CIT 592 Fall 2019 Homework 12

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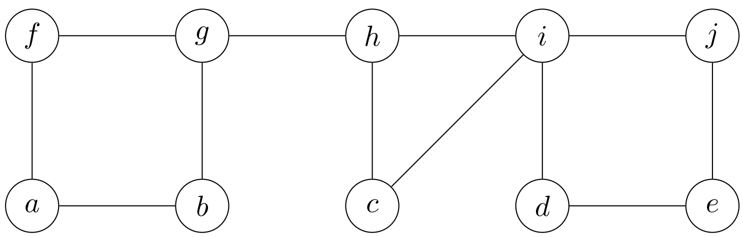
1. [10 pts]



In the graph above, how many connected components are in the subgraph induced by each of the following subsets of the vertices?

(a) **[2 pts]** $\{a, b, c, d, e\}$

(b) **[2 pts]** $\{f, g, h, i, j\}$

(c) **[2 pts]** $\{a, b, e, h, i\}$

(d) **[2 pts]** $\{c, f, g, i, j\}$

(e) **[2 pts]** $\{a, c, d, g, j\}$

Solution.

A

Three CC's: a to b, c, and d to e

B

One CC: f to g to h to i to j

C

Three CC's: e, h to i, a to b

D

Two CC's: f to g, c to i to j

E

Five CC's: a,c,d,g,j

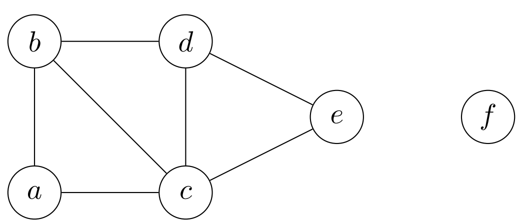
2. **[10 pts]** Recall that we proved in lecture that in a tree, any two nodes are connected by a *unique* path. Here you will prove the converse. From now on, you can use the statement of this converse in the same way you use any of the statements proven or stated in lecture.

Prove that if G is a graph in which any two nodes are connected by a unique path then G is a tree.

Solution.

With every pair of vertices there exists a path. Thus, G is connected.
G is not a cycle graph because there is only one path between every pair of vertices since any two nodes are connected by a unique path.
Therefore, G is a tree.

3. **[10 pts]** This problem and the next one will use the concept of a graph's *degree sequence*. This is a list of the degrees of all the vertices in the graph, **in descending order of degree**. For example, the graph



has degree sequence $(4, 3, 3, 2, 2, 0)$ because there is one node with degree 4 (c), two nodes with degree 3 (b and d), two nodes with degree 2 (a and e), and one node with degree 0 (f).

For each of the following, either list the set of edges of a **tree** with vertex set $\{a, b, c, d, e, f\}$ that has the stated degree sequence, or show that no such tree exists.

(a) **[4 pts]** $(4, 2, 1, 1, 1, 1)$

(b) **[6 pts]** $(4, 2, 2, 2, 1, 1)$

Solution.

A.
{a-b,a-c,a-d,a-e,b-f}
B.
Sum of degrees = 2*number of edges. So we have the following:
4+2+2+2+1+1 =12
vertex= 2(n-1) = 2(6-1) = 10
$12 \neq 10$
Therefore, B is not a tree.

4. [10 pts] We state without proof the following

Proposition. If $\beta : V_1 \to V_2$ is an isomorphism between the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then for every vertex $u \in V_1$, the degree of u (in G_1) is the same as the degree of $\beta(u)$ (in G_2).

(From now on you can use this statement in the same way you use any of the statements proven or stated in lecture.)

Using this statement, give an example of two trees that have the same degree sequence but are not isomorphic. Justify your answer. You can either draw the graphs or list the vertices and edges.

Solution.

Tree One
Vertices with degree one = 4
$\{\mathrm{V}_1, V_4, V_5, V_7\}$
Vertices with degree two = 2
$\{V_2, V_6\}$
Vertices with degree four = 1
$\{V_3\}$
Edges for Tree One:
$\{1-2, 2-3, 3-4, 3-5, 3-6, 6-7\}$

Tree Two
Vertices with degree one = 4

$\{\mathrm{V}_1, V_3, V_4, V_7\}$

Vertices with degree two $= 2$

$\{V_5, V_6\}$

Vertices with degree four $= 1$

$\{V_2\}$

Edges for Tree Two:

$\{1-2, 2-3, 2-4, 2-5, 5-6, 6-7\}$

These two tree have the same degree sequence and are not isomorphic because:

Tree one: V_3 has four edges $3-2, 3-6, 3-4, 3-5$

Tree two:V_2 has four edges $2-1, 2-3, 2-4, 2-5$

Because these graph's vertices are not connected in the same way they are not isomorphic.

5. [10 pts] Let $T = (V, E)$ be a tree, and suppose that some node $u \in V$ has degree d. Prove that T has at least d leaves. *Hint:* Consider the induced subgraph with vertex set $V \setminus \{u\}$

Solution.

Sum of degrees $=$ 2*edges $\to 1$

Sum of degrees $= 2(n-1) \to 2$

Let $|V| =$ n and $|E| = n-1$

Let L $=$ number of leaves in T

$n - L - 1$ vertices ≥ 2

Sum of degrees where l is of degree one $\geq L(1) + (n-L-1)(2) + d$

$2(n-1) \geq L + 2(n-L-1) + d$

$2n - 2 \geq L + 2n - 2L - 2 + d$

$0 \geq -L + d$

$L \geq d$

Graph T has at least d leaves

6. [10 pts] In lecture, we proved that any tree with n vertices must have $n-1$ edges. Here, you will prove the *converse* of this statement. From now on, you can use this converse in the same way as you use statements from lecture.

Prove that if $G = (V, E)$ is a connected graph such that $|E| = |V| - 1$, then G is a tree. **Solution.**

If G is a tree, then G is a connected graph without a cycle.
Let G have n-1 edges.
Prove by induction:
BC:
When n=1,2,3 T is a tree with n-1 edges.
IS:
IH:
Let T have n vertices and let E be the edge with ends points u and V. The only path between u and v is E.
Now, if we remove E we have two components T_1, T_2 where each component is a tree
$n_1 + n_2 = n$
$n_1 < n, n_2 < n$
Conclusion:
The edges in T_1 and T_2 are $n_1 - 1$ and $n_2 - 1$.
Therefore,
$T = n_1 - 1 + n_2 - 1 + 1$
$= n_1 + n_2 - 1$
$= n - 1$

7. [6 pts] OPTIONAL EXTRA CREDIT PROBLEM

Suppose that we generate a *random graph* $G = (V, E)$ on the vertex set $V = \{1, 2, \ldots, n\}$ in the following way. For each pair of vertices $i, j \in V$ with $i < j$, we flip a fair coin, and we include the edge i–j in E if and only if the coin comes up heads. How many edges should we expect G to contain? How many cycles of length 3 should we expect G to contain?

Solution.

pr[edge inclusion] = 1/2
n = number of vertices
n(n-1) = pairs of vertices in random graph
We can add edges with pr[1/2] by picking 2 from n.Therefore:
E =n/2*1/2

There are $\binom{n}{2}$ edges in the graph