Part I

Prove that the second derivative at b_0 of a Bezier cubic of control points (b_0, b_1, b_2, b_3) is $6(b_0 - 2b_1 + b2)$ and the second derivative at b_3 becomes $6(b_1 - 2b_1 + b3)$

Answer

$$C(t) = (1-t)^3 b_0 + 3(1-t)^2 t b_1 + 3(1-t)t^2 b_2 + t^3 b_3$$

$$C'(t) = -3(1-t)^2 b_0 + 3 * 2(1-t) * (-1)t * b_1 + 3(1-t)^2 b_1 + (-3)t^2 * b_2 + 3(1-t) * 2t * b_2 + 3b_3 * t^2$$
$$C'(t) = -3(1-t)^2 b_0 - 6(1-t)t * b_1 + 3(1-t)^2 b_1 - 3t^2 b_2 + 6t(1-t)b_2 + 3b_3 t^2$$
$$C'(t) = 3(1-t)^2(b_1 - b_0) + 6t(1-t)(b_2 - b_1) + 3t^2(b_3 - b_2)$$

$$C''(t) = -6(1-t)(b_1 - b_0) + 6(1-t)(b_2 - b_1) + 6t * (-1)(b_2 - b_1) + 6t)b_3 - b_2)$$
$$C''(t) = 6(1-t)(b_2 - 2b_1 + b_0) + 6t(b_3 - 2b_2 + b_1)$$

$$C''(0) = 6(b_2 - 2b_1 + b_0)$$
$$C''(1) = 6(b_3 - 2b_2 + b_1)$$

Prove that our system becomes:

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} 6_x 1 - x_0 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} - x_N \end{pmatrix}$$

Answer:

$$d_0 = \frac{2}{3}x_0 + \frac{1}{3}d_1$$
$$d_n = \frac{1}{3}d_{N-1} + \frac{2}{3}x_N$$

$$x_1 = b_2^1 * \frac{1}{2} + b_1^2 * \frac{1}{2}$$
$$x_1 = (\frac{1}{2}d_0 + \frac{1}{2}d_1) * \frac{1}{2} + (\frac{2}{3}d_1 + \frac{1}{3}d_2) * \frac{1}{2}$$
$$x_1 = \frac{1}{4}d_0 + \frac{1}{4}d_1 + \frac{1}{3}d_1 + \frac{1}{6}d_2$$
$$x_1 = \frac{1}{4}d_0 + \frac{7}{12}d_1 + \frac{1}{6}d_2$$
$$x_1 = \frac{1}{4}(\frac{2}{3}x_0 + \frac{1}{3}d_1) + \frac{7}{12}d_1 + \frac{1}{6}d_2$$

$$
\begin{aligned}
x_1 &= \frac{1}{6}x_0 + \frac{1}{12}d_1 + \frac{7}{12}d_1 + \frac{1}{6}d_2 \\
x_1 &= \frac{1}{6}x_0 + \frac{2}{3}d_1 + \frac{1}{6}d_2
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= b_2^2 * \frac{1}{2} + b_1^3 * \frac{1}{2} \\
x_2 &= (\frac{1}{3}d_1 + \frac{2}{3}d_2) * \frac{1}{2} + (\frac{2}{3}d_1 + \frac{1}{3}d_2) * \frac{1}{2} \\
x_2 &= \frac{1}{6}d_1 + \frac{1}{3}d_2 + \frac{1}{3}d_2 + \frac{1}{6}d_3 \\
x_2 &= \frac{1}{6}d_1 + \frac{4}{6}d_2 + \frac{1}{6}d_3
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= b_2^3 * \frac{1}{2} + b_1^4 * \frac{1}{2} \\
x_3 &= (\frac{1}{3}d_2 + \frac{2}{3}d_3) * \frac{1}{2} + (\frac{2}{3}d_3 + \frac{1}{3}d_4) * \frac{1}{2} \\
x_3 &= \frac{1}{6}d_2 + \frac{1}{3}d_3 + \frac{1}{3}d_3 + \frac{1}{6}d_4 \\
x_3 &= \frac{1}{6}d_2 + \frac{4}{6}d_3 + \frac{1}{6}d_4 \\
&\vdots \\
x_{N-2} &= \frac{1}{6}d_{N-3} + \frac{4}{6}d_{N-2} + \frac{1}{6}d_{N-1}
\end{aligned}
$$

$$
\begin{aligned}
x_{N-1} &= (\frac{1}{3}d_{N-2} + \frac{2}{3}d_{N-1}) * \frac{1}{2} + (\frac{1}{2}d_{N-1} + \frac{1}{2}d_N) * \frac{1}{2} \\
x_{N-1} &= \frac{1}{6}d_{N-2} + \frac{1}{3}d_{N-1} + \frac{1}{4}d_{N-1} + \frac{1}{4} * (\frac{1}{3}d_{N-1} + \frac{2}{3}x_N) \\
x_{N-1} &= \frac{1}{6}d_{N-2} + \frac{2}{3}d_{N-1} + \frac{1}{6}x_N
\end{aligned}
$$

Now, we have the following equations:

$$
\begin{aligned}
x_1 &= \frac{1}{6}x_0 + \frac{2}{3}d_1 + \frac{1}{6}d_2 \\
x_2 &= \frac{1}{6}d_1 + \frac{4}{6}d_2 + \frac{1}{6}d_3 \\
x_3 &= \frac{1}{6}d_2 + \frac{4}{6}d_3 + \frac{1}{6}d_4 \\
&\vdots \\
x_{N-2} &= \frac{1}{6}d_{N-3} + \frac{4}{6}d_{N-2} + \frac{1}{6}d_{N-1} \\
x_{N-1} &= \frac{1}{6}d_{N-2} + \frac{2}{3}d_{N-1} + \frac{1}{6}x_N
\end{aligned}
$$

These can be written as:

$$\begin{aligned}
4d_1 + d_2 &= 6x_1 - x_0 \\
d1 + 4d_2 + d_3 &= x_2 \\
d_2 + 4d_3 + d_4 &= x_3 \\
&\vdots \\
d_{n-3} + 4d_{n-2} + d_{N-1} &= x_{N-2} \\
d_{n-2} + 4d_{n-1} &= 6x_{N-1} + x_N
\end{aligned}$$

When written in matrix form we arrive back at:

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} 6_x 1 - x_0 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} - x_N \end{pmatrix}$$