Part III — Further Pure Math

Based on lectures by Brian

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Lent 2017-2018

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1 Complex numbers

2 Numerical solutions of equations

3 Coordinate systems

4 Matrix algebra

5 Series

6 Proof by mathematical induction

6.1 Summation of series

Example. Prove by the method of mathematical induction , that, for $n \in \mathbb{Z}^+, \sum_{r=1}^{n}(2r-1) = n^2$.

Proof.

$$n = 1; LHS = \sum_{r=1}^{1}(2r-1) = 2(1) - 1 = 1$$
$$RHS = 1^2 = 1$$

As $LHS = RHS$, the summation formula is true for $n = 1$,

Assume hat the summation formula is true for $n = k$,

$$i.e. \sum_{r=1}^{k}(2r-1) = k^2$$

With n=k+1 terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1) &= 1 + 2 + 3 + \cdots + (2k-1) + (2(2k+1)-1)\\ &= k^2 + (2(k+1)-1)\\ &= k^2 + (2k+2-1)\\ &= k^2 + 2k + 1\\ &= (k+1)^2\end{aligned}$$

Therefore , summation formula is true when $n = k + 1$
If the summation formula is true for $n = k$ thenit is shown to be true for $n = k+1$.
As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.. □

Example. Prove by the method of mathematical induction, that , for $n \in \mathbb{Z}^+, \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof.

$$n = 1; LHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$$
$$RHS = \frac{1}{6}(1)(2)(3) = \frac{6}{6} = 1$$

As $LHS = RHS$, the summation formula is true for n=1

Assume that the summation formula is true for $n = k$

$$i.e. \sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)[2k^2 + k + 6k + 6] \\ &= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$
If the summation formula is true for $n = k$ thenit is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.. □

Example. Prove by the method of mathematical induction, that , for $n \in \mathbb{Z}^+, \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof.

$$\begin{aligned} n = 1; LHS &= \sum_{r=1}^{1} r^2 = 1^2 = 1 \\ RHS &= \frac{1}{6}(1)(2)(3) = \frac{6}{6} = 1\end{aligned}$$

As $LHS = RHS$, the summation formula is true for n=1

Assume that the summation formula is true for $n = k$

$$i.e. \sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1} r^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
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\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$
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6.2 Divisibility

6.3 General term of a recurrence relation

6.4 Matrix multiplication

7 Inequalities

8 Further Series

9 Further complex numbers

10 First order differential equations

11 Second order differential equations

12 Maclaurin and Taylor series

13 Polar coordinates

14 Hyperbolic functions

15 Further coordinate systems

16 Differentiation

17 Integration

18 Vectors - Cross product and vector equation of plane

19 Further matrix algebra

20 Extension topics - STEP