Part III — Mechanics

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2 Dynamics of a particle moving in a straight line

3 Statics of a particle

4 Moments

5 Vectors

6 Kinematics of a particle moving in a straight line or plane

7 Centres of mass

8 Work , energy and power

9 Collisions

10 Statics of rigid bodies 1

11 Further kinematics

11.1 Forces which vary with speed

Proposition.

$$\mathbf{a} = \mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}$$

Proof.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{t}} \times \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}} = \mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}$$

□

12 Elastic strings and springs

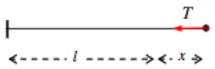
12.1 Hooke's Law

Law (Hooke's Law)**.** There are two cases for using Hooke's Law

(i) Elastic strings: The tension T in an elastic string is

$$T = \frac{\lambda x}{l}$$

where
l is the natural (unstretched) length of the string,
x is the extension and
λ is the modulus of elasticity



When the string is slack there is no tension.

(ii) Elastic springs: The tension, or thrust, T is an elastic spring is

$$T = \frac{\lambda x}{l}$$

where
l is the natural (unstretched) length of the string,
x is the extension or compression and
λ is the modulus of elasticity

Tension (stretched) *Thrust (compressed)*

12.2 Energy stored in an elastic string or spring

Like kinematics,If there is force F and displacement traveled δs, the Work done is $\delta W = F\delta s$. Similarly, If the tension force is T and string/spring extended/stretched, then

$$\delta W \approx T\delta x$$

Total work done in exrending from $x = 0$ to $x = X$ is approximately

$$\sum_{0}^{X} T\delta x$$

and , as $\delta x \to 0$, the total work done:

$$W = \int_0^X T\mathrm{d}x = \int_0^X \frac{\lambda x}{l}\mathrm{d}x = \frac{\lambda x^2}{2l}$$

The expression of Total work done is also called the Elastic Potential Energy

13 Further dynamics

13.1 Impulse of a variable force

$$\delta I \approx F(t)\delta t$$

The total impulse from time t_1 to t_2 is

$$I \approx \sum_{t_1}^{t_2} F(t)\delta t$$

and as $\delta t \to 0$, the total impulse is

$$I = \int_{t_1}^{t_2} F(t)\mathrm{d}t$$

Also, as $F(t) = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}$

$$\int_{t_1}^{t_2} F(t)\mathrm{d}t = \int_{U}^{V} m\mathrm{d}v = mV - mU$$

13.2 Work done by a variable force

$$\delta W \approx G(x)\delta x$$

and the total work done in moving from a displacement x_1 to x_2 is

$$W \approx \sum_{x_1}^{x_2} G(x)\delta x$$

and as $\delta x \to 0$, the total work done is

$$W = \int_{x_1}^{x_2} G(x)\mathrm{d}x$$

Also $G(x) = ma = m\frac{\mathrm{d}v}{\mathrm{d}x} = m\frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}v}{\mathrm{d}x} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$

$$\int_{x_1}^{x_2} G(x)\mathrm{d}x = \int_{U}^{V} mv\mathrm{d}v = \frac{1}{2}mV^2 - \frac{1}{2}mU^2$$

13.3 Newton's Law of Gravitation

Law. The force of attraction between two bodies of masses M_1 and M_2 is directly proportional to the product of their masses and inversely proportional to the square of the distance, d , between them:

$$F = \frac{GM_1M_2}{d^2}$$

where G is a constant known as the constant of Gravitation

13.4 Finding k in $F = \frac{k}{x^2}$

$$F = ma = \frac{k}{d^2}$$

13.5 Simple harmonic motion S.H.M.

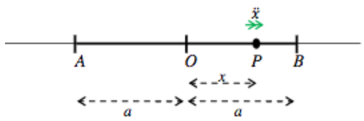
Definition (S.H.M. equation)**.** If a particle , P , moves in a straight line so that its acceleration is proportional to its distance from a fixed point O , and directed towards O , then

$$\ddot{x} = -\omega^2 x$$

and the particle will oscillate between two points, A and B , with simple harmonic motion.

The amplitude of the oscillation is $OA = OB = a$.

Notice that $\ddot{x}$ is marked in the direction of x increasing n the diagram , and, since ω^2 is positive, $\ddot{x}$ is negative , so the acceleration acts towards O.



Proposition (Solving equation)**.** A.E. is

$$m^2 = -\omega^2 \to m = i\omega$$

G.S. is

$$x = \lambda \sin \omega t + \mu \cos \omega t$$

If x starts from O, $x = O$ when $t = 0$,
then

$$x = a \sin \omega t$$

If x starts from B, $x = a$ when $t = 0$,
then

$$x = a \cos \omega t$$

Definition (Period and amplitude)**.** From the equations $x = a \sin \omega t$ and $x = a \cos \omega t$
we can see that the period, the time for one complete oscillation, is

$$T = \frac{2\pi}{\omega}$$

The period is the time taken to go from $O \to B \to A \to O$, or from $B \to A \to B$ and that the amplitude, maximum distance from the central point, is a.

Proposition (Alternative equation of S.H.M.)**.**

$$v^2 = \omega^2(a^2 - x^2)$$

Proof. Consider the basic S.H.M. equation $\ddot{x} = -\omega^2 x$ and $\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 x$$

$$\int v \mathrm{d}v = \int -\omega^2 x \mathrm{d}x$$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}c$$

But $v =)$ when x at its maxumum, $x = a \to c = a^2\omega^2$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2x^2 + \frac{1}{2}a^2\omega^2$$
$$v^2 = \omega^2(a^2 - x^2)$$

□

Horizontal

Example.

Vertical (relate to mg)

Example.

14 Motion in a circle

14.1 Angular velocity

A particle moves in a circle of radius r with constant speed, v.

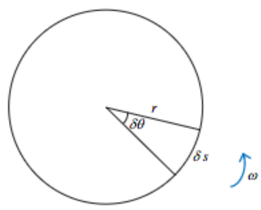
Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, then the distance moved will be $\delta s = r\delta\theta$
and its speed $v = \frac{\delta s}{\delta t} = r\frac{\delta\theta}{\delta t}$

and , as $\delta t \to 0$, $v = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\theta$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta$$

is the angular velocity, usually written as the Greek letter omega, ω, and so, for a particle moving in a circle with radius r, its speed is

$$v = r\omega$$



14.2 Acceleration

A particle moves in a circle of radius r with constant speed , v.

Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, and that its velocity changes from v_1 to v_2,

then its change in velocity is $\delta v = v_2 - v_1$, which is shown in the second diagram.

The lengths of both v_1 and v_2 are v, and the angle between v_1 and v_2 is $\delta\theta$.

$$\delta v = 2 \times v \sin\frac{\delta\theta}{2} \approx 2v \times \frac{\delta\theta}{2} = v\delta\theta\frac{\delta v}{\delta t} \qquad \approx v\frac{\delta\theta}{\delta t}$$

as $\delta t \to 0$, acceleration:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}\theta}{\mathrm{d}t} = v\theta$$

But

$$\theta = \omega = \frac{v}{r} \to a = \frac{v^2}{r} = r\omega^2$$

Notice that as $\delta\theta \to 0$, the direction of δv becomes perpendicular to both v_1 and v_2, and so is directed towards the centre of the circle.

The acceleration of a particle moving in a circle with speed v is $a = r\omega^2 = \frac{v^2}{r}$, and is directed towards the centre of the circle.

Alternative proof

Proof. If a particle moves , with constant speed, in a circle of radius r and centre O, then its position vector can be written:

$$\mathbf{r} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \to \dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta\dot{\theta} \\ \cos\theta\dot{\theta}\end{pmatrix}$$

Particle moves with constant speed $\to \dot{\theta} = \omega$ is constant

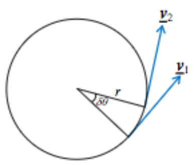
$$\dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} \to v = r\omega$$

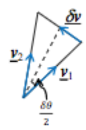
$$\ddot{\mathbf{r}} = r\omega\begin{pmatrix}-\cos\theta\dot{\theta} \\ -\sin\theta\dot{\theta}\end{pmatrix} = -\omega^2 r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = -\omega^2\mathbf{r}$$

acceleration is

$$r\omega^2 \text{ or } \frac{v^2}{r}$$

directed towards O. □

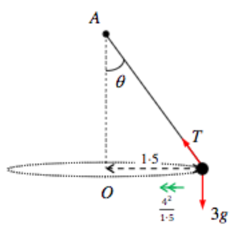




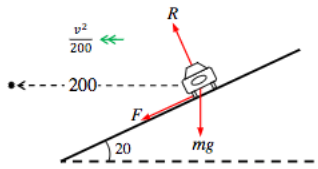
Types of problems:

(i) Horizontal

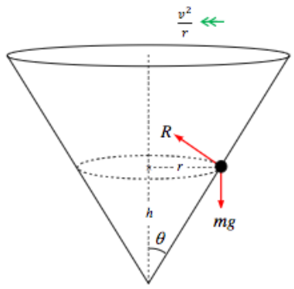
(ii) Conical pendulum



(iii) Banking



(iv) Inside an inverted vertical cone



14.3 Motion in a vertical circle

Proposition.

$$a = \frac{v^2}{r}$$

Proof. If a particle moves in a circle of radius r and centre O, then its position vector can be written:

$$\mathbf{r} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\dot{\mathbf{r}} = r\begin{pmatrix}-\sin\theta\dot{\theta} \\ \cos\theta\dot{\theta}\end{pmatrix} = r\dot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

$$\ddot{\mathbf{r}} = r\begin{pmatrix}-\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta} \\ -\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\end{pmatrix} = -r\dot{\theta}^2\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + r\ddot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

From this we can see that the speed is $v = r\dot{\theta} = r\omega$,
and is perpendicular to the radius since $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$

We can also see that the acceleration has two components

$$r\dot{\theta}^2 = r\omega^2 = \frac{v^2}{r}$$

towards the centre opposite direction to $\mathbf{r}$

and $r\ddot{\theta}$ perpendicular to the radius which is what we should expect since $v = r\dot{\theta}$ and r is constant.

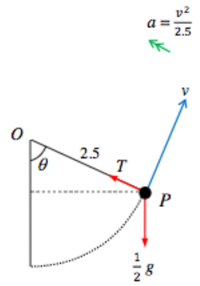
In practice we shall onlu use

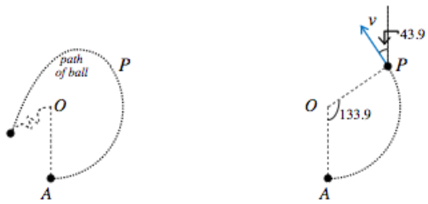
$$a = r\omega^2 = \frac{v^2}{r}$$

directed towards the centre of the circle □

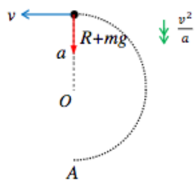
Types of problems

(i) A particle attached to an inextensible string

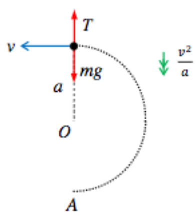




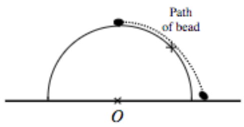
(ii) A particle moving on the indside of a smooth, hollow sphere

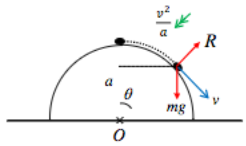


(iii) A particle attached to a rod



(iv) A particle moving on the outside of a smooth sphere





15 Statics of rigid bodies 2

15.1 Centre of mass

When finding a centre of mass

Centres of mass depend on the formula :

$$M\bar{x} = \sum m_i x_i$$

or Similar, Remember that

$$\lim_{\delta x \to 0} \sum f(x_i)\delta x = \int f(x)\mathrm{d}x$$

15.2 Centre of mass of geometric shapes

15.2.1 Sector

In this case we can find a nice method, using the result for the centre of mass of a triangle.
We take a sector of angle 2α and divide it into many smaller sectors.

Mass of whole sector

$$M = \frac{1}{2}r^2 \times 2\alpha \times \rho = r^2\alpha\rho$$

Consuder each small sector as approximately a triangle, with centre of mass, G_1, $2/3$ along the median from O

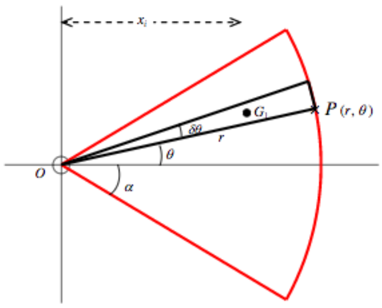
Working in polar coordinates for one small sector,

$$m_i = \frac{1}{2}r^2\rho\delta\theta$$

$$OP = r \to OG_1 \cong \frac{2}{3}r \to x_i \cong \frac{2}{3}r\cos\theta$$

$$\begin{aligned}
\lim_{\delta\theta \to 0} \sum_{\theta=-\alpha}^{\alpha} m_i x_i &= \int_{-\alpha}^{\alpha} \frac{1}{2}r^2\rho \times \frac{2}{3}r\cos\theta\mathrm{d}\theta \\
&= \frac{2}{3}r^3\rho\sin\alpha \\
\bar{x} = \frac{\sum m_i x_i}{M} &= \frac{\frac{2}{3}r^3\rho\sin\alpha}{r^2\alpha\rho} = \frac{2r\sin\alpha}{3\alpha}
\end{aligned}$$

By symmetry, $\bar{y} = 0$
centre of mass is at $(\frac{2r\sin\alpha}{3\alpha}, 0)$



15.2.2 Circular arc

For a circular arc of radius r which subtends an angle of 2α at the centre.

The length of the arc is $r \times 2\alpha$
The mass of the arc is $M = 2\alpha r\rho$

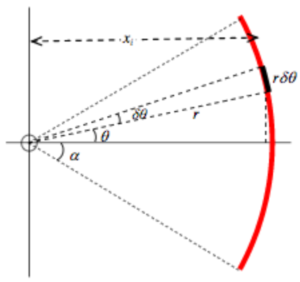
First divide the arc into several small pieces, each subtending an angle of $\delta\theta$ at the centre

The length of each piece is $r\delta\theta \to m_i = r\rho\delta\theta$

We now think of each small arc as a point mass at the centre of the arc, with x-corrdinate $x_i = r\cos\theta$

$$\begin{aligned}\lim_{\delta\theta\to 0}\sum_{\theta=-\alpha}^{\alpha} m_i x_i &= \int_{-\alpha}^{\alpha} r\rho \times r\cos\theta \mathrm{d}\theta \\ &= 2r^2\rho\sin\alpha \\ \bar{x} &= \frac{\sum m_i x_i}{M} = \frac{2r^2\rho\sin\alpha}{2r\alpha\rho} = \frac{r\sin\alpha}{\alpha}\end{aligned}$$

By symmetry, $\bar{y} = 0$
centre of mass is at $(\frac{r\sin\alpha}{\alpha}, 0)$



15.2.3 Others

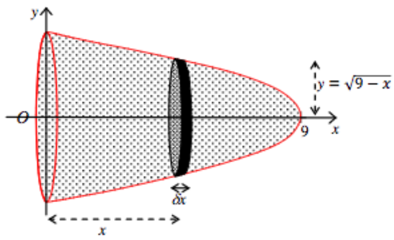
Standard results for centre of mass of uniform laminas and arcs

Triangle	$\frac{2}{3}$ of the way along the median, from the vertex.
Semi-circle, radius r	$\frac{4r}{3\pi}$ from centre, along axis of symmetry
Sector of circle, radius r, angle 2α	$\frac{2r\sin\alpha}{3\alpha}$ from centre, along axis of symmetry
Circular arc, radius r, angle 2α	$\frac{r\sin\alpha}{\alpha}$ from centre, along axis of symmetry

15.2.4 Solid of revolution

Example: A machine component has the shape of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$, $x \geq 0$, about the x-axis. Find the position of the centre of mass.

Solution:



Mass, M, of the solid $= \rho \int_0^9 \pi y^2 \, dx = \rho \int_0^9 \pi(9-x) \, dx$

$\Rightarrow \quad M = \frac{81}{2}\rho\pi.$

The diagram shows a typical thin disc of thickness δx and radius $y = \sqrt{9-x}$.

$\Rightarrow \quad$ Mass of disc $\approx \rho\pi y^2 \delta x = \rho\pi(9-x)\,\delta x$

Note that the x coordinate is the same (nearly) for all points in the disc

$$\Rightarrow \sum m_i x_i \approx \sum_0^9 \rho\pi(9-x_i)x_i \,\delta x$$

$$\lim_{\delta x \to 0} \sum m_i x_i = \int_0^9 \rho\pi(9-x)x \, dx = \frac{243}{2}\rho\pi$$

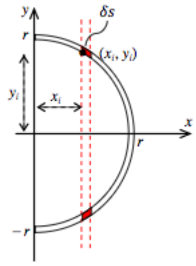
$$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\frac{243}{2}\rho\pi}{\frac{81}{2}\rho\pi} = 3$$

By symmetry, $\bar{y} = 0$

$\Rightarrow \quad$ the centre of mass is on the x-axis, at a distance of 3 from the origin.

15.2.5 Hemispherical shell

Mass of shell



Let the density of the shell be ρ, radius r

In the xy-plane, the curve has equation

$$x^2 + y^2 = r^2$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow \quad \left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{y^2 + x^2}{y^2}}$$

Take a slice perpendicular to the x-axis through the point (x_i, y_i) to form a ring with arc length δs.

Area of the ring $\cong 2\pi y_i \delta s \quad \Rightarrow \quad$ mass of ring $m_i \cong 2\pi y_i \rho\, \delta s$

$$\Rightarrow \text{ Total mass } \cong \sum 2\pi y_i \rho\, \delta s$$

$$\Rightarrow \text{ Total mass } M = \lim_{\delta s \to 0} \sum 2\pi y_i \rho\, \delta s = \int 2\pi y \rho\, ds$$

$$\Rightarrow \quad M = \int_0^r 2\pi y \rho \frac{ds}{dx}\, dx = \int_0^r 2\pi y \rho \sqrt{\frac{y^2 + x^2}{y^2}}\, dx$$

$$\Rightarrow \quad M = \int_0^r 2\pi \rho \sqrt{r^2}\, dx = 2\pi \rho r \Big[x\Big]_0^r = 2\pi \rho r^2$$

To find $\sum m_i x_i = \sum 2\pi y_i\, \rho\, \delta s\, x_i$

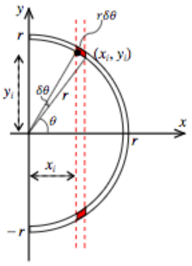
$$\Rightarrow \quad \lim_{\delta s \to 0} \sum 2\pi y_i\, \rho\, \delta s\, x_i = \int_0^r 2\pi \rho\, yx \frac{ds}{dx}\, dx$$

$$= \int_0^r 2\pi \rho\, yx \sqrt{\frac{y^2 + x^2}{y^2}}\, dx = 2\pi \rho r \left[\frac{x^2}{2}\right]_0^r = \pi \rho r^3$$

$$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\pi \rho r^3}{2\pi \rho r^2} = \frac{r}{2}$$

$\Rightarrow$ the centre of mass is on the line of symmetry at a distance of $\frac{1}{2}r$ from the centre.

Mass of shell



Let the density of the shell be ρ, radius r

Take a slice perpendicular to the x-axis through the point (x_i, y_i) to form a ring with arc length $r\delta\theta$, and circumference $2\pi y$. This can be 'flattened out' to form a rectangle of length $2\pi y$ and height $r\delta\theta$

Area of the ring $\cong$

$\Rightarrow$ mass of ring $m_i \cong 2\pi\,\rho\, y x_i r\delta\theta$

$$\Rightarrow \text{Total mass } \cong \sum 2\pi y r\rho\,\delta\theta$$

$$\Rightarrow \text{Total mass } M = \lim_{\delta\theta\to 0}\sum 2\pi y r\rho\,\delta\theta = \int 2\pi y r\rho\, d\theta$$

But $y = r\sin\theta$

$$\Rightarrow M = \int_0^{\frac{\pi}{2}} 2\pi r^2 \sin\theta\,\rho\, d\theta = 2\pi\rho r^2\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = 2\pi\rho r^2$$

$$\text{To find } \sum m_i x_i = \sum 2\pi\, y_i r\,\rho\,\delta\theta\, x_i$$

$$\Rightarrow \lim_{\delta\theta\to 0}\sum 2\pi\, y_i r\,\rho\,\delta s\, x_i = \int_0^{\frac{\pi}{2}} 2\pi\rho\, r\, yx \quad d\theta$$

But $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow \sum m_i x_i = \int_0^{\frac{\pi}{2}} 2\pi\rho r^3 \sin\theta\cos\theta \quad d\theta$$

$$= \pi\rho r^3 \left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \pi\rho r^3$$

$$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\pi\rho r^3}{2\pi\rho r^2} = \frac{r}{2}$$

$\Rightarrow$ the centre of mass is on the line of symmetry at a distance of $\frac{1}{2}r$ from the centre.

15.2.6 conical

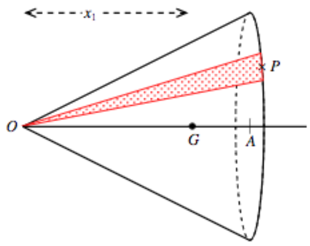
Centre of mass of a conical shell

To find the centre of mass of a conical shell, or the surface of a cone, we divide the surface into small sectors, one of which is shown in the diagram.

We can think the small sector as a triangle with centre of mass at G_1, where $OG_1 = \frac{2}{3}OP$.

This will be true for all the small sectors, and the x-coordinate, x_1, of each sector will be the same

$\Rightarrow$ the x-coordinate of the shell will also be x_1



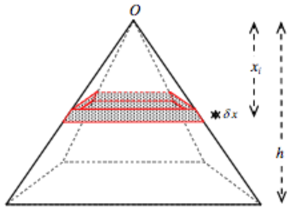
As the number of sectors increase, the approximation gets better, until it is exact,

and as $OG_1 = \frac{2}{3}OP$ then $OG = \frac{2}{3}OA$ (similar triangles)

$\Rightarrow$ the centre of mass of a conical shell is on the line of symmetry, at a distance of $\frac{2}{3}$ of the height from the vertex.

15.2.7 Square based pyramid

Centre of mass of a square based pyramid



A square based pyramid has base area A and height h

The centre of mass is on the line of symmetry

$\Rightarrow$ volume $= \frac{1}{3}Ah$

$\Rightarrow$ mass $M = \frac{1}{3}Ah\rho$

Take a slice of thickness δx at a distance x_i from O

The base of the slice is an enlargement of the base of the pyramid with scale factor $\frac{x_i}{h}$

$\Rightarrow$ ratio of areas is $\left(\frac{x_i}{h}\right)^2$

$\Rightarrow$ area of base of slice is $\frac{x_i^2}{h^2}A$

$\Rightarrow$ mass of slice $m_i = \delta x$

$$\Rightarrow \lim_{\delta x \to 0} \sum_{x=0}^{h} m_i x_i = \int_0^h \frac{x^3}{h^2} A\rho \; dx = \tfrac{1}{4}h^2 A\rho$$

$$\Rightarrow \bar{x} = \frac{\sum m_i x_i}{M} = \frac{\frac{1}{4}h^2 A\rho}{\frac{1}{3}Ah\rho} = \tfrac{3}{4}h$$

The centre of mass lies on the line of symmetry at a distance $\frac{3}{4}h$ from the vertex.

The above technique will work for a pyramid with any shape of base.

The centre of mass of a pyramid with any base has centre of mass $\frac{3}{4}$ of the way along the line from the vertex to the centre of mass of the base (considered as a lamina).

There are more examples in the book, but the basic principle remains the same:

- *find the mass of the shape, M*
- *choose, carefully, a typical element, and find its mass (involving δx or δy)*
- *for solids of revolution about the x-axis (or y-axis), choose a disc of radius y and thickness δx, (or radius x and thickness δy).*
- *find* $\sum m_i x_i$ or $\sum m_i y_i$
- *let* δx *or* $\delta y \to 0$, *and find the value of the resulting integral*
- $\bar{x} = \frac{1}{M}\sum m_i x_i$, $\bar{y} = \frac{1}{M}\sum m_i y_i$

15.2.8 The standard results

Standard results for centre of mass of uniform bodies

Solid hemisphere, radius r	$\frac{3r}{8}$ from centre, along axis of symmetry
Hemispherical shell, radius r	$\frac{r}{2}$ from centre, along axis of symmetry
Solid right circular cone, height h	$\frac{3h}{4}$ from vertex, along axis of symmetry
Conical shell, height h	$\frac{2h}{3}$ from vertex, along axis of symmetry

Centre of mass of a hemispherical shell – method 2

Note: if you use this method in an exam question which asks for a calculus technique, you would have to use calculus to prove the results for a solid hemisphere first.

The best technique for those who have not done FP3 is method 1b.

We can use the theory for compound bodies to find the centre of mass of a hemispherical shell.

From a hemisphere with radius $r + \delta r$ we remove a hemisphere with radius r, to form a hemispherical shell of thickness δr and inside radius r.

radius	$r + \delta r$	r	
Mass	$\frac{2}{3}\pi(r+\delta r)^3\rho$	$\frac{2}{3}\pi r^3\rho$	$\frac{2}{3}\pi(r+\delta r)^3\rho - \frac{2}{3}\pi r^3\rho$
centre of mass above base	$\frac{3}{8}(r+\delta r)$	$\frac{3}{8}r$	$\bar{y}$

$$\Rightarrow \quad \tfrac{2}{3}\pi(r+\delta r)^3\rho \times \tfrac{3}{8}(r+\delta r) \quad - \quad \tfrac{2}{3}\pi r^3\rho \times \tfrac{3}{8}r \quad = \quad \{\tfrac{2}{3}\pi(r+\delta r)^3\rho - \tfrac{2}{3}\pi r^3\rho\}\bar{y}$$

$$\Rightarrow \quad \frac{1}{4}\pi\rho\,(r^4 + 4r^3\delta r \ldots - r^4) \;=\; \tfrac{2}{3}\pi\rho\,(r^3 + 3r^2\delta r \ldots - r^3)\,\bar{y} \qquad \text{ignoring } (\delta r)^2 \text{ and higher}$$

$$\Rightarrow \quad r^3\delta r \cong 2r^2\,\delta r\,\bar{y}$$

and as $\delta r \to 0$, $\bar{y} = \frac{1}{2}r$

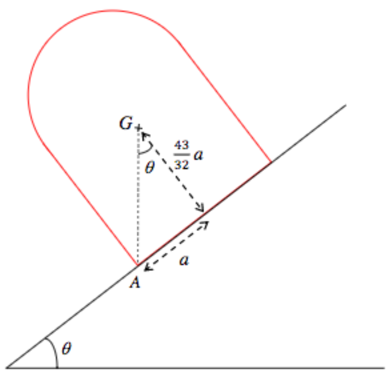
The centre of mass of a hemispherical is on the line of symmetry, $\frac{1}{2}r$ from the centre.

15.2.9 Tilting and hanging freely

Tilting and hanging freely

Tilting

Example: The compound body of the previous example is placed on a slope which makes an angle θ with the horizontal. The slope is sufficiently rough to prevent sliding. For what range of values of θ will the body remain in equilibrium.



Solution: The body will be on the point of tipping when the centre of mass, G, lies vertically above the lowest corner, A.

Centre of mass is $2a - \frac{21}{32}a$

$= \frac{43}{32}a$ from the base

At this point

$\tan\theta = \frac{a}{43a/32} = \frac{32}{43}$

$\Rightarrow \theta = 36{\cdot}65610842$

The body will remain in equilibrium for

$\theta \le 36{\cdot}7^\circ$ to the nearest $0{\cdot}1^\circ$.

Hanging freely under gravity

This was covered in M2. For a body hanging freely from a point A, you should always state, or show clearly in a diagram, that AG is vertical – this is the only piece of mechanics in the question!

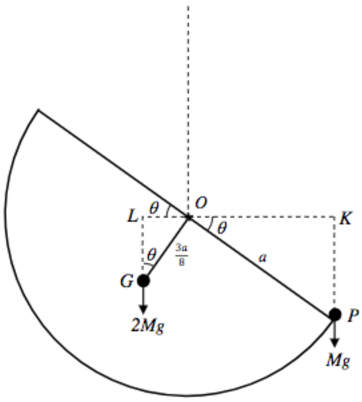
Body with point mass attached hanging freely

The best technique will probably be to take moments about the point of suspension.

Example: A solid hemisphere has centre O, radius a and mass $2M$. A particle of mass M is attached to the rim of the hemisphere at P.

The compound body is freely suspended under gravity from O. Find the angle made by OP with the horizontal.

Solution: As usual a *good, large diagram* is essential.



Let the angle made by OP with the horizontal be θ, then $\angle OGL = \theta$.

We can think of the hemisphere as a point mass of $2M$ at G, where $OG = \frac{3a}{8}$.

The perpendicular distance from O to the line of action of $2Mg$ is $OL = \frac{3a}{8}\sin\theta$, and

the perpendicular distance from O to the line of action of Mg is $OK = a\cos\theta$

Taking moments about O

$$2Mg \times \frac{3a}{8}\sin\theta = Mg \times a\cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{4}{3}$$

$$\Rightarrow \quad \theta = 53{\cdot}1^\circ.$$

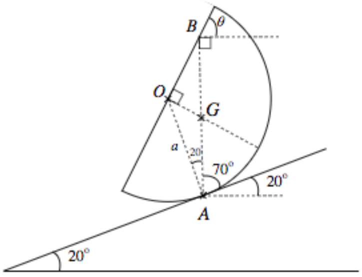
15.2.10 Hemisphere in equilibrium on a slope

Hemisphere in equilibrium on a slope

Example: A uniform hemisphere rests in equilibrium on a slope which makes an angle of 20° with the horizontal. The slope is sufficiently rough to prevent the hemisphere from sliding. Find the angle made by the flat surface of the hemisphere with the horizontal.

Solution: *Don't forget the basics.*

The centre of mass, G, must be vertically above the point of contact, A. If it was not, there would be a non-zero moment about A and the hemisphere would not be in equilibrium.



BGA is a vertical line, so we want the angle θ.

OA must be perpendicular to the slope (radius $\perp$ tangent), and with all the 90° angles around A, $\angle OAG = 20^\circ$.

Let a be the radius of the hemisphere

then $OG = \frac{3a}{8}$ and, using the sine rule

$$\frac{\sin \angle OGA}{a} = \frac{\sin 20}{{}^{3a}/_{8}} \Rightarrow \angle OGA = 65{\cdot}790\ldots \text{ or } 114{\cdot}209\ldots$$

Clearly $\angle OGA$ is obtuse $\Rightarrow$ $\angle OGA = 114{\cdot}209\ldots$

$\Rightarrow \angle OBG = 114{\cdot}209\ldots - 90 = 24{\cdot}209\ldots$

$\Rightarrow \theta = 90 - 24{\cdot}209\ldots = 65{\cdot}8^\circ$ to the nearest $0{\cdot}1^\circ$.

16 Relative motion

Definition (relative displacement and velocity)**.** which is direction vector of two position vectors.
Velocity is diifferentiating the direction vector

There are four types of questions

(i) Collisions

(ii) Closest distance

(iii) Best course

(iv) Change in apparent direction of wind or current

Example: A cyclist, *C*, travelling at 6 m s^{-1} sights a walker, *W*, 500 m due east. The walker is travelling at 2 m s^{-1} on a bearing of 310°. There are no obstacles and both the cyclist and the walker can travel anywhere.

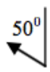
What course should the cyclist set in order to meet the walker, and how long will it take for them to meet?

Solution:

*Imagine that W is **fixed**, then C will travel directly towards W, in this case due east.*

So the direction of ${}_C\mathbf{v}_W$ *will be due **east**.*

$\mathbf{v}_C$ 6 ?

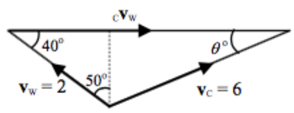
$\mathbf{v}_W$ 2 

${}_C\mathbf{v}_W$? 

Draw a vector triangle,

${}_C\mathbf{v}_W + \mathbf{v}_W = \mathbf{v}_B$

draw $\mathbf{v}_W$ *first, as we know all about* $\mathbf{v}_W$



Sine Rule $\frac{\sin\theta}{2} = \frac{\sin 40}{6}$

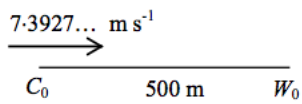
$\Rightarrow \sin\theta = 0{\cdot}2142\ldots \quad \Rightarrow \theta = 12{\cdot}372\ldots{}^\circ$ so *C* travels on bearing of $90 - 12{\cdot}4 = 77{\cdot}6$.

From the triangle $|{}_C\mathbf{v}_W| = 2\cos 40 + 6\cos\theta = 7{\cdot}3927\ldots$

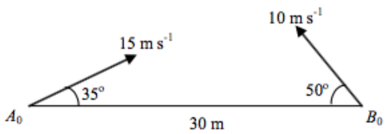
Considering the walker as fixed, the cyclist has to travel at 7·3927… m s^{-1} for a distance of 500 m.

$\Rightarrow$ time to meet $= 500 \div 7.3927\ldots = 67.63\ldots$

Cyclist travels on a bearing of 078° and they meet after 68 seconds.



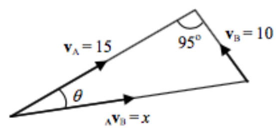
Example: Two ice-skaters, Alice and Bob, start 30 m apart and travel on converging courses. Alice travels at 15 m s^{-1} at an angle of $35°$ to the initial line and Bob travels at 10 m s^{-1} at an angle of $50°$ to the initial line, as shown in the diagram.



Find the closest distance between the two skaters, and the time for A to reach that position.

Solution: We consider B as fixed and A moving with the velocity ${}_A\mathbf{v}_B$.

We draw a vector triangle, starting with $\mathbf{v}_A - \mathbf{v}_B$, noting that the angle between $\mathbf{v}_A$ and $\mathbf{v}_B$ is $180 - 35 - 50 = 95°$.



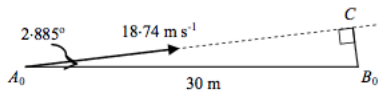
Cosine rule $x^2 = 225 + 100 - 300 \cos 95$

$\Rightarrow$ $x = 18{\cdot}7389\ldots$

Sine rule $\frac{\sin\theta}{10} = \frac{\sin 95}{x}$

$\Rightarrow$ $\sin\theta = 0{\cdot}5316\ldots$ $\Rightarrow$ $\theta = 32{\cdot}114\ldots$

We now think of A moving with speed $18{\cdot}7389\ldots$ m s-1



at an angle of $35 - 32.114\ldots = 2{\cdot}885\ldots°$ to the initial line, A_0B_0, with ***B* fixed**.

The closest distance is $B_0C = 30 \sin 2{\cdot}885\ldots = 1.51$ m to 3 S.F.

and A 'moves' with speed $18{\cdot}7389\ldots$ through a distance $A_0C = 30 \cos 2.885\ldots$

$\Rightarrow$ A takes $\frac{30 \cos 2{\cdot}885\ldots}{18{\cdot}7389\ldots} = 1{\cdot}60$ seconds to 3 S.F.

B, of course, takes the same time to reach the position where they are closest (just in case you did not realise!).

Example: A man, who can swim at 2 m s^{-1} in still water, wishes to cross a river which is flowing at 3 m s^{-1}. The river is 40 metres wide, and he wants to drift downstream as little as possible before landing on the other bank.

(*a*) What course should he take?

(*b*) How far downstream does he drift?

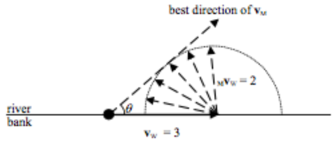
(*c*) How long does it take for him to cross the river.

Solution: The velocity of the man, v_M, must be directed at as big an angle to the downstream bank as possible.

(*a*) ${}_M\mathbf{v}_W + \mathbf{v}_W = \mathbf{v}_M$

First draw $\mathbf{v}_W$ of length 3.

${}_M\mathbf{v}_W$ is of length 2, but can vary in direction.

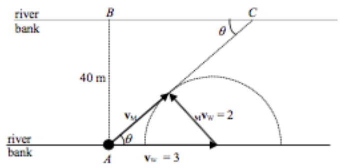


We can choose any direction for ${}_M\mathbf{v}_W$, to give $\mathbf{v}_M = {}_M\mathbf{v}_W + \mathbf{v}_W$

The best direction of $\mathbf{v}_M$ is tangential to the circle of radius 2

$\Rightarrow$ direction of $\mathbf{v}_M$ is at an angle $\theta = \sin^{-1}\left(\frac{2}{3}\right) = 41{\cdot}8\ldots^\circ$, with the bank downstream

and the direction of ${}_M\mathbf{v}_W$ is at an angle $90 - \theta = 48{\cdot}18\ldots^\circ$ with the bank upstream.



The man should swim upstream at an angle of 48° to the bank.

(*b*) Distance downstream is BC

$= \frac{40}{\tan\theta} = 20\sqrt{5}$

$= 44{\cdot}7$ m to 3 S.F.

(*c*) $|\mathbf{v}_M| = \sqrt{3^2 - 2^2} = \sqrt{5}$

$AC = \frac{40}{\sin\theta} = 60$

$\Rightarrow$ time taken $= \frac{60}{\sqrt{5}} = 12\sqrt{5} = 26{\cdot}8$ s to 3 S.F.

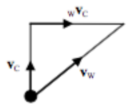
Example: A cyclist travelling due north at 10 m s^{-1}, feels that the wind is coming **from** the west. When travelling in the opposite direction at the same speed the wind appears to be coming **from** a bearing of 210°. What is the true velocity of the wind?

Solution:

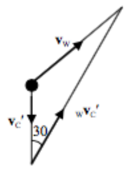
Case 1			Case 2		
$\mathbf{v}_C$	10	↑	$\mathbf{v}_C'$	10	↓
$\mathbf{v}_W$	?	?	$\mathbf{v}_W$	?	?
${}_W\mathbf{v}_C$	?	→	${}_W\mathbf{v}_C'$	?	↗ 60°

Note that $\mathbf{v}_W$ is the same in both cases

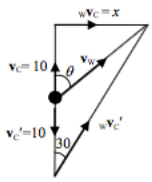
$\mathbf{v}_C + {}_W\mathbf{v}_C = \mathbf{v}_W$



$\mathbf{v}_C' + {}_W\mathbf{v}_C' = \mathbf{v}_W$



Combining the two diagrams



$x = 20 \tan 30 = \frac{20}{\sqrt{3}}$

$\Rightarrow \quad \tan\theta = \frac{x}{10} = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad \theta = 49{\cdot}1066\ldots$

and $|\mathbf{v}_W| = \sqrt{\left(\frac{20}{\sqrt{3}}\right)^2 + 10^2} = \sqrt{\frac{700}{3}} = 15{\cdot}275\ldots$

The true velocity of the wind is 15·3 m s^{-1} blowing **in** the direction 049°.

Note. In this sort of question look for 'nice symmetry', right angled triangles, isosceles triangles or equilateral triangles etc.

17 Elastic collisions in two dimensions

17.1 Impulse

17.2 Momemtum

18 Resisted motion of a particle moving in a straight line

19 Damped and forced harmonic motion

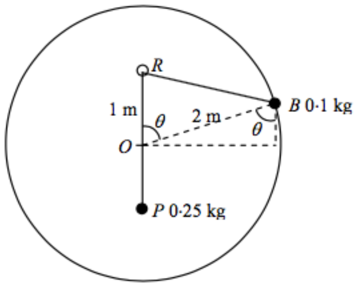
19.1 Damped (Homogenuous)

19.2 Forced (In homogenuous)

20 Stability

Example: A smooth circular wire of radius 2 metres is fixed in a vertical plane. A bead, B, of mass 0·1 kg is threaded onto the wire. A small smooth ring, R, is fixed 1 metre above the centre of the wire. An inextensible string of length 4 metres, with one end attached to the bead, passes through the ring; a particle, P, of mass 0·25 kg, attached to the other end of the string, and hangs vertically below the ring. Find the positions of equilibrium and investigate their stability.

Solution: Calculate the potential energy of the system relative to the centre of the circle, O.



The bead, B, is $2\cos\theta$ above O,

$\Rightarrow$ P.E. of bead is $0{\cdot}1\,g \times 2\cos\theta = 0{\cdot}2\,g\cos\theta$

Note that if $\theta > 90^o$, B is below O and $\cos\theta$ is negative, so P.E. is negative as it should be.

For the P.E. of the particle, P, we need the length RB.

Cosine rule $\Rightarrow RB^2 = 1^2 + 2^2 - 2 \times 1 \times 2 \times \cos\theta$

$\Rightarrow \quad RB = \sqrt{5 - 4\cos\theta}$

Length of string is 4 m $\Rightarrow \quad OP = 4 - 1 - \sqrt{5 - 4\cos\theta}$

$\Rightarrow \quad$ P.E. of particle is $-0{\cdot}25 \times g \times (3 - \sqrt{5 - 4\cos\theta})$ negative as P is below O

Note that if P is above O then OP is negative and the P.E. is positive, as it should be.

Thus the P.E. of the system is

$$V = 0{\cdot}2\,g\cos\theta - 0{\cdot}25 \times g \times (3 - \sqrt{5 - 4\cos\theta})$$

$$\Rightarrow \quad V = 0{\cdot}2\,g\cos\theta + 0{\cdot}25 \times g \times \sqrt{5 - 4\cos\theta} + \text{constant}$$

Note that the 'constant' allows the P.E. to be measured from ***any*** *fixed level.*

To find the positions of equilibrium

$$\frac{dV}{d\theta} = -0{\cdot}2\, g \sin\theta \;+\; 0{\cdot}25 \times g \times \frac{1}{2}(5 - 4\cos\theta)^{\frac{-1}{2}} \times 4\sin\theta$$

$$\frac{dV}{d\theta} = 0 \;\Rightarrow\; g\sin\theta\left(-0{\cdot}2 + \frac{1}{2}(5 - 4\cos\theta)^{\frac{-1}{2}}\right)$$

$$\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad 5 - 4\cos\theta = \tfrac{25}{4} \Rightarrow \cos\theta = \tfrac{-5}{16}$$

$$\Rightarrow \quad \theta = 0^{\circ}, 180^{\circ}, 108{\cdot}2^{\circ} \text{ or } 251{\cdot}8^{\circ}$$

To investigate the stability

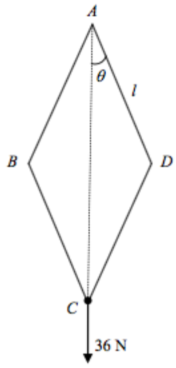
$$\frac{d^2V}{d\theta^2} = g\cos\theta\left(-0{\cdot}2 + \frac{1}{2}(5 - 4\cos\theta)^{\frac{-1}{2}}\right) + g\sin\theta \times \frac{-1}{4}(5 - 4\cos\theta)^{\frac{-3}{2}} \times 4\sin\theta$$

$$\theta = 0 \quad \Rightarrow \frac{d^2V}{d\theta^2} = g \times (-0{\cdot}2 + 0{\cdot}5) \quad > 0 \quad \Rightarrow \min V \quad \Rightarrow \text{STABLE}$$

$$\theta = 180 \quad \Rightarrow \frac{d^2V}{d\theta^2} = -g \times \left(-0{\cdot}2 + \tfrac{1}{2} \times \tfrac{1}{3}\right) > 0 \quad \Rightarrow \min V \quad \Rightarrow \text{STABLE}$$

$$\theta = 108.2 \Rightarrow \frac{d^2V}{d\theta^2} = g \times (0 - 0{\cdot}\,0577\ldots) \quad < 0 \quad \Rightarrow \max V \quad \Rightarrow \text{UNSTABLE}$$

$$\theta = 251.8 \Rightarrow \frac{d^2V}{d\theta^2} = g \times (0 - 0{\cdot}\,0577\ldots) \quad < 0 \quad \Rightarrow \max V \quad \Rightarrow \text{UNSTABLE}$$



Example: A framework consists of 4 identical light rods of length l m. The rods are smoothly joined at their ends. The framework is suspended from a fixed point, A, and a weight of 36 N is attached at C. B and D are connected by a light elastic spring of natural length l m, and modulus of elasticity 65 N.

Show that the framework can rest in equilibrium when each rod makes and angle of $\sin^{-1}\left(\frac{5}{13}\right)$ and investigate the stability.

Solution: Take A as the fixed level from which P.E. is measured. Let $\angle CAD = \theta$.

P.E. of the weight is $-2l\cos\theta \times 36$

$BD = 2\times l\sin\theta$

$\Rightarrow$ E.P.E. $= \frac{1}{2}\times 65\times\frac{(l-2l\sin\theta)^2}{l}$

$\Rightarrow$ $V = \frac{1}{2}\times 65\times\frac{(l-2l\sin\theta)^2}{l} - 72l\cos\theta$ (+ const)

$\Rightarrow$ $\frac{dV}{d\theta} = \frac{1}{2}\times 65\times\frac{2\times(l-2l\sin\theta)\times(-2l\cos\theta)}{l} + 72l\sin\theta$

$= -130l\,(1-2\sin\theta)\cos\theta + 72l\sin\theta$

When $\theta = \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{12}{13}\right)$ 5, 12, 13 triangle

$\frac{dV}{d\theta} = -130l\times\frac{3}{13}\times\frac{12}{13} + 72l\times\frac{5}{13} = 0$

$\Rightarrow$ Equilibrium when $\theta = \sin^{-1}\left(\frac{5}{13}\right)$

$\frac{dV}{d\theta} = -130l\cos\theta + 130l\sin 2\theta + 72l\sin\theta$

$\frac{d^2V}{d\theta^2} = 130l\sin\theta + 260l\cos 2\theta + 72l\cos\theta$

$= 130l\sin\theta + 260l\,(1-2\sin^2\theta) + 72l\cos\theta$

$= 130l\times\frac{5}{13} + 260l\times\left(1-2\times\left(\frac{5}{13}\right)^2\right) + 72l\times\frac{12}{13}$ = a positive number

$\Rightarrow$ minimum of V, $\Rightarrow$ equilibrium is stable.

I did try putting AC = 2x, instead of using θ – not a good idea!!

21 Applications of vectors in mechanics

Solution of simple vector differential equations

It's the same as in previous modules (M3 and M4), where you had to solve scalar differential equations, but here you have vectors: all what you have to do is substitute

$\mathbf{v} = u\mathbf{i} + w\mathbf{j}$

where u an w are scalars, so

$\frac{d\mathbf{v}}{dt} = \frac{du}{dt}\mathbf{i} + \frac{dw}{dt}\mathbf{j}$, and

$\frac{d^2\mathbf{v}}{dt^2} = \frac{d^2u}{dt^2}\mathbf{i} + \frac{d^2w}{dt^2}\mathbf{j}$.

Now equate coefficients of **i** and **j**, solve a if it is a scalar equation, hence you can find u and w.

Work done by a constant force

$\text{Work done} = \mathbf{F}.\mathbf{d}$

All what you have to do is to find the scalar product of the Force vector and the Displacement vector. Of course to find the distance: position vector of final point - position vector of initial point.

Vector moment of a force

$\text{Vector moment of a force} = \mathbf{r} \times \mathbf{F}$

Where **r** is the position vector of any point on the line of action of **F** relative to the point where the moment is to be taken about.

Resultant Force and Couples

Resultant force $= \sum_{i=1}^{n} F_i$

Couples of moment $G = \sum_{i=1}^{n} r_i \times F_i$

If a system of forces can be reduced to a resultant force, then **G** = 0

If a system of forces can be reduced into a couple of moment, then $\mathbf{F_R}$ = 0

A system s l equilibrium when both **G** and $\mathbf{F_R}$ are equal to zero.

It doesn´t matter about what point the resultant force is about. It´s the same about any point.

22 Variable mass

22.1 Examples

Motion with variable mass

You only need to know the impulse-momentum principle:

Impulse = Change in momentum = Force $\times$ Time.

From this you can set up a differential equation that describes the motion of the system.

e.g. A particle is moving upward against gravity and is losing mass at a rate k mass/sec. The lost mass is ejected vertically downwards with speed u relative to the body.

To start off, the particle is moving against gravity. So the force acting on it is its weight. Thus:

$I = Ft = -(m + \delta m)g\delta t$, where δt is the time interval through which this impulse occurs and δm is the increase in mass (see below).

Now we want to find the change in momentum.

(i) particle

$(m + \delta m)(v + \delta v) - mv$, supposing that the body gains δm of mass and δv of speed (the fact that it loses mass will be accounted for later)

(ii) ejected mass

$-\delta m(v + \delta v - u) - 0 = -\delta m(v - u)$, we always ignore $\delta m \delta v$ since it's small.

So the change in momentum of the whole system is:

$(m + \delta m)(v + \delta v) - mv - \delta m(v + \delta v - u)$

Hence by the impulse-momentum principle:

$(m + \delta m)(v + \delta v) - mv - \delta m(v - u) = -(m + \delta m)g\delta t$

Some rearrangement (remember, we ignore $\delta m \delta v$):

$m\delta v + u\delta m = -(m + \delta m)g\delta t$

Divide by δt and take limits:

$m(\frac{dv}{dt}) + u(\frac{dm}{dt}) = -mg$

This is the differential equation that describes this motion. Now we take into account one last piece of information:

The body <u>loses</u> mass at constant rate, i.e. $\frac{dm}{dt} = -k$.

This transforms the differential equation into:

$m(\frac{dv}{dt}) - ku = -mg$, which you can now solve easily.

Some things you should know when solving questions related to this chapter:

- Limiting speed is achieved when $\frac{dv}{dt} = 0$.
- When dealing with rocket problems, sometimes the time of burnout helps. e.g. suppose that a rocket has variable mass m and that the mass of the fuel is Mf and of the rocket is Mr. The time of burnout T is when $m = Mr$. This fact is simple (and obvious), but easily over-looked.
- When dealing with rain-drop type problems, use mass=density*volume, i.e. $m = \rho v$. This is helpful because sometimes you're told that the rain-drop is spherical, and this helps you get enough information to find an expression for $\frac{dm}{dt}$. That's because you know that p is constant, $v = (\frac{4}{3})\pi r^3$, and you're usually given that $\frac{dr}{dt}$ is constant. In other words: $\frac{dm}{dt} = 4p\pi r^2 \frac{dr}{dt}$.

23 Moments of inertia of a rigid body

Moments of inertia

Calculating moments of inertia

$M.I._{\text{of a point}} = mr^2$

$M.I._{\text{total}} = \sum_i m_i r_i^2$

To find moments of inertia:

Take a small part of length δx which is at distance x from axis, the mass of that small part is: M/[length or area of the body] . [length or area of the small part].

Substitute the mass of the small part in the equation of M.I., r will be x,

$\sum m_i r_i^2$ will contain δx,

So you let $\delta x \longrightarrow 0$ then:

$M.I. = \lim_{\delta x \to 0} \sum m_i r_i^2$, so you integrate to get M.I.

eg.:

Show that the moments of inertia of a uniform rod f mass M and length 2a about an axis through its centre perp. to its length to be $\frac{1}{3}Ma^2$.

Take a small part of length δx, which is at distance x from the axis.

Mass of rod per length $= \frac{M}{2a}$

So mass of the small piece which is of length $\delta x = \frac{M}{2a}.\delta x$

So $M.I._{\text{small piece}} = mr^2 = \frac{M}{2a}.\delta x.x^2$

So total $M.I. = lim_{\delta x \to 0} \sum_{x=-a}^{x=a} \frac{M}{2a}.\delta x.x^2$

Which means:

$$M.I. = \int_{-a}^{a} \frac{M}{2a}.x^2 \delta x$$

$$= \frac{M}{2a} \int_{-a}^{a} x^2 \delta x$$

$$= \frac{M}{2a}[\frac{x^3}{3}]_{-a}^{a}$$

$$= \frac{M}{2a}[\frac{a^3}{3} - \frac{(-a)^3}{3}] = \frac{M}{2a}(\frac{2a^3}{3}) = \frac{1}{3}Ma^2.$$

Standard moments of inertia that are given in the formula sheet

For uniform bodies of mass m:

- Thin rod, length 2l, about perpendicular axis through centre: $\frac{1}{3}ml^2$
- Rectangular lamina about axis in plane bisecting edges of length 2l: $\frac{1}{3}ml^2$
- Thin rod, length 2l, about perpendicular axis through end: $\frac{4}{3}ml^2$
- Rectangular lamina about edge perp. to edges of length 2l: $\frac{4}{3}ml^2$
- Rectangular lamina, sides 2a and 2b, about perpendicular axis through centre: $\frac{1}{3}m(a^2+b^2)$
- Hoop or cylindrical shell of radius r about axis through centre: mr^2
- Hoop of radius r about a diameter: $\frac{1}{2}mr^2$
- Disc or solid cylinder of radius r about axis through centre: $\frac{1}{2}mr^2$
- Disc of radius r about a diameter: $\frac{1}{4}mr^2$
- Solid sphere, radius r, about diameter: $\frac{2}{5}mr^2$
- Spherical shell of radius r about a diameter: $\frac{2}{3}mr^2$

Additive rule

If two bodies have moments of inertia I_1 and I_2 about the same axis, then the moments of inertia of the composite body about that axis is $I_1 + I_2$.

eg. A uniform rod of mass 2m and length a, has a particle of mass m at distance 2a/3 from one of its ends. Find M.I. of the system.

$$I_{\text{rod}} = \frac{1}{3}(2m)(\frac{a}{2})^2 = \frac{1}{6}.ma^2$$

$$I_{\text{particle}} = m(\frac{1}{3}a)^2 = \frac{1}{9}.ma^2$$

$$I_{\text{system}} = ma^2(\frac{1}{6}+\frac{1}{9}) = \frac{5}{18}ma^2$$

Stretching rule

If one body can be obtained from another body by stretching parallel to the axis without altering the distribution of mass relative to the axis, then the moments of inertia of the two bodies about the axis is the same.

For example: a solid cylinder is a disc stretched parallel to the axis through its centre perp. to it. A uniform rectangular lamina is a rod stretched parallel to the axis through its centre perp. to it.

Radius of Gyration

$I = mk^2$

Or $k = \sqrt{\frac{I}{m}}$

Where k is the radius of gyration.

Parallel axis theorem

If a body of mass m has moments of inertia I about the axis through the centre of mass, then the moments of inertia of that body about an axis PARALLEL to and is at distance d from the first axis is $I_{CM} + md^2$.

Perpendicular axes theorem

If a lamina lies on the plane xy, where Ox and Oy are perp., and has moments of inertia I_x and I_y about Ox and Oy respectively, and Oz is an axis perp to Ox, Oy and the lamina, then moments of inertia about Oz is I_z, where $I_z = I_x + I_y$.

24 Rotation of a rigid body about a fixed smooth axis

Revision:Motion of a rigid body

These notes are based on the requirement of the M5 A Level mathematics module.

Rotation of a rigid body

$K.E._{\text{of a body}} = \frac{1}{2}I\omega^2$

$P.E._{\text{of a body}} = gh_{\text{of centre of mass}}$

You should know that conservation of energy should be used with such things.

Moment of the resultant force about an axis:

$L = I_{\text{about the axis}}.\theta''$(proof to that is in Heinemann M5 book, page 85, if you don't have the book, and still need the proof, please contact me). θ'' is the angular acceleration.

The components of acceleration of centre of mass (G) which is rotating about axis through O are:

- $r\theta'^2$ along GO
- $r\theta''$ perpendicular to OG

To find the force that the body is doing on the axis: put force X at the axis in the direction GO, and force Y at O perpendicular to OG. Then resolve any other forces on the body in the directions OY and OX, then use:

- radial force$_{\text{resultant}} = mr\theta'^2$
- transverse force$_{\text{resultant}} = mr\theta''$

hence find X and Y, and then find their resultant force (if required to find the magnitude of the force acting on the axis).

Angular momentum

Angular momentum = moment of momentum $= I\theta'$

(How to derive that is in the same book page 94, also contact me if needed.)

As angular momentum = moment of momentum, then angular momentum for a body moving in a straight line is: its momentum × the distance from the axis.

Conservation of angular momentum

Same as conservation of momentum, angular momentum$_{\text{before}}$ = angular momentum$_{\text{after}}$

Example

A rod (mass m, length 4a) is free to rotate about a vertical axis through its centre, it rests on a smooth horizontal table, a particle of mass 3m moving in a straight line perpendicular to the rod with speed u ms[sup]-1[/sup] hits the rod at distance a from one of its ends. The particle sticks to the rod. Find the angular speed of the body after collision.

Initial angular momentum$_{\text{rod}}$ = 0

Initial angular momentum$_{\text{particle}} = (3m.u).a$

Final angular momentum$_{\text{rod}} = I_{\text{rod}}\omega$

Final angular momentum$_{\text{particle}} = I_{\text{particle}}\omega$

$$I_{\text{rod}} = \frac{1}{3}.m.(2a)^2 = \frac{4}{3}ma^2$$

$$I_{\text{particle}} = (3m)a^2$$

Initial Angular momentum$_{\text{total}}$ = Final angular momentum$_{\text{total}}$

$$3mua = \frac{4}{3}ma^2\omega + 3ma^2\omega$$

$$3u = \frac{13}{3}a\omega$$

$$\omega = \frac{9u}{13a}$$

Effect of an Impulse on a rigid body that is free to rotate about an axis

You know that:

$$L = I\theta''$$

Integrate with respect to t:

$$\int_{t_1}^{t_2} L\ dt = \int_{\omega_1}^{\omega_2} I\theta''\ dt$$

$$\int_{t_1}^{t_2} L\ dt = [I\theta']_{\omega_1}^{\omega_2}$$

$$\int_{t_1}^{t_2} L\ dt = I\Delta\omega$$

usually $L = F.r$ so:

$$\int_{t_1}^{t_2} F.r\ dt = I\Delta\omega$$

$$F.r(\Delta t) = I\Delta\omega$$

$$F.(\Delta t) = \frac{I\Delta\omega}{r}$$

Impulse $= \dfrac{I\Delta\omega}{r}$

Pendulums

Simple pendulum

Period $= 2\pi\sqrt{\dfrac{l}{g}}$

Compound pendulum

Period $= 2\pi\sqrt{\dfrac{I}{mgh}}$

where h is the distance between axis and C.M. and I is moments of inertia about the axis.

Or it might be written in another form:

Period $= 2\pi\sqrt{\dfrac{k^2+h^2}{gh}}$

where k is the radius of gyration.

To find the simple pendulum with the same period of a certain compound pendulum: use T = T,

i.e. $2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{\dfrac{I}{mgh}}$ which is simplified to $l = \dfrac{I}{mh}$.

Or

$2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{\dfrac{k^2+h^2}{gh}}$ which is simplified to $l = \dfrac{k^2+h^2}{h}$.