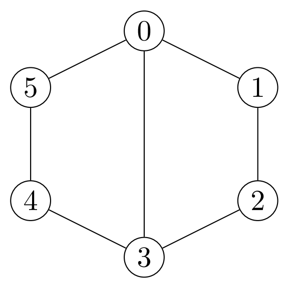
Problem Set

1. **[10 pts]** Let $p \geq 2$. Consider the graph G_p whose set of $2p$ vertices is $[0..(2p-1)]$ and whose set of $2p+1$ edges is

$$\{0\text{-}1, 1\text{-}2, \ldots, (p-1)\text{-}p, p\text{-}(p+1), \ldots, (2p-2)\text{-}(2p-1), (2p-1)\text{-}0\} \cup \{0\text{-}p\}$$

For illustration, here is how G_3 looks like:



Count the number of distinct spanning trees of G_p. Express your answer in terms of p. Justify your answer.

Solution:

We know from lecture that a spanning tree of G_p has one less edge than the number of vertices hence $2p-1$ edges. Since G_p has $2p+1$ edges, exactly two of its edges must be deleted in order to obtain a spanning tree, provided the resulting graph is connected. Define
$A = \{0\text{-}1, 1\text{-}2, \ldots, (p-1)\text{-}p\}$
$B = \{p\text{-}(p+1), \ldots, (2p-2)\text{-}(2p-1), (2p-1)\text{-}0\}$
and note that both A and B have cardinality p.

If we delete two or more edges from A the result is disconnected so it cannot be a spanning tree. The same holds for B. Therefore we could delete at most one edge from A. Similarly, we could delete at most one edge from B. Finally, if we delete the edge 0–p then deleting any other edge will result in a connected graph.

Putting these observations together we have three *disjoint* cases in which we delete two edges and obtain a connected graph (hence a spanning tree):

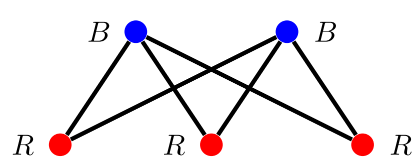
Case 1. Delete one edge of A and one edge of B. This can be done in $p \cdot p = p^2$ ways.

Case 2. Delete one edge of A and the edge 0–p. This can be done in $p \cdot 1 = p$ ways.

Case 3. Delete one edge of B and the edge 0–p. This can be done in $p \cdot 1 = p$ ways.

By the addition rule there are $p^2 + 2p$ ways to delete two edges of G_p and still have a connected graph so there are $p^2 + 2p$ distinct spanning trees.

2. [**15 pts**] The **complete bipartite graph** $K_{m,n}$ where $m, n \geq 1$, has m red nodes, n blue nodes and has an edge between every red node and every blue node. There are no edges between nodes of the same color. For illustration, here is $K_{3,2}$:



(a) Let D be the number of edges of $K_{m,n}$ that we have to delete from the graph to obtain a spanning tree. Prove that D is divisible by $m - 1$.

(b) Assume that $m > n$. Prove that the length of a longest path in $K_{m,n}$ is even.

Solution:

(a) We have $m+n$ total vertices, and as every red node has an edge to every blue node, we have mn total edges. Any spanning tree of this graph will have a number of edges that is one less than the number of nodes, that is, $m+n-1$ edges. Thus, the number of edges, D, we must delete is equal to:

$$\begin{aligned} D &= mn - (m+n-1) \\ &= mn - m - n + 1 \\ &= m(n-1) - 1(n-1) \\ &= (m-1)(n-1) \end{aligned}$$

It follows that $m-1$ is a factor of D. (This proof is valid even when $n=1$ or when $m=1$ because 0 is divisible by any integer.)

Note. This was not required for the solution but note that, since $K_{m,n}$ is connected, we know from lecture that there is some (here, more than one!) way to delete D edges and obtain a spanning tree.

(b) Let P be a longest path in $K_{m,n}$, and let p_R, respectively p_B, be the number of red, respectively blue, vertices in P. (The length of P is $p_R + p_B - 1$ and we wish to show that this number is even.) Because edges only link nodes of distinct colors, the colors of the vertices along P *alternate*. Therefore, we either have $p_R = p_B$ or $p_R = p_B + 1$, or $p_B = p_R + 1$.

Claim $p_R = p_B$ cannot happen.

Proof of Claim Suppose, toward a contradiction, that $p_R = p_B$. Then, the first and last node in P (its *end* vertices) have different colors, so one of them must be blue. Even if all the blue vertices occur in P, i.e., $p_B = n$, we still must have have a red vertex v that does not occur in P because $m > n \geq p_B = p_R$. Then we can extend P with an edge from its blue end to v, which contradicts the assumption that P is longest.

It follows that the only possible cases are $p_R = p_B + 1$ or $p_B = p_R + 1$. In either of these cases the end vertices have the same color. We have, more generally:

Claim If a path Q in $K_{m,n}$ (or in any bipartite graph) has ends of the same color then its length is even.
Proof of Claim WLOG assume that both ends of Q are blue. It takes two edges of Q to get from a blue vertex in Q to the next blue vertex. So if there are q_B blue vertices in Q then it has $2(q_B - 1)$ edges. The length is the number of edges so we are done.

Therefore, the longest path has ends of the same color and any path with ends of the same color has even length, which proves this part.

Note. We are not asked to show this, but it is easy to prove, using similar arguments as in the proof of the first claim, that the longest path contains all the blue nodes and its ends are red.

3. **[10 pts]** Let G be the graph obtained by erasing one edge from K_5. What is the chromatic number of G? Prove your answer.

Solution:

K_5 is the graph defined by 5 vertices and edges between any two of the vertices. WLOG let $V = \{u, v, w, x, y\}$ be the set of vertices of this graph. Again WLOG we obtain the graph $G = (V, E)$ by removing, say, the edge $\{u, v\}$. We will prove that the chromatic number of G is 4 by showing that it's 4-colorable and that 4 is the minimum number of colors for which a proper coloring can be found.

We we will start from a proper coloring of K_5, which we know is 5-colorable. Now we change the color of v to the same color that u has.

This results in a 4-coloring of G. We claim that this coloring is proper. Since we changed only the color of v we only need to check the edges incident to v. w–v is one of these edges. In the coloring of K_5 the color of w was different from the color of u, therefore different from the color of v. Similarly for the edges x–v and y–v. It follows that every edge incident to v in G has distinct colors at its ends. Therefore, the coloring is proper.

To prove that 4 is the minimum, we will consider the vertices u, w, x and y. (Also works if you consider v, w, x and y.) Each of these vertices must be colored a different color since any pair of these vertices are still adjacent in G. Thus we need at least 4 colors.

4. **[10 pts]** Let G be an $m \times n$ grid graph for some $m, n \geq 2$. Prove that G is bipartite.

Solution:

For each $i \in [1..m]$ and $j \in [1..n]$, let $v_{i,j}$ be the vertex in the i^{th} row and j^{th} column. Color $v_{i,j}$ blue if $i + j$ is even and red if $i + j$ is odd.

Each $v_{i,j}$ has at most four neighbors: $v_{i-1,j}$, $v_{i,j-1}$, $v_{i+1,j}$, and $v_{i,j+1}$. If $v_{i,j}$ is blue, then $i + j$ is even, then $i - 1 + j$, $i + j - 1$, $i + 1 + j$, and $i + j + 1$ are all odd, so all of $v_{i,j}$'s neighbors are red. Similarly, if $v_{i,j}$ is red, then all of its neighbors are blue. Since this is a proper coloring, we conclude that G is bipartite.

5. **[10 pts]** Prove that C_5 is isomorphic to its complement.

Solution:

C_5 is the cycle graph $(\{1, 2, 3, 4, 5\}, \{1{-}2, 2{-}3, 3{-}4, 4{-}5, 5{-}1\}$. Its complement is the graph

$$\overline{C_5} = (\{1, 2, 3, 4, 5\}, \{1{-}3, 3{-}5, 5{-}2, 2{-}4, 4{-}1\}) .$$

Let $\beta : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ be the bijection with $\beta(1) = 1$,

$\beta(2) = 3$, $\beta(3) = 5$, $\beta(4) = 2$, and $\beta(5) = 4$. This has the property that u–v is an edge in C_5 if and only if $\beta(u)$–$\beta(v)$ is an edge in $\overline{C_5}$, so it is an isomorphism.

6. **[10 pts]** Prove that for all $k > 5$, C_k has fewer edges than its complement.

Solution:

C_k has k edges, and there are $\binom{k}{2}$ possible edges on the vertex set $[1..k]$, so $\overline{C_k}$ has $\binom{k}{2} - k$ edges. Hence, we have

$$\begin{aligned} k &> 5 \\ k - 1 &> 4 \\ \frac{k * (k-1)}{2} &> 2k \\ \binom{k}{2} &> 2k \\ \binom{k}{2} - k &> k \end{aligned}$$

7. **[6 pts] EXTRA CREDIT CHALLENGE PROBLEM**

Suppose that $f : V \to [1..k]$ is a proper k-coloring of a graph $G = (V, E)$, for some $k \geq 1$. Prove that G has some independent set $S \subseteq V$ with $|S| \geq |V|/k$.

Solution:

For $i = 1, 2, \ldots, k$, let $V_i = \{v \in V \mid f(v) = i\}$. That is, V_i is all the vertices with the color i. Then each V_i is an independent set, by the definition of a proper coloring. Furthermore, $|V_1| + |V_2| + \ldots + |V_k| = |V|$.

Now, let S be the V_i with the greatest cardinality. (If there are multiple V_i of the same cardinality, choose any one of them.) Then $\underbrace{|S| + |S| + \ldots + |S|}_{k \text{ times}} \geq |V_1| + |V_2| + \ldots + |V_k|$, so $k \cdot |S| \geq |V|$ and $|S| \geq |V|/k$.