Game Theory and Applications (博弈论及其应用)

Chapter 12: Repeated Games II

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Recap on Previous Chapter

- Repeated game: many real interactions have an ongoing structure; players consider short- and long-term payoffs.
- A repeated game $G^{\mathrm{T}}(\delta)$ consists of stage game G, terminal date T and discount factor δ
- Folk Theorem
  - An infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \ldots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \ldots, u_N^*)$.
  - Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N)$, where, $\hat{u}_i \geq u_i^*$ for every player i
  - There is a Subgame Perfect Nash Equilibrium for some discount factor δ

Construct SPNE in Repeated Games

1. Solve all equilibria of the stage game (Competition)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (Cooperation)
3. Design trigger strategies that support cooperation and punish with competition
4. Compute the maximum discount factor so that cooperation is an equilibrium
5. The trigger strategies are an SPEN of the infinitely repeated game for some larger discount factor

Repeated Cournot Competition

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a-(q_1+q_2))$$

  - Costs $(i=1,2)$

$$c_i(q_i) = 0$$

这里简化为
cost为0

  - Payoffs $(i=1,2)$

$$u_i(q_1,q_2) = (\max\big(0, a-(q_1+q_2)\big))q_i$$

  - Condition $a > 0, q_1 \geq 0, q_2 \geq 0$

Step 1: Nash Equilibrium for One Stage

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - q_{-i})/2)$$

The Nash equilibria is give by

$$q^* = (q_1^*, q_2^*) = \left(\frac{a}{3}, \frac{a}{3}\right)$$

The payoff is

$$u^* = (u_1^*, u_2^*) = \left(\frac{a^2}{9}, \frac{a^2}{9}\right)$$

What happens if two firms cooperate for their profits?

Maximal Payoff for Cooperation

Summing the firms' profits, we get

$$u_1 + u_2 = (a - q_1 - q_2)q_1 + (a - q_1 - q_2)q_2$$
$$= (a - q_1 - q_2)(q_1 + q_2)$$

Maximizing the above gives

$$q_1 + q_2 = a/2$$

The total payoff for cooperation: $a^2/4 = 2a^2/8$

The total payoff for completion: $2a^2/9$

Cooperation is potentially profitable

Step 2: Cooperation

Suppose the two firms are playing the Cournot game an infinite number of times, and they share a discount factor δ.

Let

$$\hat{q} = (\hat{q}_1, \hat{q}_2) = \left(\frac{a}{4}, \frac{a}{4}\right)$$

$$\hat{u} = (\hat{u}_1, \hat{u}_2) = \left(\frac{a^2}{8}, \frac{a^2}{8}\right)$$

In competitive model,

$$\hat{q} = (\hat{q}_1, \hat{q}_2) = \left(\frac{a}{3}, \frac{a}{3}\right)$$

$$\hat{u} = (\hat{u}_1, \hat{u}_2) = \left(\frac{a^2}{9}, \frac{a^2}{9}\right)$$

Step 3: Trigger Strategy

Consider the strategy:

- If the two firms have both used $\hat{q} = (a/4, a/4)$ in all previous periods, use $\hat{q}_j = a/4$ this period

- If either firm ever did anything besides $\hat{q}$, play the stage Cournot quantity $q_j^* = a/3$

Is this a subgame perfect Nash equilibrium of the infinitely repeated game?

Check the NE of Cooperative Strategy

To check whether $\hat{q} = (\hat{q}_1, \hat{q}_2) = (a/4, a/4)$ is a NE?

By symmetry, it is sufficient to check player 1. We solve

$$\max_{q_2}(a - \hat{q}_1 - q_2)q_2 = \max_{q_2}(a - a/4 - q_2)q_2$$

Maximizing the above gives

$$q_2' = \frac{3a}{8}, u_2'\left(\frac{a}{4}, \frac{3a}{8}\right) = \left(\frac{3a}{8}\right)^2$$

9/64 = 0.140625
大于1/9=0.1111111

$\boldsymbol{\hat{q} = (\hat{q}_1, \hat{q}_2) = (a/4, a/4)}$ **is not a NE**

Step 4: Select Discounting Factor

For cooperating case, all players keep the cooperation model, and the payoff for player 2 is

$$\hat{u}_2(1+\delta+\delta^2+\cdots)=\frac{a^2}{8}\frac{1}{1-\delta}$$

For competitive case, deviating optimally in some period t after a history, and all players cooperated switches the game to competition. The pay off for player 2 is

贴现因子只和次数有关

$$u_2'+u_2^*(\delta+\delta^2+\cdots)=\left(\frac{3a}{8}\right)^2+\left(\frac{a}{3}\right)^2\frac{\delta}{1-\delta}$$

原来，越轨后是进入NE状态，之前都为0是因为NE时的支付刚好为0

Step 5: SPNE

- The cooperating is better than deviating if

$$\frac{a^2}{8}\frac{1}{1-\delta} \geq \left(\frac{3a}{8}\right)^2 + \left(\frac{a}{3}\right)^2 \frac{\delta}{1-\delta}$$

This implies $\delta \geq 9/17$.

If $\delta \geq 9/17$, then the strategy:

- If the two firms have both used $\hat{q} = (a/4, a/4)$ in all previous periods, use $\hat{q}_j = a/4$ this period

- If either firm ever did anything besides $\hat{q}$, play the stage Cournot quantity $q_j^* = a/3$

is a SPNE of the infinitely repeated game

Convex Hull

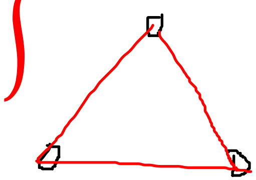
先将一些凸集的性质　局部最小就是全局最小

- A set is said to be convex if it contains the line segments connecting each pair of its points

凸包

- The convex hull of set $S = \{x_1, \dots, x_n\}$ is defined as

$$\mathrm{Conv}(S) = \left\{ \sum_i a_i x_i \mid a_i \in [0,1], \sum_i a_i = 1 \right\}$$



  - The set of all convex combinations of points in S
  - The (unique) minimal convex set containing S
  - The intersection of all convex sets containing S

包含这n个点的最小的凸集

所有包含这n个点的凸集的交

可行的

Feasible Payoffs

- Consider stage game $G = \{N, \{A_i\}, \{u_i\}\}$ and infinitely repeated game $G^\infty(\delta)$.

- Let us introduce the set of feasible payoffs:

$$U = \text{conv}\left\{ \begin{matrix} \mathrm{u} \in R^N: & \text{there exists } a = (a_1, \dots, a_N) \\ & \text{s.t.} \quad u = (u_1(a), \dots, u_N(a)) \end{matrix} \right\}$$

That is, U is the convex hull of all N-dimensional vectors that can be obtained by some (possibly mixed) strategy outcome.

Minmax Payoffs 最小化最大（和二人零和博弈无关，但理论神似）

- Minmax payoff of player i: the lowest payoff that player i's opponent can hold him to:

$$\underline{u}_i = \min_{a_{-i}} \left[\max_{a_i} u_i(a_i, a_{-i}) \right]$$

- The player can never receive less than this amount.

直观：玩家i不遵守规则
对手们选一个策略惩罚他
使得{max}最小

- Minmax strategy outcome against to i

$$a_{-i}^{i} = \arg\min_{a_{-i}} \left[\max_{\mathrm{a}} u_i(a_i, a_{-i}) \right]$$

- Let a_i^i denote the strategy of player i such that

$$u_i\left(a_i^i, a_{-i}^i\right) = \underline{u}_i$$

Notice that a_i may be a mixed strategy for each player i

Example

Player 2

q 1−q

Player 1

		L	R
q1	U	-2, 2	1, -2
q2	M	1, -2	-2, 2
1−q1−q2	D	0, 1	0, 1

How to find the minmax payoff for player 1

The payoffs of player 1 for different strategies are

‘U’ : $1 - 3q$ for player2

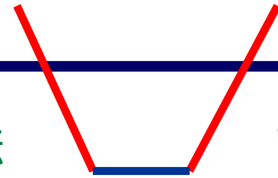
L: 2q1−2q2+1−q1−q2=1+q1−3q2

‘M’: $-2 + 3q$

R: −2q1+2q2+1−q1−q2=1−3q1+q2

‘D’: 0

Example (cont.)



We have 画图法 “最大”的min是0（蓝色那段）

$$\underline{u}_1 = \min_{q\in[0,1]} [\max\{1-3q, -2+3q, 0\}]$$

Then, $\underline{u}_1 = 0$, and $a^{1}_{-1} = a^{1}_{2}$ is the mixed strategy with probability $q \in [\frac{1}{3}, \frac{2}{3}]$ over strategy ‘L’

两平面交于一直线，大脑画图就难了
线性规划用计算机可解（LINGO），好在这里只有
两项，左右比一下大小

For player 2, we have

$$\underline{u}_2 = \min_{\substack{q_1,q_2\in[0,1] \\ q_1+q_2\leq 1}} [\max\{1+q_1-3q_2, 1+q_2-3q_1\}]$$

1–q1–3q2 ⩾ 1+q2–3q1

q1 ⩾ q2

Then, $\underline{u}_2 = 0$, and a^{2}_{-1} is the mixed strategy with probability $q_1 = q_2 = 1/2$ over strategy ‘U’ and ‘M’

Minmax Payoff Lower Bounds

Theorem

- Let $a' = (a'_1, a'_2, \dots, a'_n)$ be a (possibly mixed) Nash Equilibrium of game G and $u_i(a')$ be its payoff. Then

$$u_i(a') \geq \underline{u}_i$$

Proof. See board.

Nash Folk Theorem

之前都是为了“制定惩罚规则”做铺垫

Definition A payoff vector $(u_1, u_2, \dots, u_N) \in R^N$ is strictly individually rational if $u_i > \underline{u}_i$ for all i

之前学的是Folk定理

Nash Folk Theorem If $(u_1, u_2, \dots, u_N) \in U$ is strictly individually rational, then there exists some $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, there is Nash equilibrium of $G^\infty(\delta)$ with payoff $(u_1, u_2, \dots, u_N)$

合作时要比最差时好

Any strictly individually rational payoff can be obtained as a Nash Equilibrium when players are patient enough

Proof. See board

Problem with Nash Folk Theorem

NE不一定是SPNE
反例如下

- Nash Folk Theorem may be not a subgame perfect

Player 2

		L		R	
Player 1	U	6	6	0	-20
	D	7	1	0	-20

- The unique NE in this game is (D,L).
- The minmax payoff are given by

$$\underline{u}_1 = 0 \text{ and } \underline{u}_2 = 1$$

and $a^2_{-1} = R$

Problem with Nash Folk Theorem

		Player 2 L	Player 2 R
Player 1	U	6, 6	0, -20
Player 1	D	7, 1	0, -20

- Nash Folk Theorem: the strategy
  - Play (U,L) as long as no one deviates
  - If Player 1 deviates, then player 2 select R
- While this will hurt player 1, it will hurt player 2 a lot.
- It is an threat, and it is not a SPNE

Subgame Perfect Folk Theorem

The first subgame perfect folk theorem shows that any payoff above the static Nash payoffs can be sustained as a subgame perfect equilibrium of the repeated game

Theorem (Friedman)

- Let a^{NE} be a static equilibrium of the stage game with payoffs u^{NE};
- For any feasible payoff u with $u_i > u_i^{\mathrm{NE}}$ for all i;

There exists some $\delta_0 < 1$ s.t. for all $\delta \geq \delta_0$, there is subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ with payoff u.

Cooperation in Finitely-Repeated Games

- There are unique stage equilibrium in previous example
- There are multiple Nash equilibria in the stage game.

Player 2

Player 1	A	B	C
A	3, 3	0, 4	-2, 0
B	4, 0	1, 1	-2, 0
C	0, -2	0, -2	-1, -1

The Nash equilibria are (B,B) and (C,C)

For cooperation, the best strategy is (A,A)

Cooperation in Finitely-Repeated Games

For T = 1, the strategy is (B, B)

For $T = 2$, the strategy is

- Each plays (A, A) in the first period, and plays (B, B) in the second period
- If some player plays B in the first period, then the other plays C in the second period

If each player agrees the strategy, then the payoff is 4 for each player

If some one deviates, then the other will play C, and the payoff is 3 先使坏，得4，然后(C, C) –1得3

Deviation is not profitable 以至于没有人有动机改变strategy

For T = 3, (1) (A, A)(A, A)(B, B)
(2) if ...

Repeated Games with Imperfect Information

- So far, we assume that in the repeated game, each player observes the actions of others at the end of each stage
  - I could observe if you stick or deviated from the agreement
- In several cases, player's actions may not be directly observable, e.g.,
  - Firm productions in Cartel
  - Antiballistic Missile treaty between the US and USSR in 1972 (ABM treaty).
  - Every country can imperfectly observe each other's compliance (despite spies, satellites, etc.)

ABMs treaty with Imperfect Information

古巴导弹危机

We introduce ABMs treaty as follows:

部署的导弹 Number of ABMs	被发现概率 Probability of detection ABMs
None	0
Low	10%
High	50%

- If a country has no ABMs, then the probability that satellite detects ABMs is zero
- If a country has a low level of ABMs, then the probability that my satellite detects ABMs is 10%
- If a country has a high level of ABMs, then the probability that my satellite detects ABMs is 50%

ABMs treaty with Imperfect Information

USA \ USSR	No		Low		High	
No	10	10	6	12	0	18
Low	12	6	8	8	2	14
High	18	0	14	2	3	3

- The unique NE is (High,High)
- (Low,Low) is more efficient, and (No,No) is the most efficient
- Can we cooperate playing (No,No) in the SPNE of the infinitely repeated game

ABMs treaty with Imperfect Information

Strategy

- In period $t = 1$, choose No ABMs (cooperation)
- For $t \geq 1$, the strategy is:
  - No ABMs if neither country has observed ABMs in other countries during the previous period, or
  - High ABMs if either country has observed ABMs in other countries during previous periods
- At any time t, if no country has detected ABMs, the payoff from sticking to the agreement is:

$$10 + 10\delta + \cdots + 10\delta^{t-1} + \cdots = \frac{10}{1-\delta}$$

ABMs treaty with imperfect monitoring google 这个标题

- In contract, the payoff from deviating to low ABMs during one period

$$12 + \delta\left(0.1 \times \frac{3}{1-\delta} + 0.9 \times \frac{10}{1-\delta}\right)$$

- The payoff from deviating to high ABMs during one period

$$18 + \delta\left(0.5 \times \frac{3}{1-\delta} + 0.5 \times \frac{10}{1-\delta}\right)$$

We need Coop ≥ Low and Coop ≥ High by

$$\delta \geq 0.74 \quad \text{and} \quad \delta \geq 0.7$$

概率分布到次数上，90%日子都是no,no，然后有一天越轨一次，之后就3，3了

Summaries

- Repeated Game
- Minmax strategy
- Nash Folk Theorem (NE, Not SPNE)
- Folk Theorem (previous chapter)
- Repeated game with imperfect information