Part III — Further Pure Math

Based on lectures by Brian
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

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6.1 Summation of series

Example. Prove by the method of mathematical induction , that, for $n \in \mathbb{Z}^+, \sum_{r=1}^{n}(2r-1) = n^2$.

Proof.

$$n = 1; LHS = \sum_{r=1}^{1}(2r-1) = 2(1) - 1 = 1$$
$$RHS = 1^2 = 1$$

As $LHS = RHS$, the summation formula is true for $n = 1$,

Assume hat the summation formula is true for $n = k$,

$$i.e. \sum_{r=1}^{k}(2r-1) = k^2$$

With n=k+1 terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1) &= 1 + 2 + 3 + \cdots + (2k-1) + (2(2k+1)-1) \\ &= k^2 + (2(k+1)-1) \\ &= k^2 + (2k+2-1) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2\end{aligned}$$

Therefore , summation formula is true when $n = k + 1$
If the summation formula is true for $n = k$ thenit is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.. □

Example. Prove by the method of mathematical induction, that , for $n \in \mathbb{Z}^+, \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof.

$$n = 1; LHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$$
$$RHS = \frac{1}{6}(1)(2)(3) = \frac{6}{6} = 1$$

As $LHS = RHS$, the summation formula is true for n=1

Assume that the summation formula is true for $n = k$

$$i.e. \sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)[2k^2 + k + 6k + 6] \\ &= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$
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Example. Prove by the method of mathematical induction, that , for $n \in \mathbb{Z}^+, \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof.

$$\begin{aligned}n = 1; LHS &= \sum_{r=1}^{1} r^2 = 1^2 = 1 \\ RHS &= \frac{1}{6}(1)(2)(3) = \frac{6}{6} = 1\end{aligned}$$

As $LHS = RHS$, the summation formula is true for n=1

Assume that the summation formula is true for $n = k$

$$i.e. \sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)[2k^2 + k + 6k + 6] \\ &= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)\end{aligned}$$

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