Part III — Mechanics

Based on lectures by Brian
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11.1 Forces which vary with speed

Proposition.

$$\mathbf{a} = \mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}$$

Proof.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{t}} \times \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}} = \mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}$$

□

12 Elastic strings and springs

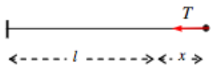
12.1 Hooke's Law

Law (Hooke's Law)**.** There are two cases for using Hooke's Law

(i) Elastic strings: The tension T in an elastic string is

$$T = \frac{\lambda x}{l}$$

where
l is the natural (unstretched) length of the string,
x is the extension and
λ is the modulus of elasticity

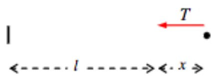


When the string is slack there is no tension.

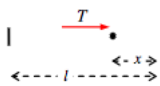
(ii) Elastic springs: The tension, or thrust, T is an elastic spring is

$$T = \frac{\lambda x}{l}$$

where
l is the natural (unstretched) length of the string,
x is the extension or compression and
λ is the modulus of elasticity



Tension (stretched)



Thrust (compressed)

12.2 Energy stored in an elastic string or spring

Like kinematics,If there is force F and displacement traveled δs, the Work done is $\delta W = F\delta s$. Similarly, If the tension force is T and string/spring extended/stretched, then

$$\delta W \approx T\delta x$$

Total work done in exrending from $x = 0$ to $x = X$ is approximately

$$\sum_{0}^{X} T\delta x$$

and , as $\delta x \to 0$, the total work done:

$$W = \int_0^X T\mathrm{d}x = \int_0^X \frac{\lambda x}{l}\mathrm{d}x = \frac{\lambda x^2}{2l}$$

The expression of Total work done is also called the Elastic Potential Energy

13 Further dynamics

13.1 Impulse of a variable force

$$\delta I \approx F(t)\delta t$$

The total impulse from time t_1 to t_2 is

$$I \approx \sum_{t_1}^{t_2} F(t)\delta t$$

and as $\delta t \to 0$, the total impulse is

$$I = \int_{t_1}^{t_2} F(t)\mathrm{d}t$$

Also, as $F(t) = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}$

$$\int_{t_1}^{t_2} F(t)\mathrm{d}t = \int_U^V m\mathrm{d}v = mV - mU$$

13.2 Work done by a variable force

$$\delta W \approx G(x)\delta x$$

and the total work done in moving from a displacement x_1 to x_2 is

$$W \approx \sum_{x_1}^{x_2} G(x)\delta x$$

and as $\delta x \to 0$, the total work done is

$$W = \int_{x_1}^{x_2} G(x)\mathrm{d}x$$

Also $G(x) = ma = m\frac{\mathrm{d}v}{\mathrm{d}x} = m\frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}v}{\mathrm{d}x} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$

$$\int_{x_1}^{x_2} G(x)\mathrm{d}x = \int_U^V mv\mathrm{d}v = \frac{1}{2}mV^2 - \frac{1}{2}mU^2$$

13.3 Newton's Law of Gravitation

Law. The force of attraction between two bodies of masses M_1 and M_2 is directly proportional to the product of their masses and inversely proportional to the square of the distance, d , between them:

$$F = \frac{GM_1M_2}{d^2}$$

where G is a constant known as the constant of Gravitation

13.4 Finding k in $F = \frac{k}{x^2}$

$$F = ma = \frac{k}{d^2}$$

13.5 Simple harmonic motion S.H.M.

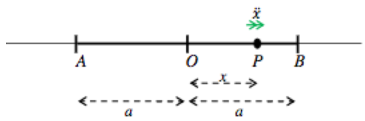
Definition (S.H.M. equation)**.** If a particle , P , moves in a straight line so that its acceleration is proportional to its distance from a fixed point O , and directed towards O , then

$$\ddot{x} = -\omega^2 x$$

and the particle will oscillate between two points, A and B , with simple harmonic motion.

The amplitude of the oscillation is $OA = OB = a$.

Notice that $\ddot{x}$ is marked in the direction of x increasing n the diagram , and, since ω^2 is positive, $\ddot{x}$ is negative , so the acceleration acts towards O.



Proposition (Solving equation)**.** A.E. is

$$m^2 = -\omega^2 \to m = i\omega$$

G.S. is

$$x = \lambda \sin \omega t + \mu \cos \omega t$$

If x starts from O, $x = O$ when $t = 0$,
then

$$x = a \sin \omega t$$

If x starts from B, $x = a$ when $t = 0$,
then

$$x = a \cos \omega t$$

Definition (Period and amplitude)**.** From the equations $x = a \sin \omega t$ and $x = a \cos \omega t$
we can see that the period, the time for one complete oscillation, is

$$T = \frac{2\pi}{\omega}$$

The period is the time taken to go from $O \to B \to A \to O$, or from $B \to A \to B$ and that the amplitude, maximum distance from the central point, is a.

Proposition (Alternative equation of S.H.M.)**.**

$$v^2 = \omega^2(a^2 - x^2)$$

Proof. Consider the basic S.H.M. equation $\ddot{x} = -\omega^2 x$ and $\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 x$$
$$\int v\mathrm{d}v = \int -\omega^2 x\mathrm{d}x$$
$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}c$$

But $v =)$ when x at its maxumum, $x = a \to c = a^2\omega^2$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}a^2\omega^2$$
$$v^2 = \omega^2(a^2 - x^2)$$

□

Horizontal

Example.

Vertical (relate to mg)

Example.

14 Motion in a circle

14.1 Angular velocity

A particle moves in a circle of radius r with constant speed, v.

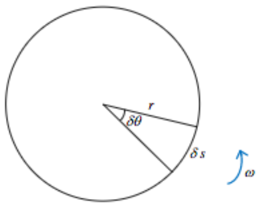
Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, then the distance moved will be $\delta s = r\delta\theta$
and its speed $v = \frac{\delta s}{\delta t} = r\frac{\delta\theta}{\delta t}$

and , as $\delta t \to 0$, $v = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\theta$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta$$

is the angular velocity, usually written as the Greek letter omega, ω, and so, for a particle moving in a circle with radius r, its speed is

$$v = r\omega$$



14.2 Acceleration

A particle moves in a circle of radius r with constant speed , v.

Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, and that its velocity changes from v_1 to v_2,

then its change in velocity is $\delta v = v_2 - v_1$, which is shown in the second diagram.

The lengths of both v_1 and v_2 are v, and the angle between v_1 and v_2 is $\delta\theta$.

$$\delta v = 2 \times v \sin\frac{\delta\theta}{2} \approx 2v \times \frac{\delta\theta}{2} = v\delta\theta\frac{\delta v}{\delta t} \qquad\qquad \approx v\frac{\delta\theta}{\delta t}$$

as $\delta t \to 0$, acceleration:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}\theta}{\mathrm{d}t} = v\theta$$

But

$$\theta = \omega = \frac{v}{r} \to a = \frac{v^2}{r} = r\omega^2$$

Notice that as $\delta\theta \to 0$, the direction of δv becomes perpendicular to both v_1 and v_2, and so is directed towards the centre of the circle.

The acceleration of a particle moving in a circle with speed v is $a = r\omega^2 = \frac{v^2}{r}$, and is directed towards the centre of the circle.

Alternative proof

Proof. If a particle moves , with constant speed, in a circle of radius r and centre O, then its position vector can be written:

$$\mathbf{r} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \to \dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta\dot{\theta} \\ \cos\theta\dot{\theta}\end{pmatrix}$$

Particle moves with constant speed $\to \dot{\theta} = \omega$ is constant

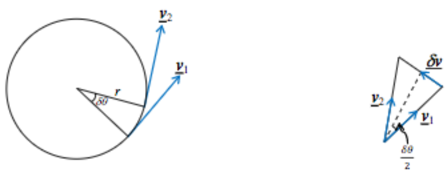
$$\dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} \to v = r\omega$$

$$\ddot{\mathbf{r}} = r\omega\begin{pmatrix}-\cos\theta\dot{\theta} \\ -\sin\theta\dot{\theta}\end{pmatrix} = -\omega^2 r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = -\omega^2\mathbf{r}$$

acceleration is

$$r\omega^2 \text{ or } \frac{v^2}{r}$$

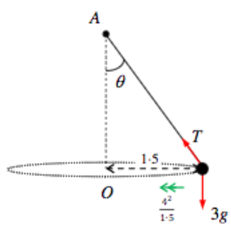
directed towards O. □



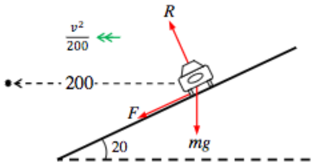
Types of problems:

(i) Horizontal

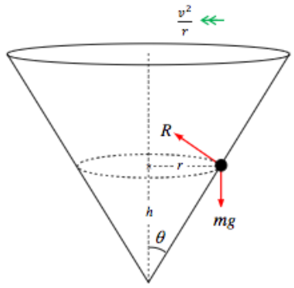
(ii) Conical pendulum



(iii) Banking



(iv) Inside an inverted vertical cone



14.3 Motion in a vertical circle

Proposition.

$$a = \frac{v^2}{r}$$

Proof. If a particle moves in a circle of radius r and centre O, then its position vector can be written:

$$\mathbf{r} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta\dot{\theta} \\ \cos\theta\dot{\theta}\end{pmatrix} = r\dot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

$$\ddot{\mathbf{r}} = r\begin{pmatrix}-\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta} \\ -\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\end{pmatrix} = -r\dot{\theta}^2\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + r\ddot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

From this we can see that the speed is $v = r\dot{\theta} = r\omega$,
and is perpendicular to the radius since $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$

We can also see that the acceleration has two components

$$r\dot{\theta}^2 = r\omega^2 = \frac{v^2}{r}$$

towards the centre opposite direction to $\mathbf{r}$

and $r\ddot{\theta}$ perpendicular to the radius which is what we should expect since $v = r\dot{\theta}$ and r is constant.

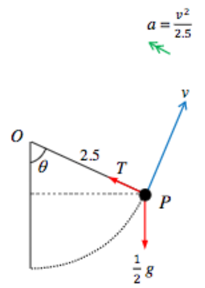
In practice we shall onlu use

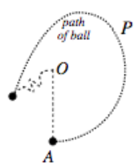
$$a = r\omega^2 = \frac{v^2}{r}$$

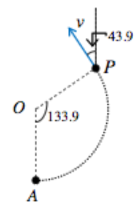
directed towards the centre of the circle □

Types of problems

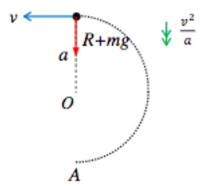
(i) A particle attached to an inextensible string



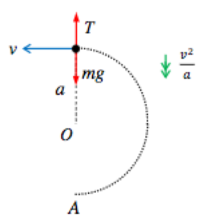




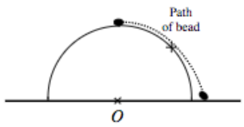
(ii) A particle moving on the indside of a smooth, hollow sphere

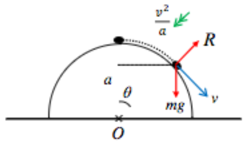


(iii) A particle attached to a rod



(iv) A particle moving on the outside of a smooth sphere





15 Statics of rigid bodies 2

15.1 Centre of mass

When finding a centre of mass

Centres of mass depend on the formula :

$$M\bar{x} = \sum m_i x_i$$

or Similar, Remember that

$$\lim_{\delta x \to 0} \sum f(x_i)\delta x = \int f(x)\mathrm{d}x$$

15.2 Centre of mass of geometric shapes

15.2.1 Sector

In this case we can find a nice method, using the result for the centre of mass of a triangle.
We take a sector of angle 2α and divide it into many smaller sectors.

Mass of whole sector

$$M = \frac{1}{2}r^2 \times 2\alpha \times \rho = r^2\alpha\rho$$

Consuder each small sector as approximately a triangle, with centre of mass, G_1, 2/3 along the median from O

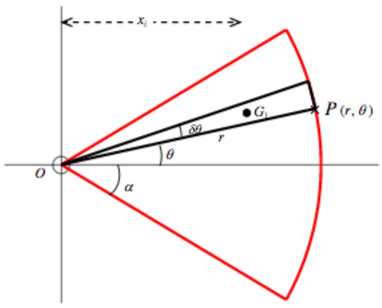
Working in polar coordinates for one small sector,

$$m_i = \frac{1}{2}r^2\rho\delta\theta$$

$$OP = r \to OG_1 \cong \frac{2}{3}r \to x_i \cong \frac{2}{3}r\cos\theta$$

$$\begin{aligned}\lim_{\delta\theta \to 0} \sum_{\theta=-\alpha}^{\alpha} m_i x_i &= \int_{-\alpha}^{\alpha} \frac{1}{2}r^2\rho \times \frac{2}{3}r\cos\theta\mathrm{d}\theta \\ &= \frac{2}{3}r^3\rho\sin\alpha \\ \bar{x} &= \frac{\sum m_i x_i}{M} = \frac{\frac{2}{3}r^3\rho\sin\alpha}{r^2\alpha\rho} = \frac{2r\sin\alpha}{3\alpha}\end{aligned}$$

By symmetry, $\bar{y} = 0$
centre of mass is at $(\frac{2r\sin\alpha}{3\alpha}, 0)$



15.2.2 Circular arc

For a circular arc of radius r which subtends an angle of 2α at the centre.

The length of the arc is $r \times 2\alpha$
The mass of the arc is $M = 2\alpha r\rho$

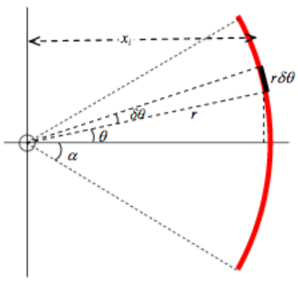
First divide the arc into several small pieces, each subtending an angle of $\delta\theta$ at the centre

The length of each piece is $r\delta\theta \rightarrow m_i = r\rho\delta\theta$

We now think of each small arc as a point mass at the centre of the arc, with x-corrdinate $x_i = r\cos\theta$

$$\begin{aligned}\lim_{\delta\theta\to 0}\sum_{\theta=-\alpha}^{\alpha} m_i x_i &= \int_{-\alpha}^{\alpha} r\rho \times r\cos\theta \mathrm{d}\theta \\ &= 2r^2\rho\sin\alpha \\ \bar{x} &= \frac{\sum m_i x_i}{M} = \frac{2r^2\rho\sin\alpha}{2r\alpha\rho} = \frac{r\sin\alpha}{\alpha}\end{aligned}$$

By symmetry, $\bar{y} = 0$
centre of mass is at $(\frac{r\sin\alpha}{\alpha}, 0)$



15.2.3 Others

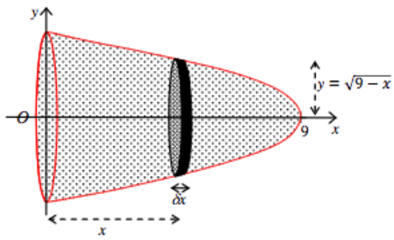
Standard results for centre of mass of uniform laminas and arcs

Triangle	$\frac{2}{3}$ of the way along the median, from the vertex.
Semi-circle, radius r	$\frac{4r}{3\pi}$ from centre, along axis of symmetry
Sector of circle, radius r, angle 2α	$\frac{2r\sin\alpha}{3\alpha}$ from centre, along axis of symmetry
Circular arc, radius r, angle 2α	$\frac{r\sin\alpha}{\alpha}$ from centre, along axis of symmetry

15.2.4 Solid of revolution

Example: A machine component has the shape of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$, $x \geq 0$, about the x-axis. Find the position of the centre of mass.

Solution:



$$\text{Mass, } M \text{, of the solid} \;=\; \rho\int_0^9 \pi y^2\,dx = \rho\int_0^9 \pi(9-x)\,dx$$

$$\Rightarrow \quad M = \tfrac{81}{2}\rho\pi.$$

The diagram shows a typical thin disc of thickness δx and radius $y = \sqrt{9-x}$.

$$\Rightarrow \quad \text{Mass of disc} \approx \rho\pi y^2\delta x = \rho\pi(9-x)\,\delta x$$

Note that the x coordinate is the same (nearly) for all points in the disc

$$\Rightarrow \sum m_i x_i \approx \sum_0^9 \rho\pi(9-x_i)x_i\,\delta x$$

$$\lim_{\delta x\to 0}\sum m_i x_i = \int_0^9 \rho\pi(9-x)x\,dx = \tfrac{243}{2}\rho\pi$$

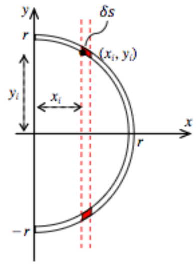
$$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\frac{243}{2}\rho\pi}{\frac{81}{2}\rho\pi} = 3$$

By symmetry, $\bar{y} = 0$

$\Rightarrow$ the centre of mass is on the x-axis, at a distance of 3 from the origin.

15.2.5 Hemispherical shell

Mass of shell



Let the density of the shell be ρ, radius r

In the xy-plane, the curve has equation

$$x^2 + y^2 = r^2$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow \quad \left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{y^2 + x^2}{y^2}}$$

Take a slice perpendicular to the x-axis through the point (x_i, y_i) to form a ring with arc length δs.

Area of the ring $\cong 2\pi y_i \delta s \quad \Rightarrow \quad$ mass of ring $m_i \cong 2\pi y_i \rho\, \delta s$

$$\Rightarrow \text{ Total mass } \cong \sum 2\pi y_i \rho\, \delta s$$

$$\Rightarrow \text{ Total mass } M = \lim_{\delta s \to 0} \sum 2\pi y_i \rho\, \delta s = \int 2\pi y \rho\, ds$$

$$\Rightarrow \quad M = \int_0^r 2\pi y \rho \frac{ds}{dx}\, dx = \int_0^r 2\pi y \rho \sqrt{\frac{y^2 + x^2}{y^2}}\, dx$$

$$\Rightarrow \quad M = \int_0^r 2\pi \rho \sqrt{r^2}\, dx = 2\pi \rho r \Big[x \Big]_0^r = 2\pi \rho r^2$$

To find $\sum m_i x_i = \sum 2\pi y_i\, \rho\, \delta s\, x_i$

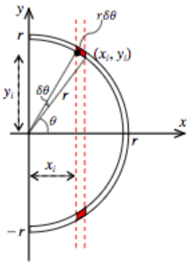
$$\Rightarrow \lim_{\delta s \to 0} \sum 2\pi y_i\, \rho\, \delta s\, x_i = \int_0^r 2\pi \rho\, y x \frac{ds}{dx}\, dx$$

$$= \int_0^r 2\pi \rho\, y x \sqrt{\frac{y^2 + x^2}{y^2}}\, dx = 2\pi \rho r \left[\frac{x^2}{2}\right]_0^r = \pi \rho r^3$$

$$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\pi \rho r^3}{2\pi \rho r^2} = \frac{r}{2}$$

$\Rightarrow$ the centre of mass is on the line of symmetry at a distance of $\frac{1}{2}r$ from the centre.

Mass of shell



Let the density of the shell be ρ, radius r

Take a slice perpendicular to the x-axis through the point (x_i, y_i) to form a ring with arc length $r\delta\theta$, and circumference $2\pi y$. This can be 'flattened out' to form a rectangle of length $2\pi y$ and height $r\delta\theta$

Area of the ring $\cong$

$\Rightarrow$ mass of ring $m_i \cong 2\pi\, \rho\, y x_i r\delta\theta$

$\Rightarrow$ Total mass $\cong \sum 2\pi y r \rho\, \delta\theta$

$\Rightarrow$ Total mass $M = \lim_{\delta\theta \to 0} \sum 2\pi y r \rho\, \delta\theta \;=\; \int 2\pi y r \rho\, d\theta$

But $y = r \sin\theta$

$$\Rightarrow \quad M = \int_0^{\frac{\pi}{2}} 2\pi r^2 \sin\theta\, \rho\, d\theta \;=\; 2\pi\rho r^2 \Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} \;=\; 2\pi\rho r^2$$

To find $\sum m_i x_i \;=\; \sum 2\pi\, y_i r\, \rho\, \delta\theta\, x_i$

$$\Rightarrow \lim_{\delta\theta \to 0} \sum 2\pi\, y_i r\, \rho\, \delta s\, x_i \;=\; \int_0^{\frac{\pi}{2}} 2\pi\rho\, r\, yx \quad d\theta$$

But $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow \sum m_i x_i \;=\; \int_0^{\frac{\pi}{2}} 2\pi\rho\, r^3 \sin\theta \cos\theta \quad d\theta$$

$$= \pi\rho\, r^3 \left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} \;=\; \pi\rho\, r^3$$

$$\Rightarrow \qquad \bar{x} \;=\; \frac{\sum m_i x_i}{M} \;=\; \frac{\pi\rho\, r^3}{2\pi\rho r^2} \;=\; \frac{r}{2}$$

$\Rightarrow$ the centre of mass is on the line of symmetry at a distance of $\frac{1}{2}r$ from the centre.

15.2.6 conical

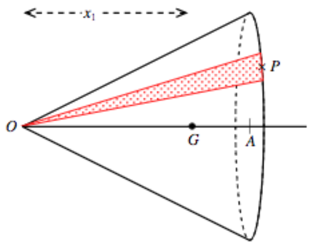
Centre of mass of a conical shell

To find the centre of mass of a conical shell, or the surface of a cone, we divide the surface into small sectors, one of which is shown in the diagram.

We can think the small sector as a triangle with centre of mass at G_1, where $OG_1 = \frac{2}{3}OP$.

This will be true for all the small sectors, and the x-coordinate, x_1, of each sector will be the same

$\Rightarrow$ the x-coordinate of the shell will also be x_1



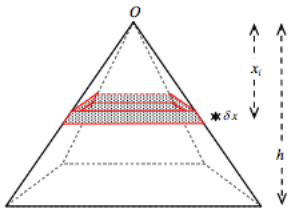
As the number of sectors increase, the approximation gets better, until it is exact,

and as $OG_1 = \frac{2}{3}OP$ then $OG = \frac{2}{3}OA$ (similar triangles)

$\Rightarrow$ the centre of mass of a conical shell is on the line of symmetry, at a distance of $\frac{2}{3}$ of the height from the vertex.

15.2.7 Square based pyramid

Centre of mass of a square based pyramid



A square based pyramid has base area A and height h

The centre of mass is on the line of symmetry

$\Rightarrow$ volume $= \frac{1}{3}Ah$

$\Rightarrow$ mass $M = \frac{1}{3}Ah\rho$

Take a slice of thickness δx at a distance x_i from O

The base of the slice is an enlargement of the base of the pyramid with scale factor $\frac{x_i}{h}$

$\Rightarrow$ ratio of areas is $\left(\frac{x_i}{h}\right)^2$

$\Rightarrow$ area of base of slice is $\frac{x_i^{\,2}}{h^2}A$

$\Rightarrow$ mass of slice $m_i = \delta x$

$$\Rightarrow \lim_{\delta x \to 0} \sum_{x=0}^{h} m_i x_i = \int_0^h \frac{x^3}{h^2} A\rho \, dx = \tfrac{1}{4}h^2 A\rho$$

$$\Rightarrow \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\frac{1}{4}h^2 A\rho}{\frac{1}{3}Ah\rho} = \tfrac{3}{4}h$$

The centre of mass lies on the line of symmetry at a distance $\frac{3}{4}h$ from the vertex.

The above technique will work for a pyramid with any shape of base.

The centre of mass of a pyramid with any base has centre of mass $\frac{3}{4}$ of the way along the line from the vertex to the centre of mass of the base (considered as a lamina).

There are more examples in the book, but the basic principle remains the same:

- *find the mass of the shape, M*
- *choose, carefully, a typical element, and find its mass (involving δx or δy)*
- *for solids of revolution about the x-axis (or y-axis), choose a disc of radius y and thickness δx, (or radius x and thickness δy).*
- *find* $\sum m_i x_i$ or $\sum m_i y_i$
- *let* δx *or* $\delta y \to 0$, *and find the value of the resulting integral*
- $\bar{x} = \frac{1}{M}\sum m_i x_i$, $\bar{y} = \frac{1}{M}\sum m_i y_i$

15.2.8 The standard results

Standard results for centre of mass of uniform bodies

Solid hemisphere, radius r	$\frac{3r}{8}$ from centre, along axis of symmetry
Hemispherical shell, radius r	$\frac{r}{2}$ from centre, along axis of symmetry
Solid right circular cone, height h	$\frac{3h}{4}$ from vertex, along axis of symmetry
Conical shell, height h	$\frac{2h}{3}$ from vertex, along axis of symmetry

Centre of mass of a hemispherical shell – method 2

Note: if you use this method in an exam question which asks for a calculus technique, you would have to use calculus to prove the results for a solid hemisphere first.

The best technique for those who have not done FP3 is method 1b.

We can use the theory for compound bodies to find the centre of mass of a hemispherical shell.

From a hemisphere with radius $r + \delta r$ we remove a hemisphere with radius r, to form a hemispherical shell of thickness δr and inside radius r.

radius	$r + \delta r$	r	
Mass	$\frac{2}{3}\pi(r+\delta r)^3\rho$	$\frac{2}{3}\pi r^3\rho$	$\frac{2}{3}\pi(r+\delta r)^3\rho - \frac{2}{3}\pi r^3\rho$
centre of mass above base	$\frac{3}{8}(r+\delta r)$	$\frac{3}{8}r$	$\overline{y}$

$$\Rightarrow \quad \tfrac{2}{3}\pi(r+\delta r)^3\rho \times \tfrac{3}{8}(r+\delta r) \; - \; \tfrac{2}{3}\pi r^3\rho \times \tfrac{3}{8}r \; = \; \{\tfrac{2}{3}\pi(r+\delta r)^3\rho - \tfrac{2}{3}\pi r^3\rho\}\overline{y}$$

$$\Rightarrow \quad \tfrac{1}{4}\pi\rho(r^4 + 4r^3\delta r \ldots - r^4) = \tfrac{2}{3}\pi\rho(r^3 + 3r^2\delta r \ldots - r^3)\,\overline{y} \qquad \text{ignoring } (\delta r)^2 \text{ and higher}$$

$$\Rightarrow \quad r^3\delta r \cong 2r^2\,\delta r\,\overline{y}$$

and as $\delta r \to 0$, $\overline{y} = \frac{1}{2}r$

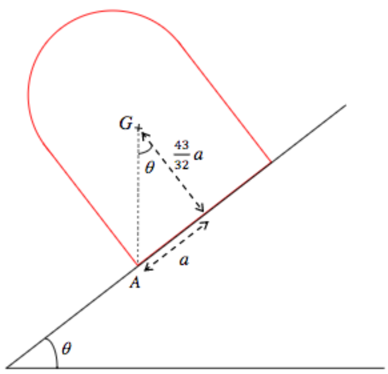
The centre of mass of a hemispherical is on the line of symmetry, $\frac{1}{2}r$ from the centre.

15.2.9 Tilting and hanging freely

Tilting and hanging freely

Tilting

Example: The compound body of the previous example is placed on a slope which makes an angle θ with the horizontal. The slope is sufficiently rough to prevent sliding. For what range of values of θ will the body remain in equilibrium.



Solution: The body will be on the point of tipping when the centre of mass, G, lies vertically above the lowest corner, A.

Centre of mass is $2a - \frac{21}{32}a$

$= \frac{43}{32}a$ from the base

At this point

$\tan\theta = \frac{a}{43a/32} = \frac{32}{43}$

$\Rightarrow \theta = 36{\cdot}65610842$

The body will remain in equilibrium for

$\theta \le 36{\cdot}7^\circ$ to the nearest $0{\cdot}1^\circ$.

Hanging freely under gravity

This was covered in M2. For a body hanging freely from a point A, you should always state, or show clearly in a diagram, that AG is vertical – this is the only piece of mechanics in the question!

Body with point mass attached hanging freely

The best technique will probably be to take moments about the point of suspension.

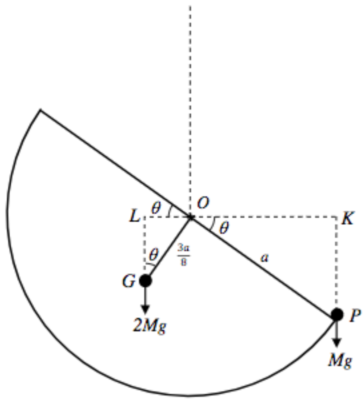
Example: A solid hemisphere has centre O, radius a and mass $2M$. A particle of mass M is attached to the rim of the hemisphere at P.

The compound body is freely suspended under gravity from O. Find the angle made by OP with the horizontal.

Solution: As usual a *good, large diagram* is essential.

Let the angle made by OP with the horizontal be θ, then $\angle OGL = \theta$.

We can think of the hemisphere as a point mass of $2M$ at G, where $OG = \frac{3a}{8}$.



The perpendicular distance from O to the line of action of $2Mg$ is $OL = \frac{3a}{8}\sin\theta$, and

the perpendicular distance from O to the line of action of Mg is $OK = a\cos\theta$

Taking moments about O

$$2Mg \times \frac{3a}{8}\sin\theta = Mg \times a\cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{4}{3}$$

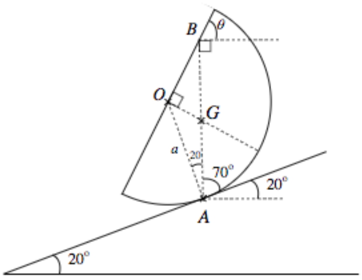
$$\Rightarrow \quad \theta = 53{\cdot}1^\circ.$$

15.2.10 Hemisphere in equilibrium on a slope

Hemisphere in equilibrium on a slope

Example: A uniform hemisphere rests in equilibrium on a slope which makes an angle of 20° with the horizontal. The slope is sufficiently rough to prevent the hemisphere from sliding. Find the angle made by the flat surface of the hemisphere with the horizontal.

Solution: *Don't forget the basics.*



The centre of mass, G, must be vertically above the point of contact, A. If it was not, there would be a non-zero moment about A and the hemisphere would not be in equilibrium.

BGA is a vertical line, so we want the angle θ.

OA must be perpendicular to the slope (radius $\perp$ tangent), and with all the 90° angles around A, $\angle OAG = 20^\circ$.

Let a be the radius of the hemisphere

then $OG = \frac{3a}{8}$ and, using the sine rule

$$\frac{\sin \angle OGA}{a} = \frac{\sin 20}{3a/8} \Rightarrow \angle OGA = 65{\cdot}790\ldots. \text{ or } 114{\cdot}209\ldots$$

Clearly $\angle OGA$ is obtuse $\Rightarrow$ $\angle OGA = 114{\cdot}209\ldots$

$\Rightarrow \angle OBG = 114{\cdot}209\ldots - 90 = 24{\cdot}209\ldots$

$\Rightarrow \theta = 90 - 24{\cdot}209\ldots = 65{\cdot}8^\circ$ to the nearest $0{\cdot}1^\circ$.

16 Relative motion

17 Elastic collisions in two dimensions

18 Resisted motion of a particle moving in a straight line

19 Damped and forced harmonic motion

20 Stability

21 Applications of vectors in mechanics

22 Variable mass

23 Moments of inertia of a rigid body

24 Rotation of a rigid body about a fixed smooth axis