Replacement algorithm

Q1, Q2 - expectation → determinant algebra or do you prefer us solve them as systems of equations for determining matrix column linear independence

(u_1, u_2, u_3) = linearly independent family
$(v_1, v_2, v_3, v_4, v_5)$ = another family, might not be linearly dependent

idea: replace some of "v's" w/ the "u's" and still span S

Step 1: How is each u_i expressed in terms of v's
textbook pg 50 λ_j's

$u_1 = v_4 + v_5$
$u_2 = v_3 + v_4 - v_5$
$u_3 = v_1 + v_2 + v_3$

Step 2: going to solve equations to write a certain # of v's as terms of u.
- Start w/ lowest index u_i rewrite to solve for lowest index v_j

$u_1 = v_4 + v_5$

$u_2 = v_3 + v_4 - v_5 \rightarrow u_2 = v_3 + u_1 - v_5 - v_5$
$= v_3 + u_1 - 2v_5$

$u_2 = v_3 + u_1 - 2v_5$

$\boxed{v_3 = u_2 - u_1 + 2v_5}$

$u_3 = v_1 + v_2 + v_3$
$u_3 = v_1 + v_2 + u_2 - u_1 + 2v_5$
$v_1 = u_3 - v_2 - v_2 + u_1 + 2v_5$

4b) dimension = $n \times n$

A is $m \times n$, B is $n \times m$. Show $I_n - BA$ is invertible iff $I_m - AB$ is invertible

write as geometric series, infinite polynomial

$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$

$1 = (1-x) \cdot (1-x)^{-1}$

Identity matrix

$(1 - x)^{-1} \Longrightarrow (I_n - BA)^{-1}$

Assuming $(I - BA)^{-1}$ is invertible

$(I_n - BA)^{-1} = I + BA + (BA)^2 + (BA)^3 + \dots$

$(I_m - AB)^{-1} = I + AB + (AB)^2 + (AB)^3 + \dots$

$A(I - BA)^{-1}B = AB + (AB)^2 + \overset{(AB)^3}{A(BA)^2B} + \dots$

we have formally shown that $A(I_n - BA)^{-1}B = AB + (AB)^2 + (AB)^3 + \dots$
$A(I_n - BA)^{-1} = (I_m - AB)^{-1} - I$

Candidate for inverse of $I_m - AB$
$(I_m - AB)^{-1} = I_m + A(I_n - BA)^{-1}B$ we used the series to get a non-series form

$(I - AB)(I - AB)^{-1} = I$

- substitute found expression

Matrix · inverse = Identity

$(I_m - AB)\left[I_m + A(I - BA)^{-1}B\right]$

$= I_m + A(I-BA)^{-1}B - AB - ABA(I-BA)^{-1}B$

$= I_m + A\left[(I-BA)^{-1}B - B - BA(I-BA)^{-1}B\right]$

$= I_m + A\left[(I_n-BA)^{-1} - I_n - BA(I_n-BA)^{-1}\right]B$

$= I_m + A\left[(I-BA)^{-1}[I_n - BA] - I_n\right]B$

$= \underbrace{I_m + A[I_n - I_n]B}$ $= (I-AB)^{-1}$

$= I_m + A[0]B$

$= I_m$

$\therefore$ Shown $I_m + A(I-BA)^{-1}B$ is correct inverse