(b) $\forall m \in \mathbb{Z}\ 10 < m$

Incorrect. In English, this says for all integers m, $10 < m$. This is not true, which we can see by letting $m = 8$, for example.

(c) $\forall m \in \mathbb{Z}\ \exists n \in \mathbb{Z}\ m < n$

Correct. In English, this says for each integer m, there is a larger integer n. This is true; no matter what m is, $n = m + 1$ always satisfies the inequality.

(d) $\exists m \in \mathbb{Z}\ \forall n \in \mathbb{Z}\ m < n$

Incorrect. In English, this says there is some integer m such that for all integers n, $m < n$. In other words, it says there is an integer m that is smaller than all other integers. This is not true; no matter what m is, $n = m$ always violates the inequality.

1. [10 pts]

Thirteen Dwarves (Thorin, Fili, Kili, Balin, Dwalin, Oin, Gloin, Dori, Nori, Ori, Bifur, Bofur, and Bombur) want to ask for Bilbo's help, but they won't all fit in his Hobbit-hole, so they need to choose a delegation to visit him. How many different groups with at least one member and at most twelve members can the thirteen Dwarves choose among themselves?

Solution:

Let D be the set of Dwarves. Then the total number of subsets of D is $|2^D| = 2^{|D|} = 2^{13}$. We can now find the answer using complementary counting. The power set 2^D includes some sets that don't fit our constraints on the number of members:

- 1 subset of cardinality 0 ($\emptyset$)
- 1 subset of cardinality 13 (D itself)

By excluding these 2 sets, we see that the total number of subsets of D

with cardinality at least 1 and at most 12 is $\boxed{2^{13} - 2}$.

2. [10 pts]

John has a string of n empty light bulb sockets hanging above his front door, where $n = 2k$ is some even positive integer. He has an unlimited supply of blue, green, red, and yellow light bulbs. He wants to place the bulbs in the sockets in a *symmetric* way, meaning the order of colors should be the same left-to-right and right-to-left. How many different symmetric ways are there for John to arrange n bulbs of those four colors on this string?

Solution:

For the arrangement to be symmetric, the colors in the right half of the string must mirror the colors on the left half of the string. In other words, the colors of the last k lights are determined by the colors of the first k lights. Thus, John can choose the colors by the following procedure.

Step 1: Pick the color for the first and the n^{th} lights. (4 ways)
Step 2: Pick the color for the second and the $(n-1)^{\text{th}}$ lights. (4 ways)

$\vdots$

Step k: Pick the color for the k^{th} and the $(n-k)^{\text{th}}$ lights. (4 ways)

Applying the multiplication rule, this means John has $4^k = \boxed{4^{n/2}}$ ways to create a symmetric string with n of lights!

3. [10 pts]

Jocelyn plans to watch all the films on a list of the 100 greatest films. She has only one constraint on the order in which she views them: Four films on the list were directed by Alfred Hitchcock, and she wants to have a "Hitchcock marathon" in which she watches those four films consecutively, in any order.

How many possible orders for the 100 movies are there that satisfy this constraint?

Solution:

Let's order the non-Hitchcock films first. There are 96 of them, so there are 96! distinct permutations.

There are 97 options for where in that order to place the block of Hitchcock films: before the first film, before the second film, ..., before the 96^{th} film, or after the 96^{th} film.

Finally, there are 4! ways to order the Hitchcock films within that block.

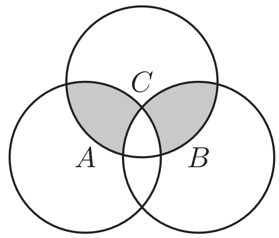
Applying the multiplication rule, there are $96! \cdot 97 \cdot 4! = \boxed{97!4!}$ valid ways for Jocelyn to order her viewing.

4. [8 pts]

Let A, B, and C be any three sets. For each Venn diagram, describe the gray set in two different ways:

- First, in terms of the set operations $\cup$ (union), $\cap$ (intersection), and $\setminus$ (set difference).
- Second, using set builder notation and the logical connectives $\wedge$ (and), $\vee$ (or), and $\neg$ (not).

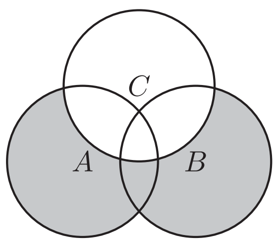
For example, consider the following Venn diagram.



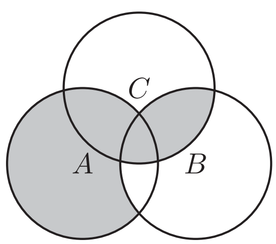
The gray set can be described in terms of set operations as $\big((A \cup B) \cap C\big) \setminus (A \cap B)$, and in set builder notation as

$$\big\{x \mid \big((x \in A \vee x \in B) \wedge x \in C\big) \wedge \neg(x \in A \wedge x \in B)\big\}.$$

(a)



(b)



Solution:

(a) The gray area is the union of A and B with C taken out. Using set operations, we can write that as $\boxed{(A \cup B) \setminus C}$. Another way to describe the gray area is that contains everything that's in A or B and also not in C. In set builder notation, we write that as $\boxed{\{x \mid (x \in A \vee x \in B) \wedge \neg(x \in C)\}}$.

(b) The gray area can be described as A with B taken away, unioned with the intersection of B and C. Using set operations, that's $\boxed{(A \setminus B) \cup (B \cap C)}$. We can equivalently describe the gray area by saying that it contains everything that is either in A and not B, or in both B and C. In set builder notation, that's $\boxed{\{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \in C)\}}$.

5. [10 pts]

Two boys are trick-or-treating on Halloween, dressed as T'Challa and Dracula. Mrs. Owens has a bowl with seven different candy bars in it (one Kit-Kat, one Milky Way, one Snickers, one Crunch, one Twix, one Baby Ruth, and one Almond Joy), and she tells the boys that they can each take two candy bars from the bowl. If T'Challa picks first, how many ways are there for them to select their candy?

Solution:

T'Challa selects 2 candy bars from a set of 7, which he can do in $\binom{7}{2}$ ways. Once he has picked, there are 5 candy bars remaining, so Dracula selects 2 candy bars from a set of of 5, which he can do in $\binom{5}{2}$ ways. Applying the multiplication rule, the boys can choose their candy in

$$\binom{7}{2} \cdot \binom{5}{2} = \frac{7!}{2!5!} \cdot \frac{5!}{2!3!} = \frac{7!}{2!2!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 = \boxed{210}$$

ways.

6. [12 pts]

Write the following statements in English. Then say whether each statement is true or false, and briefly explain why.

(a) $\forall m \in \mathbb{Z}^+ \; \exists n \in \mathbb{Z} \; (n > m \wedge n < 2m)$

(b) $\forall m \in \mathbb{Z} \; \exists n \in \mathbb{Z} \; \forall p \in \mathbb{Z} \; p + n > m$

(c) $\exists m \in \mathbb{Z} \; \forall n \in \mathbb{Z} \; mn = m$

Solution:

(a) For all positive integers m, there exists an integer n such that $n > m$ and $n < 2m$.

FALSE. If $m = 1$, then there is no integer n that satisfies "$n > m$ and $n < 2m$."

(b) For all integers m, there exists an integer n such that for all integers p, $p + n > m$.

FALSE. If $m = 0$, then no matter what n is, choosing $p = -n$ violates "$p + n > m$".

(c) There exists an integer m such that for all integers n, $mn = m$.

TRUE. If $m = 0$, then all integers n satisfy "$mn = m$."

7. [6 pts] EXTRA CREDIT CHALLENGE PROBLEM

Ms. Frizzle is planning to give her class a multiple-choice exam with kn problems, each with k choices, for some positive integers n and k. She wants to make sure that each of the k choices is the correct answer for exactly n of the problems. How many possible answer keys are there that satisfy this constraint?

Solution:

We call the multiple-choice options $c_1, c_2, \ldots, c_k$. Ms. Frizzle can design the answer key by the following procedure:

Step 1: Pick n of the kn problems to have answer c_1. ($\binom{kn}{n}$ ways)
Step 2: Pick n of the $(k-1)n$ remaining problems to have answer c_2. ($\binom{(k-1)n}{n}$ ways)
Step 3: Pick n of the $(k-2)n$ remaining problems to have answer c_3. ($\binom{(k-2)n}{n}$ ways)

$\vdots$

Step $k-1$: Pick n of the $2n$ remaining problems to have answer c_{k-1}. ($\binom{2n}{n}$ ways)
Step k: Pick n of the n remaining problems to have answer c_k. ($\binom{n}{n}$ ways)

By the multiplication rule, the total number of ways to design the answer key is

$$\binom{kn}{n}\binom{(k-1)n}{n}\binom{(k-2)n}{n}\cdots\binom{2n}{n}\binom{n}{n}.$$

Applying the formula $nr = \frac{n!}{r!(n-r)!}$, this is

$$\frac{(kn)!}{n!((k-1)n)!}\cdot\frac{((k-1)n)!}{n!((k-2)n)!}\cdot\frac{((k-3)n)!}{n!((k-2)n)!}\cdot\dots\cdot\frac{(2n)!}{n!n!}\cdot\frac{n!}{n!0!}.$$

Notice that the second term in each denominator cancels out the next numerator:

$$\frac{(kn)!}{n!((k-1)n)!}\cdot\frac{((k-1)n)!}{n!((k-2)n)!}\cdot\frac{((k-3)n)!}{n!((k-2)n)!}\cdot\dots\cdot\frac{(2n)!}{n!n!}\cdot\frac{n!}{n!0!}.$$

Since $0! = 1$, we are left with

$$\boxed{\frac{(kn)!}{(n!)^k}}.$$