Phase shifts and hard sphere scattering

(a) Find the phase shifts for scattering by a hard sphere

$$V(r) = \begin{cases} \infty & r<a \\ 0 & r>a \end{cases}$$

(b) Find the total cross section for an incoming energy

$$E = \hbar^2 k^2/2m$$

in the two limits $k \to \infty$, $k \to 0$.

Give a physical interpretation of the factors 4 and 2 in your answers.

Hint 1: For $k \to \infty$ use the asymptotic form of j_l and n_l to obtain a simple form for $\sin^2\delta_l$. Furthermore, replace the sum over l by an integral so that

$$\sigma = \sum_{l=0}^{l=ka} \sigma_l \approx \frac{4\pi}{k}\int_0^{ka} dl\,(2l+1)\sin^2\delta_l .$$

Hint 2: Look at Zettili, problem 11.3

Solution (a) In this problem we need not even evaluate β_l (which is actually ∞). All we need to know is that the wave function must vanish at $r=R$ because the sphere is impenetrable.

β_l, logarithmic derivative at $r=R$.

$$\beta_l \equiv \left(\frac{r}{A_l}\frac{dA_l}{dr}\right)_{r=R} = kR\left[\frac{j_l'(kR)\cos\delta_l - n_l'(kR)\sin\delta_l}{j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l}\right]$$

知道了在 R 处的对数导数 β_l，就可以得到相移

$$\tan\delta_l = \frac{kR\, j_l'(kR) - \beta_l\, j_l(kR)}{kR\, n_l'(kR) - \beta_l\, n_l(kR)}.$$

Therefore, $A_l(r)\big|_{r=R} = 0$.

or, from $A_l(r) = e^{i\delta_l}\left[\cos\delta_l\, j_l(kr) - \sin\delta_l\, n_l(kr)\right]$,

$$j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l = 0,$$

or $$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}.$$

Thus the phase shifts are known for any l. Notice that no approximations have been made so far.

(b) $k \to 0$, $kR \ll 1$,

use $$j_l(kr) \overset{kr\to 0}{\simeq} \frac{(kr)^l}{(2l+1)!!}, \qquad n_l(kr) \overset{kr\to 0}{\simeq} -\frac{(2l-1)!!}{(kr)^{l+1}}$$

to obtain $$\tan\delta_l = \frac{-(kR)^{2l+1}}{\{(2l+1)[(2l-1)!!]^2\}}.$$

It is therefore all right to ignore δ_l with $l \neq 0$.

In other words, we have S-wave scattering only, which is actually expected for almost any finite-range potential at low energy.

$$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)},$$

for $l=0$, $$\tan\delta_0 = \frac{j_0(kR)}{n_0(kR)} = \frac{\sin kR/kR}{-\cos kR/kR} = -\tan kR.$$

$$\delta_0 = -kR.$$

$$f(\theta) = \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta), \qquad \frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

for $l=0$, $$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2}.$$

because $\delta_0 = -kR$ regardless of whether k is large or small.

$$\approx R^2 \text{ for } kR \ll 1.$$

the total cross section, given by

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega}\, d\Omega = 4\pi R^2,$$

is four times the geometric cross section πR^2. Low-energy scattering means a very large-wavelength scattering, and we do not necessarily expect a classically reasonable result.

$\lambda k = 2\pi$

there is a factor 4 between the quantum mechanical cross section at low energies and the classical cross section for scattering from a hard sphere. ($\sigma_{classical} = \pi a^2$).

This can be explained by the fact that in quantum mechanics one consider probabilities, i.e. the square of wave functions. At low energies the wave length is considerably larger than the ~~siz~~ size of the target, thus the scattering can be viewed as scattering by on the entire surface of the sphere, not only the cross section area of the sphere.

$k \to \infty$

$ka \gg 1$, the number of partial waves contributing to the scattering is large. We may regard l as a continuous variable.

as an aside, we note the semiclassical argument that $l = bk$, (because angular momentum $l\hbar = bp$, where b is the impact parameter and $p = \hbar k$).

$l\hbar = b\hbar k$, $l = bk$. $\hat{L} = \hat{r} \times \hat{p}$, $l(l+1)\hbar = b\hbar k$, $l(l+1) = bk$.

we take $l_{max} = kR$.

then we make the following substitutions in expression

$$f(\theta) = \sum_{l=0} (2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right) P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \quad (6.4.40)$$

At high ~~energy~~ energies many l-values contribute, up to $l_{max} \simeq kR$, a reasonable assumption. The total cross section is therefore given by

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{l\simeq kR}(2l+1)\sin^2\delta_l.$$

using $\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$, we have

$$\sin^2\delta_l = \tan^2\delta_l \cdot \cos^2\delta_l = \frac{\tan^2\delta_l}{\sec^2\delta_l} = \frac{\tan^2\delta_l}{1+\tan^2\delta_l} = \frac{[j_l(kR)]^2}{[j_l(kR)]^2 + [n_l(kR)]^2} \simeq$$

$\sec^2\delta_l = 1 + \tan^2\delta_l$

$\sin^2(kR - \frac{\pi l}{2})$,

where we have used

$$j_l(kr) \overset{kr\to\infty}{\sim} \frac{1}{kr}\sin(kr - \frac{l\pi}{2}) \qquad j_l(\rho) \xrightarrow[\rho\to\infty]{} \frac{1}{\rho}\sin(\rho - \frac{l\pi}{2})$$

$$n_l(kr) \overset{kr\to\infty}{\sim} \frac{1}{-kr}\cos(kr - \frac{l\pi}{2}). \qquad n_l(\rho) \xrightarrow[\rho\to\infty]{} -\frac{1}{\rho}\cos(\rho - \frac{l\pi}{2}).$$

$$\frac{[j_l(kR)]^2}{[j_l(kR)]^2 + [n_l(kR)]^2} \simeq \frac{\sin^2(kR - \frac{l\pi}{2})}{\sin^2 + \cos^2} = \sin^2(kR - \frac{l\pi}{2})$$

each time l increases by one unit. δ_l decreases by $\frac{\pi}{2}$.

Thus, for an adjacent (相邻的) pair of partial waves,

$$\sin^2\delta_l + \sin^2\delta_{l+1} = \sin^2\delta_l + \sin^2(\delta_l - \frac{\pi}{2}) = \sin^2\delta_l + \cos^2\delta_l = 1.$$

and with so many l-values contributing to

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{l\simeq kR}(2l+1)\sin^2\delta_l.$$

it is legitimate to replace $\sin^2\delta_l$ by its average value, $\frac{1}{2}$.

The number of terms in the l-sum is roughly kR,

the average of $2l+1$ is roughly kR,

$$\sigma_{tot} \approx \frac{4\pi}{k^2}\cdot(kR)^2\cdot\frac{1}{2} = 2\pi R^2.$$

求和公式

$$\frac{1}{2}\cdot\frac{4\pi}{k^2}\sum_{l=0}^{l\simeq kR}(2l+1) = \frac{2\pi}{k^2}[1 + (2kR+1)]\cdot(kR+1)\cdot\frac{1}{2} =$$

$$\frac{2\pi}{k^2}(kR+1)^2, \quad (kR \gg 1.) \approx 2\pi R^2.$$

也不是几何散射截面 πR^2.

To see the origin of the factor of 2, we may split

$$f(\theta) = \sum_{l=0} (2l+1) \left(\frac{e^{2i\delta_l}-1}{2ik}\right) P_l(\cos\theta) = \frac{1}{k}\sum_{l=0} (2l+1)\, e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta)$$

into two parts:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{kR} (2l+1)\, e^{2i\delta_l} P_l(\cos\theta) + \frac{i}{2k}\sum_{l=0}^{kR} (2l+1) P_l(\cos\theta)$$

$$= f_{\text{reflection}} + f_{\text{shadow}}$$

In evaluating $\int |f_{\text{refl}}|^2 d\Omega$, the orthogonality of the $P_l(\cos\theta)$'s ensure that there is no inference among contributions from different l, and we obtain the sum of the square of partial-wave contributions:

$$\int |f_{\text{refl}}|^2 d\Omega = \frac{2\pi}{4k^2} \sum_{l=0}^{l_{max}} \int_{-1}^{+1} (2l+1)^2 [P_l(\cos\theta)]^2 d(\cos\theta) = \frac{\pi l_{max}^2}{k^2} = \pi R^2 \triangle$$

Turning our attention to f_{shad}, it is pure imaginary. It is particularly strong in the forward direction because $P_l(\cos\theta) = 1$ for $\theta = 0$ $P_l(1) = 1$.

and the contributions from various l-values all add up coherently —— that is, with the same phase, pure imaginary and positive in our case.

we can use the small-angle approximation for P_l to obtain

$$(l = bk,\ dl = k\,db)$$

$$bk \gg 1$$

$$\begin{aligned} f_{shad} &\simeq \frac{i}{2k} \sum (2l+1) J_0(l\theta) \\ &\simeq \frac{i}{2k} \int (2bk+1) J_0(kb\theta) \cdot k\,db \\ &\simeq ik \int_0^R b\,db\, J_0(kb\theta) \\ &= \frac{iR\, J_1(kR\theta)}{\theta}. \end{aligned}$$

this is just the formula for Fraunhofer diffraction in optics with a strong peaking near $\theta \simeq 0$. Letting $\xi = kR\theta$ and $d\xi/\xi = d\theta/\theta$, we can evaluate

$$\int |f_{shad}|^2 d\Omega = 2\pi \int_{-1}^{+1} \frac{R^2 [J_1(kR\theta)]^2}{\theta^2} d(\cos\theta)$$

$$\approx 2\pi R^2 \int_{\pi}^{0} \frac{[J_1(\xi)]^2}{\theta^2} (-\sin\theta)\, d\theta$$

$$\approx -2\pi R^2 \int_{\infty}^{0} \frac{[J_1(\xi)]^2}{\xi} d\xi$$

$$\approx 2\pi R^2 \int_{0}^{\infty} \frac{[J_1(\xi)]^2}{\xi} d\xi$$

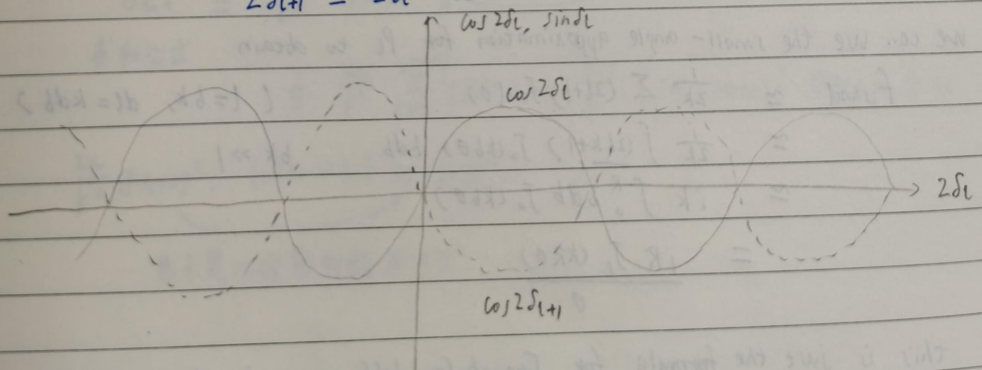
$$\approx \pi R^2.$$

Finally, the interference between f_{shad} and f_{refl} vanishes. because the phase of f_{refl} oscillates $(2\delta_{l+1} = 2\delta_l - \pi)$, approximately averaging to zero, while f_{shad} is pure imaginary.

$$f(\theta) = \underbrace{\frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)\, e^{2i\delta_l} P_l(\cos\theta)}_{f_{refl}} + \underbrace{\frac{i}{2k}\sum_{l=0}^{kR}(2l+1)\, P_l(\cos\theta)}_{f_{shad}}$$

$$e^{2i\delta_l} = \cos 2\delta_l + i\sin 2\delta_l$$

$$2\delta_{l+1} = 2\delta_l - \pi$$



pure imaginary × oscillates → oscillates

Σ oscillates → 0.

$$\mathrm{Re}\,(f_{shad}^{*}\, f_{refl}) \approx 0.$$

the interference between f_{shad} and f_{refl} vanishes.

Thus , $\sigma_{tot} = \underset{\sigma_{refl}}{\pi R^2} + \underset{\sigma_{shad}}{\pi R^2}$

第二项（朝前方的相干贡献）叫做阴影，因为对于高能硬球散射，碰撞参数小于R的波一定会偏转，所以，就在靶的后方找到粒子的概率是0，一定会产生一个阴影。

从波动力学来说，这个阴影是由于最初的入射波和新散射的波之间的非常强的干涉（destructive interference）。因此我们需要散射来产生一个阴影。

阴影散射振幅一定是纯虚数可以从下面看出：

$$\langle \vec{x} | \psi^{(+)} \rangle \xrightarrow{\text{large } r} \frac{1}{(2\pi)^{3/2}}\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] =$$

(f_{shad})

$$\frac{1}{(2\pi)^{3/2}} \sum_l (2l+1)\frac{P_l}{2ik}\left[\underline{[1+2ikf_l(k)]}\frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right]$$

that the coefficient of $e^{ikr}/2ikr$ for the lth partial wave behaves like $1+2ikf_l(k)$, where the 1 would be present even without the scatterer. hence <u>there must be a positive imaginary term in f_l to get cancellation.</u> f_l包含一个正虚数项。

f_l —— f_{shad}

$$f_l = \frac{e^{2i\delta_l}-1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k}, \quad f(\theta) = \sum_{l=0} (2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right) P_l(\cos\theta) =$$

$$\frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta), \quad f(\theta) = \sum_{l=0}(2l+1) f_l P_l(\cos\theta).$$

$$f_{shad} = \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta).$$

$$f(\theta) = f_{refl} + f_{shad},$$

$$f_{refl} = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(\cos\theta).$$

In fact, this gives a physical interpretation of the optical theorem, which can be checked explicitly. First note that

$$\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(\theta=0).$$

$$\frac{4\pi}{k} \operatorname{Im} f(0) \simeq \frac{4\pi}{k} \operatorname{Im}[f_{shad}(0)]$$

because $\operatorname{Im}[f_{refl}(0)]$ averages to zero due to oscillating phase.

Using $$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(\cos\theta) + \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)$$

$$= \quad f_{refl} \quad + \quad f_{shad}.$$

we obtain $$\frac{4\pi}{k} \operatorname{Im} f_{shad}(0) = \frac{4\pi}{k}\frac{1}{2k}\sum_{l=0}^{kR}(2l+1) =$$

$$\frac{2\pi}{k^2}(kR+1)^2 \overset{(kR \gg 1)}{\approx} 2\pi R^2.$$