Module 3

Haar Matrices, Haar Wavelets
Solutions

Problem Haar Extravaganza!

Consider the matrix

$$W_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1) (5 points) Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,3}c$ of applying $W_{3,3}$ to c is

$$W_{3,3}c = (c_1 + c_5, c_1 - c_5, c_2 + c_6, c_2 - c_6, c_3 + c_7, c_3 - c_7, c_4 + c_8, c_4 - c_8),$$

the last step in reconstructing a vector from its Haar coefficients.

(2) (10 points) Prove that the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^{\top}$. Prove that the columns and the rows of $W_{3,3}$ are orthogonal.

(3) (20 points) Let $W_{3,2}$ and $W_{3,1}$ be the following matrices:

$$W_{3,2} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_{3,1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,2}c$ of applying $W_{3,2}$ to c is

$$W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),$$

the second step in reconstructing a vector from its Haar coefficients, and the result $W_{3,1}c$ of applying $W_{3,1}$ to c is

$$W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),$$

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

$$W_{3,3}W_{3,2}W_{3,1} = W_3,$$

the Haar matrix

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Hint. First check that

$$W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

where

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

(4) (25 points) Prove that the columns and the rows of $W_{3,2}$ and $W_{3,1}$ are orthogonal. Deduce from this that the columns of W_3 are orthogonal, and the rows of W_3^{-1} are orthogonal. Are the rows of W_3 orthogonal? Are the columns of W_3^{-1} orthogonal? Find the inverse of $W_{3,2}$ and the inverse of $W_{3,1}$.

Solution (1). Multiply $W_{3,3}$ by c; this gives the desired result.

Solution (2). Multiply $W_{3,3}^{\top}$ by $W_{3,3}$ and you find that

$$W_{3,3}^{\top}W_{3,3} = 2I_8.$$

This shows that $(1/2)W_{3,3}^{\top}$ is a left inverse of $W_{3,3}$. But in the case of square matrices, the existence of a left inverse implies that the matrix is invertible and that its inverse is equal to the left inverse. The equation

$$W_{3,3}^{\top}W_{3,3} = 2I_8$$

shows that the columns of $W_{3,3}$ are orthogonal. Since $(1/2)W_{3,3}^\top$ is the inverse of $W_{3,3}$, we also have

$$I_8 = W_{3,3}(1/2)W_{3,3}^\top = (1/2)W_{3,3}W_{3,3}^\top,$$

which shows that the rows of $W_{3,3}$ are orthogonal.

Solution (3). Multiply the matrices $W_{3,2}$ and c, and the matrices $W_{3,1}$ and c; you find the desired results.

If you multiply $W_{3,2}$ and $W_{3,1}$ you find that

$$W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix},$$

where

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Then multiplying on the left by $W_{3,3}$ yields W_3, which proves that

$$W_{3,3}W_{3,2}W_{3,1} = W_3.$$

Solution (4). Let

$$W_{1,1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$W_{2,2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

and note that

$$W_{3,2} = \begin{pmatrix} W_{2,2} & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix}$$

and

$$W_{3,1} = \begin{pmatrix} W_{1,1} & 0_{2,6} \\ 0_{6,2} & I_6 \end{pmatrix},$$

so we find that

$$W_{3,2}^\top W_{3,2} = \begin{pmatrix} 2I_4 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix}$$

and that

$$W_{3,1}^\top W_{3,1} = \begin{pmatrix} 2I_2 & 0_{2,6} \\ 0_{6,2} & I_6 \end{pmatrix}.$$

This shows that the columns of $W_{3,2}$ and $W_{3,1}$ are orthogonal, and that

$$W_{3,2}^{-1} = \begin{pmatrix} (1/2)W_{2,2}^\top & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix}$$

and

$$W_{3,1}^{-1} = \begin{pmatrix} (1/2)W_{1,1}^\top & 0_{2,6} \\ 0_{6,2} & I_6 \end{pmatrix}.$$

We also have

$$W_{3,2}W_{3,2}^\top = \begin{pmatrix} 2I_4 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix}$$

and

$$W_{3,1}W_{3,1}^\top = \begin{pmatrix} 2I_2 & 0_{2,6} \\ 0_{6,2} & I_6 \end{pmatrix},$$

so the rows of $W_{3,2}$ and $W_{3,1}$ are orthogonal. Since

$$W_3 = W_{3,3}W_{3,2}W_{3,1}$$

and $W_{3,3}^\top W_{3,3} = 2I_8$, we have

$$\begin{aligned} W_3^\top W_3 &= (W_{3,3}W_{3,2}W_{3,1})^\top W_{3,3}W_{3,2}W_{3,1} \\ &= W_{3,1}^\top W_{3,2}^\top W_{3,3}^\top W_{3,3}W_{3,2}W_{3,1} \\ &= 2W_{3,1}^\top W_{3,2}^\top W_{3,2}W_{3,1} \\ &= 2W_{3,1}^\top \begin{pmatrix} 2I_4 & 0_{4,4} \\ 0_{4,4} & I_4 \end{pmatrix} W_{3,1} \\ &= \begin{pmatrix} 8I_2 & & \\ & 4I_2 & \\ & & 2I_4 \end{pmatrix}. \end{aligned}$$

The above shows that the columns of W_3 are orthogonal. By taking inverses, we get

$$W_3^{-1}(W_3^{-1})^\top = W_3^{-1}(W_3^\top)^{-1} = (W_3^\top W_3)^{-1} = \begin{pmatrix} (1/8)I_2 & & \\ & (1/4)I_2 & \\ & & (1/2)I_4 \end{pmatrix},$$

which shows that the rows of W_3^{-1} are orthogonal. The rows of W_3 are not all orthogonal, for example row 1 and row 2 are not. Similarly, the columns of W_3^{-1} are not all orthogonal.

Total: 60 points