This problem set will refer to a *standard deck of cards*. Each card in such a deck has a *rank* and a *suit*. The 13 ranks, ordered from lowest to highest, are *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, *10*, *jack*, *queen*, *king*, and *ace*. The 4 suits are *clubs* ($\clubsuit$), *diamonds* ($\diamondsuit$), *hearts* ($\heartsuit$), and *spades* ($\spadesuit$). Cards with clubs or spades are *black*, while cards with diamonds or hearts are *red*. The deck has exactly one card for each rank–suit pair: *4 of diamonds*, *queen of spades*, etc., for a total of $13 \cdot 4 = 52$ cards.

Problem Set

1. [10 pts]

Suppose you select a card at random from a standard deck, and then without putting it back, you select a second card at random from the remaining 51 cards. What is the probability that both cards have the same rank, or both have the same suit, or one is red and one is black?

Solution:

There are $52 \cdot 51$ possible outcomes. These outcomes are our sample space Ω. Note that Ω has a uniform probability distribution which will allow us to find probability by simply dividing event space size by sample space size. Let R be the event that both cards have the same rank, S be the event that both cards have the same suit, and C be the event that one is red and one is black. We are interested in $\Pr[R \cup S \cup C]$, which we will calculate using the principle of inclusion-exclusion for three events.

There are $52 \cdot 3$ ways to choose the cards such that both cards have the same rank: 52 ways to choose the first card and then $4 - 1 = 3$ ways to choose a second card of the same rank from the remaining cards. Thus, by the multiplication rule, $|R| = 52 \cdot 3$. Similarly, $|S| = 52 \cdot 12$ (there are 52 ways to choose the first card and 12 ways to choose a second card of the same suit). There are 52 ways to choose the first card, and then 26 ways to choose a second card of a different color from the remaining cards, so $|C| = 52 \cdot 26$. We have $\Pr[R] = |R|/|\Omega| = 52 \cdot 3/(52 \cdot 51) = 3/51$, $\Pr[S] = 52 \cdot 12/(52 \cdot 51) = 12/51$, and $\Pr[C] = 52 \cdot 26/(52 \cdot 51) = 26/51$.

Once the first card has been drawn, there are no remaining cards with the same rank and suit, so $|R \cap S| = 0$. There is also no way for the two cards to have the same suit but different colors, so $|S \cap C| = 0$. Once the first card has been drawn, there are 2 ways to choose a second card of the same rank but a different color, so $|R \cap C| = 52 \cdot 2$. Finally, $|R \cap S \cap C| = 0$ because $R \cap S \cap C \subseteq R \cap S = \emptyset$. We have $\Pr[R \cap C] = |R \cap C|/|\Omega| = 52 \cdot 2/(52 \cdot 51) = 2/51$ and $\Pr[R \cap S] = \Pr[S \cap C] = \Pr[R \cap S \cap C] = 0$.

We can now apply the principle of inclusion-exclusion for three events:

$$\begin{aligned}\Pr[R \cup S \cup C] &= \Pr[R] + \Pr[S] + \Pr[C] - \Pr[R \cap S] - \Pr[S \cap C] - \Pr[R \cap C] + \Pr[R \cap S \cap C] \\ &= \frac{3}{51} + \frac{12}{51} + \frac{26}{51} - 0 - 0 - \frac{2}{51} + 0 \\ &= \frac{39}{51} = \boxed{\frac{13}{17}}.\end{aligned}$$

2. [**10 pts**]

For each of the three golfers g_1, g_2, g_3 the probabilities of hitting a ball on the green (the desired play!), in a bunker (sand trap), or in a water hazard are given by the following table (we make the simplifying assumption that these are the only three results of a hit):

golfer	green	bunker	water
g_1	1/2	1/3	1/6
g_2	1/3	1/6	1/2
g_3	1/6	1/2	1/3

The three golfers, playing together. hit one ball each, mutually independently.

(a) [**4 pts**] What is the probability that all three balls end up in the water?

(b) [**6 pts**] What is the probability that at least one of the balls ends up on the green?

Solution:

(a) Let $W_i,\ i = 1, 2, 3$ be the event that golfer $g_i,\ i = 1, 2, 3$ hits their ball in the water. The event that all three balls end up in the water is $W_1 \cap W_2 \cap W_3$. By mutual independence

$$\Pr[W_1 \cap W_2 \cap W_3] = \Pr[W_1] \cdot \Pr[W_2] \cdot \Pr[W_3] = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{3} = \boxed{\frac{1}{36}}$$

(b) Let $G_i,\ i = 1, 2, 3$ be the event that golfer $g_i,\ i = 1, 2, 3$ hits their ball on the green. The event that at least one of the balls ends up on the green is $G_1 \cup G_2 \cup G_3$. By using probability properties, De Morgan's laws, mutual independence and properties of independence we obtain:

$$\begin{aligned}\Pr[G_1 \cup G_2 \cup G_3] &= 1 - \Pr[\overline{G_1 \cup G_2 \cup G_3}] \\ &= 1 = \Pr[\overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3] \\ &= 1 - \Pr[\overline{G}_1] \cdot \Pr[\overline{G}_2] \cdot \Pr[\overline{G}_3] \\ &= 1 - (1 - \Pr[G_1]) \cdot (1 - \Pr[G_2]) \cdot (1 - \Pr[G_3]) \\ &= 1 - (1 - \frac{1}{2}) \cdot (1 - \frac{1}{3}) \cdot (1 - \frac{1}{6}) = \boxed{\frac{13}{18}}\end{aligned}$$

Instead of the derivation shown above we could have directly used the proposition on unions of mutually independent events that we have stated in the lecture segment entitled "Pairwise and mutual independence".

3. [10 pts]

Suppose you roll three distinguishable fair dice and call the resulting numbers a, b, and c. Define events X = "$a + b$ is even," Y = "$b + c$ is even," and Z = "$a + c$ is even." Prove that these three events are pairwise independent but not mutually independent.

Solution:

There are 216 possible outcomes in our sample space Ω. Note that Ω has a uniform probability distribution. X happens when a and b are both even or both odd. There are 6 possible values for a, and once a has been determined there are 3 possibilities for b that have the same parity and 6 possibilities for c, so $|X| = 6 \cdot 3 \cdot 6 = 108$. Similarly, $|Y| = |Z| = 108$. Thus, $\Pr[X] = \Pr[Y] = \Pr[Z] = 108/216 = 1/2$.

Now consider $X \cap Y$. If $a + b$ is even, then a and b are either both odd or both even. If $b + c$ is even, then b and c are either both odd or both even. This means that for all outcomes $(a, b, c) \in X \cap Y$, a, b, and c are either all odd or all even. This also implies that $a + c$ is even, so $X \cap Y = X \cap Y \cap Z$.

Consider the situation when a has been determined. If a is odd, then there are 3 possibilities for b and 3 possibilities for c which are also odd. Similarly, if a is even, then there are 3 possibilities for b and 3 possibilities for c which are also even. So $|X \cap Y| = |X \cap Y \cap Z| = 6 \cdot 3 \cdot 3 = 54$. Thus, $\Pr[X \cap Y] = \Pr[X \cap Y \cap Z] = 54/216 = 1/4$. So

$$\Pr[X] \cdot \Pr[Y] = 1/2 \cdot 1/2 = 1/4 = \Pr[X \cap Y]\,,$$

i.e., X and Y are independent. Similarly, Y and Z are independent and X and Z are independent, so X, Y, and Z are pairwise independent. But

$$\Pr[X] \cdot \Pr[Y] \cdot \Pr[Z] = 1/2 \cdot 1/2 \cdot 1/2 = 1/8 \neq 1/4 = \Pr[X \cap Y \cap Z]\,,$$

so X, Y, and Z are not mutually independent. $\square$

4. [10 pts]

John and Annie have two distinguishable ponds. Initially, each of the ponds contains four ducks and five geese. John first picks a bird uniformly at random from the left pond and moves it to the right pond. Then, Annie picks a bird uniformly at random from the right pond.

What is the probability that Annie picks a duck?

Solution:

The sample space here is the possibilities of what John and Annie pic. $\Omega = DD, DG, GD, GG$ where D represents "picks a duck" and G represents "picks a goose." Let D_1 = "John picks a duck," G_1 = "John picks a goose," D_2 = "Annie picks a duck," and G_2 = "Annie picks a goose." Then we are looking for $\Pr[D_2]$. D_2 consists of two outcomes: John picks a duck and Annie picks a duck, or John picks a goose and Annie picks a duck. More formally, $D_2 = (D_2 \cap D_1) \cup (D_2 \cap G_1)$. And since $D_2 \cap D_1$ and $D_2 \cap G_1$ are disjoint, the addition rule applies. This gives us

$$\begin{aligned}\Pr[D_2] &= \Pr[D_2 \cap D_1] + \Pr[D_2 \cap G_1] \\ &= \Pr[D_2 \mid D_1] \cdot \Pr[D_1] + \Pr[D_2 \mid G_1] \cdot \Pr[G_1] \,.\end{aligned}$$

Now, four out of the nine birds initially in the left pond are ducks, so $\Pr[D_1] = 4/9$ and $\Pr[G_1] = 5/9$. If John moves a duck to the right pond, then five out of the ten birds there when Annie makes her choice are ducks. If he instead moves a goose, then four of the ten birds are ducks. Thus, $\Pr[D_2 \mid D_1] = 5/10 = 1/2$ and $\Pr[D_2 \mid D_1] = 4/10 = 2/5$. We have

$$\Pr[D_2] = \frac{1}{2} \cdot \frac{4}{9} + \frac{2}{5} \cdot \frac{5}{9} = \frac{2}{9} + \frac{2}{9} = \boxed{\frac{4}{9}}.$$

5. [10 pts]

Consider the random permutations of [1..6]. Let A be the event that the number in the first position of such a random permutation is 6, and let B be the event that the number in the last position of such a random permutation is 1. Calculate $\Pr[A \mid B]$.

Solution:

Recall that

$$\Pr[A \mid B] \;=\; \frac{\Pr[A \cap B]}{\Pr[B]}$$

therefore we need $\Pr[A \cap B]$ and $\Pr[B]$.

There are 6! possible outcomes in the uniform probability space of random permutations of [1..6]. In general, we have seen in lecture that the probability of a particular element being in a particular location in a random permutation on n elements is $1/n$. Therefore $\Pr[B] = 1/6$.

Now consider the event $A \cap B$. Once the first and last elements are set there are four elements left to place in the middle four positions and this can be done in 4! ways. Therefore, $\Pr[A \cap B] = |A \cap B|/6! = 4!/6! = 1/30$.

Using these values and the definition of conditional probability, we have

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{1/30}{1/6} = \boxed{\frac{1}{5}}.$$

6. [10 pts]

You draw a card uniformly at random from a standard deck, then remove all cards of strictly higher rank (e.g., if you draw a 4, then the remaining deck consists of four 2s, four 3s, and three 4s). You repeat this process three times on the same deck, without putting the removed cards back. What is the probability that the three cards you've drawn are a 3 and two 5s?

Solution:

This can only happen if the first two cards are 5s and the third is a 3, so let A be the event that the first card is a 5, B be the event that the second card is a 5, and C be the event that the third card is a 3. Then we want $\Pr[A \cap B \cap C]$.

The chain rule tells us that this is equal to $\Pr[A] \cdot \Pr[B \mid A] \cdot \Pr[C \mid B \cap A]$.

There are 52 possibilities for the first card, 4 of which are 5s, so $\Pr[A] = 4/52 = 1/13$. Given A, there are 15 possibilities for the second card, and 3 of them are 5s, so $\Pr[B \mid A] = 3/15 = 1/5$. Given $A \cap B$, there are 14 possibilities for the third card, and 4 of them are 3s, so $\Pr[C \mid A \cap B] = 4/14 = 2/7$. So

$$\Pr[A \cap B \cap C] = \frac{1}{13} \cdot \frac{1}{5} \cdot \frac{2}{7} = \boxed{\frac{2}{455}}.$$

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Two events A, B are *conditionally independent* given a third event C, written $A \perp B \mid C$, if

$$\Pr[A \cap B \mid C] = \Pr[A \mid C] \cdot \Pr[B \mid C]\,.$$

Give an example of events A, B, C such that $A \perp B$ and $A \not\perp B \mid C$. Make sure to define your set Ω of outcomes, and justify your answer.

Solution:

Consider flipping a fair coin twice. Let Ω be the results of the flips: $\Omega = \{\texttt{HH}, \texttt{HT}, \texttt{TH}, \texttt{TT}\}$. Let $A = \{\texttt{HH}, \texttt{TH}\}$, $B = \{\texttt{HH}, \texttt{TT}\}$, and $C = \{\texttt{HH}, \texttt{HT}, \texttt{TH}\}$. Observe that we have $A \cap B = \{\texttt{HH}\}$, $A \cap C = \{\texttt{HH}, \texttt{TH}\}$, $B \cap C = \{\texttt{HH}\}$, and $A \cap B \cap C = \{\texttt{HH}\}$.

Since Ω is uniform, we have the following probabilities:

$$\Pr[A] = \frac{1}{2} \qquad \Pr[B] = \frac{1}{2} \qquad \Pr[C] = \frac{3}{4}$$

$$\Pr[A \cap B] = \frac{1}{4} \qquad \Pr[A \cap C] = \frac{1}{2} \qquad \Pr[B \cap C] = \frac{1}{4}$$

$$\Pr[A \cap B \cap C] = \frac{1}{4}\,.$$

First, we show that $A \perp B$. We have:

$$\Pr[A] \cdot \Pr[B] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \Pr[A \cap B]\,.$$

We now show that $A \not\perp B \mid C$. By the definition of conditional probability, we find:

$$\Pr[A \mid C] \cdot \Pr[B \mid C] = \frac{\Pr[A \cap C]}{\Pr[C]} \cdot \frac{\Pr[B \cap C]}{\Pr[C]} = \frac{1/2}{3/4} \cdot \frac{1/4}{3/4} = \frac{2}{9}$$

$$\Pr[A \cap B \mid C] = \frac{\Pr[A \cap B \cap C]}{\Pr[C]} = \frac{1/4}{3/4} = \frac{1}{3}$$

Since $\frac{1}{3} \neq \frac{2}{9}$, $\Pr[A \cap B \mid C] \neq \Pr[A \mid C] \cdot \Pr[B \mid C]$. Therefore, $A \perp B$, but $A \not\perp B \mid C$. $\square$