Dirichlet-Neumann Iteration

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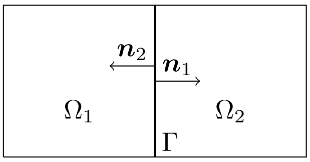
1 Continuous version



Base Problem: Poisson equation (stationary heat equation, think $t \to \infty$)

$$\begin{aligned} \Delta u(x,y) &= 0, \quad (x,y) \in \Omega = \Omega_1 \cup \Omega_2 \\ u(x,y) &= g(x,y), \quad (x,y) \in \partial\Omega \end{aligned} \tag{1}$$

Here, $\partial\Omega$ is the boundary of Ω and $g(x,y)$ is the prescribed (Dirichlet) boundary condition.

Now, consider $u(x,y)$ split into $u_1(x,y)$ and $u_2(x,y)$, according to their respective domains Ω_1 and Ω_2

Then, we add the following redundant conditions to the above PDE:

$$\nabla u_1(x,y) \cdot \boldsymbol{n}_1 = -\nabla u_2(x,y) \cdot \boldsymbol{n}_2, \quad (x,y) \in \Gamma, \tag{2}$$
$$u_1(x,y) = u_2(x,y), \quad (x,y) \in \Gamma. \tag{3}$$

That is, the solution and its derivative are continuous at Γ.

The domain decomposition approach is to solve the PDE, here the Poisson equation (1), on subdomains. That is, we solve (1) on Ω_1 and Ω_2 separately. Now, we lack a boundary condition at Γ. The solution is to enforce (3) for the problem corresponding to Ω_1 and (2) for the problem corresponding to Ω_2.

The iteration is then as follows: Given an initial guess $u_\Gamma^{(0)}$, solve

$$\begin{aligned} \Delta u_1^{(k+1)}(x,y) &= 0, \quad (x,y) \in \Omega_1, \\ u_1^{(k+1)}(x,y) &= g(x,y), \quad (x,y) \in \partial\Omega_1 \setminus \Gamma, \\ u_1^{(k+1)}(x,y) &= u_\Gamma^{(k)}, \quad (x,y) \in \Gamma, \end{aligned} \tag{4}$$

i.e., given a Dirichlet boundary condition at Γ, solve the Poisson equation on Ω_1. Based on the solution $u_1^{(k+1)}(x,y)$, we can compute the heat flux $q_\Gamma^{(k+1)}(x,y) = -\nabla u_1(x,y) \cdot \boldsymbol{n}_1$. Using this, we can solve the problem on Ω_2, using the heat flux $q_\Gamma^{(k+1)}(x,y)$ as boundary condition:

$$\begin{aligned} \Delta u_2^{(k+1)}(x,y) &= 0, \quad (x,y) \in \Omega_2, \\ u_2^{(k+1)}(x,y) &= g(x,y), \quad (x,y) \in \partial\Omega_2 \setminus \Gamma, \\ \nabla u_2^{(k+1)}(x,y) \cdot \boldsymbol{n}_2 &= q_\Gamma^{(k+1)}(x,y), \quad (x,y) \in \Gamma. \end{aligned} \tag{5}$$

Here, the interface temperature $u_\Gamma^{(k+1)}(x,y)$ is part of the solution $u_2^{(k+1)}(x,y)$, i.e., an interior unknown.

The relaxation step is

$$u_\Gamma^{(k+1)} \leftarrow \Theta u_\Gamma^{(k+1)} + (1-\Theta) u_\Gamma^{(k)}, \tag{6}$$

which is necessary for the iteration to converge.

The iteration can be terminated after a fixed number of iterations or if $\|u_\Gamma^{(k+1)} - u_\Gamma^{(k)}\| < TOL$.

2 Discrete version

(As a bit of background)

Discretize (1) to get a system of the form

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{7}$$

where $\boldsymbol{x}$ is the solution of $u(x,y)$ at a set of discrete points and $\boldsymbol{b}$ includes the boundary conditions. Now, split $\boldsymbol{A} = \boldsymbol{M}_A - \boldsymbol{N}_A$ and given an initial guess $\boldsymbol{x}^{(0)}$, perform the iteration

$$\boldsymbol{M}_A \boldsymbol{x}^{(k+1)} = \boldsymbol{N}_A \boldsymbol{x}^{(k)} + \boldsymbol{b}. \tag{8}$$

The splitting $\boldsymbol{A} = \boldsymbol{M}_A - \boldsymbol{N}_A$ corresponds to the splitting on domains above and assures that $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ is fixed point to the iteration (8).