Quantum Computing

Abstract

This brief introduction to quantum computing is designed for readers familiar with high-school algebra and basic vectors and matrices. Every new mathematical concept is introduced as it arises in the text, motivated by physical or computational need, defined clearly, and illustrated immediately with examples. The presentation builds understanding step by step, from classical bits to quantum algorithms, using only necessary mathematics at each stage.

Contents

1 Classical Bits and the Need for a New Model

A classical computer uses **bits**, each in one of two states: 0 or 1. A system of n bits has 2^n possible configurations. For example, 3 bits can be in any of the 8 states: $000, 001, \ldots, 111$.

Note 1.1. At any given time, the system is in *exactly one* of these 2^n states.

In quantum computing, we replace bits with **qubits**, which can exist in *multiple states simultaneously* — a property called **superposition**. To describe this, we introduce vectors.

2 Vectors: The Language of Qubit States

We represent the two basic states of a qubit using **column vectors**:

Notation 2.1.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The symbol $|\cdot\rangle$ is part of **Dirac notation** — it simply labels a vector. Think of it as a name for a column of numbers.

A general qubit state is a **linear combination** of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The coefficients $\alpha, \beta \in \mathbb{C}$ are called **amplitudes**.

Example 2.1. The state

$$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

is a valid qubit state.

3 Normalization: Ensuring Probabilities Sum to 1

When we **measure** a qubit in the $\{|0\rangle, |1\rangle\}$ basis:

- Outcome 0 occurs with probability $|\alpha|^2$
- Outcome 1 occurs with probability $|\beta|^2$

Therefore, we require:

$$|\alpha|^2 + |\beta|^2 = 1.$$

Definition 3.1 (Normalized State)**.** A state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is *normalized* if $|\alpha|^2 + |\beta|^2 = 1$.

Example 3.1. Check normalization of $|+\rangle$:

$$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

4 Superposition

A qubit in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with both $\alpha, \beta \neq 0$ is in **superposition**: it has nonzero amplitude to be found in either basis state upon measurement.

Note 4.1. Superposition is not probabilistic uncertainty — it is a coherent combination of possibilities that can interfere.

5 Inner Products: Computing Overlap and Probability

To compute measurement probabilities, we need the **inner product**.

Notation 5.1. For vectors $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}$, the inner product is:

$$\langle\psi|\phi|\psi|\phi\rangle = \alpha^*\gamma + \beta^*\delta,$$

where z^* denotes the complex conjugate of z.

The probability of measuring $|\psi\rangle$ and obtaining $|0\rangle$ is:

$$p(0) = |\langle 0|\psi|0|\psi\rangle|^2 = |\alpha|^2.$$

Example 5.1. For $|\psi\rangle = 0.6|0\rangle + 0.8|1\rangle$:

$$p(0) = |0.6|^2 = 0.36, \quad p(1) = |0.8|^2 = 0.64.$$

6 Matrices: Changing Quantum States

Quantum operations are represented by **matrices** that act on state vectors via matrix multiplication.

Example 6.1 (Pauli-X Gate — Quantum NOT)**.**

$$X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} = |1\rangle.$$

Definition 6.1 (Unitary Matrix)**.** A matrix U is **unitary** if $U^\dagger U = I$, where $U^\dagger$ is the conjugate transpose (transpose and complex conjugate).

All valid quantum operations are unitary — they preserve normalization.

Exercise 6.1. Verify that X is unitary by computing $X^\dagger X$.

7 Hadamard Gate: Creating Superposition

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) \equiv \left|+\right\rangle$$

$$H \left|1\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \equiv \left|-\right\rangle$$

Note 7.1. $H^2 = I$, so H is its own inverse.

8 Two Qubits: Tensor Products

For two qubits, the state space is formed using the **tensor product**.

Notation 8.1.

$$\left|a\right\rangle \otimes \left|b\right\rangle \equiv \left|ab\right\rangle$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

The computational basis for two qubits is:

$$\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle.$$

A general two-qubit state is:

$$\left|\Psi\right\rangle = c_{00}\left|00\right\rangle + c_{01}\left|01\right\rangle + c_{10}\left|10\right\rangle + c_{11}\left|11\right\rangle, \quad \sum_{ij} |c_{ij}|^2 = 1.$$

9 Entanglement

Definition 9.1 (Product State)**.** A two-qubit state is a **product state** if it can be written as $\left|\Psi\right\rangle = \left|\psi\right\rangle \otimes \left|\phi\right\rangle$.

Definition 9.2 (Entangled State)**.** A state that is not a product state is **entangled**.

Example 9.1 (Bell State)**.**

$$\left|\Phi^+\right\rangle = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}}$$

is entangled: measuring the first qubit in $\left|0\right\rangle$ forces the second to be $\left|0\right\rangle$, and vice versa.

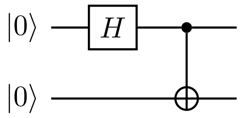


Figure 1: Circuit to prepare $|\Phi^+\rangle$.

10 CNOT Gate: Controlled Operations

The **CNOT** (controlled-NOT) gate is:

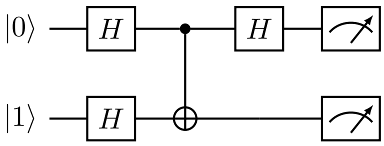
$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It flips the target qubit if the control is $|1\rangle$:

$$\text{CNOT}\,|10\rangle = |11\rangle\,, \quad \text{CNOT}\,|11\rangle = |10\rangle\,.$$

11 Quantum Circuits

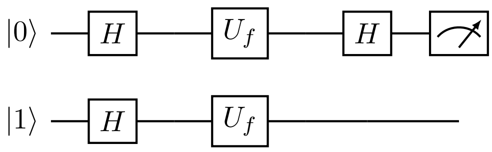
A **quantum circuit** is a sequence of unitary gates applied to qubits, followed by measurement.



12 Deutsch's Algorithm

Problem: Given $f : \{0,1\} \to \{0,1\}$ that is either constant or balanced, determine which with one query.

Circuit:



The oracle is $U_f\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle$.
Final measurement of first qubit:

- $0 \to$ constant
- $1 \to$ balanced

Note 12.1. Interference from the Hadamard gates reveals the global property of f.

13 Grover's Algorithm

Searches $N = 2^n$ items for a marked one in $O(\sqrt{N})$ steps using **amplitude amplification**.

14 No-Cloning Theorem

No unitary operation can copy an arbitrary unknown qubit state.

Sketch. Suppose $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$. Then for any $|\psi\rangle, |\phi\rangle$:

$$\langle\psi|\phi|\psi|\phi\rangle = (\langle\psi|\phi|\psi|\phi\rangle)^2.$$

This holds only if $\langle\psi|\phi|\psi|\phi\rangle \in \{0, 1\}$, contradicting general superpositions. □

15 Quantum Error Correction: 3-Qubit Bit-Flip Code

Encode logical qubit as:

$$|0\rangle_L = |000\rangle, \quad |1\rangle_L = |111\rangle.$$

A single bit flip produces a state with detectable parity error. Measuring parity between qubit pairs identifies and corrects the error.

16 Physical Implementations

- **Superconducting qubits**: Microwave-controlled circuits at mK temperatures.
- **Trapped ions**: Laser-manipulated atomic states.
- **Photonic qubits**: Polarization or path encoding of light.

All suffer from **decoherence** — interaction with the environment destroys quantum behavior.

17 The NISQ Era

Current devices are **NISQ** (Noisy Intermediate-Scale Quantum):

- 50–1000 qubits
- Gate errors approximately 0.1–1%
- Limited circuit depth

Useful for hybrid algorithms: VQE, QAOA.

18 Summary of Key Mathematical Concepts

19 Exercises

Exercise 19.1. Compute $H|+\rangle$. What is the result?

Exercise 19.2. Apply CNOT to $|+0\rangle$. Is the output entangled?

Exercise 19.3. Normalize the state $|\psi\rangle = 3|0\rangle + 4|1\rangle$.

Concept	**Symbol**	**Meaning**
Qubit state	$\lvert\psi\rangle = \alpha \lvert 0\rangle + \beta \lvert 1\rangle$	Superposition
Normalization	$\lvert\alpha\rvert^2 + \lvert\beta\rvert^2 = 1$	Total probability = 1
Inner product	$\langle\psi\lvert\phi\rvert\psi\lvert\phi\rangle$	Overlap amplitude
Unitary gate	$U^\dagger U = I$	Reversible evolution
Tensor product	$\lvert a\rangle \otimes \lvert b\rangle = \lvert ab\rangle$	Multi-qubit states
CNOT	4×4 matrix	Entangling gate

Table 1: Core mathematical objects introduced.

20 Further Reading

- Yanofsky and Mannucci, *Quantum Computing for Computer Scientists*
- Qiskit Textbook: https://qiskit.org/textbook
- Preskill's Lecture Notes: http://theory.caltech.edu/~preskill/ph229/