CIT 592 Summer 2019 **Homework 6** **Solutions**

Problem Set

1. [10 pts]

Use ordinary induction to prove that for every positive integer n, $n^2 + n$ is even.

Solution:

Let $P(n)$ be "$n^2 + n$ is even."

(BC) $1^2 + 1 = 2$ is even, so $P(1)$ holds.

(IS) Assume $P(k)$ for some arbitrary positive integer k; this is our induction hypothesis (IH). Then

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k = (k^2 + k) + 2(k+1)\,.$$

Our IH says that (k^2+k) is even, $2(k+1)$ is even by definition, and even plus even is even, so $(k+1)^2 + (k+1)$ is even, which is $P(k+1)$. Thus, we have shown that $P(k)$ implies $P(k+1)$.

By ordinary induction, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$, i.e., for every $n \in \mathbb{Z}^+$, $n^2 + n$ is even. $\square$

2. [10 pts]

Use ordinary induction to prove that for every positive integer n, $n^3 - n$ is a multiple of 6.

Solution:

Let $P(n)$ be "$n^3 - n$ is a multiple of 6."

(BC) $1^3 - 1 = 0 = 0 \cdot 6$, so $P(1)$ holds.

(IS) Assume $P(k)$ for some arbitrary positive integer k; this is our IH. Then

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = (k^3 - k) + 3(k^2 + k)$$

Our IH tells us that $(k^3 - k)$ is a multiple of 6, and the previous problem tells us that $(k^2 + k)$ is even, so $3(k^2 + k)$ is also a multiple of 6. Thus, their sum is a multiple of 6, and $P(k+1)$ holds. We have shown that $P(k)$ implies $P(k+1)$.

By ordinary induction, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$, i.e., for every $n \in \mathbb{Z}^+$, $n^3 - n$ is a multiple of 6. $\square$

3. [10 pts]

Use strong induction to prove that for every integer $n \geq 6$ there are non-negative integers a and b such that $n = 3a + 4b$.

Solution:

Let $P(n)$ be "there are positive integers a and b such that $n = 3a + 4b$."

(BC) $6 = 3 \cdot 2 + 4 \cdot 0$, $7 = 3 \cdot 1 + 4 \cdot 1$, and $8 = 3 \cdot 0 + 4 \cdot 2$, so $P(6)$, $P(7)$, and $P(8)$ all hold.

(IS) Assume for some arbitrary integer $k \geq 8$ that $P(j)$ holds for all $6 \leq j \leq k$ this is our IH. Then in particular, since $k - 2 \geq 6$, $P(k-2)$ says that there exist $a, b \in \mathbb{N}$ such that $k - 2 = 3a + 4b$. Now,

$$k + 1 = k - 2 + 3 = 3a + 4b + 3 = 3(a+1) + 4b\,,$$

so $P(k+1)$ holds. We have shown that $(\forall j \in [6..k]\ P(j))$ implies $P(k+1)$.

By strong induction, we conclude that $P(n)$ holds for all integers $n \geq 6$, i.e., for every integer $n \geq 6$, there exist $a, b \in \mathbb{N}$ such that $n = 3a + 4b$. $\square$

4. [10 pts]

Use ordinary induction to prove that $C(n) = 2^n + 3$ is a solution to the recurrence $C(0) = 4$ and $C(n) = 2 \cdot C(n-1) - 3$ for all $n \in \mathbb{Z}^+$.

Solution:

Let $P(n)$ be "$C(n) = 2^n + 3$."

(BC) $C(0) = 4 = 2^0 + 3$, so $P(0)$ holds.

(IS) Assume $P(k)$ for some arbitrary $k \in \mathbb{N}$; this is our IH. Then the recurrence tells us that $C(k+1) = 2C(k+1-1) - 3 = 2C(k) - 3$, which by our IH is

$$2 \cdot (2^k + 3) - 3 = 2^{k+1} + 6 - 3 = 2^{k+1} + 3\,,$$

so $P(k+1)$ holds. We have shown that $P(k)$ implies $P(k+1)$.

By ordinary induction, we conclude that $P(n)$ holds for all $nin\mathbb{N}$, i.e., $C(n) = 2^n + 3$ for all $n \in \mathbb{N}$. $\square$

5. [10 pts]

Use the recursion tree method or the telescopic method to solve the recurrence relation $f(0) = 4$ and $f(n) = f(n-1) + 2n - 1$ for all $n \in \mathbb{Z}^+$.

Solution:

We use the telescopic method.

$$\begin{aligned}
f(n) &= f(n-1) + 2n - 1 \\
f(n-1) &= f(n-2) + 2(n-1) - 1 \\
&\vdots \\
f(2) &= f(1) + 2(2) - 1 \\
f(1) &= f(0) + 2(1) - 1 \\
f(0) &= 4\,.
\end{aligned}$$

Looking at the terms that appear on only one side of these equations, we can see that

$$\begin{aligned}
f(n) &= (2n-1) + (2(n-1)-1) + \ldots + (2(2)-1) + (2(1)-1) + 4 \\
&= 2n + 2(n-1) + \ldots + 2(2) + 2(1) - n + 4 \\
&= 2(n + (n-1) + \ldots + 2 + 1) - n + 4 \\
&= 2(1 + 2 + \ldots + (n-1) + n) - n + 4 \\
&= 2\frac{n^2+n}{2} - n + 4 && \text{(using the summation formula)} \\
&= n^2 + n - n + 4 \\
&= n^2 + 4\,.
\end{aligned}$$

Therefore our closed formula for f is $\boxed{f(n) = n^2 + 4}$.

6. [10 pts]

Use strong induction to prove that $F_n < 2^n$ for all $n \in \mathbb{N}$.

Solution:

Let $P(n)$ be "$F_n < 2^n$."

(BC) $F_0 = 0 < 1 = 2^0$ and $F_1 = 1 < 2 = 2^1$, so $P(0)$ and $P(1)$ hold.

(IS) For some arbitrary positive integer $k \in \mathbb{Z}$, assume that $P(k-1)$ and $P(k)$ hold; this is our IH. Then

$$F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}\,.$$

Thus, $P(k-1)$ and $P(k)$ together imply $P(k+1)$.

By strong induction, we conclude that $P(n)$ holds for all $n \in \mathbb{N}$, i.e., that $F_n < 2^n$ for all $n \in \mathbb{N}$. $\square$

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

There are some real numbers x such that $x + \frac{1}{x}$ is an integer. For example, $2 + \sqrt{3} + \frac{1}{2+\sqrt{3}} = 4$, $1 + \frac{1}{1} = 2$, and $2\sqrt{6} - 5 + \frac{1}{2\sqrt{6}-5} = -10$.

Prove for all $x \in \mathbb{R}$ that if $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ also is an integer for all $n \in \mathbb{N}$.

Solution:

We prove this by strong induction. Assume that $x + \frac{1}{x}$ is an integer, and let $P(n)$ be "$x^n + \frac{1}{x^n}$ is an integer."

(BC) $P(0) = x^0 + \frac{1}{x^0} = 2$ is an integer, and $P(1)$ holds by our assumption.

(IS) Assume that $P(k)$ and $P(k-1)$ hold for some arbitrary $k \in \mathbb{N}$; this is our IH. Now,

$$\left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) = x^{k+1} + x^{k-1} + \frac{1}{x^{k-1}} + \frac{1}{x^{k+1}}\,.$$

Rearranging, we have

$$x^{k+1} + \frac{1}{x^{k+1}} = \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right)\,.$$

Our IH tells us that $x^k + \frac{1}{x^k}$ and $x^{k-1} + \frac{1}{x^{k-1}}$ are integers, and our BC tells us that $x + \frac{1}{x}$ is an integer. Hence, the entire RHS of the equation is an integer, and therefore the LHS is as well. We have shown that $P(k-1)$ and $P(k)$ together imply $P(k+1)$.

By strong induction, we conclude that $P(n)$ holds for all $n \in \mathbb{N}$, i.e., that $x^n + \frac{1}{x^n}$ is an integer for all $n \in \mathbb{N}$. $\square$