Método TMM:

Assume-se que $E_y(x,y,t) = E_y(x)\, e^{\omega t - \beta z}$

The structure and fields are taken not to be a function of the y coordinate. ω is the frequency and $\underset{\sim}{\beta} = \beta_{re} + i\beta_{im}$ is the complex propagation constant of the modes. The layers have complex refractive indexes $\underset{\sim}{n} = n_{re} + in_{im}$, with n_{re} being the real and n_{im} being the imaginary part, which is due to gain or loss.

The absorption coefficient α_{WG} is given by

$$\alpha_{WG} = -2k_0 n_{im} \tag{2}$$

where $k_0 = 2\pi/\lambda$ is the free space wavenumber and $\underset{\sim}{n}$ is defined such that positive values of n_{im} denote gain and negative values denote losses.

With these assumptions the wave equations for the jth layer reduces to

$$\frac{\partial}{\partial x} E_{y_j}(x) - (\underbrace{\beta^2 - k_0^2 n_j^2}_{\alpha_j^2}) E_{y_j}(x) = 0$$

$$h^2 - \alpha_j^2 = 0 \Rightarrow h = \pm \alpha_j$$

$$E_{y,j}(x) = A_j e^{\alpha_j (x - t_j)} + B_j e^{-\alpha_j (x - t_j)} \quad (*)$$

com $\alpha_j = \sqrt{\beta^2 - k_0^2 n_j^2}$

Obs.: ((*) descreve a solução geral pois α_j pode ser complexo.)

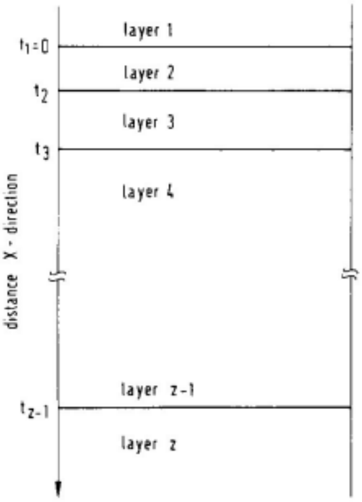

Fig. 1. The geometry of the layer structures. The z coordinate is taken to be the direction of propagation and parallel to the interfaces. The x coordinate is perpendicular to the layers, and t_j are the positions of the interfaces.

$\underset{\sim}{A}_j$, $\underset{\sim}{B}_j$ are the complex field coefficients of the field in layer j.

The real part of $\underset{\sim}{\beta}$ gives the effective refractive index of the waveguide $N = \beta_{re}/k_0$ and the imaginary part the waveguide absorption as $\alpha_{WG} = 2\beta_{im}$. The boundary conditions determine the coefficients. For TE modes, the electrical field and its derivative must be equal at the boundaries:

Considerando os campos em j e $t+j$:

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$$E_{y_{1,j}}(x) = A_j e^{\alpha_j(x-t_j)} + B_j e^{-\alpha_j(x-t_j)}$$

e

$$E_{y_{1,j+1}}(x) = A_{j+1} e^{\alpha_{j+1}(x-t_{j+1})} + B_{j+1} e^{-\alpha_{j+1}(x-t_{j+1})}$$

• Aplicando as condições de fronteira em $x = t_{j+1}$, tem-se:

$$E_j(t_{j+1}) = E_{j+1}(t_{j+1})$$

$$A_j e^{\alpha_j(t_{j+1}-t_j)} + B_j e^{-\alpha_j(t_{j+1}-t_j)} = A_{j+1} e^{\alpha_{j+1}(\overset{t_{j+1}}{x}-t_{j+1})} + B_{j+1} e^{\alpha_{j+1}(\overset{t_{j+1}}{x}-t_{j+1})}$$

onde $t_{j+1} - t_j = d_j$ espessura da j-ésima camada.
adota-se também $\delta_j = \alpha_j d_j$.

Logo,

$$A_j e^{\delta_j} + B_j e^{-\delta_j} = A_{j+1} + B_{j+1}$$

Por outro lado:

$$E_j(t_{j+1}) = E_{j+1}(t_{j+1})$$

$$\frac{\partial E_j}{\partial x}(t_{j+1}) = \frac{\partial}{\partial x} E_{j+1}(t_{j+1})$$

$$\alpha_j A_j e^{\alpha_j(x-t_j)} - \alpha_j B_j e^{-\alpha_j(x-t_j)} = \alpha_{j+1} A_{j+1} e^{\alpha_{j+1}(x-t_{j+1})} - \alpha_{j+1} B_{j+1} e^{-\alpha_{j+1}(x-t_{j+1})}$$

Para $x = t_{j+1}$

$$\alpha_j A_j e^{\delta_j} - \alpha_j B_j e^{-\delta_j} = \alpha_{j+1} A_{j+1} - \alpha_{j+1} B_{j+1}$$

$$A_{j+1} - B_{j+1} = \frac{\alpha_j}{\alpha_{j+1}} A_j e^{\delta_j} - \frac{\alpha_j}{\alpha_{j+1}} B_j e^{-\delta_j}$$

Assim,

$$\begin{cases} A_{j+1} + B_{j+1} = A_j e^{\delta_j} + B_j e^{-\delta_j} & (1) \\ A_{j+1} - B_{j+1} = \dfrac{\alpha_j}{\alpha_{j+1}} A_j e^{\delta_j} - \dfrac{\alpha_j}{\alpha_{j+1}} B_j e^{-\delta_j} & (2) \end{cases}$$

(1) + (2) : $$2A_{j+1} = \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} A_j + \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} B_j$$

$$A_{j+1} = \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} \frac{A_j}{2} + \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \frac{B_j}{2}$$

(1) - (2) : $$2B_{j+1} = \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} A_j + \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} B_j$$

$$B_{j+1} = \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} \frac{A_j}{2} + \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \frac{B_j}{2}$$

Em resumo:

$$\begin{cases} A_{j+1} = \left[1 + \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} \dfrac{A_j}{2} + \left[1 - \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \dfrac{B_j}{2} \\ B_{j+1} = \left[1 - \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} \dfrac{A_j}{2} + \left[1 + \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \dfrac{B_j}{2} \end{cases}$$

Pode-se escrever na forma matricial:

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \left[1 + \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} & \left[1 - \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \\ \left[1 - \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} & \left[1 + \dfrac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

$$\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \begin{bmatrix} T_j \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$; T_j é um operador matricial

em que a matrix complexa T_j descreve a transformação dos coeficientes entre a camada j e $j+1$. Assim, os coeficientes a camada j são calculados de forma recursiva a partir dos coeficientes da primeira camada

Dessa forma, observa-se que:

$$\begin{bmatrix} A \\ B \end{bmatrix}_j = T_{j-1} \begin{bmatrix} A \\ B \end{bmatrix}_{j-1} = T_{j-1} \cdot T_{j-2} \begin{bmatrix} A \\ B \end{bmatrix}_{j-2} = T_{j-1} \cdot T_{j-2} \dots T_1 \begin{bmatrix} A \\ B \end{bmatrix}_1$$

Para determinar os elementos da última camada z, temos:

$$\begin{bmatrix} A \\ B \end{bmatrix}_z = T_{WG} \begin{bmatrix} A \\ B \end{bmatrix}_1$$

em que $\mathbf{T}_{\mathrm{WG}} = \mathbf{T}_{z-1} \cdot \mathbf{T}_{z-2} \dots \mathbf{T}_1 = \Pi_{k=z-1}^{1} \mathbf{T}_k.$

Agora, para a primeira camada tem-se: $E_{y,1}(x) = A\,e^{\alpha_j(x-t_1)} + B\,e^{-\alpha_j(x-t_1)}$
Fora dessa camada $(x \to -\infty)$ o campo deve ser nulo. Logo, $B=0$, assim:

$$E_{y,1}(x) = A\,e^{\alpha_j(x-t_1)}$$

Portanto,

$$\begin{bmatrix} A \\ B \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} a$$; $\underline{a}$ é um número complexo, pois α_j pode ser complexo.

De forma similar, para a última camada z $(x \to \infty)$:

$$E_{y,1}(x) = A\,e^{\alpha_j(x-t_1)} + B\,e^{-\alpha_j(x-t_1)} \ ; \ A=0$$

$$\begin{bmatrix} A \\ B \end{bmatrix}_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} b \ ; \ b \in \mathbb{C}$$

Pode-se relacionar essas camadas por meio de $[T_{WG}]$. Assim:

$$\begin{bmatrix} A \\ B \end{bmatrix}_z = \begin{bmatrix} T_{WG} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}_1 \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} T_{WG} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a$$

O produtório de matrizes 2x2 resulta também em uma matriz 2x2.
Portanto,

$$\begin{bmatrix} T_{WG} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

Com isso, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} a$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} a$$

isso resulta em:

$$a t_{11} = 0 \quad e \quad a t_{21} = b$$

$$t_{11} = 0 \qquad t_{21} = b/a$$

↳ Ñ se conhece

logo a constante de propagação β é determinada por:

$$\boxed{t_{11}(\beta) = 0}$$

Como visto a matriz de TWG depende dos seguintes parâmetros

$$\frac{1}{2}\begin{bmatrix} \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} & \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \\ \left[1 - \frac{\alpha_j}{\alpha_{j+1}}\right] e^{\delta_j} & \left[1 + \frac{\alpha_j}{\alpha_{j+1}}\right] e^{-\delta_j} \end{bmatrix}$$

- δ_j : espessura da camada, conhecido via projeto.
- α_j e α_{j+1} : $\alpha_j = \sqrt{\beta^2 - k_0^2 n_j^2}$; k_0 valor conhecido
  n_j valor conhecido

Conclui-se que a única variável desconhecida é β. Desse modo, a matriz TWG tem a variável β e, consequentemente, o índice t_{11} também. Então, t_{11} depende de β.

=> Para o modo TM:

$$\begin{cases} \dfrac{d^2E_z}{dx^2} + k_i^2 E_z = 0 \\ E_x = -j\dfrac{\beta}{k_i^2}\dfrac{dE_z}{dx} \\ H_y = -j\dfrac{\omega\varepsilon}{k_i^2}\dfrac{dE_z}{dx} \end{cases}$$

$$\frac{d^2 E_z}{dx^2} - \alpha_i^2 E_z = 0 \qquad (k_i^2 = -\alpha_i^2)$$

Assim, $h^2 - \alpha_i^2 = 0 \Rightarrow h = \pm\alpha_i$

Portanto,

$$E_{z_i}(x) = A_i e^{\alpha_i x} + B_i e^{-\alpha_i x}$$

- Condições de fronteira em $x = t_{i+1}$:

$$E_{z_i}(t_{i+1}) = E_{z_{i+1}}(t_{i+1})$$

$$A_i e^{\alpha_i(x-t_i)} + B_i e^{-\alpha_i(x-t_i)} = A_{i+1} e^{\alpha_{i+1}(x-t_{i+1})} + B_{i+1} e^{\alpha_{i+1}(x-t_{i+1})}, \quad (x = t_{i+1})$$

Obs.: $x - t_i \Rightarrow t_{i+1} - t_i = d$ e $\alpha_i . d = \delta$

$$A_i e^{\delta_i} + B_i e^{-\delta_i} = A_{i+1} + B_{i+1} \quad (*)$$

Para a segunda condição:

$$H_{y_i}(t_{i+1}) = H_{y_{i+1}}(t_{i+1})$$

$$-j\frac{\omega\varepsilon_i}{k_i^2}\frac{dE_{z_i}(t_{i+1})}{dx} = -j\frac{\omega\varepsilon_{i+1}}{k_{i+1}^2}\frac{dE_{z_{i+1}}(t_{i+1})}{dx}$$

$$\boxed{H_{y_i} = -j\frac{\omega\varepsilon}{k_i^2}\frac{dE_{z_i}}{dx}}$$

$$\frac{\varepsilon_{n_i}}{k_i^2}\frac{d}{dx}E_{z_i}(t_{i+1}) = \frac{\varepsilon_{n_{i+1}}}{k_{i+1}^2}\frac{d}{dx}E_{z_{i+1}}(t_{i+1}) \quad ; \quad \boxed{n_i^2 = \varepsilon_{n_i}}$$

$$\frac{n_i^2}{k_i^2}\left[\alpha_i A_i e^{\alpha_i(x-t_i)} - \alpha_i B_i e^{-\alpha_i(x-t_i)}\right] =$$

$$\frac{n_{i+1}^2}{k_{i+1}^2}\left[\alpha_{i+1} A_{i+1} e^{\alpha_{i+1}(x-t_{i+1})} - \alpha_{i+1} B_{i+1} e^{-\alpha_{i+1}(x-t_{i+1})}\right]$$

$(x = t_{i+1})$

$$\frac{n_i^2}{k_i^2}\left[\alpha_i A_i e^{\delta_i} - \alpha_i B_i e^{-\delta_i}\right] = \frac{n_{i+1}^2}{k_{i+1}^2}\left[\alpha_{i+1} A_{i+1} - \alpha_{i+1} B_{i+1}\right]$$

$$\alpha_{i+1} A_{i+1} - \alpha_{i+1} B_{i+1} = \frac{n_i^2}{n_{i+1}^2}\cdot\frac{k_{i+1}^2}{k_i^2}\left[\alpha_i A_i e^{\delta_i} - \alpha_i B_i e^{-\delta_i}\right]$$

$$A_{i+1} - B_{i+1} = \frac{n_i^2}{n_{i+1}^2}\cdot\frac{k_{i+1}^2}{k_i^2}\left[\frac{\alpha_i}{\alpha_{i+1}} A_i e^{\delta_i} - \frac{\alpha_i}{\alpha_{i+1}} B_i e^{-\delta_i}\right] \quad ; \quad \boxed{k_i^2 = -\alpha_i^2}$$

$$A_{i+1} - B_{i+1} = \frac{n_i^2}{n_{i+1}^2}\cdot\left[\frac{\alpha_{i+1}}{\alpha_i} A_i e^{\delta_i} - \frac{\alpha_{i+1}}{\alpha_i} B_i e^{-\delta_i}\right] \quad (\#)$$

Em resumo:

$$\begin{cases} A_{i+1} + B_{i+1} = A_i e^{\delta_i} + B_i e^{-\delta_i} & (*) \\ A_{i+1} - B_{i+1} = \dfrac{n_i^2}{n_{i+1}^2}\left[\dfrac{\alpha_i}{\alpha_{i+1}} A_i e^{\delta_i} - \dfrac{\alpha_i}{\alpha_{i+1}} B_i e^{-\delta_i}\right] & (\#) \end{cases}$$

Manipulando as equações:

- $(*) + (\#)$; $2A_{i+1} = \dfrac{n_i^2}{n_{i+1}^2}\dfrac{\alpha_i}{\alpha_{i+1}}\left[A_i e^{\delta_i} - B_i e^{-\delta_i}\right] + A_i e^{\delta_i} + B_i e^{-\delta_i}$

$$A_{i+1} = \frac{1}{2}\frac{n_i^2}{n_{i+1}^2}\frac{\alpha_i}{\alpha_{i+1}}\left[A_i e^{\delta_i} - B_i e^{-\delta_i}\right] + \frac{1}{2}\left[A_i e^{\delta_i} + B_i e^{-\delta_i}\right]$$

$$A_{i+1} = \frac{A_i}{2} e^{\delta_i}\left[\frac{n_i^2}{n_{i+1}^2}\frac{\alpha_i}{\alpha_{i+1}} + 1\right] + \frac{B_i}{2} e^{-\delta_i}\left[1 - \frac{n_i^2}{n_{i+1}^2}\frac{\alpha_i}{\alpha_{i+1}}\right]$$

- (*) − (#) :

$$\begin{cases} A_{i+1} + B_{i+1} = A_i e^{\delta_i} + B_i e^{-\delta_i} & (*) \\ A_{i+1} - B_{i+1} = \dfrac{n_i^2}{n_{i+1}^2}\left[\dfrac{\alpha_i}{\alpha_{i+1}} A_i e^{\delta_i} - \dfrac{\alpha_i}{\alpha_{i+1}} B_i e^{-\delta_i}\right] & (\#) \end{cases}$$

$$B_{i+1} = \frac{A_i}{2} e^{\delta_i}\left[1 - \frac{n_i^2}{n_{i+1}^2}\frac{\alpha_i}{\alpha_{i+1}}\right] + \frac{B_i}{2} e^{-\delta_i}\left[\frac{n_i^2}{n_{i+1}^2}\frac{\alpha_i}{\alpha_{i+1}} + 1\right]$$

Logo,

$$\begin{cases} A_{i+1} = \dfrac{A_i}{2} e^{\delta_i}\left[\dfrac{n_i^2}{n_{i+1}^2}\dfrac{\alpha_i}{\alpha_{i+1}} + 1\right] + \dfrac{B_i}{2} e^{-\delta_i}\left[1 - \dfrac{n_i^2}{n_{i+1}^2}\dfrac{\alpha_i}{\alpha_{i+1}}\right] \\ B_{i+1} = \dfrac{A_i}{2} e^{\delta_i}\left[1 - \dfrac{n_i^2}{n_{i+1}^2}\dfrac{\alpha_i}{\alpha_{i+1}}\right] + \dfrac{B_i}{2} e^{-\delta_i}\left[\dfrac{n_i^2}{n_{i+1}^2}\dfrac{\alpha_i}{\alpha_{i+1}} + 1\right] \end{cases}$$

Adotando $p_i = \left[\dfrac{n_i}{n_{i+1}}\right]^2$

$$\begin{cases} A_{i+1} = \dfrac{A_i}{2} e^{\delta_i}\left[\dfrac{p_i\alpha_i}{\alpha_{i+1}} + 1\right] + \dfrac{B_i}{2} e^{-\delta_i}\left[1 - \dfrac{p_i\alpha_i}{\alpha_{i+1}}\right] \\ B_{i+1} = \dfrac{A_i}{2} e^{\delta_i}\left[1 - \dfrac{p_i\alpha_i}{\alpha_{i+1}}\right] + \dfrac{B_i}{2} e^{-\delta_i}\left[\dfrac{p_i\alpha_i}{\alpha_{i+1}} + 1\right] \end{cases}$$

$$A_{i+1} = \frac{A_i}{2} e^{\delta_i} \left[\frac{p_i \alpha_i}{\alpha_{i+1}} + 1 \right] + \frac{B_i}{2} e^{-\delta_i} \left[1 - \frac{p_i \alpha_i}{\alpha_{i+1}} \right]$$

$$B_{i+1} = \frac{A_i}{2} e^{\delta_i} \left[1 - \frac{p_i \alpha_i}{\alpha_{i+1}} \right] + \frac{B_i}{2} e^{-\delta_i} \left[\frac{p_i \alpha_i}{\alpha_{i+1}} + 1 \right]$$

• Da forma matricial:

$$\begin{bmatrix} A \\ B \end{bmatrix}_{i+1} = \frac{1}{2} \begin{bmatrix} \left(\frac{p_i \alpha_i}{\alpha_{i+1}} + 1 \right) e^{\delta_i} & \left(1 - \frac{p_i \alpha_i}{\alpha_{i+1}} \right) e^{-\delta_i} \\ \left(1 - \frac{p_i \alpha_i}{\alpha_{i+1}} \right) e^{\delta_i} & \left(\frac{p_i \alpha_i}{\alpha_{i+1}} + 1 \right) e^{-\delta_i} \end{bmatrix}_i \begin{bmatrix} A \\ B \end{bmatrix}_i$$

$$\begin{bmatrix} A \\ B \end{bmatrix}_{i+1} = [T]_i \begin{bmatrix} A \\ B \end{bmatrix}_i$$

• De maneira análoga para o modo TE, temos que:

$$\begin{bmatrix} A \\ B \end{bmatrix}_2 = \begin{bmatrix} T_{WG} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}_1$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} T_{WG} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a$$

O produtório de matrizes 2x2 resulta também em uma matriz 2x2. Portanto,

$$\begin{bmatrix} T_{WG} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

Com isso, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} a$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} b = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} a$$

isso resulta em:

$$a t_{11} = 0 \quad e \quad a t_{21} = b$$

$$t_{11} = 0 \qquad t_{21} = b/a$$

↳ Ñ se conhece

logo a constante de propagação β é determinada por:

$$\boxed{t_{11}(\beta) = 0}$$

Como visto a nova matriz T_{WG} depende dos seguintes parâmetros

$$\frac{1}{2} \begin{bmatrix} \left(p_i \frac{\alpha_i}{\alpha_{i+1}} + 1\right) e^{\delta_i} & \left(1 - p_i \frac{\alpha_i}{\alpha_{i+1}}\right) e^{-\delta_i} \\ \left(1 - p_i \frac{\alpha_i}{\alpha_{i+1}}\right) e^{\delta_i} & \left(p_i \frac{\alpha_i}{\alpha_{i+1}} + 1\right) e^{-\delta_i} \end{bmatrix}$$

- δ_j : espessura da camada, conhecido via projeto.
- p_i : $\left(\frac{n_i}{n_{i+1}}\right)^2$ índices de refração conhecidos.

- α_j e α_{j+1}: $\alpha_j = \sqrt{\beta^2 - k_0^2 n_j^2}$; k_0 valor conhecido
  n_j valor conhecido

Conclui-se que a única variável desconhecida é β. Desse modo, a matriz TWG tem a variável β e, consequentemente, o índice t_{11} também. Portanto, t_{11} é função de β.

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