Game Theory and Applications (博弈论及其应用)

# Chapter 6: Strategic Games with Incomplete Information

南京大学

高尉



# Strategy Game with Complete Information

Previous strategy game with complete information

- Who are playing
- All players are rational
- What strategies are possible for each player
- What are the payoffs
- All players know the complete information
- All players knows that all players know complete information

. . .

## Games with Incomplete Information

- In many games, some players are not sure the payoff
- Players may have incomplete inform. about components
- Bidder does not know value of other bidders in auction
- Some players may have private information
- Despite of different types of incomplete information, we consider incomplete payoffs

Incomplete information motivates additional interaction

### Battle of Sexes with Incomplete Information

### Complete information:

both boy and girl like to meet

neet ,		Ba	llet	Football		
) 	Ballet	1	2	0	0	
B B	Football	0	0	2	1	

Girl

There are 2 pure strategy NE and 1 mixed strategy NE

男孩1/3概率选芭蕾, 2/3概率选足球

(B, B) (F, F)

Girl select the mixed strategy
P2=(2/3, 1/3) over (Ballet, Football)

Both boy and girl know each other in a short time.

The boy is not sure if the girl wishes to meet or not

**Incomplete information** 

# Battle of Sexes with Incomplete Information (cont.)

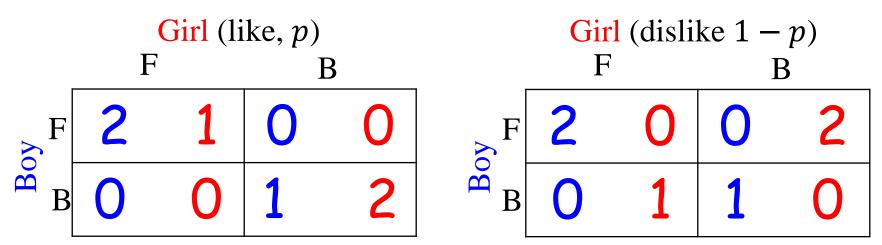
The girl has two types: like or dislike Attention: 这两个表不一样 The boy assumes that girl like with probability p

	Girl (like, p)					Girl (dislike $1 - p$ )				
,	F	₹ 	]	В		F		В		
F	2	1	0	0	F	2	0	0	2	
В	0	0	1	2	В	0	1	1	0	

The girl knows the complete information The boy does not know

What strategies in this game? how to reason?

Battle of Sexes with Incomplete Information (cont.)



男孩选B, 女孩喜欢就选F, 不喜欢就选E

If the boy selects B, then the girl selects (B,F).  $\{B, (B,F)\}$ 

$$U_1(B,(B,F)) = p + (1-p) * 0 = p$$

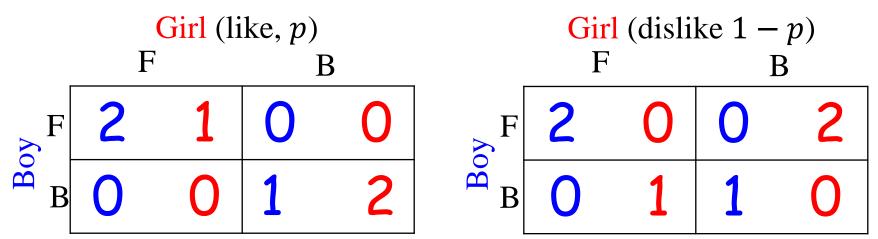
$$U_1(F,(B,F)) = 0 + (1-p) * 2 = 2(1-p)$$

$$U_1(F,(B,F)) = 0 + (1-p) * 2 = 2(1-p)$$

if  $p \ge 2/3$ , {B, (B,F)} is a NE;

if p < 2/3, {B, (B,F)} is not a NE;

Battle of Sexes with Incomplete Information (cont.)



男孩选F,女孩喜欢就选F,不喜欢就选B

If the boy selects F, then the girl selects (F,B).  $\{F, (F,B)\}$ 

$$U_1(F,(F,B)) = 2p + (1-p) * 0 = 2p$$
  
$$U_1(B,(F,B)) = 0 + (1-p) = 1-p$$

if  $p \ge 1/3$ , {F, (F,B)} is a NE;

if p < 1/3, {F, (F,B)} is not a NE;

# A Bayesian game consists of

- A set of players N
- A set of strategies  $A_i$  for each player i
- A set of types  $\Theta_i$  for each player i
  - Type set  $\Theta_i$  includes all private information for player i
  - The types on payoff are adequate (Payoff types)
- Probability distribution  $p = p(\theta_1, ..., \theta_N)$  on  $\times_{i=1...n} \Theta_i$

### Bayesian Games (cont.)

• For player i, a pure strategy is a map  $a_i: \Theta_i \to A_i$ , which prescribes an strategy for each type

$$a_i = \left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i})\right)$$

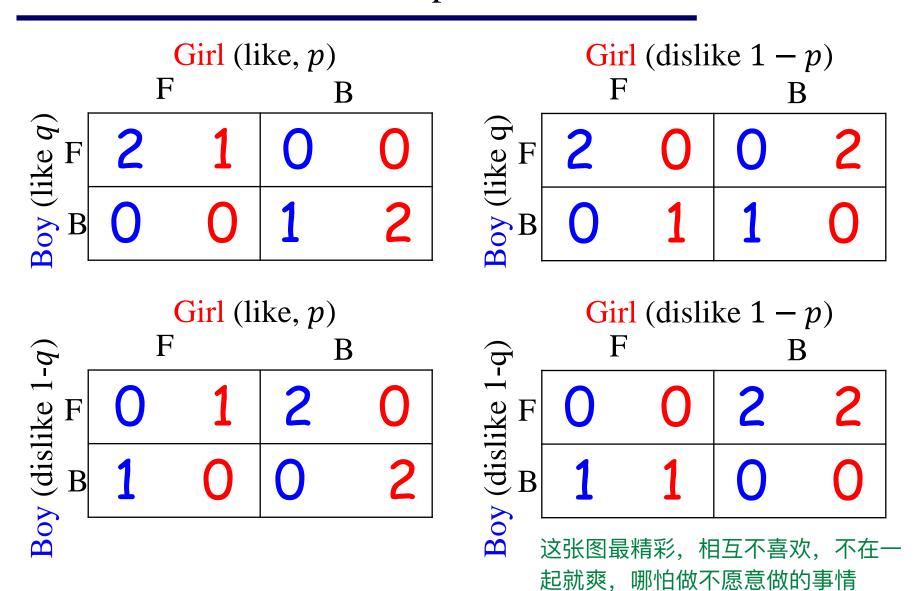
与纯策略不同的是,女孩即使dislike也一定要选一个

• A payoff function  $u_i:\times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \to R$ 

$$u_i(a_1, ..., a_N, \theta_1, ..., \theta_N)$$
 for  $a_i \in A_i$  and  $\theta_i \in \Theta_i$ 

$$G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$$

# Battle of Sexes with Incomplete Information



### Bayesian Games (cont.)

- The set of types  $\Theta_i$  for each player i
  - Player *i* does not know the selection of  $\Theta_i$
  - All types are drawn from the prior dis.  $p(\theta_1, ..., \theta_N)$
  - $p(\theta_1, ..., \theta_N) = p(\theta_1)p(\theta_2) ... p(\theta_N)$  (independent types)
- Given  $p(\theta_1, ..., \theta_N)$ , we have, by Bayes rule,

$$p(\theta_{-i}|\theta_i) = p(\theta_i, \theta_{-i})/p(\theta_i)$$

where 
$$\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1} \dots, \theta_N)$$

Such game is called Bayes Game

## Outcome and Payoff Functions

A pure strategy for player i

$$\left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i})\right)$$

An outcome of Bayes game is given by

$$\left(\left(a_1(\theta_1^1), \dots, a_1(\theta_1^{n_1})\right), \dots, \left(a_N(\theta_N^1), \dots, a_N(\theta_N^{n_N})\right)\right)$$

Given  $a_{-i}$ , the expected payoff of player i and type  $\theta_i$  is

$$U_i(a_i(\theta_i), a_{-i}) = \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(a_{-i}(\theta_{-i}), a_i, \theta_{-i}, \theta_i)$$

**Definition** The outcome  $(a_1, a_2, ..., a_N)$  is a **Bayesian** Nash Equilibrium if for each type  $\theta_i$ , we have

$$U_i(a_i(\theta_i), a_{-i}) \ge U_i(a'_i(\theta_i), a_{-i})$$
 for all  $a'_i(\theta_i) \in A_i$ 

Given  $a_{-i}$  and type  $\theta_i$ , the best response for player *i* is

$$B_i(a_{-i}, \theta_i) = \{a_i(\theta_i) : U_i(a_i(\theta_i), a_{-i}) \}$$
  
 
$$\geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \}$$

**Theorem** The outcome  $(a_1, a_2, ..., a_N)$  is a Bayesian NE if and only if for every player i and each type  $\theta_i$ , we have

$$a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$$

# How to find Bayesian Nash Equilibrium

How to find Bayesian Nash Equilibrium

- 1) Find the best response function for each player and type
- 2) Find Bayesian NE by  $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

# Bank Runs (银行挤脱)

- Both players 1 and 2 have a deposit of \$100 in bank
- If the bank manager is good, each player get \$150; if the manager is bad, then they lose all money
- Players can withdraw money but bank has only \$100
  - If only one player withdraws, he get \$100 and the other gets 0
  - If two players both withdraw, each get \$50
- Player 1 believes a good manager with probability p
- Player 2 knows whether the manager is good or bad
- Two players simultaneously make strategy: withdraw/not

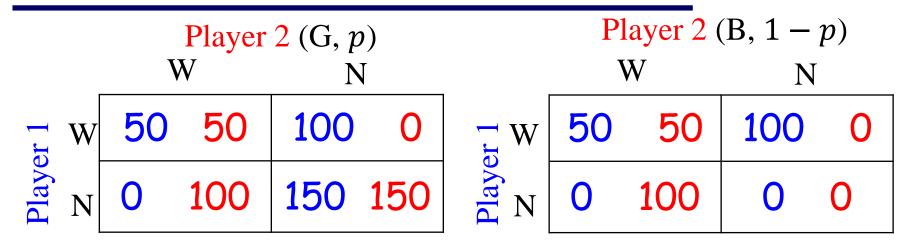
### Bank Runs (cont.)

- Two players
- Strategies  $A_1 = A_2 = \{W, N\}$
- Types  $\Theta_1 = \{1\}; \Theta_2 = \{G, B\}$
- A probability distribution  $p_1(\theta_2 = G) = p$
- Payoffs

	Player 2 $(G, p)$							
	1	V	N					
T W	50	50	100	0				
Playe Z	0	100	150	150				

	Player 2 (B, $1-p$ )								
	7	W	N						
W W	50	50	100	0					
Playe N	0	100	0	0					

### Bank Runs (cont.)



Player 1: W or N

Player 2: W(G) N(G) W(B) N(B)

### Bayesian Nash Equilibrium of Bank Runs

	Player 2 (G, p)					Player 2 (B $1 - p$ )				
	_	W		N			$\mathbf{W}$		N	
yr 1	W	50	50	100	0	T W	50	50	100	0
Playe	N	0	100	150	150	Playe Z	0	100	0	0

If Player 1 selects W, then

$$B_2(W,G) = \{W\}, B_2(W,B) = \{W\}$$

Outcome (W, (W(G), W(B))): best strategy for Player 2

Is W a best response to W(G), W(B)?

$$U_1(W, (W(G), W(B))) = 50p + 50(1 - p) = 50$$
  
 $U_1(N, (W(G), W(B))) = 0p + 0(1 - p) = 0$   
(W,(W(G), W(B))) is a Bayesian NE

### Bayesian Nash Equilibrium of Bank Runs

If Player 1 selects N, then

$$B_2(N, G) = \{N\}, B_2(N, B) = \{W\}$$

Outcome (N,(N(G),W(B))): Player 2 makes best strategy

Is N a best response to 
$$(W(G), W(B))$$
? (see board)
$$U_1(N, N(G), W(B)) = 150p + 0(1-p) = 150p$$

$$U_1(W, N(G), W(B)) = 100p + 50(1-p) = 50(1+p)$$

$$U_1(W, N(G), W(B)) = 100p + 50(1-p) = 50(1+p)$$

$$U_1(W, N(G), W(B)) = 100p + 50(1-p) = 50(1+p)$$

# Cournot Duopoly with Incomplete Information

Two firms  $N = \{1,2\}$ 

Firm 1 has a cost  $c_H$ ;

Firm 2 has two costs  $c_L$  and  $c_H$ 

Firm 1 believes that Firm 2 selects  $c_H$  with probability p

Firm 1's strategy  $\{q_1: q_1 \ge 0\}$ 

Firm 2's strategy  $\{q_{2,L}, q_{2,H}: q_{2,L} \ge 0 \text{ and } q_{2,H} \ge 0\}$ 

Output:  $a - q_1 - q_{2,L}$  or  $a - q_1 - q_{2,H}$ 

# Cournot Duopoly with Incomplete Information

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) +$$

$$+ (1 - p)q_1(a - q_1 - q_{2,L}) - c_Hq_1$$

For player 2, the expected payoff function of type  $c_H$  is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

For player 2, the expected payoff function of type  $c_L$  is

$$U_2(q_1, q_{2,L}, c_L) = (a - q_1 - q_{2,L})q_{2,L} - c_L q_{2,L}$$

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) +$$

$$+ (1 - p)q_1(a - q_1 - q_{2,L}) - c_H q_1$$

Maximizing  $U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H)$  gives

$$B_1(q_{2,L},q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1-p)q_{2,L} - c_H}{2} \right\}$$

For player 2, the expected payoff function of type  $c_H$  is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

Maximizing  $U_2(q_1, q_{2,H}, c_H)$  gives

$$B_2(q_1, c_H) = \left\{ \frac{a - q_1 - c_H}{2} \right\}$$

Similarly, we have

$$B_2(q_1, c_L) = \left\{ \frac{a - q_1 - c_L}{2} \right\}$$

# Bayesian Nash Equilibrium

$$B_{2}(q_{1}, c_{H}) = \left\{ \frac{a - q_{1} - c_{H}}{2} \right\}$$

$$B_{2}(q_{1}, c_{L}) = \left\{ \frac{a - q_{1} - c_{L}}{2} \right\}$$

$$B_{1}(q_{2,L}, q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1 - p)q_{2,L} - c_{H}}{2} \right\}$$

We solve the Bayesian Nash Equilibrium by

$$q_{2,H} \in B_2(q_1, c_H)$$

$$q_{2,L} \in B_2(q_1, c_L)$$

$$q_1 \in B_1(q_{2,L}, q_{2,H})$$

# Bayesian Nash Equilibrium

The Bayesian Nash Equilibrium is  $(q_1, (q_{2,L}, q_{2,H}))$ 

$$q_{1} = \frac{a - c_{H} - (1 - p)(c_{H} - c_{L})}{3}$$

$$q_{2,H} = \frac{a}{3} - \frac{c_{H} + c_{L}}{6} - p \frac{c_{H} - c_{L}}{6}$$

$$q_{2,L} = \frac{a}{3} - \frac{c_{L}}{3} - p \frac{c_{H} - c_{L}}{6}$$

### **Discussions**

- ➤ Incomplete information affects the outputs of players
- $> q_{2,L} > q_{2,H}$  implies player 2 produce more for lower price

# Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects  $c_H$  (p=1) then  $q_1 = q_{2,H} = \frac{a c_H}{3}$
- If player 1 knows that player 2 selects  $c_L$  (p = 0) then  $q_1 = q_{2,L} = \frac{a c_L + c_H c_L}{3}$
- If player 1 does not know the choices of player 2

$$q_{1} = \frac{a - c_{H} - (1 - p)(c_{H} - c_{L})}{3}$$

$$q_{1} = \frac{a - c_{L} + c_{H} - c_{L} - (3 - p)(c_{H} - c_{L})}{3}$$

Player 1 produces less with the incomplete information

# Discussions on Bayesian Nash Equilibrium

• If player 1 knows that player 2 selects  $c_H$  (p=1) then

$$q_{2,H}=\frac{a-c_H}{3};$$

otherwise,

$$q_{2,H} = \frac{a - c_H}{3} + (1 - p) \frac{c_H - c_L}{6}$$

Player 2 benefits from the incomplete information

The firm will benefit by keeping cost secrets

# First Price Auction with Incomplete information

 $N = \{1,2\}$ : players bid a building

 $v_i$ : the true value for player i

 $v_i$ : a uniform distribution over [0,1]

 $b_i \ge 0$ : the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ v_i/2 - b_i & \text{if } b_i = b_j \\ 0 & \text{otherwise} \end{cases}$$

# Bayesian NE of First Price Auction

For player *i*, the expected payoff function is

$$U_{i}(b_{i}, b_{j}(v_{j}), v_{i}) = (v_{i} - b_{i}) \Pr[b_{i} > b_{j}(v_{j})] + (v_{i}/2 - b_{i}) \Pr[b_{i} = b_{j}(v_{j})]$$

$$\Pr[b_{i} > b_{j}(v_{j})] = \Pr[b_{i} > av_{j}] = \Pr[b_{i}/a > v_{j}]$$

 $v_i$  is a uniform distribution over [0,1]

$$U_i(\mathbf{b}_i, \mathbf{b}_j, v_i) = (v_i - b_i)b_i/a$$

Maximizing  $U_i(b_i, b_j, v_i)$  with respect to  $b_i$  gives

$$\mathbf{b_i}(v_i) = \frac{v_i}{2}$$

# First Price Auction with Incomplete information

 $N = \{1, ..., N\}$ : players bid a building

 $v_i$ : the true value for player i

求a的值 2个人算出来是1/2 n个人(n-1)/n

 $v_i$ : a uniform distribution over [0,1]

 $b_i \ge 0$ : belief that the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player *i* 

$$u_i(\mathbf{b}_1, \dots \mathbf{b}_N) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ v_i/2 - b_i & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

# Bayesian NE of First Price Auction (N Players)

For player *i*, the expected payoff function is

$$U_{i}(b_{i}, b_{j}(v_{j}), v_{i}) = (v_{i} - b_{i}) \Pr[b_{i} > \max_{j \neq i} b_{j}(v_{j})]$$
$$U_{i}(b_{i}, b_{j}, v_{i}) = (v_{i} - b_{i})(b_{i}/a)^{N-1}$$

Maximizing  $U_i(b_i, b_j, v_i)$  with respect to  $b_i$  gives

$$\mathbf{b_i}(v_i) = \frac{\mathbf{N} - 1}{N} v_i$$