Module 6

Euclidean Spaces
Solutions

Problem 1: 5 points

Let $\mathcal{C}[a,b]$ denote the set of continuous functions $f : [a,b] \to \mathbb{R}$. Given any two functions $f, g \in \mathcal{C}[a,b]$, let

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt.$$

Prove that the above form is bilinear, symmetric and positive (that is, $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{C}[a,b]$). You **do not** have to prove that this bilinear form is definite (that is, if $\langle f, f \rangle = 0$, then $f = 0$).

Solution. The definite integral has the property (this is basic calculus) that for any continuous functions $f_1, f_2, f, g, g_1, g_2 \in \mathcal{C}[a,b]$ and any two scalars $\lambda\,\mu$, we have

$$\int_a^b (\lambda f_1 + \mu f_2)(t)g(t)dt = \int_a^b (\lambda f_1(t)g(t) + \mu f_2(t)g(t))dt = \lambda \int_a^b f_1(t)g(t)dt + \mu \int_a^b f_2(t)g(t)dt,$$

and

$$\int_a^b f(t)g(t)dt = \int_a^b g(t)f(t)dt,$$

so we also have

$$\int_a^b f(t)(\lambda g_1 + \mu g_2)(t)dt = \int_a^b (\lambda f(t)g_1(t) + \mu f(t)g_2(t))dt = \lambda \int_a^b f(t)g_1(t)dt + \mu \int_a^b f(t)g_2(t)dt,$$

which shows that the map $f, g \mapsto \langle f, g \rangle = \int_a^b f(t)g(t)dt$ is bilinear and symmetric.

Given any continuous function $f \in \mathcal{C}[a,b]$, the function $t \mapsto (f(t))^2$ is continuous, and since $(f(t))^2 \geq 0$ on $[a,b]$, by a standard property of the integral,

$$\langle f, f \rangle = \int_a^b (f(t))^2 dt \geq 0.$$

Problem 2: 5 points

Prove that the following matrix is orthogonal and skew-symmetric:

$$M = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

Solution. Since

$$M^\top = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}^\top = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} = -M,$$

M is skew symmetric. We verify immediately that

$$\begin{aligned} M^\top M = -M^2 &= -\frac{1}{3} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = I_3. \end{aligned}$$

Therefore, M is orthogonal.

Problem 3: 25 points

(1) (**10 points**) Prove that if an upper triangular matrix A is invertible, then its inverse A^{-1} is also upper triangular and its diagonal entries are nonzero. Conversely, if an upper triangular matrix A has nonzero diagonal entries, then it is invertible. Furthermore, if the diagonal entries of A are positive, then so are the diagonal entries of A^{-1}.

(2) (**3 points**) An orthogonal matrix which is also upper triangular is a diagonal matrix with entries ± 1.

(3) (**7 points**) The product $A_1 A_2$ of two upper triangular matrices A_1 and A_2 is also upper triangular. Furthermore if the diagonal entries of A_1 and A_2 are strictly positive, then so are the diagonal entries of $A_1 A_2$.

(4) (**5 points**) Let A be an invertible matrix. Prove that if $A = Q_1 R_1 = Q_2 R_2$ are two QR-decompositions of A and if the diagonal entries of R_1 and R_2 are positive, then $Q_1 = Q_2$ and $R_1 = R_2$.

Solution.

(1) Fact (1) is proven by induction on the dimension n of the matrix A.

The base case $n = 1$ is trivial ($A = (a)$ for a nonzero scalar a, so $A^{-1} = (a^{-1})$).

For the induction step ($n \geq 1$), write the $(n+1) \times (n+1)$ matrix A in block form as

$$A = \begin{pmatrix} T & U \\ 0_{1,n} & \alpha \end{pmatrix},$$

where T is an $n \times n$ upper triangular matrix, U is an $n \times 1$ matrix and $\alpha \in \mathbb{R}$. Assume that A is invertible and let B be its inverse, written in block form as

$$B = \begin{pmatrix} C & V \\ W & \beta \end{pmatrix},$$

where C is an $n \times n$ matrix, V is an $n \times 1$ matrix, W is a $1 \times n$ matrix, and $\beta \in \mathbb{R}$. Since B is the inverse of A, we have $AB = I_{n+1}$, which yields

$$\begin{pmatrix} T & U \\ 0_{1,n-1} & \alpha \end{pmatrix} \begin{pmatrix} C & V \\ W & \beta \end{pmatrix} = \begin{pmatrix} I_n & 0_{n,1} \\ 0_{1,n} & 1 \end{pmatrix}.$$

By block multiplication we get

$$\begin{aligned} TC + UW &= I_n \\ TV + \beta U &= 0_{n,1} \\ \alpha W &= 0_{1,n} \\ \alpha\beta &= 1. \end{aligned}$$

From the above equations we deduce that $\alpha, \beta \neq 0$ and $\beta = \alpha^{-1}$. Furthermore, if $\alpha > 0$, then $\alpha^{-1} > 0$. Since $\alpha \neq 0$, the equation $\alpha W = 0_{1,n}$ yields $W = 0_{1,n}$, and so

$$TC = I_n, \quad TV + \beta U = 0_{n,1}.$$

It follows that T is invertible and C is its inverse, and since T is upper triangular, by the induction hypothesis, C is also upper triangular and T has nonzero diagonal entries. Also, if the diagonal entries of T are positive, so are the diagonal entries of C. Since we also have $W = 0_{1,n}$, we conclude that C is upper triangular with positive diagonal entries if the diagonal entries of T are positive, which establishes the induction step.

We now prove by induction on the dimension n of A that if A is upper triangular with nonzero diagonal entries, then A is invertible.

The base case $n = 1$ is trivial since $A = (a)$ with $a \neq 0$, so $A^{-1} = (a^{-1})$.

For the induction step, write

$$A = \begin{pmatrix} T & U \\ 0_{1,n} & \alpha \end{pmatrix},$$

with $\alpha \neq 0$, T is upper triangular and all diagonal entries of T nonzero, and let us look for an upper triangular inverse of the form

$$B = \begin{pmatrix} C & V \\ 0_{1,n} & \beta \end{pmatrix}.$$

Since $AB = I_{n+1}$ must hold, we must have

$$\begin{aligned} TC &= I_n \\ TV + \beta U &= 0_{n,1} \\ \alpha W &= 0_{1,n} \\ \alpha\beta &= 1. \end{aligned}$$

Since $\alpha \neq 0$, we have $\beta = \alpha^{-1}$, and by the induction hypothesis, since T is upper triangular and has nonzero diagonal entries, T is invertible. From $TV + \beta U = 0_{n,1}$, we get

$$V = -\beta T^{-1} U,$$

so B is uniquely determined as the inverse of A.

For this part, we can also prove by induction using the Laplace expansion rule with respect to the last row that if A is upper triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$, and that an $n \times n$ matrix A is invertible iff $\det(A) \neq 0$ iff all diagonal entries a_{ii} are nonzero.

(2) Assume R is an orthogonal matrix which is also upper triangular. Then, $R^{-1} = R^{\top}$ is lower triangular. By (1), since R is invertible and upper triangular, R^{-1} is also upper triangular. Since R^{-1} is simultaneously upper and lower diagonal, it must be diagonal, so R is a diagonal matrix. But then, since R is orthogonal, the entries of R are ± 1.

(3) This is verified by a direct computation using induction on the dimension n of the matrices. The base case $n = 1$ is trivial. For the induction step, if A_1 and A_2 are $(n+1) \times (n+1)$ upper triangular matrices written in block form as

$$A_1 = \begin{pmatrix} T_1 & U_1 \\ 0_{1,n} & \alpha_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} T_2 & U_2 \\ 0_{1,n} & \alpha_2 \end{pmatrix},$$

where T_1 and T_2 are $n \times n$ upper triangular matrices, U_1, U_2 are $n \times 1$ matrices and $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$$A_1A_2 = \begin{pmatrix} T_1 & U_1 \\ 0_{1,n} & \alpha_1 \end{pmatrix} \begin{pmatrix} T_2 & U_2 \\ 0_{1,n} & \alpha_2 \end{pmatrix} = \begin{pmatrix} T_1T_2 & T_1U_2 + \alpha_2 U_1 \\ 0_{1,n} & \alpha_1\alpha_2 \end{pmatrix},$$

by the induction hypothesis T_1T_2 is upper triangular, so A_1A_2 is upper triangular.

If A_1 and A_2 have positive diagonal entries, then $\alpha_1, \alpha_2 > 0$ and the diagonal entries of T_1 and T_2 are positive, so by the induction hypothesis, T_1T_2 has positive diagonal entries. Since in this case $\alpha_1\alpha_2 > 0$, the diagonal entries of A_1A_2 are also positive.

(4) Now assume that $Q_1R_1 = Q_2R_2$, where R_1 and R_2 are upper triangular with positive diagonal entries and Q_1, Q_2 are orthogonal. By (1), as R_1 and R_2 are upper triangular with nonzero diagonal entries, R_1 and R_2 are invertible. From $Q_1R_1 = Q_2R_2$ we deduce that

$$R_2R_1^{-1} = Q_2^{\top}Q_1.$$

Now from (1), as R_1 is upper triangular with positive diagonal entries, R_1 is invertible and R_1^{-1} is also upper triangular with positive diagonal entries, and thus by (3), $R_2R_1^{-1}$ is upper triangular with positive diagonal entries. On the other hand, $Q_2^\top Q_1 = R_2R_1^{-1}$ is orthogonal and upper triangular, so by (2), $Q_2^\top Q_1$ is a diagonal matrix. Furthermore, since $Q_2^\top Q_1 = R_2R_1^{-1}$ has positive entries and as the entries in $Q_2^\top Q_1$ are ± 1, we must have $Q_2^\top Q_1 = I$ and so

$$Q_1 = Q_2,$$

from which we deduce that $R_1 = R_2$ (since $Q_2^\top Q_1 = R_2R_1^{-1}$ and $Q_1 = Q_2$ implies that $R_2R_1^{-1} = I$).

Problem 4: 25 points total

Let $\varphi \colon E \times E \to \mathbb{R}$ be a bilinear form on a real vector space E of finite dimension n. Given any basis $(e_1, \ldots, e_n)$ of E, let $A = (a_{i\,j})$ be the matrix defined such that

$$a_{i\,j} = \varphi(e_i, e_j),$$

$1 \leq i, j \leq n$. We call A *the matrix of* φ *w.r.t. the basis* $(e_1, \ldots, e_n)$.

(1) (5 points) For any two vectors x and y, if X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^{\mathrm{T}} A Y.$$

(2) (5 points) Recall that A is a *symmetric* matrix if $A = A^{\mathrm{T}}$. Prove that φ is symmetric if A is a symmetric matrix.

(3) (5 points) If $(f_1, \ldots, f_n)$ is another basis of E and P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of φ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^{\mathrm{T}} A P.$$

The common rank of all matrices representing φ is called the *rank* of φ.

Solution. If we write $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$, by bilinearity, we have

$$\begin{aligned}
\varphi(x, y) &= \varphi\left(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n x_i \varphi(e_i, e_j) y_j \\
&= X^\top A Y,
\end{aligned}$$

with $A = (\varphi(e_i, e_j))$, and where X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$.

(b) If $A = (\varphi(e_i, e_j))$ is a symmetric matrix, since $\varphi(x, y)$ and $\varphi(y, x)$ are scalars, we have

$$\varphi(y, x) = Y^\top AX = (Y^\top AX)^\top = X^\top A^\top Y = X^\top AY = \varphi(x, y).$$

(c) If P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, we know that

$$X = PX', \quad Y = PY',$$

where X, Y are the coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, and X', Y' are the coordinates of x and y w.r.t. the basis $(f_1, \ldots, f_n)$. Then,

$$\varphi(x, y) = X^\top AY = (PX')^\top APY' = X'^\top P^\top APY',$$

which shows that the matrix of φ w.r.t the basis $(f_1, \ldots, f_n)$ is

$$P^\top AP.$$

Since P is invertible, all these matrices have the same rank.

Total: 60 points