Discounted Cash Flows & Rates of Return

Future Value

Why is a dollar today worth more than a dollar tomorrow?

Example:

Receive \$100 today. Invest at 2% per annum. Calculate Future Value:

$$100 + 2\% \times 100 = 100\,(1 + 0.02) = \$102$$

And in two years' time?

$$102 + (2\% \times 102) = 104.04$$

Generalize:

Present Value *PV* dollars, *t* years, *r* is interest rate

Future Value *FV* is:

$$\boxed{FV = PV\,(1 + r)^t}$$

Future Value Example

If you have \$100 today, and invest it at 2% per year, what will be its future value in three years?

$$FV = PV\,(1 + r)^t$$

End of: Year 1: $FV = \$100\,(1 + 0.02)^1 = \$102$

Year 2: $FV = \$100\,(1 + 0.02)^2 = \$102\,(1.02) = \$104.04$

Year 3: $FV = \$100\,(1 + 0.02)^3 = \$104.04\,(1.02) = \$106.12$

Worked Examples: Future Value of a Single Cashflow

Example 1*:* Suppose that a wealthy relative gives you $20,000 to help provide for your newborn child's university fees. You are able to invest this money at 5% p.a. (per annum) until your child is ready to begin college. How much will be in the account 18 years from now?

$$FV = PV(1 + r)^t$$

$$FV = 20{,}000\,(1 + 5\%)^{18}$$

$$= \$48{,}132.38$$

Example 2*:* A woman invests $1,000 at 10% p.a. and plans to hold this investment for 5 years. How much will she have at the end of the holding period?

$$FV = PV(1 + r)^t$$

$$FV = 1{,}000\,(1 + 10\%)^5$$

$$= \$1{,}610.51$$

Future Value with Compounding

If you have \$100 today, and invest it at 2% per year ***with semi-annual compounding***, what will be its future value in two years?

"Compounding" means that you receive partial interest *more often* than just once per year.

2% per year with semi-annual compounding means 1% every 6 months

End of: 1st 6 months: $FV = \$100\ (1 + 0.01)^1 = \$101$

2nd 6 months: $FV = \$101\ (1 + 0.01)^1 = \$102.01$

3rd 6 months: $FV = \$102.01\ (1 + 0.01)^1 = \$103.03$

4th 6 months: $FV = \$103.03\ (1 + 0.01)^1 = \$104.06$

Equivalently: $FV = 100\ (1.01)^4 = 104.06$

Future Value with Compounding

General formula for Future Value with Compounding:

$$FV = PV\,(1 + r/m)^{t*m} \quad = \; PV\,(1 + q)^n$$

Where: *r* is the annual (or "quoted") rate,

t is the number of years

m is the number of compounding periods per year

q is the interest rate per compounding period (= r/m)

n is the number of compounding periods (= t*m)

y is the "Effective Annual Rate" = $(1 + q)^m - 1$

Examples: Future value of $100 after 1 year with 4% *quoted* rate, with different *compounding* periods:

no compounding	FV = 100 (1.04) = $104	y = 4.00%
semi-annual compounding	FV = 100 $(1.02)^2$ = $104.04	y = 4.04%
Monthly compounding	FV = 100 $(1.0033)^{12}$ = $104.07	y = 4.07%

Future Value with Compounding

In some settings, we use *continuous* compounding:

$FV = PV\ (1 + r/m)^{t*m}$ where $m \rightarrow \infty$ (∞ = *infinity,* i.e., very large!)

$= PV\ e^{r}$

For continuous compounding over *t* years:

$FV = PV\ e^{rt}$

Example: \$500 invested for 3 years at 5% with continuous compounding:

$FV = 500\ e^{0.05*3}$

$= 580.92$

Compare with \$500 invested for 3 years at 5% with *no* compounding:

$FV = 500\ (1.05)^{3} = \$578.81$

Future Value of a Series of Identical Cashflows

Some financial investments pay a stream of cash flows at equally spaced intervals, and of an equal amount.

Example: You put $100 per month into a vacation fund at 6% per year, with monthly compounding. How much will you have available after one year?

$$FV = 100[(1 + 0.5\%)^{12} - 1] / 0.5\% = \$1{,}233.56$$

More generally: the future value of a series of identical cash flows of $\$C$ for t years at $r\%$ per year with m compounding periods per year is:

$$\begin{aligned} FV &= C\,[(1 + r/m)^{t*m} - 1] / (r/m) \\ &= C\,[(1 + q)^{n} - 1]/q \end{aligned}$$

where: C = size of each cash flow
q = periodic interest rate (= r / m)
n = total number of periods (t * m)

Future Value of a Series of Identical Cashflows

Example: Your financial advisor tells you that you'll need to have $2 million to fund your retirement. You plan to work for another 30 years before retiring. You will make 30 annual contributions to a pension plan. The first contribution will be made one year from now, and the last will be made 30 years from now, on the day you retire. How much will each contribution have to be to ensure that you have $2 million in your pension plan account on your retirement day if the pension plan guarantees a return of 5% p.a.?

$FV = 2{,}000{,}000$

$r = 5\%$

$C = ??$

so $2{,}000{,}000 = C\ [(1.05)^{30} - 1]/0.05$

Hence $C = \$30{,}102.87$

Present Value

Example:
Receive \$100 in two years time. What is that worth today?

$FV = 100$, $t = 2$, $r = 2\%$

$$PV\,(1 + 0.02)^2 = 100$$

hence $$PV = \frac{100}{(1.02)^2}$$

$$= 96.12$$

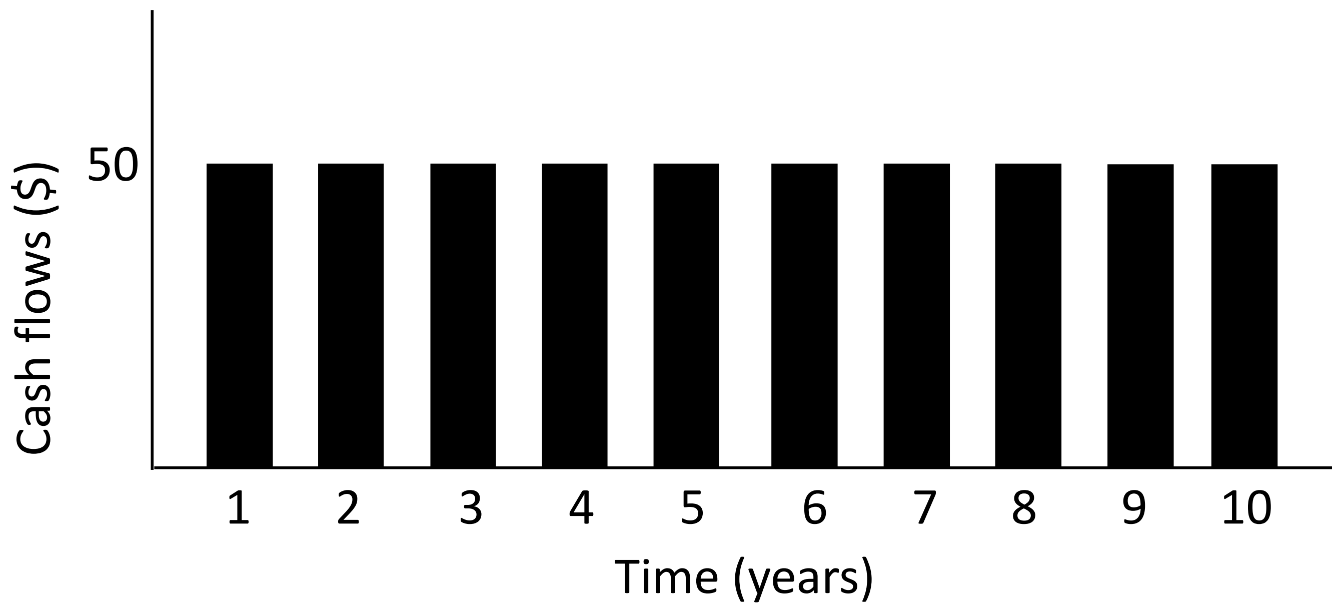
Generalize:

$$\boxed{PV = \frac{FV}{(1 + r)^t}}$$

We call r the "discount rate" since we are *discounting* the future cash flow to today

Present Value with Multiple Cash Flows

Example: identical annual payment of $50 each year for 10 years
Annual discount rate = 6%



$$P = \frac{50}{(1.06)} + \frac{50}{(1.06)^2} + \frac{50}{(1.06)^3} + ... + \frac{50}{(1.06)^{10}} = 368.00$$

Multiple Cash Flows

$$P = \frac{50}{(1.06)} + \frac{50}{(1.06)^2} + \frac{50}{(1.06)^3} + \dots + \frac{50}{(1.06)^{10}} = 368$$

$$= \frac{50}{0.06}\left[1 - (1.06)^{-10}\right]$$

$$= 368.00$$

Formula for identical cash flows *C* every year for *t* years with discount rate *r*:

$$\boxed{PV = \frac{C}{r}\left[1 - (1 + r)^{-t}\right]}$$

Multiple Cash Flows and Compounding

Example: identical payment of \$20 every six months for 5 years
Annual discount rate = 6% *with semi-annual compounding*

$$P = \frac{20}{(1.03)} + \frac{20}{(1.03)^2} + \frac{20}{(1.03)^3} + \ldots + \frac{20}{(1.03)^{10}}$$

$$= \frac{20}{0.03}[1 - (1.03)^{-10}]$$

$$= 170.60$$

$$PV = \frac{C}{r/m}\left[1 - (1 + r/m)^{-t*m}\right] = \frac{C}{q}\left[1 - (1+q)^{-n}\right]$$

where $q = r/m$, $n = t * m$

Example: Mortgage Payments

Suppose you take out a 30 year mortgage loan of \$400,000. The loan has a annual rate of 7.45%, fixed over the life of the loan, and compounded *monthly*. What is the fixed monthly payment required to pay off the loan?

For this example, we wish to solve for *C* given

$q = \frac{7.45\%}{12} = 0.62\%$ $\quad$ $n = 30 * 12 = 360$ $\quad$ PV = 400,000.

$$400{,}000 = \frac{C}{0.0062}\left[1 - (1.0062)^{-360}\right]$$

Hence C = \$2,780.44

Present Value of a Perpetuity

A perpetuity is a stream of cashflows that continue forever. The formula for the Present Value of a perpetuity is:

$$PV = C / r$$

Example 1: In the early 1900's the Canadian Government issued $100 par value 2% Consol bonds. The holder of these bonds is entitled to receive a coupon (or interest) payment of $2 per year forever. If the current appropriate discount rate is 5% p.a. and the next coupon is due one year from now, how much is one of the Consols worth?

$$PV = \$2 / 5\%$$
$$= \$40$$

Example 2: Your company anticipates the introduction of environmental protection laws in three years time. Under these laws you will have to pay a yearly environmental tax of $5,000, an obligation that will continue indefinitely. If the prevailing interest rate is 6% p.a. what is the present value of your company's obligations under this law (the first payment will be four years from now)?

PV_3 (*as of 3 years from now*) = 5,000/0.06 = $83,333.33

Hence PV_0 (*as of today*) $= 83,333.33 / (1.06)^3$

$= 69,968.27$

Future Value and Present Value Formulas

Future value of a single cash flow: $FV = PV(1 + r)^t$

Future value of a cash flow when interest rates are compounded: $FV = PV(1 + r/m)^{t*m} = PV(1 + q)^n$

Future value of a cash flow with continuous compounding: $FV = PV e^{rt}$

Future value of a stream of identical cash flows *C* each year until time *t*: $FV = C[(1 + r)^n – 1]/r$

Present Value of a single future cash flow: $PV = FV / (1 + r)^t$

Present value of a future cash flow when interest rates are compounded: $PV = FV/ (1 + r/m)^{t*m}$
$= FV (1 + q)^n$

Present value of a cash flow with continuous compounding: $PV = FV e^{-rt}$
Present value of a perpetuity: $PV = C/r$

r is the annual (or "quoted") rate,
t is the number of years
m is the number of compounding periods per year
q is the interest rate per compounding period (= r/m)
n is the number of compounding periods (= t*m)
y is the "Effective Annual Rate" = $(1 + q)^m - 1$