Module 7

Eigenvalues, Eigenvectors, and Spectral Theorems
Solutions

Problem 1: 10 points

Find the eigenvalues of the following matrices by hand **without** using `Matlab`

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \quad C = A + B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Check that the eigenvalues of $A + B$ are not equal to the sums of eigenvalues of A plus eigenvalues of B.

Solution. The eigenvalues of A are the zeros of the charactetristic polynomial of A,

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 4\lambda + 3.$$

The zeros of the equation

$$\lambda^2 - 4\lambda + 3 = 0$$

are

$$\lambda = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2},$$

so $\lambda_1 = 1$ and $\lambda_2 = 3$.

The eigenvalues of B are the zeros of the charactetristic polynomial of B,

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda + 3.$$

Thus B has the same eigenvalues as A, namely $\lambda_1 = 1$ and $\lambda_2 = 3$.

The eigenvalues of $C = A + B$ are the zeros of the charactetristic polynomial of C,

$$\det(\lambda I - C) = \begin{vmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{vmatrix} = \lambda^2 - 8\lambda + 15.$$

The zeros of the equation

$$\lambda^2 - 8\lambda + 15 = 0$$

are

$$\mu = \frac{8 \pm \sqrt{64 - 60}}{2} = \frac{8 \pm 2}{2},$$

so $\mu_1 = 3$ and $\mu_2 = 5$.

However $\lambda_1 + \lambda_1 = 1 + 1 = 2 \neq 3 = \mu_1$ and $\lambda_2 + \lambda_2 = 3 + 3 = 6 \neq 5 = \mu_2$.

Problem 2: 20 points total

Let A be the real symmetric 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(1) (5 points) Prove that the eigenvalues of A are real and given by

$$\lambda_1 = \frac{a + c + \sqrt{4b^2 + (a-c)^2}}{2}, \quad \lambda_2 = \frac{a + c - \sqrt{4b^2 + (a-c)^2}}{2}.$$

(2) (5 points) Prove that A has a double eigenvalue ($\lambda_1 = \lambda_2 = a$) if and only if $b = 0$ and $a = c$; that is, A is a diagonal matrix.

(3) (5 points) Prove that the eigenvalues of A are nonnegative iff $b^2 \leq ac$ and $a + c \geq 0$.

(4) (5 points) Prove that the eigenvalues of A are positive iff $b^2 < ac$, $a > 0$ and $c > 0$.

Solution.

(1) The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} = \lambda^2 - (a + c)\lambda + ac - b^2.$$

The eigenvalues of A are the zeros of the equation

$$\lambda^2 - (a + c)\lambda + ac - b^2 = 0.$$

They are given by

$$\begin{aligned} \lambda &= \frac{a + c \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2} \\ &= \frac{a + c \pm \sqrt{4b^2 + (a-c)^2}}{2}, \end{aligned}$$

and since $4b^2 + (a-c)^2 \geq 0$, these roots are real and given by

$$\lambda_1 = \frac{a + c + \sqrt{4b^2 + (a-c)^2}}{2}, \quad \lambda_2 = \frac{a + c - \sqrt{4b^2 + (a-c)^2}}{2}.$$

(2) The eigenvalues λ_1 and λ_2 are identical and equal to $(a + c)/2$ iff $4b^2 + (a - c)^2 = 0$, and since a, b, c are real, this is equivalent to $a = c$ and $b = 0$. In this case, A is the diagonal matrix aI.

(3) It is well known that $\lambda_1 + \lambda_2 = a + c$ (the trace of A) and $\lambda_1\lambda_2 = ac - b^2$ (the determinant of A), so λ_1 and λ_2 are nonnegative iff $\lambda_1 + \lambda_2 \geq 0$ and $\lambda_1\lambda_2 \geq 0$, that is, $b^2 \leq ac$ and $a + c \geq 0$.

(4) λ_1 and λ_2 are positive iff $\lambda_1 + \lambda_2 > 0$ and $\lambda_1\lambda_2 > 0$, that is, $b^2 < ac$ and $a + c > 0$. However, the condition $b^2 < ac$ implies that a, c have the same sign and are nonzero, and then $a + c > 0$ implies that $a, c > 0$. Conversely, $a, c > 0$ implies that $a + c > 0$, so λ_1 and λ_2 are positive iff $b^2 < ac$, $a > 0$, and $c > 0$.

Problem 3: 10 points

Let A be a complex $n \times n$ matrix. Prove that if A is invertible and if the eigenvalues of A are $(\lambda_1, \ldots, \lambda_n)$, then the eigenvalues of A^{-1} are $(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$. Prove that if u is an eigenvector of A for λ_i, then u is an eigenvector of A^{-1} for λ_i^{-1}.

Solution. A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of A if there is some $u \neq 0$ such that

$$Au = \lambda u. \tag{eq1}$$

If $\lambda \neq 0$ and A is invertible, by multiplying both sides of (eq1) by $\lambda^{-1}A^{-1}$ (which is an invertible matrix), we see that (eq1) is equivalent to

$$A^{-1}u = \lambda^{-1}u. \tag{eq2}$$

However, (eq2) means that λ^{-1} is an eigenvalue of A for the eigenvector u, so λ is eigenvalue of A for the eigenvector u iff λ^{-1} is eigenvalue of A^{-1} for the same eigenvector u, which proves both parts of Problem 3.

Problem 4: 25 points total

Let $f \colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear map.

(1) (15 points) Prove that if f is diagonalizable and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of f, then $\lambda_1^2, \ldots, \lambda_n^2$ are the eigenvalues of $f^2 = f \circ f$, and if $\lambda_i^2 = \lambda_j^2$ implies that $\lambda_i = \lambda_j$, then f and f^2 have the same eigenspaces.

Hint. Consider the direct sum decomposition of the eigenspaces and a dimension argument.

(2) (10 points) Let f and g be two real self-adjoint linear maps $f, g \colon \mathbb{R}^n \to \mathbb{R}^n$. Prove that if f and g have nonnegative eigenvalues (f and g are positive semidefinite) and if $f^2 = g^2$, then $f = g$.

Solution.

(1) If f is diagonalizable, then for any basis $(u_1, \ldots, u_n)$ of eigenvectors of f, where u_i is associated with the eigenvalue λ_i, if A is the matrix representing f over the canonical basis, we have

$$A = PDP^{-1}$$

where D is the diagonal matrix

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

and P is an invertible matrix whose columns are eigenvectors of A. Then we have

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1},$$

with

$$D^2 = \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2),$$

which shows that A^2 is diagonalizable and has the eigenvalues $(\lambda_1^2, \ldots, \lambda_n^2)$. Thus f^2 is diagonalizable and has the eigenvalues $(\lambda_1^2, \ldots, \lambda_n^2)$. Note that we could have $\lambda_i^2 = \lambda_j^2$ even though $\lambda_i \neq \lambda_j$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of f. For any eigenvector u associated with the eigenvalue λ_i, we have $f(u) = \lambda_i u$, so by applying f to both sides we get

$$f(f(u)) = f(\lambda_i u) = \lambda_i f(u) = \lambda_i^2 u,$$

which shows that u is also an eigenvector of f^2 for the eigenvalue λ_i^2. Consequently we get the following inclusions of eigenspaces:

$$E_{\lambda_i}(f) \subseteq E_{\lambda_i^2}(f^2), \quad i = 1, \ldots, m. \tag{$*_1$}$$

Since f is diagonalizable, we have

$$\mathbb{C}^n = E_{\lambda_1}(f) \oplus \cdots \oplus E_{\lambda_m}(f). \tag{$*_2$}$$

If we assume that $\lambda_i^2 = \lambda_j^2$ implies that $\lambda_i = \lambda_j$, then $(\lambda_1^2, \ldots, \lambda_m^2)$ are the distinct eigenvalues of f^2, so

$$\mathbb{C}^n = E_{\lambda_1^2}(f^2) \oplus \cdots \oplus E_{\lambda_m^2}(f^2). \tag{$*_3$}$$

But $(*_1)$ implies that

$$\dim(E_{\lambda_i}(f)) \leq \dim(E_{\lambda_i^2}(f^2)), \quad 1 \leq i \leq m, \tag{$*_4$}$$

$(*_2)$ implies that

$$n = \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_m}(f)), \tag{$*_5$}$$

and $(*_3)$ implies that

$$n = \dim(E_{\lambda_1^2}(f^2)) + \cdots + \dim(E_{\lambda_m^2}(f^2)), \tag{$*_6$}$$

and we conclude that

$$\dim(E_{\lambda_i}(f)) = \dim(E_{\lambda_i^2}(f^2)), \quad 1 \leq i \leq m, \tag{$*_7$}$$

which implies that

$$E_{\lambda_i}(f) = E_{\lambda_i^2}(f^2), \quad 1 \leq i \leq m. \tag{$*_8$}$$

(2) We know from the spectral theorem that a self-adjoint linear map f from $\mathbb{R}^n$ to itself has real eigenvalues and is diagonalizable. If in addition the eigenvalues of f are nonnegative, then $\lambda_i^2 = \lambda_j^2$ implies that $\lambda_i = \lambda_j$. It follows from (1) that if $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of f, then $\lambda_1^2, \ldots, \lambda_m^2$ are the distinct eigenvalues of f^2, and

$$E_{\lambda_i}(f) = E_{\lambda_i^2}(f^2), \quad 1 \leq i \leq m. \tag{$*_9$}$$

The same fact holds for g. But if $f^2 = g^2$, then f^2 and g^2 have the same distinct eigenvalues $\lambda_1^2, \ldots, \lambda_m^2$, and since by hypothesis f and g have nonnegative eigenvalues, then f and g have the same distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. The maps f^2 and g^2 also have the same eigenspaces $E_{\lambda_i^2}(f^2) = E_{\lambda_i^2}(g^2)$, so by $(*_9)$ and the fact that f and g have the same distinct eigenvalues $\lambda_1, \ldots, \lambda_m$,

$$E_{\lambda_i}(f) = E_{\lambda_i}(g), \quad 1 \leq i \leq m. \tag{$*_{10}$}$$

Consequently, f and g agree on all the eigenspaces, and since

$$\mathbb{C}^n = E_{\lambda_1}(f) \oplus \cdots \oplus E_{\lambda_m}(f),$$

we conclude that $f = g$.

Total: 65 points