Game Theory and Applications (博弈论及其应用)

Chapter 5.2： Correlated Equilibrium

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Correlated Strategies

- In a Nash equilibrium, players choose strategies (or randomize over strategies) independently.
- For games with multiple NE, one may want to allow for randomizations in NE by some form of communication prior.

Battle of the Sexes game

Boy \ Girl	Ballet	Football
Ballet	1 4	0 0
Football	0 0	4 1

For fairness, randomization between two pure strategy NE: flip a coin and go to the Ballet for heads coin; the Football otherwise.

Payoffs: (5/2, 5/2) that is not a Nash equilibrium payoff

Traffic Intersection Game

Consider two cars arrive at an intersection simultaneously.

- Car A has the options U(up) or D(down)
- Car B has the options L(left) or R (right)

		Car B: L		Car B: R	
Car A	U	5	1	0	0
Car A	D	4	4	1	5

Two pure strategy NE: (U, L) and (D, R)

To find a mixed strategy NE, assume car A plays U with probability p and player 2 plays L with probability q. We have

$$5q = 4q + (1 - q) \;\Rightarrow\; q = 1/2$$

$$5p = 4p + (1 - p) \;\Rightarrow\; p = 1/2$$

There is a unique mixed strategy NE with expected payoff (5/2, 5/2)

Traffic Intersection Game

- Assume there is a observable random variable (traffic light)
  - with probability 1/2 (Green): car A plays U and car B plays L
  - with probability 1/2 (Red): car A plays D and car B plays R.
- The expected payoff for this play of the game increases to (3,3)
- No player has an incentive to deviate from the "recommendation"
  - if car A sees a Head, he believes that car B will play L, and thus playing U is his best response (similar argument when he sees a Tail).
  - if car B sees a Head, he believes that car A will play U, and thus playing L is his best response (similar argument when he sees a Tail).
- When the recommendation of the traffic light is part of a Nash equilibrium, no player has an incentive to deviate

Traffic Intersection Game

- With a observable random variable, we can get any payoff vector in the convex hull of **Nash equilibrium payoffs**
  - The convex hull of a finite number of vectors $a_1, \ldots, a_k$ is given by

$$\mathrm{Conv}(a_1, \ldots, a_k) = \{a : a = \sum a_i \lambda_i, \ \ \lambda_i \geq 0, \ \ \sum \lambda_i = 1\}$$

- The traffic light is one way of communication prior to the play.

- A more elaborate signal scheme: suppose the players find a mediator who chooses $\xi \in \{1, 2, 3\}$ with equal probability 1/3.
  - If $\xi = 1$, then car A plays U, car B plays L.
  - If $\xi = 2$, then car A plays D, car B plays L.
  - If $\xi = 3$, then car A plays D, car B plays R.

Traffic Intersection Game

- We show that no player has an incentive to deviate from the "recommendation" of the mediator:
  - If car A gets the recommendation U, he believes car B will play L, so his best response is to play U.
  - If car A gets the recommendation D, he believes car B will play L, R with equal probability, so playing D is a best response.
  - If car B gets the recommendation L, he believes car A will play U, D with equal probability, so playing L is a best response.
  - If car B gets the recommendation R, he believes car A will play D, so his best response is to play R.
- The players will follow the mediator's recommendations.
- The expected payoffs are (10/3, 10/3), strictly higher than that of randomization in NEs

Correlated Strategy

Correlated strategies

- Let $\Delta(A)$ denote set of probability over $A = \mathrm{A}_1 \times A_2 \times \cdots \times A_n$
- Let π be a distribution over A
- Let R be a random variable taking values in A according to π.

What's the difference among correlated strategy, pure strategy and mixed Strategy?

Example

		Girl Ballet	Girl Football
Boy	Ballet	1 4	0 0
Boy	Football	0 0	4 1

	Ballet	Football
Ballet	p_1	p_2
Football	p_3	p_4

Correlated strategy

$$\pi = (p_1, p_2, p_3, p_4)$$

Pure strategy

$$\pi = (1,0,0,0)$$

Mixed strategy

$$\pi = \big(pq, (1-p)q, p(1-q), (1-p)q, (1-p)(1-q)\big)$$

The set of correlated strategy pairs is an extension of the set of mixed strategy pairs

Correlated Equilibrium

Definition: A **correlated equilibrium** of game $G = \{N, \{A_i\}, \{u_i\}\}$ is **a joint probability distribution $\boldsymbol{\pi} \in \boldsymbol{\Delta}(\boldsymbol{A})$** such that if R is a random variable distributed according to π then

$$\sum_{a_{-i}} \Pr(R = a | R_i = a_i) \left[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) \right] \geq 0$$

for all players i, all $a_i, a_i' \in A_i$ such that $\Pr(R_i = a_i) > 0$

No player can ever expect to unilaterally gain by deviating from his recommendation, assuming the other players play according to their recommendations.

Characterization of Correlated Equilibrium

Proposition: A joint distribution $\pi \in \Delta(A)$ is a correlated equilibrium if and only if

$$\sum_{a_{-i}} \Pr(R = a)\,[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})] \geq 0$$

for all player i and all $a_i, a_i' \in A_i$

Proof. See board

Conclusions

Theorem: Every mixed NE is a correlated equilibrium

Theorem: The set of correlated equilibria is a convex set

Corollary: The set of correlated equilibria contains the convex hull of the set of Nash equilibria

How to Compute?

		Girl B	Girl F
Boy	B	2, 5	0, 0
Boy	F	0, 0	5, 2

p_1	p_2
p_3	p_4

The condition for a correlated equilibrium are

$$2 \times p_1 + 0 \times p_2 \geq 0 \times p_1 + 5 \times p_2 \rightarrow p_1 \geq 5p_2/2$$

$$0 \times p_3 + 5 \times p_4 \geq 2 \times p_3 + 0 \times p_4 \rightarrow p_4 \geq 2p_3/5$$

$$5 \times p_1 + 0 \times p_3 \geq 0 \times p_1 + 2 \times p_3 \rightarrow p_1 \geq 2p_3/5$$

$$0 \times p_2 + 2 \times p_4 \geq 5 \times p_2 + 0 \times p_4 \rightarrow p_4 \geq 5p_2/2$$

$$p_1 + p_2 + p_3 + p_4 = 1$$

Find appropriate solutions by solving the LP problem

Compute Correlated Equilibrium

For every player i, $a_i, a'_i \in \text{A}$

$$\sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p(a_i, a_{-i}) u(a'_i, a_{-i})$$

subject to

$$p(a_i, a_{-i}) \geq 0 \quad \text{and} \quad \sum p(a_i, a_{-i}) = 1$$

variable $p(a_i, a_{-i})$ constant $u(a_i, a_{-i})$

Comparisons

$$\sum_{a_{-i}\in A_{-i}} p(a_i, a_{-i})u(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} p(a_i, a_{-i})u(a'_i, a_{-i})$$

Subject to $\sum p(a_i, a_{-i}) = 1$ and $p(a_i, a_{-i}) \geq 0$

Correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities

$$p(a_i, a_{-i}) = \prod_j p_j(a_j)$$

which is not a linear programming