Module 2

Matrices and Linear Maps
Solutions

Problem 1: 20 points total

(1) (5 points) Prove that the column vectors of the matrix A_2 given by

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

are linearly independent.

(2) (5 points) Prove that the column vectors of the matrix B_2 given by

$$B_2 = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 3 & -5 & 5 & -4 \\ 3 & -4 & 4 & -4 \end{pmatrix}$$

are linearly independent.

(3) (10 points) Prove that the coordinates of the column vectors of the matrix B_2 over the basis consisting of the column vectors of A_2 are the columns of the matrix P_2 given by

$$P_2 = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Check that $A_2 P_2 = B_2$. Prove that

$$P_2^{-1} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

What are the coordinates over the basis consisting of the column vectors of B_2 of the vector whose coordinates over the basis consisting of the column vectors of A_2 are $(2, -3, 0, 0)$?

Solution. (1) We begin by describing a general method to prove that the columns $A^1, \ldots, A^n$ of an $n \times n$ matrix A (with coefficients in any field K; you may assume that $K = \mathbb{R}$ or $K = \mathbb{C}$) are linearly independent. Recall that $A^1, \ldots, A^n$ are linearly independent iff for any scalars $x_1, \ldots, x_n \in K$,

$$\text{if } x_1 A^1 + \cdots + x_n A^n = 0, \text{ then } \quad x_1 = \cdots = x_n = 0. \tag{$*_1$}$$

If we form the column vector x whose coordinates are $x_1, \ldots, x_n \in K$, then by definition of Ax,

$$x_1 A^1 + \cdots + x_n A^n = Ax,$$

so $(*_1)$ is equivalent to

$$\text{if } Ax = 0, \text{ then } \quad x = 0. \tag{$*_2$}$$

In other words, the columns $A^1, \ldots, A^n$ of the matrix A are linearly independent iff the linear system $Ax = 0$ has the unique solution $x = 0$ (the trivial solution).

The above can typically be demonstrated by solving the system $Ax = 0$ by variable elimination, and verifying that the only solution obtained is $x = 0$.

Another way to prove that the linear system $Ax = 0$ only has the trivial solution $x = 0$ is to show that A is invertible by *by finding explicity* the inverse A^{-1} of A. Indeed, if A has an inverse A^{-1}, we have $A^{-1}A = AA^{-1} = I$, so multiplying both sides of the equation $Ax = 0$ on the left by A^{-1}, we obtain

$$A^{-1}Ax = A^{-1}0 = 0,$$

and since $A^{-1}Ax = Ix = x$, we get $x = 0$.

In many practical situations, the inverse of A can be found using the `Matlab` command `inv`. You may use this method, but make sure that if A consists of integer entries, since in this case it can be shown that A^{-1} has rational entries (fractions), you express A^{-1} using rational numbers and you check by hand that indeed $A^{-1}A = I$.

Using the second method, to prove that the columbs of A_2 are linearly independent it suffices to show that A_2 is invertible. Running the command `inv` of `Matlab` on the matrix A_2 suggests that

$$A_2^{-1} = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -1/2 & 0 & 1 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

It is easily checked by direct multiplication that $A_2 A_2^{-1} = I_4$.

(2) We explained in (1) that the columns of a square matrix A are linearly independent if A is invertible. Thus it suffices to show that B_2 is invertible. Running the command `inv` of `Matlab` on the matrix B_2 suggests that

$$B_2^{-1} = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 9/2 & -1 & -1 & -1/2 \\ 9/2 & -1 & 0 & -3/2 \\ -3/2 & 0 & 1 & -1/2 \end{pmatrix}.$$

It is easily checked by direct multiplication that $B_2 B_2^{-1} = I_4$.

(3) If we denote the column vectors of the matrix A_2 by $A_2^1, A_2^2, A_2^3, A_2^4$ and similarly the column vectors of the the matrix B_2 by $B_2^1, B_2^2, B_2^3, B_2^4$, the coordinates $z^i = (z_1^i, z_2^i, z_3^i, z_4^i)$ of B_2^i over the basis $(A_2^1, A_2^2, A_2^3, A_2^4)$ $(1 \leq i \leq 4)$ are given by the equation

$$z_1^i A_2^1 + z_2^i A_2^2 + z_3^i A_2^3 + z_4^i A_2^4 = B_2^i,$$

and this is equivalent to solving the linear system

$$A_2 z^i = B_2^i, \quad i = 1, \ldots, 4.$$

Putting z^1, z^2, z^3, z^4 into the 4×4 matrix P_2 (as columns), solving the above four linear systems is equivalent to solving the matrix equation

$$A_2 P_2 = B_2,$$

thus we get

$$P_2 = A_2^{-1} B_2.$$

Carrying out the matrix multiplication we get

$$P_2 = A_2^{-1} B_2 = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -1/2 & 0 & 1 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 3 & -5 & 5 & -4 \\ 3 & -4 & 4 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Since $P_2 = A_2^{-1} B_2$, we have

$$P_2^{-1} = (A_2^{-1} B_2)^{-1} = B_2^{-1} (A_2^{-1})^{-1} = B_2^{-1} A_2,$$

so we get

$$P_2^{-1} = B_2^{-1} A_2 = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 9/2 & -1 & -1 & -1/2 \\ 9/2 & -1 & 0 & -3/2 \\ -3/2 & 0 & 1 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

By construction the matrix P_2 is the change of basis matrix from the basis $(A_2^1, A_2^2, A_2^3, A_2^4)$ (the old basis) to the basis $(B_2^1, B_2^2, B_2^3, B_2^4)$ (the new basis). If $x = (2, -3, 0, 0)$ are the coordinates of a vector over the old basis $(A_2^1, A_2^2, A_2^3, A_2^4)$, we know from a result of the course that the coordinates x' of this vector over the new basis $(B_2^1, B_2^2, B_2^3, B_2^4)$ are given by

$$x' = P_2^{-1} x,$$

and we find that

$$x' = P_2^{-1} x = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 2: 30 points total

Consider the polynomials

$$
\begin{array}{llll}
B_0^2(t) = (1-t)^2 & B_1^2(t) = 2(1-t)t & B_2^2(t) = t^2 & \\
B_0^3(t) = (1-t)^3 & B_1^3(t) = 3(1-t)^2 t & B_2^3(t) = 3(1-t)t^2 & B_3^3(t) = t^3,
\end{array}
$$

known as the *Bernstein polynomials* of degree 2 and 3.

(1) (10 points) Show that the Bernstein polynomials $B_0^2(t), B_1^2(t), B_2^2(t)$ are expressed as linear combinations of the basis $(1, t, t^2)$ of the vector space of polynomials of degree at most 2 as follows:

$$
\begin{pmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.
$$

Prove that

$$
B_0^2(t) + B_1^2(t) + B_2^2(t) = 1.
$$

(2) (10 points) Show that the Bernstein polynomials $B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)$ are expressed as linear combinations of the basis $(1, t, t^2, t^3)$ of the vector space of polynomials of degree at most 3 as follows:

$$
\begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}.
$$

Prove that

$$
B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) = 1.
$$

(3) (10 points) Prove that the Bernstein polynomials of degree 2 are linearly independent and that the Bernstein polynomials of degree 3 are linearly independent.

Solution. (1) By expanding the polynomials

$$
B_0^2(t) = (1-t)^2 \qquad B_1^2(t) = 2(1-t)t \qquad B_2^2(t) = t^2
$$

we get

$$
\begin{aligned}
B_0^2(t) &= 1 - 2t + t^2 \\
B_1^2(t) &= 2(1-t)t = 2t - 2t^2 \\
B_2^2(t) &= t^2,
\end{aligned}
$$

which yields

$$\begin{pmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}. \tag{eq1}$$

It should be noted that the change of basis matrix from the basis $(1, t, t^2)$ to the basis $(B_0^2(t), B_1^2(t), B_2^2(t))$ is actually the *transpose* of the above matrix, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix},$$

since the columns of the above matrix are the coordinates of $(B_0^2(t), B_1^2(t), B_2^2(t))$ over the basis $(1, t, t^2)$. Indeed we can write

$$\begin{pmatrix} B_0^2(t) & B_1^2(t) & B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

We have

$$B_0^2(t) + B_1^2(t) + B_2^2(t) = 1 - 2t + t^2 + 2t - 2t^2 + t^2 = 1.$$

(2) By expanding the polynomials

$$B_0^3(t) = (1-t)^3 \qquad B_1^3(t) = 3(1-t)^2 t \qquad B_2^3(t) = 3(1-t)t^2 \qquad B_3^3(t) = t^3,$$

we get

$$\begin{aligned} B_0^3(t) &= 1 - 3t + 3t^2 - t^3 \\ B_1^3(t) &= 3(1 - 2t + t^2)t = 3t - 6t^2 + 3t^3 \\ B_2^3(t) &= 3(1-t)t^2 = 3t^2 - 3t^3 \\ B_3^3(t) &= t^3, \end{aligned}$$

which yields

$$\begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}. \tag{eq2}$$

As in the previous case it should be noted that the change of basis matrix from the basis $(1, t, t^2, t^3)$ to the basis $(B_0^3(t), B_1^3(t), B_2^3(t)), B_3^3(t)$ is actually the *transpose* of the above matrix, namely

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix},$$

since the columns of the above matrix are the coordinates of $(B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t))$ over the basis $(1, t, t^2, t^3)$. Indeed we can write

$$\begin{pmatrix} B_0^3(t) & B_1^3(t) & B_2^3(t) & B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

We have

$$B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) = 1 - 3t + 3t^2 - t^3 + 3t - 6t^2 + 3t^3 + 3t^2 - 3t^3 + t^3 = 1.$$

(3) Equation (eq1) shows that $B_0^2(t), B_1^2(t), B_2^2(t)$ are expressed in terms of the basis $1, t, t^2$ by the upper-triangular matrix with nonzero diagonal entries

$$Be_2 = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We check easily that the inverse of Be_2 is

$$Be_2^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $(1, t, t^2)$ is a basis and since the matrix Be_2^{-1} shows that the vectors in the basis $(1, t, t^2)$ are expressed in terms of the vectors $(B_0^2(t), B_1^2(t), B_2^2(t))$, by the replacement lemma, $(B_0^2(t), B_1^2(t), B_2^2(t))$ must be linearly independent (otherwise $(B_0^2(t), B_1^2(t), B_2^2(t))$ would not span a space of dimension 3).

Another proof consists in observing that the matrix Be_2 defines a linear map that sends the basis $(1, t, t^2)$ to the basis $(B_0^2(t), B_1^2(t), B_2^2(t))$. Since the matrix Be_2 is invertible, it defines an invertible linear map, but an invertible linear maps sends a basis to a basis.

Similarly, by Equation (eq2), $B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)$ are expressed in terms of the basis $1, t, t^2, t^3$ by the upper-triangular matrix with nonzero diagonal entries

$$Be_3 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We check easily that the inverse of Be_3 is

$$Be_3^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $(1, t, t^2, t^3)$ is a basis and since the matrix Be_3^{-1} shows that the vectors in the basis $(1, t, t^2, t^3)$ are expressed in terms of the vectors $(B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t))$, by the replacement lemma, $(B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t))$ must be linearly independent (otherwise $(B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t))$ would not span a space of dimension 4).

Another proof consists in observing that the matrix Be_3 defines a linear map that sends the basis $(1, t, t^2, t^3)$ to the basis $(B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t))$. Since the matrix Be_3 is invertible, it defines an invertible linear map, but an invertible linear maps sends a basis to a basis.

Problem 3: 10 points

Prove that for every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

so that f is the projection onto its image $\mathrm{Im}\, f$.

Solution. Since $f \circ f = f$, we have

$$f(u - f(u)) = f(u) - f(f(u)) = f(u) - f(u) = 0$$

for all $u \in E$. Therefore, $u - f(u) \in \mathrm{Ker}\, f$, and since

$$u = u - f(u) + f(u)$$

with $u - f(u) \in \mathrm{Ker}\, f$ and $f(u) \in \mathrm{Im}\, f$, we have

$$E = \mathrm{Ker}\, f + \mathrm{Im}\, f.$$

If $u \in \mathrm{Ker}\, f \cap \mathrm{Im}\, f$, then there is some $v \in E$ such that $u = f(v)$ and $f(u) = 0$. But then, since $f \circ f = f$, we get

$$0 = f(u) = f(f(v)) = f(v),$$

so $f(v) = 0$ and since $u = f(v)$, we also have $u = 0$, which means that $\mathrm{Ker}\, f \cap \mathrm{Im}\, f = 0$. Therefore, we have a direct sum

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

and since $f \circ f = f$, for every $u \in \mathrm{Im}\, f$, since $u = f(v)$ for some $v \in E$, we have

$$f(u) = f(f(v)) = f(v) = u,$$

so the restriction of f to $\mathrm{Im}\, f$ is the identity I and f is indeed the projection onto $\mathrm{Im}\, f$.

Problem 4: 20 points plus 15 points Extra Credit

Given any vector space E, a linear map $f\colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) (10 points) Prove that an involution f is invertible. What is its inverse?

(2) (10 points) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$\begin{aligned} E_1 &= \{u \in E \mid f(u) = u\} \\ E_{-1} &= \{u \in E \mid f(u) = -u\}. \end{aligned}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

(3) **Extra credit** (15 points) If E is finite-dimensional and f is an involution, prove that there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of f (especially when $k = n-1$)?

Solution. (1) Since $f \circ f = \mathrm{id}$, the map f is both a left and a right inverse for itself, so f is invertible and $f^{-1} = f$.

(2) As suggested, pick any $u \in E$ and write $u = u_1 + u_{-1}$, with

$$u_1 = \frac{u + f(u)}{2}, \quad u_{-1} = \frac{u - f(u)}{2}.$$

Since $f^2 = \mathrm{id}$, we have

$$f(u_1) = f\left(\frac{u + f(u)}{2}\right) = \frac{f(u) + f^2(u)}{2} = \frac{f(u) + u}{2} = u_1,$$

and

$$f(u_{-1}) = f\left(\frac{u - f(u)}{2}\right) = \frac{f(u) - f^2(u)}{2} = \frac{f(u) - u}{2} = -u_{-1}.$$

Therefore, $u_1 \in E_1$ and $u_{-1} \in E_{-1}$, and since $u = u_1 + u_{-1}$, we have

$$E = E_1 + E_{-1}.$$

If $u \in E_1 \cap E_{-1}$, then $f(u) = u$ and $f(u) = -u$, so $u = -u$, which means that $u = 0$. Therefore, $E_1 \cap E_{-1} = (0)$, and we have the direct sum

$$E = E_1 \oplus E_{-1}.$$

(3) Since $E = E_1 \oplus E_{-1}$ is a direct sum, we can pick a basis consisting of $k = \dim(E_1)$ vectors $(u_1, \ldots, u_k)$ from E_1 and $n - k$ vectors $(u_{k+1}, \ldots, u_n)$ from E_{-1} (with $n = \dim(E)$), and with respect to this basis $(u_1, \ldots, u_n)$, the matrix of f is indeed of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

The linear map f is a reflection about the subspace E_1. When $k = n - 1$, the subspace E_1 is a hyperplane, and f is a hyperplane reflection.

Total: 70 points **Extra Credit: 15 points**