Game Theory and Applications (博弈论及其应用)

Chapter 13: Extensive Game with Imperfect Information-II

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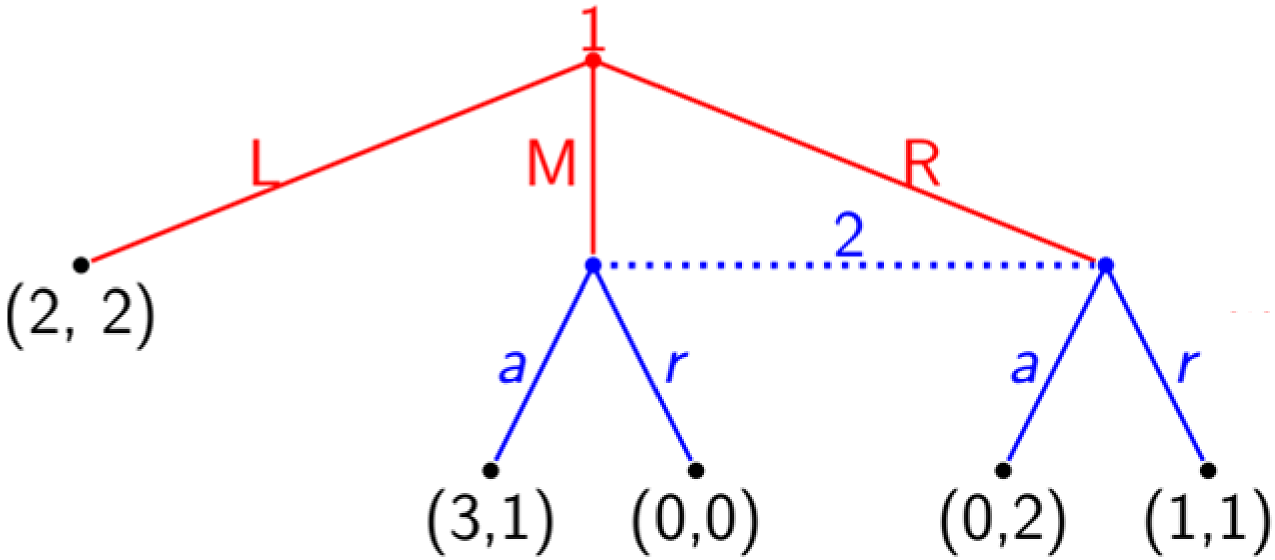
Recap on Previous Chapter

- Extensive game with imperfect information
- Formal definition $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \ldots I_N\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- Transformation of strategic game and extensive game with imperfect information
- Perfect recall and imperfect recall

Example



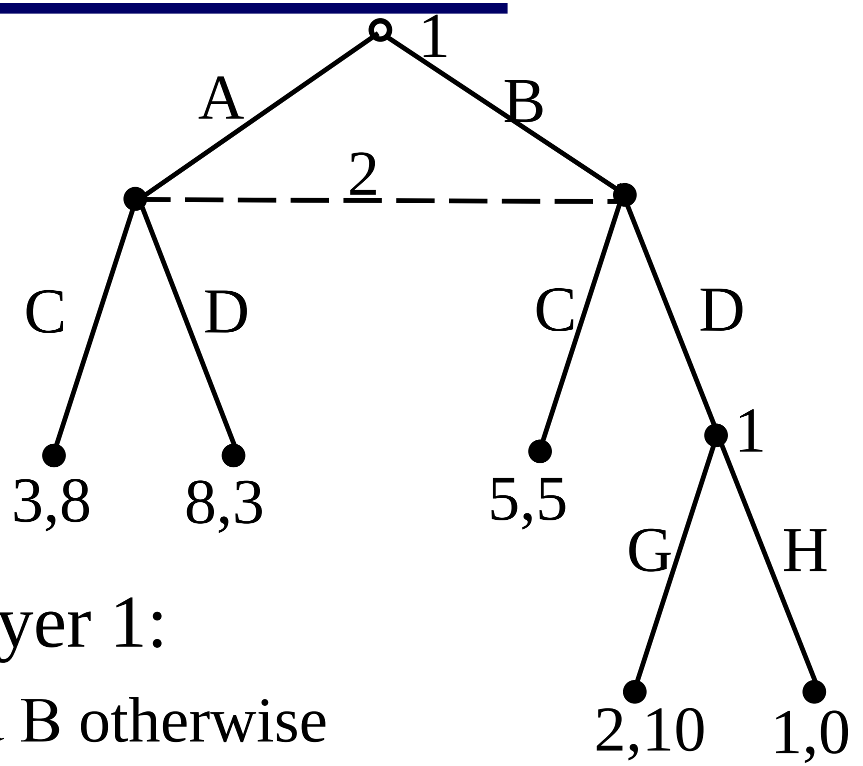
Definition of Mixed and Behavioral Strategies

- Mixed Strategies: A mixed strategy of player i in an extensive game is a probability over the set of player i's pure strategy

- Behavioral strategies: A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik} \in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $A(I_{ik})$

Behavioral vs. Mixed Strategies

Behavioral strategies distinguish from mixed strategies



A behavioral strategy for player 1:

- Selects A with prob. 0.5, and B otherwise
- choose G with prob. 0.3, and H otherwise

Here's a mixed strategy that isn't a behavioral strategy

- Pure Strategy AG with probability 0.6, pure strategy BH 0.4
- The choices at the two nodes are not independent

Behavioral vs. Mixed Strategies

In imperfect-information games, mixed and behavioral strategies produce different sets of equilibria

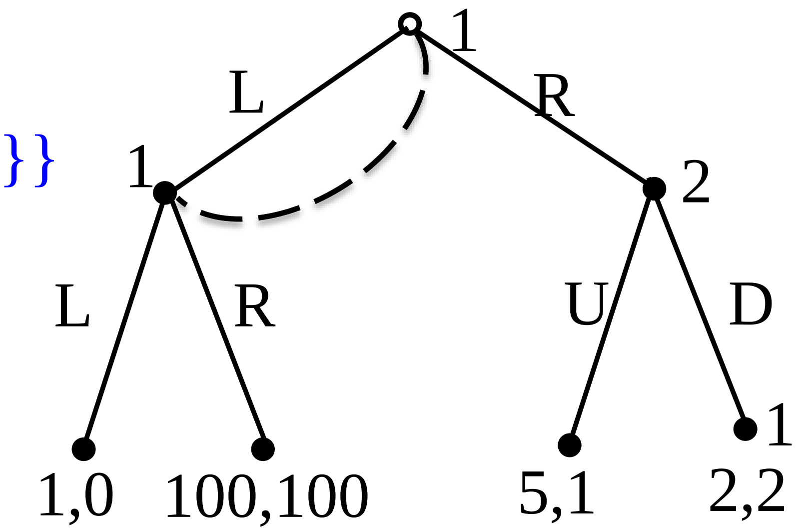
- In some games, mixed strategies can achieve equilibria that aren’t achievable by any behavioral strategy

- In some games, behavioral strategies can achieve equilibria that aren’t achievable by any mixed strategy

Behavioral vs. Mixed Strategies

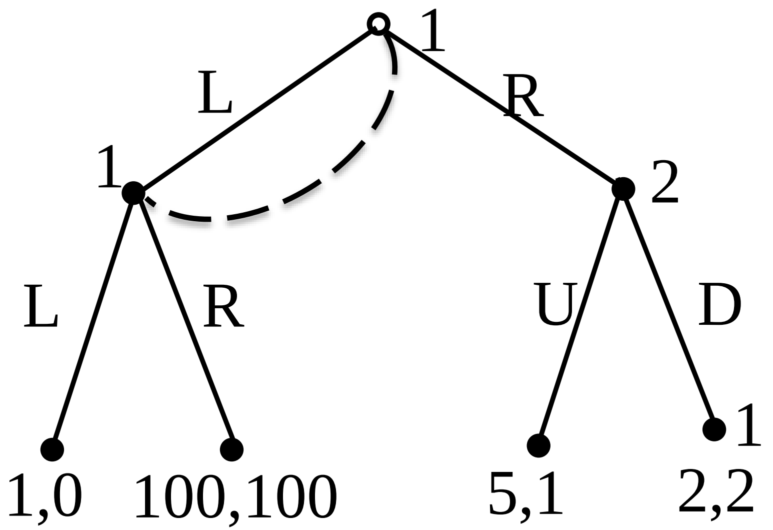
Consider game

Player 1 inform. set: {{Ø, L}}



- Player 1: R is a strictly dominant strategy
- Player 2: D is a strictly dominant strategy
  - (R, D) is the unique Nash equilibrium for mixed strategy

Behavioral vs. Mixed Strategies



1: the information set is {(∅,L)}

2: D is a strictly dominant strategy

Player 2's best response to D:

- Player 1's the behavioral strategy $[L, p; R, 1 - p]$ i.e., choose L with probability p
- The expected payoff of player 1 is
- $U_1 = p^2 + 100p(1 - p) + 2(1 - p) = -99p^2 + 98p + 2$
- To find the maximum, we have $p = 49/99$

(R,D) is not an equilibrium for behavioral strategy

Formal Definition of Perfect Recall

Player i has **perfect recall** in game G if for any two history h and h' that are in the same information set for player i, for any path $h_0, h_1, \ldots h_n, h$ and $h_0', h_1', \ldots h_m', h'$ from the root to h and $h'$with $P(h_k) = P(h_k') = i$, we have

- $n = m$
- $h_i = h_i'$ for $1 \leq i \leq n$

G is **a game of perfect recall** if every player has perfect recall in it.

Kuhn Theorem (1953)

Theorem In an finite extensive game with perfect recall

- any mixed strategy of a player can be replaced by an equivalent behavioral strategy
- any behavioral strategy can be replaced by an equivalent mixed strategy
- Two strategies are equivalent

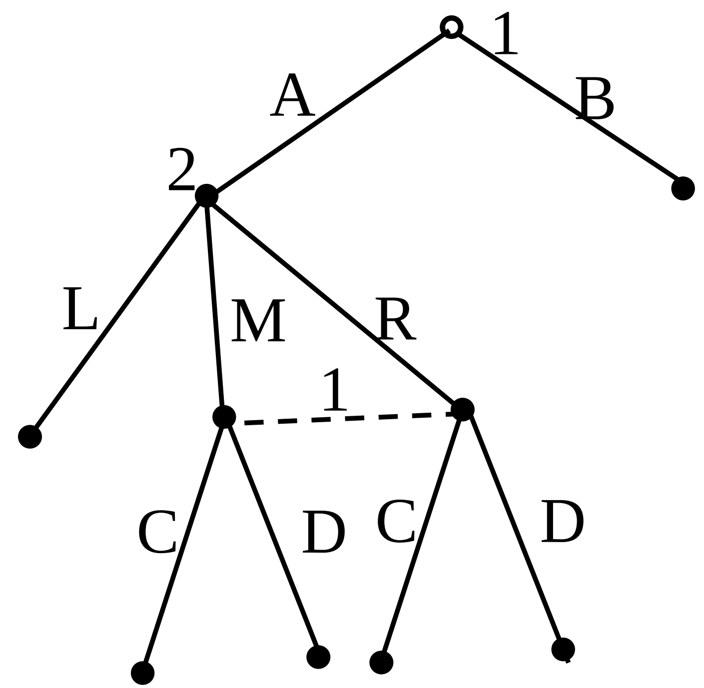
Corollary In an finite extensive game with perfect recall, the set of Nash equilibrium does not change if we restrict ourselves to behavior strategies

Proof. See board.

Example

What behavioral strategy is equivalent to mixed strategy $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$

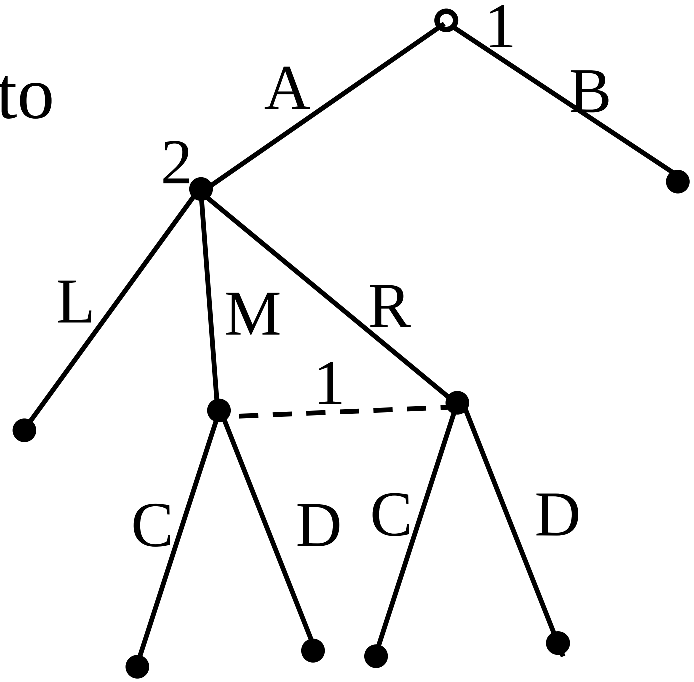
- $I_{11} = \{\emptyset\}$ $I_{12} = \{AM, AR\}$
- $A(I_{11}) = \{A, B\}$
- $A(I_{12}) = \{C, D\}$



- $\beta_{11}(I_{11})(A) = p_{AC} + p_{AD}$ $\beta_{11}(I_{11})(B) = p_{BC} + p_{BD}$
- $\beta_{12}(I_{12})(C) = \frac{p_{AC}}{p_{AC}+p_{AD}}$ $\beta_{12}(I_{12})(D) = \frac{p_{AD}}{p_{AC}+p_{AD}}$

Example

What mixed strategy is equivalent to behavioral strategy of prob. p over A and q over C



$$(p_{AC}, p_{AD}, p_{BC}, p_{BD})$$
$$= (pq, p(1-q), (1-p)q, (1-p)(1-q))$$

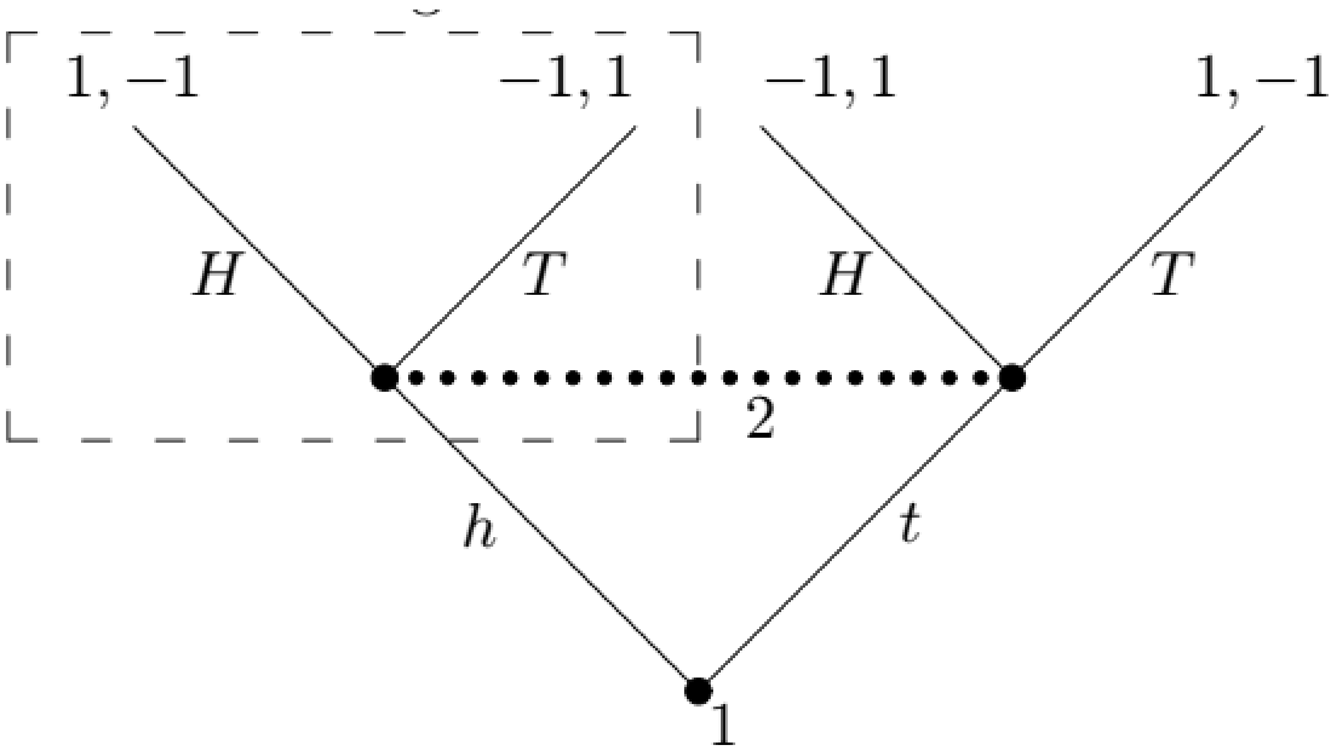
How to Compute Nash Equilibria of Perfect Recall Game

How can we find an equilibrium of an imperfect information extensive form game?

- One idea: convert to normal-form game
  - General game: exponential blow up in game size
  - Zero-sum game: LP formulation

Extensive Imperfect Subgame

Definition A subgame of an extensive imperfect game G is some node in the tree G and all the nodes that follow it, with the properties that any information set of G is either completely in or outside the subgame



Subgame Perfect Nash Equilibrium

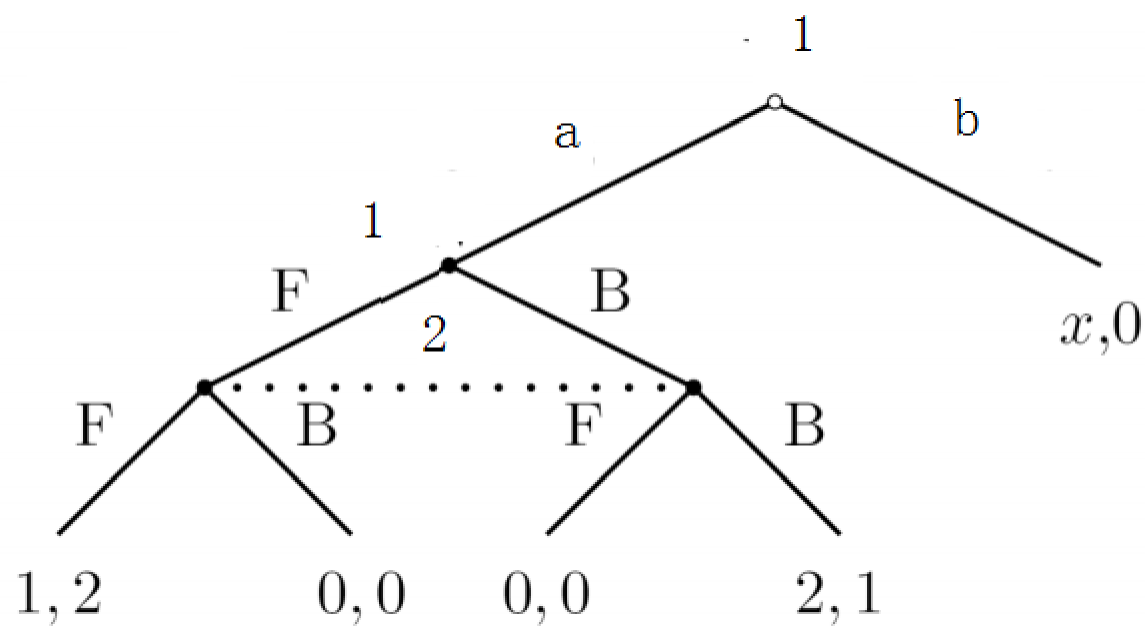
Definition A subgame perfect Nash equilibrium of an extensive form game G with perfect recall is a outcome of behavior strategies $(\beta_1, \beta_2, \ldots, \beta_N)$ such that it is a Nash Equilibrium for every subgame

Theorem Every finite extensive game with perfect recall has at least one subgame perfect Nash Equilibrium

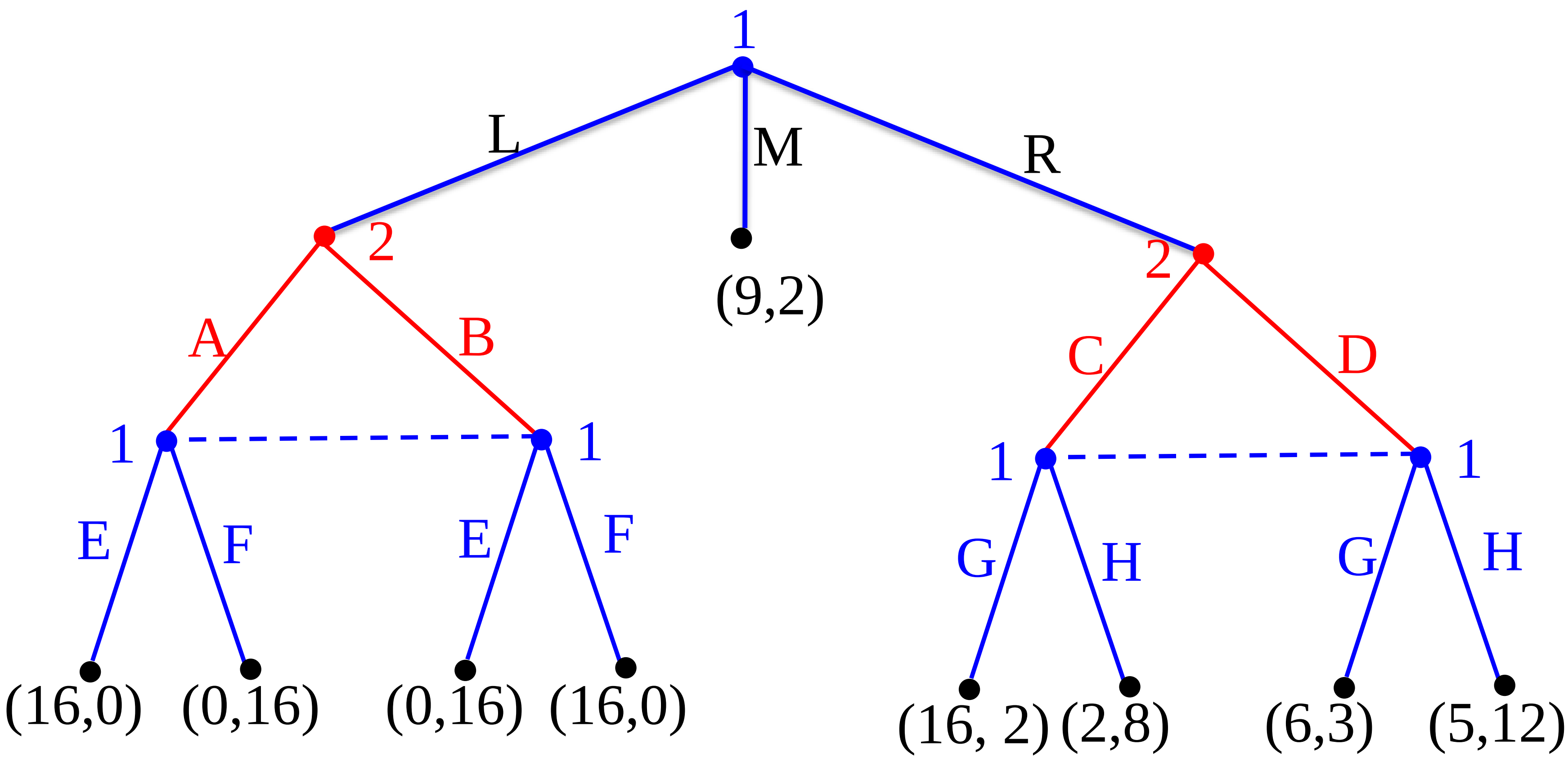
How to find SPNE

Backwards Induction

Example

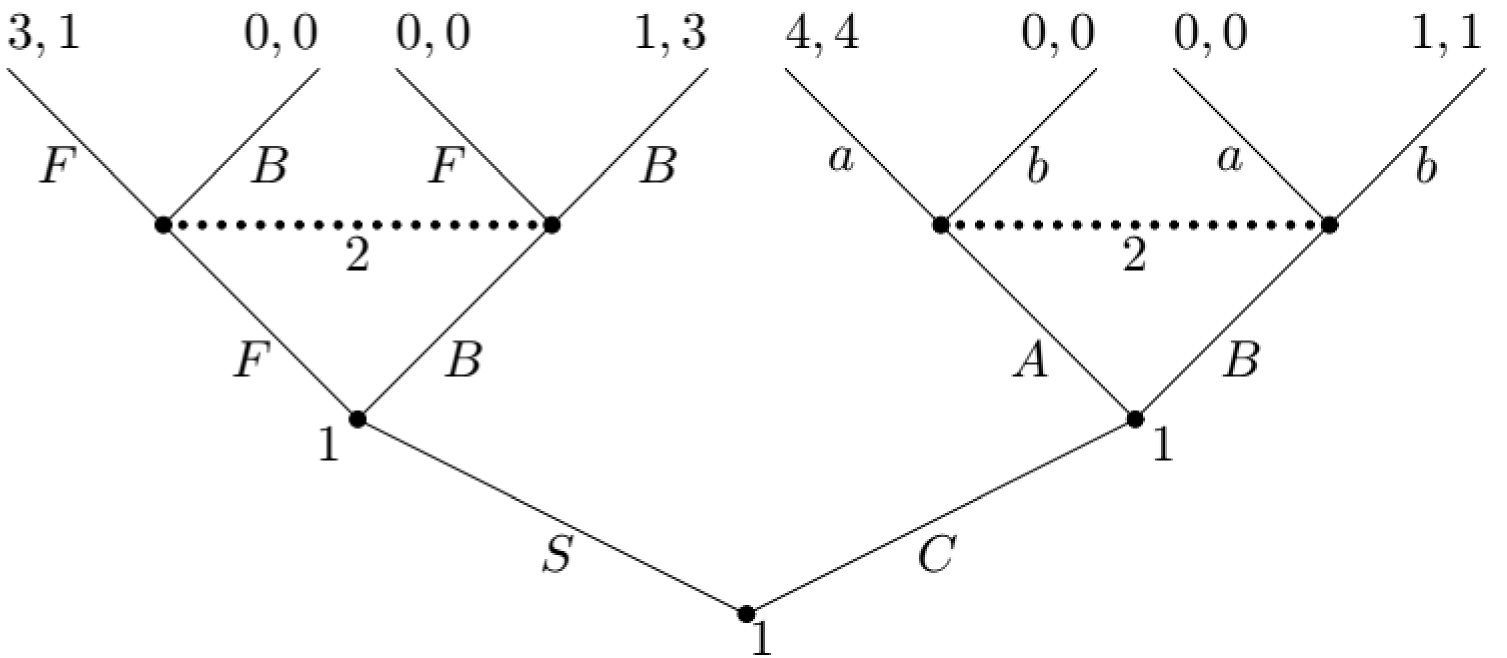


Example



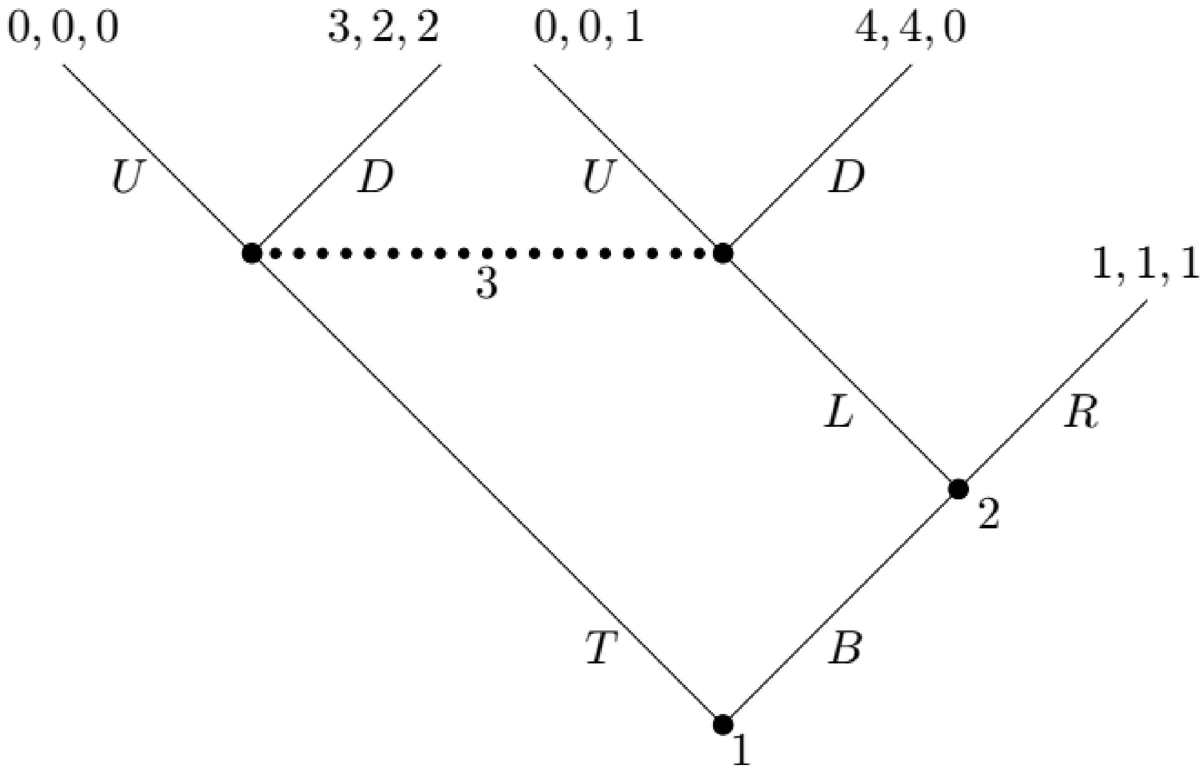
How to solve SPNE?

Exercise



- How many SPNE for this game?

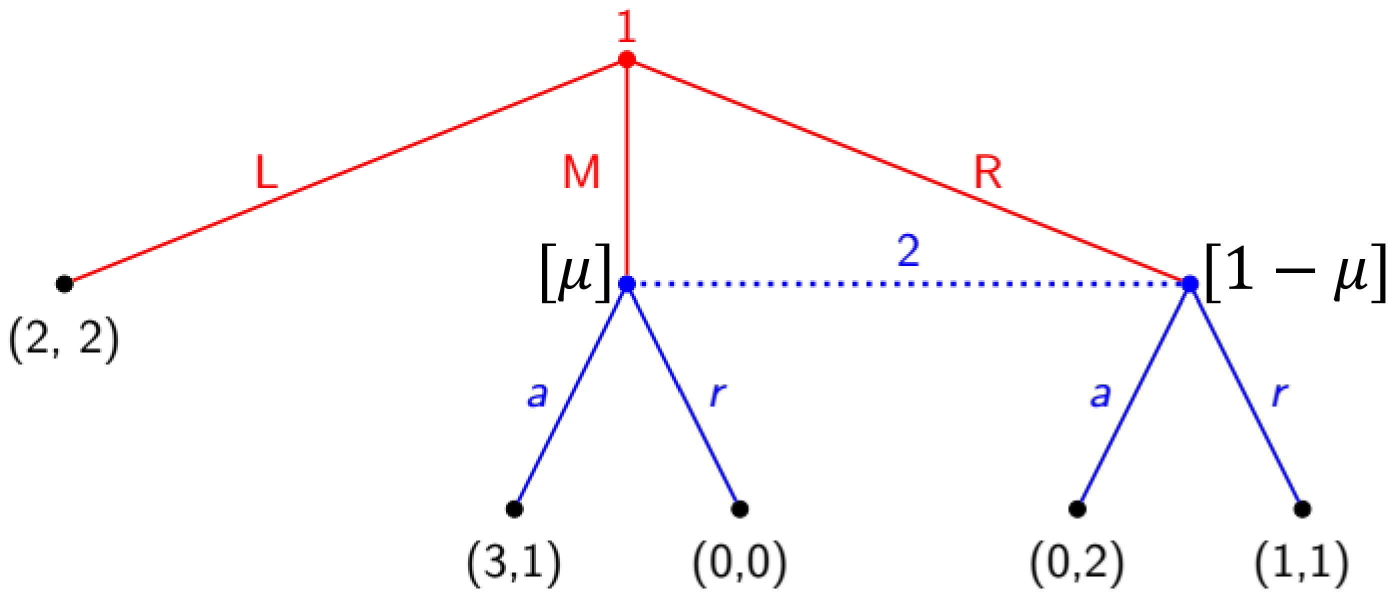
Exercise



How many SPNE?

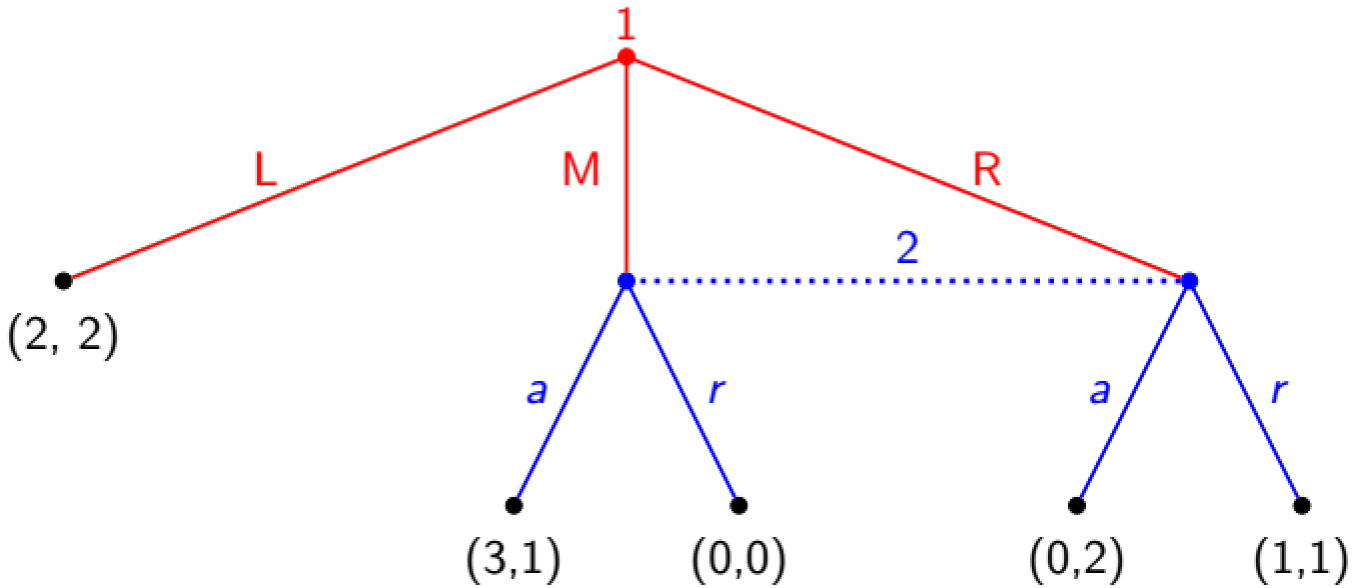
Beliefs

- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- The probability is 1 for the information set of size 1

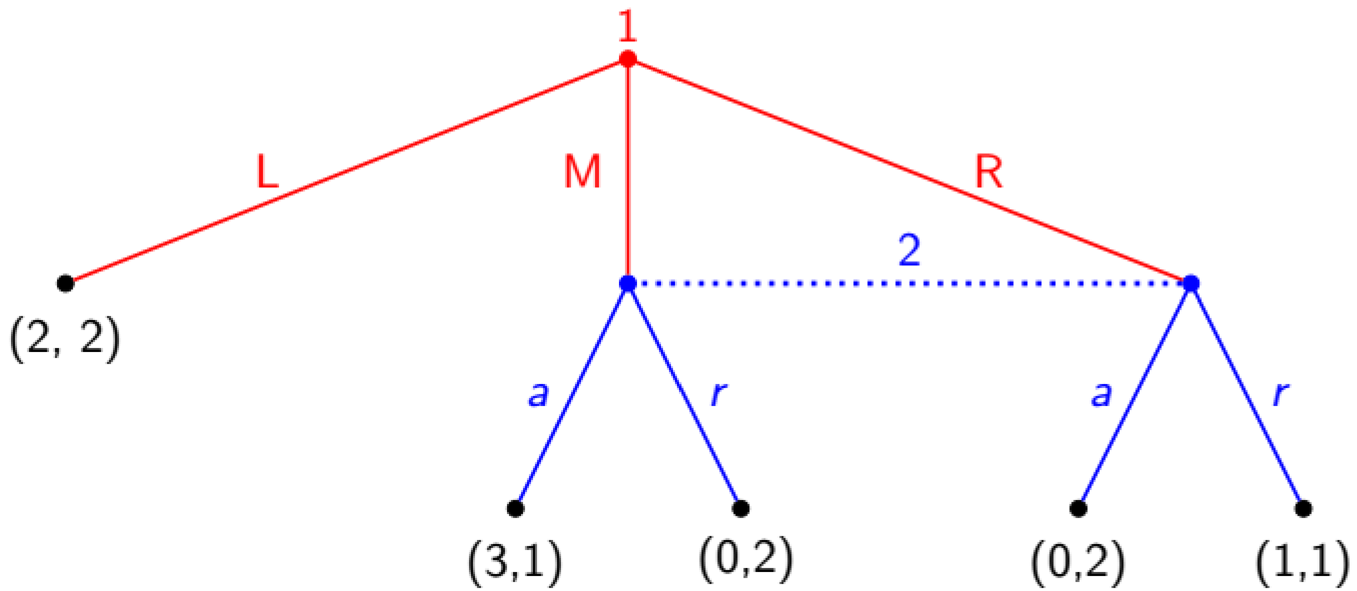


Behavioral Strategies

- A behavior strategy β a collection of independent probability measure over the actions after information set



Beliefs and Optimal Behavior Strategies

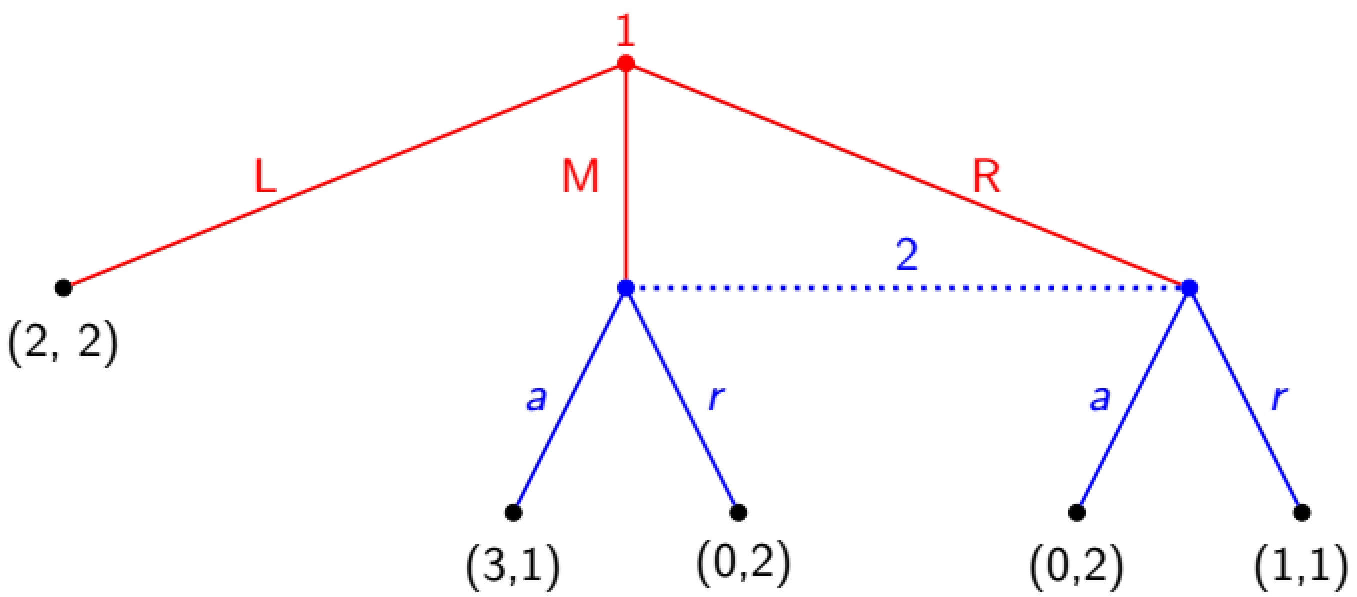


- Beliefs affect optimal strategies: For 2, a is the best strategies iff 2 assigns a belief $\mu(M) \leq 1/2$
- Strategies affect reasonable beliefs: If 1 assigns to action (L,M,R) prob. (0.1,0.3,0.6), then Bayes rule requires the belief (1/3,2/3) of 2
- What are reasonable beliefs if 1 select L with prob. 1

Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

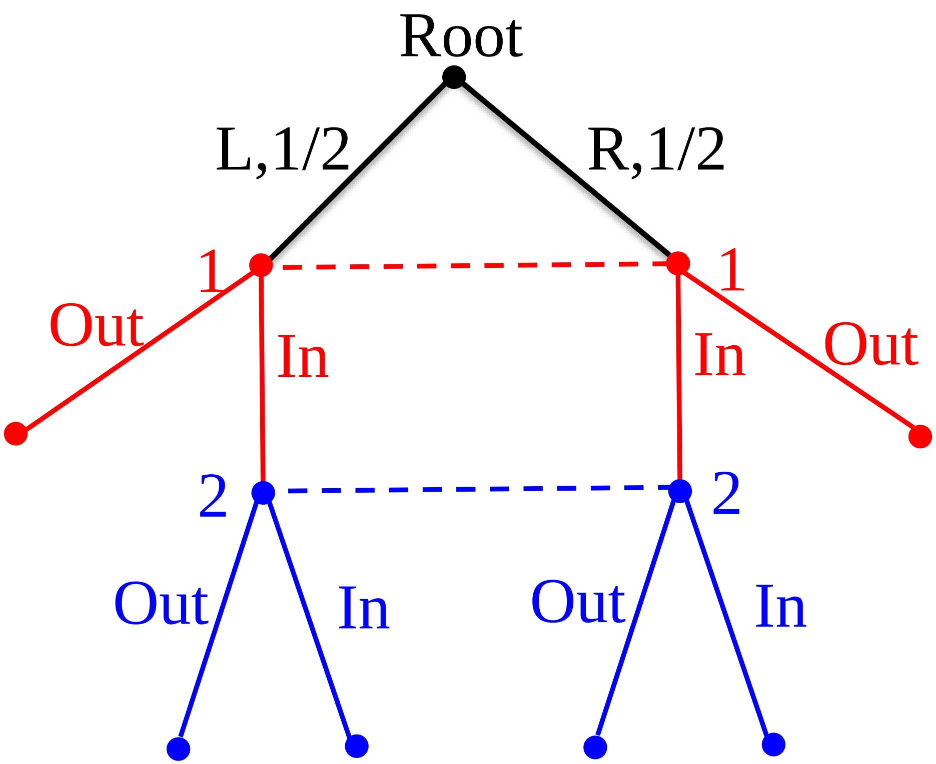
Assessment (评估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h}, \mathrm{a}) = \beta_{h,a}(h,a) / \sum_a \beta_{h,a}(h,a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent $(\beta^n, \mu^n) \to (\beta, \mu)$ as $n \to \infty$
- (β, μ) is consistent $\rightarrow$ (β, μ) Bayesian consistent

Example



- The payoffs are omitted since they are irrelevant
- Find all Bayesian consistent assessments
- Find all consistent assessments

Bayesian consistency

An assessment (β, μ) by a 4-tuple $(\beta_1, \beta_2, \mu_1, \mu_2) \in [0,1]^4$

- β_1 is the probability that 1 chooses In
- β_2 is the probability that 2 chooses In
- μ_1 is the belief assigns to the left node in 1's info set
- μ_2 is the belief assigns to the left node in 2's info set

Two cases:

- i) If $\beta_1 \in (0,1]$, 2's information set is reached with positive probability. Bayes' Law dictates that $\mu_1 = \mu_2 = 1/2$.

  $$(\beta_1, \beta_2, \mu_1, \mu_2) = (0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

  are Bayesian consistent

- ii) If $\beta_1 = 0$, then 2's information set is reached with zero probability and $\mu_2 \in [0,1]$

  $$(\beta_1, \beta_2, \mu_1, \mu_2) = \{0\} \times [0,1] \times \{1/2\} \times [0,1]$$

  are Bayesian consistent

Consistency

- Every complete outcome of behavioral strategies leads to $\mu_1 = \mu_2 = 1/2$.

- 2's information set, both nodes are reached with equal probability.

- Conclusion:

$$(\beta_1, \beta_2, \mu_1, \mu_2) = [0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

are consistent