# Chapter 3

# The Wave Properties of Particles

## 3.1 De Broglie Matter Waves

According to the theory of relativity, energy and matter are inter-convertible. Thus, if radiant energy has dual character which is established by experimental facts, it is logical to conclude that matter also should have a dual character. This was the chain of reasoning that led de Broglie to postulate the existence of matter waves in 1924. De Broglie reasoned that since nature loves symmetry, the duality of radiation points to the duality of matter, and *no laws of physics prohibit it.*

Since the wavelength of a photon is:

$$\lambda = \frac{h}{p} \qquad 3.1$$

de Broglie extended Eq. 3.1 to apply to any particle. The momentum of a particle of mass $m$ and velocity $v$ is $p = mv$ and its de Broglie wavelength is

$$\lambda = \frac{h}{mv} \qquad 3.2$$

where $m = \frac{m_o}{\sqrt{1 - v^2/c^2}}$ is the relativistic mass of the particle.

Eq. 3.2 has been verified by a number of experiments. The experiment of Davisson and Germer, who showed that a beam of electrons reflected from the surface of a nickel crystal forming diffraction patterns exactly analogous to the diffraction of light by grating, is one among the others.

**Example 1**

Calculate the de Broglie wavelength of

a. A rock of mass 2 kg moving at a speed of 5 m/s
b. An electron of energy 50 eV.

## 3.2 Wave Motion

A plane wave whose amplitude varies with time $t$ can be represented as:

$$\Psi(x,t)\big|_{x=0} = A\cos\omega t \qquad 3.3$$

where $\omega = 2\pi f$ is the angular frequency.

If the wave is traveling in a given direction at a velocity *v* m/s, then it will take time *x/v* to reach a point *x* m from its point of origin, i.e., the variation of amplitude at the point *x* will lag, in time, behind the variation at the origin. Thus:

$$\Psi(x,t) = A\cos\omega(t - x/v)$$

Since $v = \lambda f,\ \omega = 2\pi f$ :

$$\Psi(x,t) = a\cos\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

The *propagation constant k* is defined as:

$$k = \frac{2\pi}{\lambda} \qquad 3.4$$

*k* is equal to the number of radians corresponding to a wave train 1 m long.

$$\boxed{\Psi(x,t) = a\cos(\omega t - kx)} \qquad 3.5$$

The traveling wave in Eq.3.5 satisfies the wave equation

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \qquad 3.6$$

Its solutions are $f(x - vt)$ and $g(x + vt)$ where *f* and *g* denote any function of $x - vt$ and $x + vt$, respectively. For example, $x \pm vt$, $\sin k(x \pm vt)$, $\cos k(x \pm vt)$, and $e^{jk(x \pm vt)}$ where *k* is a constant.

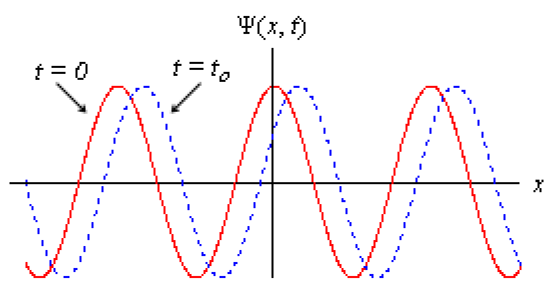


fig. 3.1

The *phase(or wave) velocity* $v_{ph}$ is the velocity of a point on the wave that has a given phase (for example, the crust)

$$\omega t - kx = \text{constant}$$

$$\Rightarrow \omega - k\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dx}{dt} = v_{ph} = \frac{\omega}{k} = \lambda f = v$$

Thus

$$v_{ph} = v = \frac{\omega}{k} \qquad 3.7$$

**The de Broglie Phase (Wave) Velocity**

Since we associate a de Broglie wave with a moving body, it is reasonable to expect that this wave travels at the same velocity *v* as that of the body. $v_{ph} = \lambda f$ but $\lambda = \frac{h}{mv}$, hence

$$v_{ph} = \frac{hf}{mv} = \frac{mc^2}{mv} = \frac{c^2}{v} > c \ !$$

This is impossible. Clearly $v_{ph}$ and $v$ are never equal for a moving body. Let us consider *group velocity.*

**Group Velocity**

How might we use waves to represent a moving particle? If we superimpose two waves of the same amplitude but of slightly differing frequencies, we obtain regions of relatively large (and small) displacement (fig. 3.2a). if we add many waves of slightly differing frequencies, it is possible to obtain what is called a *wave group (packet)*. The important property of the wave group is that its net amplitude differs from zero only over a small region $\Delta x$ (fig. 3.2b).

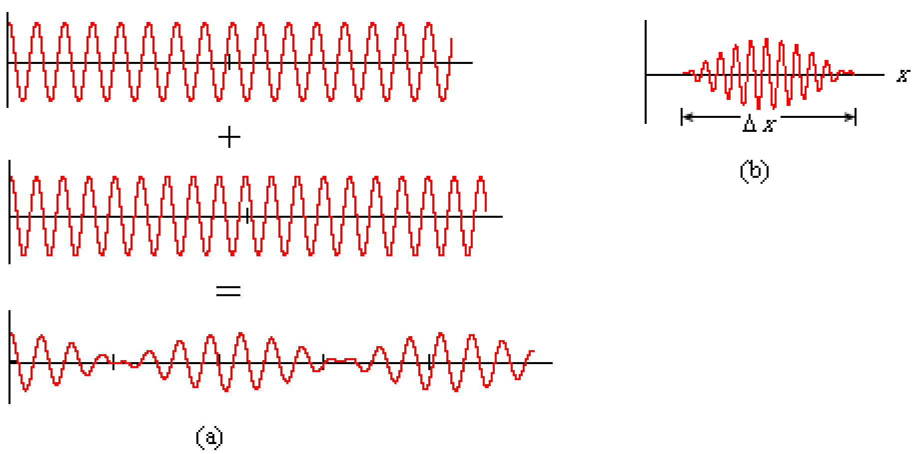


fig. 3.2

The wave group moves at the *group velocity* $v_{gr.}$

Let

$$\Psi_1(x,t) = A\cos(\omega t - kx)$$
$$\Psi_2(x,t) = A\cos[(\omega + d\omega)t - (k + dk)x]$$

The resultant wave is:

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$$
$$= A\cos(\omega t - kx) + A\cos[(\omega + d\omega)t - (k + dk)x]$$

Since $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta), \quad \cos(-\theta) = \cos\theta$,

$$\Psi(x,t) = 2A\cos\tfrac{1}{2}\left[(2\omega + d\omega)t - (2k + dk)x\right]\cos\tfrac{1}{2}(d\omega\, t - dk\, x)$$

$2\omega + d\omega \approx 2\omega,\ \ 2k + dk \approx 2k$ , therefore,

$$\Psi(x,t) = 2A\cos(\omega t - kx)\cos\left(\frac{dw}{2}t - \frac{dk}{2}x\right) \qquad 3.8$$

Eq.3.8 represents a wave of angular frequency $\omega$ and propagation constant $k$ that has superimposed upon it a modulation of angular frequency ½$d\omega$ and propagation constant ½$dk$. The modulation produces wave groups.

$$v_{ph} = \frac{\omega}{k}$$

$$v_{gr} = \frac{d\omega}{dk} \qquad 3.9$$

For a de Broglie wave associated with a body of rest mass $m_o$ moving with a velocity $v$:

$$\omega = 2\pi f = \frac{2\pi fh}{h} = \frac{2\pi mc^2}{h} = \frac{2\pi m_o c^2}{h\sqrt{1 - v^2/c^2}}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h} = \frac{2\pi m_o v}{h\sqrt{1 - v^2/c^2}}$$

$$v_{gr} = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} = \frac{\dfrac{2\pi m_o v}{h}\left(1 - v^2/c^2\right)^{-\frac{3}{2}}}{\dfrac{2\pi m_o}{h}\left(1 - v^2/c^2\right)^{-\frac{3}{2}}}$$

$$v_{gr} = v \qquad 3.9$$

Thus, the de Broglie wave group associated with a moving body travels with the same velocity as the body.

The phase velocity $v_{ph}$ of the de Broglie wave has no simple physical significance in itself.

The group velocity is related to the phase velocity by:

$$v_{gr} = \frac{d\omega}{dk} = \frac{d}{dk}\left(v_{ph}k\right) = v_{ph} + k\frac{dv_{ph}}{dk} \qquad 3.10$$

Thus, the group velocity may be greater or less than the phase velocity. A medium is called *nondispersive* when the phase velocity is the same for all frequencies and $v_{gr} = v_{ph}$. An example is electromagnetic waves in vacuum. Water wave is a good example of waves in dispersive media.

**Exercice 1**

Newton showed that deep-water waves have a phase velocity of $\sqrt{g\lambda/2\pi}$ . Find the group velocity.

*Answer*: $v_{gr}=\frac{1}{2}\sqrt{g/k}=\frac{1}{2}v_{ph}$

**The Wave Function Ψ and Probability**

The function Ψ(*x,t*) is used to denote the superposition of many waves to describe the wave group. Ψ(*x,t*) is called the *wave function.*

For matter waves having de Broglie wavelength, the wave function determines the likelihood or probability of finding the particle at a particular position in space at a given time.

The value of the wave function Ψ has no physical significance itself. The quantity $|\Psi|^2$ (or $\Psi\Psi^*$ if Ψ is complex) is called the *probability density* and represents the probability of experimentally finding the particle in a given unit volume at a given instant of time.

For Ψ(*x,t*), $\Psi\Psi^* dx=|\Psi(x,t)|^2 dx$ is the probability of observing the particle in the interval between *x* and *x* + *dx* at a given time.

$$p(x)dx=|\Psi(x,t)|^2 dx$$

In general, for a particle in a volume *V* enclosing the point (*x,y,z*) at time *t*,

$$\iiint_V \Psi(x,y,z,t)\Psi^*(x,y,z,t)\,dxdydz=1 \qquad 3.11$$

This is called the *condition of normalization* of the wave function.

Since the particle exists in the volume, it has a probability of unit of being observed *somewhere* in the volume.

## 3.3 The Uncertainty Principle

If a particle has a wave associated with it which is spread out in space, what exactly is the position of the particle at any given time? The answer to this question was given by Heisenberg in 1927. He gave a principle, called the *principle of uncertainty*, which states that “it is impossible to determine precisely and simultaneously the values of both of the members of a pair of physical variables which describe the motion of an atomic system.” Such pairs of variables are called *canonically conjugate variables*, for example, position and momentum, kinetic energy and time, angular momentum *L* and angle $\theta$, rotational inertia *I* and angular velocity $\omega$.

Consider an electron of mass *m* and wavelength $\lambda$. This electron can be found somewhere with-in this wavelength and, therefore, the maximum uncertainty in its position measurement is equal to its wavelength, i.e., $\Delta x = \lambda$ . Also $\Delta p_x = h/\lambda$ , therefore, $\Delta x \Delta p_x = h$ . If $\Delta x = 0$ , then $\Delta p_x \rightarrow \infty$ i.e., the momentum becomes completely indeterminate and vice versa.

The principle is expresses as:

$$\Delta x \Delta p_x \geq \frac{h}{2\pi} = \hbar \qquad 3.12$$

The uncertainty relation is a strict law of nature, which is in no way related to some imperfections of the measuring devices. Eq. 3.12 states that it is *in principle impossible* to determine simultaneously both the position and momentum more accurately than is allowed by the inequality, just as it is impossible to exceed the velocity of light or reach absolute zero temperature.

**Example 2**

Calculate the momentum uncertainty of

a. A tennis ball constrained to be in a fence enclosure of length 35 m surrounding the court
b. An electron within the smallest radius of a hydrogen atom ($r = 0.53 \times 10^{-10}$ m)

**Example 3**

Can electrons exist in atomic nuclei ?