Problem Set

1. [10 pts]

Show that each of the following functions is not a bijection by giving either

- an element of the codomain that is not in the range, or
- two elements of the domain that map to the same element in the range.

(a) $a : \mathbb{N} \to \mathbb{N}$ given by $a(x) = 5x$

(b) $b : \mathbb{N} \to \mathbb{N}$ given by $b(x) = x + 5$

(c) $c : [1..6] \to [4..8]$ given by $c(x) = \begin{cases} 6 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$

(d) $d : \mathbb{Z} \to \mathbb{Z}$ given by $d(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ -x & \text{if } x \text{ is odd} \end{cases}$

(e) $e : [-7, 10] \to [0, 12]$ given by $e(x) = |x + 2|$

Solution:

(a) Not surjective because there is no $x \in \mathbb{N}$ such that $a(x) = 2$.

(b) Not surjective because there is no $x \in \mathbb{N}$ such that $b(x) = 4$.

(c) Not injective because $c(1) = c(6)$.

(d) Not surjective because there is no $x \in \mathbb{N}$ such that $d(x) = 0$.

(e) Not injective because $e(-4) = e(0)$.

2. [10 pts]

In a class of 77 students, 29 of them were born in months that are spelled with a `u` (January, February, June, July, August), 41 were born in months with an `e` (February, June, September, October, November, December), 38 were born in months with an `a` (January, February, March, April, May, August), 6 were born in January, 6 were born in February, and 5 were born in June. How many of the students were born in August?

Solution:

Let A be the set of all students in the class who were born in months that are spelled with an `a`, and similarly define the sets E and U. Then we have $|A| = 38$, $|E| = 41$, and $|U| = 29$.

Furthermore, all months have `a`, `e`, or `u` in their spelling, so $|A \cup E \cup U| = 77$. Only February has both `a` and `e`, and it also has `u`, so $|A \cap E| = |A \cap E \cap U| = 6$. Only February and June have `e` and `u` so $|E \cap U| = 6 + 5 = 11$. Only January, February, and August have `a` and `u`, so $|A \cap U| = 6 + 6 + x = 12 + x$, where x is the number of students born in August.

The principle of inclusion-exclusion for three sets tells us that

$$|A \cup E \cup U| = |A| + |E| + |U| - |A \cap E| - |E \cap U| - |A \cap U| + |A \cap E \cap U|.$$

Plugging in the values above gives

$$\begin{aligned} 77 &= 38 + 41 + 29 - 6 - 11 - (12 + x) + 6 \\ 77 &= 108 - 11 - 12 - x \\ 23 + x &= 108 - 77 \\ x &= \boxed{8}. \end{aligned}$$

3. [10 pts]

Recall that a derangement is a permutation where no element ends up in its original position. In this problem we consider a different, related concept: *deranged anagrams*. We say that an anagram is deranged if no letter ends up in its original position and no letter ends up in the original position of an identical letter. For example, `ffeeco` is a deranged anagram of `coffee`, but `eefcof` is not.

There are $\frac{(2+2+1)!}{2!2!1!} = 30$ anagrams of `cocoa`. How many of them are deranged?

Solution:

Since neither `c` can be in its original position, there are only three possibilities for which two letters are `c` in a deranged anagram:

(i) the second and fourth letter are `c`,

(ii) the second and fifth letter are `c`, or

(iii) the fourth and fifth letter are `c`.

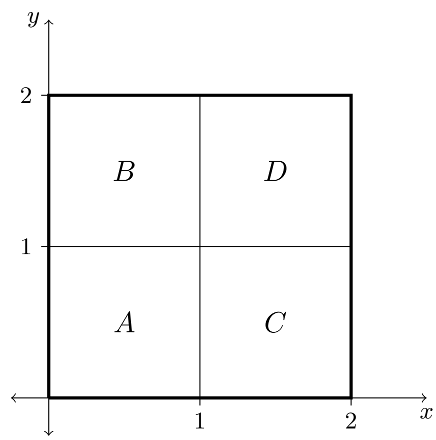
In the case (i), the `a` can be the first or third letter, so this case includes two anagrams. In case (ii), the `a` must be the fourth letter, since the fourth letter cannot be `o`, so this case includes one anagram. In case (iii), the `a` must be the second letter, since the second letter cannot be `o`, so this case includes one anagram. The cases are disjoint, so we can use the addition rule to conclude that there are $2 + 1 + 1 = \boxed{4}$ deranged anagrams. They are `acoco`, `ocaco`, `ocoac`, and `oaocc`.

4. [10 pts]

Let $S = [0, 2] \times [0, 2]$, and consider any five points in S. That is, five points inside (or on the edge of) a square with sides of length 2. Prove that among these points, there is some pair that is of distance at most $\sqrt{2}$ from each other.

Solution:

We use the pigeonhole principle. The "pigeons" are the five points, and the "pigeonholes" are the four smaller squares $A = [0, 1] \times [0, 1]$, $B = [0, 1] \times (1, 2]$, $C = (1, 2] \times [0, 1]$, and $D = (1, 2] \times (1, 2]$. Notice that $S = A \cup B \cup C \cup D$, and that A, B, C, and D are pairwise disjoint.



Since $5 > 4$, the pigeonhole principle tells us that at least two of the points must be in the same smaller square. Let (x_1, y_1) and (x_2, y_2) be the coordinates of two such points. The distance between them is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \sqrt{1^2 + 1^2} = \sqrt{2}$. $\square$

5. [10 pts]

Since Omcitville started keeping daily climate records in 1923, the highest temperature ever recorded was 39 °C, on August 12, 2011, and the lowest was 4 °C, on January 21, 1954. Prove that there were at least 11 days in 1997 on which Omcitville had the same recorded average temperature, rounded to the nearest degree Celsius.

Solution:

The average temperature cannot be less than the lowest temperature or greater than the highest temperature, so the range of possible temperatures is $[4..39]$, which has cardinality 36. There were 365 days in 1997, so the generalized pigeonhole principle tells us that since $365 > 10 \cdot 36$, there must have been $10 + 1 = 11$ days in 1997 with the same average temperature. $\square$

6. [10 pts]

Recall the *theorem of friends and strangers:* In any group of six people, there are three that are pairwise friends or there are three that are pairwise not friends.

Prove that the following claim is false.

> In any group of five people, there are three that are pairwise friends or there are three that are pairwise not friends.

Solution:

Let Alice, Bob, Carla, Dan, and Eve be the five people, and suppose that the only friendships are between Alice and Bob, Bob and Carla, Carla and Dan, Dan and Eve, and Eve and Alice. There are $\binom{5}{3} = 10$ trios, which we abbreviate by the initials of their members. The trios ABC, ABE, ADE, BCD, and CDE each have two friendships and one non-friendship, while ABD, ACD, ACE, BCE, and BDE each have one friendship and two non-friendships. Thus, this group is a counterexample to the the claim, and the claim is therefore false.

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

The Association of Tennis Professionals (ATP) ranks 200 men's tennis players and separately ranks 200 women's tennis players. There are no ties in either ranking.

Suppose that 20 of the 200 ATP-ranked men and 20 of the 200 ATP-ranked women register for a mixed doubles tournament (i.e., a tournament in which each team will consist of one man and one woman). Prove that among these 40 players the tournament organizers can form two non-overlapping mixed doubles teams with the same average ranking. That is, prove that there are two distinct men m_1 and m_2 and two distinct women w_1 and w_2 in the tournament such that

$$\frac{\text{rank}(m_1) + \text{rank}(w_1)}{2} = \frac{\text{rank}(m_2) + \text{rank}(w_2)}{2}.$$

Solution:

Each team is formed from 1 of 20 men and 1 of 20 women, so there are $20 \cdot 20 = 400$ possible mixed doubles teams.

The average ranking of a team is determined by adding two numbers in the interval $[1..200]$ and dividing by 2. So the numerator will be an integer in the interval $[2..400]$. In other words, there are $|[2..400]| = 399$ possible average rankings. Thus, by the pigeonhole principle, two of the 400 possible teams must share the same average ranking.

We have shown that there are two different teams with the same ranking, but we still need to show that they are non-overlapping. Let the two teams be (m_1, w_1) and (m_2, w_2), and assume for the sake of contradiction that $m_1 = m_2$. Then since $\frac{\text{rank}(m_1)+\text{rank}(w_1)}{2} = \frac{\text{rank}(m_2)+\text{rank}(w_2)}{2}$, we must have $\text{rank}(w_1) = \text{rank}(w_2)$, which implies that $w_1 = w_2$ because there are no ties in the rankings. But this contradicts the premise that (m_1, w_1) and (m_2, w_2) are different teams, so we conclude that $m_1 \neq m_2$. Similarly, $w_1 \neq w_2$. $\square$

8. [6 pts] ADDITIONAL EXTRA CREDIT CHALLENGE PROBLEM

Observe that $1 \mid (77 - 7)$, $2 \mid (77 - 7)$, $3 \mid (7777 - 7)$, $4 \mid (777 - 77)$, $5 \mid (77 - 7)$, $6 \mid (7777 - 7)$, $7 \mid (77 - 7)$, $8 \mid (7777 - 777)$, and $9 \mid (7777777777 - 7)$.

Prove that for every positive integer n, there exist i and j with $1 \le i < j \le n + 1$ such that n is a divisor of $\underbrace{77\ldots7}_{j\text{ digits}} - \underbrace{77\ldots7}_{i\text{ digits}}$.

Solution:

Let n be any positive integer, and consider the remainders when n is divided into each of the $n+1$ numbers 7, 77, 777, $\ldots$, $\underbrace{77\ldots7}_{n+1\text{ digits}}$. All of these $n+1$ remainders are in the range $[0..n-1]$, which has cardinality n. Thus, by the pigeonhole principle, two of the remainders are equal. That is, there are integers i, j, k, ℓ, and r with $1 \le i < j \le n + 1$ and $r \in [0..n - 1]$ such that $\underbrace{77\ldots7}_{i\text{ digits}} = kn + r$ and $\underbrace{77\ldots7}_{j\text{ digits}} = \ell n + r$. Now,

$$(\ell - k)n = \ell n + r - (kn + r) = \underbrace{77\ldots7}_{j\text{ digits}} - \underbrace{77\ldots7}_{i\text{ digits}},$$

so we have shown that n is a divisor of $\underbrace{77\ldots7}_{j\text{ digits}} - \underbrace{77\ldots7}_{i\text{ digits}}$. $\square$