

Game Theory and Applications (博弈论及其应用)

Chapter 6 : Strategic Games with Incomplete Information

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Recap on Previous Chapter

- Auctions
 - Open bid auctions
 - Sealed bid auctions (first/second price)
- Election
- Correlated Equilibrium
 - Multiple NEs
 - Correlated strategy (pure/mixed strategy)
 - Correlated equilibrium

Strategy Game with Complete Information

Previous strategy game with complete information

- Who are playing
- All players are rational
- What strategies are possible for each player
- What are the payoffs
- All players know the complete information
- All players knows that all players know complete information
- ...

Games with Incomplete Information

- In many games, some players are not sure the payoff
- Players may have incomplete inform. about components
- Bidder does not know value of other bidders in auction
- Some players may have private information
- Despite of different types of incomplete information, we consider incomplete payoffs

Incomplete information motivates additional interaction

Battle of Sexes with Incomplete Information

Complete information:

- both boy and girl like to meet

		Girl	
		Ballet	Football
Boy	Ballet	1 2	0 0
	Football	0 0	2 1

There are 2 pure strategy NE and 1 mixed strategy NE

Both boy and girl know each other in a short time.

The boy is not sure if the girl wishes to meet or not

Incomplete information

Battle of Sexes with Incomplete Information (cont.)

The girl has two types: like or dislike

The boy assumes that girl like with probability p

		Girl (like, p)	
		F	B
Boy	F	2 1	0 0
	B	0 0	1 2

		Girl (dislike $1 - p$)	
		F	B
Boy	F	2 0	0 2
	B	0 1	1 0

The girl knows the complete information

The boy does not know

What strategies in this game? how to reason?

Battle of Sexes with Incomplete Information (cont.)

		Girl (like, p)	
		F	B
Boy	F	2 1	0 0
	B	0 0	1 2

		Girl (dislike $1 - p$)	
		F	B
Boy	F	2 0	0 2
	B	0 1	1 0

If the boy selects B, then the girl selects (B,F). $\{B, (B,F)\}$

$$U_1(B, (B, F)) = p + (1 - p) * 0 = p$$

$$U_1(F, (B, F)) = 0 + (1 - p) * 2 = 2(1 - p)$$

if $p \geq 2/3$, $\{B, (B,F)\}$ is a NE;

if $p < 2/3$, $\{B, (B,F)\}$ is not a NE;

Battle of Sexes with Incomplete Information (cont.)

		Girl (like, p)	
		F	B
Boy	F	2 1	0 0
	B	0 0	1 2

		Girl (dislike $1 - p$)	
		F	B
Boy	F	2 0	0 2
	B	0 1	1 0

If the boy selects F, then the girl selects (F,B). $\{F, (F,B)\}$

$$U_1(F, (F, B)) = 2p + (1 - p) * 0 = 2p$$

$$U_1(B, (F, B)) = 0 + (1 - p) = 1 - p$$

if $p \geq 1/3$, $\{F, (F,B)\}$ is a NE;

if $p < 1/3$, $\{F, (F,B)\}$ is not a NE;

Bayesian Games

A Bayesian game consists of

- A set of players N
- A set of strategies A_i for each player i
- A set of types Θ_i for each player i
 - Type set Θ_i includes all private information for player i
 - The types on payoff are adequate (Payoff types)
- Probability distribution $p = p(\theta_1, \dots, \theta_N)$ on $\times_{i=1..n} \Theta_i$

Bayesian Games (cont.)

- For player i , **a pure strategy** is a map $a_i: \Theta_i \rightarrow A_i$, which prescribes an strategy for each type

$$a_i = \left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i}) \right)$$

- A payoff function $u_i: \times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \rightarrow R$

$$u_i(a_1, \dots, a_N, \theta_1, \dots, \theta_N) \text{ for } a_i \in A_i \text{ and } \theta_i \in \Theta_i$$

$$G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$$

Battle of Sexes with Incomplete Information

Girl (like, p)

	F	B
Boy (like q)	F	B
	2 1	0 0
	B	0 0
	0 1	1 2

Girl (dislike $1 - p$)

	F	B
Boy (like q)	F	B
	2 0	0 2
	B	0 1
	0 1	1 0

Girl (like, p)

	F	B
Boy (dislike $1 - q$)	F	B
	0 1	2 0
	B	1 0
	1 0	0 2

Girl (dislike $1 - p$)

	F	B
Boy (dislike $1 - q$)	F	B
	0 0	2 2
	B	1 1
	1 1	0 0

Bayesian Games (cont.)

- The set of types Θ_i for each player i
 - Player i does not know the selection of Θ_j
 - All types are drawn from the prior dis. $p(\theta_1, \dots, \theta_N)$
 - $p(\theta_1, \dots, \theta_N) = p(\theta_1)p(\theta_2) \dots p(\theta_N)$ (independent types)

- Given $p(\theta_1, \dots, \theta_N)$, we have, by **Bayes rule**,

$$p(\theta_{-i}|\theta_i) = p(\theta_i, \theta_{-i})/p(\theta_i)$$

where $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_N)$

Such game is called Bayes Game

Outcome and Payoff Functions

A pure strategy for player i

$$\left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i}) \right)$$

An outcome of Bayes game is given by

$$\left(\left(a_1(\theta_1^1), \dots, a_1(\theta_1^{n_1}) \right), \dots, \left(a_N(\theta_N^1), \dots, a_N(\theta_N^{n_N}) \right) \right)$$

Given a_{-i} , **the expected payoff** of player i and type θ_i is

$$U_i(a_i(\theta_i), a_{-i}) = \sum_{\theta_{-i}} p(\theta_{-i} | \theta_i) u_i(a_{-i}(\theta_{-i}), a_i, \theta_{-i}, \theta_i)$$

Bayesian Nash Equilibrium

Definition The outcome (a_1, a_2, \dots, a_N) is a **Bayesian Nash Equilibrium** if for each type θ_i , we have

$$U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \in A_i$$

Given a_{-i} and type θ_i , the **best response for player i** is

$$B_i(a_{-i}, \theta_i) = \{a_i(\theta_i) : U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i)\}$$

Theorem The outcome (a_1, a_2, \dots, a_N) is a Bayesian NE if and only if for every player i and each type θ_i , we have

$$a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$$

How to find Bayesian Nash Equilibrium

How to find Bayesian Nash Equilibrium

- 1) Find the best response function for each player and type
- 2) Find Bayesian NE by $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

Bank Runs (銀行挤脱)

- Both players 1 and 2 have a deposit of \$100 in bank
- If the bank manager is good, each player get \$150; if the manager is bad, then they lose all money
- Players can withdraw money but bank has only \$100
 - If only one player withdraws, he get \$100 and the other gets 0
 - If two players both withdraw, each get \$50
- Player 1 believes a good manager with probability p
- Player 2 knows whether the manager is good or bad
- Two players simultaneously make strategy: withdraw/not

Bank Runs (cont.)

- Two players
- Strategies $A_1 = A_2 = \{W, N\}$
- Types $\Theta_1 = \{1\}$; $\Theta_2 = \{G, B\}$
- A probability distribution $p_1(\theta_2 = G) = p$
- Payoffs

		Player 2 (G, p)			
		W		N	
Player 1	W	50 50	100 0		
	N	0 100	150 150		

		Player 2 (B, $1 - p$)			
		W		N	
Player 1	W	50 50	100 0		
	N	0 100	0 0		

Bank Runs (cont.)

		Player 2 (G, p)			
		W		N	
Player 1	W	50	50	100	0
	N	0	100	150	150

		Player 2 (B, $1 - p$)			
		W		N	
Player 1	W	50	50	100	0
	N	0	100	0	0

Player 1: W or N

Player 2: W(G) N(G) W(B) N(B)

Bayesian Nash Equilibrium of Bank Runs

		Player 2 (G, p)	
		W	N
Player 1	W	50 50	100 0
	N	0 100	150 150

		Player 2 (B $1 - p$)	
		W	N
Player 1	W	50 50	100 0
	N	0 100	0 0

If Player 1 selects W, then

$$B_2(W, G) = \{W\}, B_2(W, B) = \{W\}$$

Outcome $(W, (W(G), W(B)))$: best strategy for Player 2

Is W a best response to $W(G), W(B)$?

$$U_1(W, (W(G), W(B))) = 50p + 50(1 - p) = 50$$

$$U_1(N, (W(G), W(B))) = 0p + 0(1 - p) = 0$$

$(W, (W(G), W(B)))$ is a Bayesian NE

Bayesian Nash Equilibrium of Bank Runs

		Player 2 (G, p)	
		W	N
Player 1	W	50 50	100 0
	N	0 100	150 150

		Player 2 (B $1 - p$)	
		W	N
Player 1	W	50 50	100 0
	N	0 100	0 0

If Player 1 selects N, then

$$B_2(N, G) = \{N\}, B_2(N, B) = \{W\}$$

Outcome (N,(N(G),W(B))): Player 2 makes best strategy

Is N a best response to (W(G),W(B))? (see board)

$$U_1(N, N(G), W(B)) = 150p + 0(1 - p) = 150p$$

$$U_1(W, N(G), W(B)) = 100p + 50(1 - p) = 50(1 + p)$$

Cournot Duopoly with Incomplete Information

Two firms $N = \{1,2\}$

Firm 1 has a cost c_H ;

Firm 2 has two costs c_L and c_H

Firm 1 believes that Firm 2 selects c_H with probability p

Firm 1's strategy $\{q_1: q_1 \geq 0\}$

Firm 2's strategy $\{q_{2,L}, q_{2,H}: q_{2,L} \geq 0 \text{ and } q_{2,H} \geq 0\}$

Output: $a - q_1 - q_{2,L}$ or $a - q_1 - q_{2,H}$

Cournot Duopoly with Incomplete Information

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) + (1 - p)q_1(a - q_1 - q_{2,L}) - c_H q_1$$

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

For player 2, the expected payoff function of type c_L is

$$U_2(q_1, q_{2,L}, c_L) = (a - q_1 - q_{2,L})q_{2,L} - c_L q_{2,L}$$

Best Response for Player 1

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) + \\ +(1 - p)q_1(a - q_1 - q_{2,L}) - c_H q_1$$

Maximizing $U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H)$ gives

$$B_1(q_{2,L}, q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1 - p)q_{2,L} - c_H}{2} \right\}$$

Best Response for Player 2

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

Maximizing $U_2(q_1, q_{2,H}, c_H)$ gives

$$B_2(q_1, c_H) = \left\{ \frac{a - q_1 - c_H}{2} \right\}$$

Similarly, we have

$$B_2(q_1, c_L) = \left\{ \frac{a - q_1 - c_L}{2} \right\}$$

Bayesian Nash Equilibrium

$$B_2(q_1, c_H) = \left\{ \frac{a - q_1 - c_H}{2} \right\}$$

$$B_2(q_1, c_L) = \left\{ \frac{a - q_1 - c_L}{2} \right\}$$

$$B_1(q_{2,L}, q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1-p)q_{2,L} - c_H}{2} \right\}$$

We solve the Bayesian Nash Equilibrium by

$$q_{2,H} \in B_2(q_1, c_H)$$

$$q_{2,L} \in B_2(q_1, c_L)$$

$$q_1 \in B_1(q_{2,L}, q_{2,H})$$

Bayesian Nash Equilibrium

The Bayesian Nash Equilibrium is $(q_1, (q_{2,L}, q_{2,H}))$

$$q_1 = \frac{a - c_H - (1 - p)(c_H - c_L)}{3}$$

$$q_{2,H} = \frac{a - c_H + a - c_L}{6} - p \frac{c_H - c_L}{6}$$

$$q_{2,L} = \frac{a - c_H + c_H - c_L}{3} - p \frac{c_H - c_L}{6}$$

Discussions

- Incomplete information affects the outputs of players
- $q_{2,L} > q_{2,H}$ implies player 2 produce more for lower price

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_H ($p = 1$) then

$$q_1 = q_{2,H} = \frac{a - c_H}{3}$$

- If player 1 knows that player 2 selects c_L ($p = 0$) then

$$q_1 = q_{2,L} = \frac{a - c_L + c_H - c_L}{3}$$

- If player 1 does not know the choices of player 2

$$q_1 = \frac{a - c_H - (1 - p)(c_H - c_L)}{3}$$
$$q_1 = \frac{a - c_L + c_H - c_L - (3 - p)(c_H - c_L)}{3}$$

Player 1 produces less with the incomplete information

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_H ($p = 1$) then

$$q_{2,H} = \frac{a - c_H}{3};$$

otherwise,

$$q_{2,H} = \frac{a - c_H}{3} + (1 - p) \frac{c_H - c_L}{6}$$

Player 2 benefits from the incomplete information

The firm will benefit by keeping cost secrets

First Price Auction with Incomplete information

$N = \{1,2\}$: players bid a building

v_i : the true value for player i

v_i : a uniform distribution over $[0,1]$

$b_i \geq 0$: the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ v_i/2 - b_i & \text{if } b_i = b_j \\ 0 & \text{otherwise} \end{cases}$$

Bayesian NE of First Price Auction

For player i , the expected payoff function is

$$U_i(b_i, b_j(v_j), v_i) = (v_i - b_i) \Pr[b_i > b_j(v_j)] + (v_i/2 - b_i) \Pr[b_i = b_j(v_j)]$$

$$\Pr[b_i > b_j(v_j)] = \Pr[b_i > av_j] = \Pr[b_i/a > v_j]$$

v_j is a uniform distribution over $[0,1]$

$$U_i(b_i, b_j, v_i) = (v_i - b_i) b_i / a$$

Maximizing $U_i(b_i, b_j, v_i)$ with respect to b_i gives

$$b_i(v_i) = \frac{v_i}{2}$$

First Price Auction with Incomplete information

$N = \{1, \dots, N\}$: players bid a building

v_i : the true value for player i

v_i : a uniform distribution over $[0,1]$

$b_i \geq 0$: belief that the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(b_1, \dots, b_N) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ v_i/2 - b_i & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

Bayesian NE of First Price Auction (N Players)

For player i , the expected payoff function is

$$U_i(b_i, b_j(v_j), v_i) = (v_i - b_i) \Pr[b_i > \max_{j \neq i} b_j(v_j)]$$

$$U_i(b_i, b_j, v_i) = (v_i - b_i)(b_i/a)^{N-1}$$

Maximizing $U_i(b_i, b_j, v_i)$ with respect to b_i gives

$$b_i(v_i) = \frac{N-1}{N} v_i$$