Problem Set

1. [**10 pts**] Suppose you roll $n \geq 1$ fair dice. Let X be the random variable for the sum of their values, and let Y be the random variable for the number of times an odd number comes up. Prove or disprove: X and Y are independent.

Solution:

We disprove this. $X \perp Y$ means that for all $x \in Val(X)$ and $y \in Val(Y)$,

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]\,.$$

But $x = n$ and $y = 0$ are a counterexample to that condition, because the only way X can be n is if every die came up as a 1, which is odd, so there cannot be 0 odd numbers.

$$\begin{aligned} \Pr[X = n \text{ and } Y = 0] &= 0 \\ &\neq \left(\frac{1}{12}\right)^n \\ &= \left(\frac{1}{6}\right)^n \cdot \left(\frac{1}{2}\right)^n \\ &= \Pr[X = n] \cdot \Pr[Y = 0]\,. \end{aligned}$$

□

2. [**10 pts**] Let X and Y be independent random variables such that $\mathrm{Var}[X] = 4.8$ and $\mathrm{Var}[Y] = 9.7$. What is the standard deviation of $2X + 3Y$?

Solution:

In general $\mathrm{Var}[c \cdot Z] = c^2\mathrm{Var}[Z]$, so we have $\mathrm{Var}[2X] = 4\mathrm{Var}[X] = 19.2$ and $\mathrm{Var}[3Y] = 9 \cdot \mathrm{Var}[Y] = 87.3$. Since X and Y are independent, $2X$ and $3Y$ are also, so $\mathrm{Var}[2X + 3Y] = \mathrm{Var}[2X] + \mathrm{Var}[3Y] = 106.5$. Finally, the standard deviation of $2X + 3Y$ is the square root of the variance, or $\boxed{\sqrt{106.5}}$.

3. [**10 pts**] Suppose that you generate a 12-character password by selecting each character independently and uniformly at random from $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\} \cup \{\mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}\} \cup \{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{9}\}$.

(a) [**4 pts**] What is the probability that exactly 6 of the characters are digits?

(b) [**4 pts**] What is the expected number of digits in a password?

(c) [**2 pts**] What is the variance of the number of digits in a password?

Solution:

(a) Note that there are 26 lowercase letters, 26 uppercase letters, and 10 digits. We can now define our sample space Ω as the set of all possible passwords we can create out of these three sets. Since there are 62 total characters and 12 locations, $|\Omega| = 62^{12}$.

Let E be the event we have exactly 6 digits in our password. Note that what we want is $\frac{|E|}{|\Omega|}$. We count $|E|$ as follows:

Step 1. Choose 6 of the 12 locations as places we will place our digits ($\binom{12}{6}$ ways)

Step 2. Assign digits to each of the chosen locations (10^6 ways)

Step 3. Assign lowercase or uppercase letters to the remaining locations (52^6 ways)

Hence, by the Multiplication Rule, $|E| = \binom{12}{6} \times 10^6 \times 52^6$.

Plugging into our expression, the probability we want $\frac{|E|}{|\Omega|} = \frac{\binom{12}{6}\times 10^6 \times 52^6}{62^{12}}$

(b) Let X be a random variable denoting the number of digits in a password.

Let S_i be the event that the ith character location contains a digit.

Let X_i be an indicator random variable that is 1 if its corresponding S_i occurs and 0 otherwise.

Note that $X = X_1 + X_2 + \cdots + X_{12}$

By Linearity of Expectation, $E[X] = E[X_1] + E[X_2] + \cdots + E[X_{12}] = \sum_{i=1}^{12} E[X_i]$

$E[X_i] = \frac{|X_i=1|}{|\Omega|} = \frac{10}{62}$ as there are 10 digits and 62 total characters and we choose our character uniformly at random.

Hence, $E[X] = \sum_{i=1}^{12} \frac{10}{62} = \frac{120}{62} = \frac{60}{31}$.

(c) Notice that our random variable X as defined in part (b) is a Binomial random variable as it is composed of the independent Bernoulli random variables X_i. In other words, our X is done over $n = 12$ trials and the ith Bernoulli trial succeeds when a digit is chosen for the ith position with success probability $p = \frac{10}{62}$. Hence, $Var[X] = np(1-p) = 12 \times \frac{10}{62} \times (1 - \frac{10}{62}) = 12 \times \frac{5}{31} \times \frac{26}{31}$.

4. [10 pts] Prove that if X and Y are non-negative independent random variables, then $X^2 \perp Y^2$.

Solution:

We want to show, assuming $X \perp Y$, that for all $a \in Val(X^2)$ and $b \in Val(Y^2)$,

$$\Pr[X^2 = a \text{ and } Y^2 = b] = \Pr[X^2 = a] \cdot \Pr[Y^2 = b]\,.$$

Notice that since $a \in Val(X^2)$, we must have $a = x^2$ for some $x \in Val(X)$, and similarly $b = y^2$ for some $y \in Val(Y)$. So we can restate the desired condition as, for all $x \in Val(X)$ and $y \in Val(Y)$,

$$\Pr[X^2 = x^2 \text{ and } Y^2 = y^2] = \Pr[X^2 = x^2] \cdot \Pr[Y^2 = y^2].$$

We not show that this holds.

$$\begin{aligned}\Pr[X^2 = x^2 \text{ and } Y^2 = y^2] &= \Pr[X = x \text{ and } Y = y] \\ &= \Pr[X = x] \cdot \Pr[Y = y] \qquad \text{(by our assumption that } X \perp Y) \\ &= \Pr[X^2 = x^2] \cdot \Pr[Y^2 = y^2].\end{aligned}$$

□

5. **[10 pts]** Let X and Y be independent Bernoulli random variables with parameter 1/2, and let Z be the random variable that returns the remainder of the division of $X + Y$ by 2.

(a) **[2 pts]** Prove that Z is also a Bernoulli random variable, also with parameter 1/2.

(b) **[4 pts]** Prove that X, Y, Z are pairwise independent but not mutually independent.

(c) **[4 pts]** By computing $\text{Var}[X+Y+Z]$ according to the alternative formula for variance and using the variance of Bernoulli r.v.'s, verify that $\text{Var}[X+Y+Z] = \text{Var}[X]+\text{Var}[Y]+\text{Var}[Z]$ (observe that this also follows from the proposition on slide 5 of the lecture segment entitled "Binomial distribution").

Solution:

(a) The values taken by Z are shown in the following table (that also includes the values taken by $X + Y + Z$ in anticipation of the last part of this problem):

X	Y	$X+Y$	Z	$X+Y+Z$
0	0	0	0	0
0	1	1	1	2
1	0	1	1	2
1	1	2	0	2

Therefore, $(Z) = \{0, 1\}$. Moreover, using $X \perp Y$,

$\Pr[Z = 1] = \Pr[(X = 0 \cap Y = 1) \cup (X = 1 \cap Y = 0)] = 1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/4 + 1/4 = 1/2$. It follows that Z is Bernoulli with parameter 1/2.

(b) We already know that $X \perp Y$. We verify $X \perp Z$ as follows:

$$\begin{aligned}\Pr[X=1 \ \cap\ Z=1] &= \Pr[(X=1) \cap ((X=0 \ \cap\ Y=1) \cup (X=1 \ \cap\ Y=0))] \\ &= \Pr[X=1 \ \cap\ Y=0] \\ &= \Pr[X=1] \cdot \Pr[Y=0] = 1/2 \cdot 1/2 = \Pr[X=1] \cdot \Pr[Z=1]\end{aligned}$$

We would also need to verify that, for example, $\Pr[X=0 \ \cap\ X=1] = \Pr[X=0] \cdot \Pr[Z=1]$ and for the other intersections. However, for Bernoulli variables, these follow from the one we just proved in view of property Ind (iv) from the lecture segment entitled "Independence".

With the same observation, here is the verification of $Y \perp Z$:

$$\begin{aligned}\Pr[Y=1 \ \cap\ Z=1] &= \Pr[(Y=1) \cap ((X=0 \ \cap\ Y=1) \cup (X=1 \ \cap\ Y=0))] \\ &= \Pr[X=0 \ \cap\ Y=1] \\ &= \Pr[X=0] \cdot \Pr[Y=1] = 1/2 \cdot 1/2 = \Pr[Y=1] \cdot \Pr[Z=1]\end{aligned}$$

Therefore X, Y, Z are pairwise independent. To show that they are not mutually independent observe that $\Pr[X=1 \ \cap\ Y=1 \ \cap\ Z=1] = \Pr[\emptyset] = 0$ but $\Pr[X=1] \cdot \Pr[Y=1] \cdot \Pr[Z=1] = 1/2 \cdot 1/2 \cdot 1/2 = 1/8$.

(c) From the table above $\mathrm{Val}(X+Y+Z) = \{0, 2\}$. Each row in the table corresponds to a probability of $1/2 \cdot 1/2 = 1/4$ therefore the distribution of $X+Y+Z$ is 0 with probability 1/4 and 2 with probability 3/4.

It also follows that $(X+Y+Z)^2$ is $0^2 = 0$ with probability 1/4 and $2^2 = 4$ with probability 3/4. We now use the alternative formula for variance:

$$\begin{aligned}\mathrm{Var}[X+Y+Z] &= \mathrm{E}[(X+Y+Z)^2] - \mathrm{E}[X+Y+Z]^2 \\ &= \left(\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 4\right) - \left(\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 2\right)^2 \\ &= 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}\,.\end{aligned}$$

Using the formula for variance of a Bernoulli variable with a given parameter we have $\mathrm{Var}[X] + \mathrm{Var}[Y] + \mathrm{Var}[Z] = 3(1/2 - (1/2)^2) = 3/4$. This verifies the proposition that says that for pairwise independent r.v.'s variance distributes over sums.

6. **[10 pts]** Suppose that each time Diana Taurasi takes a 3-point shot, she has a 37% probability of success, independent of all other attempts. (Success yields 3 points; failure yields 0.) If Taurasi takes seven 3-point shots in a game, what is the variance of the total number of points she scores from these shots?

Solution:

Let X be the number of successful 3-point shots Taurasi makes. Then X is a binomial random variable with 7 trials and success probability 0.37, so its variance is given by $7 \cdot 0.37 \cdot (1 - 0.37) = 1.6317$. The number of points she scores from these shots is $3X$, and $\text{Var}[3X] = 3^2 \text{Var}[X] = 9 \cdot 1.6317 = \boxed{14.6853}$.

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

You go to an alumni party with your pal Professor Val Tannen, and you notice that there are 20 men and 20 women. Each man is married to one woman and each woman is married to one man (bijection of men to women). However, it has been a while since you've seen your classmates and you don't know who is married to who. Professor Tannen assigns you the job of figuring out who is married to who.

To do this, you make several attempts. In each attempt, you match each man to a woman such that each man is paired with exactly one woman. Then, the couples you properly guessed tell you that you have gotten them correct. If you don't get all 20 couples correct, you guess again.

Your strategy is simple: You pair the couples at random. When you guess a couple correctly, in all future attempts you pair up that couple. For everyone else, you guess them at random again.

What is the expected number of guesses it will take for you to get all 20 couples correct?

Solution:

For the first guess, let us pose this question: how many couples do you expect to guess correctly on the first turn?

For the sake of the question, let's line the men up in an arbitrary order. Then for each guess, randomly permute the 20 women in a line and see how many couples we guess correctly. If you notice, this is a derangement. As we know from lecture, if we randomly permute n items, the expected number of items (in this case women and thus couples) that are in the "correct place" is 1.

This holds for every guess, that is: the expected number of new couples you get on each guess is 1.

Now view the question such that you remove a couple once you guess them correctly. For example after one guess, the expected number of couples you guess correctly is 1. For the next guess, it is like randomly guessing among 19 couples.

Because each of the guesses described like this is independent we can say that each guess reduces the number of couples you need to guess by 1, it will take an expected 20 guesses to guess all 20 couples correctly.