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1. [**10 pts**] Recall from lecture that an $m \times n$ grid graph has m rows of n vertices with adjacent vertices connected by an edge. Find the greatest length of any path in such a graph, and prove that it is maximum.

Solution.

V= total number of vertices
Greatest length of path = V-(n-1)
There are no strictly bigger set of vertices than V-(n-1).
Minimum degree = degree of vertex with the least number of degree incident to it
Since v-(n-1) is the minimum degree, this greatest path is maximum.

2. [**10 pts**] As in the previous problem we consider the $m \times n$ grid graph and assume $m, n \geq 2$. Recall that we showed in lecture that this graph has mn vertices and $2mn - m - n$ edges. This graph is clearly connected. Calculate D, the maximum number of edges that can be deleted from this graph without disconnecting it. Justify your answer. Then describe (informally) *which* D edges of the graph you can delete without disconnecting it (there is more than one such set of D edges). **Solution.**

m*n-1 = number of edges connecting vertices
2mn-m-n = number of edges
number of edges - number of edges connecting vertices = max number of edges
(2mn-m-n) - (m*n-1) = D

Informal Description:
let G = 3x2 grid
The following are the edges of the graph that can be deleted: Left most edge and center edge.
This will leave an outside line connecting the remaining 5 edges.

3. [**10 pts**] Consider the graph whose edges are the seams of a standard soccer ball and whose vertices are the places where these seams meet.

Each vertex in this graph lies at the corner of one of the 12 black pentagons and has degree 3. How many edges are in this graph?

Solution.

12 pentagons * 5 sides on each pentagon = 60
60 + (60/2) (adjusting for over counting)
= 90 edges

4. [**10 pts**] A *d-regular* graph is a graph in which every vertex has the same degree d, where d is some natural number. Prove that there is no 9-regular graph with 42 edges.

Solution.

V = vertex
9*V/2 = number of edges=42
V = 84/9
V = 9.3
Vertex's cannot be negative and must be natural numbers. Therefore,
V $\neq$ 9.3
Thus, there are no 9 regular graphs with 42 edges.

5. [**10 pts**] Prove that every graph with at least two vertices contains two vertices with the same degree.

Solution.

We will prove this by contradiction.

Let G = (V,E) with n vertices and m edges

Suppose that no two vertices of G have the same degree. deg(u) $\neq deg(v)$

Highest degree $= n-1$ Therefore,

$v \in V, 0 \leq deg(v) \leq n-1$

Now, there must be a one-to-one correspondence between V and D because G cannot contain two vertices with the same degree.

Assume this graph is connected

Therefore, one vertex in G has degree 0 and another degree n-1

A vertex of 0 is not connected to any other vertex

Since this is impossible, there a fewer elements in D that can be assigned to V.

By PHP, two or more elements in V must map to the same element in D, thus there are two vertices with the same degree

We have arrived at a contradiction.

6. **[10 pts]** Let $n \geq 2$ and $p \geq 1$ be two positive integers. Let G be a graph with n vertices such that each vertex has *p or more* incident edges. Prove that if $p > \frac{n-2}{2}$ then G is connected. **Solution.**

Suppose by contradiction that G is not connected.

Then, G contains at least two disjoint graphs with no edge between them.

Let these two graphs be G_1 and G_2,

Let $x_i =$ number of vertices in G_i

Let $X_1 =$ number of V in G_1

Let $X_2 =$ number of V in G_2

Each vertex has at least P edges, thus $x \in G_i \geq p+1, i = 1,2$

$x_i \geq p+1 > n-2/2+1 = n/2$ for $i = 1,2$

$X_1 + X_2 =$ total number of V $= n$

$X_1 + X_2 > n/2$ for $i = 1,2$

$x_i > n/2 + n/2 = n$

$x_i > n$

We now have a contradiction thus G is connected

7. [6 pts] OPTIONAL EXTRA CREDIT PROBLEM Li and Rose are married. They invite three other married couples to a party at their house. Various people shake hands during the party, but no one shakes hands with their own spouse (or with themselves, of course). At the end of the party, Li asks everybody else how many people they shook hands with and receives seven different responses. How many people did Rose shake hands with?

Solution.

YOUR SOLUTION HERE