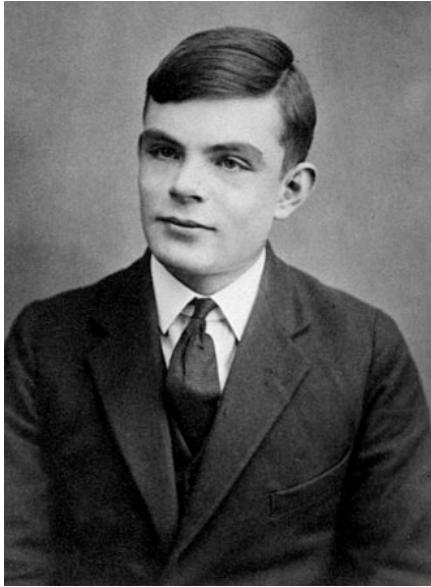


Turing Machines



Alan M. Turing (1912-1954)

- English mathematician, computer scientist
- providing the concepts of algorithm and computation with the Turing machine, which can be considered a model of a computer.
- During the 2nd World War, Turing devised a technique for breaking German ciphers machine (Enigma).
- Turing was prosecuted in 1952 for homosexual acts.
- He accepted chemical treatment, as an alternative to prison. However, it was unsuccessful.
- In 1954, he committed suicide by cyanide poisoning.
- In 2017, UK retroactively apologize for an outlawed homosexual acts.

The Language Hierarchy

$a^n b^n c^n$?

ww ?

Context-Free Languages

$a^n b^n$

ww^R

Regular Languages

a^*

$a^* b^*$

The diagram consists of three concentric ellipses. The outermost ellipse is labeled 'Languages accepted by Turing Machines'. Inside it is an ellipse labeled 'Context-Free Languages'. Inside that is the innermost ellipse labeled 'Regular Languages'. Each level contains specific language examples.

Languages accepted by
Turing Machines

$a^n b^n c^n$

ww

Context-Free Languages

$a^n b^n$

ww^R

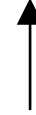
Regular Languages

a^*

$a^* b^*$

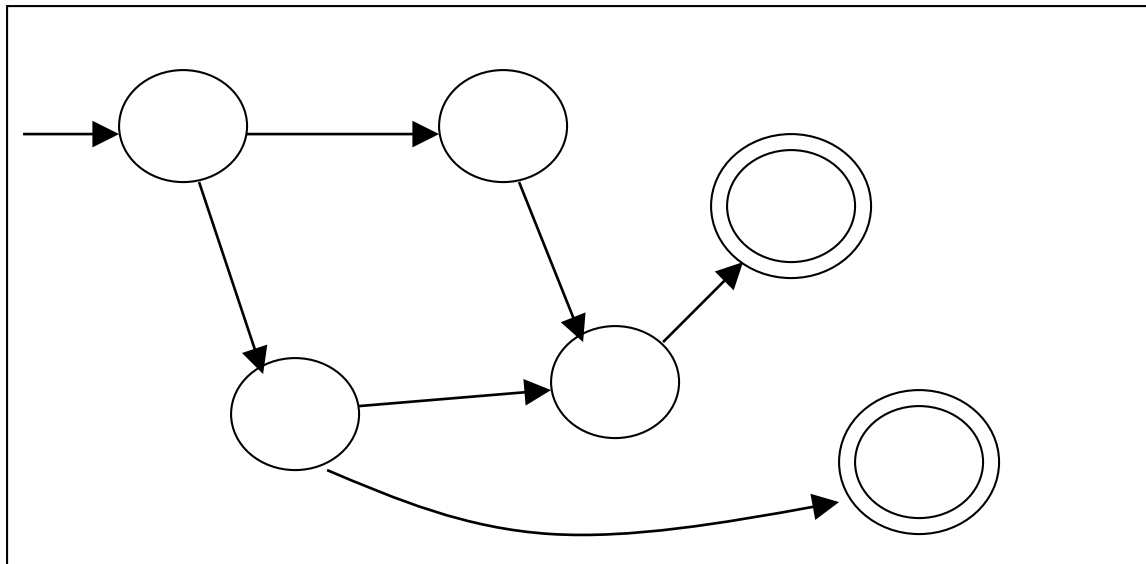
A Turing Machine

Tape



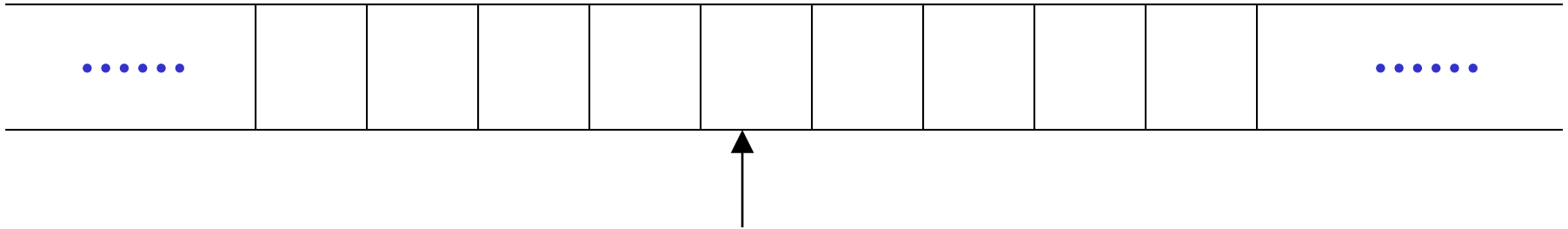
Read-Write head

Control Unit



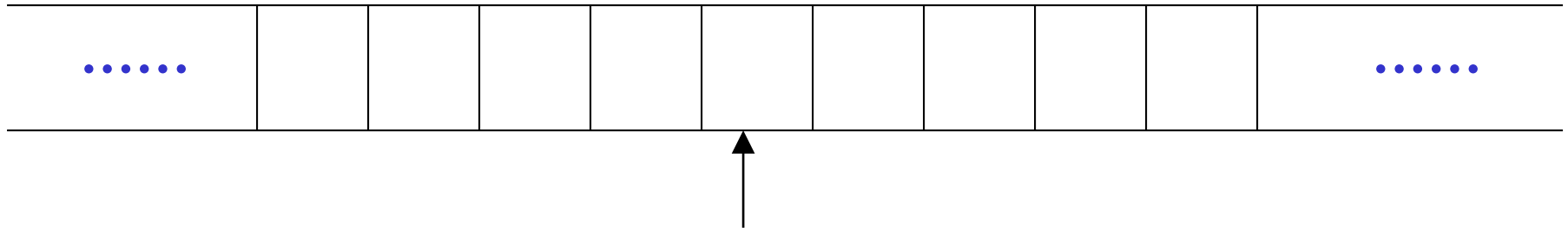
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



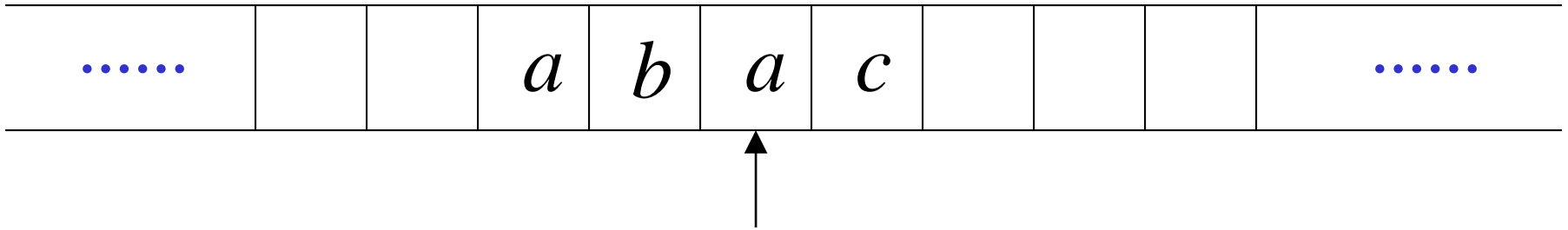
Read-Write head

The head at each time step:

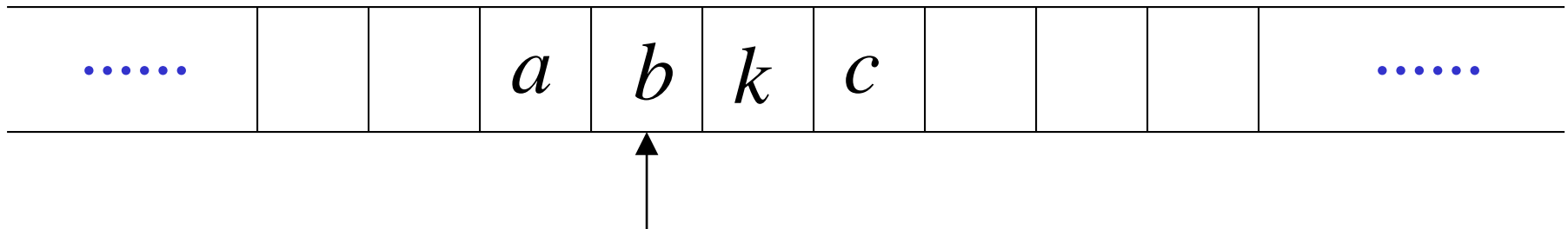
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

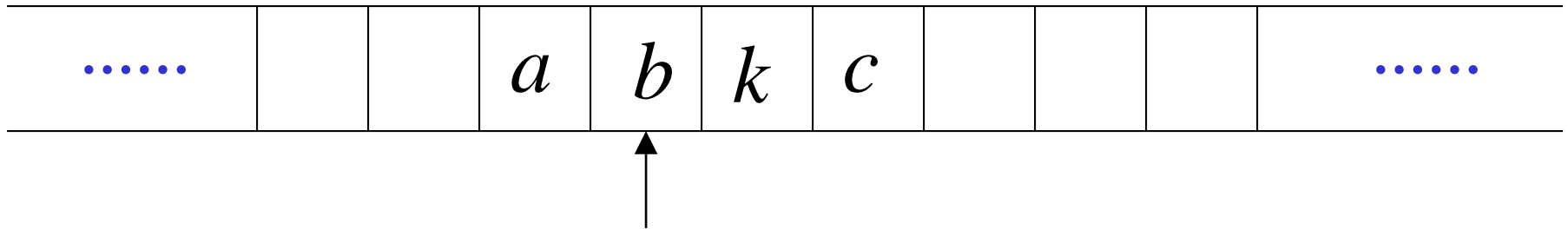


Time 1

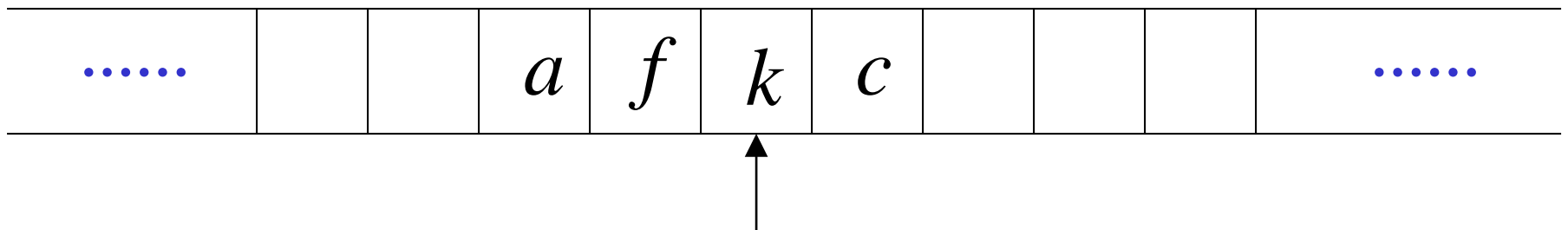


1. Reads *a*
2. Writes *k*
3. Moves Left

Time 1

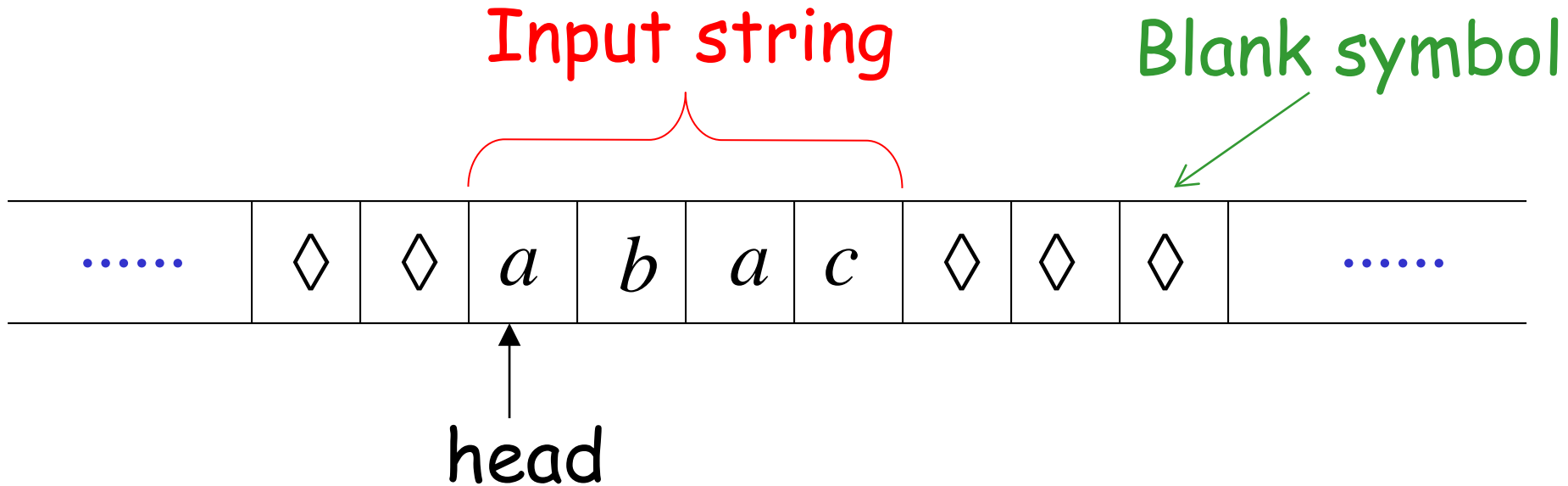


Time 2

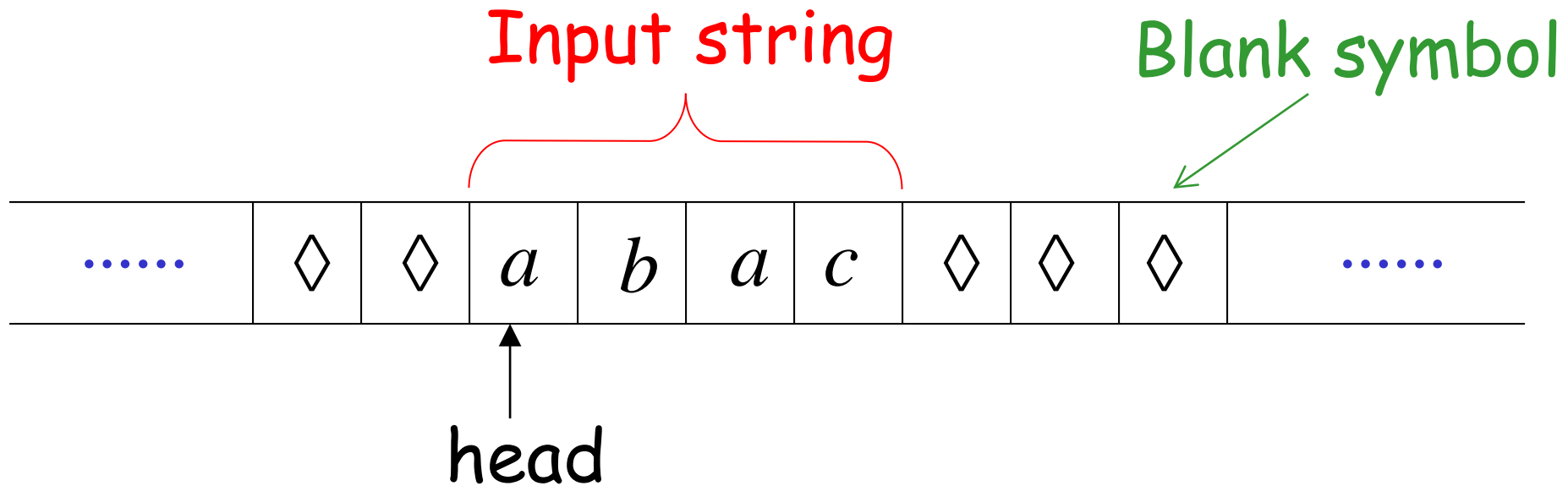


1. Reads b
2. Writes f
3. Moves Right

The Input String

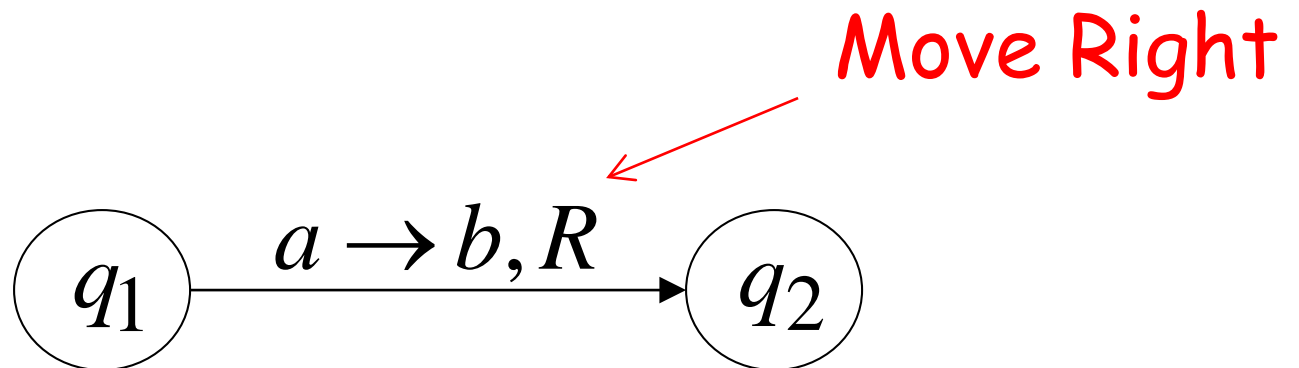
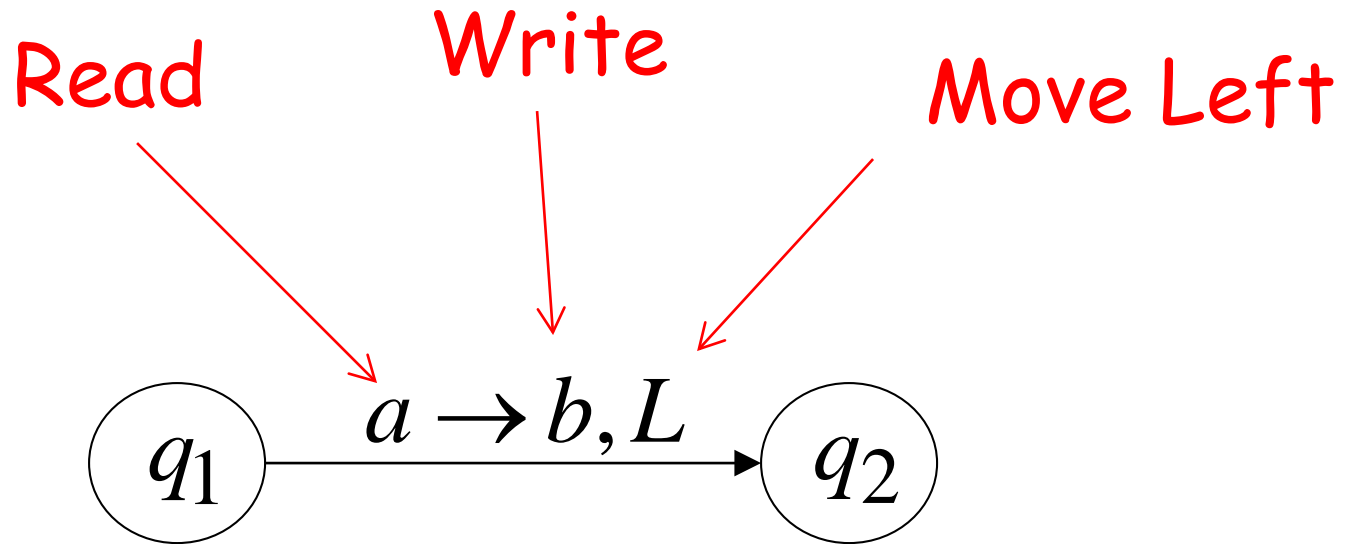


Head starts at the leftmost position of the input string



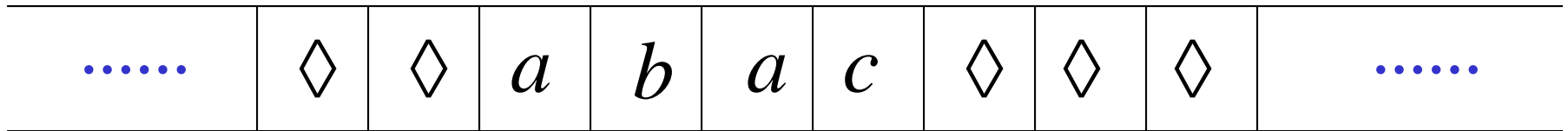
Remark: the input string is never empty

States & Transitions



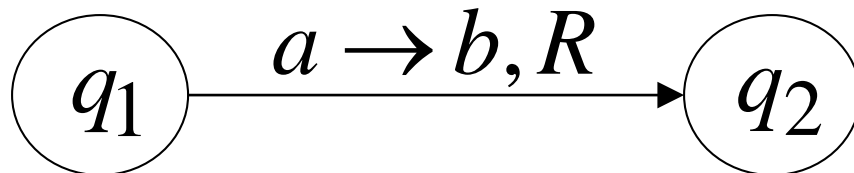
Example:

Time 1

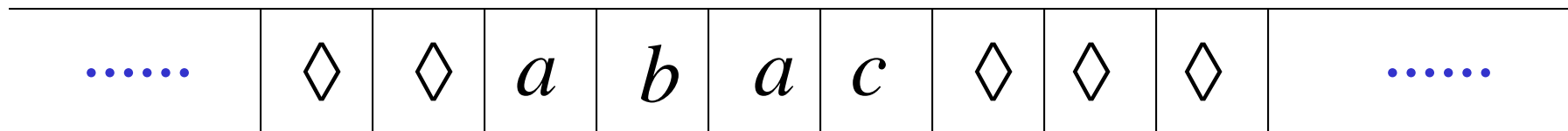


q_1

current state

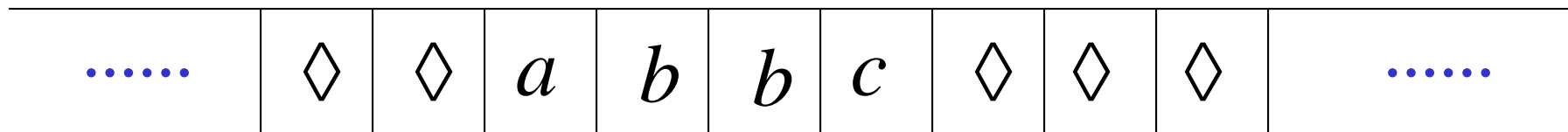


Time 1

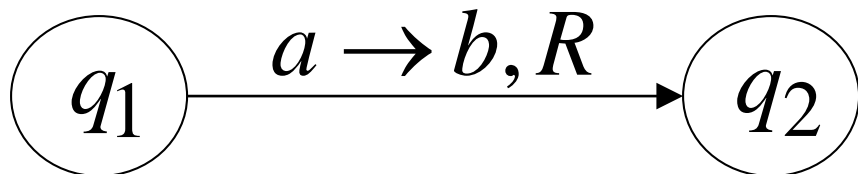


q_1

Time 2

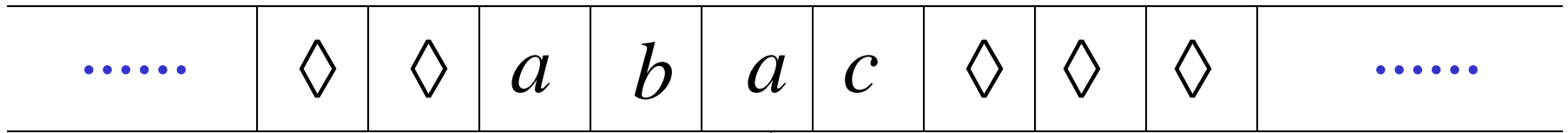


q_2



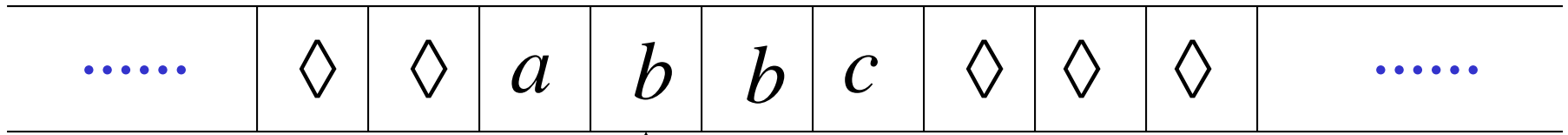
Example:

Time 1

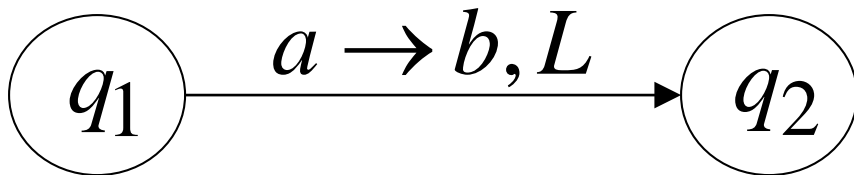


q_1

Time 2

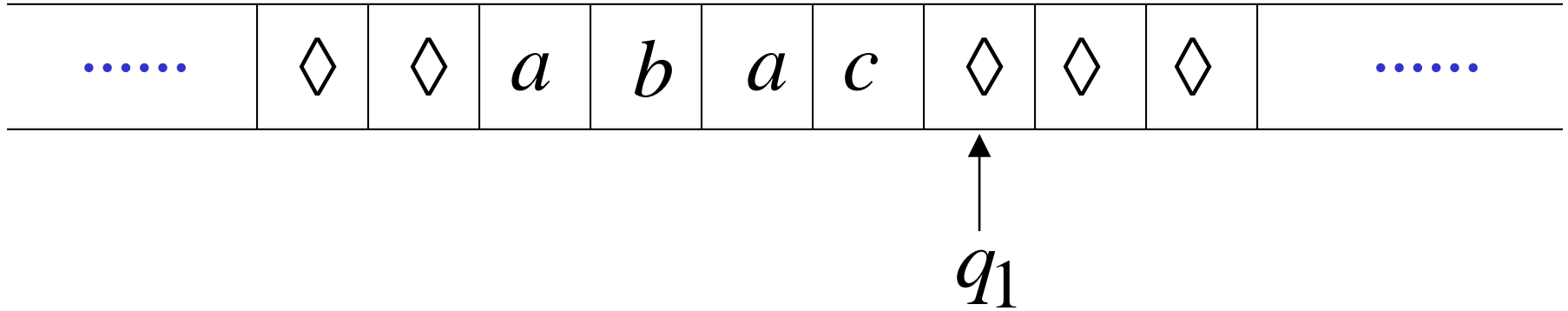


q_2

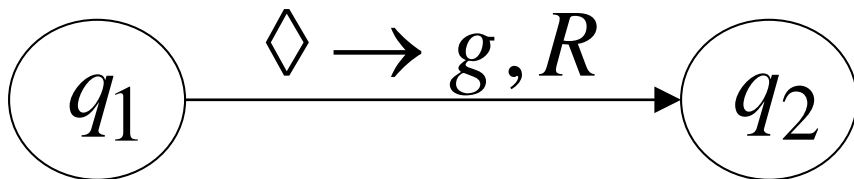
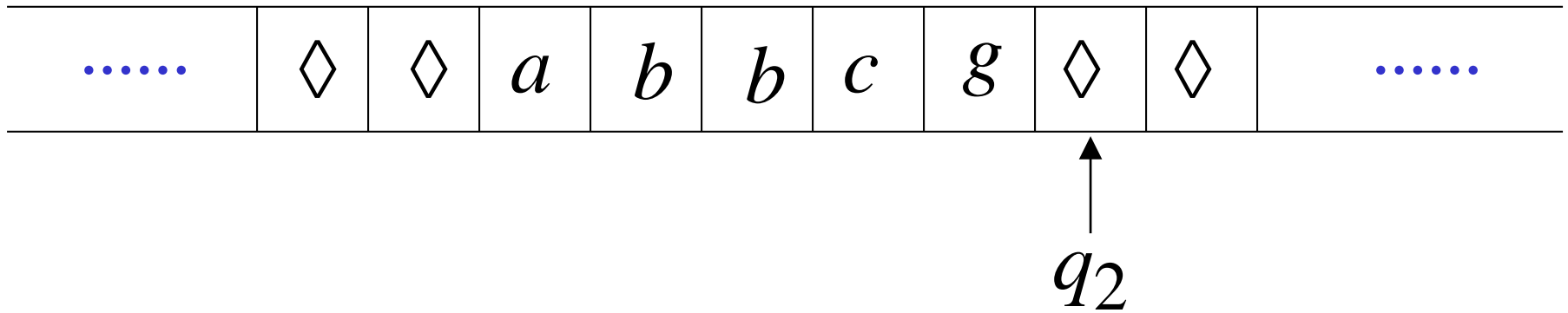


Example:

Time 1



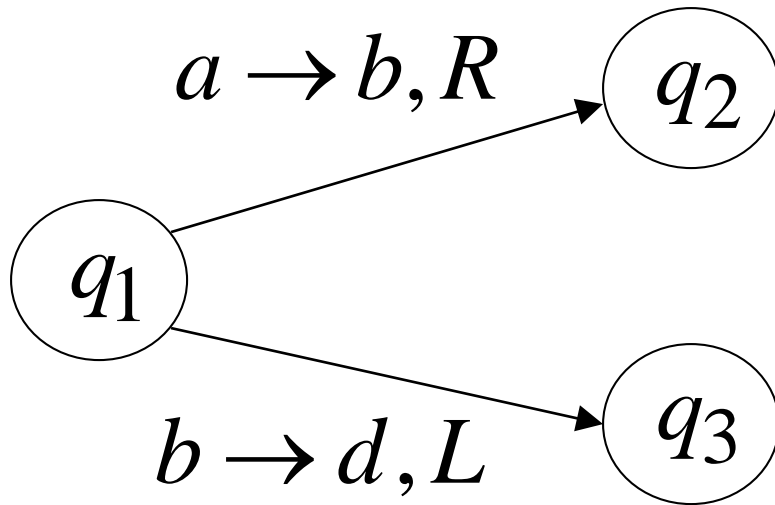
Time 2



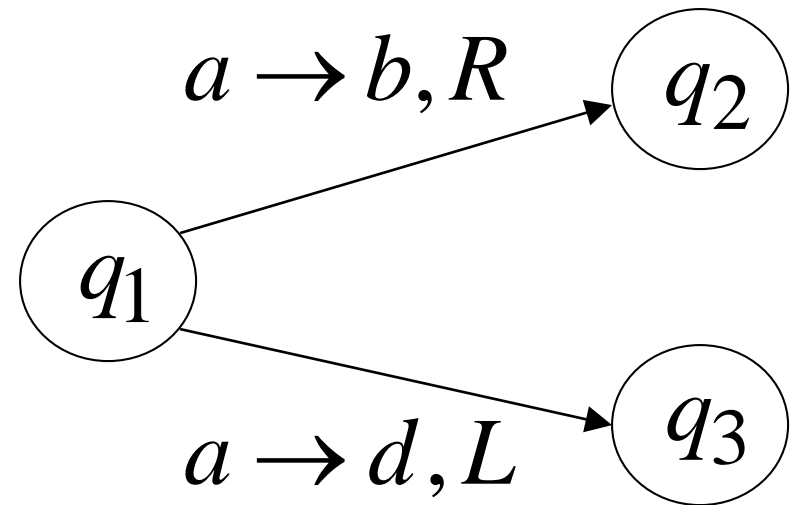
Determinism

Turing Machines are deterministic

Allowed



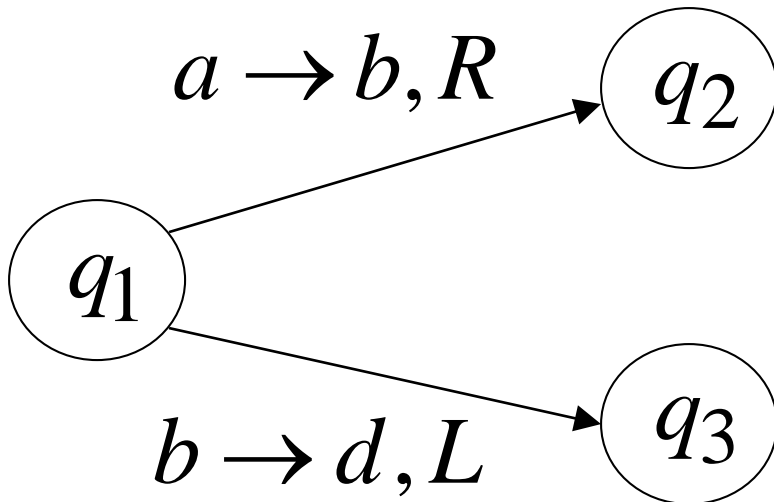
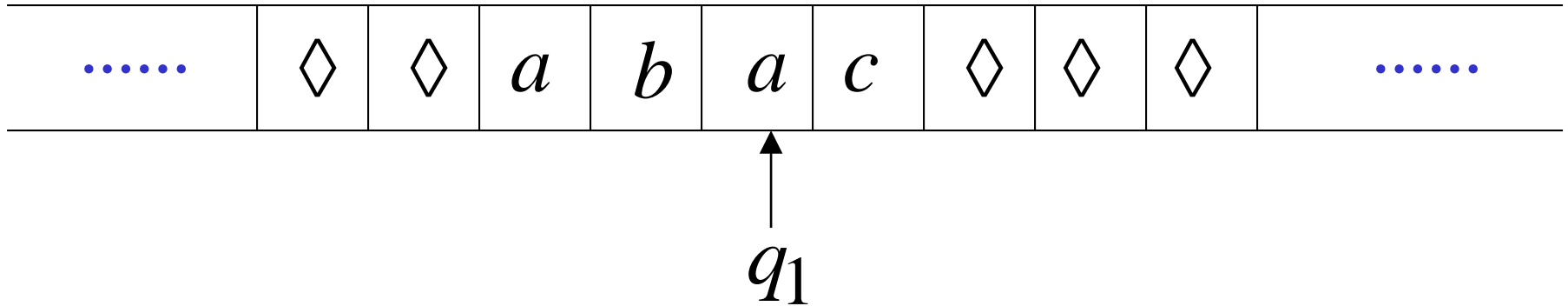
Not Allowed



No lambda transitions allowed

Partial Transition Function

Example:



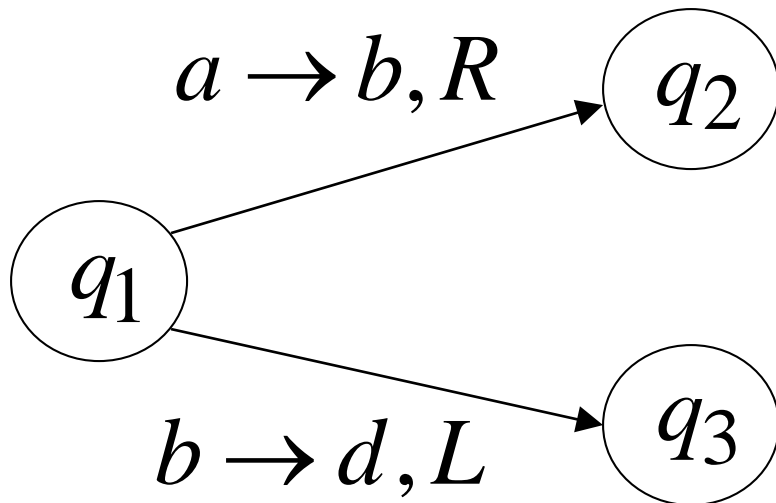
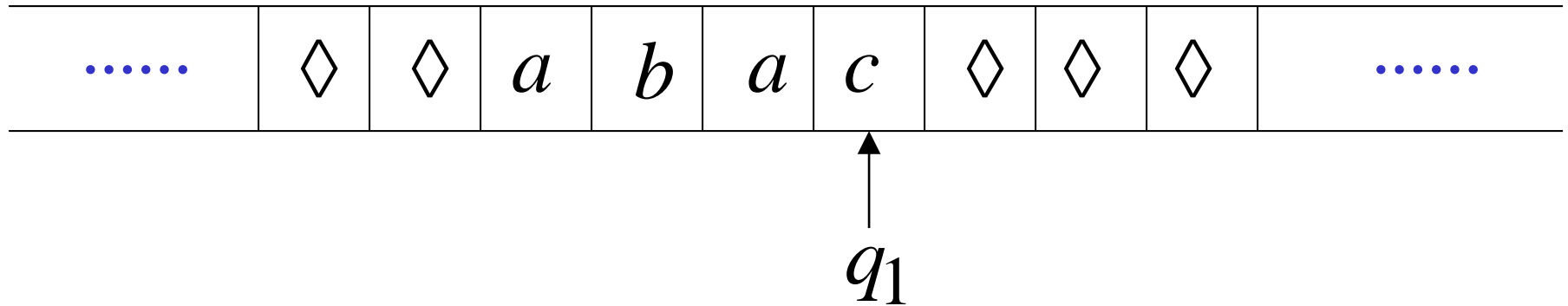
Allowed:

No transition
for input symbol c

Halting

The machine *halts* if there are
no possible transitions to follow

Example:



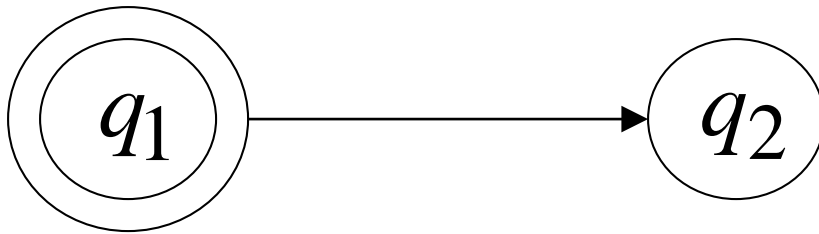
No possible transition

HALT!!!

Final States



Allowed



Not Allowed

- Final states have no outgoing transitions
- In a final state the machine halts

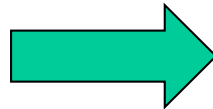
Acceptance

Accept Input



If machine halts
in a final state

Reject Input



If machine halts
in a non-final state

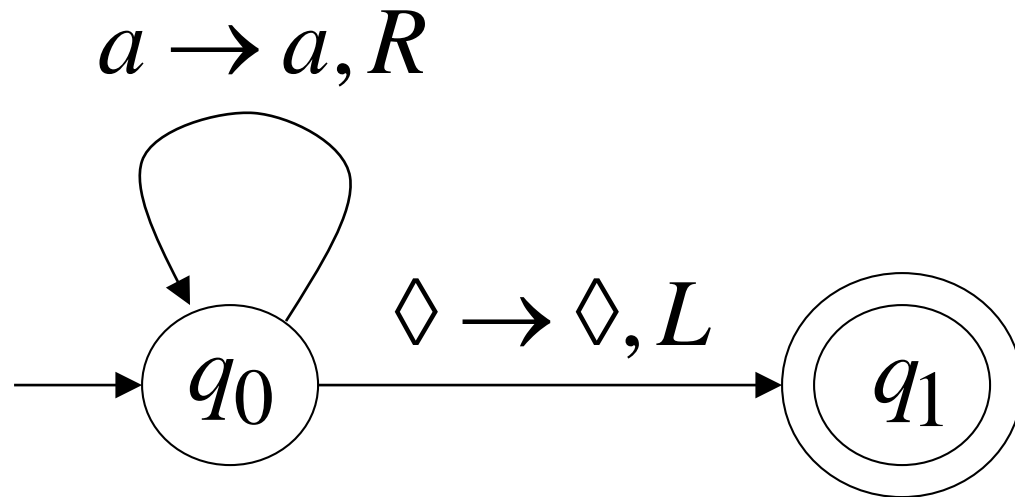
or

If machine enters
an *infinite loop*

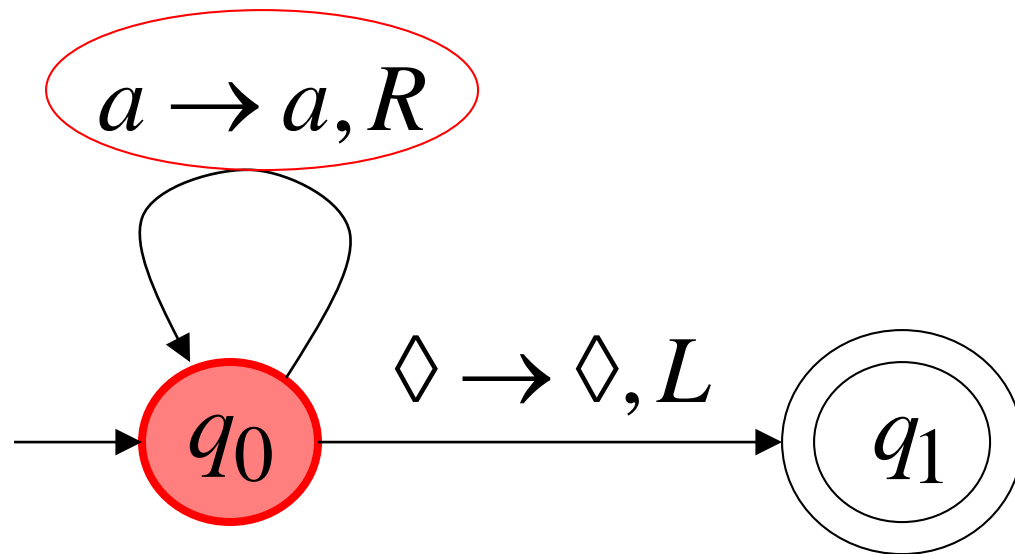
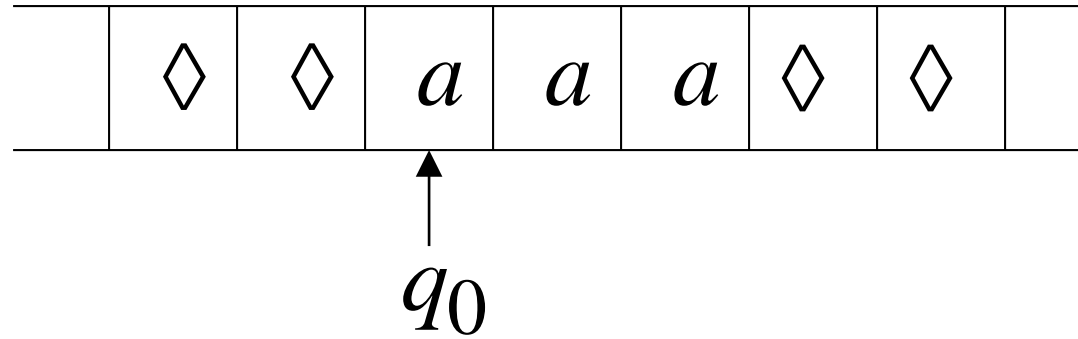
Turing Machine Example

A Turing machine that accepts the language:

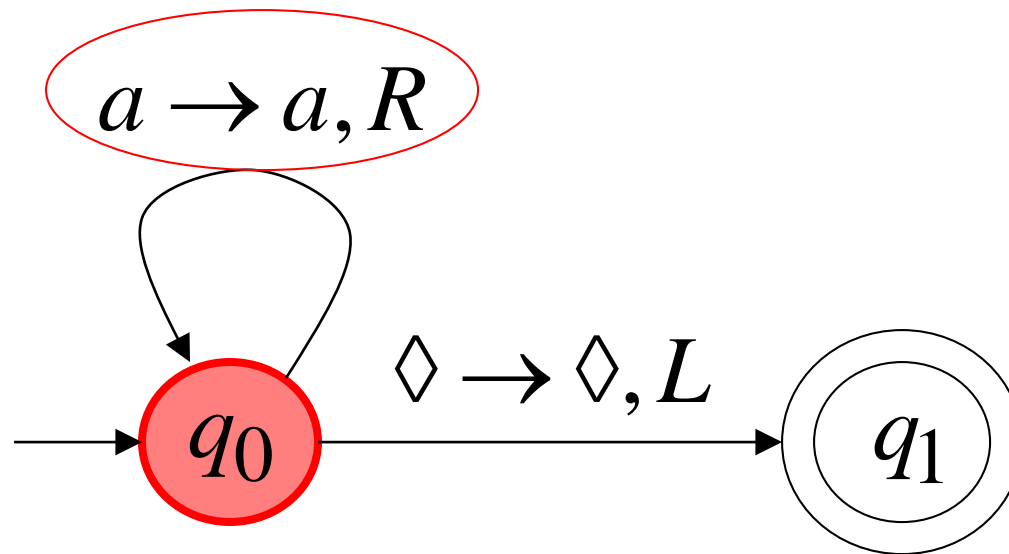
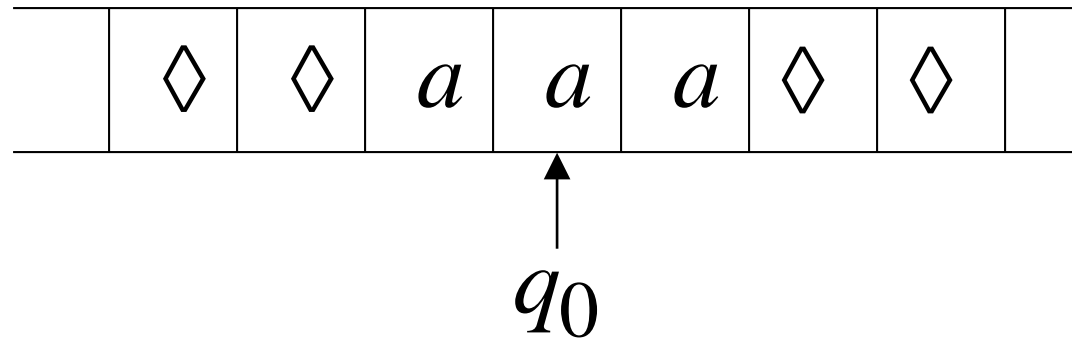
a^*



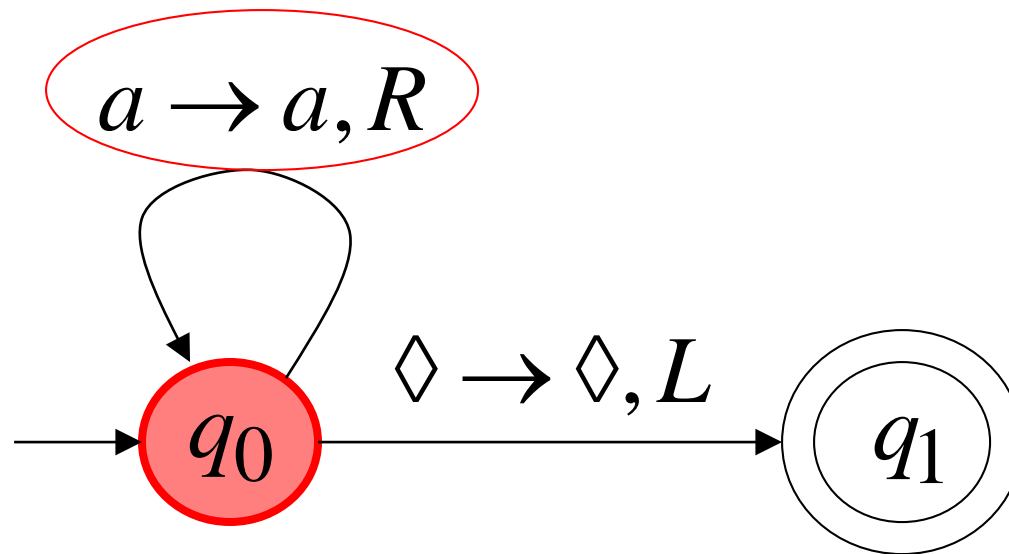
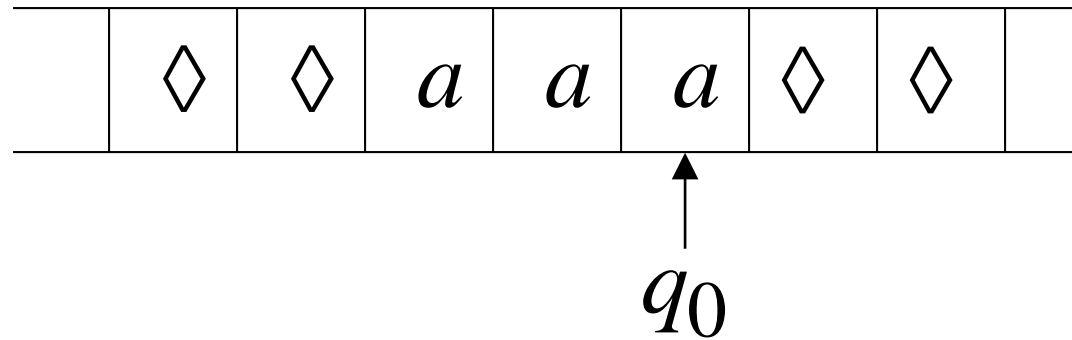
Time 0



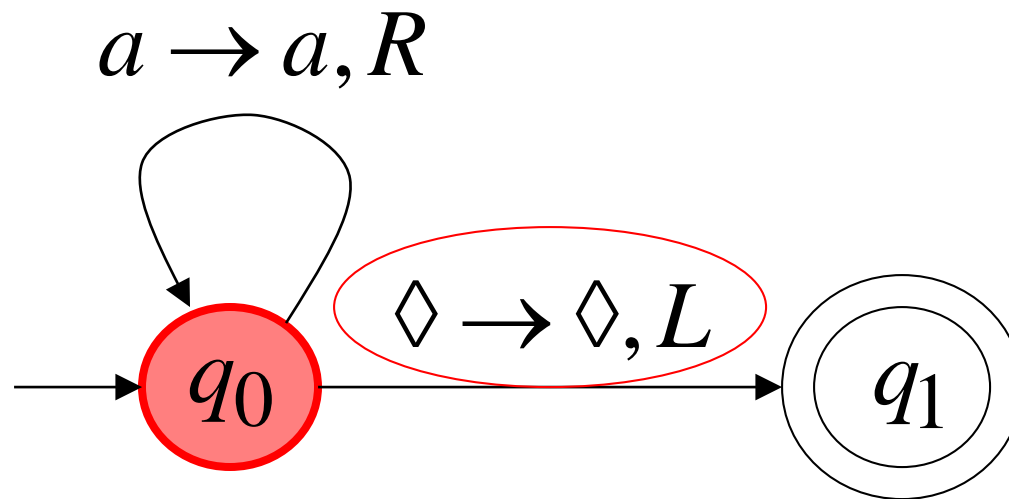
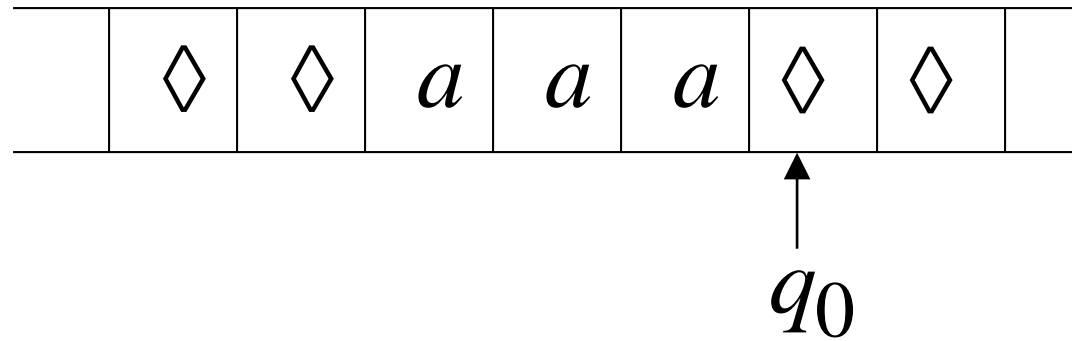
Time 1



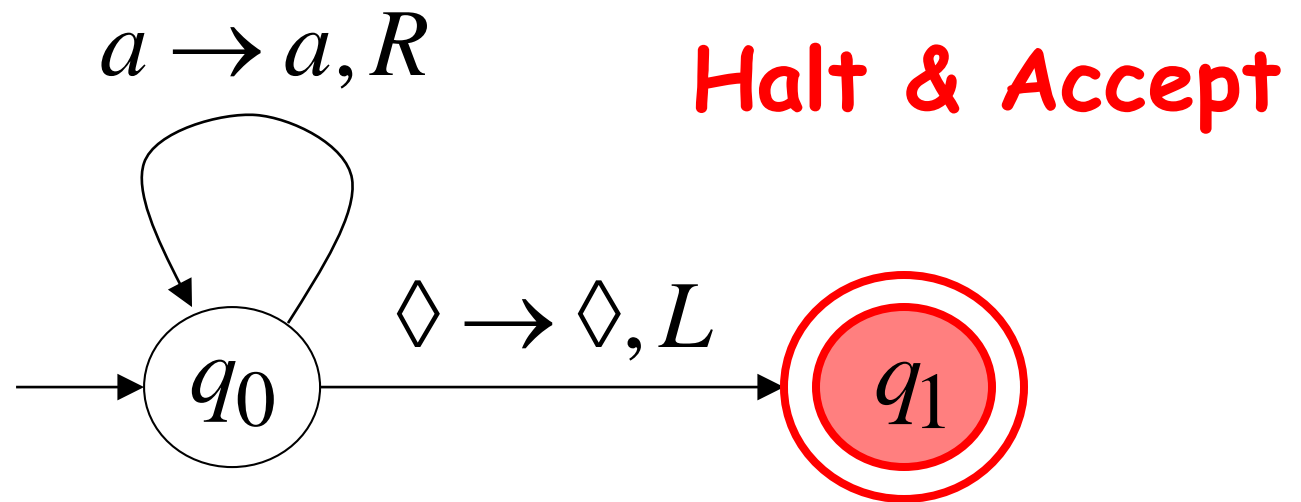
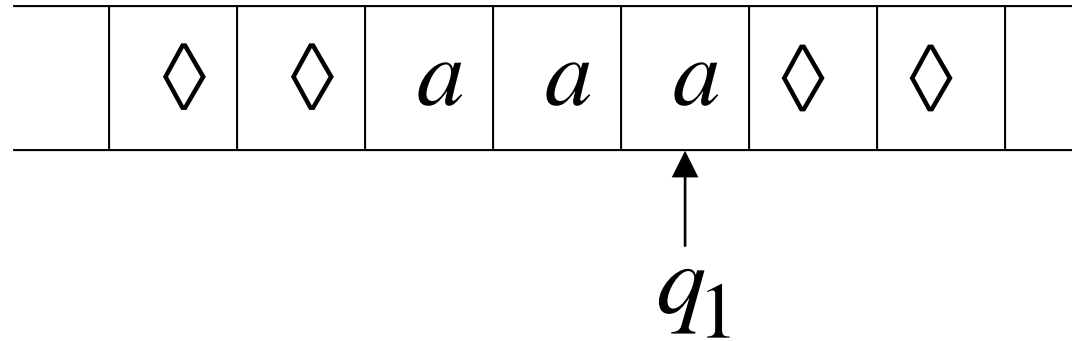
Time 2



Time 3

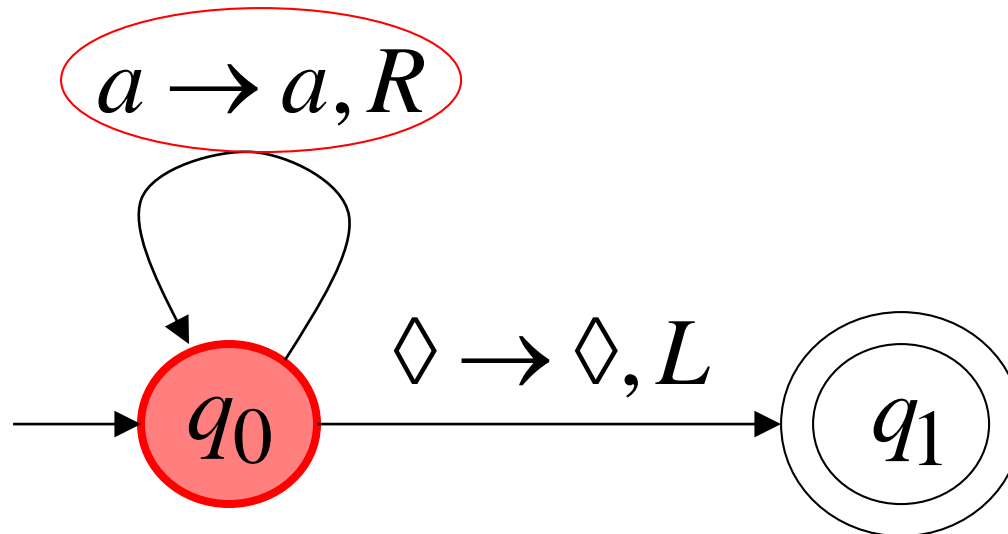
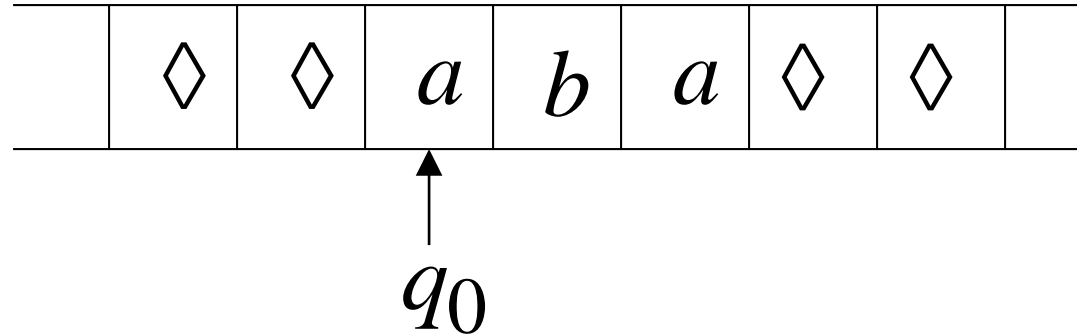


Time 4

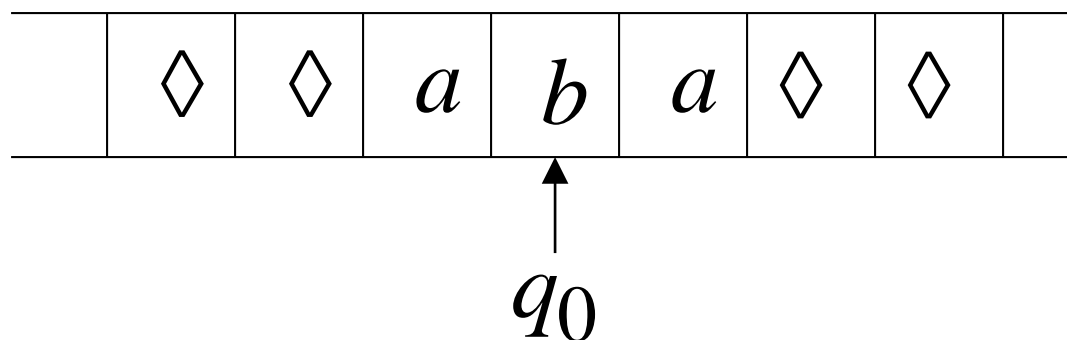


Rejection Example

Time 0

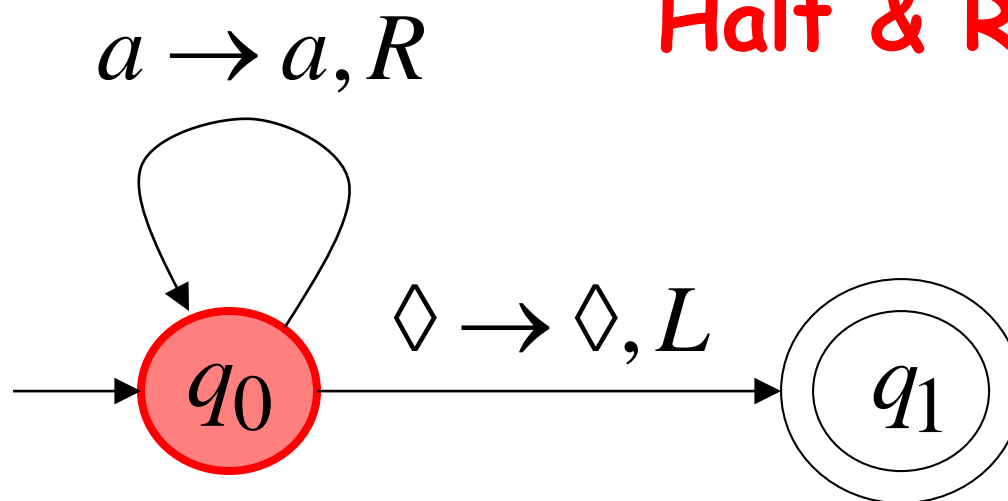


Time 1

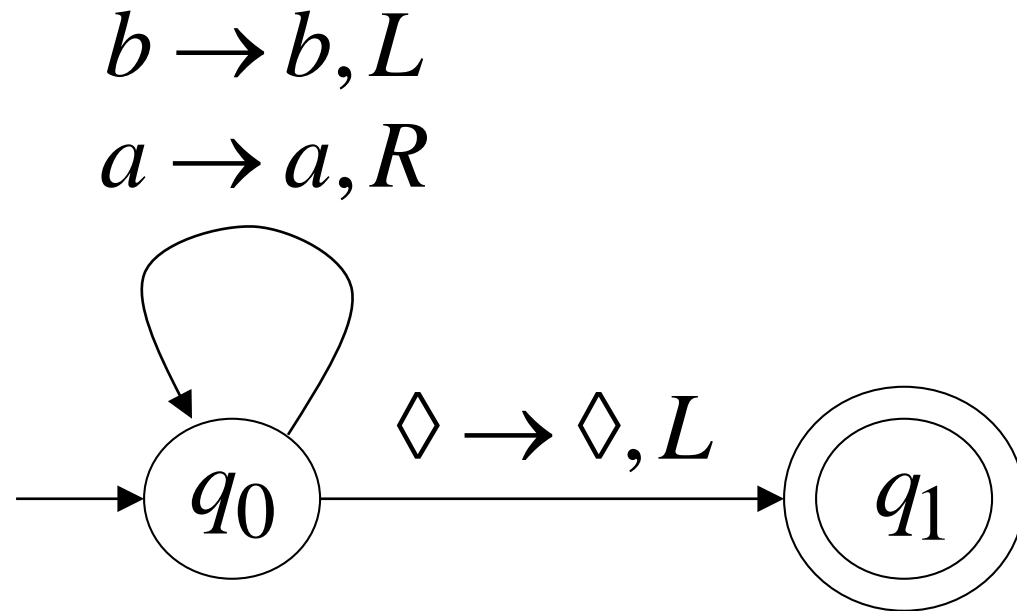


No possible Transition

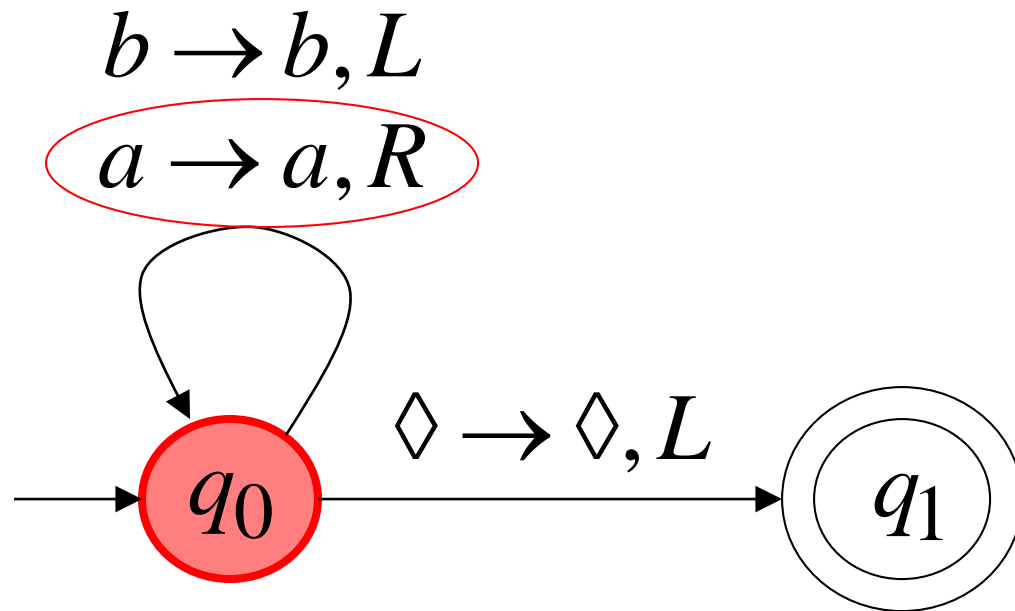
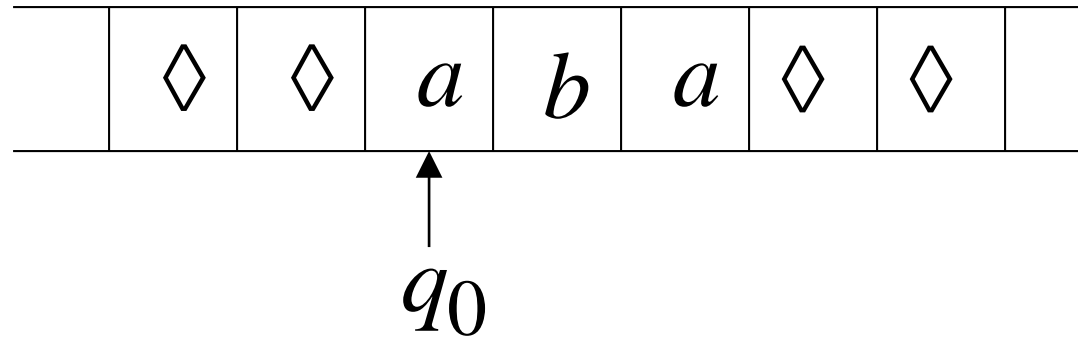
Halt & Reject



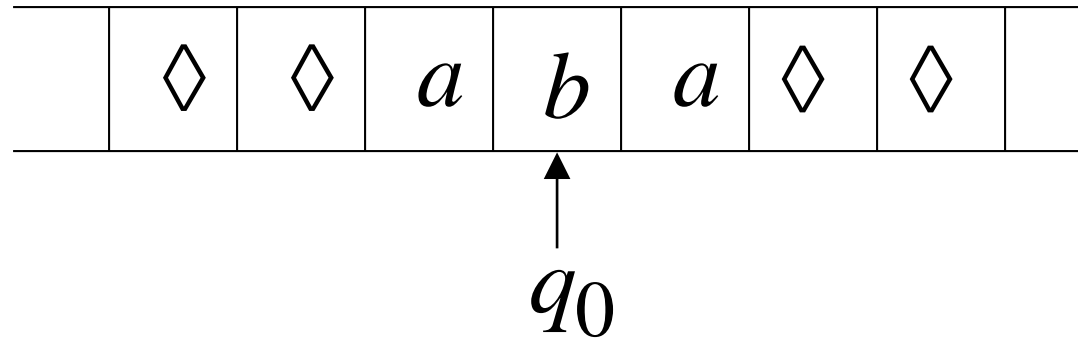
Infinite Loop Example



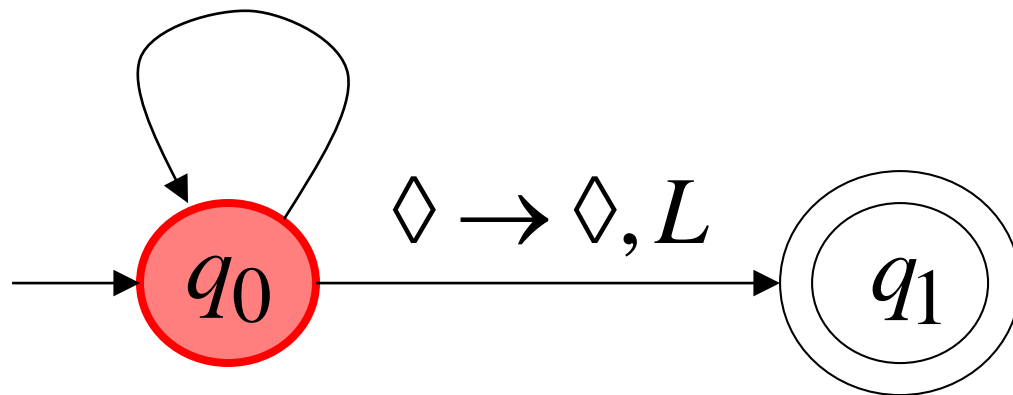
Time 0



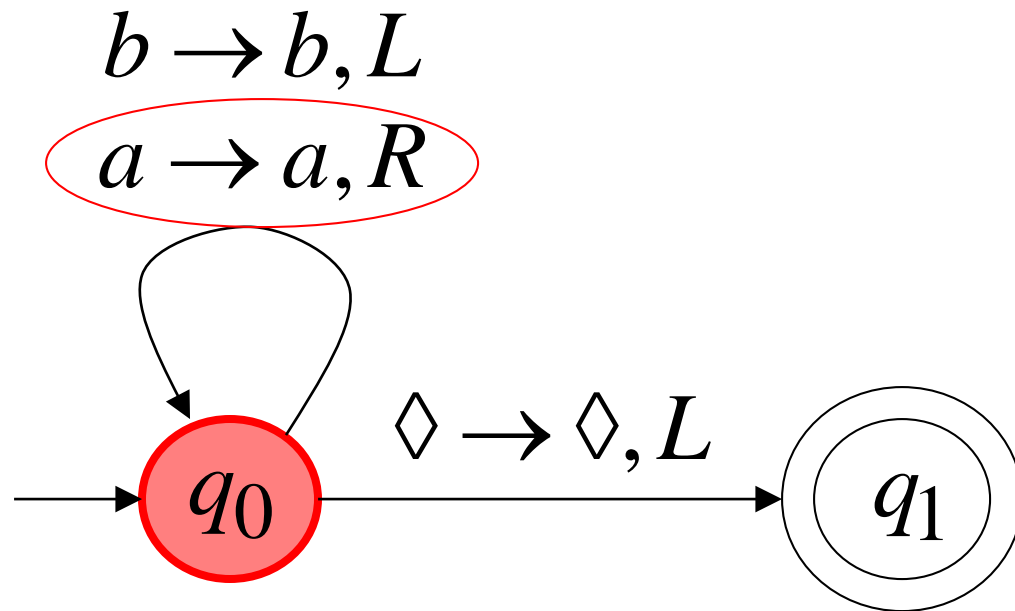
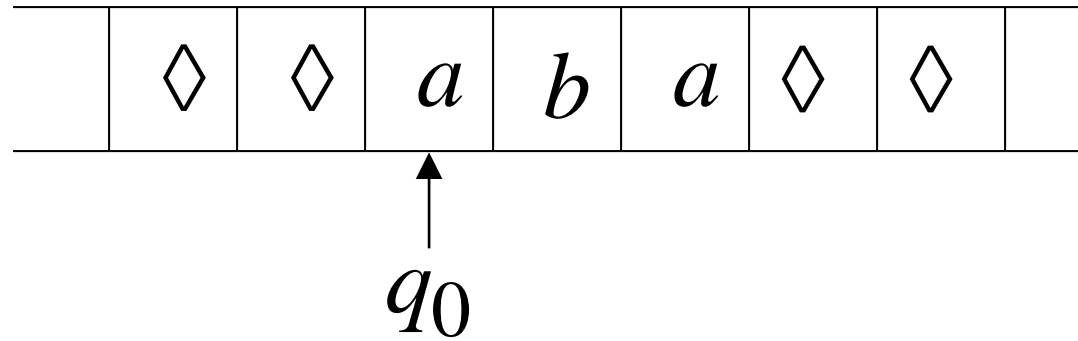
Time 1



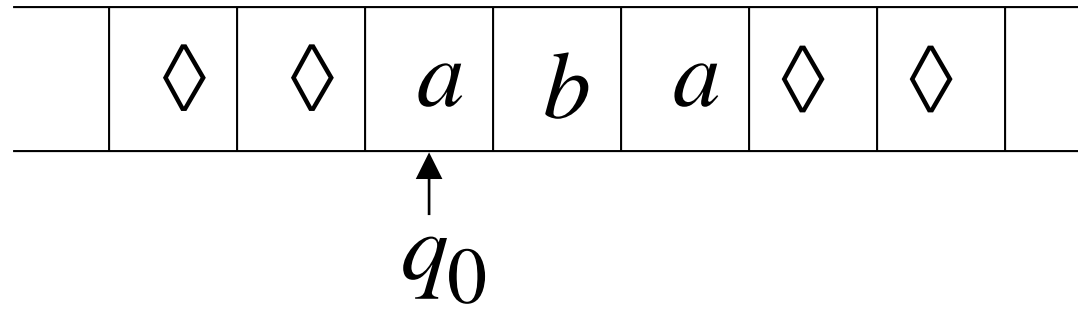
$b \rightarrow b, L$
 $a \rightarrow a, R$



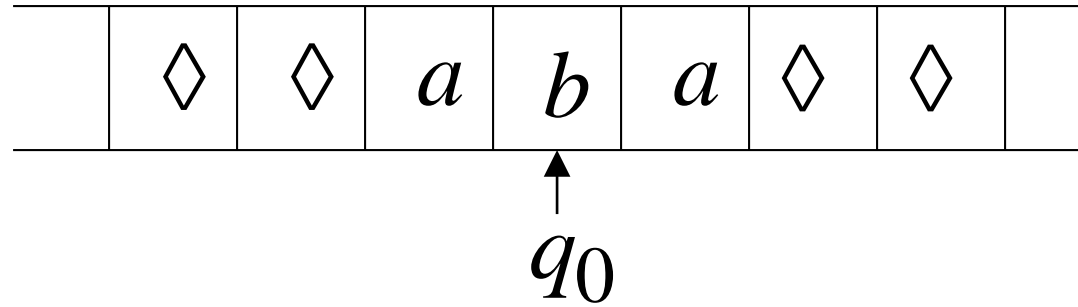
Time 2



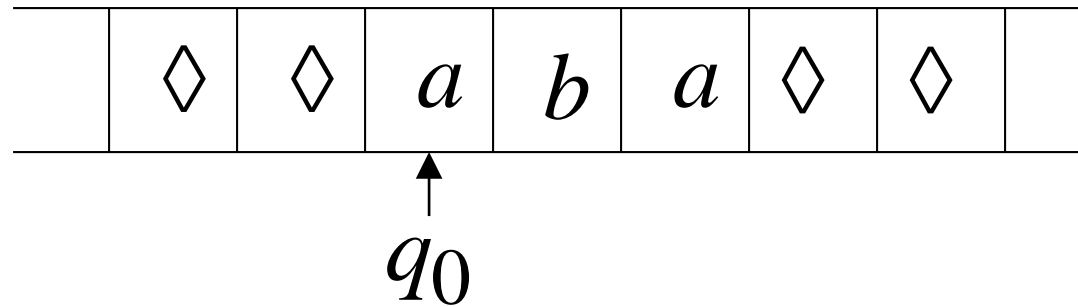
Time 2



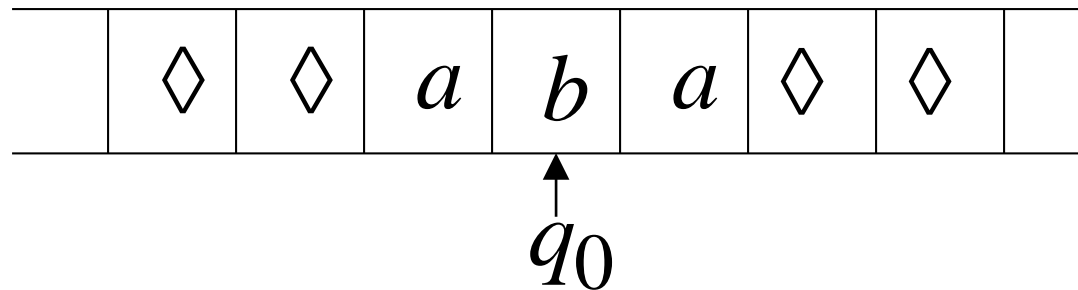
Time 3



Time 4



Time 5



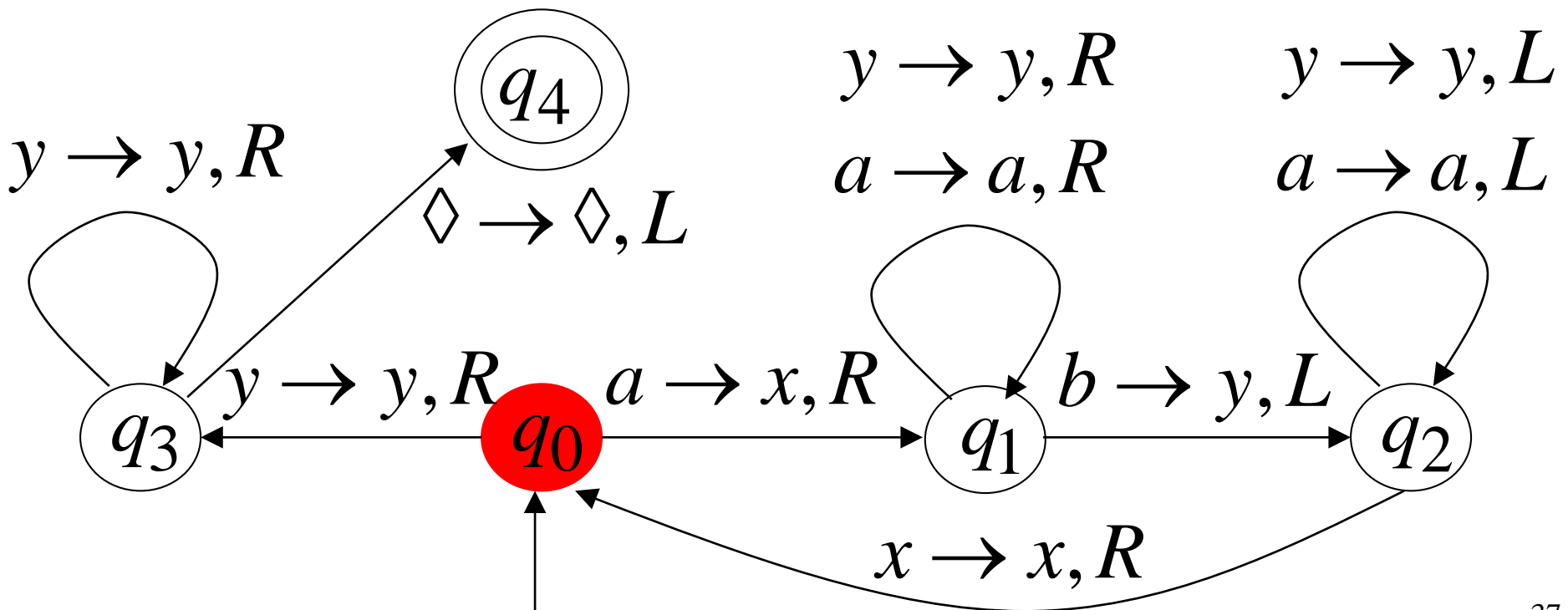
Infinite loop

Because of the **infinite loop**:

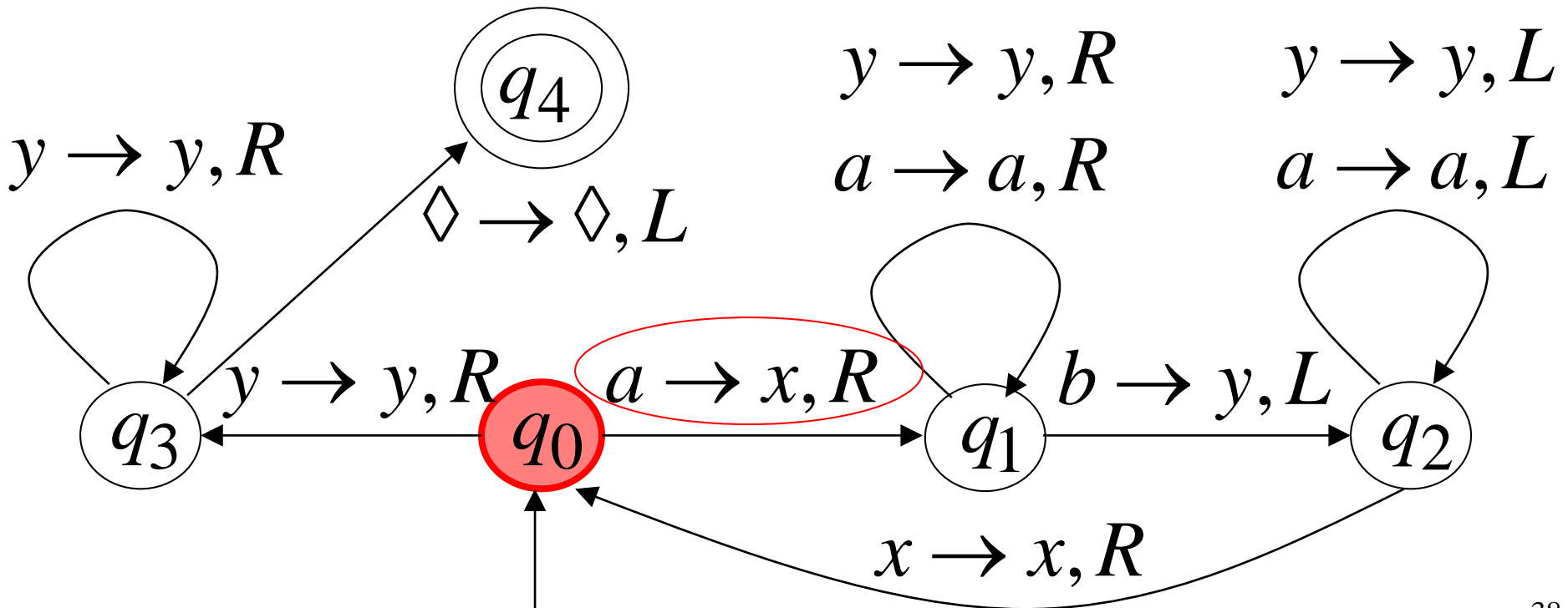
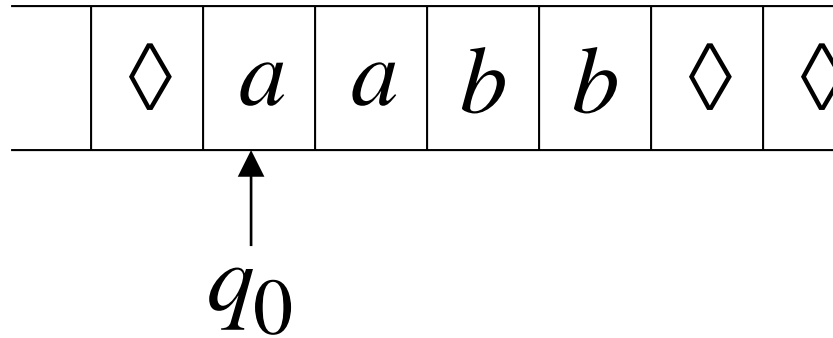
- The final state cannot be reached
- The machine never halts
- The input is **not accepted**

Another Turing Machine Example

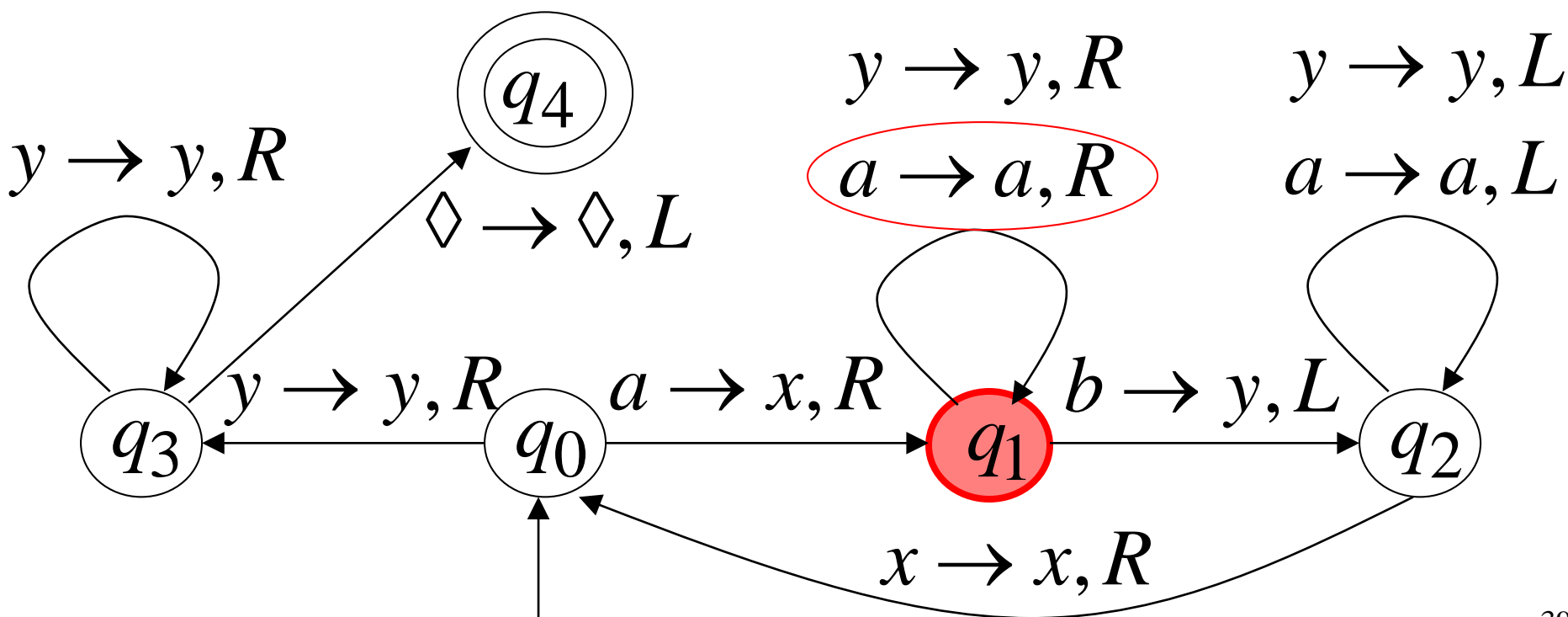
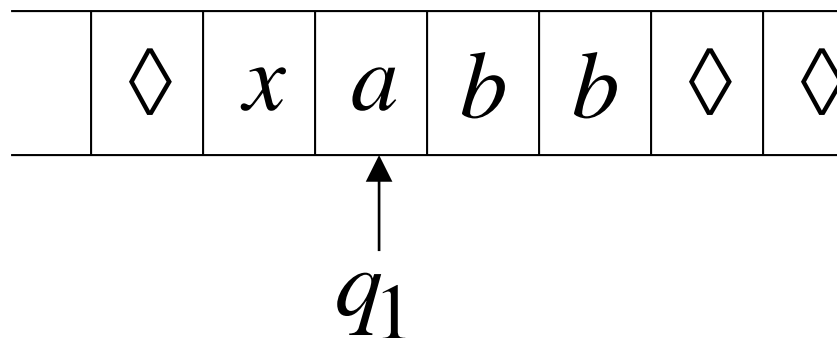
Turing machine for the language $\{a^n b^n\}$



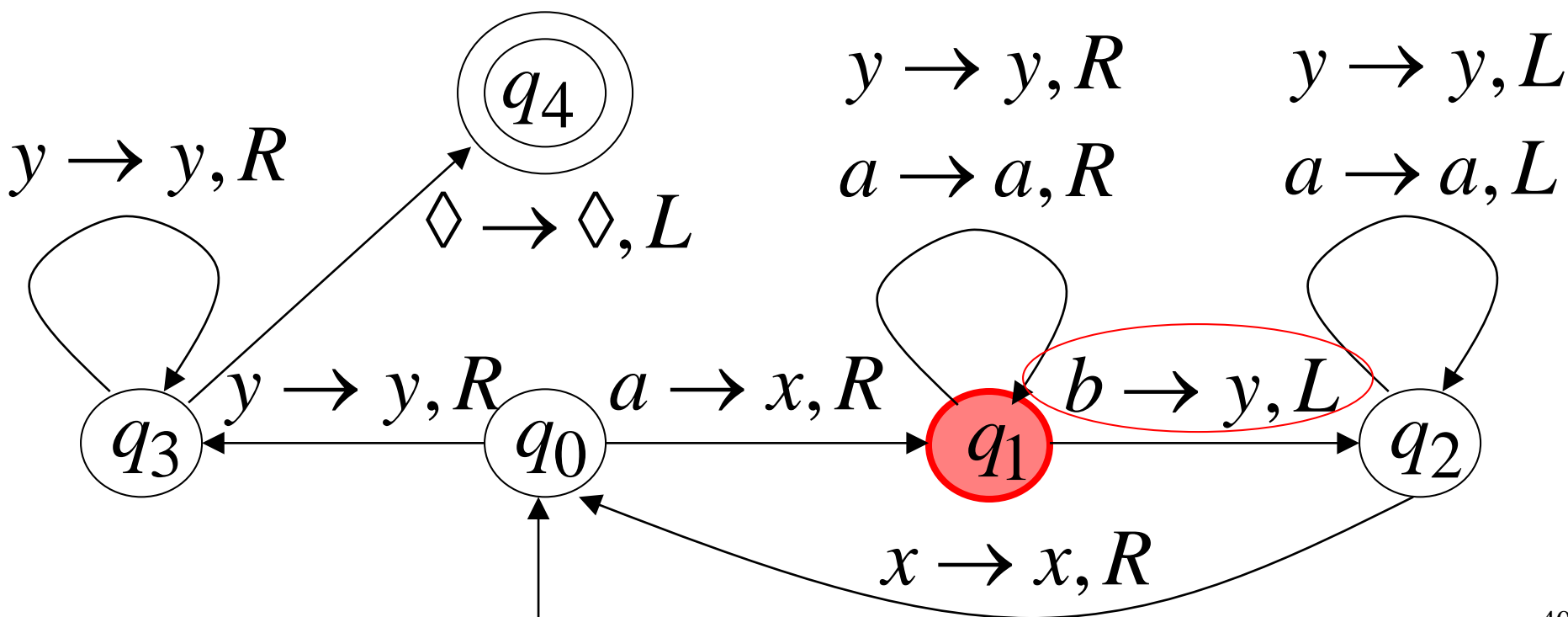
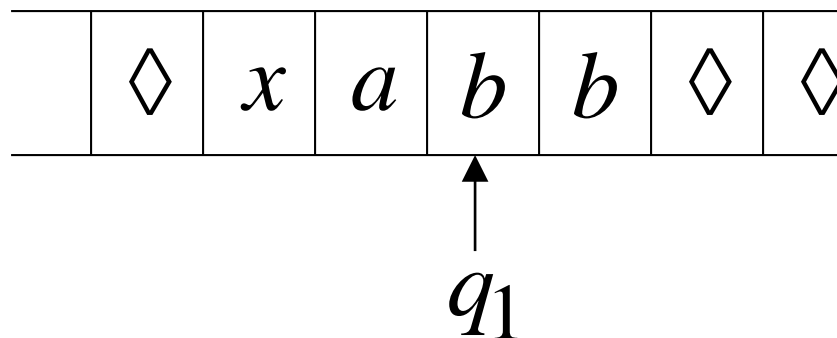
Time 0



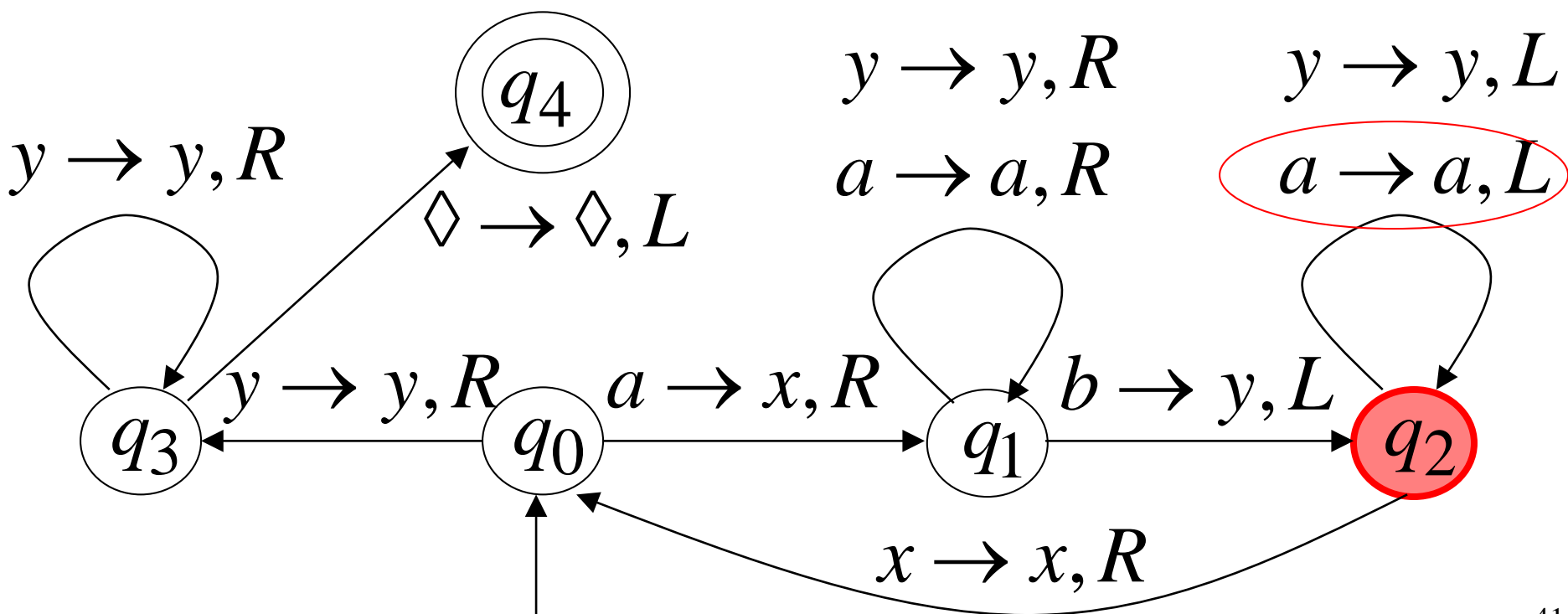
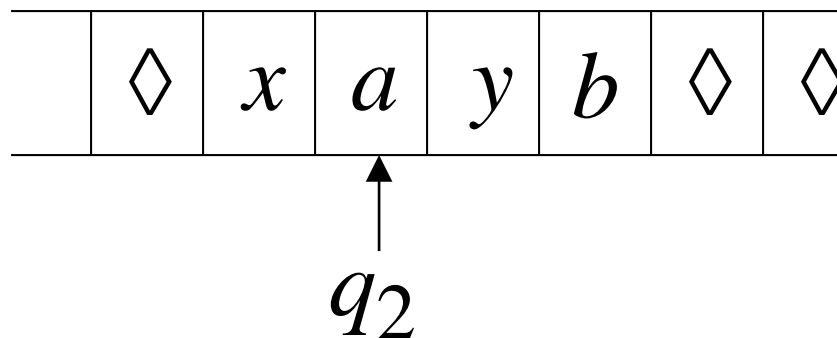
Time 1



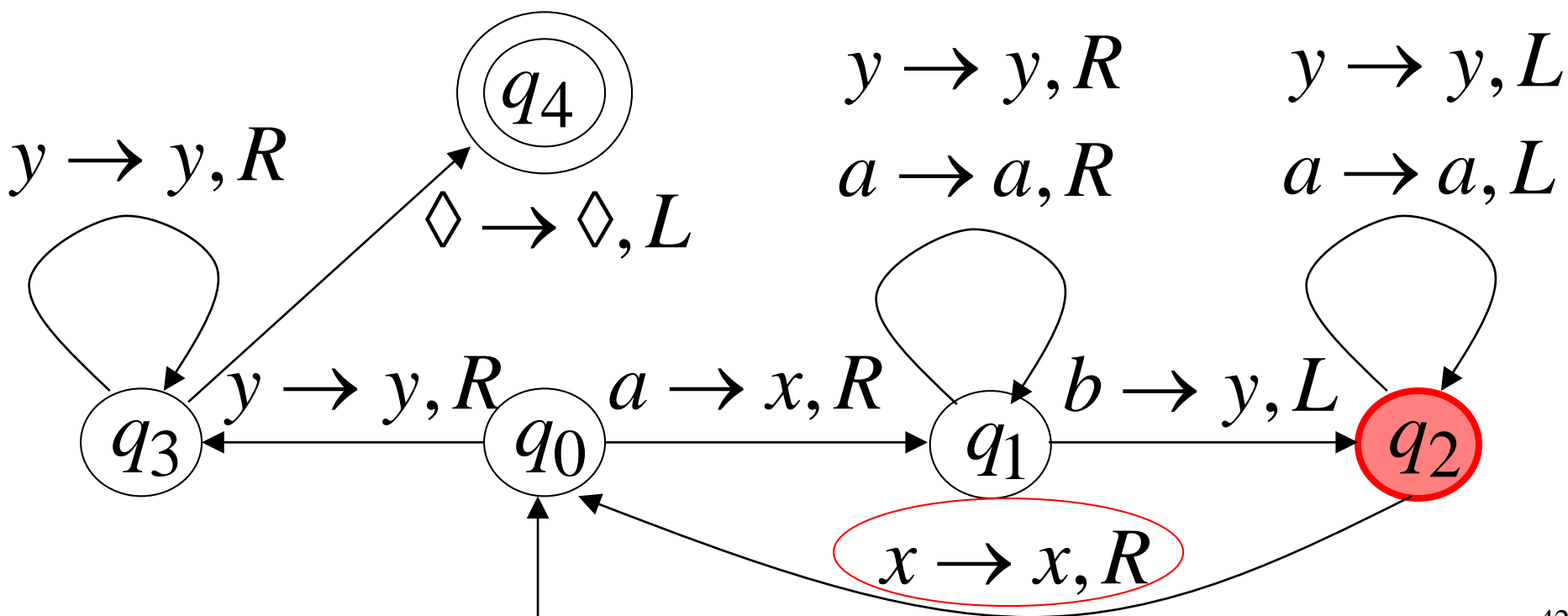
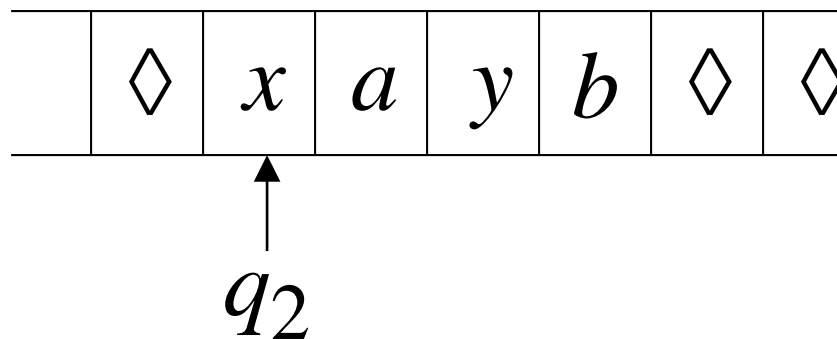
Time 2



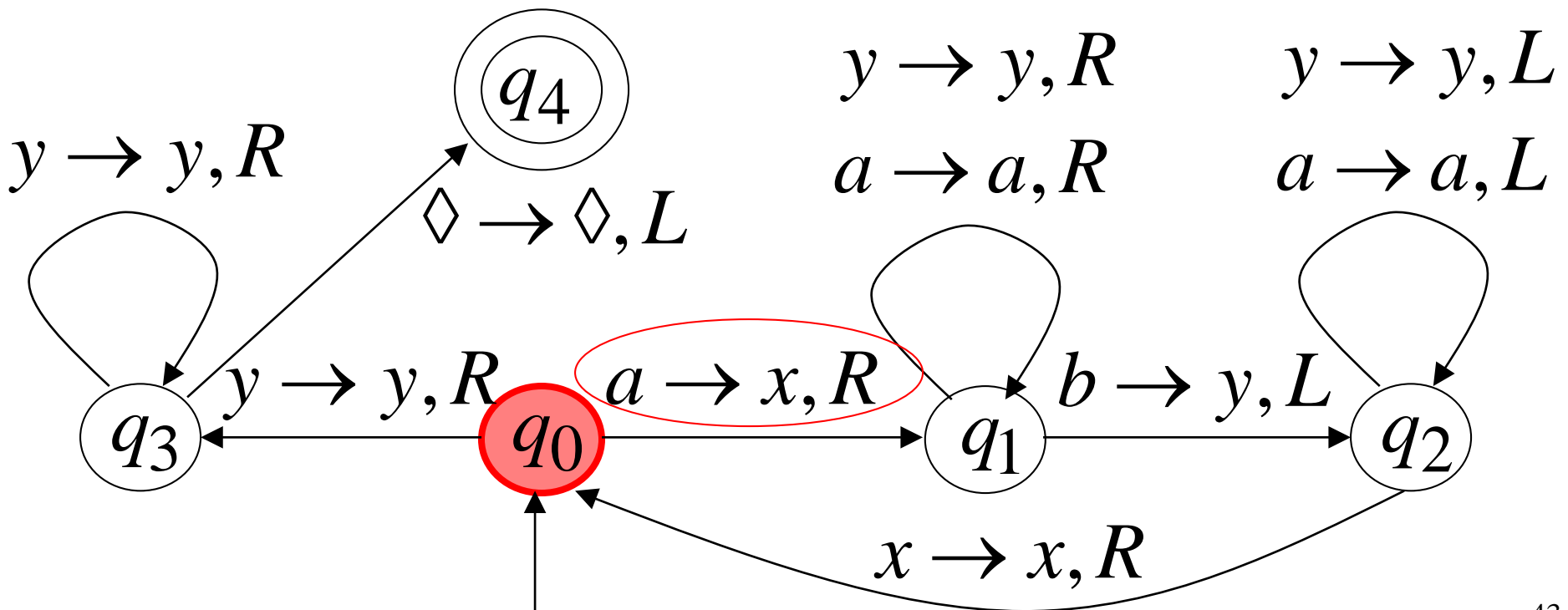
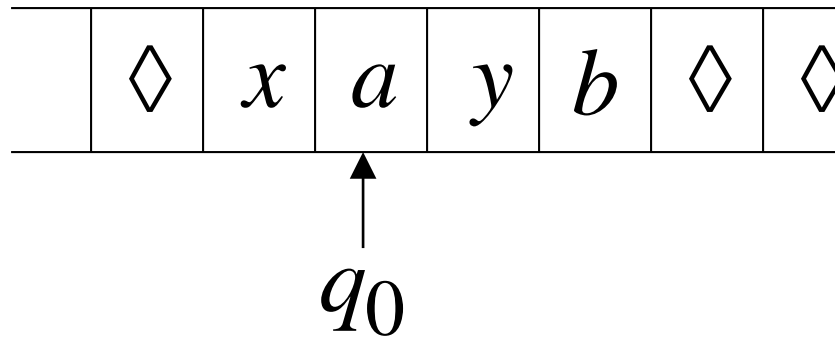
Time 3



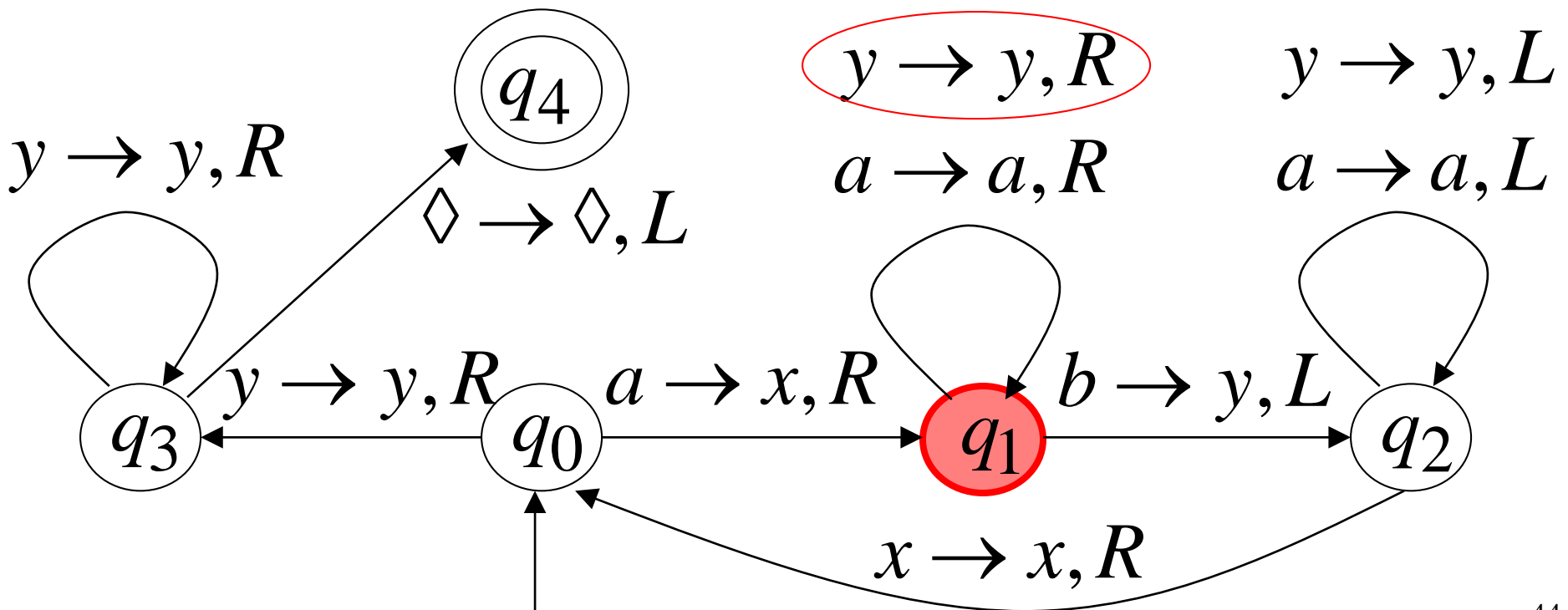
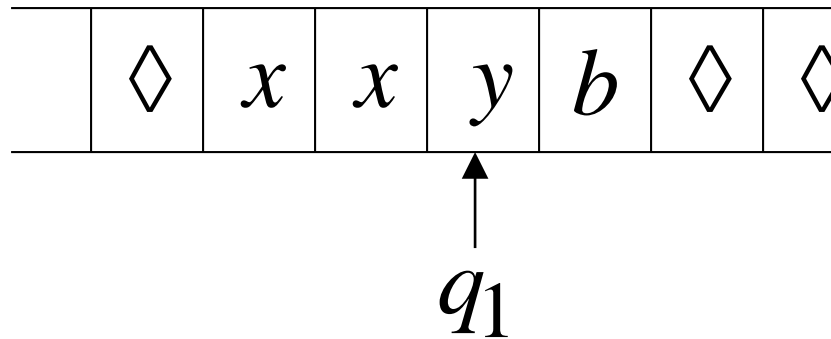
Time 4



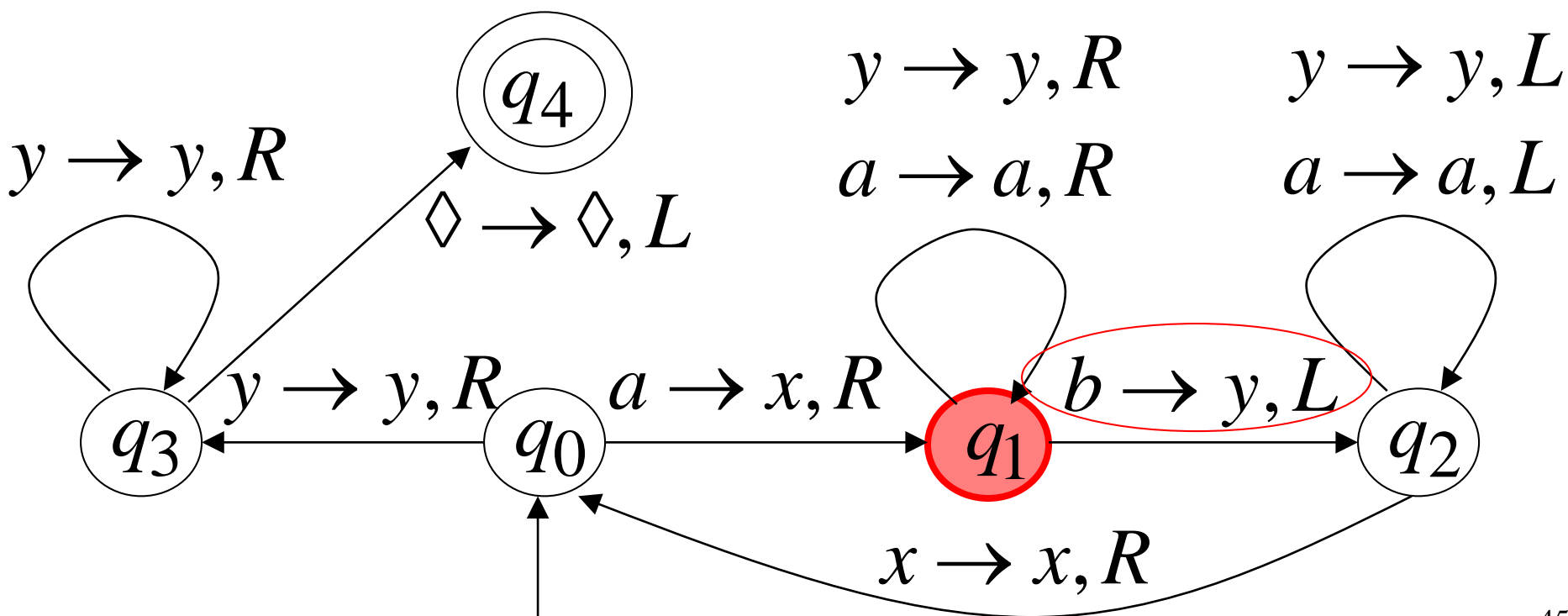
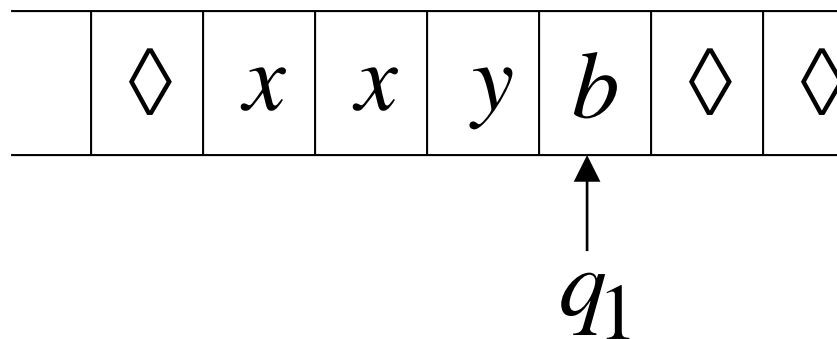
Time 5



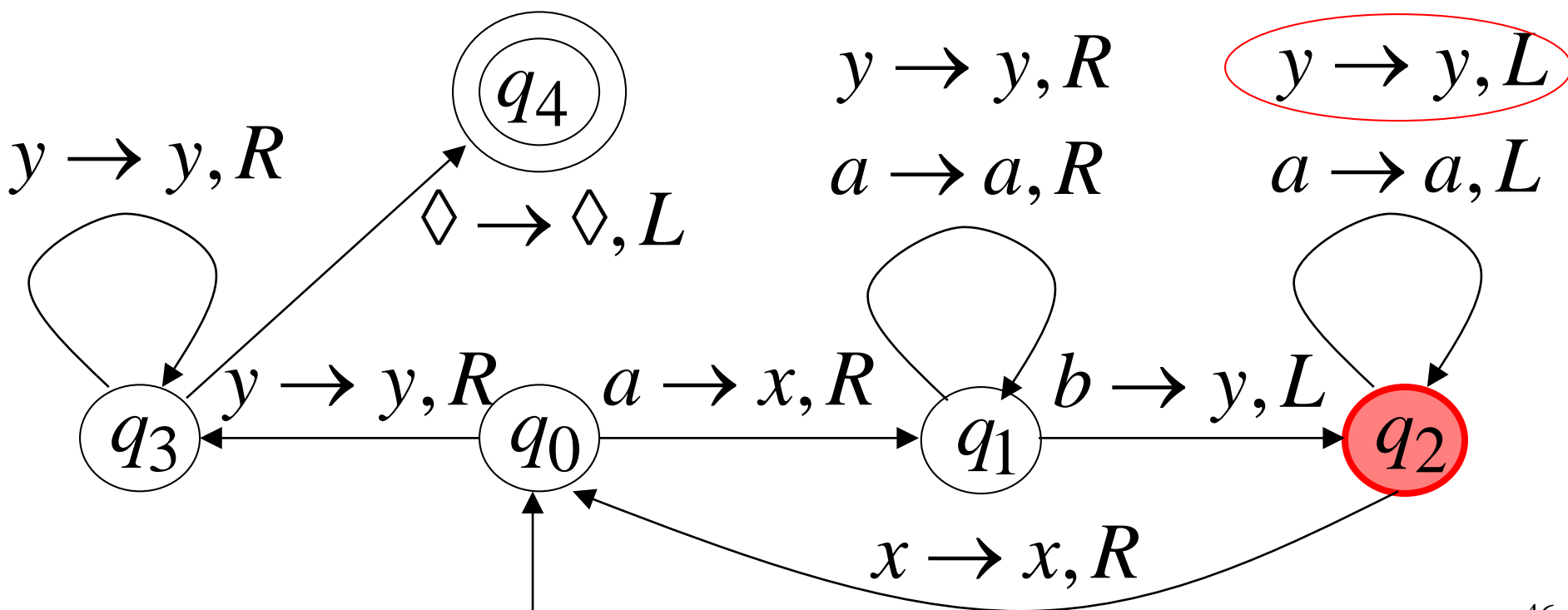
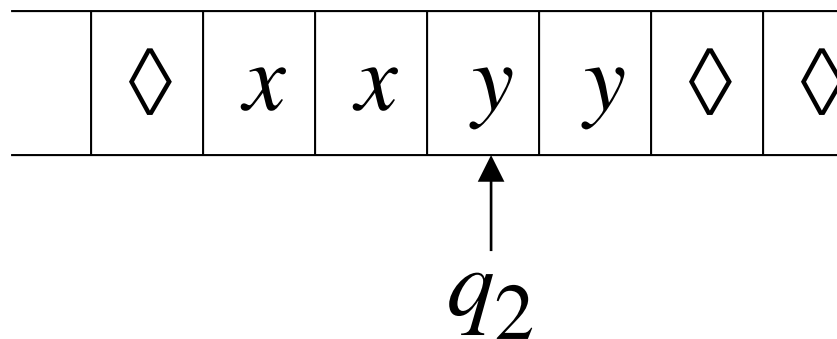
Time 6



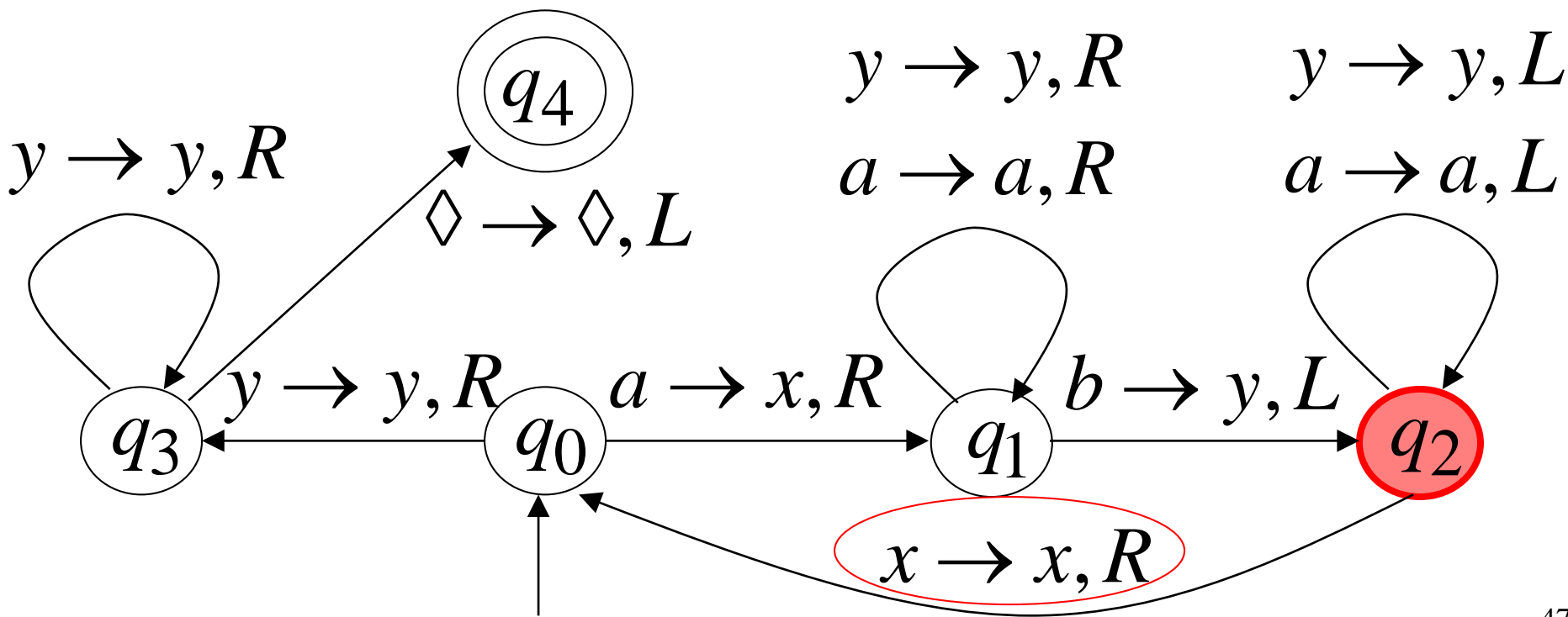
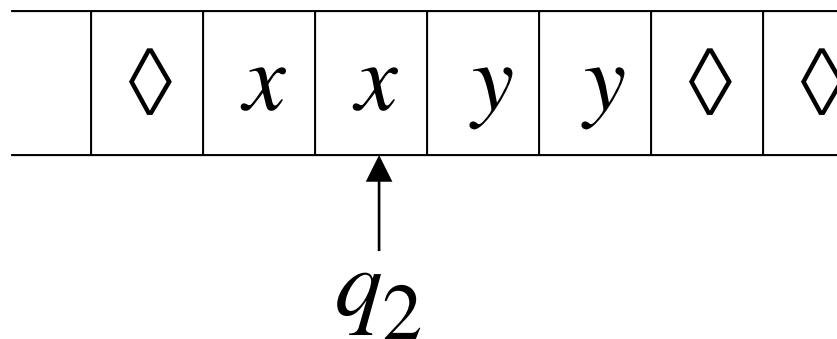
Time 7



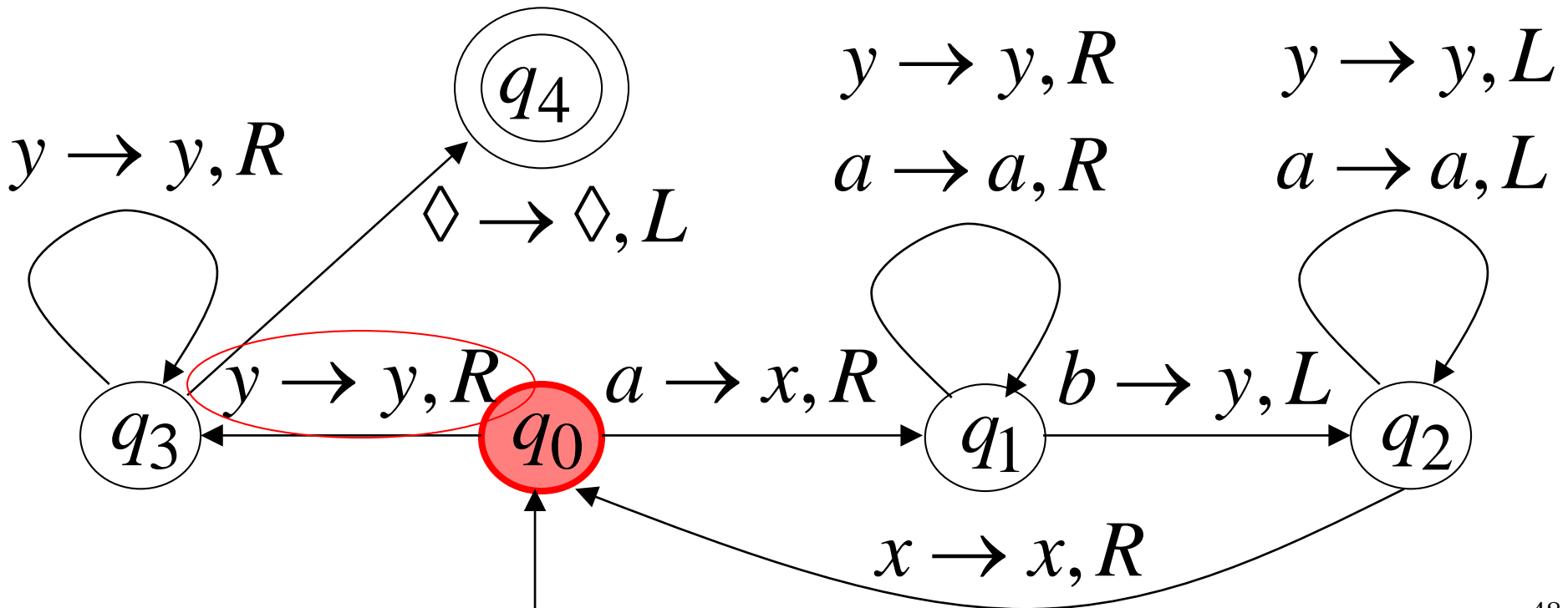
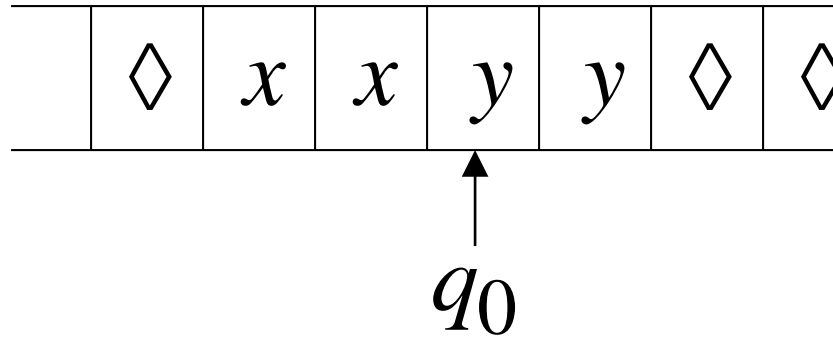
Time 8



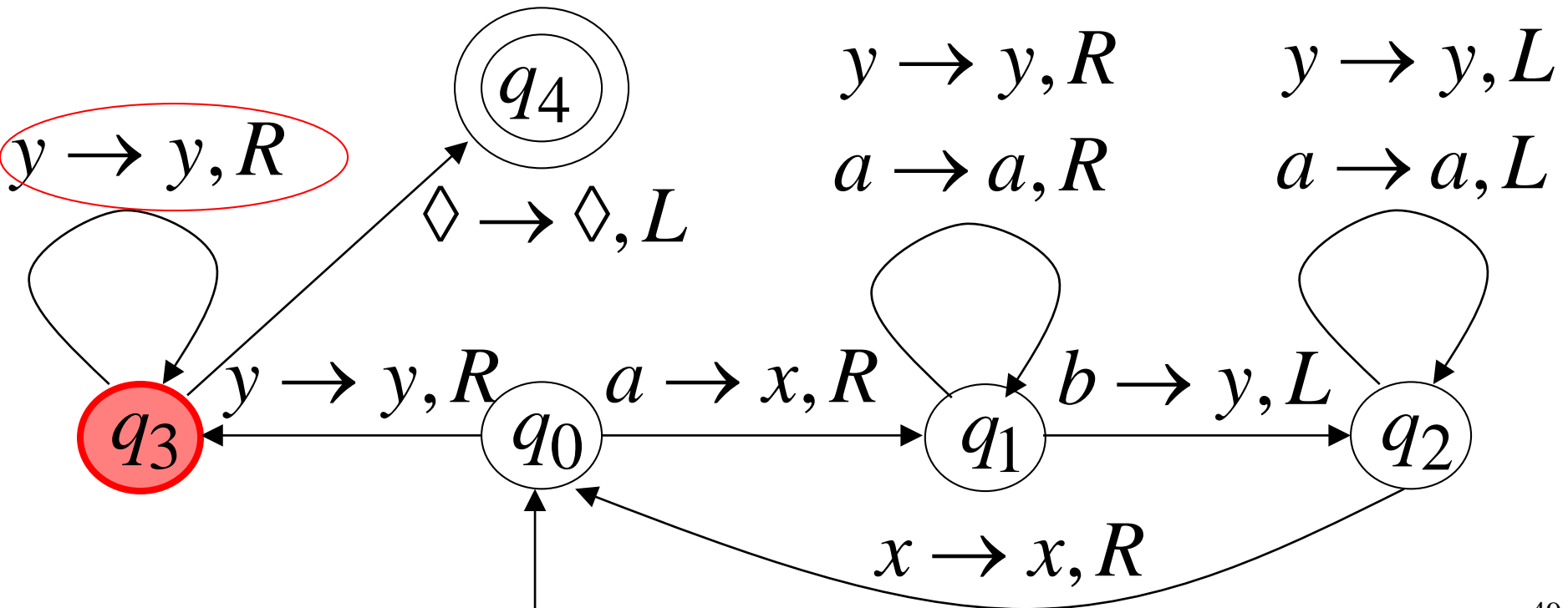
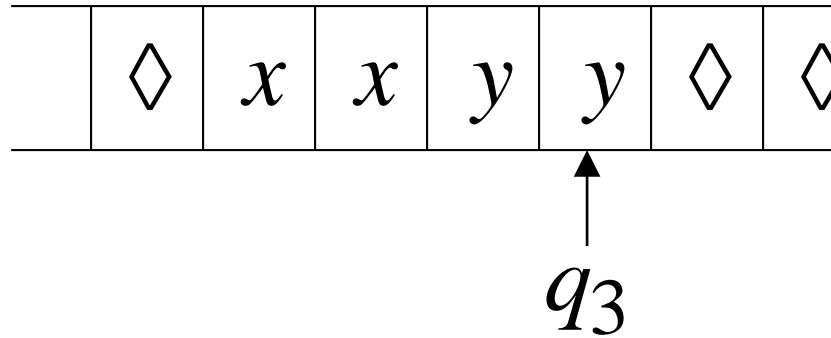
Time 9



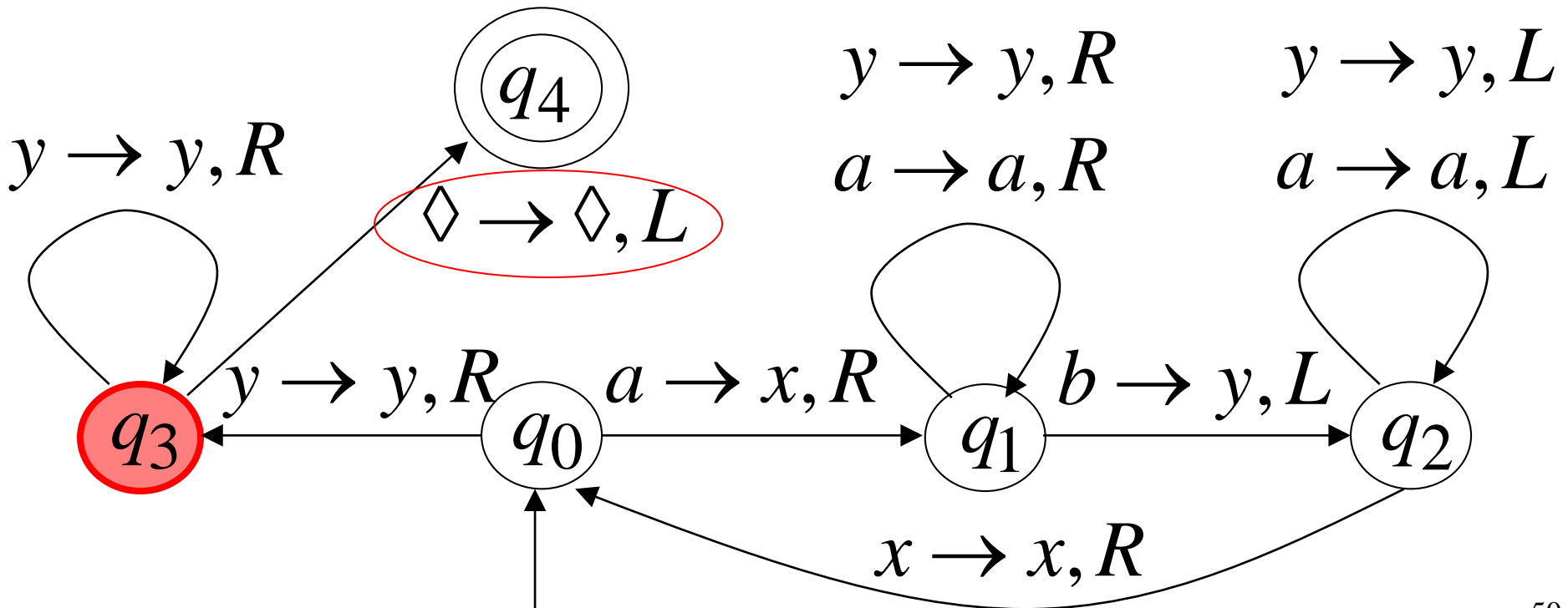
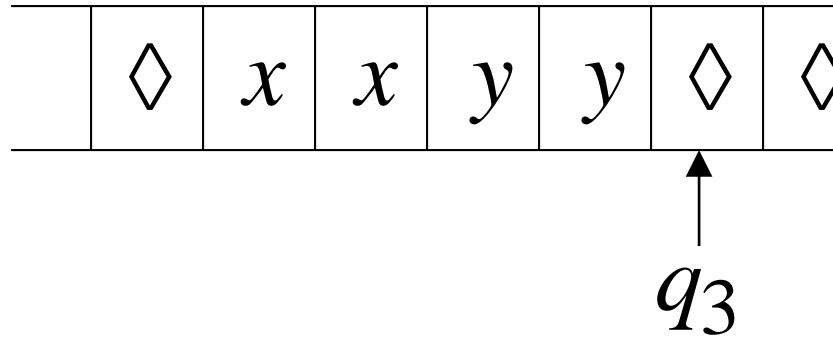
Time 10



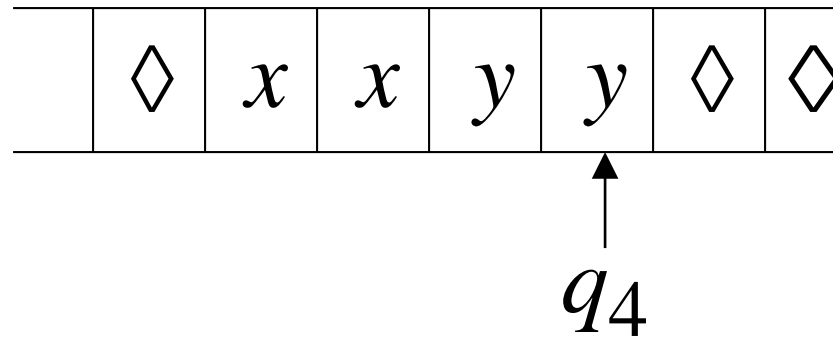
Time 11



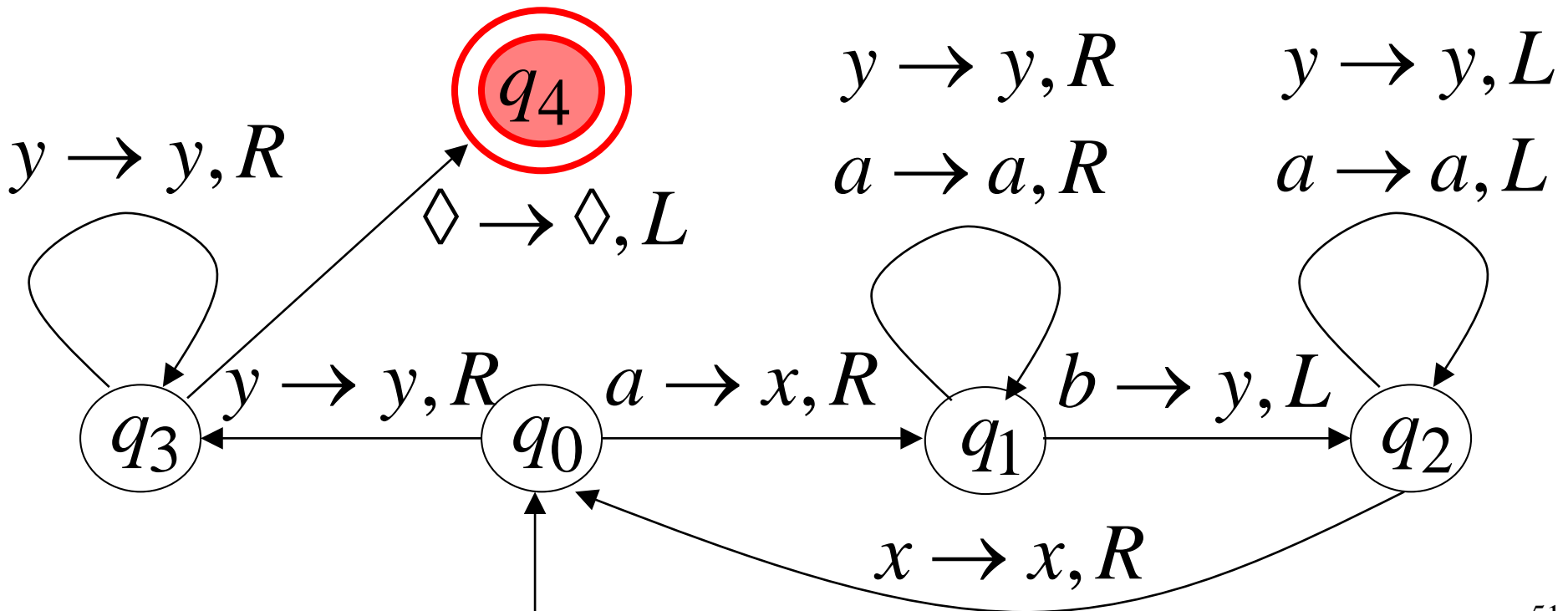
Time 12



Time 13



Halt & Accept



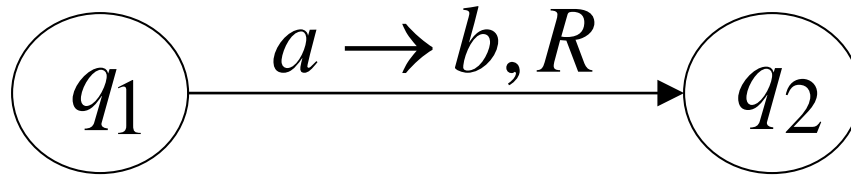
Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

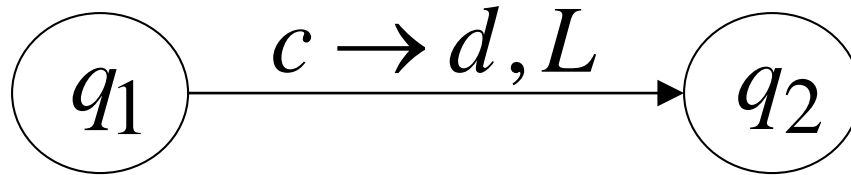
Formal Definitions for Turing Machines

Transition Function



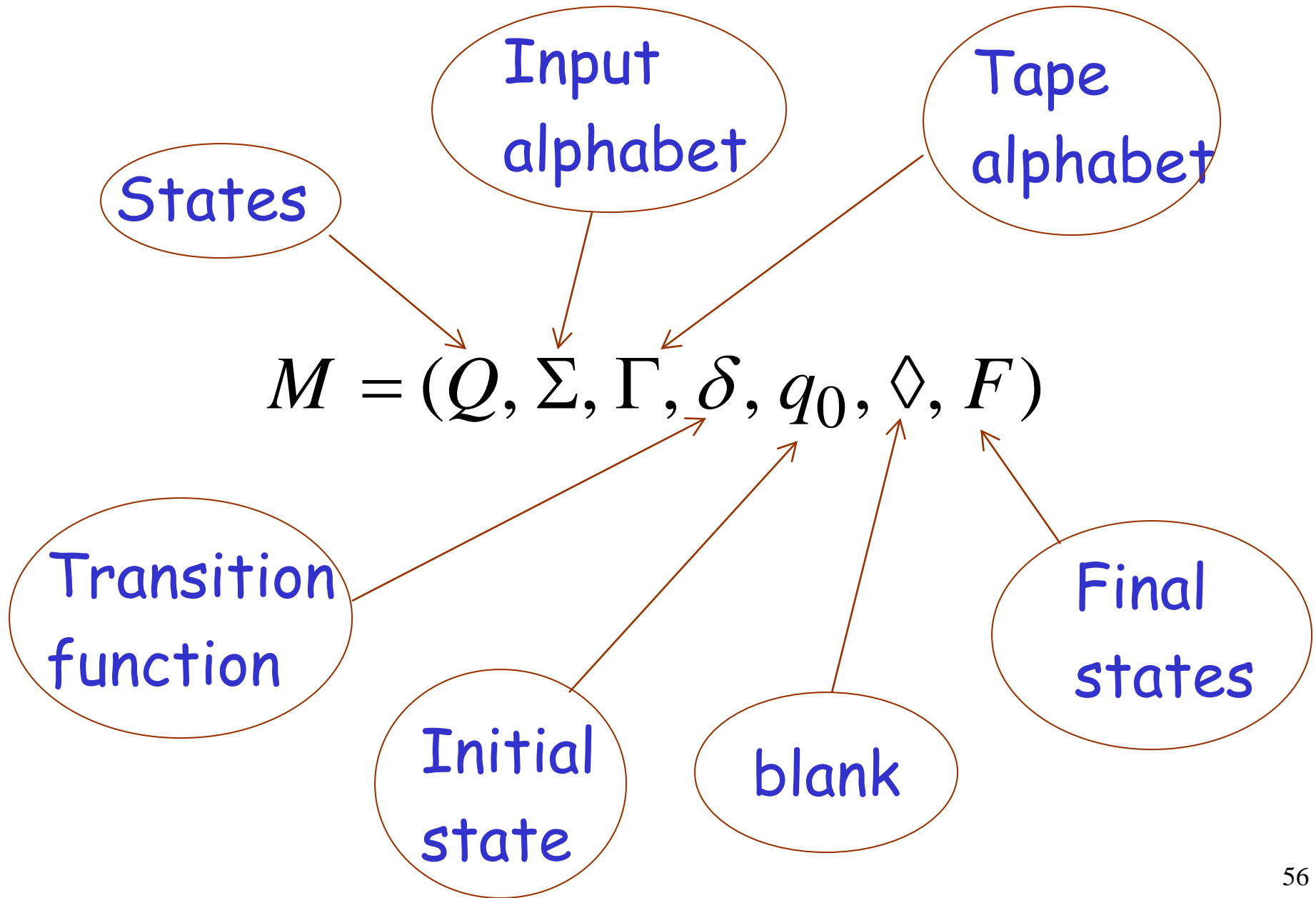
$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function

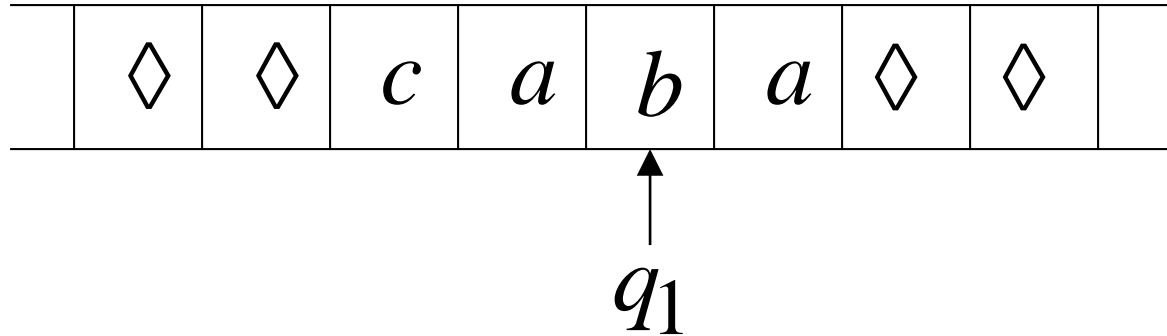


$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

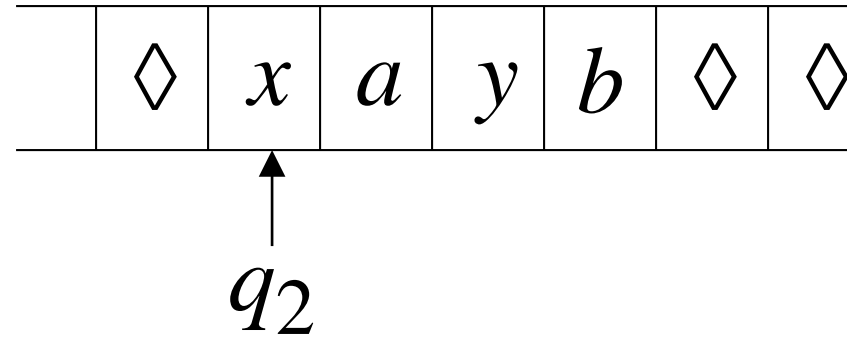


Configuration

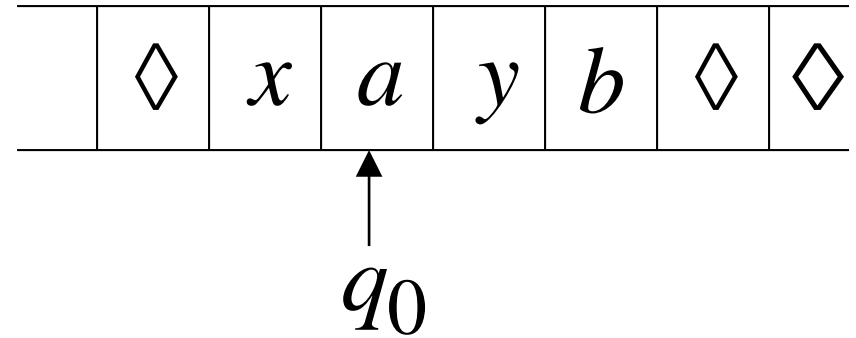


Instantaneous description: $ca\ q_1\ ba$

Time 4

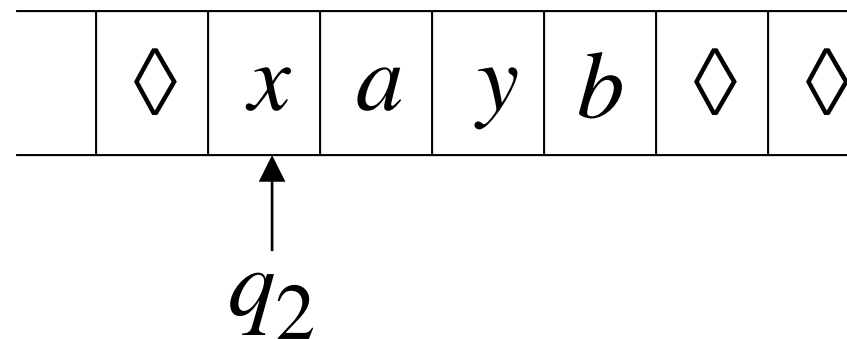


Time 5

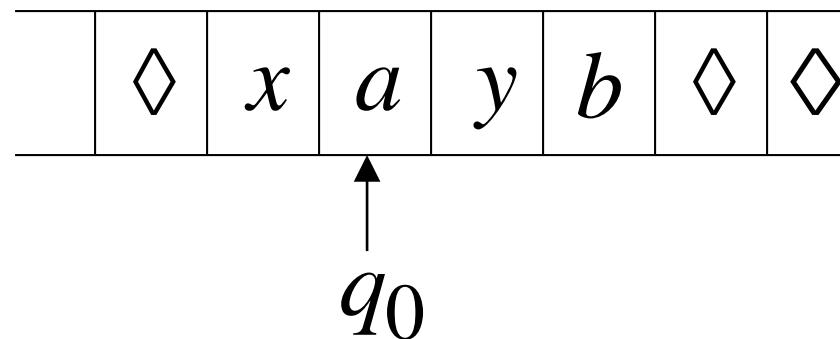


A Move: $q_2 xayb \succ x q_0 ayb$

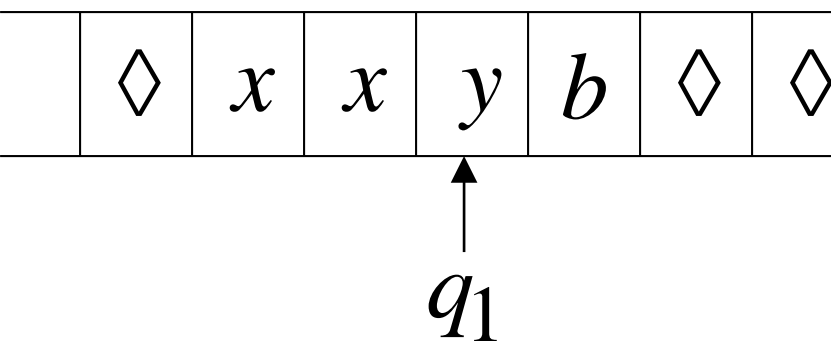
Time 4



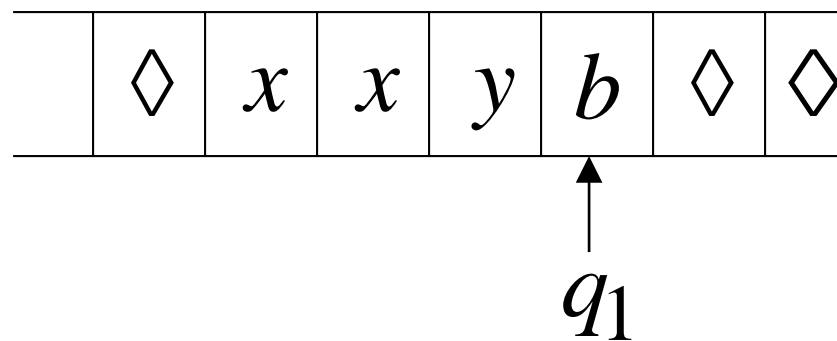
Time 5



Time 6



Time 7



$$q_2 \ x a y b \succ x \ q_0 \ a y b \succ x x \ q_1 \ y b \succ x x y \ q_1 \ b$$

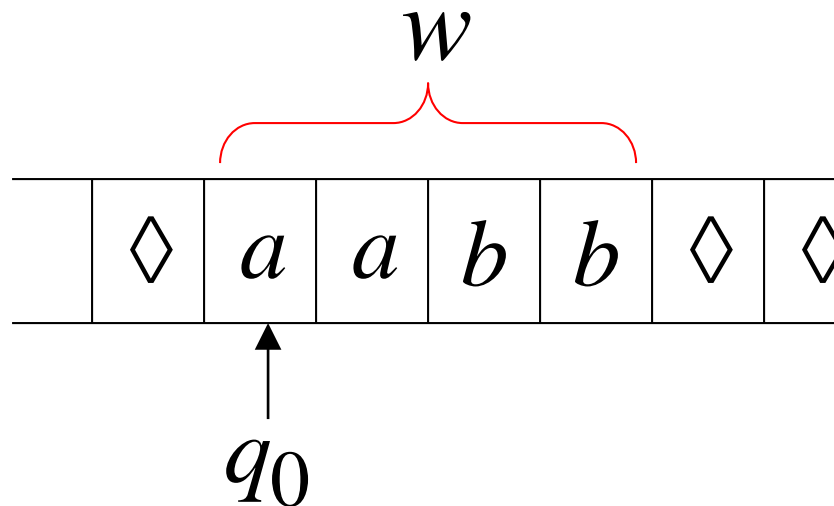
$$q_2 \ x a y b \succ x \ q_0 \ a y b \succ x x \ q_1 \ y b \succ x x y \ q_1 \ b$$

Equivalent notation:

$$q_2 \ x a y b \overset{*}{\succ} x x y \ q_1 \ b$$

Initial configuration: $q_0 w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : q_0 w \stackrel{*}{\rightarrow} x_1 q_f x_2\}$$



Initial state



Final state

Standard Turing Machine

The machine we described is the standard:

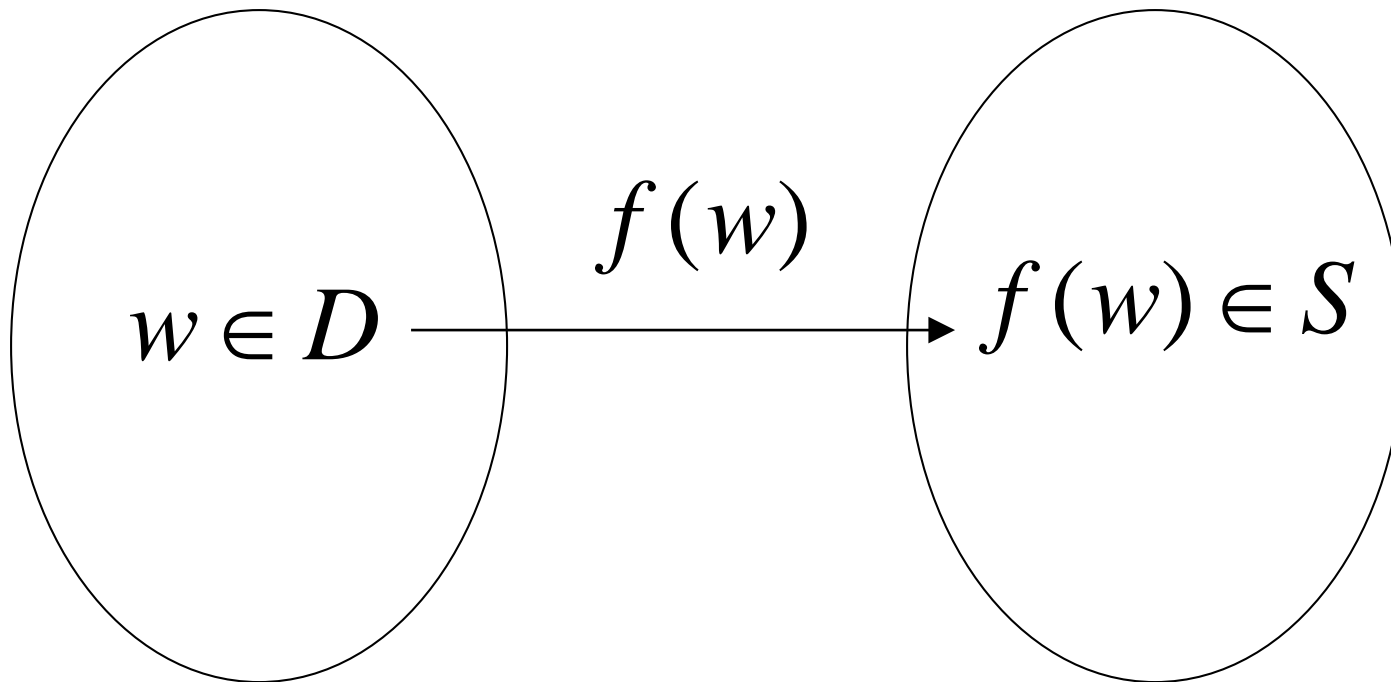
- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 1111

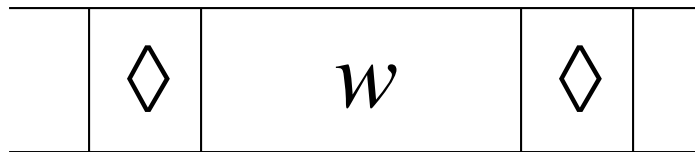
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

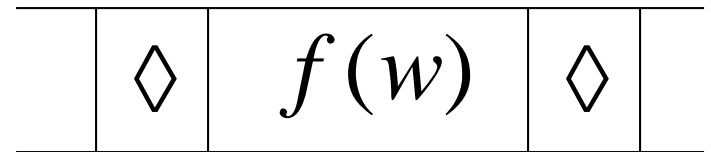
A function f is computable if
there is a Turing Machine M such that:

Initial configuration



q_0 initial state

Final configuration



q_f final state

For all $w \in D$ Domain

In other words:

A function f is computable if
there is a Turing Machine M such that:

$$q_0 w \xrightarrow{*} q_f f(w)$$

Initial

Configuration

Final

Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

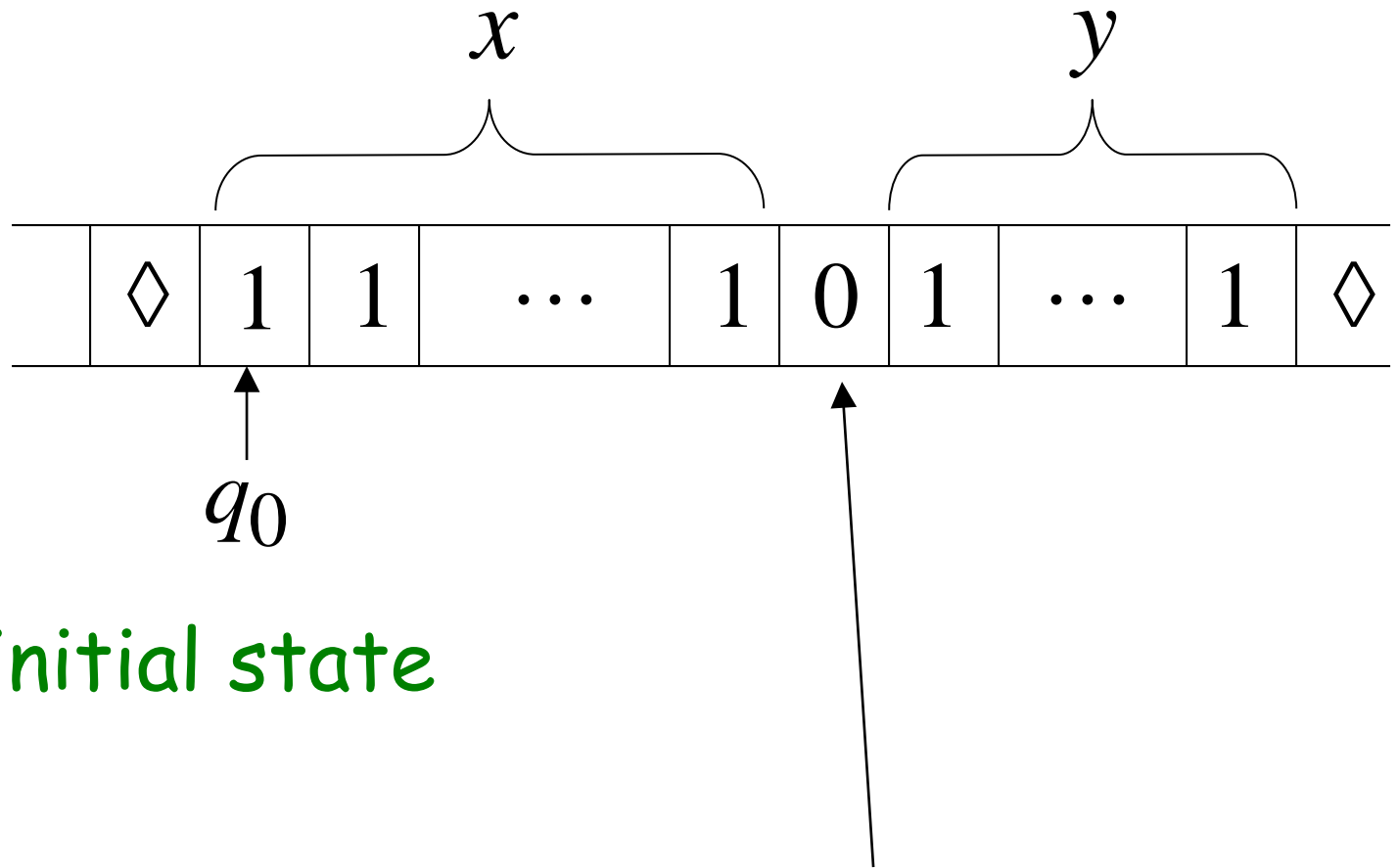
x, y are integers

Turing Machine:

Input string: $x0y$ unary

Output string: $xy0$ unary

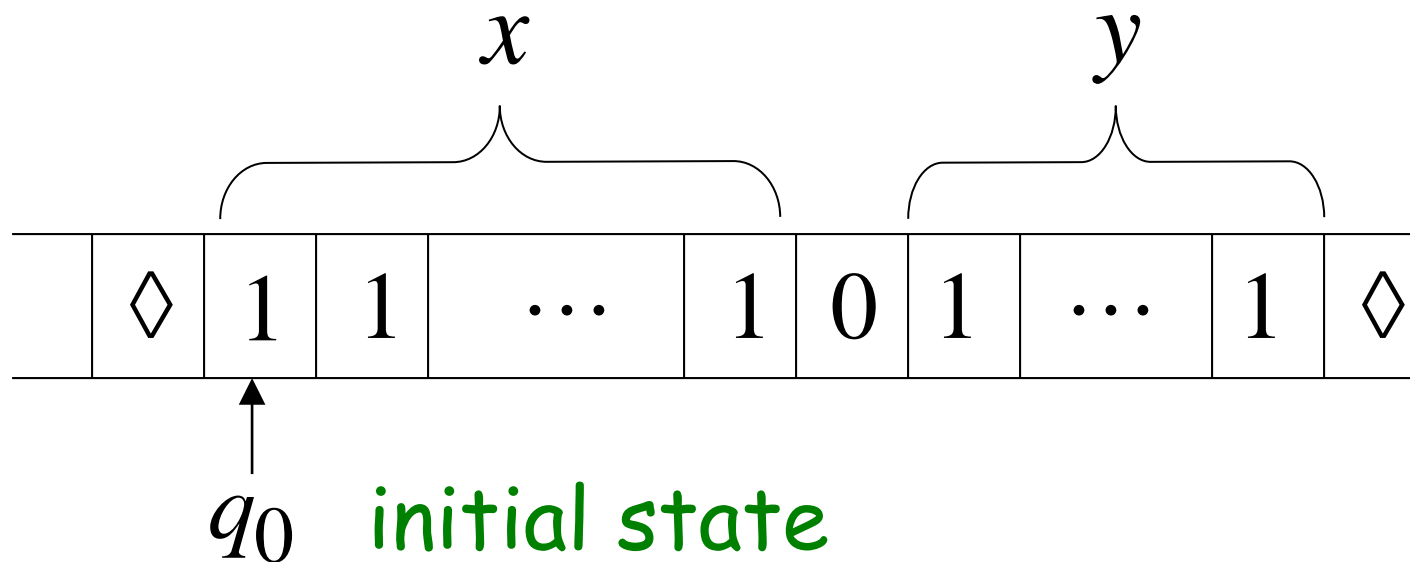
Start



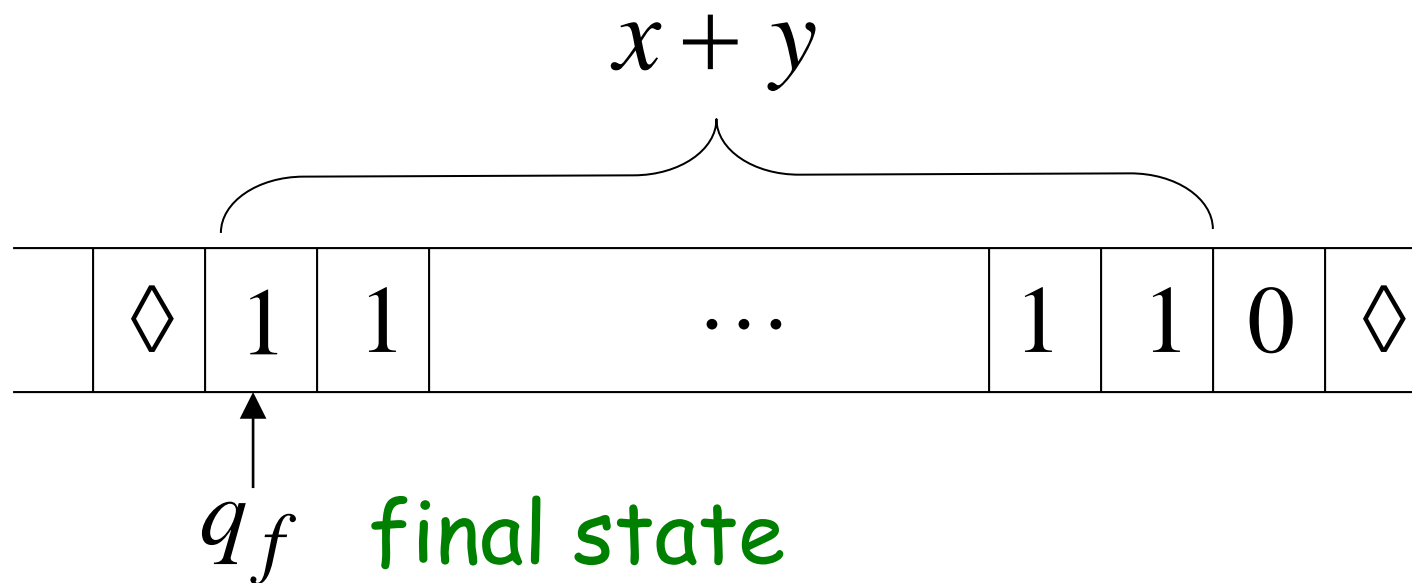
initial state

The 0 is the delimiter that separates the two numbers

Start

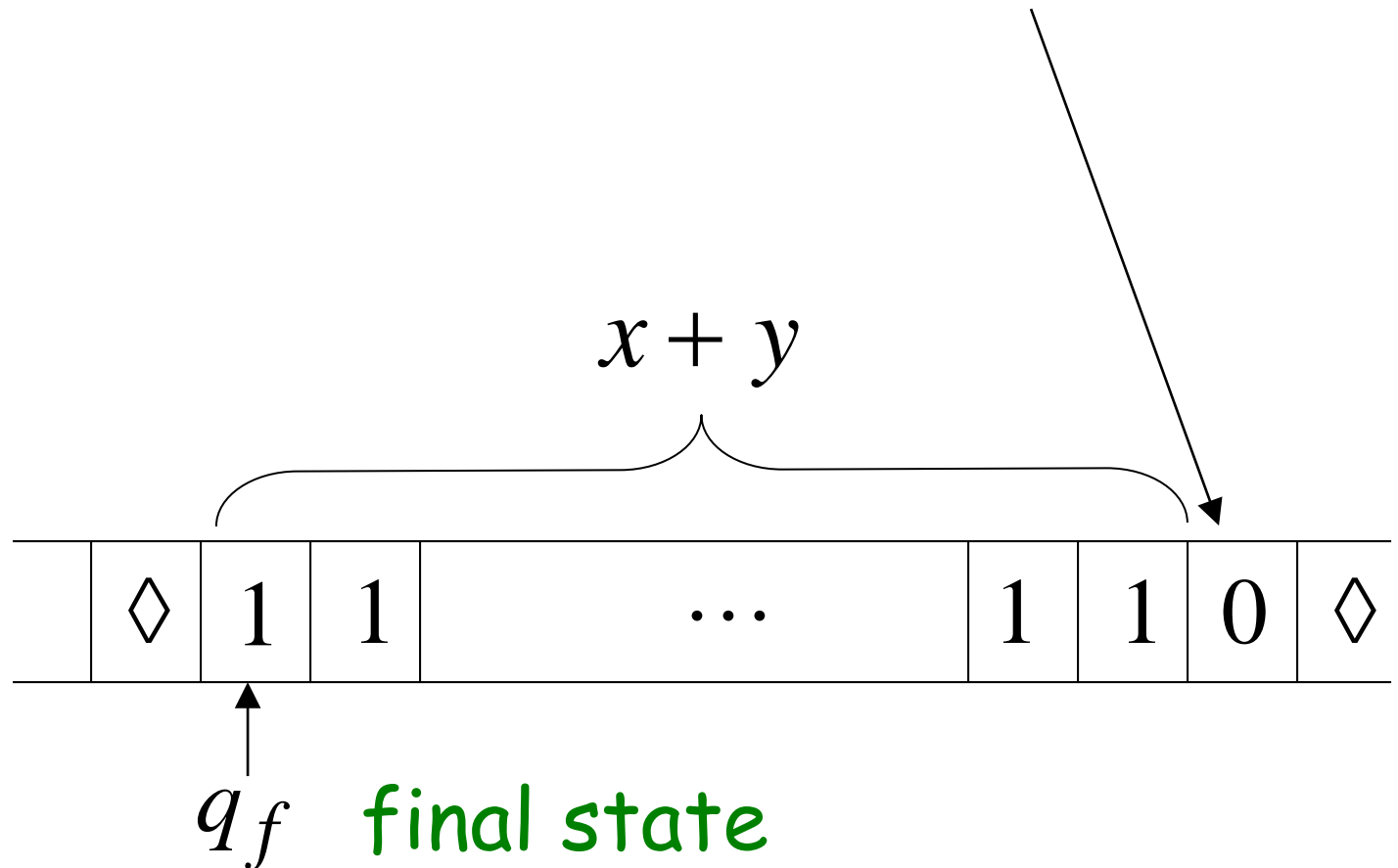


Finish

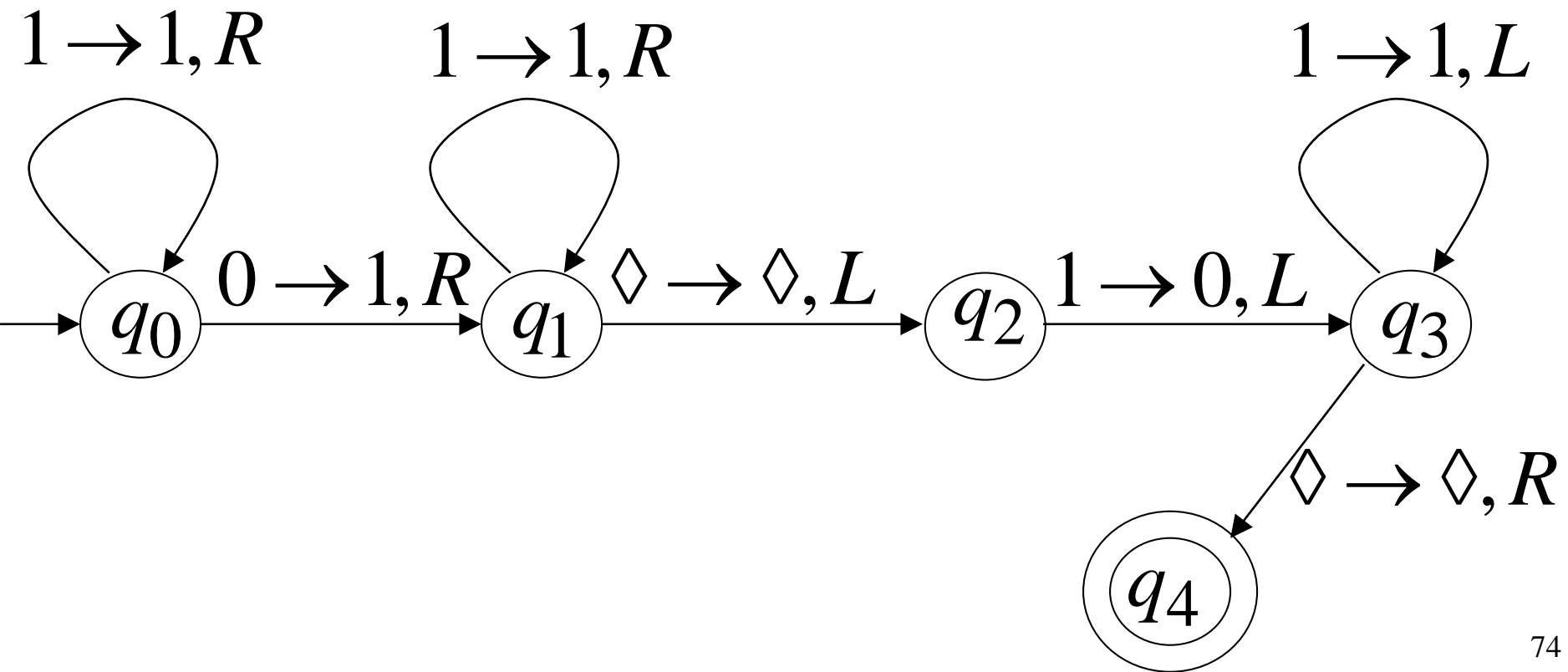


The 0 helps when we use
the result for other operations

Finish



Turing machine for function $f(x, y) = x + y$

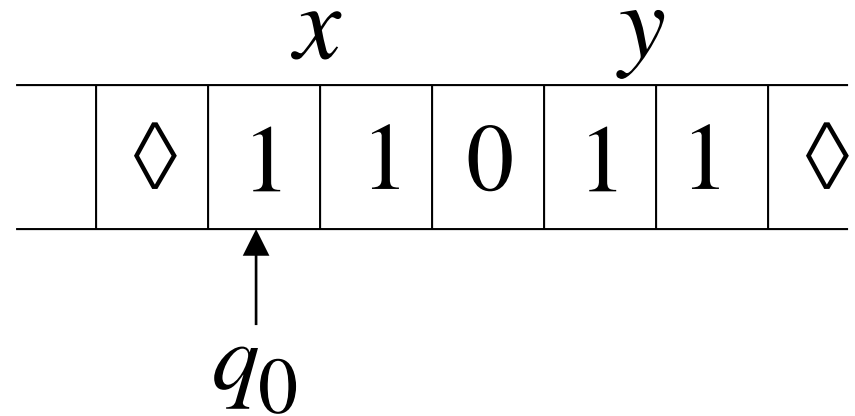


Execution Example:

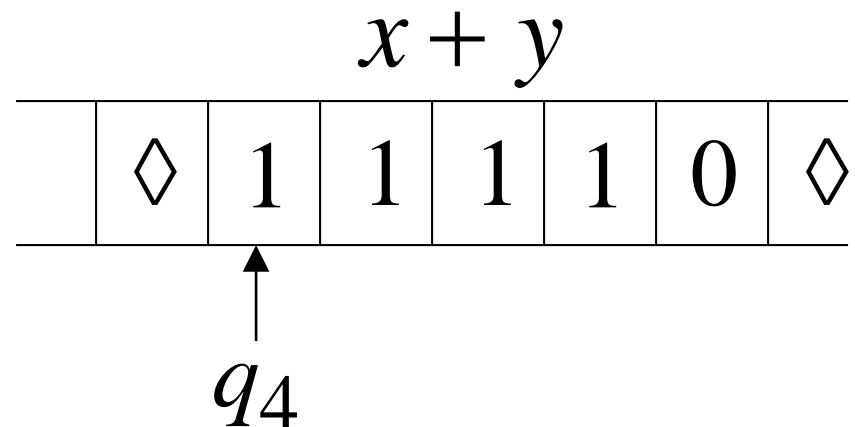
$$x = 11 \quad (2)$$

$$y = 11 \quad (2)$$

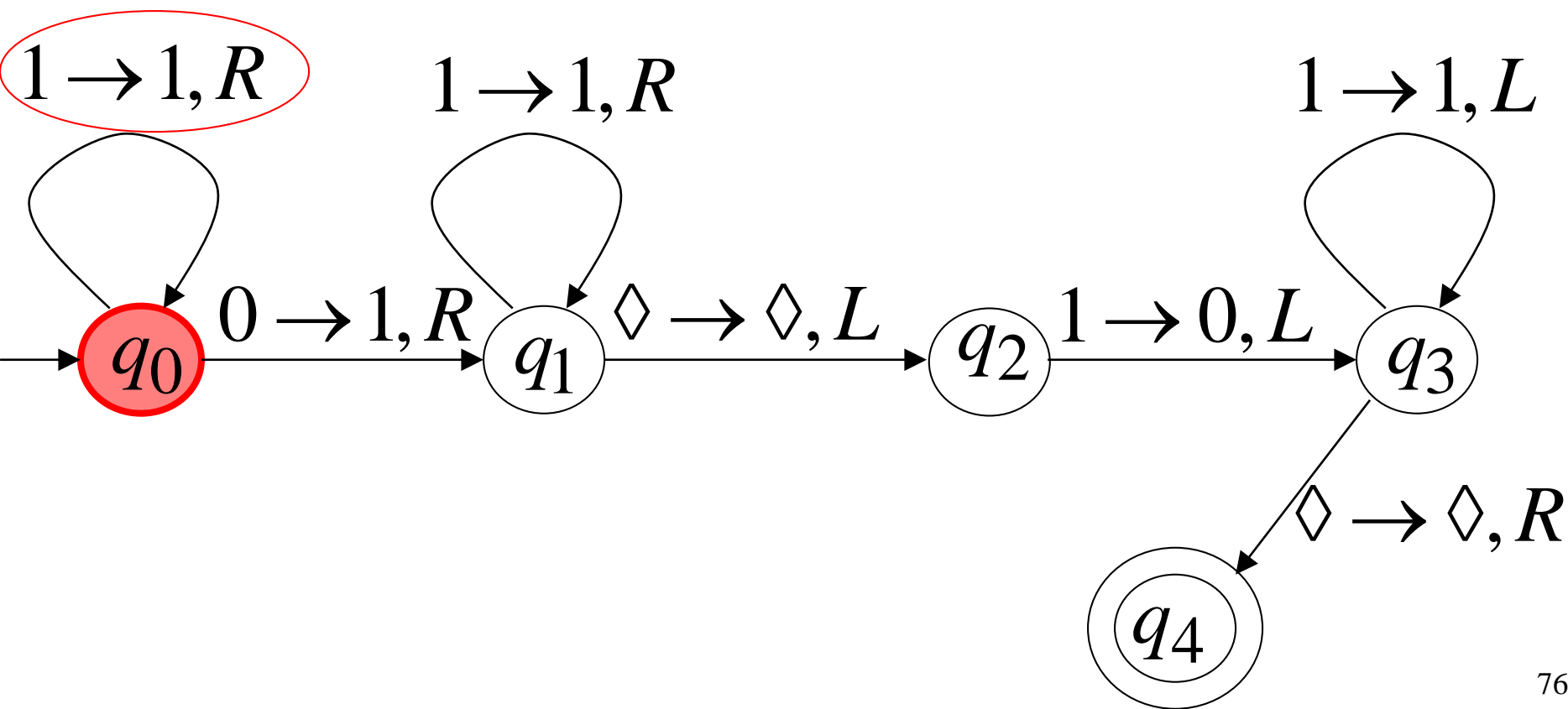
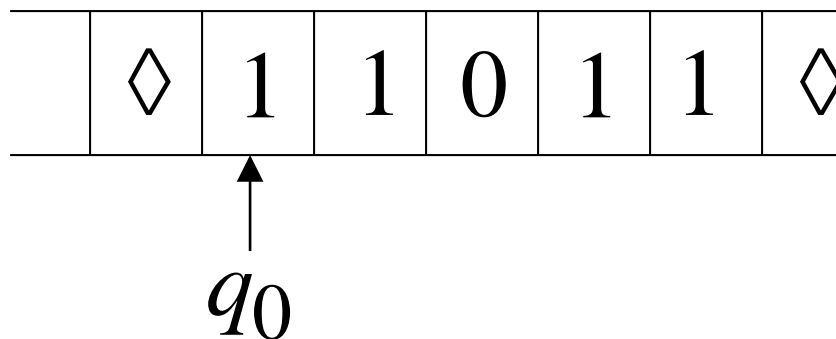
Time 0



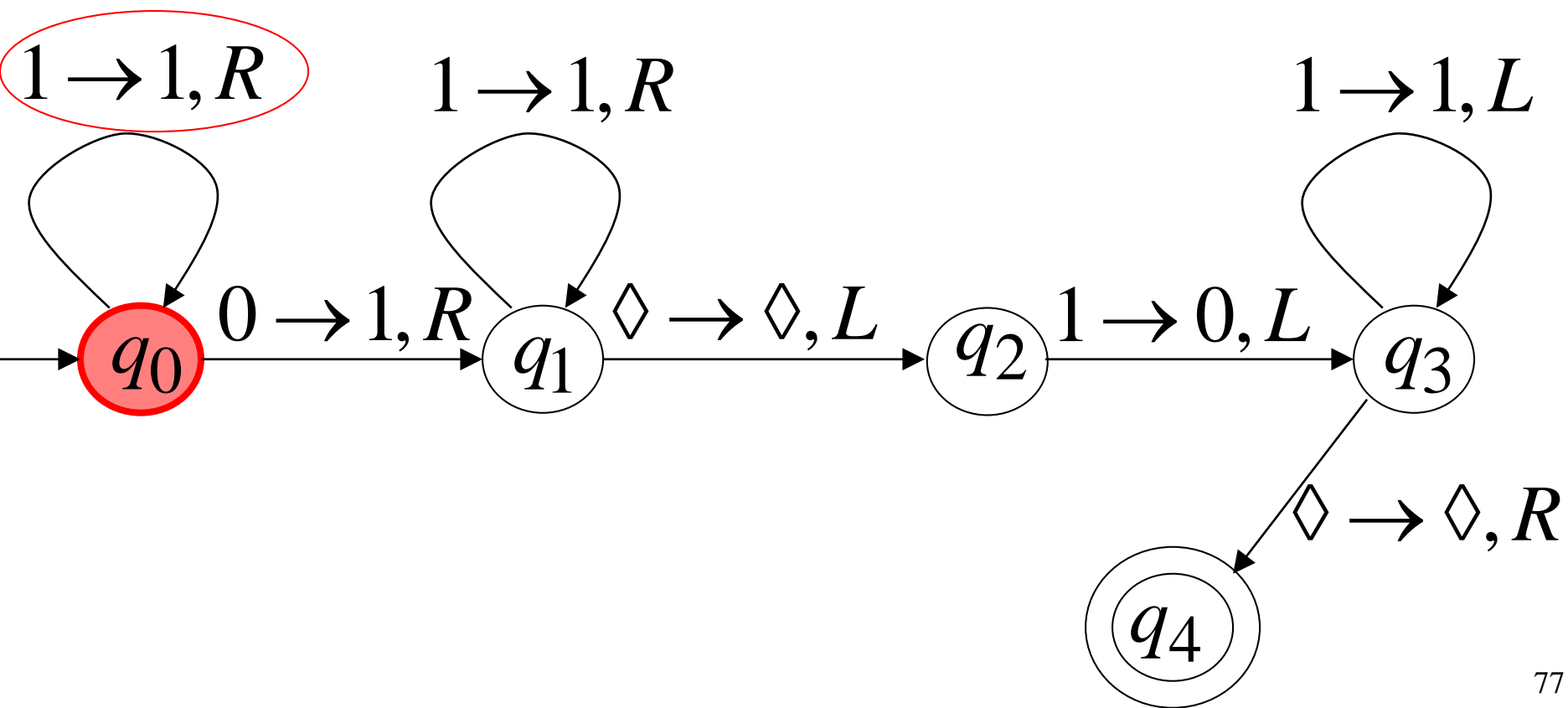
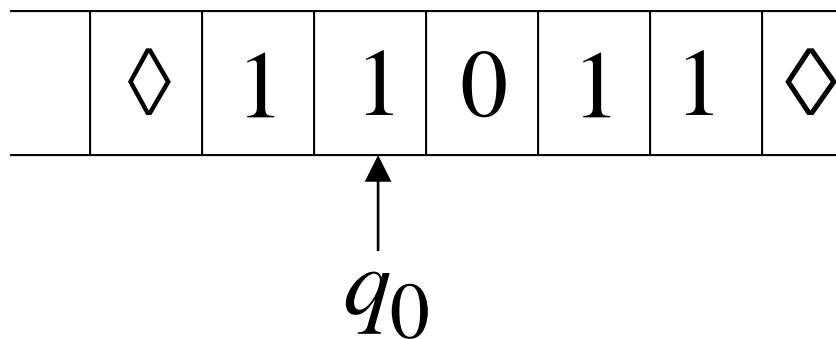
Final Result



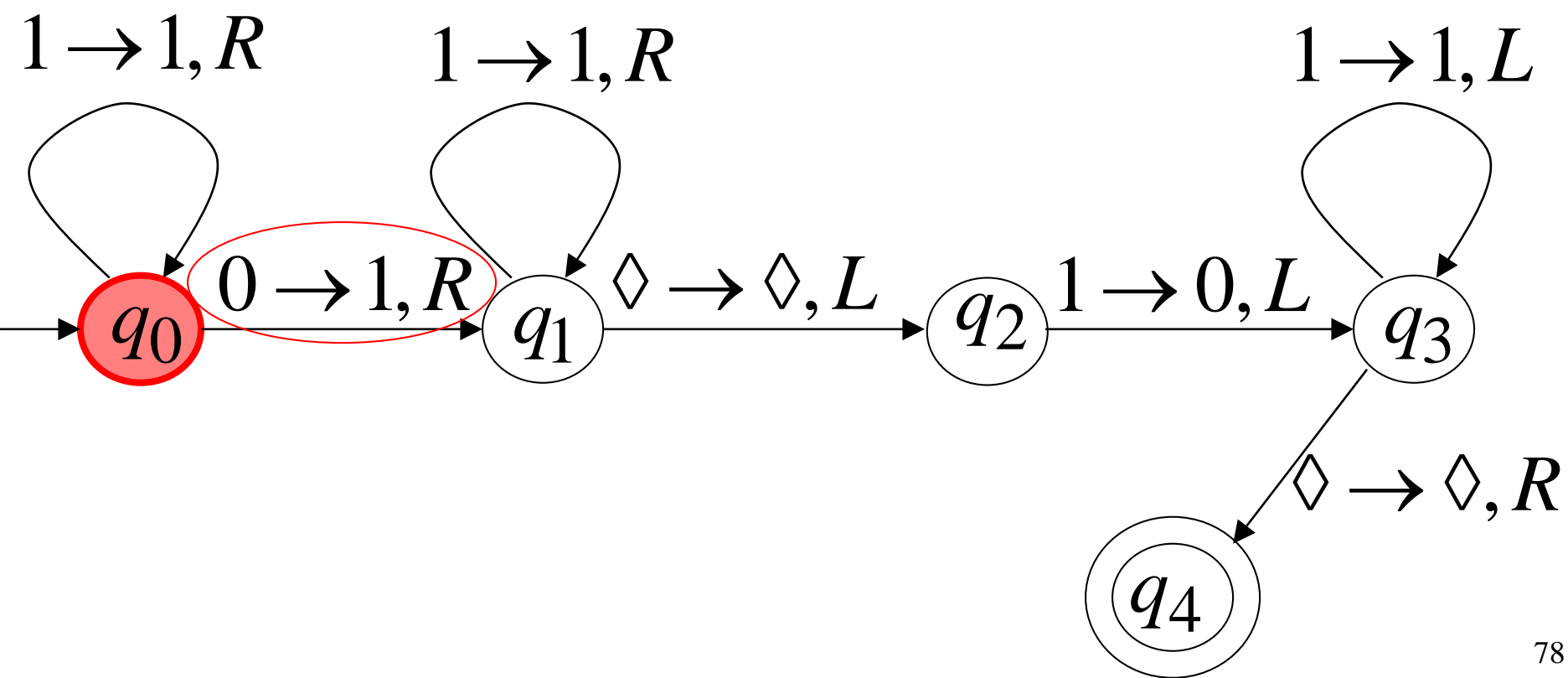
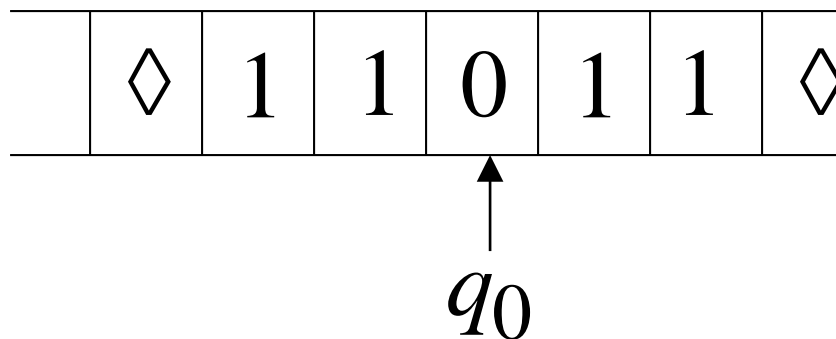
Time 0



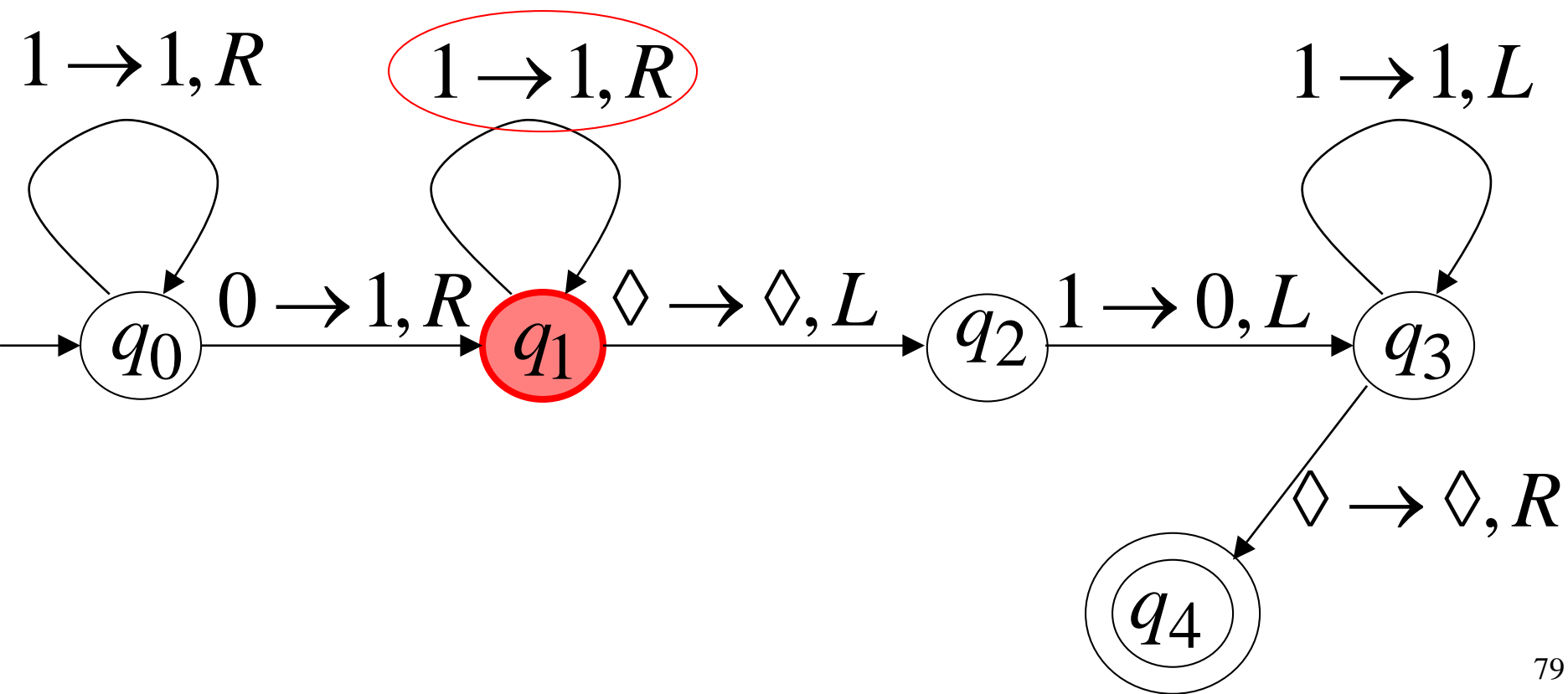
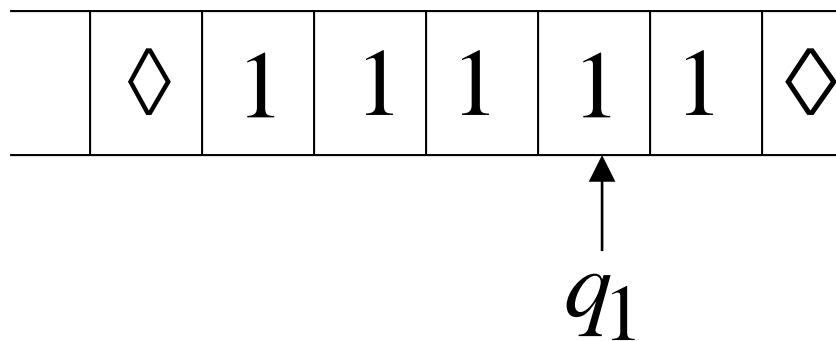
Time 1



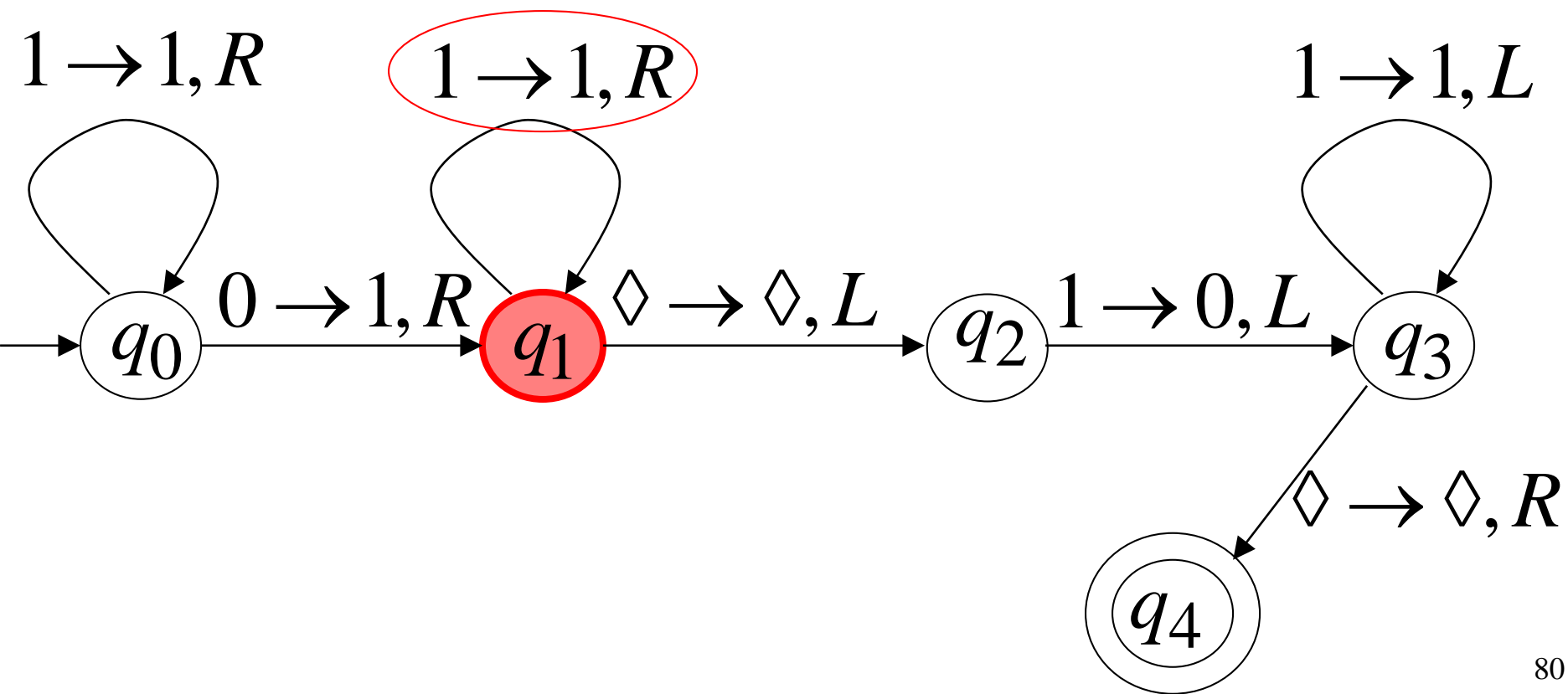
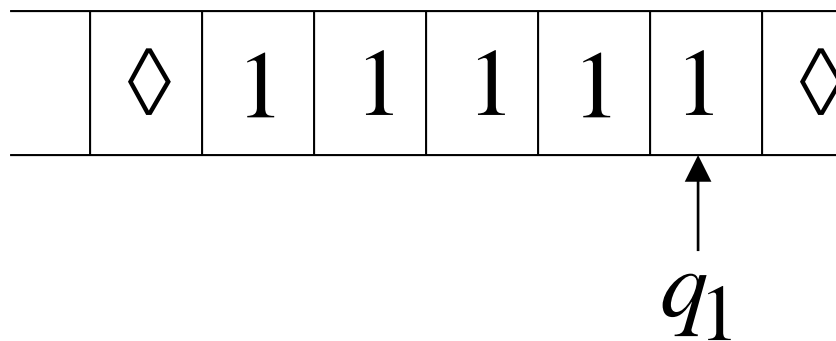
Time 2



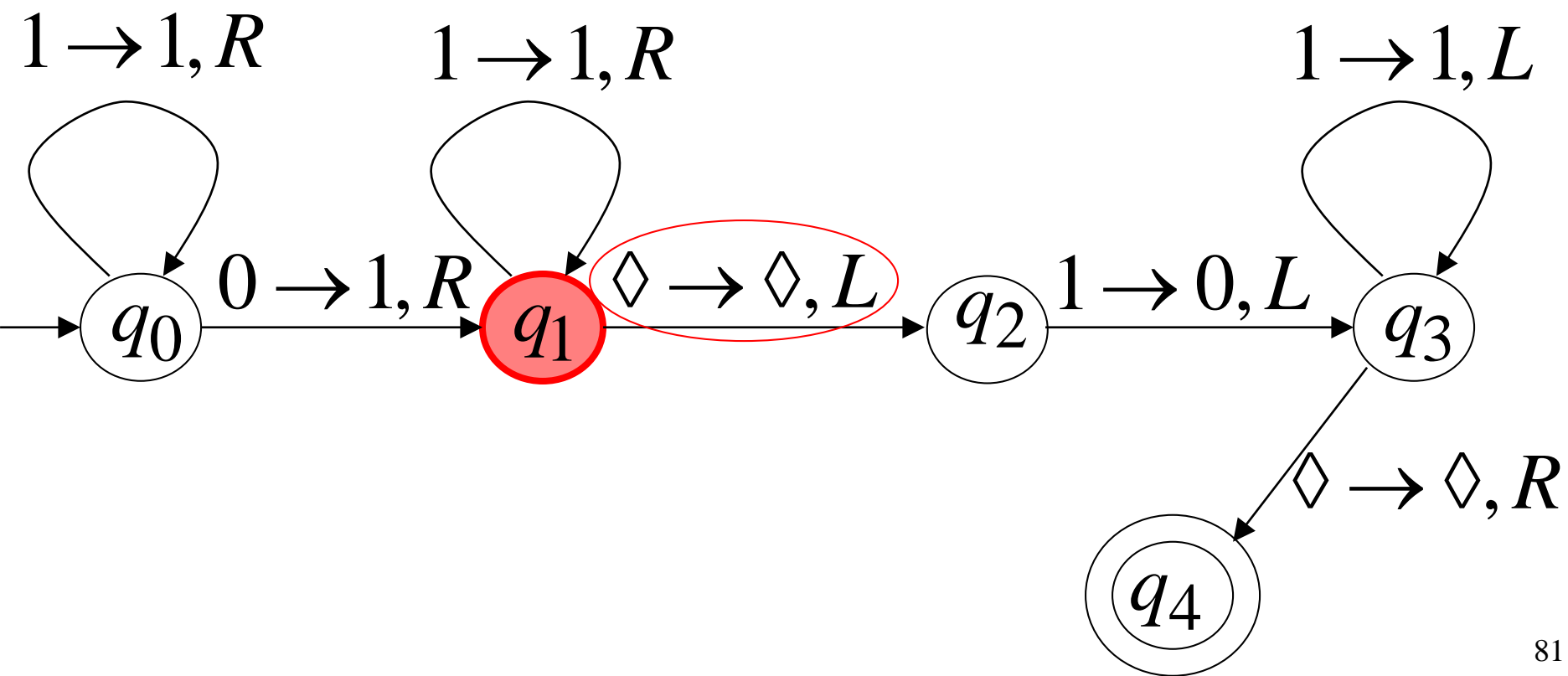
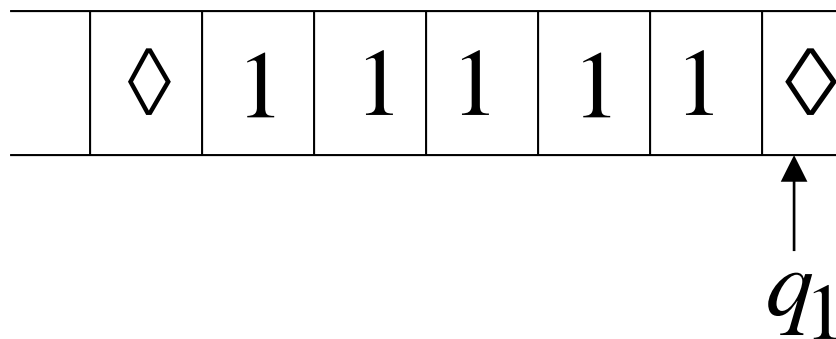
Time 3



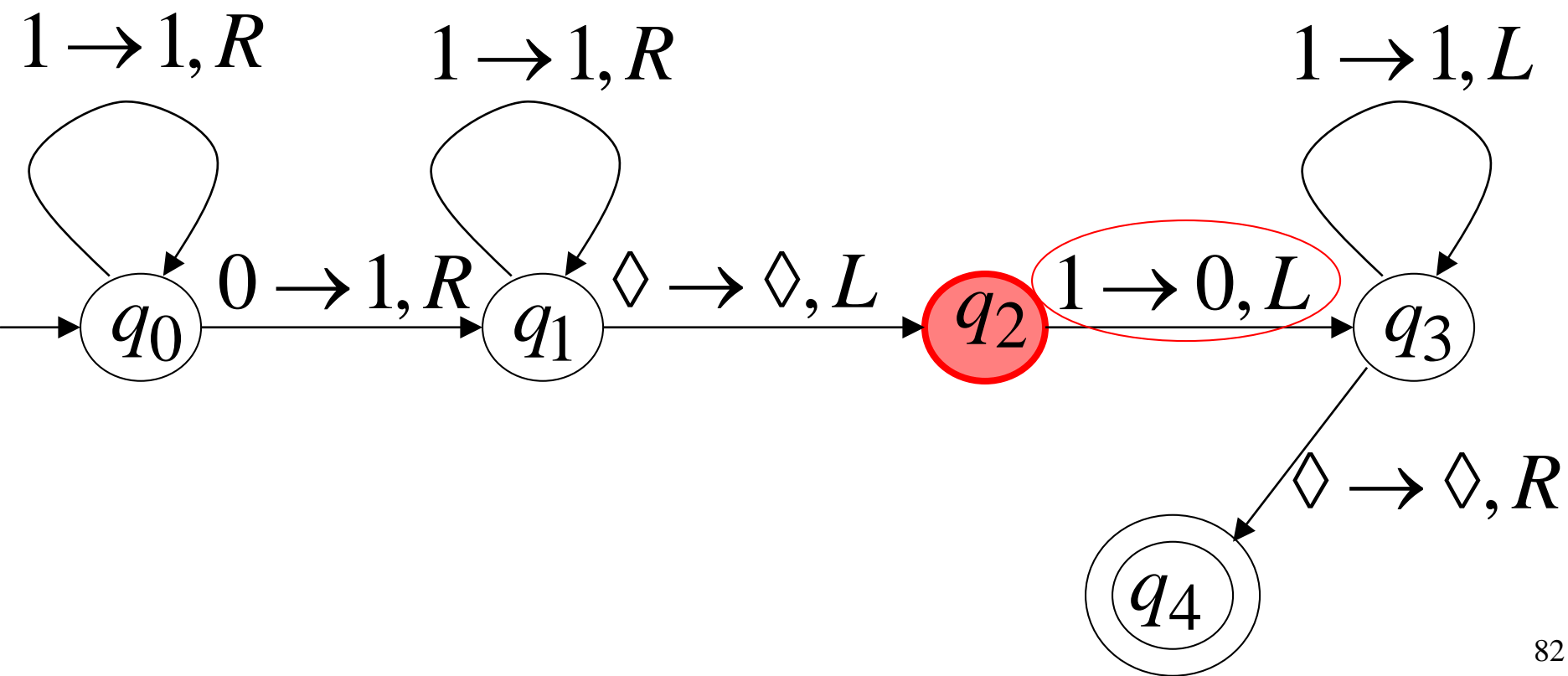
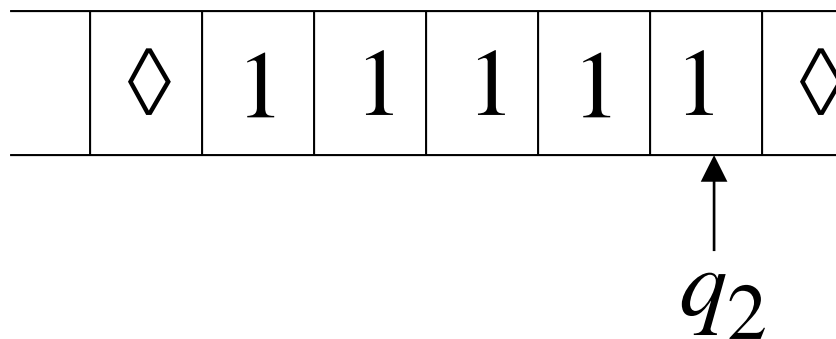
Time 4



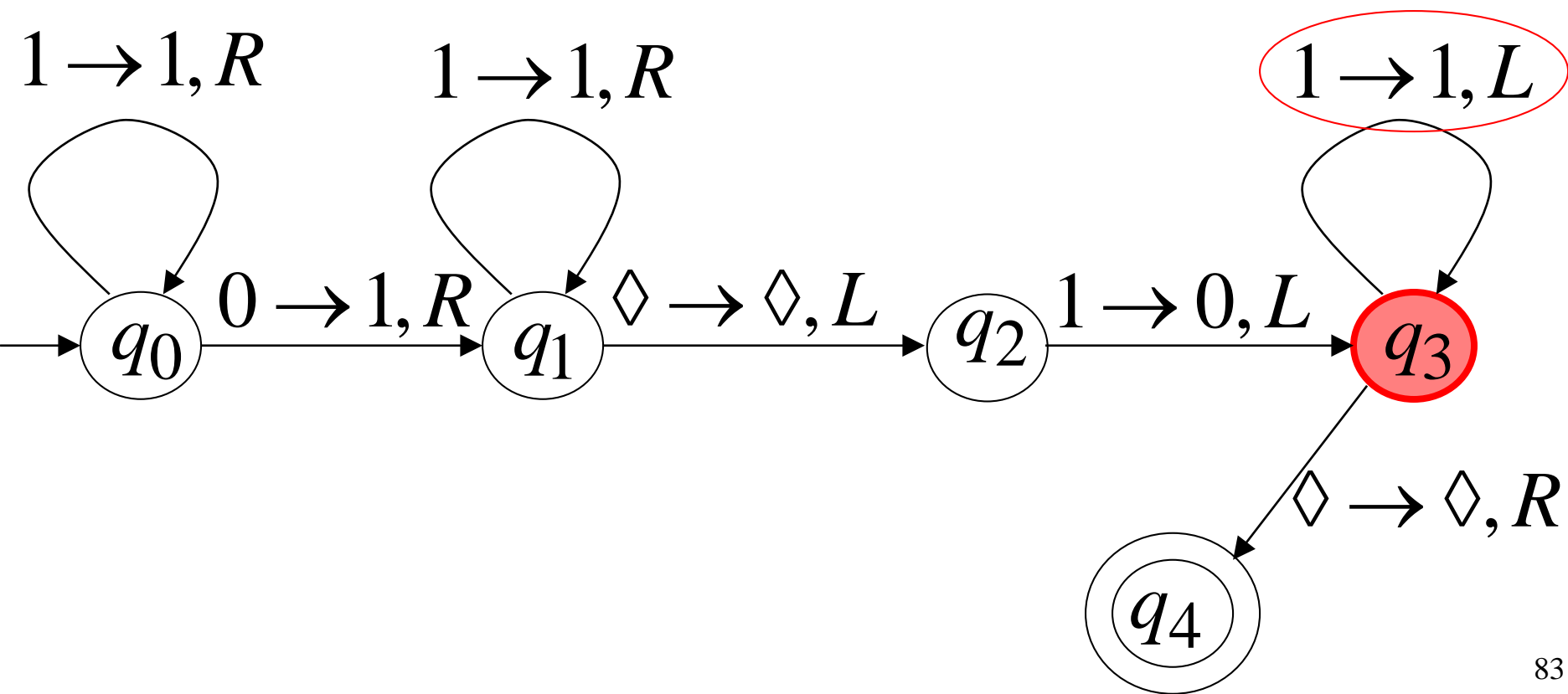
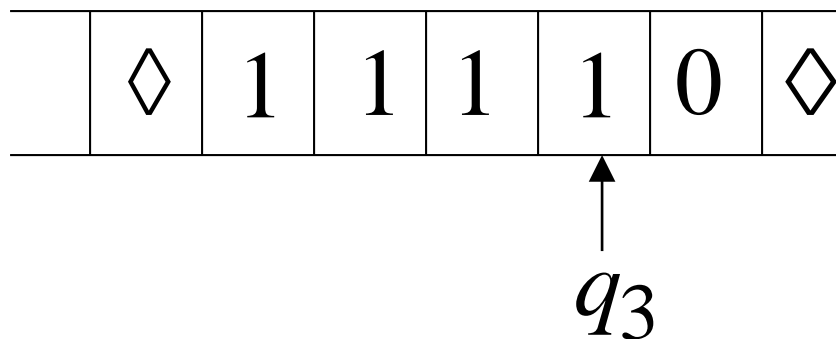
Time 5



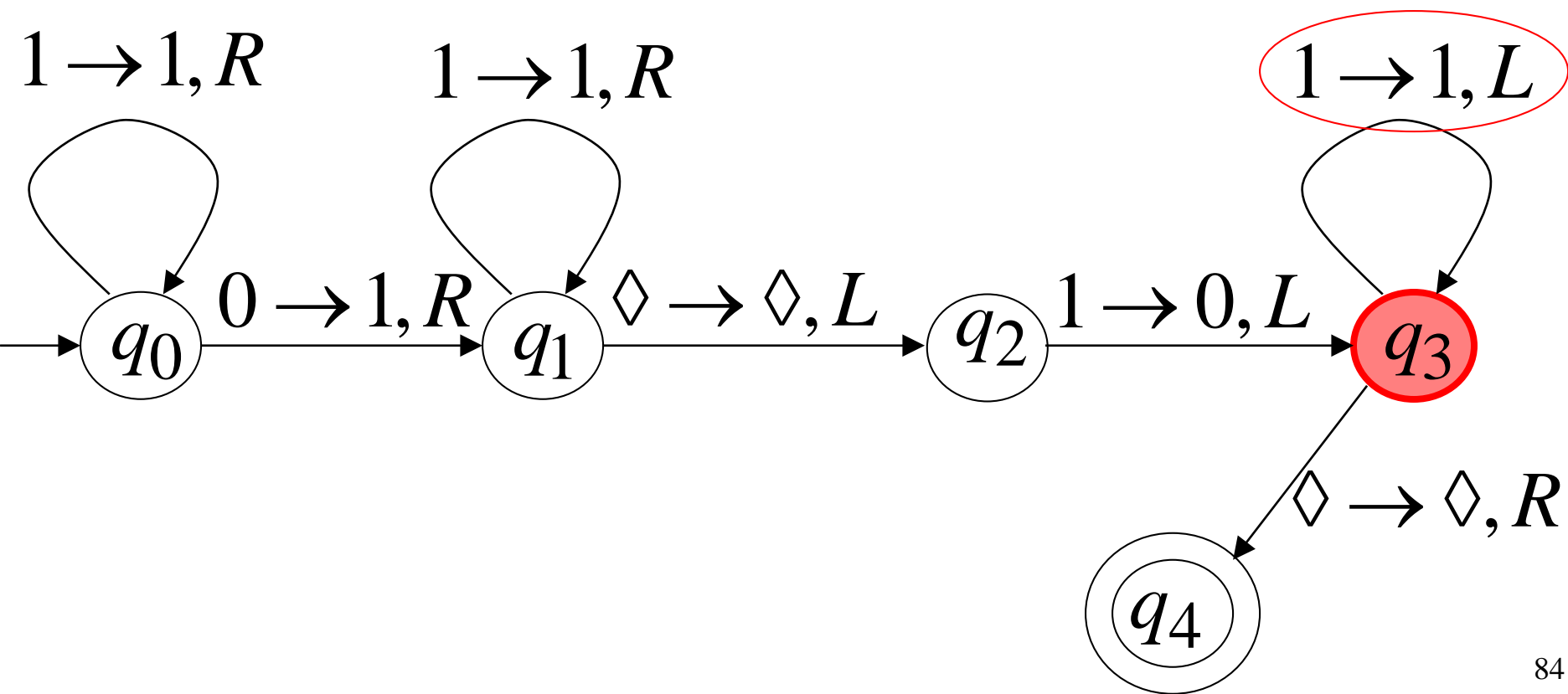
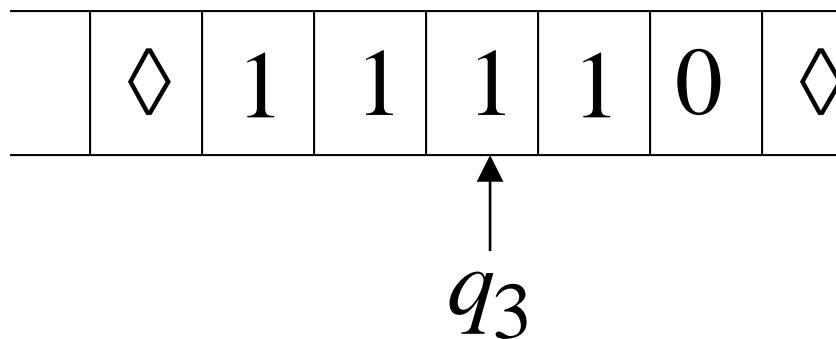
Time 6



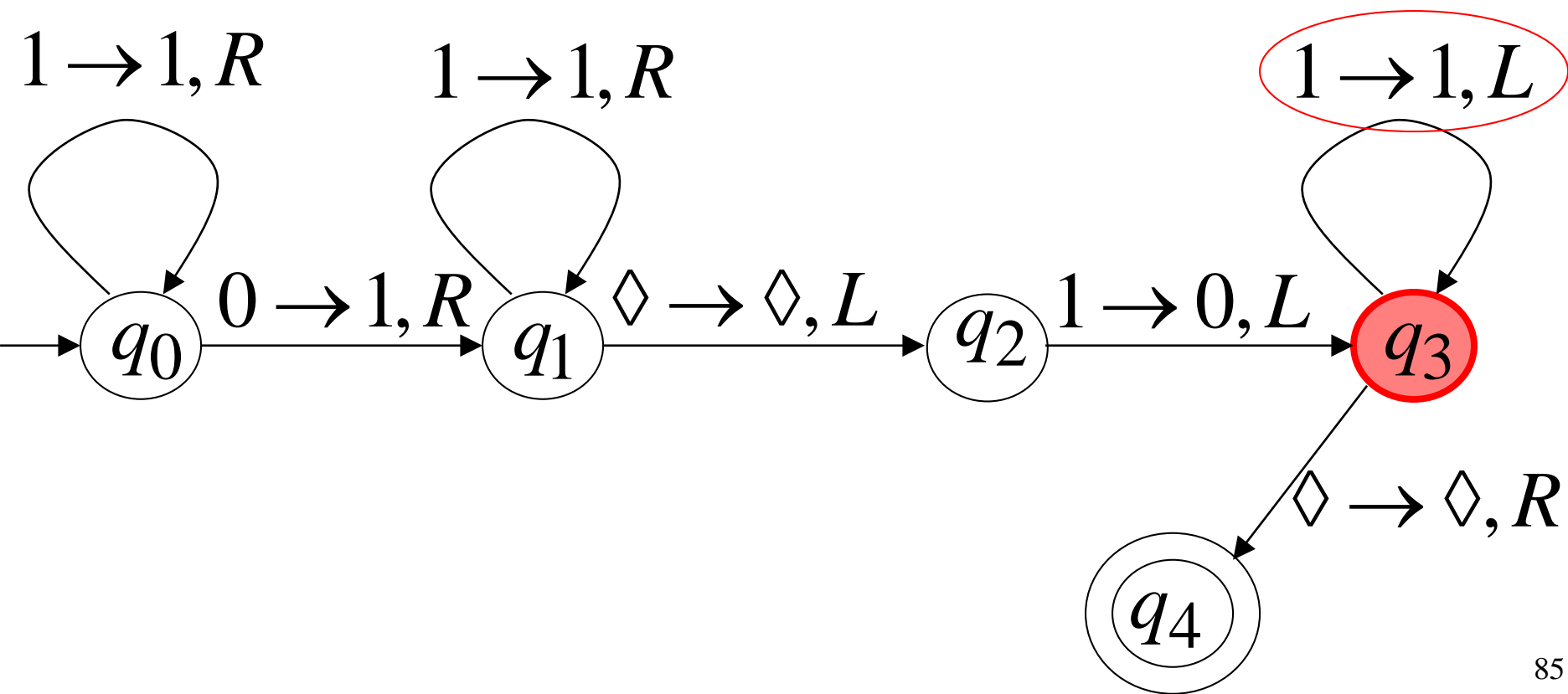
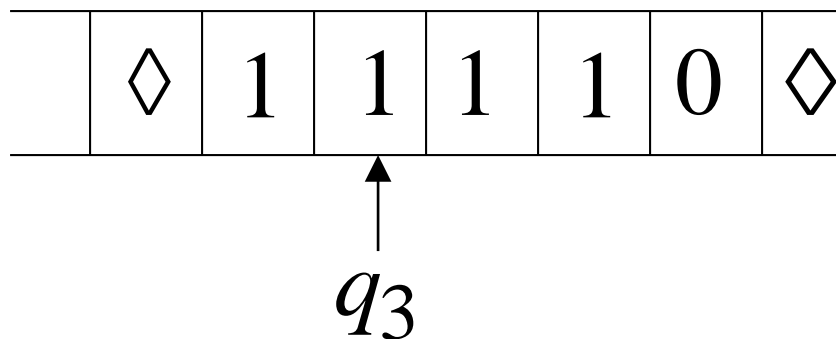
Time 7



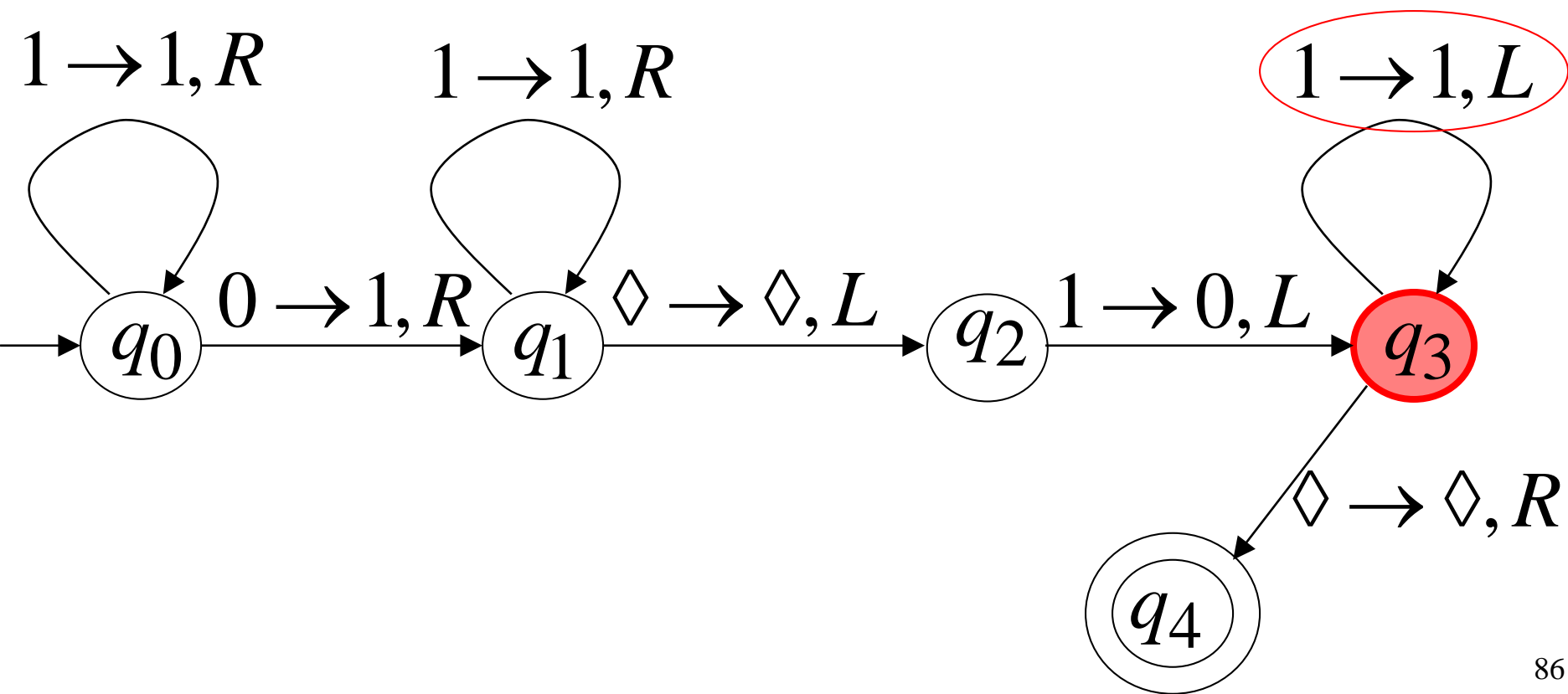
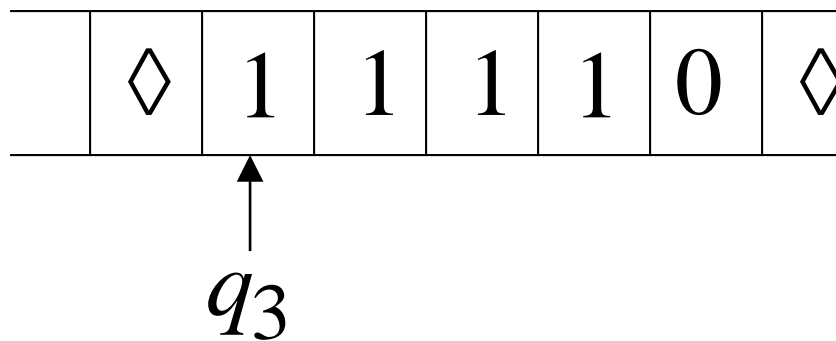
Time 8



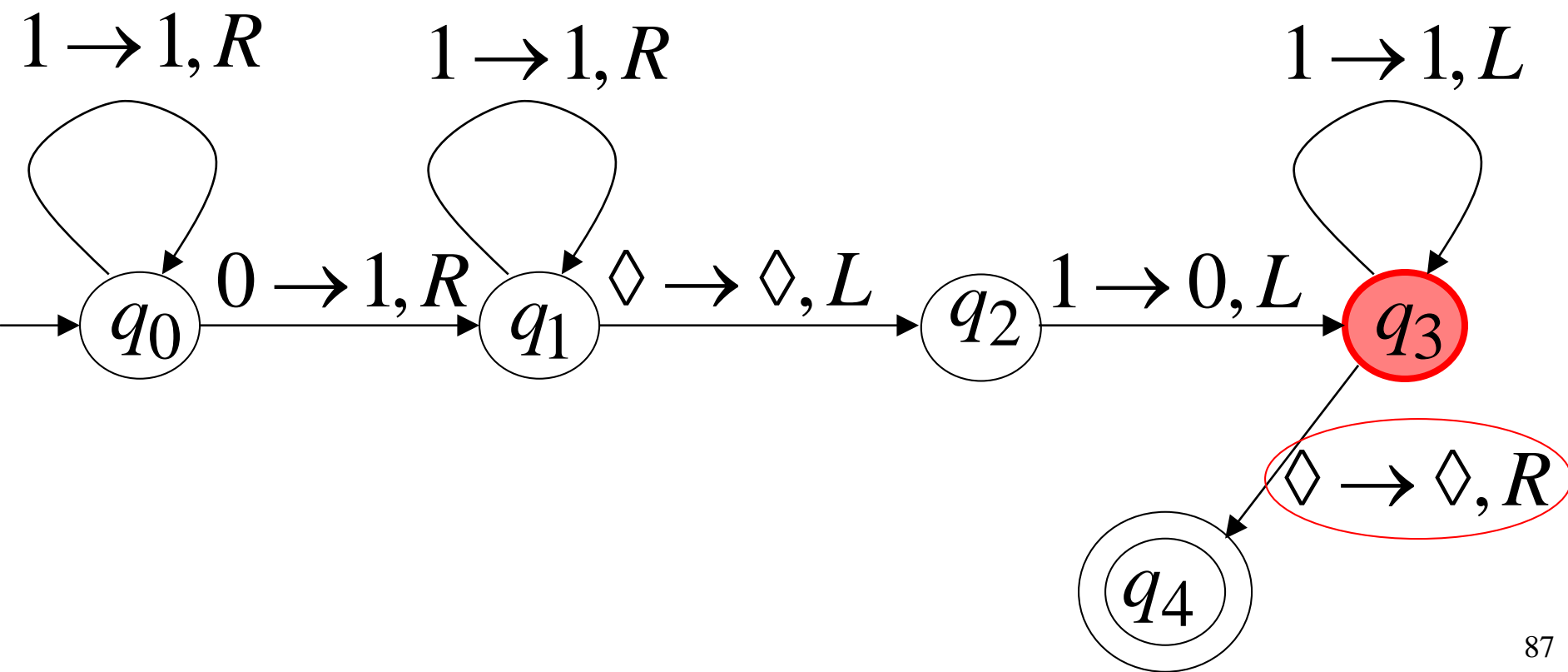
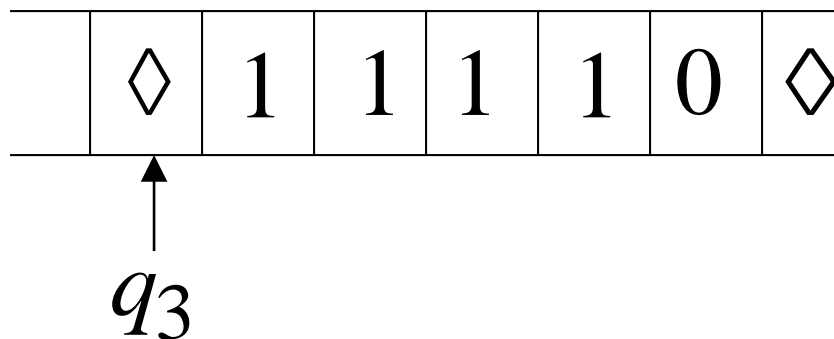
Time 9



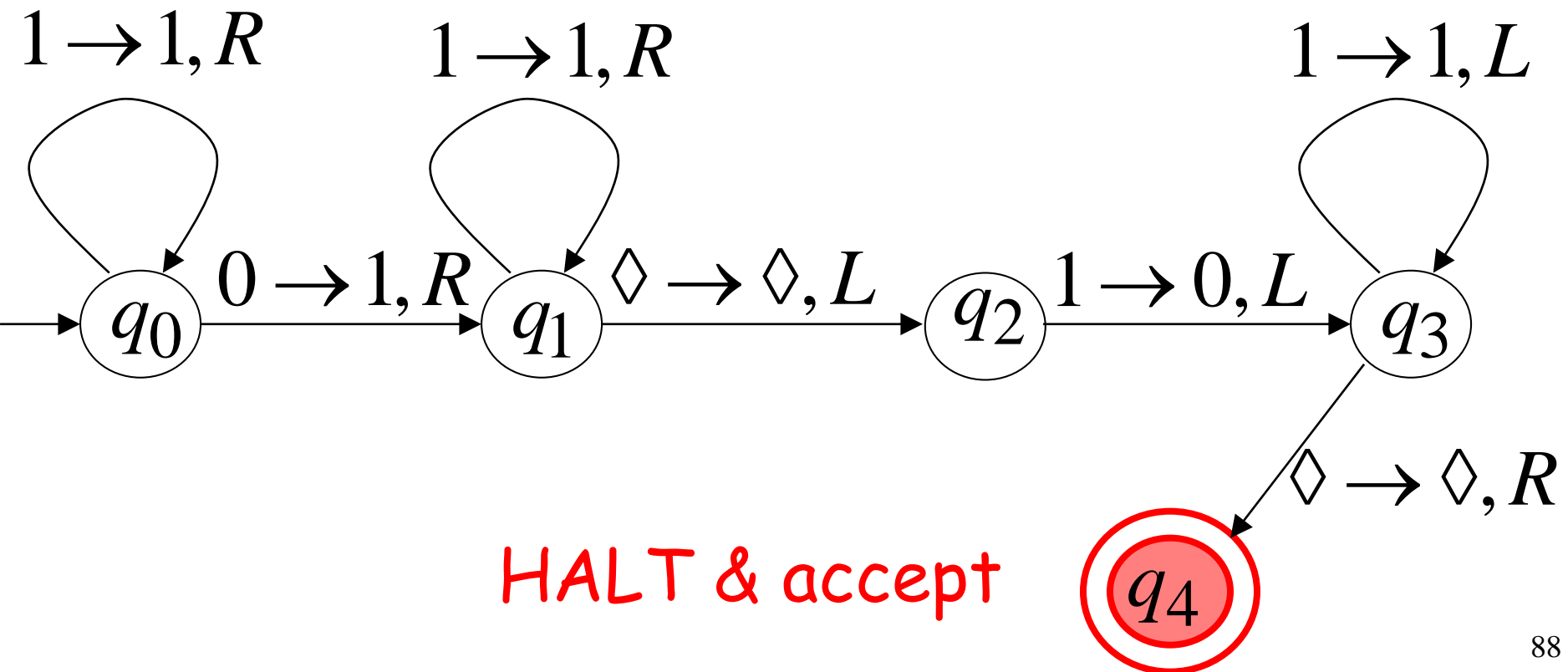
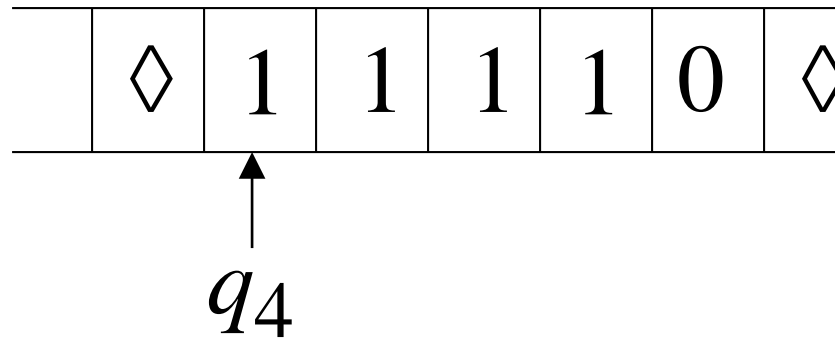
Time 10



Time 11



Time 12



Another Example

The function $f(x) = 2x$ is computable

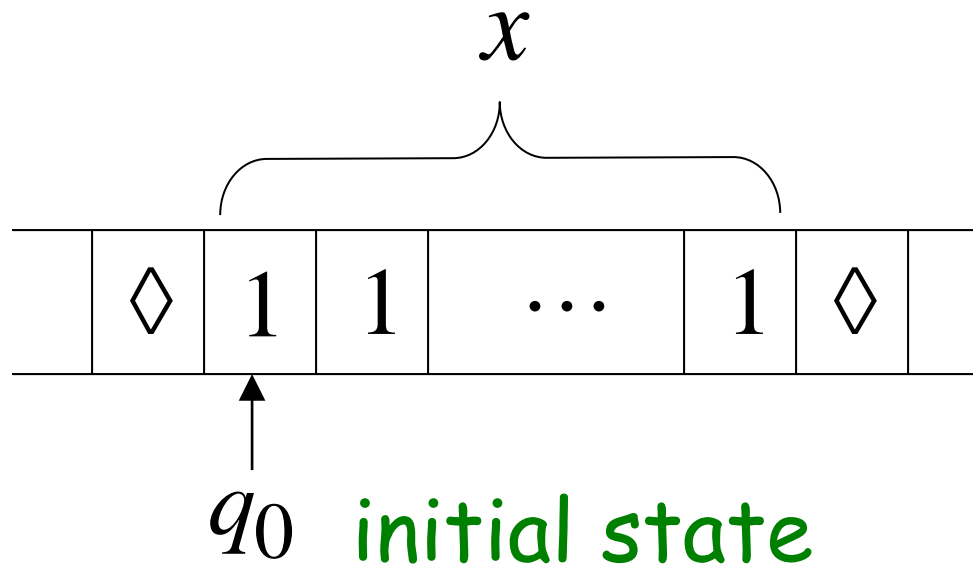
x is integer

Turing Machine:

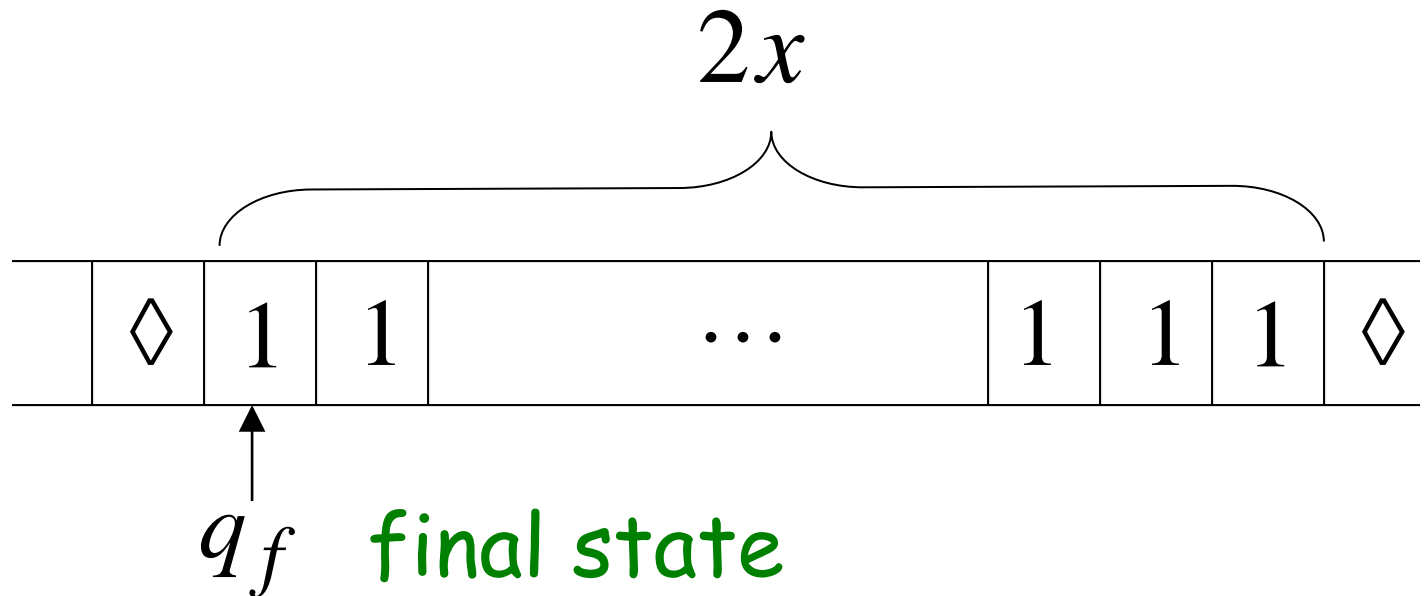
Input string: x unary

Output string: xx unary

Start



Finish

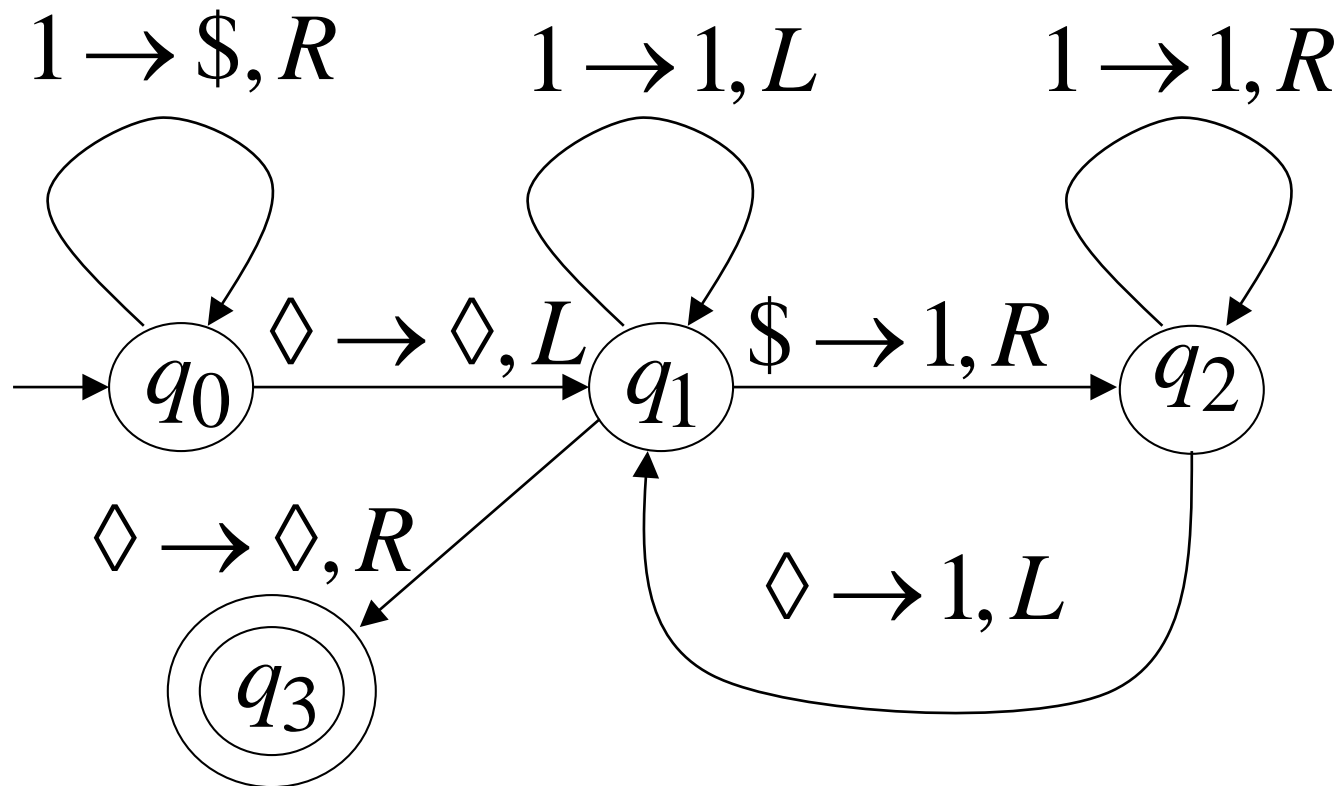


Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with \$
- Repeat:
 - Find rightmost \$, replace it with 1
 - Go to right end, insert 1

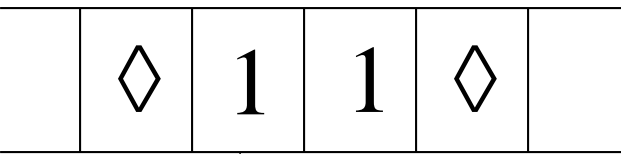
Until no more \$ remain

Turing Machine for $f(x) = 2x$



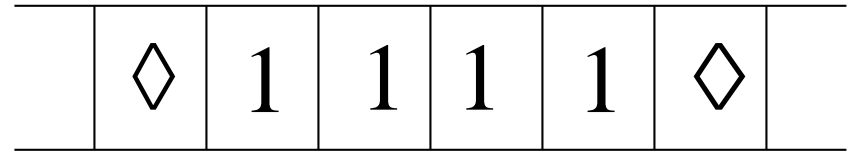
Example

Start

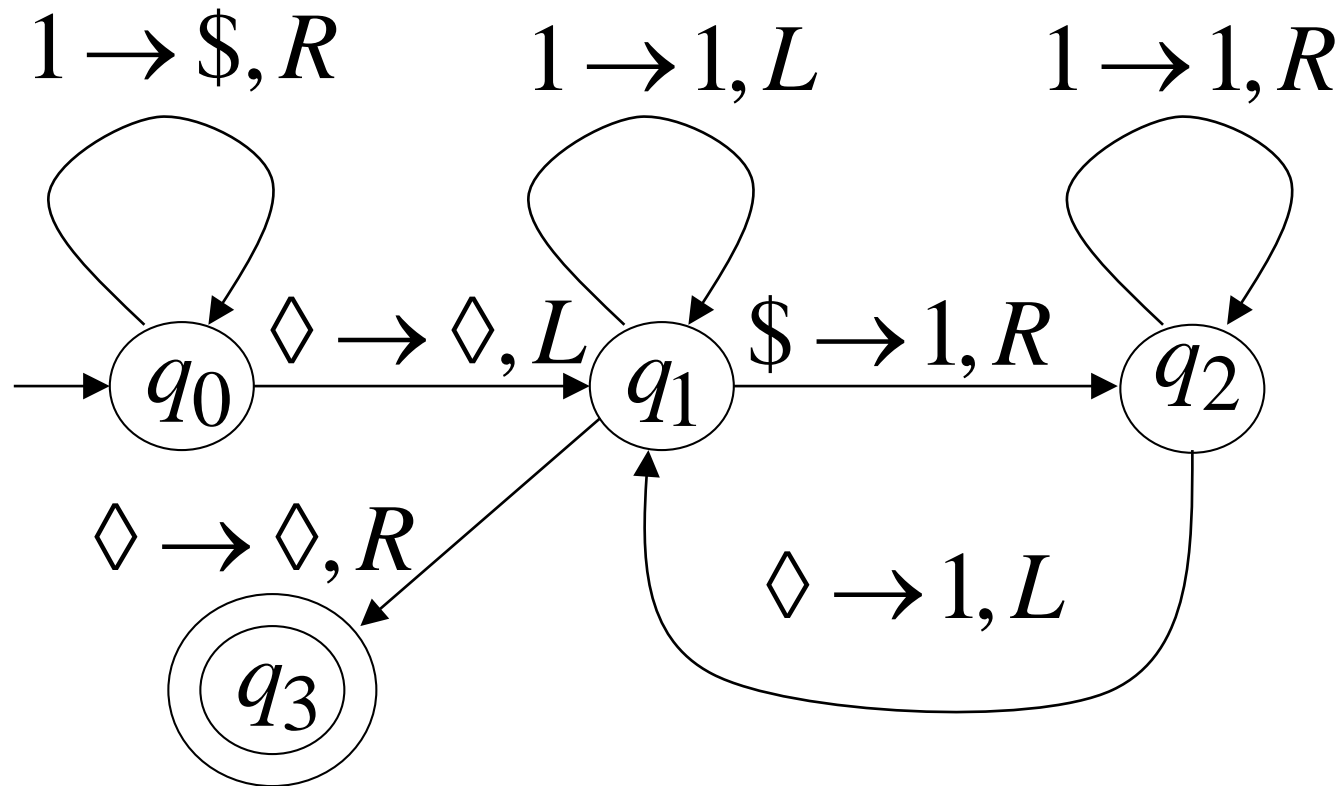


q_0

Finish



q_3



Another Example

The function
is computable

$$f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Turing Machine for

$$f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

Match a 1 from x with a 1 from y

Until all of x or y is matched

- If a 1 from x is not matched

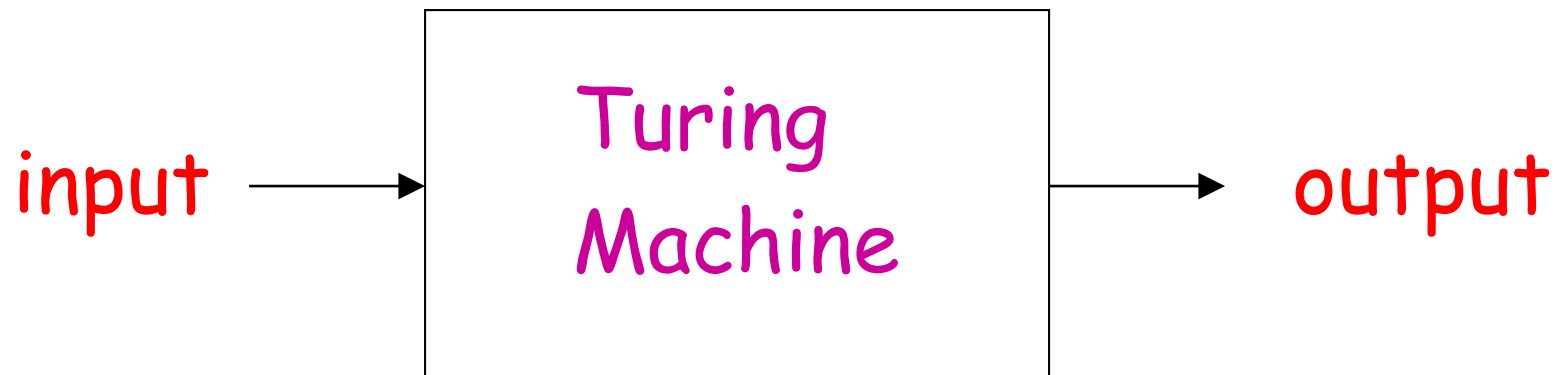
erase tape, write 1 $(x > y)$

else

erase tape, write 0 $(x \leq y)$

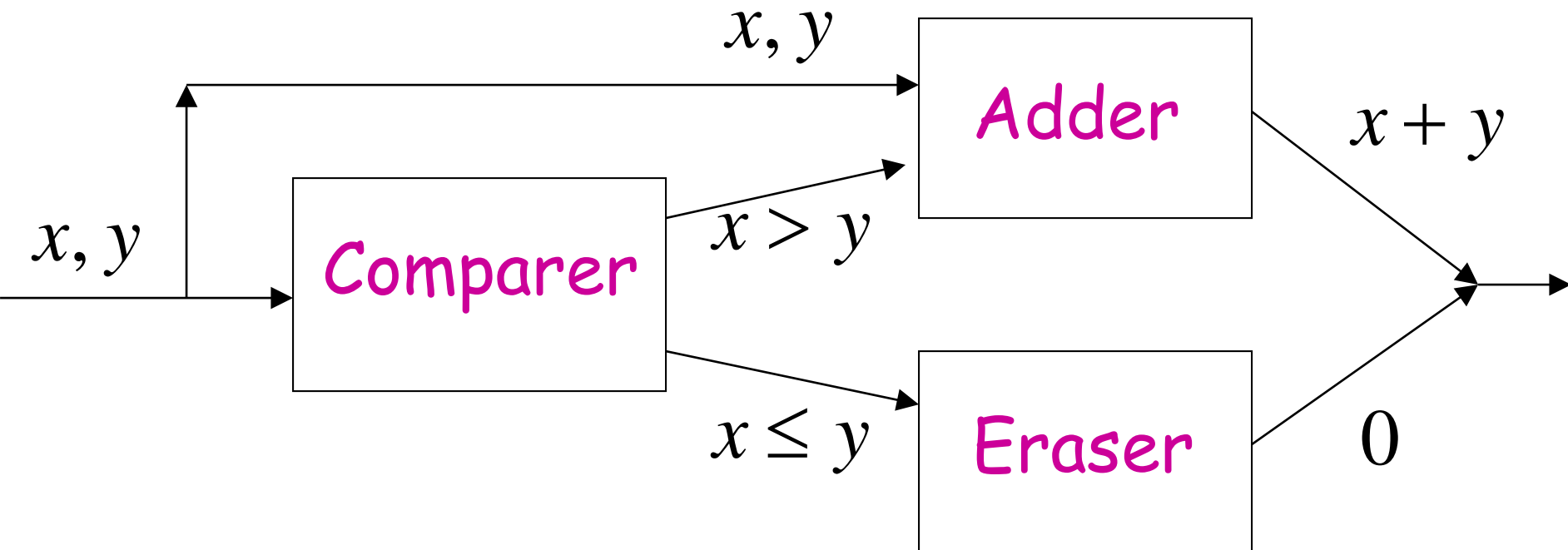
Combining Turing Machines

Block Diagram



Example:

$$f(x, y) = \begin{cases} x + y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$





That's all Folks!