function** u_i
- **Action set** $A(h) = \{a: (h, a) \in H\}$

Ultimatum Game

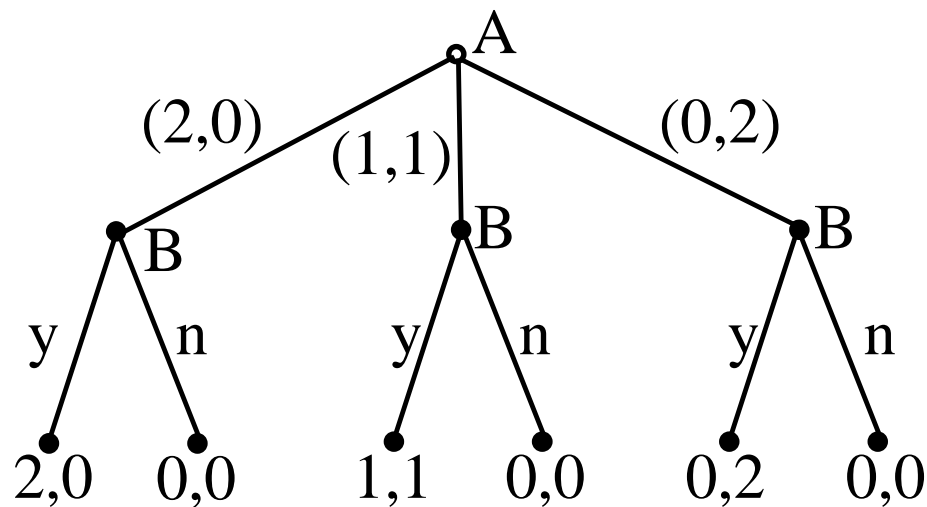
$$G = \{N, H, P, \{u_i\}\}$$

$$N = \{A, B\}$$

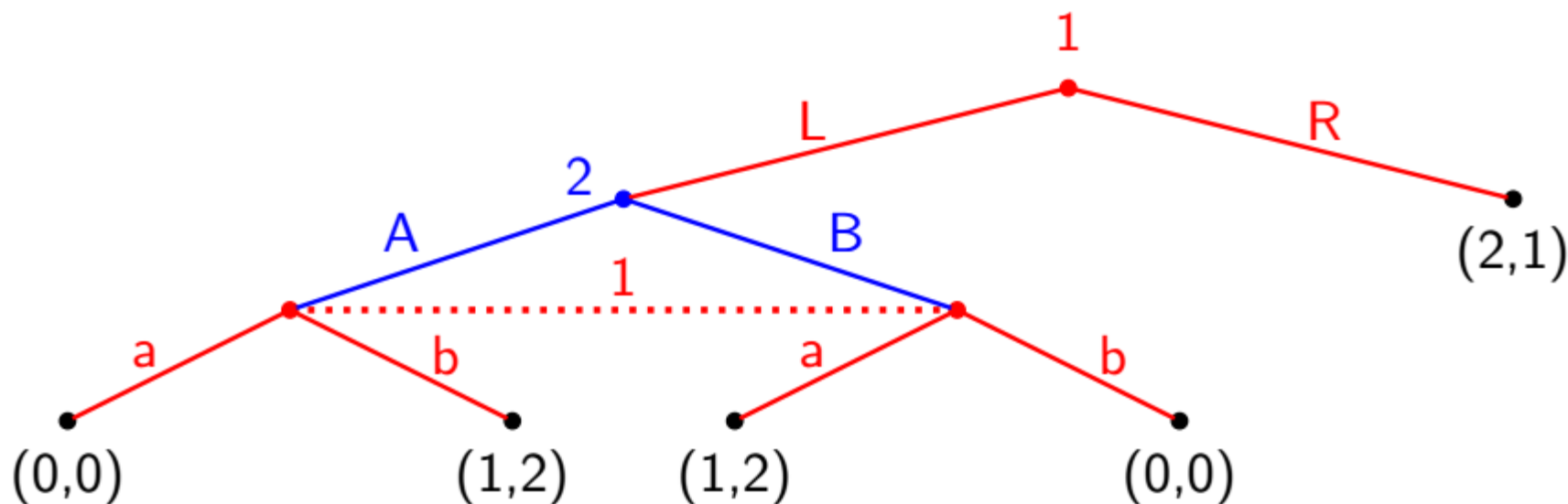
$$\begin{aligned} H = & \{ \emptyset, (2,0), (1,1), (0,2), ((2,0),y) \} \\ & \cup \{ ((2,0),n), ((1,1),y), ((1,1),n) \} \\ & \cup \{ ((0,2),y), ((0,2),n) \} \end{aligned}$$

$$P: P(\emptyset) = A; P((2,0)) = B; P((1,1)) = B; P((0,2)) = B$$

$$A: A(\emptyset) = \{(2,0), (0,2), (1,1)\}; A((2,0)) = A((0,2)) = A((1,1)) = \{y, n\}$$



Extensive Game with Imperfect Information



Player 1 does not know the choice of player 2 over LA or LB

Nonterminal histories: $\{\emptyset, L, LA, LB\}$

➤ Player 1 has information set $I_1 = \{\emptyset, \{LA, LB\}\}$,

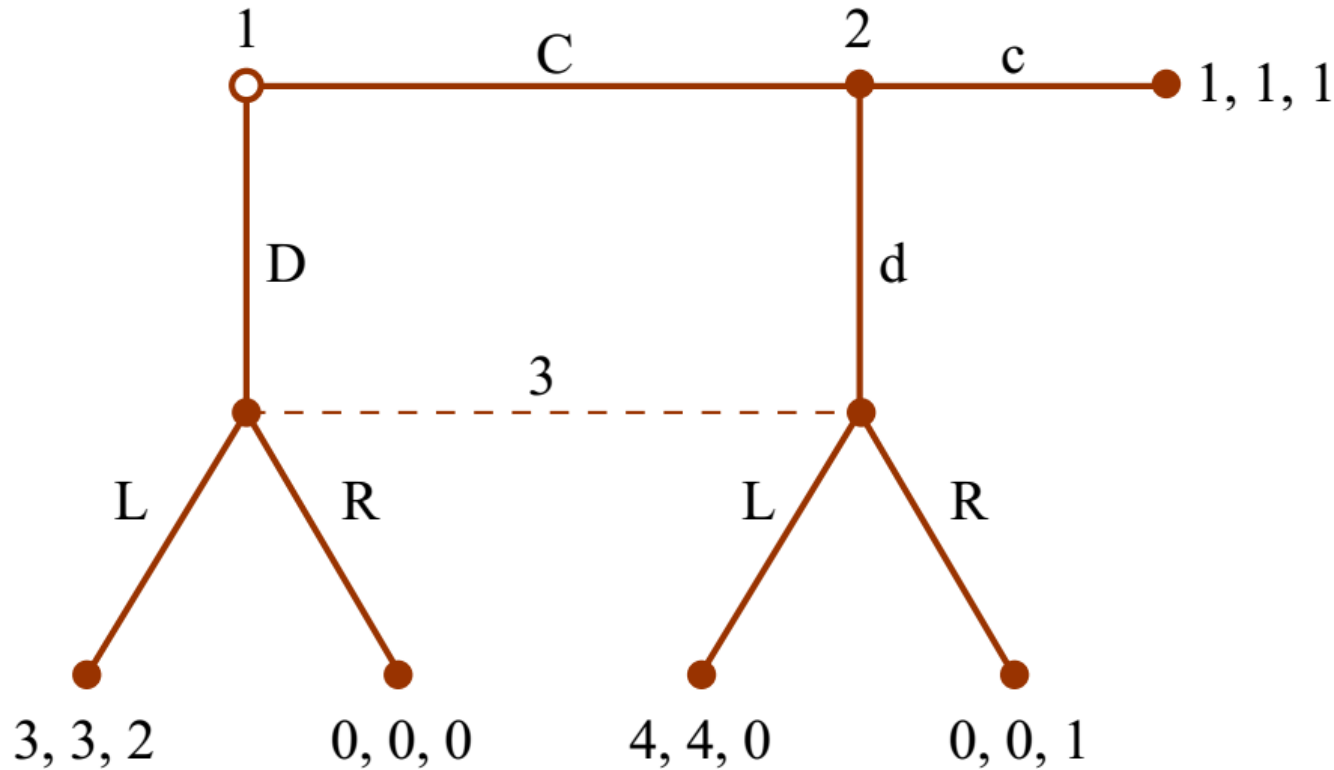
➤ Player 2 has information set $I_2 = \{L\}$

Definition of Extensive Game with Imperfect Information

An **extensive game** with **imperfect information** is defined by $G = \{N, H, P, I, \{u_i\}\}$

- **Information set** $I = \{I_1, I_2, \dots, I_N\}$ is the set of information partition of all players' strategy nodes, where the nodes in an information set are **indistinguishable** to player
 - $I_i = \{I_{i1}, \dots, I_{ik_i}\}$ is the information partition of player i
 - $I_{i1} \cup \dots \cup I_{ik_i} = \{\text{all nodes of player } i\}$
 - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq k$
 - **Action set** $A(h) = A(h')$ for $h, h' \in I_{ij}$, denote by $A(I_{ij})$
 - $P(I_{ij})$ be the player who plays at information set I_{ij}
- An **extensive game with perfect information** is a special case where each I_{ij} contains **only one node**

Example



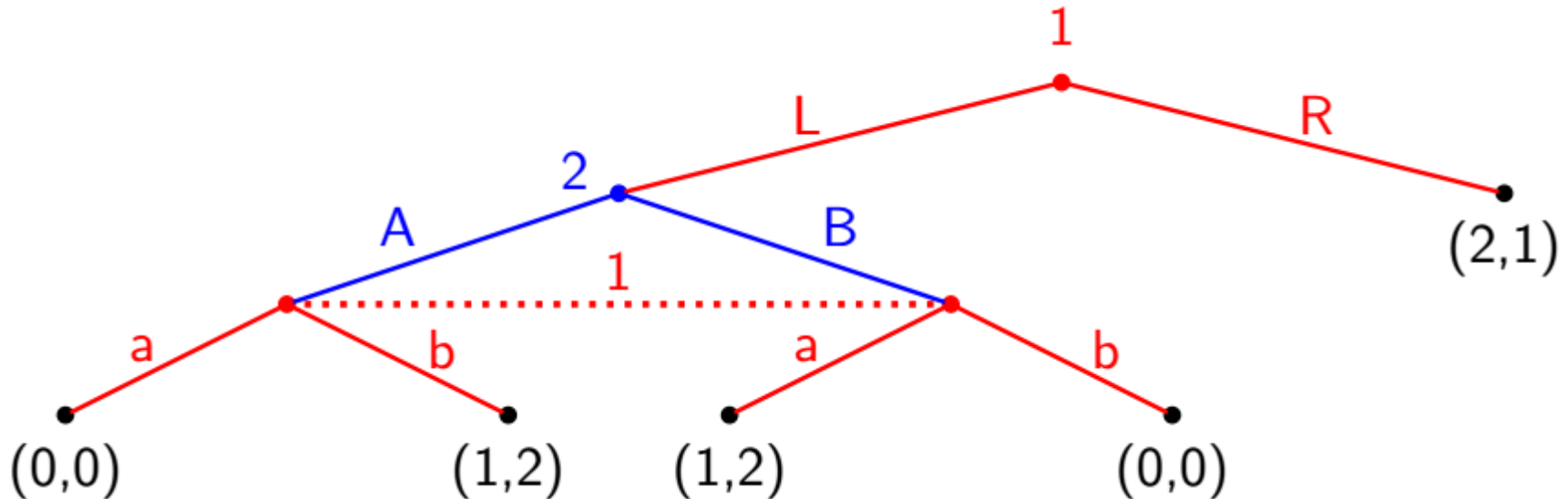
- Player 1 has information set $I_{11} = \{\emptyset\}$
- Player 2 has information set $I_{21} = \{C\}$
- Player 3 has the information set $I_{31} = \{D, Cd\}$

Pure Strategies

- A pure strategy for player i selects an available action at each of i 's information sets I_{i1}, \dots, I_{im}
- All pure strategies for player i is

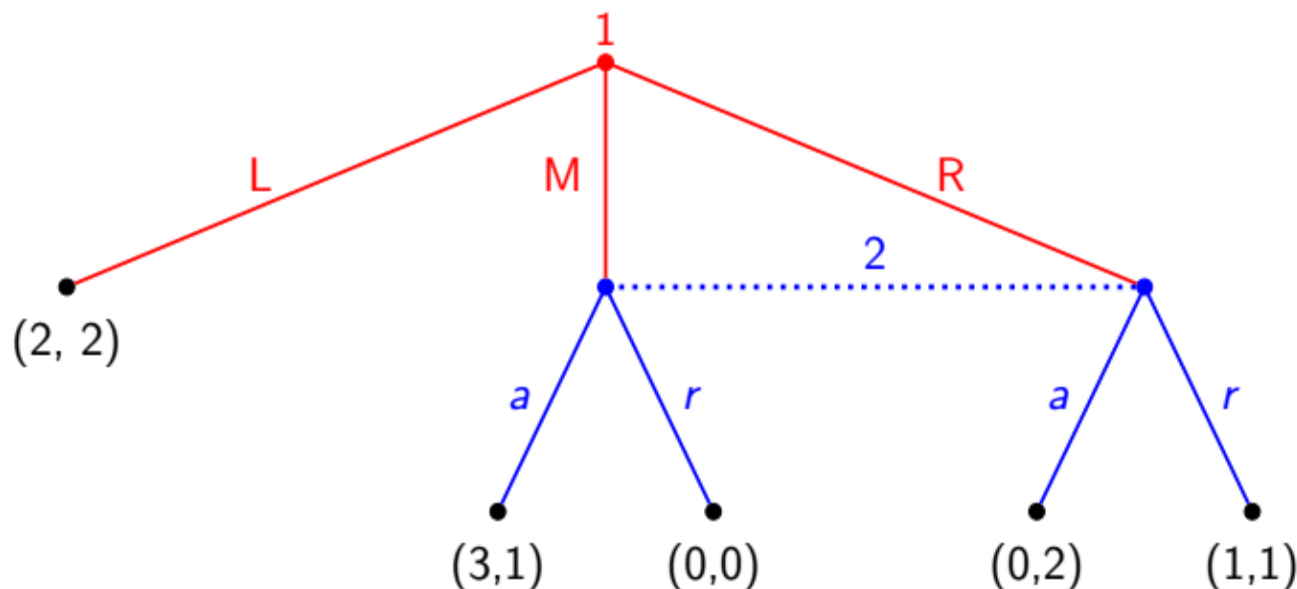
$$A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$$

where $A(I_{ij})$ denotes the strategies available in I_{ij}



What's the pure strategies for players 1 and 2?

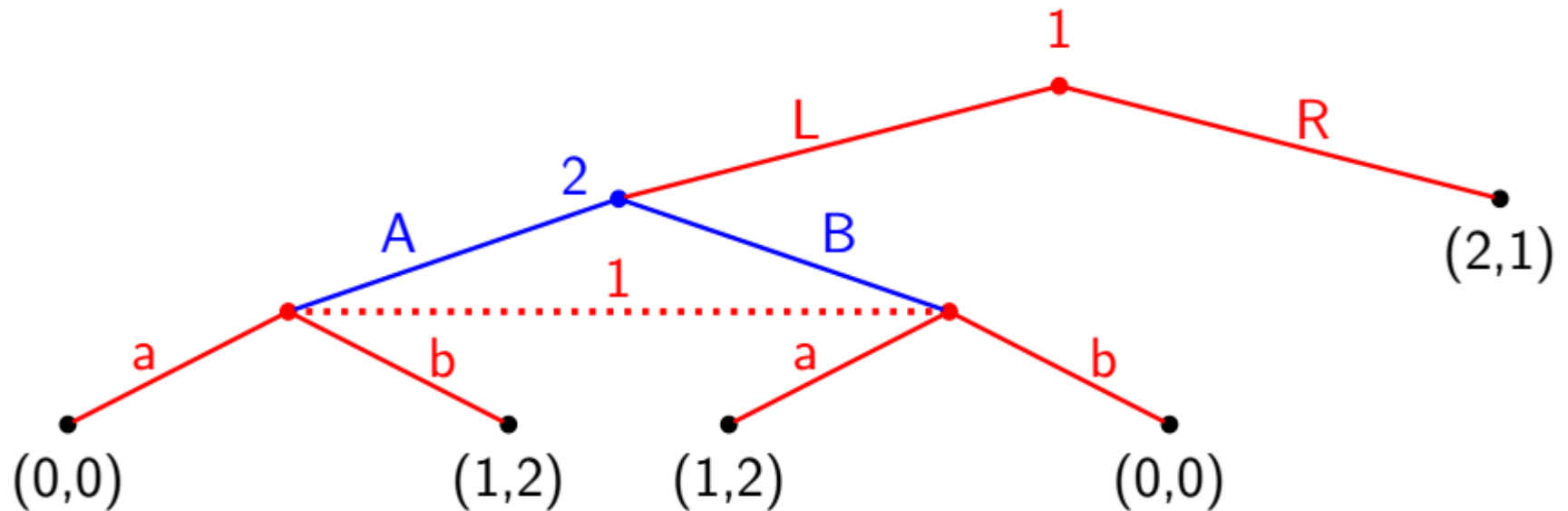
Normal-Form Representation of Extensive Imperf. Game



	a	r
L	2,2	2,2
M	3,1	0,0
R	0,2	1,1

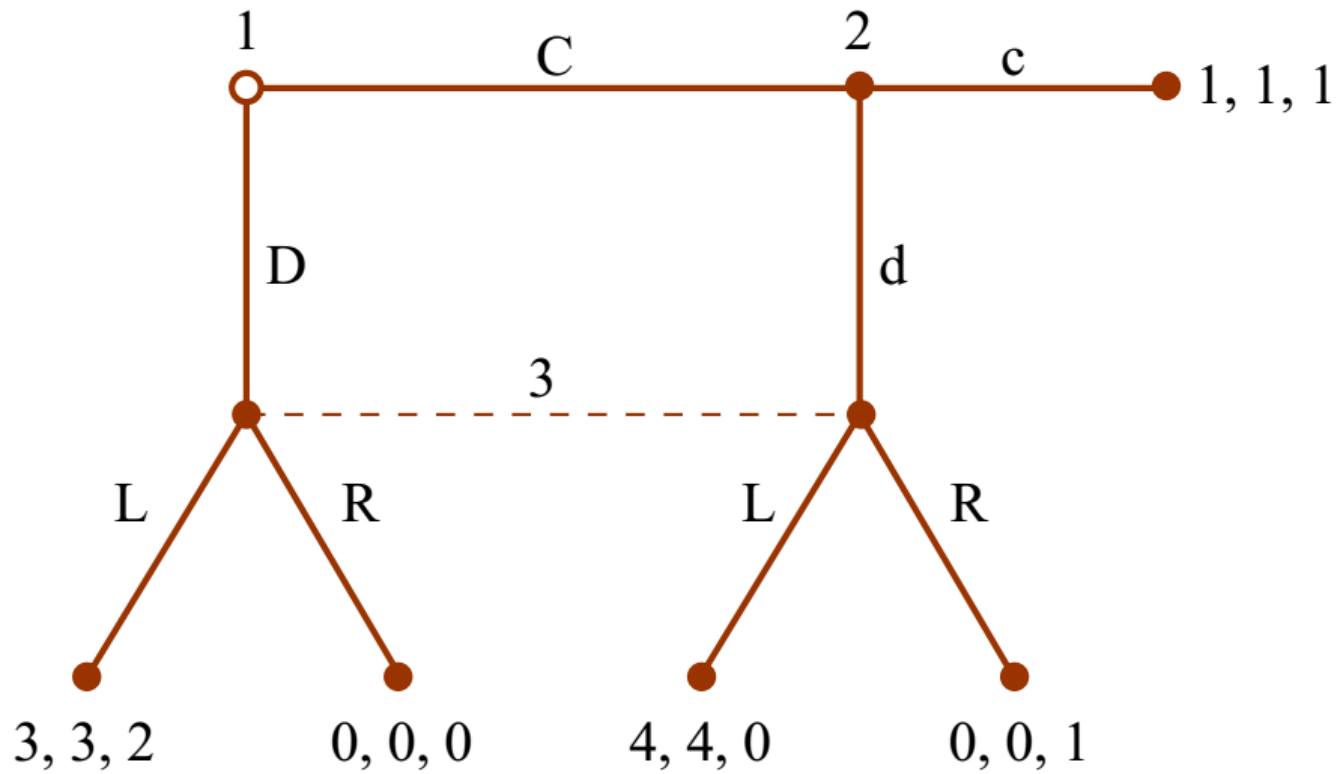
- The pure and mixed strategy Nash Equilibrium remains?
- What's the difference from the extensive game with perfect information game?

Exercise



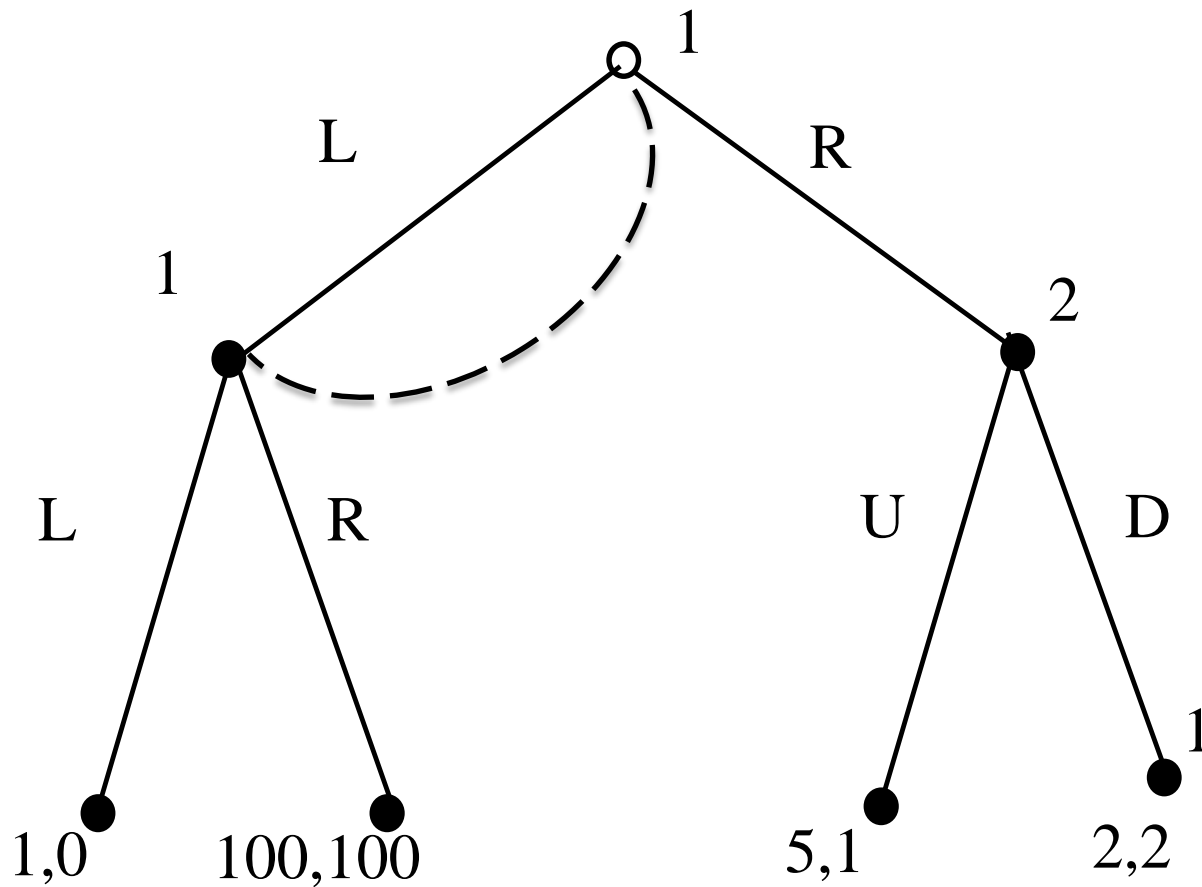
What are Nash Equilibria

Exercise



What are Nash Equilibria

Exercise



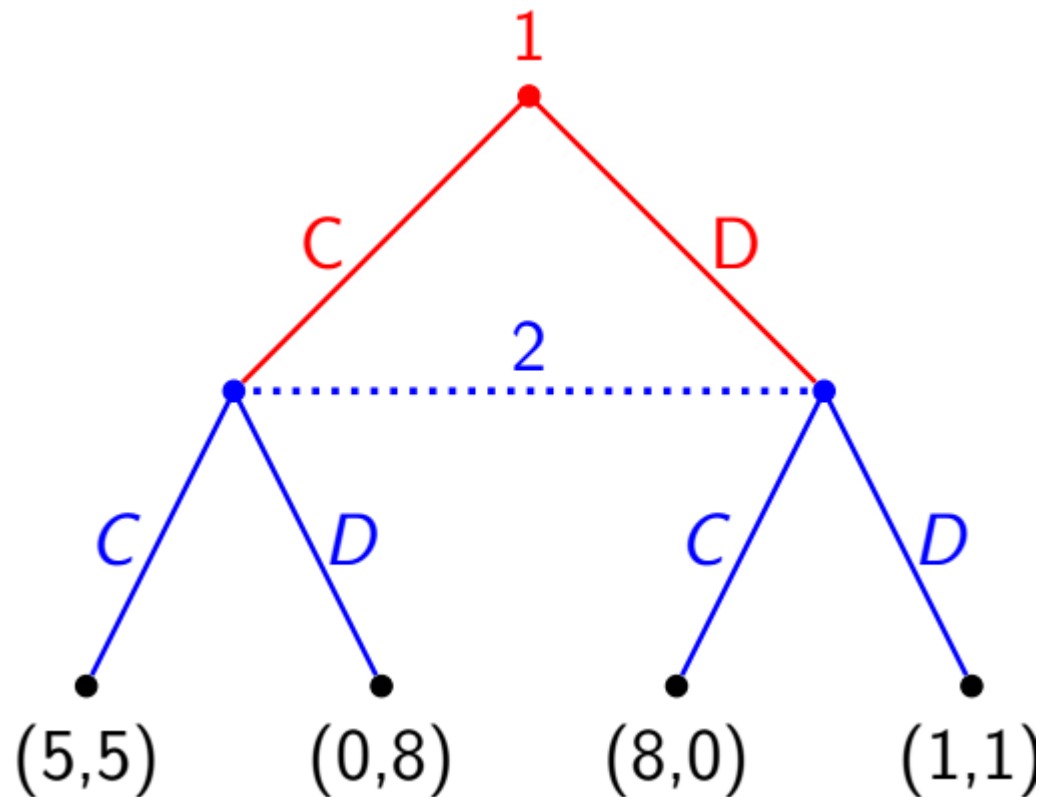
What are Nash Equilibria

Extensive Representation of Normal-Form Game

A strategy game \implies An extensive game with imp.

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Prisoner's Dilemma



Exercise: 3-Players Game

$$G = \{\{1, 2, 3\}, \{\{a, b, c\}, \{x, y, z\}, \{L, R\}\}, \{u_i\}_{i=1}^3\}$$

P3 chooses L

P2

P1

	x			y			z		
a	8	7	4	2	9	1	4	1	8
b	4	6	5	7	2	6	1	3	7
c	6	2	2	5	1	7	4	4	2

P3 chooses R

P2

P1

	x			y			z		
a	5	3	2	6	5	4	1	2	4
b	8	6	2	2	8	10	5	2	6
c	6	9	4	1	1	3	9	7	8

Perfect Recall (完美回忆) and Imperfect Recall

- An extensive game has **perfect information** if each information set consist of only one nodes
- An extensive game has **perfect recall** if each player recalls exactly what he did in the past
 - otherwise, this game has **imperfect recall**

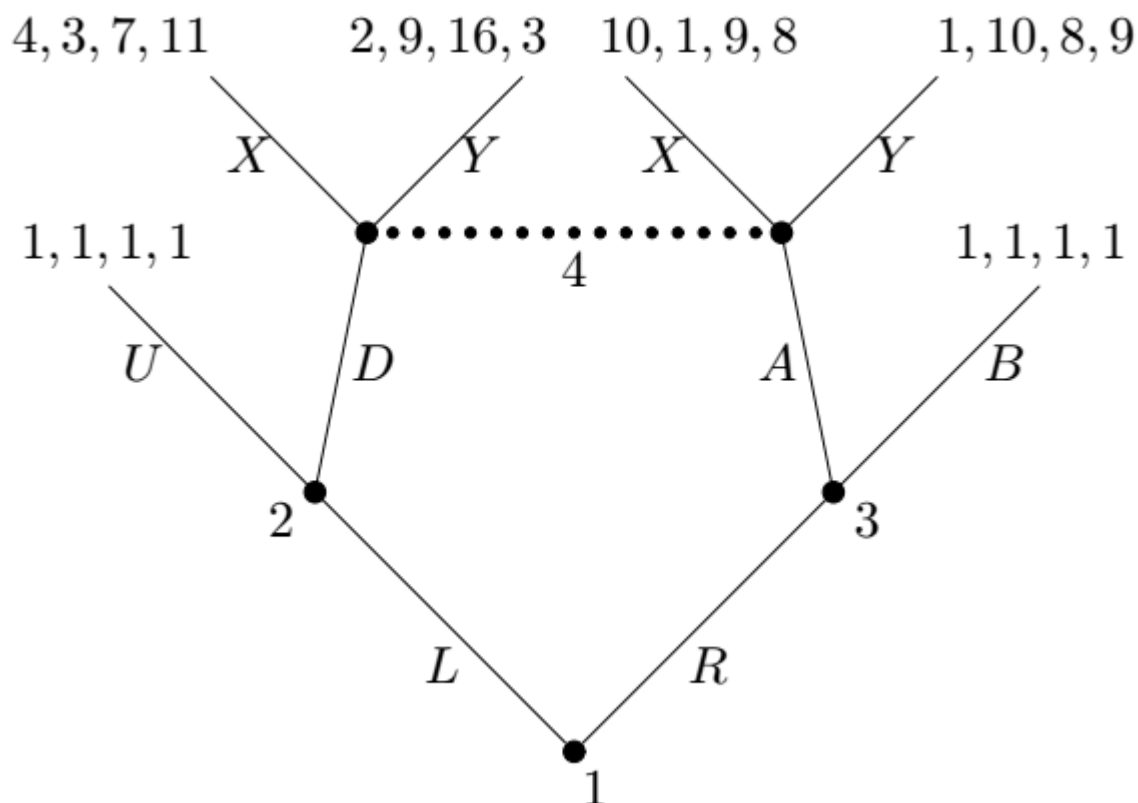
Formal Definition of Perfect Recall

Player i has **perfect recall** in game G if for any two history h and h' that are in the same information set for player i , for any path h_0, h_1, \dots, h_n, h and $h'_0, h'_1, \dots, h'_m, h'$ from the root to h and h' with $P(h_k) = P(h'_k) = i$, we have

- $n = m$
- $h_i = h'_i$ for $1 \leq i \leq n$

G is **a game of perfect recall** if every player has perfect recall in it.

Example



Perfect recall

If we change player 4 by player 1, is it a perfect recall

Example of Imperfect Recall

