Problem Set

1. [10 pts]

The digits 1, 2, and 3 are randomly arranged to form a one digit number and a two-digit number. Each digit can only be used once; for example, if the one-digit number is 3, then the two-digit number is either 12 or 21. What is the expected value of the product of the two numbers?

Solution:

Let the sample space be a random permutation of $\{1, 2, 3\}$, where the first number is the single digit number and the following numbers are the two digit number. Thus, there are six outcomes of equal probability.

The expected value is the value of each case times the probability that it occurs. That is:

$$\frac{1}{6}(1 \cdot 23) + \frac{1}{6}(1 \cdot 32) + \frac{1}{6}(2 \cdot 13) + \frac{1}{6}(2 \cdot 31) + \frac{1}{6}(3 \cdot 12) + \frac{1}{6}(3 \cdot 21)$$
$$= \frac{23 + 32 + 26 + 62 + 36 + 63}{6} = \frac{242}{6} = \boxed{\frac{121}{3}}.$$

2. [10 pts]

Jay has \$500 in the bank when he decides to try a savings experiment. On each day $i \in [1..30]$, Jay flips a fair coin. If it comes up heads, he deposits i dollars into the bank; if it comes up tails, he withdraws \$10. How many dollars should he expect to have in the bank after 30 days?

Solution:

Let X_i be the random variable for the amount Jay deposits on day i (this will negative if he withdraws). Then the amount in the bank after 30 days is $500 + X = 500 + \sum_{i=1}^{30} X_i$. By the definition of expectation, $\mathrm{E}[X_i] = \frac{i-10}{2}$. By linearity of expectation,

$$\begin{aligned}
\mathrm{E}[500 + X] &= \mathrm{E}\left[500 + \sum_{i=1}^{30} X_i\right] \\
&= 500 + \sum_{i=1}^{30} \mathrm{E}[X_i] \\
&= 500 + \sum_{i=1}^{30} \frac{i - 10}{2} \\
&= 350 + \frac{1}{2}\sum_{i=1}^{30} i \\
&= 350 + \frac{1}{2}\frac{30^2 + 30}{2} \\
&= \boxed{582.50}
\end{aligned}$$

3. [10 pts]

A group of 36 people, consisting of 18 married couples, are put into random teams of three for a scavenger hunt. The teams are numbered 1 to 12. How many of the teams should we expect to contain a married couple?

Solution:

Let X_k be the indicator random variable for the team k. It equals 1 if team k has a couple on it and 0 otherwise. Define X as the number of teams that contain a married couple, and it follows that $X = \sum_{k=1}^{12} X_k$. We are interested in finding $E[X]$. Using linearity of expectation and the fact that the expected value of indicator random variables is just the probability they equal 1:

$$E[X] = \sum_{k=1}^{12} E[X_k] = \sum_{k=1}^{12} Pr[X_k = 1]$$

To find $Pr[X_k = 1]$, first define the sample space Ω as the possible sets of three different people out of the 36. This space is uniform since the three people are chosen randomly. Thus, the probability that a team has a couple can be calculated as: $Pr[X_k = 1] = |X_k = 1|/|\Omega|$. Since teams are formed by choosing three people, $|\Omega| = \binom{36}{3} = 7140$. Use the multiplication rule to determine the number of teams with a couple. First, choose a couple to be part of that team. Then, choose one more random person. That is, $|X_k = 1| = \binom{18}{1}34 = 612$. Thus, $Pr[X_k = 1] = 612/7140 = 3/35$. To finish the question:

$$E[X] = \sum_{k=1}^{12} Pr[X_k = 1] = 12 \times \frac{3}{35} = \boxed{\frac{36}{35}}$$

4. [10 pts]

You draw b balls at random (without replacement) from an urn that initially contains N balls, of which r are red and the rest are yellow. What is the expected number of yellow balls drawn?

Solution:

Number the yellow balls 1 through $N-r$, and for each $i \in [1..N-r]$, let X_i be the indicator random variable for yellow ball i being drawn. Then we want $\mathrm{E}\left[\sum_{i=1}^{N-r} X_i\right]$, which by linearity of expectation is equal to $\sum_{i=1}^{N-r} \mathrm{E}[X_i]$.

For each i, $\mathrm{E}[X_i] = \Pr[X_i = 1]$ since it's a random variable. To solve this probability, define the sample space as a set of b balls chosen uniformly at random. Thus, $\Pr[X_i = 1] = |X_i = 1|/|\Omega|$. First, $|\Omega| = \binom{N}{b}$. To find $|X_i = 1|$, just first state that the set must have yellow ball i. Then, choose the remaining balls. That is, $|X_i = 1| = \binom{N-1}{b-1}$. So, $\Pr[X_i = 1] = \binom{N-1}{b-1}/\binom{N}{b} = b/N$

Thus, the expected number of balls drawn is $\mathrm{E}[X] = \sum_{i=1}^{N-r} \mathrm{E}[X_i] = \boxed{\dfrac{(N-r)\cdot b}{N}}$.

5. [10 pts]

Suppose we draw a random card from a 52-card deck, put the card back in the deck, shuffle the deck, and repeat this process a total of k times. What is the expected number of cards that have been drawn at least once?

Solution:

Number the cards 1 through 52, and for each $i \in [1..52]$ let X_i be the indicator random variable for card i being drawn at least once. The probability that card i was never drawn is $(51/52)^k$, so $\Pr[X_i = 1] = 1 - (51/52)^k$. Let X be the random variable representing the number of cards that were drawn at least once. Then $X = \sum_{i=1}^{52} X_i$, and we want to find

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{52} X_i\right] \\
&= \sum_{i=1}^{52} \mathrm{E}[X_i] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{52} \Pr[X_i = 1] \\
&= \boxed{52\left(1 - (51/52)^k\right)}.
\end{aligned}$$

6. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Suppose you start with $z = 1$ and append random digits in $[0..9]$ to the end of z until z is even or z has 6 digits. What is the expected number of digits in z at the end of this process?

For example, one possible instance of this process would be

$$1 \to 17 \to 173 \to 1731 \to 17310\,,$$

at which point the process stops because $z = 17310$ is even.

Solution:

For all $i \in [1..6]$, let z_i be the i^{th} digit of z, if it exists. Let X_i be the indicator random variable for z_i existing, and let $X = \sum_{i=1}^{6} X_i$ be the random variable for the number of digits in z. Notice that $\Pr[X_1 = 1] = 1$. An integer is even if and only if it ends in an even number, so for each $i \in [2..6]$,

$$\Pr[X_i = 1] = \Pr[z_1 \text{ is odd} \cap \ldots \cap z_{i-1} \text{ is odd}]\,.$$

These are independent events with $\Pr[z_i \text{ is odd}] = 1/2$ for all $i \in [2..6]$, so $\Pr[X_i = 1] = 1/2^{i-2}$ for all $i \in [2..6]$. ($i-2$ because the first digit is always 1). We now have:

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{6} X_i\right] \\
&= \sum_{i=1}^{6} \mathrm{E}[X_i] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{6} \Pr[X_i = 1] \\
&= 1 + \sum_{i=2}^{6} \frac{1}{2^{i-2}} \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\
&= \boxed{\frac{47}{16}}\,.
\end{aligned}$$