Part III — Statistics

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Contents

1 Representation and summary of data - location

1.1 Basic Concepts of Variable

Definition (Quantitative variables and Qualitative variables)**.** Quantitative variable associated with numerical observation. Qualitative variables associated with non-numerical observations.

Definition (Continuous variable and discrete variable)**.** Continuous variable can take ant value in given range. Discrete can take only specific values in a given range.

1.2 Grouped data

Definition (Grouped data)**.** The groups are more commonly known as classes.

- class boundaries.
- mid-point of a class.
- class width.

Example. Example 5-6

Definition (Frequency and cumulative frequency)**.** Number of anything; example is how many sheeps. It is sometimes helpful to add a column to the table showing the running total of the frequencies. This is called the cumulative frequency

Definition (Ungrouped data)**.** Show all data

1.3 Mean , mode and median

Definition (Mode)**.** The mode is the value that occurs most often

Definition (Median)**.** n/2 term or 1 term above

Definition (Mean)**.**

$$\bar{x} = \frac{\sum_i^n x_i}{n}$$

1.4 Linear interpolation

Example. Example 14-15

1.5 Coding

Example. pick 1 example

2 Representation and summary of data - measures of dispersion

2.1 Range and interquartile range

The list of formula:

- Range = Upper value − Lowest value

Example. example 3

2.2 Percentiles split the data into 100 parts

Example. example 4

2.3 Range and Interquartile range

Example (Linear Interpolation)**.**

2.4 Variance and standard deviation

Definition (Variance)**.** Let f stand for the frequency, then $n = \sum f$ and

$$\text{Variance} = \frac{\sum f(x-\bar{x})^2}{\sum f} \text{ or } \frac{\sum fx^2}{\sum f} - (\frac{\sum fx^2}{\sum f})$$

2.5 Variance and standard deviation for grouped data

Definition.

Example. example 7-8

2.6 Coding

Example. example 9-11

3 Representation of data

3.1 Stem and Leaf diagrams

3.2 Outlier

Definition. An outlier is an extreme value that lies outside the overall pattern of the data.

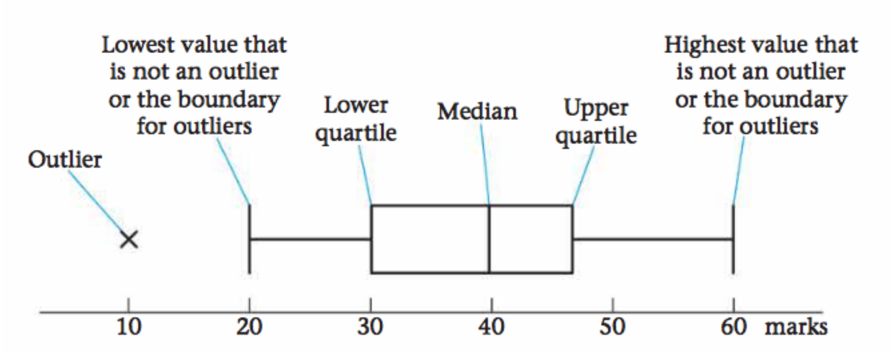
An outlier is any value, which is

$$\text{greater than the upper quartile} + 1.5 \times \text{interquartile range}$$

OR

$$\text{less than the lower quartile} + 1.5 \times \text{interquartile range}$$

3.3 Box plot



3.4 Histogram

Definition (Frequency density)**.**

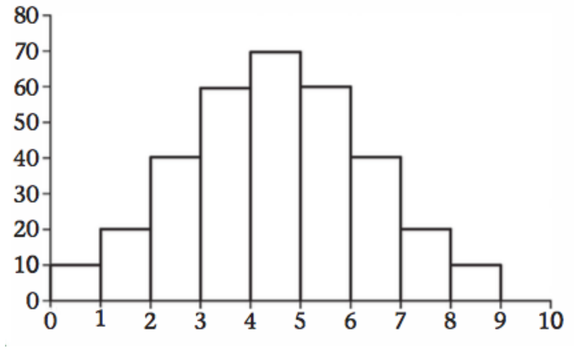
$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Example. 7

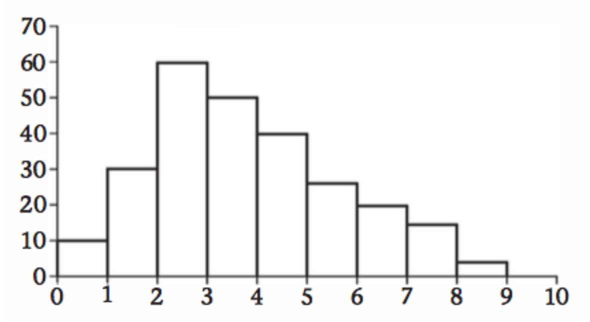
3.5 Skewness (Shape)

A distribution can be symmetrical , have positive skew or have negative skew

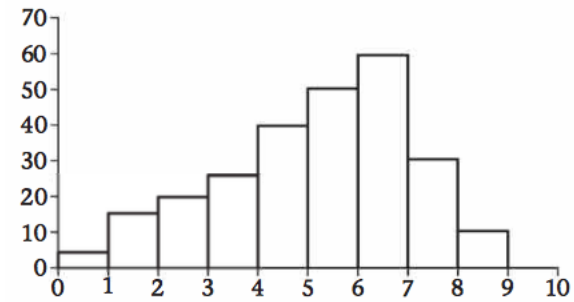
symmetrical $Q_2 - Q_1 = Q_3 - Q_2$ or mode=median=mean



positive :$Q_2 - Q_1 < Q_3 - Q_2$ or mode<median<mean



negative :$Q_2 - Q_1 > Q_3 - Q_2$ or mode>median>mean



Or you can calculate:

$$\frac{3(\text{mean} - \text{median})}{\text{SD}}$$

3.6 What!?

Example. example 10-12

4 Probability

4.1 Classical Probability

4.2 Venn diagram and their rules

Definition (Complementary Probability).

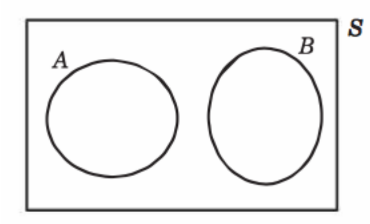
4.3 Conditional Probabilites

4.3.1 Vann diagram

4.3.2 Tree diagram

4.4 Special Events of Probabilites

Definition (Mutually exclusive). When events have no outcomes in common, they are mutually exclusive.



There is no intersection of A and B, so $P(A \cap B) = 0$

We can use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
result is

$$P(A \cup B) = P(A) + P(B)$$

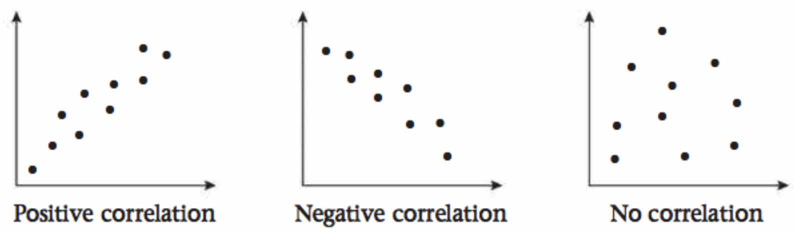
Definition (Independent events). When one event has no effect on another, they are independent so $P(A|B) = P(A)$

by $\frac{P(A \cap B)}{P(B)} = P(A)$ we have:

$$P(A \cap B) = P(B) \times P(A)$$

5 Correlation

5.1 Correlation



5.2 Bivariate data

Recall this formula :

$$\text{Variance} = \frac{\sum(x-\bar{x})^2}{n}$$

In correlation we write:

$$S_{xx} = \sum(x-\bar{x})^2$$

$$S_{yy} = \sum(y-\bar{y})^2$$

so

$$\text{Variance} = \frac{S_{xx}}{n}$$

Definition (Co-Variance)**.**

$$S_{xy} = \frac{\sum(x-\bar{x})(x-\bar{y})}{n}$$

5.3 Product moment Correlation coefficient r

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

The value of r varies between -1 and 1

If $r = 1$, positive linear correlation

If $r = -1$, nagative linear correlation

If $r = 0$, no linear correlation

limitation:

5.4 Coding

does not effect r

6 Regression

6.1 Linear

let $y = a + bx$ be a regression line
where

$$b = \frac{S_{xy}}{S_{xx}} \text{ and } a = \bar{y} - b\bar{x}$$

6.2 Coding

6.3 Interpolation and Extrapolation

7 Discrete random variables

7.1 Probability distribution

Definition (Mean / Expected value)**.**

$$E(X) = \sum xp(x)$$

when we find $E(X^n)$:

$$E(X^n) = \sum x^n p(x)$$

Definition (Variable)**.**

$$Var(X) = E(X^2) - (E(X))^2$$

The constant a and b affect on $E(X)$ and $Var(X)$

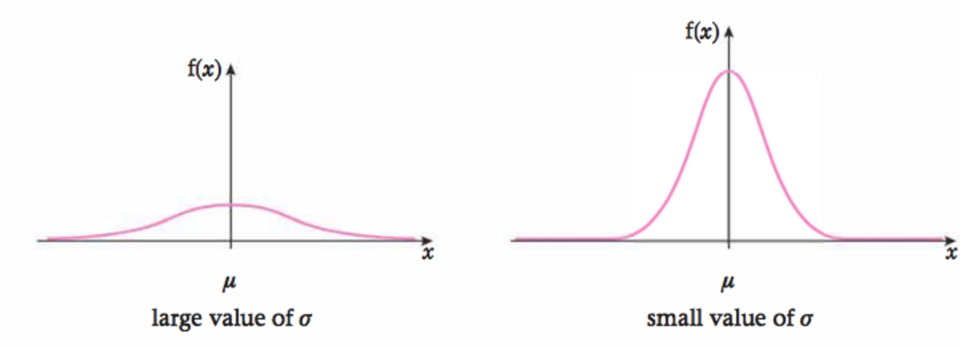
$$E(aX + b) = aE(x) + b$$

$$Var(aX + b) = a^2 Var(X)$$

Definition (Uniform distribution)**.** The distribution is uniform when all the probabilities is the same of all values.

8 The normal distribution

$Z \sim N(\mu, \sigma^2)$ represent the normal distribution.



The random variable X can be written as $X \sim N(\mu, \sigma^2)$

you can transformed X to Z by this formula

$$z = \frac{X - \mu}{\sigma}$$

Example. Example 8-9

9 Binomial distribution

9.1 Basic Concept

$X \sim B(n, p)$ represent the Binomial distribution, then

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

9.2 Mean and Variance

If $X \sim B(n, p)$ then

$$E(X) = \mu = np$$
$$Var(X) = \sigma^2 = np(1-p)$$

Example. example 9-14

10 Poisson distribution

10.1 Basic Concepts

Recall the exponential function

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + + \frac{x^r}{r!} + \cdots$$

If you let $x = \lambda$ and remember that $\lambda^0 = 1$ this gives

$$e^\lambda = \lambda^0 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots + + \frac{\lambda^r}{r!} + \cdots$$

Dividing by e^λ gives

$$\frac{e^\lambda}{e^\lambda} = \lambda^0 + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots + \frac{\lambda^r e^{-\lambda}}{r!} + \cdots$$

And the probability function is

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

We say that X has a Poisson distribution with parameter λ abd write

$$X \sim Po(\lambda)$$

10.2 Mean and Variance

$$Var(X) = E(X) = \mu = \sigma^2 = \lambda$$

Lemma. If mean and standard deviation square is same, we usually use Poisson distribution.

Example. example 5-6

10.3 Approximate a Binomial with Poisson

If $X \sim B(n, p)$ and

- n is large
- p is small

then X can be approximated by

$$Po(np)$$

Example. example 7-8 9-10

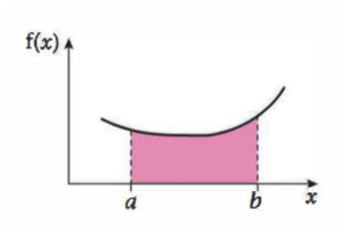
11 Continuous random variables

11.1 Continous random variable

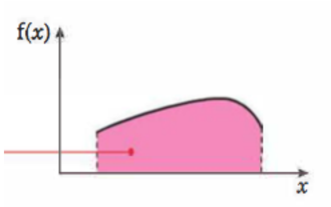
The Continuous random variables with p.d.f $f(x)$ satisfied the following properties:

(i) $f(x) \geq 0$ since we cannot have negative probabilities

(ii) $P(a < X < b) =$ shaded area $= \int_a^b f(x)\,\mathrm{d}x$



(iii) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$ since the area under the curve $= 1$.



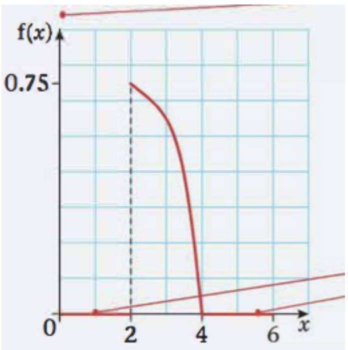
Example. The random variable X has probability density function

$$f(x) = \begin{cases} kx(4-x) & 2 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of k and sketch the p.d.f

Proof.

$$\int_2^4 k(4x - x^2)\,\mathrm{d}x = 1$$
$$k[2x^2 - \frac{x^3}{3}]_2^4 = 1$$
$$k = (\frac{3}{16})$$



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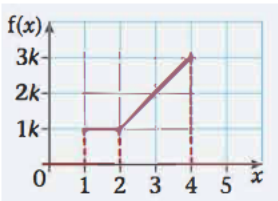
Example. The random variable X has probability density function

$$f(x) = \begin{cases} k & 1 < x < 2 \\ k(x-1) & 2 \le x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of k and sketch the p.d.f

Proof.

$$\int_1^2 k \,\mathrm{d}x + \int_2^4 k(x-1)\,\mathrm{d}x = 1$$
$$k = \frac{1}{5}$$



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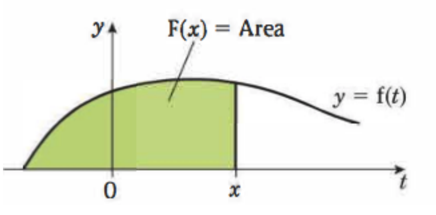
11.2 Cumulative distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

where $F(x) = P(X \le x) = 1$



If X us a Continous random variable with c.d.f. $F(x)$ and p.d.f $f(x)$

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}t}F(x) \text{ and } F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

Example. example 5-6

11.3 Mean and Variance

If X is a Continuous random variable with p.d.f $f(x)$

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x$$

$$\sigma^2 = E(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$$

Remark. The range is the range of that function instead of negative infinity to infinity.

11.4 Mode, median and quartiles

The median m or Q_2 satisfies $F(m) = F(Q_2) = 0.5$
The lower quartile Q_1 satisfies $F(Q_1) = 0.25$
The lower quartile Q_1 satisfies $F(Q_3) = 0.75$

The mode is the x value at the highest point of the p.d.f $f(x)$

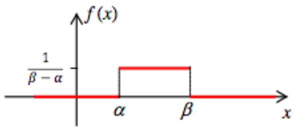
12 Continuous uniform distribution

12.1 Continuous uniform distribution

Definition

A continuous uniform distribution has **constant** probability density over a fixed interval.

Thus $f(x) = \dfrac{1}{\beta - \alpha}$ is the continuous uniform p.d.f. over the interval $[\alpha, \beta]$ and has a rectangular shape.



Median

By symmetry the median is $\dfrac{\alpha + \beta}{2}$

Mean and Variance

The expected mean is $\mathrm{E}[X] = \mu = \dfrac{\alpha + \beta}{2}$, which is the same as the median.

and the expected variance is $\mathrm{Var}[X] = \sigma^2 = \dfrac{(\beta - \alpha)^2}{12}$.

These formulae are proved in the appendix

12.2 Mean and Variance

Example. example 4-7

12.3 Choosing the right model

Example. example 8-10

13 Normal approximation

13.1 Approximating binomial by normal

If $X \sim B(n, p)$ and

- n is large
- p is close to 0.5

Then X can be approximated by

$$Y \sim N(np, np(1-p))$$

Example. $X \sim B(120, 0.25)$ approximated to $Y \sim N(30, (\sqrt{22.5})^2)$

Example. example 4

13.2 Approximating Poisson by normal

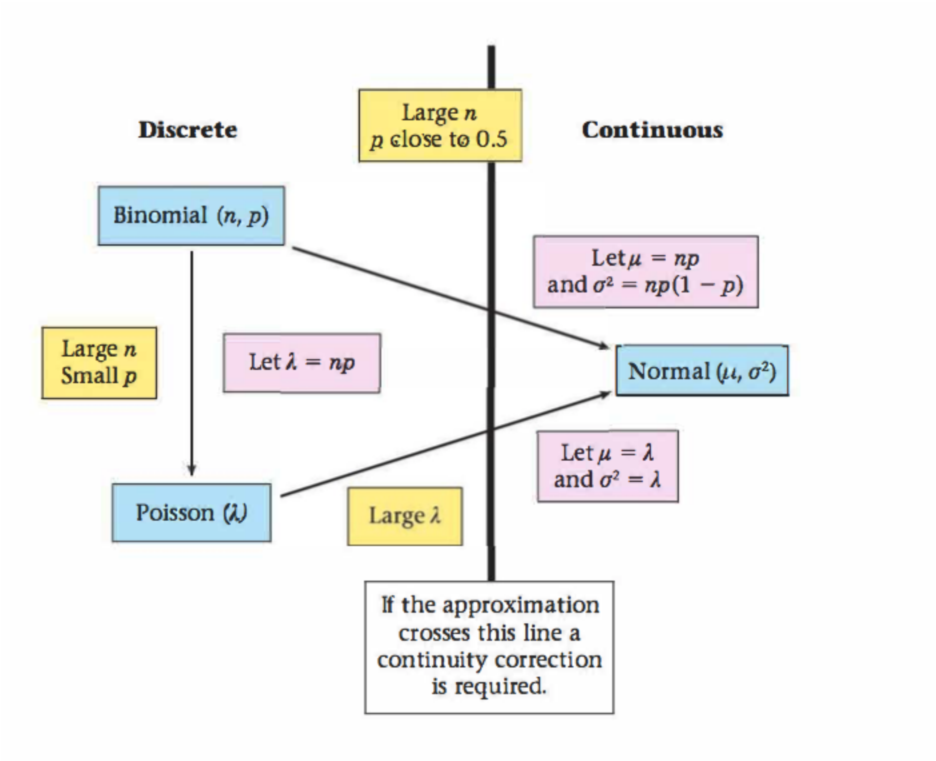
If λ is large

$$X \sim Po(\lambda) \text{to} Y \sim N(\lambda, (\sqrt{\lambda})^2)$$

Example. $X \sim Po(25)$ transformed to $Y \sim N(25, 5^2)$

Example. example 6

13.3 Choosing the appropriate approximation



Example. example 7

14 Population and samples

14.1 The Concept of population and samples

List of the possible samples and find their probabilities and distribution.

Example. example 5 6

15 Hypothesis testing

15.1 Concept of hypothesis testing

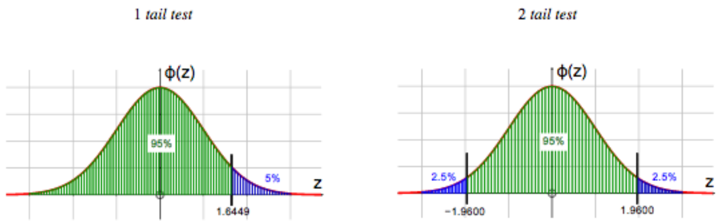
Definition (Null hypothesis H_0)**.** The hypothesis which is assumed to be correct unless shown otherwise.

Definition (Alternarive hypothesis H_1)**.** This is the conclusion that should be made if H_0 is rejected

Definition (Critical region)**.** The range of values which would lead you to reject the null hypothesis, H_0

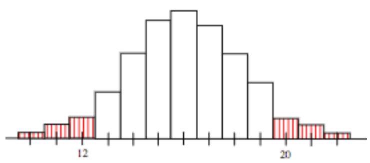
Definition (Significance level)**.** The actual significance level is the probability of rejecting H_0 when it is in fact true.

From your observed result (test statistic) you decide whether to reject or not to reject the null hypothesis H_0



The test statistic is significantat 5%, or that we reject H_0. Thus H_0could actually be true but we still reject it. Thus, the significance level, 5%, is the probability that we reject H_0 when it is in fact true, or the probability of incorrectly rejecting H_0.
When we reject the null hypothesis, H_0, we use the alternative hypothesis to write the conclusion.

The Poisson and Binomial distributions are discrete, and we look at probability histograms.



In the diagram, the critical region (shown by the shaded areas) is $X \leq 12$ or $X \geq 20$.

We include the whole bar around $X = 12$ and around $X = 20$

So $P(X \leq 12)$ is the area to the left of 12.5,
and $P(X \geq 20)$ is the area to the right of 19.5,

If $P(X \leq 12) = 0.0234$ and $P(X \geq 20) = 0.0217$, then
the actual signifaicance level is $0.0234 + 0.0217 = 0.0451 = 4.51\%$
Thus the probability of incorrectly rejecting H_0 is 0.0451.

15.2 One- and two-tailed tests

The One-tail test is

$$\begin{aligned} H_0 :& a = b \\ H_1 :& a > b \text{ or } a < b \end{aligned}$$

The Two-tail test is

$$\begin{aligned} H_0 :& a = b \\ H_1 :& a \neq b \end{aligned}$$

Example. 3

Example. 4-13

16 Combination of random variables

If X_1 and X_2 are independent normal random variables

$$X_1 \sim \mathbb{N}(\mu_1, \sigma_1^2) \text{ and } X_2 \sim \mathbb{N}(\mu_2, \sigma_2^2)$$

then $X_1 + X_2$ and $X_1 - X_2$ are also normal random variables

$$X_1 + X_2 \sim \mathbb{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \text{ and } X_1 - X_2 \sim \mathbb{N}(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

17 Sampling

18 Estimation , confidence intervals and tests

18.1 Estimation

Definition (Biased and unbiased estimator)**.** If X (usually found from a sample) is used to estimate the value of a population parameter, t, then X is an unbiased estimator of t if $E[X] =$ the true value of the parameter t.

If an estimator, X, is biased, then the bias is the difference between $E[X]$ and the true value of the parameter t.

Definition (Unbiased estimators of μ and σ^2)**.**

18.2 Confidence intervals and significance tests

Definition (Sampling distribution of the mean)**.**

$$\mathbb{N}(\mu, \frac{\sigma^2}{n})$$

Theorem (Central limit theorem)**.** If $\{X_1, X_2, \cdots, X_n\}$ is a random sample of size n drawn any population with mean μ and variance σ^2 then the population of sample means.

(i) has expected mean μ

(ii) has expected variance $\frac{\sigma^2}{n}$

(iii) forms a normal distribution if n is 'large enough' i.e. $\bar{X} \sim \mathbb{N}(\mu, \frac{\sigma^2}{n})$

The standard error of the sample mean is $\frac{\sigma}{\sqrt{n}}$

Example.

Example:

A biscuit manufacturer makes packets of biscuits with a nominal weight of 250 grams. It is known that over a long period the variance of the weights of the packets of biscuits produced is 25 grams^2. A sample of 10 packets is taken and found to have a mean weight of 253·4 grams. Find 95% confidence limits for the mean weight of all packets produced by the machine.

Solution:

First assume that the machine is still producing packets with the same variance, 25.

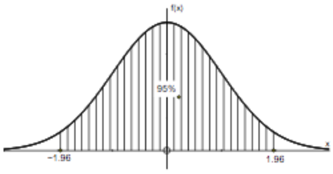
Suppose that the mean weight of all packets of biscuits is μ grams then the population of all packets has mean μ and standard deviation 5.

From the central limit theorem we can assume that the sample means form an approximately normal population with mean μ and standard error (standard deviation) $\frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{10}} = 1{\cdot}5811$

95% of the samples will have a mean in the region

$$-1{\cdot}96 \quad < \quad Z \quad < \quad 1{\cdot}96$$

We assume that the mean of this sample, 253·4, lies in this region



$\Rightarrow \quad -1{\cdot}9600 < \frac{253{\cdot}4-\mu}{1{\cdot}5811} < 1{\cdot}9600$

$\Rightarrow \quad -1{\cdot}9600 < \frac{253{\cdot}4-\mu}{1{\cdot}5811}$ and $\frac{253{\cdot}4-\mu}{1{\cdot}5811} < 1{\cdot}9600$

$\Rightarrow \quad \mu - 1{\cdot}9600 \times 1{\cdot}5811 < 253{\cdot}4$ and $253{\cdot}4 < \mu + 1{\cdot}9600 \times 1{\cdot}5811$

$\Leftrightarrow \quad \mu < 253{\cdot}4 + 1{\cdot}9600 \times 1{\cdot}5811$ and $253{\cdot}4 - 1{\cdot}9600 \times 1{\cdot}5811 < \mu$

$\Leftrightarrow \quad 253{\cdot}4 - 1{\cdot}9600 \times 1{\cdot}5811 < \mu < 253{\cdot}4 + 1{\cdot}9600 \times 1{\cdot}5811$

$\Leftrightarrow \quad 250{\cdot}3 < \mu < 256{\cdot}5$

This means that 95% of the samples will give an interval which contains the mean and we say that [250·3 g, 256·5 g] is a 95% *confidence interval* for μ.

This means that there is a 0·95 probability that **this interval contains the true mean**.

It *does not* mean that there is a probability of 0·95 that the true mean lies in this interval - the true mean is a fixed number, and either *does* or *does not* lie in the interval so the probability that the true mean lies in the interval is either 1 or 0.

In practice we go straight to the last line of the example:

95% confidence limits are $\mu \pm 1{\cdot}9600 \times \frac{\sigma}{\sqrt{n}}$ since $P(Z -1{\cdot}9600 < z < 1{\cdot}9600) = 0{\cdot}95$

tables give $P(Z > 1{\cdot}9600) = 0{\cdot}025$

90% confidence limits are $\mu \pm 1{\cdot}6449 \times \frac{\sigma}{\sqrt{n}}$ since $P(Z -1{\cdot}6449 < z < 1{\cdot}6449) = 0{\cdot}90$

tables give $P(Z > 1{\cdot}6449) = 0{\cdot}05$

Other confidence limits can be found using the Normal Distribution tables.

Example: A sample of 64 packets of cornflakes has a mean weight $\overline{X} = 510$ grams and a variance $S^2 = 36$ grams^2. Find 90% confidence limits for the mean weight of all packets.

(Note that the 'sample variance' is taken as the unbiased estimate of σ^2.)

Solution: We **assume** that the sample variance = the variance of the population of all packets

$\Rightarrow \quad S^2 = 36 = \sigma^2$.

Now find standard deviation (standard error) of the sampling distribution of the mean (population of sample means), standard error = $\frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{64}} = 0 \cdot 75$

For 90% confidence limits $z = \pm 1{\cdot}6449$ (remember to use the 4 D.P. tables after the Normal Dist. tables), using the sample mean $\overline{X} = 510$ grams

$\Rightarrow$ 90% confidence limits are $510 \pm 1{\cdot}6449 \times 0.75 = 510 \pm 1{\cdot}234$

$\Rightarrow$ a 90% confidence interval is $[508{\cdot}8,\ 511{\cdot}2]$ to 4 S.F.

Note that we have assumed that the unbiased estimate, S^2 (=36), is the actual variance, σ^2, of the population.

This is a reasonable assumption as the number in the sample, 64, is large and the error introduced is therefore small.

Example.

Significance testing– variance of population known

Mean of normal distribution

Example:

A machine, when correctly set, is known to produce ball bearings with a mean weight of 84 grams with a standard deviation of 5 grams. The production manager decides to test whether the machine is working correctly and takes a sample of 120 ball bearings. The sample has mean weight 83.2 grams. Would you advise the production manager to alter the setting of his machine? Use a 5% significance level.

Solution:

1) H_0: $\mu = 84$ grams

2) H_1: $\mu \neq 84$ grams $\Rightarrow$ 2 tail test

(Note that the machine is not working correctly if the test result is too high *or* too low)

3) *5% Significance level*

4) *The Test*

We assume that the machine is still working with a standard deviation of $\sigma = 5\ g$.

From H_0, the mean weight of all ball bearings is assumed to be $\mu = 84\ g$.

These are the parameters for the population of **all** ball bearings.

We want to test a sample mean and therefore need the mean and standard deviation of the population of sample means (the sampling distribution of the sample mean, $\bar{X}$).

Expected mean of the sample means = $\mu = 84\ g$. and

expected standard deviation of the sample means = standard error = $\frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{120}} = 0{\cdot}456435...$

.

We have an observed mean of 83·2

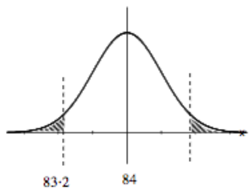
For a two-tailed test at 5%, we take 2·5% at each end

$P(\bar{X} < 83{\cdot}2) = \Phi\left(\frac{83{\cdot}2-84}{0{\cdot}456435...}\right) = \Phi(-1{\cdot}7527)$

$= (1 - \Phi(1{\cdot}75)) = 0{\cdot}0401$

$= 4{\cdot}01\% > 2{\cdot}5\%$

and so not significant at the 5% level.



5) *Conclusion*

Do not reject H_0 at the 5% level and advise the production manager that there is evidence that he should not change his setting, or that there is evidence that the machine is working correctly, etc.

Fortunately the formula for testing the difference between sample means

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}}$$

18.3 Combination of sampling distribution

Same

19 Goodness of fit and contingency tables

20 Regression and correlation

21 Quality of tests and estimators

22 One-sample procedures

23 Two-sample procedures