CIT 592 Summer 2019 **Homework 3** **Solutions**

Problem Set

1. [10 pts]

Kiara is hosting a s'mores bonfire for ten of her friends. In how many ways can she distribute 20 marshmallows, 30 graham crackers, and 15 chocolate bars (all indistinguishable) to her ten (distinguishable) friends so that each one of them gets at least one marshmallow, at least one graham cracker, and at least one chocolate bar?

Solution:

First, she must ensure that each friend gets at least one of each ingredient. After giving each friend the minimum amount, she has 10 marshmallows, 20 graham crackers, and 5 chocolate bars. Ways to distribute the remaining ingredients can be counted with stars and bars: The number of ways to distribute n indistinguishable objects to k distinguishable people is $\binom{n+k-1}{k-1}$. So Kiara can distribute all the ingredients with the following procedure.

Step 1: Give 1 marshmallow, 1 graham cracker, and 1 chocolate bar to each of the 10 friends. (1 way)
Step 2: Distribute the remaining 10 marshmallows. ($\binom{19}{9}$ ways)
Step 3: Distribute the remaining 10 graham crackers. ($\binom{29}{9}$ ways)
Step 4: Distribute the remaining 10 chocolate bars. ($\binom{14}{9}$ ways)

By the multiplication rule, there are

$$1 \cdot \binom{19}{9} \cdot \binom{29}{9} \cdot \binom{14}{9} = \boxed{\frac{19!29!14!}{(9!)^3 10!20!5!}}$$

ways for Kiara to distribute the s'mores ingredients.

2. [10 pts]

Consider the following statement.

> There exist integers a and c such that for all integers x if $x \geq a$ then $x^2 < c \cdot x$.

Write the negation of this statement, and then disprove the statement by counterexample.

Solution:

The negation is "For all integers a and c there exists an integer x such that $x \geq a$ and $x^2 \geq c \cdot x$."

Let a and c be any integers, and let $x = \max\{a, c\}$ (i.e., either let $x = a$ or let $x = c$, whichever is greater). Then x is an integer, $x \geq a$, and $x^2 = x \cdot x \geq c \cdot x$. Thus, this x proves the negation and is a counterexample to the statement. $\square$

3. [10 pts]

In solving this problem it will be useful to remember the rules of thumb about even, odd, and arithmetical operations, for example, if m and n are odd then mn is odd, etc.

Let x, y, z be three integers. Consider the following implication.

> If $x^2 = y^2 + z^2$, then xyz is even.

Write down the contrapositive of this implication, and then prove it.

Solution:

The contrapositive of this implication is: "If xyz is odd, then $x^2 \neq y^2 + z^2$."

Assume that xyz is odd. We show that x, y, and z are all odd. Assume for contradiction that at least one of x, y, z is even. Without loss of generality, assume that x is even. Then, xyz is the product of an even integer x and another integer yz, so xyz is even. Since xyz was assumed to be odd, this is a contradiction. Therefore, x, y, and z are all odd.

Furthermore, since x^2, y^2, and z^2 are each the product of two odd integers, they are all odd. Hence, $y^2 + z^2$ is even, and therefore it cannot be equal to the odd integer x^2. Thus, we have $x^2 \neq y^2 + z^2$, as desired. $\square$

4. [10 pts]

In solving this problem, you may find it helpful to remember the *rules of thumb* for adding and multiplying odd and even numbers (e.g., an odd plus an odd is even).

Prove by contradiction that if x is rational, then $x^2 \neq x + 1$.

Solution:

Let x be any rational number, and assume for contradiction that $x^2 = x + 1$. There exist integers a and b such that $\frac{a}{b} = x$, and we can assume without loss of generality that the fraction is in lowest terms, meaning that a and b have no common factors. We can rewrite the equation as $\frac{a^2}{b^2} = \frac{a}{b} + 1$ and rearrange it as $a^2 = b \cdot (a + b)$. Since a and b have no common factors, they cannot both be even. This leaves three possibilities.

Case 1: a is even and b is odd. Then a^2 is even, $a + b$ is odd, and $b \cdot (a+b)$ is odd. But an even cannot equal an odd, so this case contradicts $a^2 = b \cdot (a + b)$.

Case 2: a is odd and b is even. Then a^2 is odd, $a+b$ is odd, and $b\cdot(a+b)$ is even. Again, this case contradicts $a^2 = b\cdot(a+b)$.

Case 3: a and b are both odd. Then a^2 is odd, $a+b$ is even, and $b\cdot(a+b)$ is even. Again, this case contradicts $a^2 = b\cdot(a+b)$.

Since we reach a contradiction in every case, we conclude that our assumption was false, and therefore $x^2 \neq x+1$. $\square$

5. [10 pts]

Using truth tables, *prove or disprove* each of the following. Please make a separate column in your truth table for each intermediate logical step. For example, to prove/disprove that $p \Rightarrow (q \wedge p) \equiv p \vee q$, you would make a column for each of: p, q, $q \wedge p$, $p \Rightarrow (q \wedge p)$, and $p \vee q$.

(a) $r \Rightarrow (p \vee \neg q) \;\equiv\; (q \wedge r) \Rightarrow p$

(b) $(p \Rightarrow q) \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$

Solution:

(a)

p	q	r	$\neg q$	$p \vee \neg q$	$r \Rightarrow (p \vee \neg q)$	$(q \wedge r)$	$(q \wedge r) \Rightarrow p$
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	T
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	F	T	F
F	T	F	F	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Since $r \Rightarrow (p \vee \neg q)$ and $(q \wedge r) \Rightarrow p$ have the same truth value for all possible assignments of p, q, and r, (a) is a logical equivalence. $\square$

(b)

p	q	r	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow r$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

As $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$ differ for $p = F$, $q = T$, and $r = F$ and for $p = F$, $q = F$, and $r = F$, (b) is not a logical equivalence. $\square$

6. [10 pts]

How many anagrams of `hippopotamus` are there that have at least two consecutive `p`'s?

Solution:

We can construct such an anagram by first choosing an anagram of the ten-character string `hiopotamus`, then deciding where in the resulting string to put `pp`. There are $10!/2!$ anagrams of `hiopotamus`, since `o` is the only repeated letter. For each of these, we could put `pp` in one of 11 places: before the first letter, before the second letter, etc., or after the last letter. Putting `pp` immediately before the other `p` will result in the same string as putting `pp` immediately after the other `p`, so there are 10 distinct ways to insert `pp`. In other words, we remove the potential spot for *pp* positioned directly after *p* in the sequence. Thus, by the multiplication rule, there are $10!/2! \cdot 10 = \boxed{5 \cdot 10!}$ such anagrams.

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Vova has invented a game with paper cups. He lines up 121 cups face down in a straight line from left to right and consecutively labels them from 1 to 121. Vova then walks from left to right down the line of cups, flipping all of the cups over. He returns to the left end of the line, then makes a second pass from left to right, this time flipping cups $2, 4, 6, 8 \ldots$. On the third pass, he flips cups $3, 6, 9, 12 \ldots$. He continues like this: On the i^{th} pass, he flips over cups $i, 2i, 3i, 4i, \ldots$. (By "flip," we mean changing the cup from face down to face up or vice versa.) After 121 passes, how many cups are face up?

Solution:

Notice that Vova flips cup i on pass j if and only if $j \mid i$. Therefore, the number of times cup i is flipped is equal to the number of divisors of i.

We are interested in finding the number of cups which are face up at the end of the process. A cup must have been flipped an odd number of times to begin face down and end face up. Hence, we wish to count the number of integers between 1 and 121 (inclusive) that have an odd number of divisors.

Most numbers have an even number of divisors because you can "pair up" its divisors. That is, for every value a which is a factor of n, $\frac{n}{a}$ is also a factor of n. (This is true by the very definition of factors). All factors can be paired in this way, and each factor can be paired with exactly one other factor. The only numbers for which this is not true are perfect squares. For all numbers m^2, m is a factor but it cannot be paired with a different factor because $\frac{m^2}{m} = m$. Thus, perfect squares (and only perfect squares) have an odd number of factors.

For example, the factors of 6 can be "paired up" as $(1, 6)$ and $(2, 3)$. The factors of 9 $(1, 3, 9)$, however, cannot be paired up because 3 cannot be paired with another distinct factor.

There are 11 perfect squares between 1 and 121 inclusive: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121$. Thus, there will be $\boxed{11}$ cups that end up face up.