Module 1

Vector Spaces, Bases, Linear Maps

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Problem 1: 10 points

Let A_4 be the following matrix:

$$A_4 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix}.$$

Prove that the columns of A_4 are linearly independent. Find the coordinates of the vector $x = (7, 14, -1, 2)$ over the basis consisting of the column vectors of A_4.

To prove that the columns of A_4 are linearly independent we set the rows s.t we create a system of equations. If each vector column in A_4 sums to zero, then we know that A_4 is linearly independent.

We arrange A_4 s.t. there exists a linear equation for each column in relation to that vector and scalar linear combination equal to zero:
let the vectors of each column $\in A_4 =$v$_1$, v_2 v_3, and v_4 respectively.

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix} v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix} v_4 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

Our system of equations is as follows:

$$\begin{aligned} 0 &= v_1 + 2v_2 + v_3 + v_4 \\ 0 &= 2v_1 + 3v_2 + 2v_3 + 3v_4 \\ 0 &= -v_1 + v_3 - v_4 \\ 0 &= -2v_1 - v_2 + 4v_3 \end{aligned}$$

Now we solve the system:

$$
\begin{aligned}
0 &= -2(v_3 - v_4) - v_1 + 4v_4 \\
0 &= v_3 - v_4 + 2v_2 + v_3 + v_4 \\
0 &= 2(v_3 - v_4) + 3v_2 + 2v_3 + 3v_4 \\
\\
0 &= -3v_2 + 2v_4 \\
0 &= v_2 + v_4 \\
\\
0 &= \frac{v_4}{3} \\
0 &= v_4
\end{aligned}
$$

Solve for each vector:

$$
\begin{aligned}
v_1 = 0, v_2 = 0, v_3 = 0, v_4 = 0 \\
v_1 + v_2 +, v_3 + v_4 = 0
\end{aligned}
$$

$\therefore A_4$ is linearly independent.

Now we find the coordinates of the vector $x = (7, 14, -1, 2)$ over basis A_4 through a similar process: We let A_4 be equal to the column X and solve as a systems of equations. let the vectors of each column $= v_1$, v_2 v_3, v_4, and X respectively.

$$
v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix} v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix} v_4 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix} X = \begin{pmatrix} 7 \\ 14 \\ -1 \\ 2 \end{pmatrix}
$$

Our system of equations is as follows:

$$
\begin{aligned}
7 &= v_1 + 2v_2 + v_3 + v_4 \\
14 &= 2v_1 + 3v_2 + 2v_3 + 3v_4 \\
-1 &= -v_1 + v_3 + v_4 \\
2 &= -2v_1 - v_2 + 4v_3
\end{aligned}
$$

Now we solve the system:

$$
\begin{aligned}
2 &= -2(v_3 - v_4 + 1) - v_2 + 4v_3 \\
7 &= 2v_3 + 2v_2 + 1 \\
14 &= 4v_3 + 3v_2 + v_4 + 2
\end{aligned}
$$

$$
\begin{aligned}
2 &= 2v_4 + 4 - 3v_2 \\
14 &= v_4 + 14 - v_2 \\
14 &= \frac{v_2 - 2}{2} + 14
\end{aligned}
$$

$$v_1 = 0, v_2 = 2, v_3 = 1, v_4 = 2$$

coordinates of $X = (0, 2, 1, 2)$

Problem 2: 10 points

Consider the following Haar matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Prove that the columns of H are linearly independent.

Hint. Compute the product $H^{\top}H$.

After computing $H^{\top}H$ we can visually depict that H is linearly independent. We will prove H is linearly independent by creating and solving the system of equations of H.

let the vectors of each column $\in H =$v$_1$, v_2 v_3, and v_4 respectively.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Our system of equations is as follows:

$$
\begin{aligned}
0 &= v_1 + v_2 + v_3 \\
0 &= v_1 + v_2 - v_3 \\
0 &= v_1 - v_2 + v_4 \\
0 &= v_1 - v_2 - v_4
\end{aligned}
$$

Now we solve the system:

$$
\begin{aligned}
0 &= -2v_3 \\
0 &= -2v_2 - v_3 - v_4 \\
0 &= -2v_2 - v_3 - v_4
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_4 - 2v_2 \\
0 &= -2v_2 - v_4
\end{aligned}
$$

$$
0 = -4v_2
$$

Solve for each vector:

$$
\begin{aligned}
v_1 = 0, v_2 = 0, v_3 = 0, v_4 = 0 \\
v_1 + v_2 +, v_3 + v_4 = 0
\end{aligned}
$$

Therefore, H is linearly independent.

Problem 3: 10 points

Let $E = \mathbb{R} \times \mathbb{R}$, and define the addition operation

$$
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R},
$$

and the multiplication operation $\cdot \colon \mathbb{R} \times E \to E$ by

$$
\lambda \cdot (x, y) = (\lambda x, y), \quad \lambda, x, y \in \mathbb{R}.
$$

Show that E with the above operations $+$ and $\cdot$ is not a vector space. Which of the axioms is violated?

The axiom that is violated is $V1 : \alpha * (u + v) = (\alpha * u) + (\alpha * v)$ which distributes α through the parenthesis. In the case above, $\lambda \cdot (x, y) = (\lambda x, y)$ fails to distribute to y. $\therefore$ it cannot be a vector space because it violates axiom $V1$.

Problem 4: 15 points total

(1) (5 points) Let A be an $n \times n$ matrix. If A is invertible, prove that for any $x \in \mathbb{R}^n$, if $Ax = 0$, then $x = 0$.

Given: A^{-1} exists
Let $Ax = 0$. We multiply both sides by A^{-1}.

$$A^{-1} * Ax = 0 * A^{-1}$$

through the associative property of the identify matrix multiplication, we get:

$$\begin{aligned} (A^{-1} * A) &= Ix \\ \text{Substituting this back into the above equation, we get: } Ix &= 0 * A^{-1} \\ Ix &= 0 \\ x &= 0 \end{aligned}$$

(2) (10 points) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that $I_m - AB$ is invertible iff $I_n - BA$ is invertible.

Hint.

If for all $x \in \mathbb{R}^n$, $Mx = 0$ implies that $x = 0$, then M is invertible.

$(I_n - BA)^{-1} = I_n + BA + (BA)^2 + (BA)^3 + \ldots$ (a geometric series)
Assume $(I_n - BA)^-1$ is invertible, w.t.s that $(I_m - AB)^-1$ is invertible.
$(I_m - AB)^{-1} = I_m + AB + (AB)^2 + (AB)^3 + \ldots$
We turn I into AB. Then $A(I_n - BA)^{-1}B = AB + (AB)^2 + (AB)^3 + \ldots = (I_m - AB)^{-1} - I_n$

So assuming $I_n - BA$ is invertible,

$$\begin{aligned}
A(I_n - BA)^{-1}B &= (I_m - AB)^{-1} - I_m \\
(I_m - AB)^{-1} &= I_m + A(I_n - BA)^{-1}B \\
(I_m - AB)(I_m - AB)^{-1} &= I_m \text{ (try to find the right inverse)} \\
(I_m - AB)(I_m + A(I_m - BA)^{-1}B) &= I_m + A(I_n - BA)^{-1}B - AB - ABA(I_n - BA)^{-1}B \\
&= I_m + A((I_n - BA)^{-1}B - B - BA(I_n - BA)^{-1}B) \\
&= I_m + A((I_n - BA)^{-1} - I_n - BA(I_n - BA)^{-1})B \\
&= I_m + AB(I_n - BA)^{-1}((I_n - BA) - I_n) \\
&= I_m + A(I_n - I_n)B \\
&= I_m + A(0)B \\
&= I_m
\end{aligned}$$

So we have found the right inverse of the matrix. $\therefore$ we have shown that $I_m + A(I_n - BA)^{-1}B$ is the right inverse of $I_n - BA$.

Assume that $I_m - AB$ is invertible:

$$\begin{aligned}
B(I_m - AB)^{-1}A &= (I_n - BA)^{-1} - I_n \\
(I_n - BA)^{-1} &= I_n + B(I_m - AB)^{-1}A \\
(I_n - BA)(I_n - BA)^{-1} &= I_n \text{ (find the right inverse)} \\
(I_n - BA)(I_n + B(I_n - AB)^{-1}A) &= I_n + B(I_m - AB)^{-1}A - BA - BAB(I_m - AB)^{-1}A \\
&= I_n + B((I_m - AB)^{-1}A - A - AB(I_m - AB)^{-1}A) \\
&= I_n + B((I_m - AB)^{-1} - I - AB(I_m - AB)^{-1})A \\
&= I_n + BA(I_m - BA)^{-1}((I_m - AB) - I_m) \\
&= I_n + B(I_m - I_m)A \\
&= I_n + B(0)A \\
&= I_n
\end{aligned}$$

$\therefore$ we have shown that $I_n + B(I - AB)^{-1}A$ is the right inverse of $I_m - AB$.
$\therefore$ $I_m - AB$ is invertible iff $I_n - BA$ is invertible

Problem 5: 10 points

Let $f : E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1} : E \to F$ is linear.

In order to prove that the inverse function is also linear, we must show that the two definitions of linear mapping remains true:

Given: f is a bijection $\longrightarrow u, v \in U$ s.t $f(u) = x, f(v) = y$

First definition: $f(x + y) = f(x) + f(y)$:
$f^{-1}(x + y) = f^{-1}(f(u) + f(v))$
$= f^{-1}(f(u + v))$
$= u + v$
$= f^{-1}(x) + f^{-1}(y)$ by linearity of f^{-1}

Second definition: $f(\lambda x) = \lambda f(x)$
$f^{-1}(\lambda x) = f^{-1}(\lambda f(u))$
$= f^{-1}(f(\lambda u))$
$= \lambda u$
$= \lambda f^{-1}(x)$

$\therefore f^{-1} : E$ is linear

Total: 55 points