Module 1

Vector Spaces, Bases, Linear Maps
Solutions

Problem 1: 10 points

Let A_4 be the following matrix:

$$A_4 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix}.$$

Prove that the columns of A_4 are linearly independent. Find the coordinates of the vector $x = (7, 14, -1, 2)$ over the basis consisting of the column vectors of A_4.

Solution. We show that for any $x \in \mathbb{R}^4$, if $A_4 x = 0$, then $x = 0$, which shows that the columns of A_4 linearly independent. This is because if we write $x = (x_1, x_2, x_3.x_4)$, then $Ax = x_1A^1 + x_2A^2 + x_3A^3 + x_4A^4$, where A^1, A^2, A^3, A^4 are the columns of A, and what we proved is that if $x_1A^1 + x_2A^2 + x_3A^3 + x_4A^4 = 0$ then $x_1 = x_2 = x_3 = x_4 = 0$, which is exactly the definition of linear independence.

The linear system $A_4 x = 0$ is written as

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + 2x_3 + 3x_4 &= 0 \\ -x_1 + x_3 - x_4 &= 0 \\ -2x_1 - x_2 + 4x_3 &= 0. \end{aligned}$$

We proceed by elimination. The first equation yields

$$x_1 = -2x_2 - x_3 - x_4, \qquad \text{(eq1)}$$

and by substituting the right-hand side for x_1 in the other equations we get

$$\begin{aligned} 2(-2x_2 - x_3 - x_4) + 3x_2 + 2x_3 + 3x_4 &= 0 \\ -(-2x_2 - x_3 - x_4) + x_3 - x_4 &= 0 \\ -2(-2x_2 - x_3 - x_4) - x_2 + 4x_3 &= 0, \end{aligned}$$

which simplifies to

$$\begin{aligned} -x_2 + x_4 &= 0 \\ 2x_2 + 2x_3 &= 0 \\ 3x_2 + 6x_3 + 2x_4 &= 0. \end{aligned}$$

The first equation yields

$$x_2 = x_4, \tag{eq2}$$

and by substituting the right-hand side for x_2 in the other equations we get

$$\begin{aligned} 2x_4 + 2x_3 &= 0 \\ 3x_4 + 6x_3 + 2x_4 &= 0. \end{aligned}$$

which simplifies to

$$\begin{aligned} 2x_4 + 2x_3 &= 0 \\ 5x_4 + 6x_3 &= 0. \end{aligned}$$

The first equation yields

$$x_3 = -x_4 \tag{eq3}$$

and by substituting the right-hand side for x_3 in the last equation we get

$$5x_4 + 6(-x_4) = 0,$$

which yields

$$-x_4 = 0,$$

that is, $x_4 = 0$. But, using (eq1), (eq2, (eq3), we deduce that $x_1 = x_2 = x_3 = x_4 = 0$.

To find the coordinates of the vector $x = (7, 14, -1, 2)$ over the basis consisting of the column vectors of A_4, we need to find $z = (z_1, z_2, z_3, z_4)$ such that

$$z_1A^1 + z_2A^2 + z_3A^3 + z_4A^4 = x,$$

which means that we need to solve the system $A_4z = x$, that is

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ -1 \\ 2 \end{pmatrix},$$

which is also expressed as

$$\begin{aligned} z_1 + 2z_2 + z_3 + z_4 &= 7 \\ 2z_1 + 3z_2 + 2z_3 + 3z_4 &= 14 \\ -z_1 + z_3 - z_4 &= -1 \\ -2z_1 - z_2 + 4z_3 &= 2. \end{aligned}$$

Again, we proceed by elimination. The first equation yields

$$z_1 = -2z_2 - z_3 - z_4 + 7 \tag{eq4}$$

and by substituting the right-hand side for z_1 in the other equations we get

$$\begin{aligned} 2(-2z_2 - z_3 - z_4 + 7) + 3z_2 + 2z_3 + 3z_4 &= 14 \\ -(-2z_2 - z_3 - z_4 + 7) + z_3 - z_4 &= -1 \\ -2(-2z_2 - z_3 - z_4 + 7) - z_2 + 4z_3 &= 2, \end{aligned}$$

which simplifies to

$$\begin{aligned} -z_2 + z_4 &= 0 \\ 2z_2 + 2z_3 &= 6 \\ 3z_2 + 6z_3 + 2z_4 &= 16. \end{aligned}$$

The first equation yields

$$z_2 = z_4, \tag{eq5}$$

and by substituting the right-hand side for z_2 in the other equations we get

$$\begin{aligned} 2z_4 + 2z_3 &= 6 \\ 3z_4 + 6z_3 + 2z_4 &= 16. \end{aligned}$$

which simplifies to

$$\begin{aligned} 2z_4 + 2z_3 &= 6 \\ 5z_4 + 6z_3 &= 16. \end{aligned}$$

The first equation yields

$$z_3 = -z_4 + 3 \tag{eq6}$$

and by substituting the right-hand side for z_3 in the last equation we get

$$5z_4 + 6(-z_4 + 3) = 16,$$

which yields

$$z_4 = 2.$$

By (eq6) we get $z_3 = -2 + 3 = 1$, by (eq5) we get $z_2 = 2$, and by (eq4) we get $z_1 = -2(2) - 1 - 2 + 7 = 0$. Therefore $z = (0, 2, 1, 2)$.

Problem 2: 10 points

Consider the following Haar matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Prove that the columns of H are linearly independent.

Hint. Compute the product $H^\top H$.

Solution. We have

$$H^\top H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

By multiplying both sides of the above equation by the inverse of the right-hand side, we obtain

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that is,

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} H^\top H = I_4.$$

The above shows that

$$H^{-1} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} H^\top = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} H^\top.$$

Since H is invertible, its columns are linearly independent, because if $Hx = 0$, then $Ix = H^{-1}Hx = H^{-1}0 = 0$, that is, $x = 0$.

We can also prove by elimination that if $Hx = 0$, then $x = 0$. The linear system $Hx = 0$ is expressed by

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_4 &= 0 \\ x_1 - x_2 - x_4 &= 0. \end{aligned}$$

Adding up the first two equations we get

$$2(x_1 + x_2) = 0,$$

which is equivalent to

$$x_1 + x_2 = 0, \tag{h1}$$

and adding up the last two equations we get

$$2(x_1 - x_2) = 0,$$

which is equivalent to

$$x_1 - x_2 = 0. \tag{h2}$$

Adding up (h1) and (h2) we get $2x_1 = 0$, that is, $x_1 = 0$, and subtracting (h2) from (h1) we get $2x_2 = 0$, that is, $x_2 = 0$. Susbtituting $x_1 = 0$ and $x_2 = 0$ in the first and the third equations of the original system, we get

$$\begin{aligned} x_3 &= 0 \\ x_4 &= 0. \end{aligned}$$

In conclusion we proved that $x_1 = x_2 = x_3 = x_4 = 0$, thus the columns of H are linearly independent.

Problem 3: 10 points

Let $E = \mathbb{R} \times \mathbb{R}$ and define the addition operation

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

and the multiplication operation $\cdot \colon \mathbb{R} \times E \to E$ by

$$\lambda \cdot (x, y) = (\lambda x, y), \quad \lambda, x, y \in \mathbb{R}.$$

Show that E with the above operations $+$ and $\cdot$ is not a vector space. Which of the axioms is violated?

Solution. Observe that

$$(\lambda + \mu) \cdot (x, y) = ((\lambda + \mu)x, y),$$

and

$$\lambda \cdot (x, y) + \mu \cdot (x, y) = (\lambda x, y) + (\mu x, y) = ((\lambda + \mu)x, 2y),$$

so if $y \neq 0$,

$$(\lambda + \mu) \cdot (x, y) = ((\lambda + \mu)x, y) \neq ((\lambda + \mu)x, 2y) = (\lambda x, y) + (\mu x, y) = \lambda \cdot (x, y) + \mu \cdot (x, y),$$

which means that Axiom (V2) fails. Thus E with the above operations is not a vector space.

Problem 4: 15 points total

(1) (5 points) Let A be an $n \times n$ matrix. If A is invertible, prove that for any $x \in \mathbb{R}^n$, if $Ax = 0$, then $x = 0$.

(2) (10 points) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that $I_m - AB$ is invertible iff $I_n - BA$ is invertible.

Hint. If for all $x \in \mathbb{R}^n$, $Mx = 0$ implies that $x = 0$, then M is invertible.

Solution. (a) If A is invertible and $Ax = 0$, by multipling both sides on the left by A^{-1}, we get

$$A^{-1}Ax = A^{-1}0,$$

and because $A^{-1}A = I$ and $A^{-1}0 = 0$, this gives us

$$Ix = x = 0,$$

so $x = 0$, as claimed.

(b) Observe that

$$(I - AB)A = A - ABA = A(I - BA),$$

and similarly

$$(I - BA)B = B - BAB = B(I - AB).$$

We prove that if $I - AB$ is invertible, then for all $x \in \mathbb{R}^n$, if $(I - BA)x = 0$, then $x = 0$.

Pick any $x \in \mathbb{R}^n$ such that $(I - BA)x = 0$. Then we have

$$A(I - BA)x = A0 = 0,$$

and because

$$(I - AB)A = A(I - BA),$$

we get

$$(I - AB)Ax = 0.$$

Because $I - AB$ is invertible, by (a) we must have $Ax = 0$. Now since we assumed that $(I - BA)x = 0$, we get

$$0 = (I - BA)x = x - BAx = x - 0 = x;$$

that is, $x = 0$.

In summary, we proved that if $I - AB$ is invertible and if $(I - BA)x = 0$, then $x = 0$. By a previous remark, this implies that $I - BA$ is invertible.

A similar argument using the fact that

$$(I - BA)B = B(I - AB).$$

shows that if $I - BA$ is invertible, then so is $I - AB$. Therefore, $I - AB$ is invertible iff $I - BA$ is invertible.

Alternate solution. It turns out that if $I - BA$ is invertible, then the inverse of $I - AB$ can be expressed in terms of $(I - BA)^{-1}$ (similarly if $I - AB$ is invertible, then the inverse of $I - BA$ can be expressed in terms of $(I - AB)^{-1}$). To see how to get such a formula, let's assume that we can express the inverse of $I - BA$ as a power series

$$(I - BA)^{-1} = I + BA + BABA + \cdots + (BA)^n + \cdots .$$

The above does not necessarily make sense but it will suggest a formula for the inverse of $I - AB$, and then we will check that the formula works. A temporary digression into nonsense sometimes helps!

Observe that

$$\begin{aligned} A(I - BA)^{-1}B &= AB + ABAB + ABABAB + \cdots + A(BA)^n B + \cdots \\ &= AB + ABAB + ABABAB + \cdots + (AB)^{n+1} + \cdots , \end{aligned}$$

and since (assuming convergence of the series!)

$$(I - AB)^{-1} = I + AB + ABAB + \cdots + (AB)^n + \cdots ,$$

we deduce that we should have

$$(I - AB)^{-1} = I + A(I - BA)^{-1}B.$$

Let us check that $I + A(I - BA)^{-1}B$ is indeed the inverse of $I - AB$, provided that $(I - BA)^{-1}$ exists. We have

$$\begin{aligned} (I - AB)[I + A(I - BA)^{-1}B] &= I + A(I - BA)^{-1}B - AB - ABA(I - BA)^{-1}B \\ &= I + A[(I - BA)^{-1} - BA(I - BA)^{-1} - I]B \\ &= I + A[(I - BA)(I - BA)^{-1} - I]B \\ &= I + A(I - I)B \\ &= I + 0 = I, \end{aligned}$$

which proves that $I + A(I - BA)^{-1}B$ is a right inverse of $I - AB$. But then we proved in class that $I - AB$ is invertible and that $I + A(I - BA)^{-1}B$ is the inverse of $I - AB$. By swapping A and B, the same computation shows that if $I - AB$ is invertible, then $I + B(I - AB)^{-1}A$ is the inverse of $I - BA$.

Problem 5: 10 points

Let $f : E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1} : F \to E$ is linear.

Solution. Since f is linear and invertible as a function, we have

$$\begin{aligned} f(f^{-1}(u) + f^{-1}(v)) &= f(f^{-1}(u)) + f(f^{-1}(v)) \\ &= u + v, \end{aligned}$$

for all $u, v \in E$. Since f is invertible, we get

$$f^{-1}(u) + f^{-1}(v) = f^{-1}(u + v).$$

Similarly,

$$\begin{aligned} f(\lambda f^{-1}(u)) &= \lambda f(f^{-1}(u)) \\ &= \lambda u, \end{aligned}$$

for all $u \in E$ and all $\lambda \in \mathbb{R}$. Since f is invertible, we get

$$\lambda f^{-1}(u) = f^{-1}(\lambda u).$$

Consequently, f^{-1} is indeed linear.

Total: 55 points