Module 4

Determinants and Matrix Norms
Solutions

Problem 1: 10 points total

(1) (5 points) Given two vectors in $\mathbb{R}^2$ of coordinates $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

(2) (5 points) Given three vectors in $\mathbb{R}^3$ of coordinates $(d_1 - a_1, d_2 - a_2, d_3 - a_3)$, $(c_1 - a_1, c_2 - a_2, c_3 - a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution. (1) Two vectors are linearly dependent iff their determinant is zero, so the vectors $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$ are linearly dependent iff

$$D_1 = \begin{vmatrix} c_1 - a_1 & b_1 - a_1 \\ c_2 - a_2 & b_2 - a_2 \end{vmatrix} = 0.$$

Recall that because determinants are multilinear alternating maps, they are unchanged if we add a multiple of a column to another column, and they change sign if we swap two columns. If we subtract the first column of the following determinant from its second and its third column and then do a Laplace expansion according to the last row, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{vmatrix} = - \begin{vmatrix} c_1 - a_1 & b_1 - a_1 \\ c_2 - a_2 & b_2 - a_2 \end{vmatrix} = -D_1.$$

Therefore, the vectors $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$ are linearly dependent iff

$$D_1 = \begin{vmatrix} c_1 - a_1 & b_1 - a_1 \\ c_2 - a_2 & b_2 - a_2 \end{vmatrix} = 0$$

iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

(2) The vectors $(d_1 - a_1, d_2 - a_2, d_3 - a_3)$, $(c_1 - a_1, c_2 - a_2, c_3 - a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ are linearly dependent iff

$$D_2 = \begin{vmatrix} d_1 - a_1 & c_1 - a_1 & b_1 - a_1 \\ d_2 - a_2 & c_2 - a_2 & b_2 - a_2 \\ d_3 - a_3 & c_3 - a_3 & b_3 - a_3 \end{vmatrix} = 0.$$

If we subtract the first column of the following determinant from its second, third, and fourth column, and then do a Laplace expansion according to the last row, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ a_3 & b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \\ 1 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \end{vmatrix}.$$

Since determinants are alternating multilinear forms, by swapping the first and third column, we get

$$- \begin{vmatrix} b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \end{vmatrix} = \begin{vmatrix} d_1 - a_1 & c_1 - a_1 & b_1 - a_1 \\ d_2 - a_2 & c_2 - a_2 & b_2 - a_2 \\ d_3 - a_3 & c_3 - a_3 & b_3 - a_3 \end{vmatrix} = D_2.$$

Therefore, $(d_1 - a_1, d_2 - a_2, d_3 - a_3)$, $(c_1 - a_1, c_2 - a_2, c_3 - a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ are linearly dependent iff $D_2 = 0$ iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

Problem 2: 15 points total

Let A be an $n \times n$ real or complex matrix.

(1) (5 points) Prove that if $A^\top = -A$ (A is *skew-symmetric*) and if n is odd, then $\det(A) = 0$.

(2) (10 points) Prove that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + dc)^2.$$

Solution. (1) (5 points) Recall that the determinant function is multilinear alternating and that

$$\det(A) = \det(A^\top)$$

for any real or complex matrix A. It follows that for any $n \times n$ matrix A, if $A^\top = -A$, then we have

$$\det(A) = \det(A^\top) = \det(-A) = (-1)^n \det(A).$$

If n is odd, then $(-1)^n = -1$, so $\det(A) = -\det(A)$, that is, $2\det(A) = 0$, namely $\det(A) = 0$.

(2) (10 points) Using the Laplace expansion rule (using the first row), we have

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = -a \begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} + b \begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix} - c \begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix}.$$

Again, using the Laplace expansion rule (using the last row), we have

$$\begin{aligned} \begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} &= (-c)\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - (-f)\begin{vmatrix} -a & e \\ -b & f \end{vmatrix} \\ &= (-c)df - (-f)(-af + be) \\ &= -cdf - af^2 + bef, \end{aligned}$$

using the Laplace expansion rule (using the first row), we have

$$\begin{aligned} \begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix} &= (-a)\begin{vmatrix} -d & f \\ -e & 0 \end{vmatrix} + e\begin{vmatrix} -b & -d \\ -c & -e \end{vmatrix} \\ &= (-a)(ef) + e(be - dc) \\ &= -aef + be^2 - cde, \end{aligned}$$

and using the Laplace expansion rule (using the first row), we have

$$\begin{aligned} \begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix} &= (-a)\begin{vmatrix} -d & 0 \\ -e & -f \end{vmatrix} + d\begin{vmatrix} -b & -d \\ -c & -e \end{vmatrix} \\ &= (-a)df + d(be - cd) \\ &= -adf + bde - cd^2. \end{aligned}$$

Combining all these equations we get

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (-a)(-cdf - af^2 + bef) + b(-aef + be^2 - cde)$$

$$\begin{aligned} &+ (-c)(-adf + bde - cd^2) \\ &= acdf + a^2f^2 - abef - abef + b^2e^2 - bcde + acdf - cbde + c^2d^2 \\ &= (af)^2 + (be)^2 + (cd)^2 - 2afbe + 2afcd - 2becd. \end{aligned}$$

If we recall that

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz,$$

with $x = af, y = -be, z = cd = dc$, we see that

$$(af - be + dc)^2 = (af)^2 + (be)^2 + (cd)^2 - 2afbe + 2afcd - 2becd,$$

and we conclude that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + dc)^2.$$

Problem 3: 5 points

Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/2 \end{pmatrix}.$$

Compute the operator 2-norm $\|A\|_2$ of A.

Solution. We know from the course that the operator 2-norm $\|A\|_2$ of A is equal to the square root of the largest eigenvalue of $A^\top A$. Since A is symmetric, we have

$$A^\top A = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/2 \end{pmatrix} = \begin{pmatrix} 3/2 & 5/(2\sqrt{2}) \\ 5/(2\sqrt{2}) & 11/4 \end{pmatrix}.$$

The eigenvalues of $A^\top A$ are the zeros of the characteristic polynomial

$$\begin{vmatrix} \lambda - 3/2 & -5/(2\sqrt{2}) \\ -5/(2\sqrt{2}) & \lambda - 11/4 \end{vmatrix} = 0,$$

that is

$$\lambda^2 - \frac{17}{4}\lambda + 1 = 0.$$

The zeros of this equation are

$$\lambda = \frac{\frac{17}{4} \pm \sqrt{\frac{289}{16} - 4}}{2} = \frac{\frac{17}{4} \pm \sqrt{\frac{225}{16}}}{2} = \frac{\frac{17}{4} \pm \frac{15}{4}}{2} = 4, 1/4.$$

Since the largest eigenvalue of $A^\top A$ is $\lambda_1 = 4$, we have

$$\|A\|_2 = \sqrt{\lambda_1} = 2.$$

Since A is a symmetric matrix, it is normal, so we can use the result from lesson 4 that states that $\|A\|_2 = \rho(A)$. This means that we have to find the eigenvalues of A, and these are the zeros of the characteristic polynomial

$$\begin{pmatrix} \lambda - 1 & -1/\sqrt{2} \\ -1/\sqrt{2} & \lambda - 3/2 \end{pmatrix} = \lambda^2 - \frac{5}{2}\lambda + 1.$$

The zeros of the equation

$$\lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

are $\lambda_1 = 2$ and $\lambda_2 = 1/2$ (these are the positive square roots of the eigenvalues of $A^\top A = A^2$ that we found ealier). Again, we find that $\|A\|_2 = 2$.

Problem 4: 40 points total

Let A be a real 2×2 matrix

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} \\ a_{2\,1} & a_{2\,2} \end{pmatrix}.$$

(1) (10 points) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of A are the roots of the quadratic equation

$$X^2 - \mathrm{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) (10 points) If we let

$$\mu(A) = \frac{a_{1\,1}^2 + a_{1\,2}^2 + a_{2\,1}^2 + a_{2\,2}^2}{2|a_{1\,1}a_{2\,2} - a_{1\,2}a_{2\,1}|},$$

prove that

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) (20 points) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side b by

$$\Delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \Delta b$$

where $y = (y_1, y_2)$. Check that

$$\Delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\Delta x\|_2$, $\|b\|_2$, $\|\Delta b\|_2$, and estimate

$$c = \frac{\|\Delta x\|_2}{\|x\|_2} \left(\frac{\|\Delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Check that

$$c \approx \mathrm{cond}_2(A_m) = 39{,}206.$$

Solution. (1) The squares σ_1^2 and σ_2^2 of the singular values of A are the roots of the characteristic polynomial $\det(XI - A^\top A)$. For any 2×2 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we know from Lesson 4 of Module 4 that

$$\det(XI_2 - M) = X^2 - (m_{11} + m_{22})X + m_{11}m_{22} - m_{12}m_{21} = X^2 - \mathrm{tr}(M)X + \det(M),$$

with $\mathrm{tr}(M) = m_{11} + m_{22}$. Apply the above to $M = A^\top A$, we get

$$\det(XI - A^\top A) = X^2 - \mathrm{tr}(A^\top A)X + \det(A^\top A).$$

However, $\det(A^\top A) = \det(A^\top)\det(A) = \det(A)^2 = |\det(A)|^2$, so σ_1^2 and σ_2^2 are the roots of the quadratic equation

$$X^2 - \mathrm{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) We have

$$\frac{\sigma_1^2}{\sigma_2^2} = \frac{\mathrm{tr}(A^\top A) + \sqrt{(\mathrm{tr}(A^\top A))^2 - 4|\det(A)|^2}}{\mathrm{tr}(A^\top A) - \sqrt{(\mathrm{tr}(A^\top A))^2 - 4|\det(A)|^2}} = \left(\frac{\mathrm{tr}(A^\top A) + \sqrt{(\mathrm{tr}(A^\top A))^2 - 4|\det(A)|^2}}{2|\det(A)|} \right)^2,$$

where we have multiplied both the numerator and the denominator of the fraction by $\operatorname{tr}(A^\top A)+\sqrt{(\operatorname{tr}(A^\top A))^2-4|\det(A)|^2}$ to make the denominator rational, which implies that

$$\frac{\sigma_1}{\sigma_2}=\frac{\operatorname{tr}(A^\top A)}{2|\det(A)|}+\sqrt{\left(\frac{\operatorname{tr}(A^\top A)}{2|\det(A)|}\right)^2-1},$$

since $\sigma_1\geq\sigma_2>0$.

Observe that $\operatorname{tr}(A^\top A)=a_{1\,1}^2+a_{1\,2}^2+a_{2\,1}^2+a_{2\,2}^2$ and $\det(A)=a_{1\,1}a_{2\,2}-a_{1\,2}a_{2\,1}$, so if we let

$$\mu(A)=\frac{a_{1\,1}^2+a_{1\,2}^2+a_{2\,1}^2+a_{2\,2}^2}{2|a_{1\,1}a_{2\,2}-a_{1\,2}a_{2\,1}|},$$

we get

$$\operatorname{cond}_2(A)=\frac{\sigma_1}{\sigma_2}=\mu(A)+(\mu(A)^2-1)^{1/2}.$$

(3) The system

$$\begin{pmatrix}100 & 99\\ 99 & 98\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}199\\ 197\end{pmatrix}$$

obviously has the solution $x_1=1, x_2=1$.

When we perturb the right-hand side b by

$$\Delta b=\begin{pmatrix}-0.0097\\ 0.0106\end{pmatrix}$$

we need to solve the new system

$$\begin{aligned}100y_1+99y_2&=198.9903\\ 99y_1+98y_2&=197,0106.\end{aligned}$$

If we substract the second equation from the first, we get

$$y_1+y_2=1.9797.$$

The first equation can also be written as

$$y_1+0.99y_2=1.989903,$$

and by eliminating y_1 we get

$$y_2=-1.0203$$

and then

$$y_1=3.$$

We get

$$\Delta x = y - x = \begin{pmatrix} 3 - 1 \\ -1.0203 - 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

We find that

$$\begin{aligned} \|b\|_2 &= 280.0 \\ \|\Delta b\|_2 &= 0.01437 \\ \|x\|_2 &= 1.414 \\ \|\Delta x\|_2 &= 2.843. \end{aligned}$$

It follows that

$$c = \frac{\|\Delta x\|_2}{\|x\|_2} \left(\frac{\|\Delta b\|_2}{\|b\|_2} \right)^{-1} = \frac{2.010}{0.000513} \approx 39{,}125$$

which confirms that

$$c \approx \mathrm{cond}_2(A_m) = 39{,}206.$$

Total: 70 points