Module 5

Gaussian Elimination
Solutions

Problem 1: 10 points

Solve the following linear system by Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ -1 \\ 2 \end{pmatrix}.$$

Solution. Let us solve the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 7 \\ 2x_1 + 3x_2 + 2x_3 + 3x_4 &= 14 \\ -x_1 + x_3 - x_4 &= -1 \\ -2x_1 - x_2 + 4x_3 &= 2 \end{aligned}$$

by Gaussian elimination. First we pick the pivot $\pi_1 = 1$ in the first row and we add -2 times the first row to the second row, $+1$ times the first row to the third row, $+2$ times the first row to the fourth row, obtaining:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 7 \\ -x_2 + x_4 &= 0 \\ 2x_2 + 2x_3 &= 6 \\ 3x_2 + 6x_3 + 2x_4 &= 16. \end{aligned}$$

The next pivot is $\pi_2 = -1$ in the second row. We add 2 times the second row to the third row and we add 3 times the second row to the fourth row, obtaining

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 7 \\ -x_2 + x_4 &= 0 \\ 2x_3 + 2x_4 &= 6 \\ 6x_3 + 5x_4 &= 16. \end{aligned}$$

The next pivot is $\pi_3 = 2$ in the third row. We add -3 times the third row to the fourth row, obtaining

$$\begin{aligned}x_1 + 2x_2 + x_3 + x_4 &= 7 \\ -x_2 + x_4 &= 0 \\ 2x_3 + 2x_4 &= 6 \\ -x_4 &= -2.\end{aligned}$$

Solving the above triangular system by back substitution (from the bottom-up), we get $x_4 = 2$, $x_3 = 1$, $x_2 = 2$, $x_1 = 0$.

Problem 2: 15 points

Consider the matrix

$$A = \begin{pmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{pmatrix}.$$

Perform Gaussian elimination by choosing the first pivot to be the entry 1 in column one and row one. Consider the second column. Which value of c yields zero for the the second element of the second column? Which value of c yields zero for the bottom element in the second column? In this case, what can your say about the matrix A?

Solution. First we pick the pivot $\pi_1 = 1$ in the first row and we add -2 times the first row to the second row and we add -3 times the first row to the third row, obtaining the matrix

$$\begin{pmatrix} 1 & c & 0 \\ 0 & 4 - 2c & 1 \\ 0 & 5 - 3c & 1 \end{pmatrix}.$$

The pivot $4 - 2c$ is 0 in the second row iff $c = 2$. The pivot $5 - 3c$ is 0 in the third row iff $c = 5/3$. In the second case we obtain the matrix

$$\begin{pmatrix} 1 & 5/3 & 0 \\ 0 & 2/3 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and Gaussian elimination has terminated after one step, since the second pivot is $\pi_2 = 2/3$ and the entry below this pivot is already 0.

Problem 3: 15 points total

(1) (5 points) Find a lower triangular matrix E such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(2) (5 points) What is the effect of the product (on the left) with

$$E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(3) (5 points) Find the inverse of the matrix Pa_3.

Solution. (1) Consider

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

Observe that mutiplying Pa_3 on the left by $E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$ has the effect of subtracting row 3 from row 4, then subtracting subtracting row 2 from row 3, and finally subtracting subtracting row 1 from row 2. The result is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

so $E = E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$.

(2) As we saw in (1), multiplying Pa_3 on the left by $E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$ yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

Multiplying the above matrix on the left by $E_{3,2;-1}E_{4,3;-1}$ subtracts row 3 from row 4 and then row 2 from row 3, which yields the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Finally, multiplying the above matrix on the left by $E_{4,3;-1}$ subtracts row 3 from row 4, which yields the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) From (2), we have

$$E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4,$$

which shows that $E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$ is the inverse of Pa_3. A direct computation shows that

$$Pa_3^{-1} = E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

Problem 4: 10 points

If A is an $n \times n$ symmetric matrix and B is any $n \times n$ invertible matrix, prove that A is positive definite iff $B^\top AB$ is positive definite.

Solution. (1) First assume that A is symmetric positive definite and that B is invertible. Since A is symmetric, we have

$$(B^\top AB)^\top = B^\top A^\top B = B^\top AB,$$

which shows that $B^\top AB$ is symmetric. Since $x^\top Ax > 0$ for all $x \neq 0$, and since $Bx \neq 0$ if $x \neq 0$ because B is invertible, we get

$$x^\top B^\top ABx = (Bx)^\top A(Bx) > 0,$$

which proves that $B^\top AB$ is also positive definite.

Conversely, if $C = B^\top AB$ is symmetric positive definite and B is invertible, we get

$$A = (B^\top)^{-1} B^\top ABB^{-1} = (B^{-1})^\top CB^{-1}$$

with C symmetric positive definite and B^{-1} invertible, so by the first part, A is symmetric positive definite.

Total: 50 points