Game Theory and Applications (博弈论及其应用)

Chapter 13: Extensive Game with Imperfect Information-II

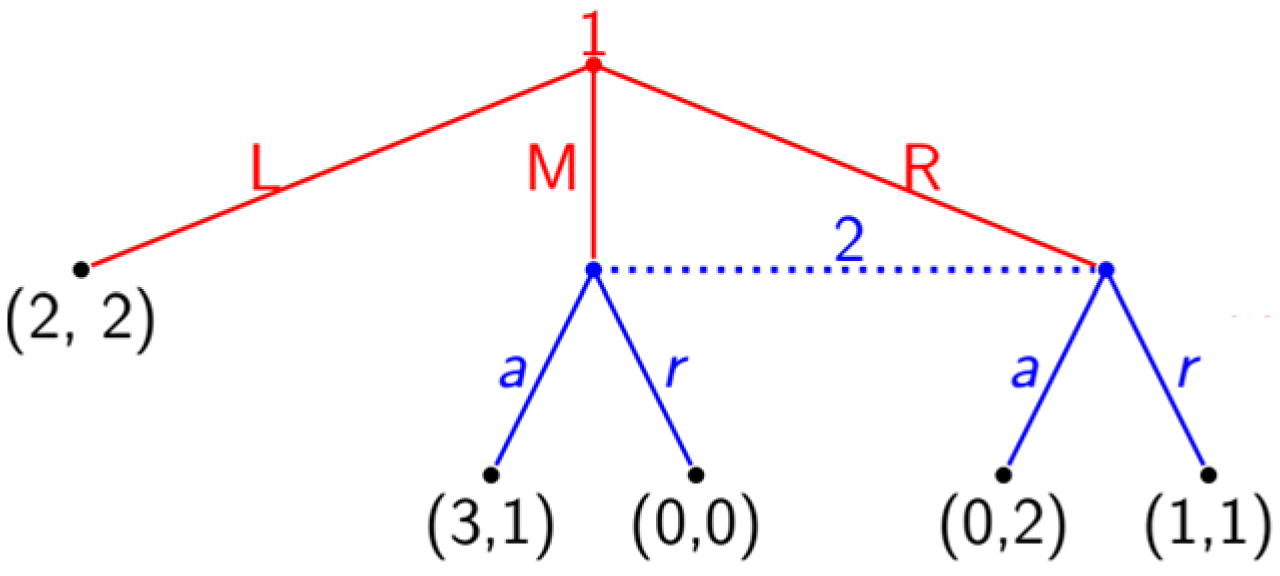
南京大学

高 尉

Recap on Previous Chapter

- Extensive game with imperfect information
- Formal definition $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \ldots I_N\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- Transformation of strategic game and extensive game with imperfect information
- Perfect recall and imperfect recall

Example



Definition of Mixed and Behavioral Strategies

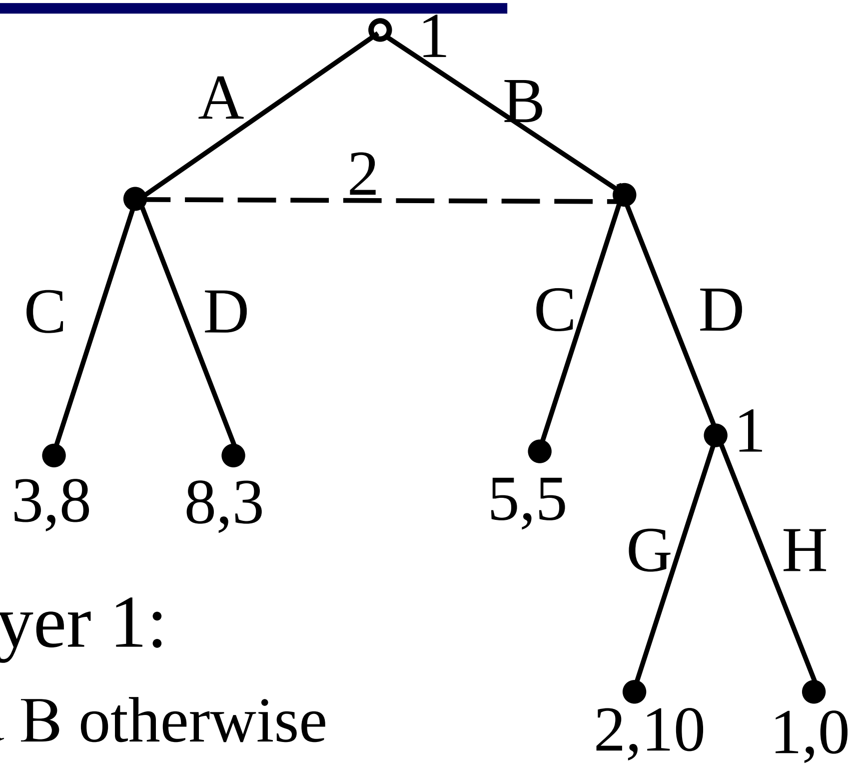
- Mixed Strategies: A mixed strategy of player i in an extensive game is a probability over the set of player i's pure strategy

- Behavioral strategies: A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik}\in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $\mathrm{A}(I_{ik})$

混合策略是在pure strategy上的概率分布
行为策略是在InfoSet的ActionSet上的概率分布，是独立的

Behavioral vs. Mixed Strategies

Behavioral strategies distinguish from mixed strategies



A behavioral strategy for player 1:

- Selects A with prob. 0.5, and B otherwise
- choose G with prob. 0.3, and H otherwise

Here's a mixed strategy that isn't a behavioral strategy

- Pure Strategy AG with probability 0.6, pure strategy BH 0.4
- The choices at the two nodes are not independent

等价的Mixed strategies：{AG,0.15; AH,0.35; BG,0.15; BH,0.35}

Behavioral vs. Mixed Strategies

In imperfect-information games, mixed and behavioral strategies produce different sets of equilibria

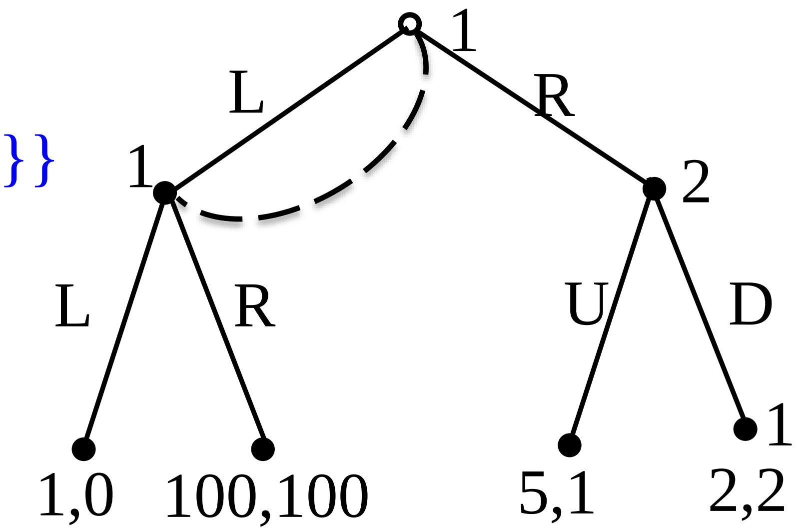
- In some games, mixed strategies can achieve equilibria that aren’t achievable by any behavioral strategy
- In some games, behavioral strategies can achieve equilibria that aren’t achievable by any mixed strategy

Behavioral vs. Mixed Strategies

和Ch12的P18同图

Consider game

Player 1 inform. set: {{Ø, L}}

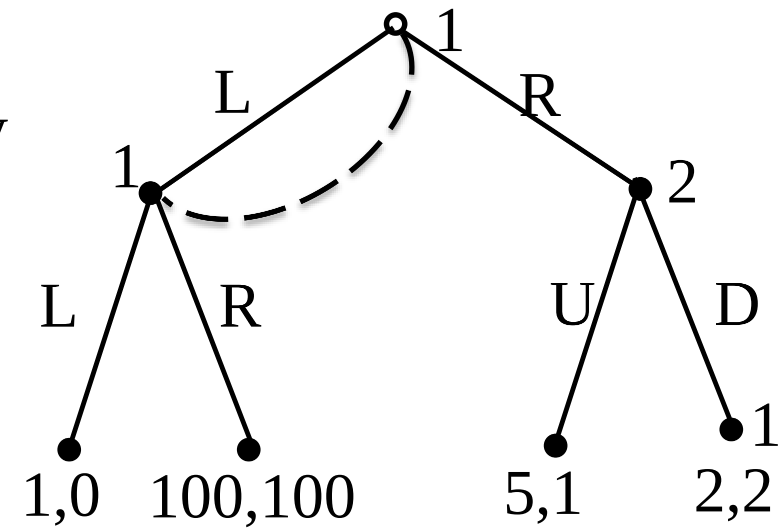


- Player 1: R is a strictly dominant strategy
- Player 2: D is a strictly dominant strategy
  - (R, D) is the unique Nash equilibrium for mixed strategy

Behavioral vs. Mixed Strategies

1: the information set is {(∅,L)}

2: D is a strictly dominant strategy



Player 2’s best response to D:

- Player 1’s the behavioral strategy $[L, p; R, 1 - p]$ i.e., choose L with probability p
- The expected payoff of player 1 is
- $U_1 = p^2 + 100p(1 - p) + 2(1 - p) = -99p^2 + 98p + 2$
- To find the maximum, we have $p = 49/99$

(R,D) is not an equilibrium for behavioral strategy

Formal Definition of Perfect Recall

Player i has **perfect recall** in game G if for any two history h and h' that are in the same information set for player i, for any path $h_0, h_1, \ldots h_n, h$ and $h'_0, h'_1, \ldots h'_m, h'$ from the root to h and $h'$with $P(h_k) = P(h'_k) = i$, we have

- $n = m$
- $h_i = h'_i$ for $1 \leq i \leq n$

G is **a game of perfect recall** if every player has perfect recall in it.

Kuhn Theorem (1953)

Theorem In an finite extensive game with perfect recall

- any mixed strategy of a player can be replaced by an equivalent behavioral strategy
- any behavioral strategy can be replaced by an equivalent mixed strategy
- Two strategies are equivalent

推论，必然的结果

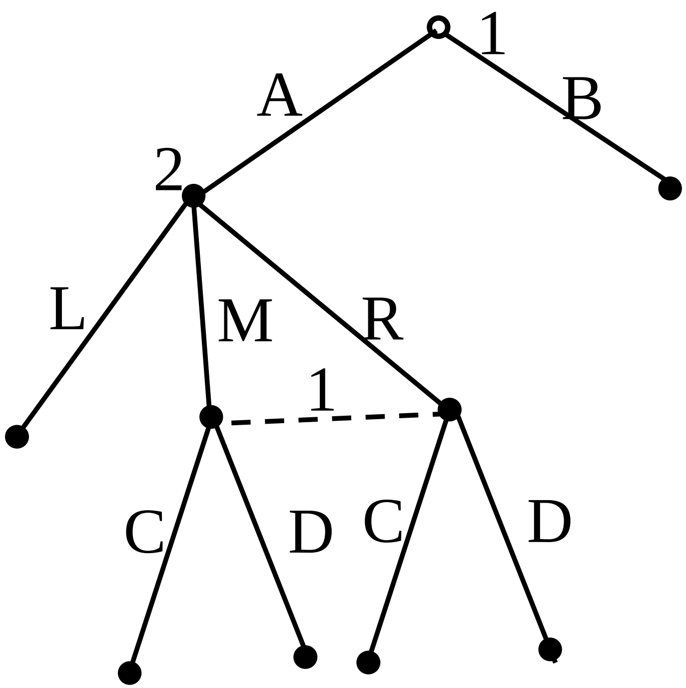
Corollary In an finite extensive game with perfect recall, the set of Nash equilibrium does not change if we restrict ourselves to behavior strategies

Proof. See board.

Example

What behavioral strategy is equivalent to mixed strategy $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$

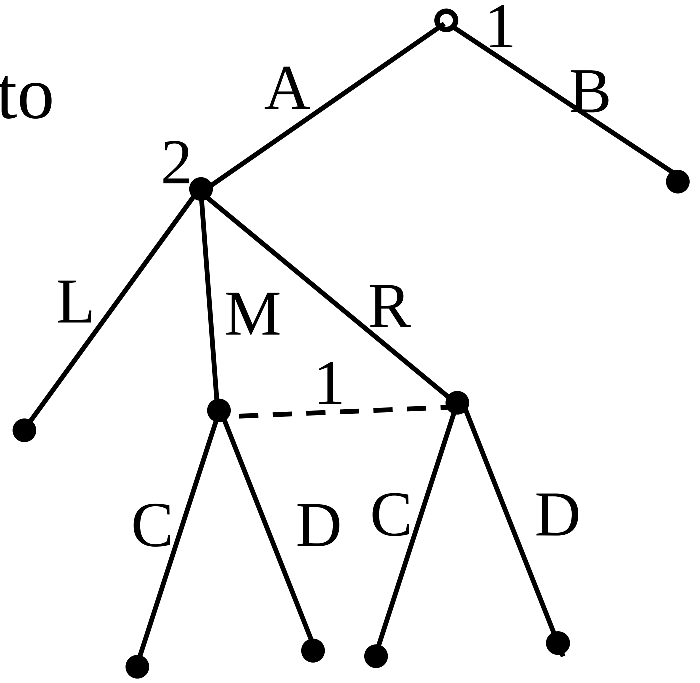
- $I_{11} = \{\emptyset\}$ $I_{12} = \{AM, AR\}$
- $A(I_{11}) = \{A, B\}$
- $A(I_{12}) = \{C, D\}$



- $\beta_{11}(I_{11})(A) = p_{AC} + p_{AD}$ $\beta_{11}(I_{11})(B) = p_{BC} + p_{BD}$
- $\beta_{12}(I_{12})(C) = \frac{p_{AC}}{p_{AC}+p_{AD}}$ $\beta_{12}(I_{12})(D) = \frac{p_{AD}}{p_{AC}+p_{AD}}$

Example

What mixed strategy is equivalent to behavioral strategy of prob. p over A and q over C



混{AC, AD, BC, BD}
P + P + P + P=1

行{A, B, C, D}
P+P=1 P+P=1

$$(p_{AC}, p_{AD}, p_{BC}, p_{BD})$$
$$= (pq, p(1-q), (1-p)q, (1-p)(1-q))$$

How to Compute Nash Equilibria of Perfect Recall Game

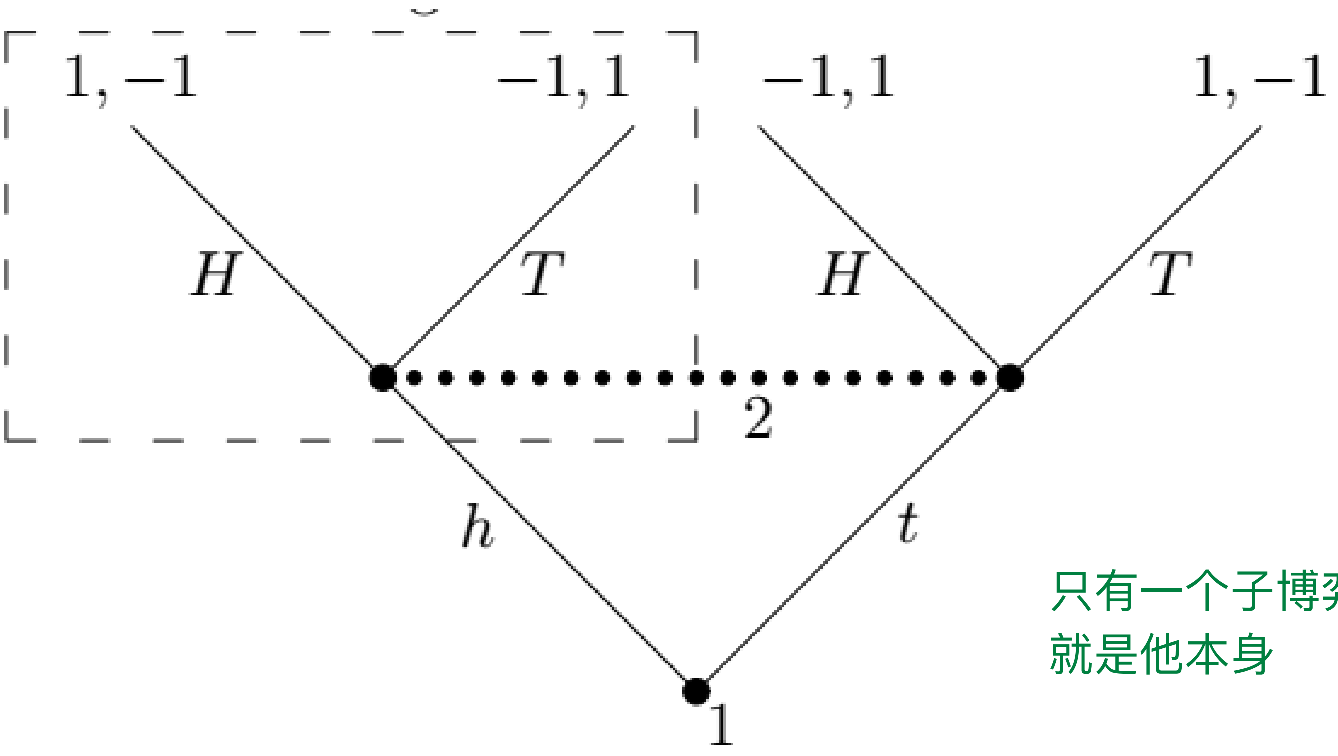
How can we find an equilibrium of an imperfect information extensive form game?

- One idea: convert to normal-form game
  - General game: exponential blow up in game size
  - Zero-sum game: LP formulation

求法都一样，只是多一个限制：分不清的点（虚线连着的点）一定要在一起

Extensive Imperfect Subgame

Definition A subgame of an extensive imperfect game G is some node in the tree G and all the nodes that follow it, with the properties that any information set of G is either completely in or outside the subgame



只有一个子博弈
就是他本身

Subgame Perfect Nash Equilibrium

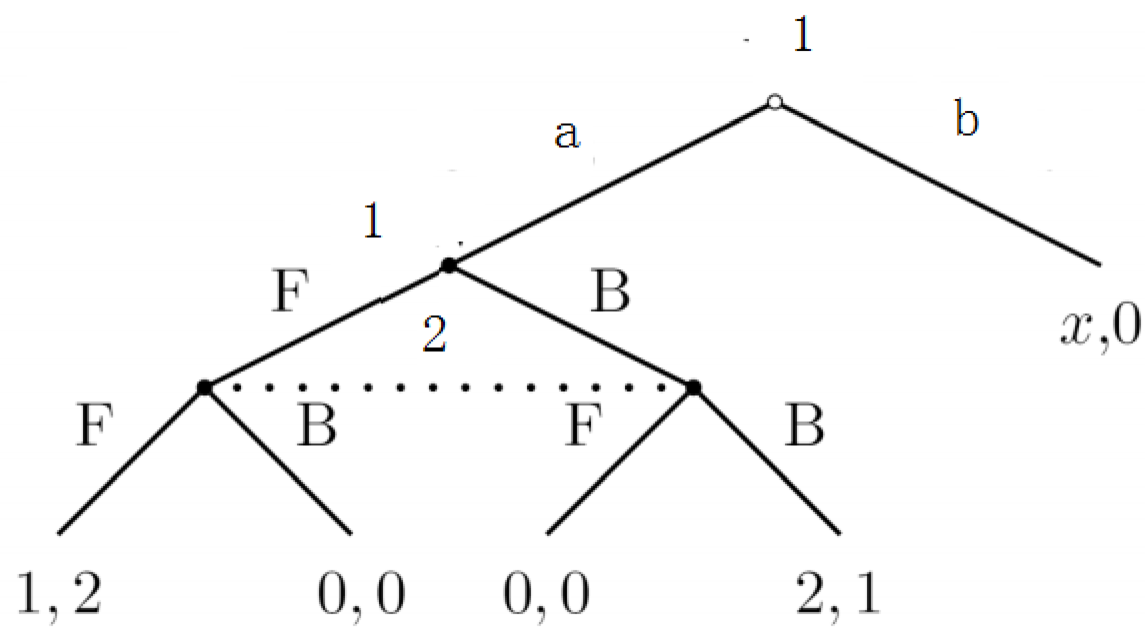
Definition A subgame perfect Nash equilibrium of an extensive form game G with perfect recall is a outcome of behavior strategies $(\beta_1, \beta_2, \dots, \beta_N)$ such that it is a Nash Equilibrium for every subgame

Theorem Every finite extensive game with perfect recall has at least one subgame perfect Nash Equilibrium

How to find SPNE

Backwards Induction

Example



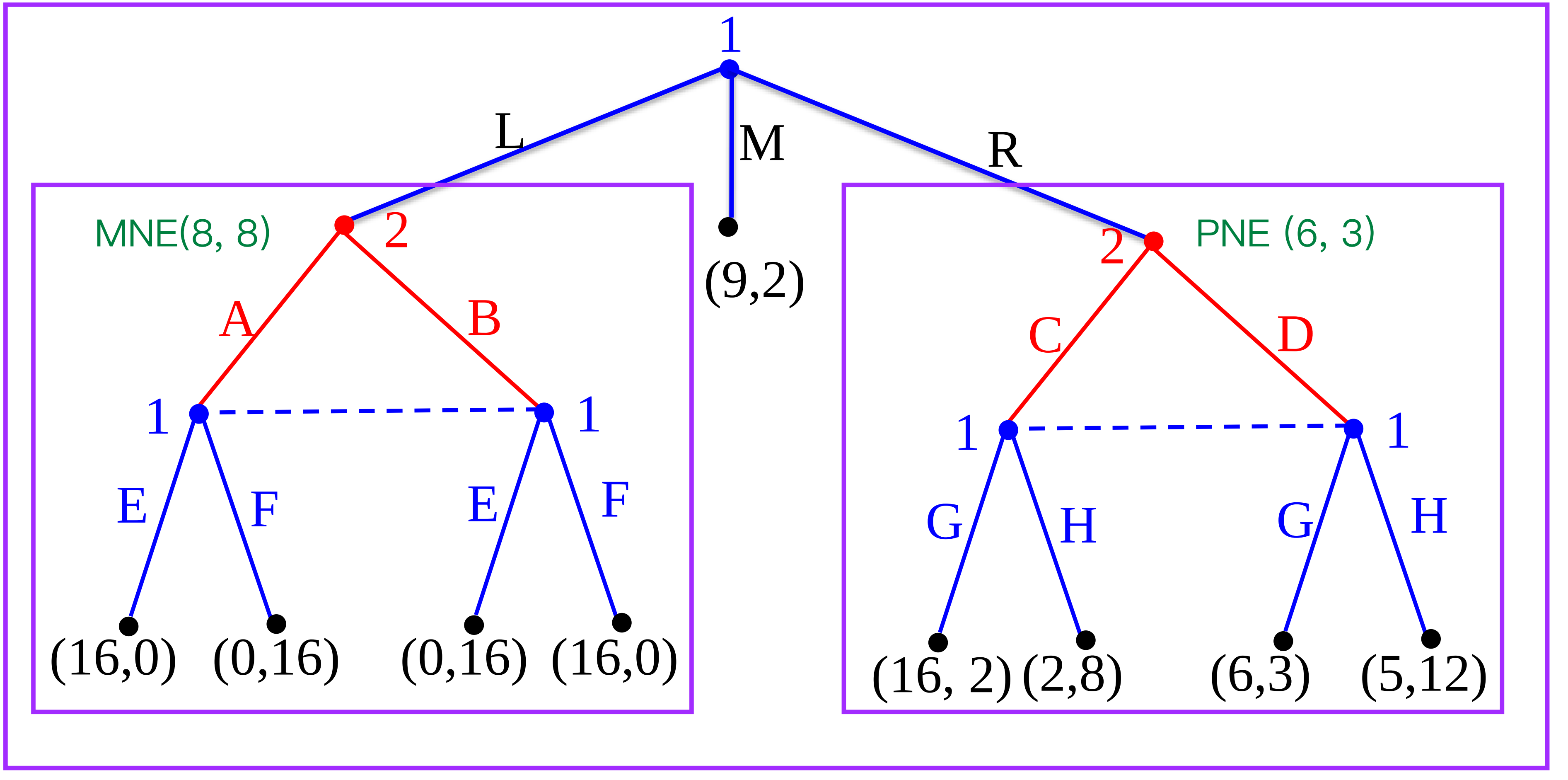
两个均衡(F, F)(B, B)，分情况讨论

（1）x>1选b

（2）x>2选b

Example 先找子博弈，再后向归纳

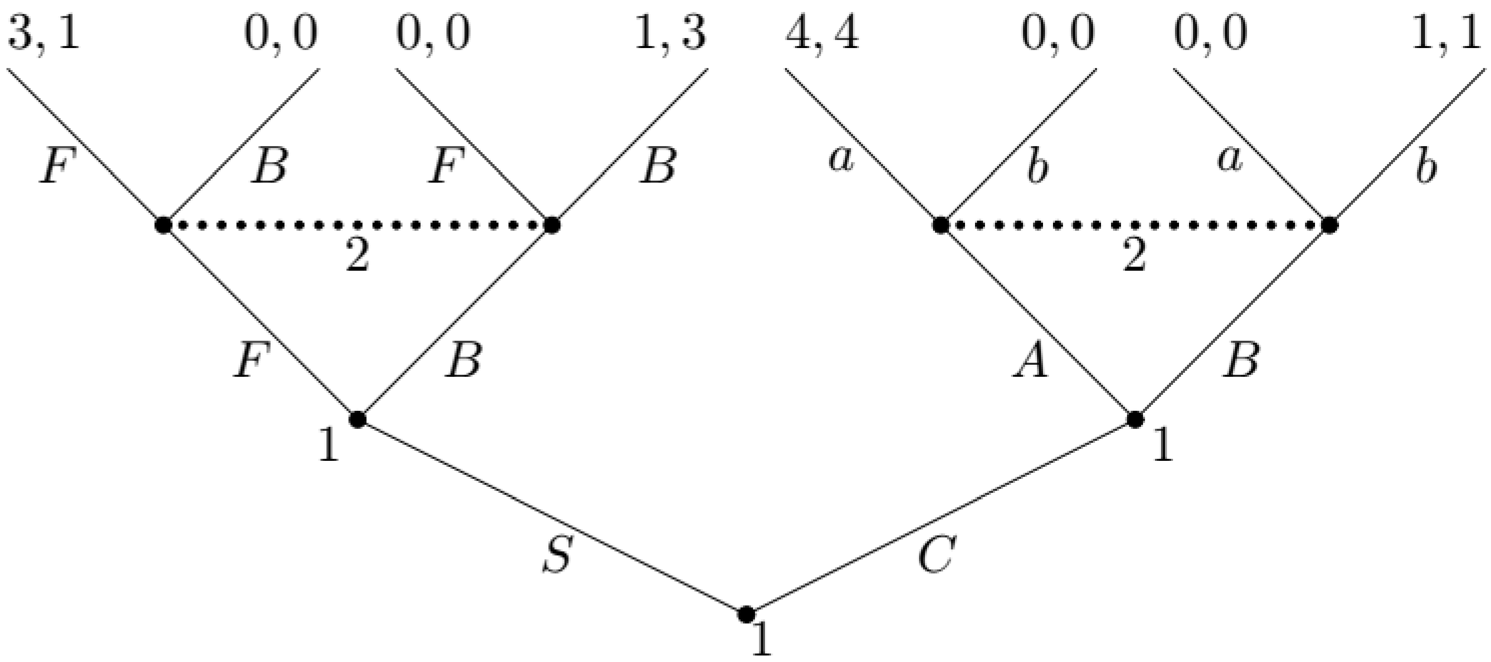
三个子博弈



How to solve SPNE?

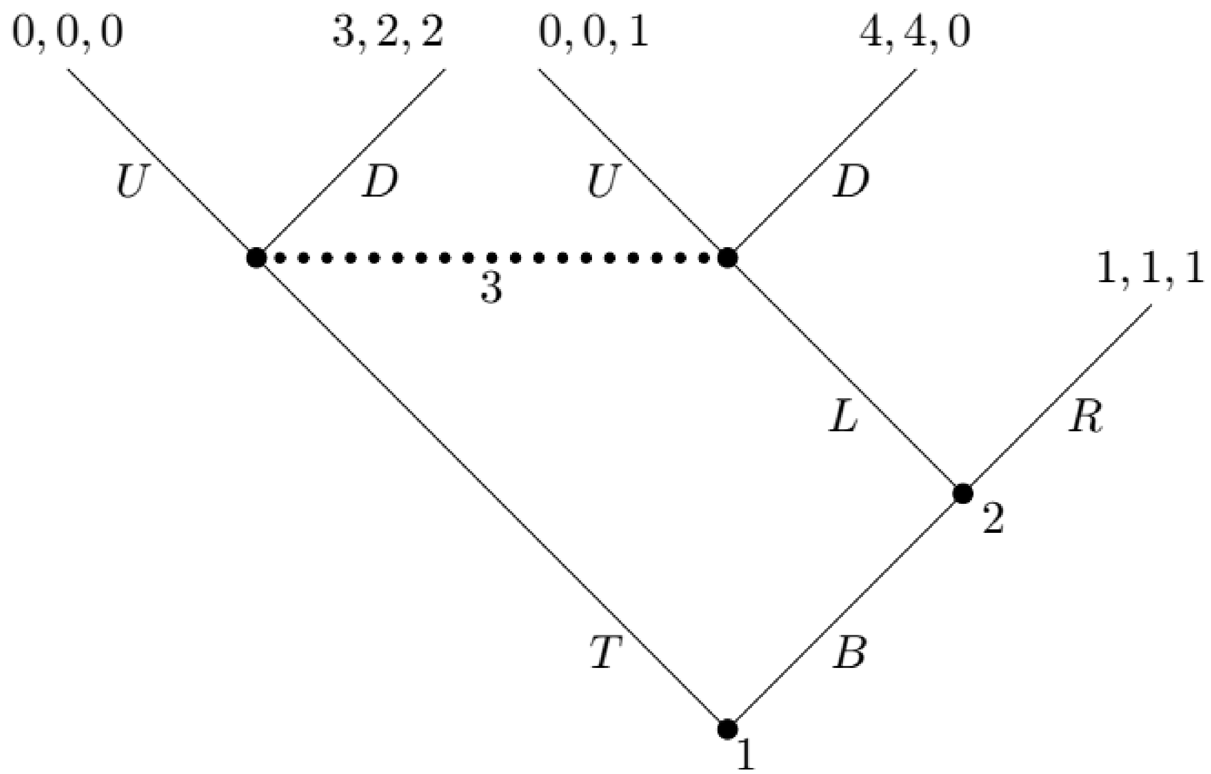
(8, 8)VS(9, 2)VS(6, 3)选9

Exercise HW1



- How many SPNE for this game?

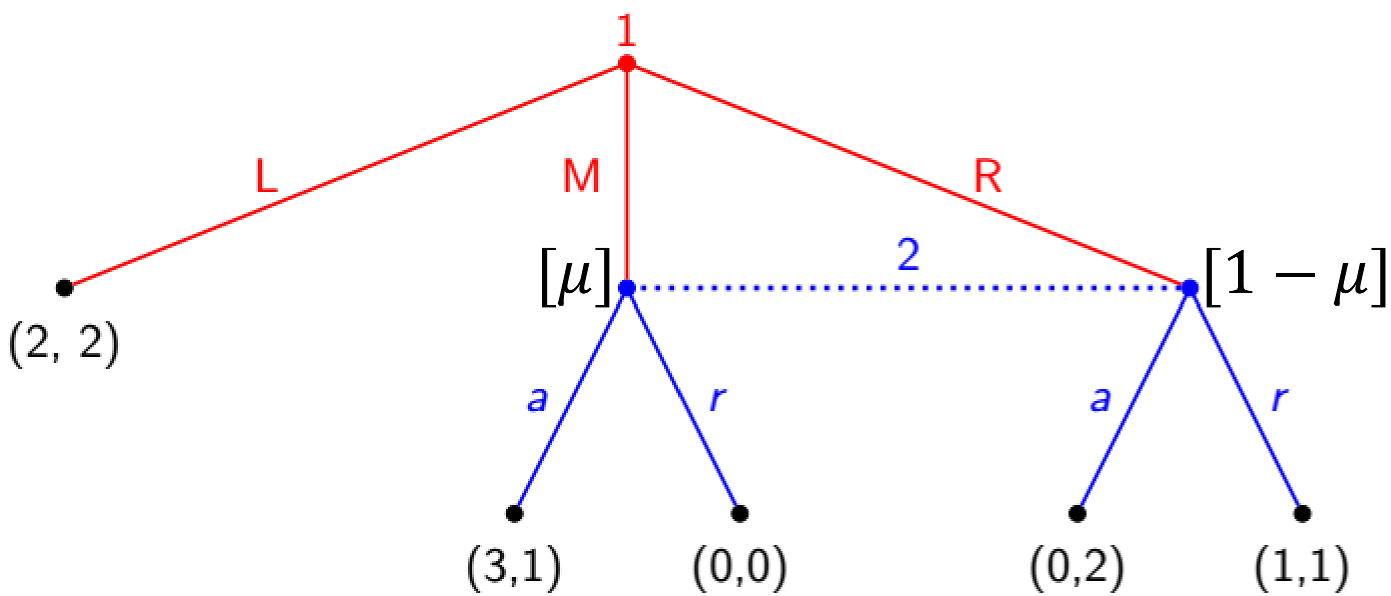
Exercise HW2



How many SPNE?

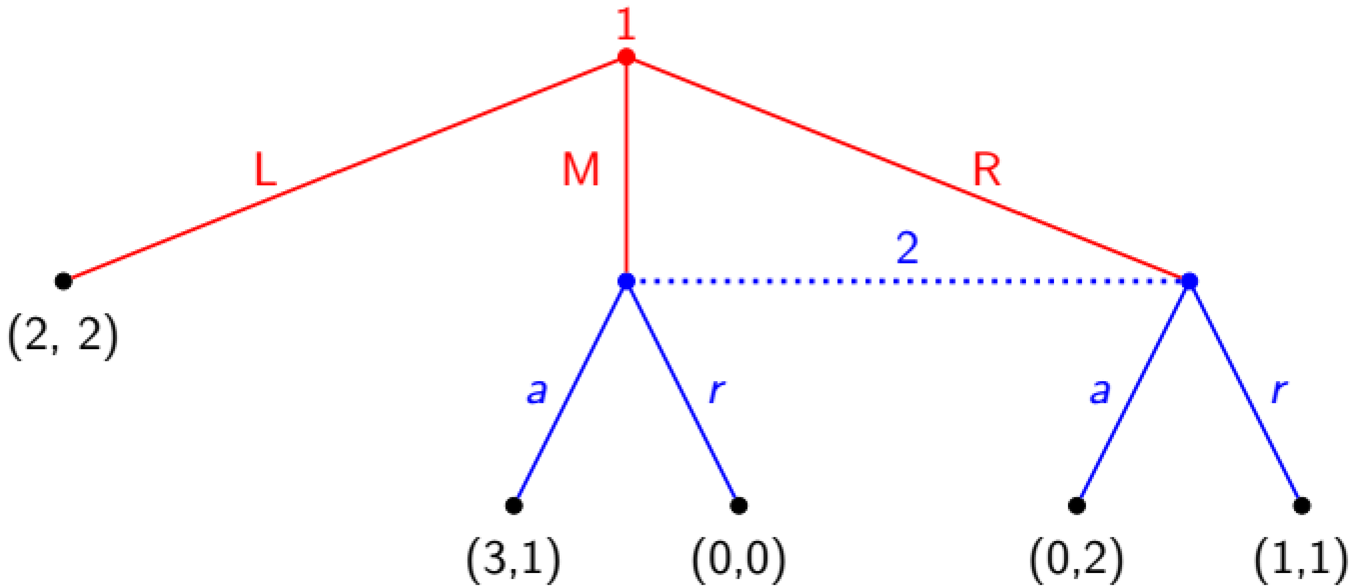
Beliefs 信念（和之前无关啦）感觉专治前两个EX里NE过多的情况

- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- The probability is 1 for the information set of size 1

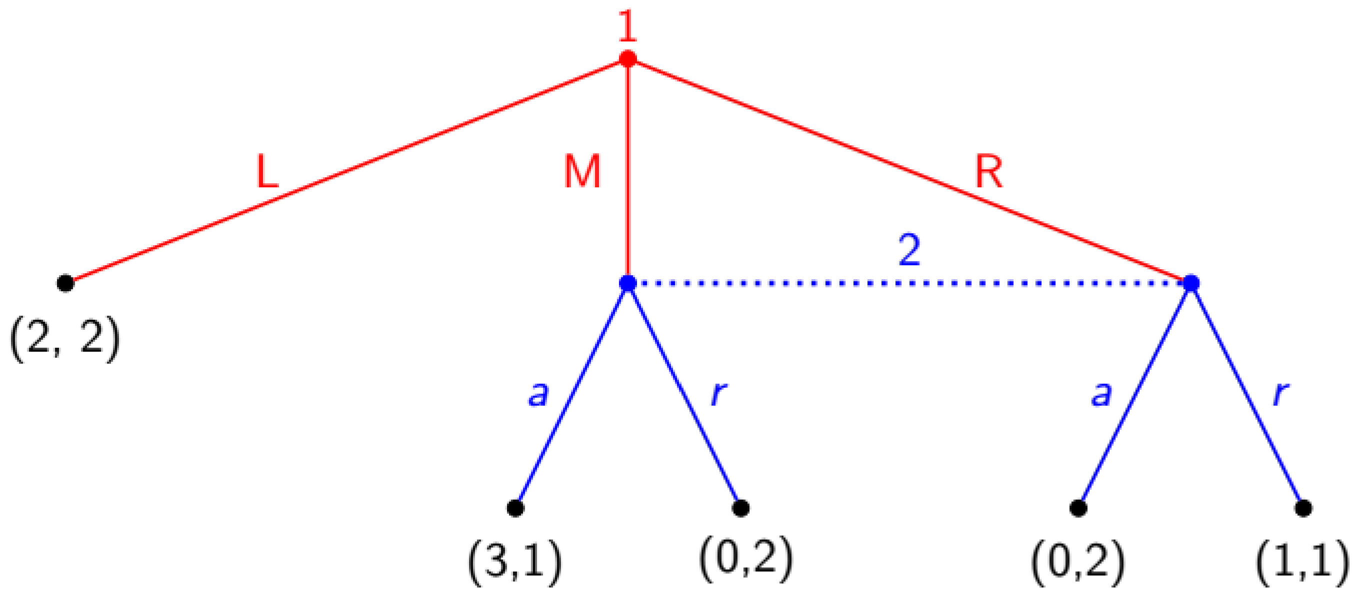


Behavioral Strategies

- A behavior strategy β a collection of independent probability measure over the actions after information set



Beliefs and Optimal Behavior Strategies 最优行为策略

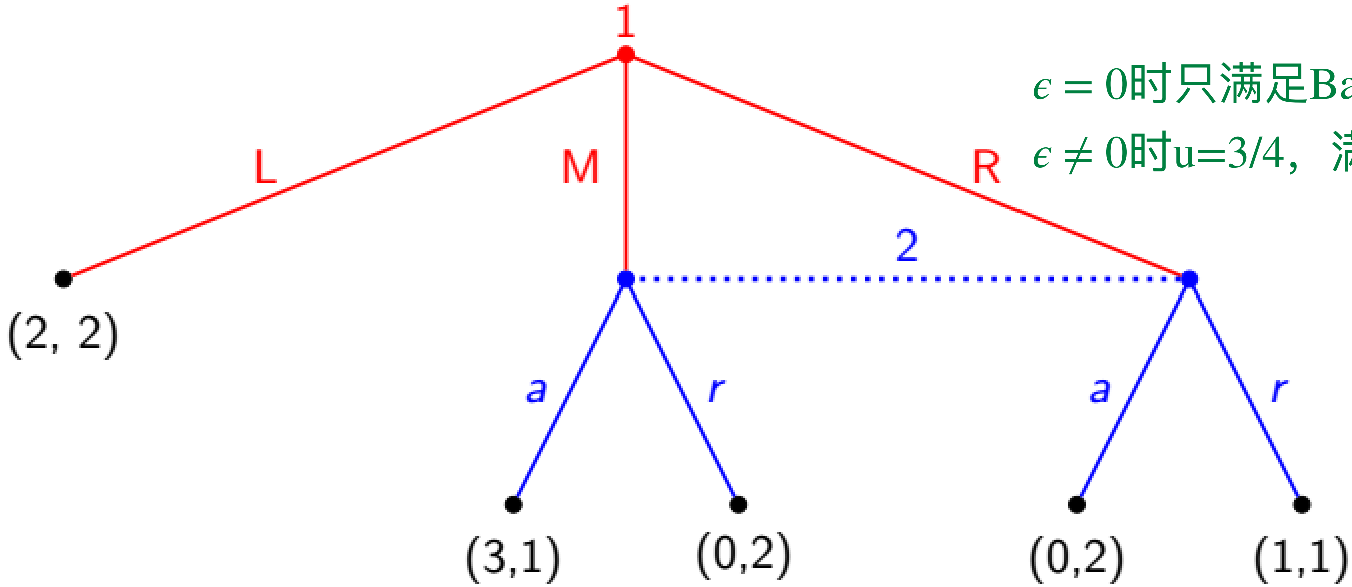


- Beliefs affect optimal strategies: For 2, a is the best strategies iff 2 assigns a belief $\mu(M) \leq 1/2$
- Strategies affect reasonable beliefs: If 1 assigns to action (L,M,R) prob. (0.1,0.3,0.6), then Bayes rule requires the belief (1/3,2/3) of 2 都是1：2
- What are reasonable beliefs if 1 select L with prob. 1

Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



$\epsilon = 0$时只满足Bayes一致性

$\epsilon \neq 0$时u=3/4，满足一致性

1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

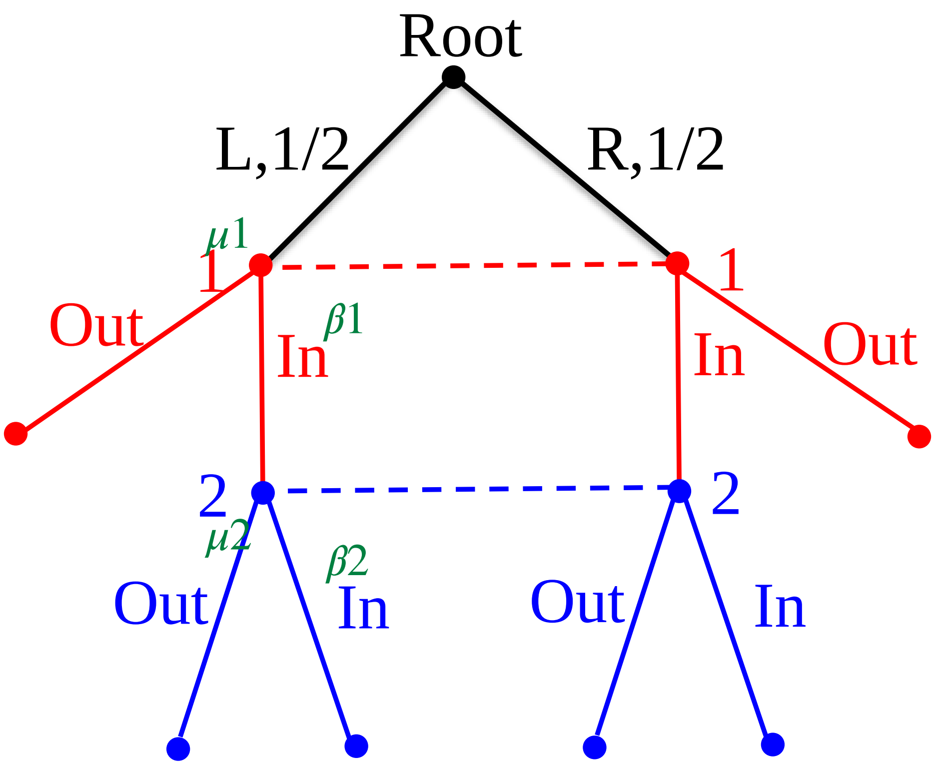
Assessment (评估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h}, \mathrm{a}) = \beta_{h,a}(h, a) / \sum_a \beta_{h,a}(h, a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent 序列 $(\beta^n, \mu^n) \rightarrow (\beta, \mu)$ as $n \rightarrow \infty$
- 自身 (β, μ) is consistent $\rightarrow$ (β, μ) Bayesian consistent

Example



- The payoffs are omitted since they are irrelevant
- Find all Bayesian consistent assessments
- Find all consistent assessments

Bayesian consistency

An assessment (β, μ) by a 4-tuple $(\beta_1, \beta_2, \mu_1, \mu_2) \in [0,1]^4$

- β_1 is the probability that 1 chooses In
- β_2 is the probability that 2 chooses In
- μ_1 is the belief assigns to the left node in 1's info set
- μ_2 is the belief assigns to the left node in 2's info set

Two cases:

- i) If $\beta_1 \in (0,1]$, 2's information set is reached with positive probability. Bayes' Law dictates that $\mu_1 = \mu_2 = 1/2$.
  $$(\beta_1, \beta_2, \mu_1, \mu_2) = (0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$
  are Bayesian consistent
- ii) If $\beta_1 = 0$, then 2's information set is reached with zero probability and $\mu_2 \in [0,1]$
  $$(\beta_1, \beta_2, \mu_1, \mu_2) = \{0\} \times [0,1] \times \{1/2\} \times [0,1]$$
  are Bayesian consistent

Consistency

- Every complete outcome of behavioral strategies leads to $\mu_1 = \mu_2 = 1/2$.

- 2's information set, both nodes are reached with equal probability.

- Conclusion:

$$(\beta_1, \beta_2, \mu_1, \mu_2) = [0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

are consistent