CIT 592 Fall 2019 **Homework 1** **Quiz + Solutions**

Preparation Quiz

A. [**Concepts: multiplication rule (seg_01_01, sp_01_01)**] A certain car can be configured in three steps as follows. The customer selects one of three trim levels, then one of six exterior colors, and then one of two interior colors. How many possible configurations are there?

(a) 11

Incorrect. Here we need to apply the multiplication rule, not the addition rule.

(b) 36

Correct. The multiplication rule applies, and $3 \cdot 6 \cdot 2 = 36$.

(c) 18

Incorrect. The multiplication rule applies here, and we need to multiply all three numbers.

B. [**Concepts: divisibility (seg_01_02, seg_01_03)**] How many (positive) factors does 36 have?

(a) 2

Incorrect. 36 has only two prime factors, but it has nine positive factors.

(b) 7

Incorrect. Don't forget that every integer has itself and 1 as factors.

(c) 9

Correct. The factors are 1, 2, 3, 4, 6, 9, 12, 18, and 36.

C. [**Concepts: primality (seg_01_03)**] Is 91 prime?

(a) Yes

Incorrect. Try dividing it by 7.

(b) No

Correct. 91 is divisible by 7 and 13, whereas a prime number is divisible by only itself and 1.

D. [**Concepts: sets (seg_01_05)**] How many elements are in the set

$$\{\ell \mid \ell \text{ is a letter in the word "octopus"}\}?$$

(a) 0

Incorrect. Although the letter "l" does not appear in "octopus", we are using ℓ as a variable here, meaning it can take any letter as a value.

(b) 6

Correct. The set is $\{\mathrm{o}, \mathrm{c}, \mathrm{t}, \mathrm{p}, \mathrm{u}, \mathrm{s}\}$.

(c) 7

Incorrect. A set only has one of each element, so we don't count "o" twice.

E. [**Concepts: sets (seg_01_05, seg_01_06)**] Which of the following are equal to $\{0\}$? Select all that apply.

(a) $\mathbb{N} \setminus \mathbb{Z}^+$

Correct. $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}^+ = \{1, 2, \ldots\}$, so when we subtract $\mathbb{Z}$ from $\mathbb{N}$, we are left with only $\{0\}$.

(b) $|\emptyset|$

Incorrect. The cardinality of $\emptyset$ is 0, but 0 is a number, and $\{0\}$ is a set.

(c) $\{-2, -1, 0\} \cap \mathbb{N}$

Correct. 0 is the only common element between $\{-2, -1, 0\}$ and $\mathbb{N}$, so their intersection is $\{0\}$.

(d) $\{0\} \cup \emptyset$

Correct. The union of any set A with the empty set is A.

(e) $\{0\} \cup \{\emptyset\}$

Incorrect. This union is $\{0, \emptyset\}$, a set with two elements.

F. [**Concepts: powerset (seg_01_07)**] What is $2^{\{0,1\}}$?

(a) $\{(0,0),(0,1),(1,0),(1,1)\}$

Incorrect. This is $\{0,1\} \times \{0,1\}$, which can be written as $\{0,1\}^2$, but that is not the same as $2^{\{0,1\}}$.

(b) $\{\emptyset, \{0\}, \{1\}, \{0,1\}, \{\{0\},\{1\}\}\}$

Incorrect. $2^{\{0,1\}}$ is the power set of $\{0,1\}$, so its elements are the subsets of $\{0,1\}$. $\{\{0\},\{1\}\} \nsubseteq \{0,1\}$.

(c) $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

Correct. $2^{\{0,1\}}$ is the power set of $\{0,1\}$, and the subsets of $\{0,1\}$ are $\emptyset$, $\{0\}$, $\{1\}$, and $\{0,1\}$.

G. [**Concepts: cartesian product (seg_01_07)**] If V represents the vowels and C represents the consonants, then which of the following belong to the cartesian product $V \times C$? Select all that apply.

(a) (g, o)

Incorrect. Since $g \notin V$ and $o \notin C$, $(g,o) \notin V \times C$.

(b) (a, a)

Incorrect. Since $a \notin C$, $(a,a) \notin V \times C$.

(c) $\emptyset$

Incorrect. The elements of $V \times C$ are ordered pairs.

(d) (i, t)

Correct. $V \times C = \{(v,c) \mid v \in V \text{ and } c \in C\}$. Since $\mathrm{i} \in V$ and $\mathrm{t} \in C$, $(\mathrm{i},\mathrm{t}) \in V \times C$.

(e) $\{\mathrm{i}, \mathrm{t}\}$

Incorrect. The elements of $V \times C$ are ordered pairs, not sets.

Problem Set

1. [**10 pts**]

Jack and Jill want to rent separate apartments on the third floor of a new building by the river. The building has nine apartments available, numbered 301, 302,. . . , 309. The odd-numbered apartments have a river view, and the even-numbered apartments do not. Jill will only rent an apartment with a river view, and Jack does not care about the view.

How many distinct possibilities exist for the pair of apartments they end up renting?

Solution:

If we consider Jill's decision first, then the number of choices available to Jack does not depend on Jill's decision, so we can apply the multiplication rule. Jill chooses among the five apartments with a river view, and Jack chooses among the eight remaining apartments, so the answer is $5 \cdot 8 = \boxed{40}$.

Notice that if we consider Jack's decision as the first step, then the multiplication rule does not apply: Depending on whether he chooses an apartment with a river view, Jill may have either four or five apartments to choose from.

2. [**10 pts**] $n \geq 2$ distinguishable Hogwarts students participate in Professor Snape's experiment. Each student is given potion A, or philter B, or neither, or both. We know that Harry and Hermione are the only students among the n that are given both A and B. In how many distinct ways could Snape have distributed his experimental liquids?

Solution:

Since Harry and Hermione are the only students given both, the remaining $n-2$ students have potion A, philter B, or neither. These options are mutually exclusive and are assigned to each student independently, so we can use the Multiplication Rule here. Consider the following steps:

Step 1: Assign student 1 to potion A, philter B, or neither. (3 ways)
Step 2: Assign student 2 to potion A, philter B, or neither. (3 ways)

$\vdots$

Step $n-2$: Assign student $n-2$ to potion A, philter B, or neither. (3 ways)
Step $n-1$: Assign Harry to both. (1 way)
Step n: Assign Hermione to both. (1 way)

By the Multiplication Rule, there are $\boxed{3^{n-2}}$ ways for Snape to distribute his experimental liquids.

3. [12 pts] Prove that for all integers x, if x is odd and $x \geq 3$, then $x^2 - 1$ is divisible by 8.

Solution:

If x is odd and $x \geq 3$, then $x = 2k+1$ for some $k \in \mathbb{Z}^+$. Thus,

$$\begin{aligned} x^2 - 1 &= (x-1)(x+1) \\ &= (2k+1-1)(2k+1+1) \\ &= 4 \cdot k \cdot (k+1). \end{aligned}$$

Now consider two cases:

Case 1: k is even. Then $k = 2\ell$ for some $l \in \mathbb{Z}^+$, and we can write

$$\begin{aligned} x^2 - 1 &= 4 \cdot 2\ell \cdot (2\ell + 1) \\ &= 4 \cdot (4\ell^2 + 2\ell) \\ &= 8 \cdot (2\ell^2 + \ell), \end{aligned}$$

so we can see that 8 is a divisor of $x^2 - 1$ in this case.

Case 2: k is odd. Then $k = 2\ell + 1$ for some $\ell \in \mathbb{N}$, and we can write

$$\begin{aligned} x^2 - 1 &= 4 \cdot (2\ell + 1) \cdot (2\ell + 2) \\ &= 4 \cdot (4\ell^2 + 6\ell + 2) \\ &= 8 \cdot (2\ell^2 + 3\ell + 1), \end{aligned}$$

therefore 8 is also divisor of $x^2 - 1$ in this case.

4. [10 pts]

(a) A number $n \in \mathbb{N}$ is called **perfect** if the sum of all of n's factors other than n itself is equal to n. For example, 6 is perfect because its factors are 1, 2, 3, and 6, and $1 + 2 + 3 = 6$. Prove that there are no perfect prime numbers.

(b) Let $m, n \in \mathbb{Z}^+$, and suppose that m is a factor of both n and $n + 1$. Prove that $m = 1$.

Solution:

(a) A prime number only has one factor other than itself: 1. So for every prime p, the sum of the factors of p other than p itself is 1, and $1 \neq p$ since 1 is not prime.

(b) Since m is a factor of n and $n + 1$, there are some positive integers

k and ℓ such that $mk = n$ and $m\ell = n + 1$. Rearranging, we have

$$\begin{aligned} m\ell - 1 &= n \\ m\ell - 1 &= mk \\ m\ell - mk &= 1 \\ m \cdot (\ell - k) &= 1\,. \end{aligned}$$

Thus, m is a factor of 1, which implies that $m \leq 1$. The fact that $m \in \mathbb{Z}^+$ tells us that $m \geq 1$, so we have shown that $m = 1$.

5. [8 pts]

(a) Give an example of three distinct (no two are the same), nonempty sets A, B, C such that

- there are elements that are common to A and B;
- every element of A that is also in B must also be in C;
- there are elements in B that are not in C.

(b) Let A be an arbitrary set such that $\{\emptyset\} \in A$ and $\{\emptyset\} \subseteq A$. List *four* distinct subsets of A.

(c) Define the following set using set builder notation: $\{0, 1, 64\}$.

(d) Give examples of three sets $A, B, C \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ such that A and B are disjoint, $A \setminus C = \{1, 3, 7\}$, $B \cup C = \{2, 4, 5, 6\}$, $|A| = 5$, and $B \setminus C \neq \emptyset$.

Solution:

(a) Consider $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$.

(b) Four distinct subsets of A are: $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$.

(c) $\{x^2 \mid x \in \{0, 1, 8\}\ \}$.

(d) $A = \{1,2,3,4,7\}$, $B = \{5,6\}$, and $C = \{2,4,6\}$ satisfy those conditions.

6. [10 pts]

(a) Let S be the set $\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}$. What is its power set 2^S?

(b) Let $A = \{2,3\}$, $B = \{3,4,5\}$, and $C = \{3,5,7,9\}$. What is $A \times (B \cap C)$?

Solution:

(a) The power set 2^S is the set of all subsets of S, which includes

- one subset of cardinality zero: the empty set $\emptyset$;
- four subsets of cardinality one: $\{\mathtt{a}\}$, $\{\mathtt{b}\}$, $\{\mathtt{c}\}$, and $\{\mathtt{d}\}$;
- six subsets of cardinality two: $\{\mathtt{a},\mathtt{b}\}$, $\{\mathtt{a},\mathtt{c}\}$, $\{\mathtt{a},\mathtt{a}\}$, $\{\mathtt{b},\mathtt{c}\}$, $\{\mathtt{b},\mathtt{d}\}$, and $\{\mathtt{c},\mathtt{d}\}$;
- four subsets of cardinality three: $\{\mathtt{a},\mathtt{b},\mathtt{c}\}$, $\{\mathtt{a},\mathtt{b},\mathtt{d}\}$, $\{\mathtt{a},\mathtt{c},\mathtt{d}\}$, and $\{\mathtt{b},\mathtt{c},\mathtt{d}\}$;
- one subset of cardinality four: $\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}$.

Putting this all together, the power set is

$$\boxed{\begin{array}{l}\{\emptyset,\{\mathtt{a}\},\{\mathtt{b}\},\{\mathtt{c}\},\{\mathtt{d}\},\{\mathtt{a},\mathtt{b}\},\{\mathtt{a},\mathtt{c}\},\{\mathtt{a},\mathtt{d}\},\{\mathtt{b},\mathtt{c}\}\{\mathtt{b},\mathtt{d}\},\\ \quad\{\mathtt{c},\mathtt{d}\},\{\mathtt{a},\mathtt{b},\mathtt{c}\},\{\mathtt{a},\mathtt{b},\mathtt{d}\},\{\mathtt{a},\mathtt{c},\mathtt{d}\},\{\mathtt{b},\mathtt{c},\mathtt{d}\},\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\}\end{array}}$$

(b) $A \times (B \cap C)$ is the set of all ordered pairs where the first element is in A and the second element is in $B \cap C$. To find this cartesian product, we first find $B \cap C$, which is the set of all elements that belong to both B and C. The elements 3 and 5 satisfy that criterion, so $B \cap C = \{3, 5\}$. Thus,

$$\begin{aligned} A \times (B \cap C) &= \{2, 3\} \times \{3, 5\} \\ &= \{(x, y) \mid x \in \{2, 3\} \text{ and } y \in \{3, 5\}\} \\ &= \boxed{\{(2, 3), (2, 5), (3, 3), (3, 5)\}}. \end{aligned}$$

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

We have a bag filled with 201 marbles, of which 100 of them are blue and 101 of them are red. Every turn, we remove 2 marbles from the bag. If the two marbles are of the same color, we remove the two marbles but add a blue marble into the bag. If the two marbles are of different colors, we remove the two marbles and add a red marble into the bag.

What is the color of the last marble in the bag?

Solution:

$\boxed{\text{Red.}}$ The only way to reduce the number of red marbles is by drawing a pair of red marbles at the same time, since every time we remove a red and a blue, we also add a red. So the number of red marbles in the bag at any time is $101 - 2k$, where k is the number of times we've removed a red pair. $101 - 2k = 2 \cdot (50 - k) + 1$ is odd no matter what k is, so it can never be zero. Thus, there is always at least one red marble in the bag, and therefore the last marble in the bag must be red.

Rubric:

(a) 2 for correct answer

(b) 3 for explaining why the number of red marbles must always be odd (or cannot be zero)

(c) 1 for explaining why that answers the problem.