Module 8

SVD and Polar Form
Solutions

Problem 1: 10 points

WITHOUT using Matlab, find the SVD of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution. The singular values of A are the nonnegative square roots of the eigenvalues of $A^\top$A, and the columns of U in the SVD $A = V\Sigma U^\top$ are eigenvectors of $A^\top A$. We have

$$A^\top A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Since $A^\top A$ is a diagonal matrix, its eigenvalues are the diagonal entries so

$$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We verify immediately that if we let U be the permutation matrix

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$A^\top A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} = U\Sigma^2 U^\top = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Observe that the columns of U are the eigenvectors corresponding to the eigenvalues $9, 4, 0$, and that $U^\top = U$. To find V, we form

$$AU = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first two columns v_1 and v_2 of V are obtained by normalizing the first two columns of AU, so we get

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

To complete (v_1, v_2) into a basis we add the third canonical basis vector so we get

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now have the SVD $A = V\Sigma U^\top$, as we easily verify below:

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 2: 20 points total

(1) (10 points) Let A be a real $n \times n$ matrix and consider the $2n \times 2n$ real symmetric matrix

$$S = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}.$$

Suppose that A has rank r. If $A = V\Sigma U^\top$ is an SVD for A, with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_r > 0$, denoting the columns of U by u_k and the columns of V by v_k, prove that σ_k is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ u_k \end{pmatrix}$ for $k = 1, \ldots, n$, and that $-\sigma_k$ is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ -u_k \end{pmatrix}$ for $k = 1, \ldots, n$.

Hint. We have $Au_k = \sigma_k v_k$ for $k = 1, \ldots, n$. Prove that $A^\top v_k = \sigma_k u_k$ for $k = 1, \ldots, n$.

(2) (10 points)

(8 points) Prove that the $2n$ eigenvectors of S in (1) are pairwise orthogonal.

(2 points) Check that if A has rank r, then S has rank $2r$.

Solution. Let $A = V\Sigma U^\top$ be an SVD of A. Then $AU = V\Sigma$, that is,

$$Au_k = \sigma_k v_k, \text{ for } k = 1, \ldots, n. \tag{$*_1$}$$

By transposition we obtain $A^\top = U\Sigma V^\top$, which yields $A^\top V = U\Sigma$, and thus

$$A^\top v_k = \sigma_k u_k, \text{ for } k = 1, \ldots, n. \tag{$*_2$}$$

By $(*_1)$ and $(*_2)$, we have

$$\begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \begin{pmatrix} v_k \\ u_k \end{pmatrix} = \begin{pmatrix} Au_k \\ A^\top v_k \end{pmatrix} = \begin{pmatrix} \sigma_k v_k \\ \sigma_k u_k \end{pmatrix} = \sigma_k \begin{pmatrix} v_k \\ u_k \end{pmatrix}.$$

The above shows that σ_k is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ u_k \end{pmatrix}$ for $k = 1, \ldots, n$.

Similarly, by $(*_1)$ and $(*_2)$, we have

$$\begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \begin{pmatrix} v_k \\ -u_k \end{pmatrix} = \begin{pmatrix} -Au_k \\ A^\top v_k \end{pmatrix} = \begin{pmatrix} -\sigma_k v_k \\ \sigma_k u_k \end{pmatrix} = -\sigma_k \begin{pmatrix} v_k \\ -u_k \end{pmatrix}.$$

Therefore, $-\sigma_k$ is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ -u_k \end{pmatrix}$ for $k = 1, \ldots, n$.

(2) Recall that U and V are orthogonal matrices and so the vectors $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are orthonormal families. For all i, j such that $1 \leq i, j \leq n$ and $i \neq j$, we have the inner products

$$\begin{pmatrix} v_i \\ u_i \end{pmatrix} \cdot \begin{pmatrix} v_j \\ u_j \end{pmatrix} = v_i \cdot v_j + u_i \cdot u_j = 0 + 0 = 0,$$

and

$$\begin{pmatrix} v_i \\ -u_i \end{pmatrix} \cdot \begin{pmatrix} v_j \\ -u_j \end{pmatrix} = v_i \cdot v_j + u_i \cdot u_j = 0 + 0 = 0.$$

We also have

$$\begin{pmatrix} v_i \\ u_i \end{pmatrix} \cdot \begin{pmatrix} v_j \\ -u_j \end{pmatrix} = v_i \cdot v_j - u_i \cdot u_j = 0 - 0 = 0,$$

and

$$\begin{pmatrix} v_i \\ u_i \end{pmatrix} \cdot \begin{pmatrix} v_i \\ -u_i \end{pmatrix} = v_i \cdot v_i - u_i \cdot u_i = 1 - 1 = 0.$$

Therefore, the $2n$ eigenvectors $\begin{pmatrix} v_i \\ u_i \end{pmatrix}$ and $\begin{pmatrix} v_i \\ -u_i \end{pmatrix}$ are pairwise orthogonal.

According to (1), the nonzero distinct eigenvalues of S are $\sigma_1, \ldots, \sigma_r$ and $-\sigma_1, \ldots, -\sigma_r$, so S has rank $2r$.

Problem 3: 10 points

Let A be an $n \times n$ real matrix. Prove that if $\sigma_1, \ldots, \sigma_n$ are the singular values of A, then $\sigma_1^3, \ldots, \sigma_n^3$ are the singular values of $AA^\top A$.

Solution. The singular values $\sigma_1, \ldots, \sigma_n$ of A are the nonnegative square roots of the eigenvalues $\sigma_1^2, \ldots, \sigma_n^2$ of $A^\top A$. Since $A^\top A$ is symmetric (positive semidefinite), it can be diagonalized as

$$A^\top A = U\Sigma^2 U^\top,$$

where $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$ and U is an orthogonal matrix. It follows that

$$(A^\top A)^3 = U\Sigma^2 U^\top U\Sigma^2 U^\top U\Sigma^2 U^\top = U\Sigma^6 U^\top.$$

We also have

$$(AA^\top A)^\top AA^\top A = A^\top AA^\top AA^\top A = (A^\top A)^3.$$

It follows that

$$(AA^\top A)^\top AA^\top A = U\Sigma^6 U^\top,$$

and the above equation shows that $\sigma_1^3, \ldots, \sigma_n^3$ are the singular values of $AA^\top A$.

Problem 4: 10 points

Let A be an real $n \times n$ matrix. Assume A is invertible. Prove that if $A = Q_1 S_1 = Q_2 S_2$ are two polar decompositions of A, then $Q_1 = Q_2$ and $S_1 = S_2$.

Hint. $A^\top A = S_1^2 = S_2^2$, with S_1 and S_2 symmetric positive definite. Then use Problem 4(2) from Module 7 homework set.

Solution. If $A = Q_1 S_1$ is a polar decomposition of A, then

$$A^\top A = (Q_1 S_1)^\top Q_1 S_1 = S_1^\top Q_1^\top Q_1 S_1 = S_1 S_1 = S_1^2,$$

since Q_1 is orthogonal and S_1 is symmetric. Similarly, $A^\top A = S_2^2$. It follows that $S_1^2 = S_2^2$. Since S_1 and S_2 are positive semidefinite, the eigenvalues of S_1 and S_2 are nonnegative, so the condition of Problem 4(2) from Module 7 homework set applies and we conclude that $S_1 = S_2$.

Since A is invertible, $S_1 = Q_1^\top A$ is also invertible, and similarly, $S_2 = Q_2^\top A$ is invertible. From $A = Q_1 S_1$, we get $Q_1 = AS_1^{-1}$, and similarly from $A = Q_2 S_2$, we get $Q_2 = AS_2^{-1}$. Since $S_1 = S_2$, we conclude that $Q_1 = Q_2$ as well.

Remark. The proof shows that S_1 is unique even if A is not invertible. But if A is not invertible, Q_1 is not unique.

Total: 50 points