Problem Set

1. [10 pts]

In a uniformly random permutation of the vowels $\{\texttt{a}, \texttt{e}, \texttt{i}, \texttt{o}, \texttt{u}\}$, what is the probability that the first letter is `i` and the last letter is `o`?

Solution:

Define a sample space of words made up of a random permutation of the vowels $\{\texttt{a}, \texttt{e}, \texttt{i}, \texttt{o}, \texttt{u}\}$ (essentially using what the problem gives us). Since the letters are uniformly distributed, the sample space is uniform and the probability that a word has i as its first letter and o as its last letter can be found by counting possibilities.

There are 5! possible outcomes total. If the first letter is `i` and the last letter is `o`, then there are 3! ways to arrange the middle three letters, so there are 3! such outcomes. Thus, the probability of this event is $3!/5! = \boxed{1/20}$.

2. [10 pts] Suppose you roll three fair six-sided dice and add up the numbers you get. What is the probability that the sum is at least 16?

Solution:

Define the sample space Ω as an ordered tuple of three numbers. The first number contains the value of the first dice, the second contains the value of the second dice, and the third the third dice. Notice that this sample space is uniform since each dice is uniformly distributed and the rolls are independent. Thus, the probability that the sum is at least 16 can be found by dividing the number of outcomes with a sum over 16 and dividing by total number of outcomes.

Since there are 6 sides to a dice, each number of the tuple has 6 possibilities. By the multiplication rule, there are $6^3 = 216$ possible outcomes. For the sum to be over 16, it can equal 16, 17, or 18. It is impossible to get a sum greater than 18 as the maximum value of each dice is 6. To count the number of possibilities:

- There are 6 outcomes with sum 16: $(6, 6, 4)$, $(6, 4, 6)$, $(4, 6, 6)$, $(6, 5, 5)$, $(5, 6, 5)$, and $(5, 5, 6)$. So the probability that the sum is exactly 16 is $6/216$.
- There are 3 outcomes with sum 17: $(6, 6, 5)$, $(6, 5, 6)$, and $(5, 6, 6)$. So the probability that the sum is exactly 17 is $3/216$.
- There is 1 outcome with sum 18: $(6, 6, 6)$. So the probability that the sum is exactly 18 is $1/216$.

Thus, by the addition rule (P2), the probability of getting sum at least 16 is

$$\frac{6}{216} + \frac{3}{216} + \frac{1}{216} = \frac{10}{216} = \boxed{\frac{5}{108}}.$$

3. [10 pts]

Suppose that each time Giannis Antetokounmpo shoots a free throw, he has a 3/4 probability of success. If Giannis shoots three free throws, what is the probability that he succeeds on at least two of them?

Solution:

Solve this problem is by defining an appropriate uniform probability space. Consider four marbles in an urn, labeled S_1, S_2, S_3, and F. We can think of each shot as drawing one of these marbles out of the urn at random, each with probability 1/4, and then replacing the marble. If we pull one of the S marbles out, then Giannis's shot succeeds; if we pull the F marble out, then Giannis shot fails. Thus, his shots (and the probability space) can be represented with an ordered tuple with three elements, each containing one of those labels.

Approach this problem using complementary counting. The size of the sample space is $4^3 = 64$. Of these possibilities, there is only 1 where Giannis misses all his shots: (F, F, F) and 9 where Giannis makes exactly one shot: (F, F, S_i), (F, S_i, F), and (S_i, F, F), for each $i \in \{1, 2, 3\}$. That means there are $64 - 1 - 9 = 54$ remaining outcomes where Giannis makes at least two shots. Thus, his probability of succeeding on at least two free throws out of three is $54/64 = \boxed{27/32}$.

While it would have taken more work to count, it is also possible to count the possibilities in which Giannis makes at least two free throws directly. There are 27 possibilities where he makes two free throws: (F, S_i, S_j), (S_i, F, S_j), and (S_i, S_j, F), for each $i, j \in \{1, 2, 3\}$ and 27 possibilities where he makes all three free throws: (S_i, S_j, S_k), for each $i, j, k \in \{1, 2, 3\}$

Another way to solve this is to notice that the three shots are Bernoulli trials with success probability 3/4. The easy way to do this is to take advantage of the fact that these trials are independent and use the product rule, which we haven't learned yet but you may have encountered elsewhere. Let H *represent success and* T *represent failure for each of Giannis's shots. For each shot, the probability of getting* H *is 3/4 and the probability of getting* T *is* $1 - 3/4 = 1/4$*. The set of possible outcomes is* $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$*, and we are interested in the probability of the event* $\{HHH, HHT, HTH, THH\}$*. By the product rule,* $\Pr[HHH] = (3/4)(3/4)(3/4) = 27/64$*,* $\Pr[HHT] = (3/4)(3/4)(1/4) = 9/64$*,* $\Pr[HTH] = (3/4)(1/4)(3/4) = 9/64$*, and* $\Pr[THH] = (1/4)(3/4)(3/4) = 9/64$*. The probability of an event is the sum of the probabilities of the outcomes in that event, so we have* $\Pr[\{HHH, HHT, HTH, THH\}] = 27/64 + 9/64 + 9/64 + 9/64 =$

$54/64 = \boxed{27/32}$. *However, since Bernoulli trials were not yet covered in this module, this method was given credit only if the final answer and all steps were correct, no partial credit.*

4. [10 pts]

Consider the uniform probability space where the set of outcomes consists of all 8-bit binary strings. What is the probability of each of the following events?

(a) The first three bits of the string are `001`, `010`, or `100`.

(b) The string begins or ends with `111`.

Solution:

(a) Define the uniform sample space as all possible 8-bit binary strings. The space is uniform as each bit can independently be 0 or 1 with equal probability. By the multiplication rule, there are 2^8 total outcomes in the sample space.

If the first three bits are `001`, then there are 2^5 ways to set the remaining bits, so there are 2^5 such outcomes, and the probability that the first three bits are `001` is $2^5/2^8 = 1/8$. Similarly, the probability that the first three bits are `010` and the probability that the first three bits are `100` are each $1/8$. These events are disjoint, so the addition rule (P2) tells us that the probability of their union is $1/8 + 1/8 + 1/8 = \boxed{3/8}$. Note: this can alternatively be reasoned using only multiplication rule. There are 3 possible choices for the first 3 bits: `001`, `010`, or `100`. Then there are 2^5 choices for the remaining 5 bits. By multiplication rule, there are $3(2^5)$ desirable outcomes, leading to a probability of $3(2^5)/2^8 = \boxed{3/8}$.

(b) Use the same sample space defined above. By the same reasoning as above, the probability that the first three bits are `111` and the probability that the last three bits are `111` are both $1/8$. Unlike above, these events are not disjoint. There are $2^2 = 4$ outcomes in which the string begins *and* ends with `111`, which means this happens with probability $2^2/2^8 = 1/64$. Thus, by the principle of inclusion-exclusion, the probability that the string begins or ends with `111` is

$$\frac{1}{8} + \frac{1}{8} - \frac{1}{64} = \boxed{\frac{15}{64}}.$$

5. [10 pts]

Among $2 \leq k \leq 7$ randomly selected people, what is the probability that at least two of them were born on the same day of the week?

Solution:

Define the sample space as a k element ordered tuple, where each element is a day, corresponding to the birthday of a different person. This space is uniform as each person independently has an equal likelihood of being born on any day. There are 7^k possible outcomes in the sample space.

Use complementary counting to approach this problem. The number of outcomes in which no two people were born on the same day of the week is the number of partial permutations of length k of 7 items, which is $7!/(7-k)!$. A partial permutation works well as it guarantees that no two elements are the same day, meaning no two people have the same birthday. So the probability that no two people were born on the same day of the week is $\frac{7!/(7-k)!}{7^k}$, and the probability that at least two of them were born on the same day is

$$\boxed{1 - \frac{7!/(7-k)!}{7^k}}.$$

6. [10 pts]

Prove that if A and B are events, then $\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1$.

Solution:

$A \cup B$ is an event, so $\Pr[A \cup B] \leq 1$. The principle of inclusion-exclusion states that $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Putting these together, we have $1 \geq \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Adding $\Pr[A \cap B] - 1$ to both sides gives the desired inequality. $\square$

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Suppose you toss a fair coin until you've gotten at heads at least twice or tails at least four times (not necessarily consecutive), and then you stop. What is the probability that your last coin toss came up tails?

Solution:

Define a sample space with 15 possible outcomes: `HH`, `HTH`, `HTTH`, `HTTTH`, `HTTTT`, `THH`, `THTH`, `THTTH`, `THTTT`, `TTHH`, `TTHTH`, `TTHTT`, `TTTHH`, `TTTHT`, `TTTT`. This space is not uniform as the probability of each outcome depends on its length. Namely, the probability of an outcome of length k is $1/2^k$. That is:

- $\Pr[\texttt{HH}] = 1/4$,
- $\Pr[\texttt{HTH}] = \Pr[\texttt{THH}] = 1/8$,
- $\Pr[\texttt{HTTH}] = \Pr[\texttt{THTH}] = \Pr[\texttt{TTHH}] = \Pr[\texttt{TTTT}] = 1/16$, and
- $\Pr[\texttt{HTTTH}] = \Pr[\texttt{HTTTT}] = \Pr[\texttt{THTTH}] = \Pr[\texttt{THTTT}] = \Pr[\texttt{TTHTH}] = \Pr[\texttt{TTHTT}] = \Pr[\texttt{TTTHH}] = \Pr[\texttt{TTTHT}] = 1/32$.

We want the probability of the event $\{\texttt{TTTT}, \texttt{HTTTT}, \texttt{THTTT}, \texttt{TTHTT}, \texttt{TTTHT}\}$. Adding up the probabilities of the individual outcomes in that event gives $1/16 + 1/32 + 1/32 + 1/32 + 1/32 = \boxed{3/16}$.