CIT 592 Fall 2019 **Homework 4** **Quiz + Solutions**

PREPARATION QUIZ

A. **[Concepts: binomial theorem (seg_04_01)]**

What is the coefficient of x^2y^6 when we expand $(x+y)^8$?

Correct answer: 28

The binomial theorem tells us that the coefficient is

$$\binom{8}{6} = \frac{8!}{6!2!} = 28\,.$$

Explanation for incorrect answers:

The binomial theorem tells us that the coefficient of $x^{n-i}y^i$ in $(x+y)^n$ is $\binom{n}{i}$. In this case $n = 8$ and $i = 6$, so the coefficient is

$$\binom{8}{6} = \frac{8!}{6!(8-6)!} = \frac{8!}{6!2!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 2\cdot 1\cdot 2\cdot 1} = \frac{56}{2} = 28\,.$$

B. **[Concepts: Pascal's triangle (seg_04_01)]**

What number is at the center of the seventh row in Pascal's triangle?

Correct answer: 20

The 7^{th} row of Pascal's triangle gives coefficients for the polynomial $(a+b)^6$. Thus, the entry at the center of the seventh row is the coefficient of a^3b^3 in $(a+b)^6$, which by the binomial theorem is

$$\binom{6}{3} = \frac{6!}{3!3!} = 20\,.$$

Explanation for incorrect answers:

The n^{th} row of Pascal's triangle gives coefficients for the polynomial $(a+b)^{n-1}$. Thus, the entry at the center of the seventh row is the coefficient of a^3b^3 in $(a+b)^6$, which by the binomial theorem is

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot3\cdot2\cdot1} = \frac{120}{6} = 20\,.$$

Alternatively, we can look at the first six rows of Pascal's triangle and, using Pascal's identity, add the two middle numbers in the sixth row to get the middle number in the seventh row:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & \mathbf{10} & & \mathbf{10} & & 5 & & 1
\end{array}$$

$10 + 10 = 20$.

Common mistake: 35

The n^{th} row of Pascal's triangle gives coefficients for $(a+b)^{n-1}$, not $(a+b)^n$, so we need the coefficient of a^3b^3 in $(a+b)^6$, not of a^3b^4 in $(a+b)^7$.

C. [Concepts: combinatorial identities (seg_04_02)]

$\binom{14}{11} = 364$, $\binom{14}{10} = 1001$, and $\binom{15}{10} = 3003$. Using this information (and *not* using a calculator), what is $\binom{16}{11}$?

Correct answer: 4368

Pascal's identity tells us that

$$\binom{16}{11} = \binom{15}{10} + \binom{15}{11}$$

and

$$\binom{15}{11} = \binom{14}{10} + \binom{14}{11}\,.$$

Putting these together, we have

$$\binom{16}{11} = \binom{15}{10} + \binom{14}{10} + \binom{14}{11} = 3003 + 1001 + 364 = 4368\,.$$

Explanation for incorrect answers:

Pascal's says that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. We can apply that here to see that

$$\binom{16}{11} = \binom{15}{10} + \binom{15}{11}\,.$$

We were given the value of the first term, but we still need to calculate the value of the second term. To do so, we apply Pascal's identity a second time:

$$\binom{15}{11} = \binom{14}{10} + \binom{14}{11}\,.$$

Putting these together with the information provided, we have

$$\binom{16}{11} = \binom{15}{10} + \binom{14}{10} + \binom{14}{11} = 3003 + 1001 + 364 = 4368\,.$$

D. [Concepts: functions (seg_04_03)]

What is the range of the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(x) = x + 1$?

(a) $\mathbb{N}$

Incorrect. $\mathbb{N}$ is the domain of f, not the range.

(b) $\mathbb{Z}$

Incorrect. $\mathbb{Z}$ is the codomain of f, not the range.

(c) $\mathbb{Z}^+$

Correct. Since the domain of f is the set of all integers ≥ 0 and f adds 1 to each input, the range of f is the set of all integers ≥ 1, which is $\mathbb{Z}^+$.

E. **[Concepts: functions (seg_04_03)]**

How many distinct functions are there whose domain is the English alphabet and whose codomain is the set of all binary strings of length 5?

(a) 26^{2^5}

Incorrect. The number of functions from a domain A to a codomain B is $|B^A| = |B|^{|A|}$, not $|A|^{|B|}$. Here the domain has cardinality 26 and the codomain has cardinality 2^5, so the number of functions is $(2^5)^{26}$.

(b) $(2^5)^{26}$

Correct. The number of functions from a domain A to a codomain B is $|B^A| = |B|^{|A|}$. In this case, the domain has cardinality 26 and the codomain has cardinality 2^5.

(c) $(2^{26})^5$

Incorrect. The number of functions from a domain A to a codomain B is $|B^A| = |B|^{|A|}$. In this case, the domain has cardinality 26 and the codomain has cardinality 2^5, so the number of functions is $(2^5)^{26}$.

(d) $2^{5^{26}}$

Incorrect. The number of functions from a domain A to a codomain B is $|B^A| = |B|^{|A|}$. In this case, the domain has cardinality 26 and the codomain has cardinality 2^5. When we raise 2^5 to the 26^{th} power, we get $(2^5)^{26} = 2^{5\cdot 26} \neq 2^{5^{26}}$

F. **[Concepts: integer intervals (seg_04_04)]**

What is the cardinality of $[1..2] \cup [4..8] \cup [16..32] \cup [64..128]$?

Correct answer: 89

The cardinality of $[m..n]$ is $n - m + 1$, so $|[1..2]| = 2$, $|[4..8]| = 5$,

$|[16..32]| = 17$, and $|[64..128]| = 65$. Since these sets are pairwise disjoint, the cardinality of their union is the sum of their cardinalities, so the answer is $2 + 5 + 17 + 65 = 89$.

Explanation for incorrect answers:
The cardinality of any integer interval $[m..n]$ is $n - m + 1$, so $|[1..2]| = 2 - 1 + 1 = 2$, $|[4..8]| = 8 - 4 + 1 = 5$, $|[16..32]| = 32 - 16 + 1 = 17$, and $|[64..128]| = 128 - 64 + 1 = 65$. Since these sets are pairwise disjoint, the cardinality of their union is the sum of their cardinalities, so the answer is $2 + 5 + 17 + 65 = 89$.

Common mistake: 85
Remember that the cardinality of the integer interval $[m..n]$ is $n-m+1$, not $n - m$.

G. [Concepts: injections and surjections (seg_04_05)]

Is the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ injective and/or surjective?

(a) injective
Incorrect. Since multiple elements from the domain map to the same element in the codomain (e.g., $f(2) = f(-2) = 4$), the function is not injective.

(b) surjective
Incorrect. The range of the f is $\mathbb{N}$, which is not equal to its codomain, so f is not surjective.

(c) both injective and surjective
Incorrect. The function is not an injection because multiple elements from the domain map to the same element in the codomain (e.g., $f(2) = f(-2) = 4$), and it is not a surjection because its range is $\mathbb{N}$, which is not equal to its codomain, $\mathbb{Z}$.

(d) neither injective nor surjective

Correct. The function is not an injection because multiple elements from the domain map to the same element in the codomain (e.g., $f(2) = f(-2) = 4$), and it is not a surjection because its range is $\mathbb{N}$, which is not equal to its codomain, $\mathbb{Z}$.

Problem Set

1. [10 pts]

Use the binomial theorem to expand $(2a + 3b)^4$.

Solution:

The binomial theorem tells us that $(2a + 3b)^4$ equals

$$\begin{aligned}
&\sum_{k=0}^{4} \binom{4}{k} (2a)^{4-k}(3b)^k \\
&= \binom{4}{0}(2a)^4(3b)^0 + \binom{4}{1}(2a)^3(3b)^1 + \binom{4}{2}(2a)^2(3b)^2 + \binom{4}{3}(2a)^1(3b)^3 + \binom{4}{4}(2a)^0(3b)^4 \\
&= 1 \cdot (2a)^4(3b)^0 + 4 \cdot (2a)^3(3b)^1 + 6 \cdot (2a)^2(3b)^2 + 4 \cdot (2a)^1(3b)^3 + 1 \cdot (2a)^0(3b)^4 \\
&= (1 \cdot 16 \cdot 1)a^4 + (4 \cdot 8 \cdot 3)a^3b + (6 \cdot 4 \cdot 9)a^2b^2 + (4 \cdot 2 \cdot 27)ab^3 + (1 \cdot 1 \cdot 81)b^4 \\
&= \boxed{16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4}.
\end{aligned}$$

2. [10 pts]

We have seen that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Without using that identity, use Pascal's identity to prove that the sum of each row of Pascal's triangle is twice the sum of the previous row.

Solution:

Intuitively, each term in the $(n-1)^{\text{th}}$ row of Pascal's triangle contributes to each of the two terms that are closest to it in the n^{th} row. Since each term in $(n-1)^{\text{th}}$ row is contributing to the total of the n^{th} row twice, the total of the n^{th} row is twice as big. Let's turn this intuition into a more formal proof.

The sum of the n^{th} row of Pascal's triangle is

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k} &= \binom{n}{0} + \sum_{k=1}^{n}\binom{n}{k} && \text{(splitting off } k=0 \text{ term)}\\
&= \binom{n}{0} + \sum_{k=1}^{n}\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] && \text{(Pascal's identity)}\\
&= \sum_{k=1}^{n}\binom{n-1}{k-1} + \left[1 + \sum_{k=1}^{n}\binom{n-1}{k}\right]. && \text{(rearranging)}
\end{aligned}$$

Now, we can let $j = k-1$ to see that

$$\sum_{k=1}^{n}\binom{n-1}{k-1} = \sum_{j=0}^{n-1}\binom{n-1}{j}.$$

Further, observe that

$$\begin{aligned}
1 + \sum_{k=1}^{n}\binom{n-1}{k} &= \binom{n-1}{0} + \sum_{k=1}^{n}\binom{n-1}{k}\\
&= \sum_{k=0}^{n}\binom{n-1}{k}\\
&= \binom{n-1}{n} + \sum_{k=0}^{n-1}\binom{n-1}{k}\\
&= \sum_{k=0}^{n-1}\binom{n-1}{k},
\end{aligned}$$

since $\binom{a}{b} = 0$ whenever $b > a$. So we have shown that the sum of the n^{th} row is

$$\sum_{j=0}^{n-1}\binom{n-1}{j} + \sum_{k=0}^{n-1}\binom{n-1}{k} = 2\sum_{k=0}^{n-1}\binom{n-1}{k},$$

which is exactly twice the sum of the $(n-1)^{\text{th}}$ row. $\square$

3. [10 pts]

Recall that a *combinatorial proof* for an identity proceeds as follows:

1. State a counting question.
2. Answer the question in two ways:
   (i) one answer must correspond to the left-hand side (LHS) of the identity
   (ii) the other answer must correspond to the right-hand side (RHS).
3. Conclude that the LHS is equal to the RHS.

With that in mind, give a combinatorial proof of the identity $\binom{2n}{2} = \binom{n}{2} + \binom{n}{n-2} + n^2$, where $n \geq 2$.

Solution:

We pose the following counting question:

> How many ways can we pick two distinct people for Anji's Arctic expedition from a set of n women and n men, where $n \geq 2$?

(LHS): One procedure to count the number of ways to choose an expedition team is to simply choose two out of the $2n$ people, which by definition has $\binom{2n}{2}$ ways, exactly the LHS.

(RHS): We will partition the outcomes into pairs with only women, pairs with only men, and pairs with exactly one woman and one man. The number of ways to pick only women is $\binom{n}{2}$, since we are choosing two of the n women. Similarly, the number of ways to pick only men is $\binom{n}{n-2}$, as instead of picking which men go on the expedition, we can pick the men to exclude. Finally, the number of ways to pick one of each is n^2 by the multiplication rule, since there are n choices for the woman and n choices for the man.

Since these cases are disjoint, we can sum them up to get a total of $\binom{n}{2} + \binom{n}{n-2} + n^2$ ways, which gives us the RHS.

Both sides of the expression answer the same question, so they must be equal.

4. [10 pts]

(a) Let p, q, r, s be integers with $p \leq q$ and $r \leq s$. How many distinct functions are there with domain $[p..q]$, and codomain $[r..s]$?

(b) Let p and q be integers with $p \leq q$. How many distinct functions are there of the form $f : [p..q] \to [p..q]$ such that $f(x) \leq x$ for all x in the domain?

Solution:

(a) Each integer in $[p..q]$ can be mapped to any integer in $[r..s]$, which can be done in $s - r + 1$ ways. Since there are $q - p + 1$ integers in $[p..q]$, the multiplication rule tells us that there are $(s - r + 1)^{q-p+1}$ such functions.

(b) Each integer $x \in [p..q]$ can be mapped to any integer in $[p..x]$, which can be done in $x - p + 1$ ways. So by the multiplication rule, there number of such functions is

$$\begin{aligned}
&(p - p + 1) \cdot ((p + 1) - p + 1) \cdot ((p + 2) - p + 1) \ldots \cdot (q - p + 1) \\
&= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (q - p + 1) \\
&= (q - p + 1)!\,.
\end{aligned}$$

5. [10 pts]

For each of the following functions, prove that the function is neither injective nor surjective. Then, show how you could restrict the domain and codomain — without changing the mapping rule — to make the function both injective and surjective. Your restricted domain and codomain should be as large as possible.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither injective nor surjective, but we can restrict the domain and codomain to define the function $\hat{f} : [0, \infty) \to [0, \infty)$ given by $\hat{f}(x) = x^2$, which is both injective and surjective.

(a) $g : \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ given by:

x	$g(x)$
a	a
b	i
c	i
d	e

(b) All **finite** subsets of $\mathbb{Z} \to \mathbb{Z}$ given by $h(S) = |S|$.

Solution:

(a) This function is not injective because $g(\mathtt{b}) = g(\mathtt{c})$. It is not surjective because there is no element of the domain that maps to $\mathtt{u}$.

The function $\hat{g} : \{\mathtt{a}, \mathtt{b}, \mathtt{d}\} \to \{\mathtt{a}, \mathtt{e}, \mathtt{i}\}$ with the same mapping rule is both injective and surjective; no two elements of $\{\mathtt{a}, \mathtt{b}, \mathtt{d}\}$ map to the same element of $\{\mathtt{a}, \mathtt{e}, \mathtt{i}\}$, and the codomain is equal to the range.

(b) This function is not injective because $h(\{1\}) = h(\{2\}) = 1$. It is not surjective because, as cardinalities are always non-negative, its range is $\mathbb{N}$, which is not equal to its codomain.

The function $\hat{h} : \emptyset \cup \{[1..n] : n \in \mathbb{Z}^+\} \to \mathbb{N}$ with the same mapping rule is both injective and surjective. The domain contains exactly one set of each cardinality $n \in \mathbb{N}$, so no two elements of the domain are mapped to the same natural number, and the range is equal to the codomain, $\mathbb{N}$.

6. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Give a combinatorial proof of the following identity.

$$\binom{n+1}{r+1} = \sum_{k=r+1}^{n+1} \binom{k-1}{r} \text{ whenever } n \geq r \geq 0.$$

Solution:

We pose the following counting question:

> How many $(n+1)$-bit binary strings are there with exactly $r+1$ 1s?

(LHS) One way to produce such a string is to simply choose $r+1$ out of the $n+1$ bits to be 1s, which can be done in $\binom{n+1}{r+1}$ ways.

(RHS) Another way to produce such a string is to first decide where the rightmost 1 will be, then decide where to put the r other 1s to the left of it.

Where can the rightmost 1 be? It must be at least at position $r+1$, or else there would not be room for the other r 1s to its left, and at most it can be at position $n+1$, since that's the length of the string.

Now, if we put the rightmost 1 at position k in the string, then we can choose r of the $k-1$ positions to the left of it, which can be done in $\binom{k-1}{r}$ ways. These outcomes are separate for different values of k, so we can use the addition rule and conclude that there are $\sum_{k=r+1}^{n+1} \binom{k-1}{r}$ to produce the string.

Since the LHS and RHS count the same quantity, we conclude that they are equal. $\square$