Artificial Intelligence

Algorithms and Applications with Python

Chapter 08

Dr. Dominik Jung
dominik.jung42@gmail.com

Outline

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▶ **What you will learn:**

- Concepts of statistics and probability theory to model agents that can act under uncertainty
- Foundations of Markov theory, Bayes theory and sequence analysis for to reason under uncertainty according to the laws of probability theory
- How to analyze and build optimal agent decision-making that can handle uncertainty

▶ **Duration:**

- 215 min

▶ **Relevant for Exam:**

- 10.1 – 10.4

8.1 Problems of Previous Agent Implementations

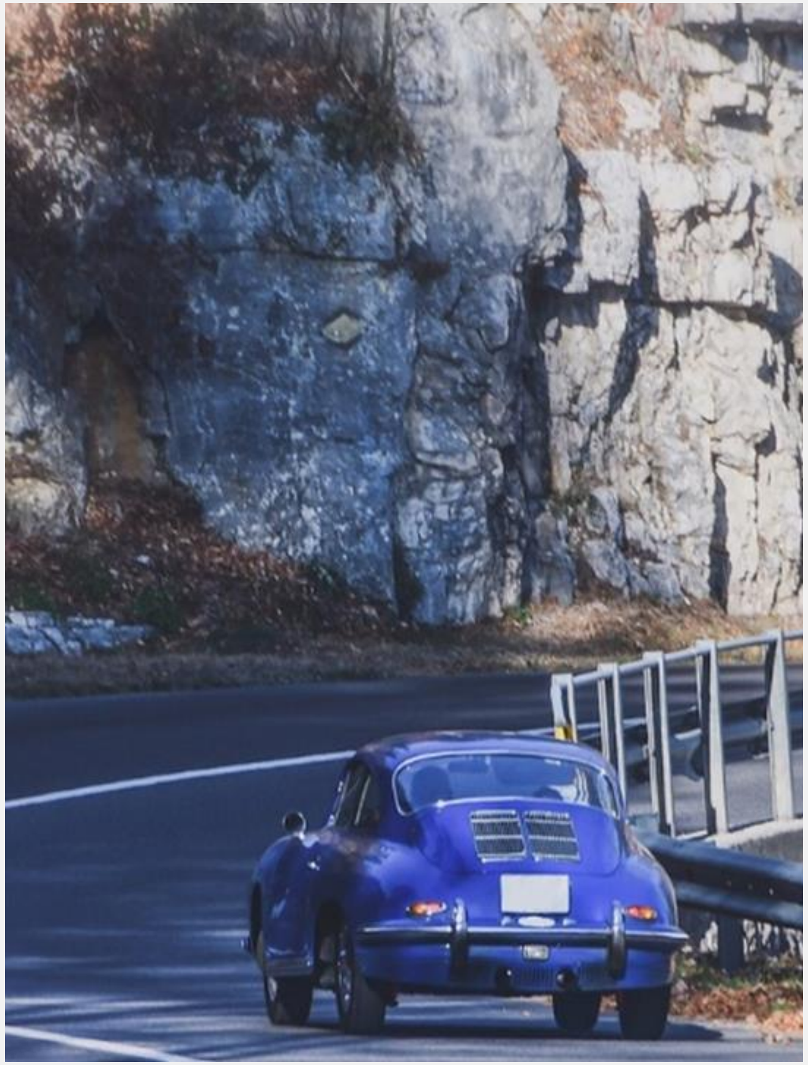
- So far, agents keep a track of a belief state, and generating a plan to act under every possible situation.

- Let action $A_t = \text{time to reach visitor center}$
  - Will A_t get me there on time?

- **Problems**:
  - Partial observability (road state, other drivers' plans)
  - Noisy sensors (traffic reports)
  - Uncertainty in action outcomes (flat tire, etc.)
  - Complexity of modeling and predicting traffic

→ Russel & Norvig: Laziness and Ignorance dilemma

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)



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8.1 Motivation: Probabilistic Reasoning in Artificial Intelligence

- Hence a purely logical approach either risks falsehood:
  - “A_{120} will get me there on time”
- Or, leads to conclusions that are too weak for decision making:
  - A_{120} will get us there on time if there's no accident on the way and it doesn't rain and my tires remain intact,…
  - A_{12000} might reasonably be said to get us there on time but I'd have to stay overnight on the road before the next national park
- **Solution**: Agents may need to handle uncertainty, whether due to partial observability, nondeterminism, or a combination of both.

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

- Suppose the agent believes the following:

$$P(A_{120}\ gets\ me\ there\ on\ time) = 5\ \%$$

$$P(A_{240}\ gets\ me\ there\ on\ time) = 60\ \%$$

$$P(A_{300}\ gets\ me\ there\ on\ time) = 95\ \%$$

$$P(A_{600}\ gets\ me\ there\ on\ time) = 99.9\%$$

$$P(A_{1000}\ gets\ me\ there\ on\ time) = 99.99\%$$

? Which action should the agent choose? What is “the best” action if you have to decide for one of these during your national park roadtrip?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Decision-Making under Uncertainty

- Which action should the agent choose?
  - Depends on preferences for missing a national park vs. time spent waiting
  - Encapsulated by a utility function

- **Utility**: Some states may be desirable, others may be undesirable

- Many environments are uncertain in the sense that it is not clear what state an action will lead to

- **Uncertainty**: Some states may be likely, others may be unlikely

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Decision-Making under Uncertainty to Decision Theory

- The agent should choose the action that maximizes the expected utility:

$$P(Success \mid A_t) \cdot U(Success \mid A_t) + P(Fail \mid A_t) \cdot U(Fail \mid A_t)$$

- Utility theory is used to represent and infer preferences

- Hence,

$$Decision\ theory = probability\ theory + utility\ theory$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Decision-Theoretic Agent

▶ A decision-theoretic agent that selects rational actions based on decision and utility theory.

- Comparable to previous agents that maintain a belief state reflecting the history of percepts to date
- Difference is that the DT-agent's belief state represents not just the possibilities for world states but also their probabilities
- Given the belief state, the agent can make probabilistic predictions of action outcomes to select the action with highest expected utility.

Algorithm: DT-Agent

function DT-AGENT(*percept*) **returns** an action
persistent:
belief_state, probabilistic beliefs about the current state of the world
action, the agent's action

update *belief_state* based on *action* and *percept*
calculate outcome probabilities for actions,
  given action descriptions and current *belief_state*
select *action* with highest expected utility
  given probabilities of outcomes and utility
  information

return *action*

Adapted from Rusell, S., & Norvig, P. (2016)

8.1 What is Probability?

Frequentism/Objectivism

- Probabilities are relative frequencies, that come from repeated experiments
- For example, if we flip a coin many times, we see experimentally that it comes out heads ½ the time
- For most events in the world, there is no history of exactly that event happening
- "Reference class" problem

Subjectivism

- Probabilities merely reflect agents' beliefs
- But then, how do we assign belief values to statements? And what would constrain agents to hold consistent beliefs
- "Assignment" problem

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020) | Image source: ↗ Pixabay (2019) / ↗ CC0

8.1 Random Variables

- We describe the (uncertain) state of the world using random variables

- Just like variables in constrain-satisfaction problems (CSP), random variables take on values in a domain (domain values must be mutually exclusive and exhaustive)

- Example:

$$S: What\ is\ the\ speed\ of\ my\ porsche?$$

$$S\ in\ [0, 350]\frac{km}{h}^{*}$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020) | Note: *350 km/h equals about 220 mph

8.1 Events and Propositions

- Probabilistic statements are defined over events, or sets of world states

$$"On\ the\ A8\ I\ can\ drive\ between\ 250\ and\ 350\ km/h"$$

- Events are described using propositions

$$250 \leq S < 350$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Notion of Probability (P-Notation)

- Hence:

$$P(proposition) = x$$

- $P(A)$ is the probability of the set of world states in which proposition A holds

- $P(X = x)$, or $P(x)$ for short, is the probability that random variable X has taken on the value x

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Kolmogorov's Axioms of Probability

D

For any propositions (events) A, B

- $0 \leq P(A) \leq 1$
- $P(True) = 1\ and\ P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

?

Based on these axioms, can you compute $P(\neg A)$?

- These axioms are sufficient to completely specify probability theory for discrete random variables (for continuous variables, we need density functions)

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Probabilities and Rationality

Bruno De Finetti (1931)

Image source: ↗ Bruno de Finetti (2019) by Ffaffff from ↗ Wikimedia

- Why should a rational agent hold beliefs that are consistent with axioms of probability?
- De Finetti postulates, if an agent has some degree of belief in proposition A, it should be able to decide whether or not to accept a bet for/against A
- If the agent believes that $P(A) = 0.4$, should it agree to bet 60€ that A will occur against 40€ that A will not occur?

D Finetti Theorem: An agent who holds beliefs inconsistent with axioms of probability can be tricked into accepting a combination of bets that are guaranteed to lose them money

Adapted from Rusell, S., & Norvig, P. (2016)

8.1 Why we need Kolmogorov's Axioms of Probability in AI

?

- Given an agent 1 with the following beliefs

$$P(A) = 0.4, \qquad P(B) = 0.3, \qquad P(A \vee B) = 0.8$$

- E.g. smart agent 2 plays against agent 1 and bets 4€ on A, 3€ on B, and 2€ on $(\neg A \vee B)$

→ Agent 1 loses money

8.1 Variables and Atomic Events

- If the world would consist of only of the following two Boolean variables then there are four related distinct atomic events:

$$Warning\ signal$$
$$Engine\ broken$$

$$Broken = false \ \wedge Warning = false$$
$$Broken = false \ \wedge Warning = true$$
$$Broken = true \wedge \ Warning = false$$
$$Broken = true \wedge \ Warning = true$$

- **Atomic event**: a complete specification of the state of the world, or a complete assignment of domain values to all random variables

- Atomic events are mutually exclusive and exhaustive

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

Image source: ↗ Pixabay (2019) / ↗ CC0

- A joint distribution is an assignment of probabilities to every possible atomic event:

Atomic Event	Probability
$Broken = false \wedge Warning = false$	0.8
$Broken = false \wedge Warning = true$	0.1
$Broken = true \wedge Warning = false$	0.05
$Broken = true \wedge Warning = true$	0.05

? Why does it follow from the axioms of probability that the probabilities of all possible atomic events must sum to 1?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Marginal Probability Distributions

- Suppose we have the joint distribution $P(X, Y)$ and we want to find the marginal distribution $P(Y)$

P(Broken, Warning)	
$Broken = false \wedge Warning = false$	0.8
$Broken = false \wedge Warning = true$	0.1
$Broken = true \wedge Warning = false$	0.05
$Broken = true \wedge Warning = true$	0.05

P(Warning)	
$Warning = false$	?
$Warning = true$	?

P(Broken)	
$Broken = false$	?
$Broken = true$	?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Marginal Probability Distributions

- Suppose we have the joint distribution $P(X, Y)$ and we want to find the marginal distribution $P(Y)$

$$P(X = x) = P\big((X = x \wedge Y = y_1) \vee \ldots \vee (X = x \wedge Y = y_n)\big)$$

$$= P\big((x, y_1) \vee \ldots \vee (x, y_n)\big) = \sum_{i=1}^{n} P(x, y_i)$$

- **General rule**: to find $P(X = x)$, sum the probabilities of all atomic events where $X = x$.

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

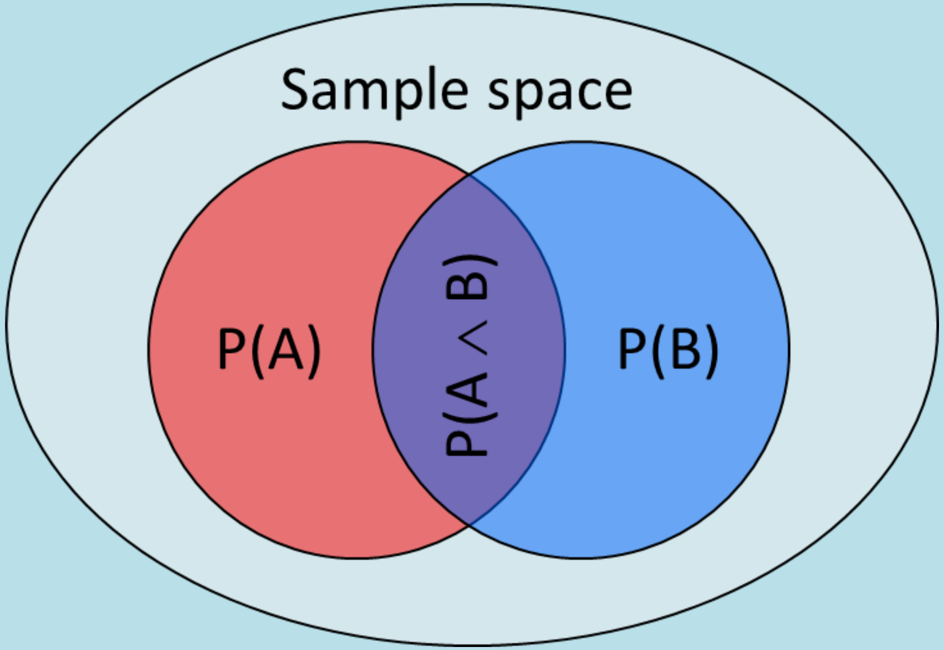
8.1 Conditional Probability Visualization

- Probability of warning given the engine is really broken

$$P(Broken = true \mid Warning = true)$$

- For any two events A and B,

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)} = \frac{P(A, B)}{P(B)}$$



Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Conditional Probability

P(Broken, Warning)	
$Broken = false \land Warning = false$	0.8
$Broken = false \land Warning = true$	0.1
$Broken = true \land Warning = false$	0.05
$Broken = true \land Warning = true$	0.05

$$P(A \mid B) = \frac{P(A \land B)}{P(B)}$$

P(Warning)	
$Warning = false$	0.85
$Warning = true$	0.15

P(Broken)	
$Broken = false$	0.9
$Broken = true$	0.1

- $P(Broken = true \mid Warning = false) = \frac{0.05}{0.85} = 0.059$
- $P(Broken = false \mid Warning = true) = \frac{0.1}{0.15} = 0.667$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

- A conditional distribution is a distribution over the values of one variable given fixed values of other variables

$P(Broken, Warning)$	
$Broken = false \wedge Warning = false$	0.8
$Broken = false \wedge Warning = true$	0.1
$Broken = true \wedge Warning = false$	0.05
$Broken = true \wedge Warning = true$	0.05

$P(Warning \mid Broken = true)$	
$Warning = false$	0.5
$Warning = true$	0.5

$P(Warning \mid Broken = false)$	
$Broken = false$	0.889
$Broken = true$	0.111

$P(Broken \mid Warning = true)$	
$Broken = false$	0.667
$Broken = true$	0.333

$P(Broken \mid Warning = false)$	
$Broken = false$	0.941
$Broken = true$	0.059

? Do you have any idea how to calculate these conditional distributions at once?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

- To get the whole conditional distribution $P(X \mid y)$ at once, select all entries in the joint distribution matching $Y = y$ and renormalize them to sum to one

$P(Broken, Warning)$	
$Broken = false \wedge Warning = false$	0.8
$Broken = false \wedge Warning = true$	0.1
$Broken = true \wedge Warning = false$	0.05
$Broken = true \wedge Warning = true$	0.05

Select

$Warning, Broken = false$	
$Warning = false$	0.8
$Warning = true$	0.1

Renormalize

$P(Warning \mid Broken = false)$	
$Broken = false$	0.889
$Broken = true$	0.111

D Renormalization

$$\frac{P(x, y)}{\sum_{a'} P(x', y)} = \frac{P(x, y)}{P(y)}$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Bayes Rule

- The product rule gives us two ways to factor a joint distribution:

$$P(A,B) = P(A|B)P(B) = P(B|A)P(A)$$

- Therefore

D Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Agent can get diagnostic probability $P(Broken|\,Warning)$ from causal probability $P(Warning\,|\,Broken)$

- Agent can now update its beliefs based on evidence

- In general, important tool for probabilistic inference

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Statistical Independence

- Two events are independent, statistically independent if the occurrence of one does not affect the probability of occurrence of the other

D Two events A and B are independent if and only if:

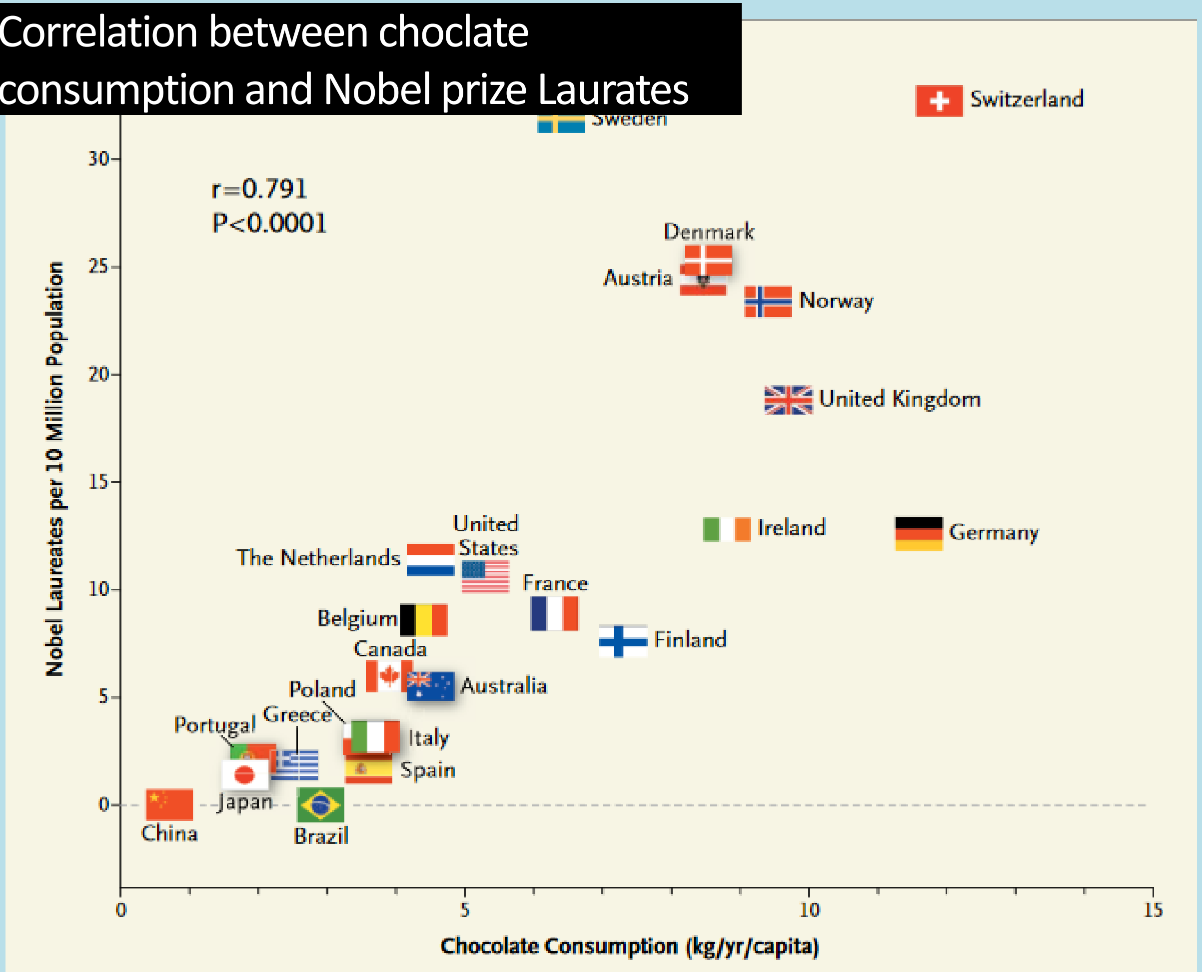
$$P(A \land B) = P(A)P(B)$$

- In other words

$$P(A \mid B) = P(A) \; and \; P(B \mid A) = P(B)$$

- Causality vs. Correlation

Correlation between choclate consumption and Nobel prize Laurates



Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020); Image Source: Messerli, F. H. et al. (2012)

8.1 Conditional independence: Example I

- If the car's engine is broken, the probability that the car stops doesn't depend on whether there is a warning message or not

$$P(Stop \mid Warning, Broken) = P(Stop \mid Broken)$$

- Therefore, that you have to interrupt your road trip and go to a mechanic is conditionally independent of a warning message given a broken engine

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Conditional independence: Example II

- Likewise, a warning message is conditionally independent of a car stop given a broken engine

$$P(Warning \mid Broken, Stop) = P(Warning \mid Broken)$$

- Equivalent statement:

$$P(Warning, Stop \mid Broken)$$

$$= P(Warning \mid Broken)\, P(Stop \mid Broken)$$

! In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Naïve Bayes Model

- Suppose we have many different types of observations (symptoms, features) that we want to use to diagnose the underlying cause

- It is usually impractical to directly estimate or store the joint distribution

$$P(Cause, Effect_1, \ldots, Effect_n).$$

- To simplify things, we can assume that the different effects are conditionally independent given the underlying cause

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Naïve Bayes Model

- To simplify things, we can assume that the different effects are conditionally independent given the underlying cause

- Then we can estimate the joint distribution as

$$P(Cause, Effect_1, \ldots, Effect_n).$$

$$P(Cause, Effect_1, \ldots, Effect_n) = P(Cause) \prod_i P(Effect_i \mid Cause)$$

- This is usually not accurate, but very useful in practice ☺

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Summary: Probabilistic inference

- **Application for us**: In general, the agent observes the values of some random variables $X_1, X_2, \dots, X_n$ and needs to reason about the values of some other unobserved random variables $Y_1, Y_2, \dots, Y_m$

- **General applications:**
  - Figuring out a diagnosis based on symptoms and test results
  - Classifying the content type of an image or a document based on some features
  - Etc.

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

8.1 Sources of Probabilities

- Statistical Experiments
- Knowledge or Experience
- Estimating
- What is the probability that the sun will rise tomorrow?

Adapted from Rusell, S., & Norvig, P. (2016)

Previous Exam Task!

Your turn!

Task

Please answer the following questions:

- How many entries do we need to represent the joint probability table of $P(Warning, Broken, Stop)$?
- Write out the joint distribution using chain rule: $P(Warning, Broken, Stop)$. How many entries do we need to represent these distributions?
- Where is the benefit of the usage of conditional independence in this example? Why do AI scientists do that?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020)

Outline

Image source: ↗ Pixabay (2019) / ↗ CC0

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▶ **Duration:**
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- 10.1 – 10.4

8.2 Probabilistic Inference

- Given:
  - Query variables: X
  - Evidence (observed) variables: $E = e$
  - Unobserved variables: Y
- If we know the full joint distribution $P(X, E, Y)$, how can we perform inference about X?

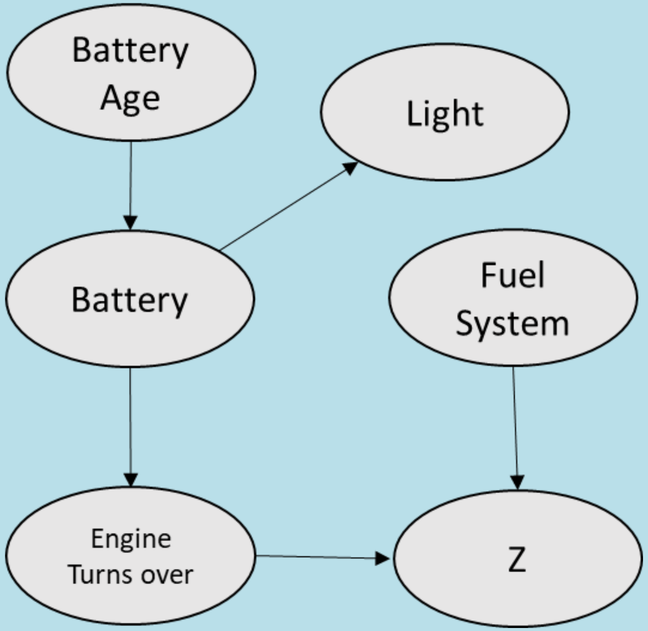
$$P(\boldsymbol{X} \mid \boldsymbol{E} = \boldsymbol{e}) = \frac{P(\boldsymbol{X}, \boldsymbol{e})}{P(\boldsymbol{e})} \propto \sum_{\boldsymbol{y}} P(\boldsymbol{X}, \boldsymbol{e}, \boldsymbol{y})$$

- Problems
  - Full joint distributions are too large
  - Marginalizing out Y may involve too many summation terms

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Bayesian Networks

▶ Represents a set of variables and their conditional dependencies via a directed acyclic graph



- More commonly called belief or probabilistic networks
- A way to depict conditional independence relationships between random variables
- A compact specification of full joint distributions

Example

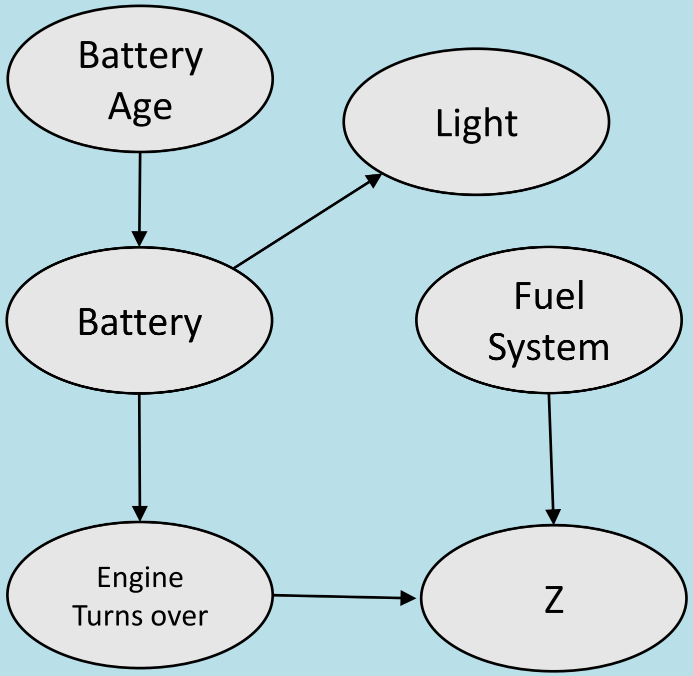
- Root Cause Analytics (RCA) in Automotive Industry

+ Natural representation for (causally induced) conditional independence, topology and conditional probability tables, generally easy for domain experts to construct

- Requires greater statistical expertise than some other methods, only as useful as this prior knowledge

8.2 Structure of Bayesian Networks

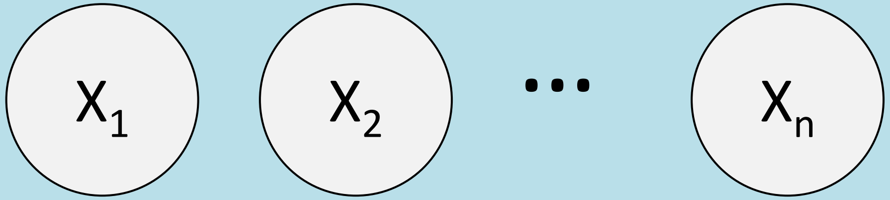
- A Bayesian network is made of nodes and arcs
- Each node represents one random variable and can be assigned (observed) or unassigned (unobserved)
- Arcs represent interactions. An arrow from one variable to another indicates direct influence from that variable upon the other variable
- You can encode conditional independence
- Must form a directed, acyclic graph



Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

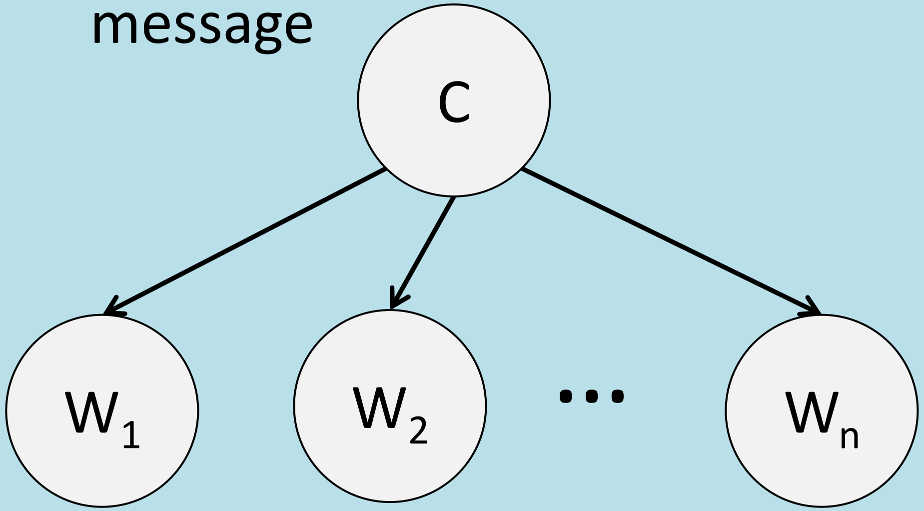
N independent Coin Flips

- Complete independence: no interactions



Naïve Bayes Spam Filter

- Random variables:
  - C: message class (spam or not spam)
  - $W_1, \ldots, W_n$: words comprising the message



Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Example: Porsche Theft Protection

- I bought a new Porsche (718) and set up a theft protection alarm that is sometimes set off by animals on the parking space. My two colleagues John and Mary promised to call me if they hear the alarm
  - Example inference task: suppose Mary calls and John doesn’t call. Is my Porsche stolen?

- What are the random variables?
  - *Theft*, *Bird attack on my car*, *Alarm*, *John calls*, *Mary calls*

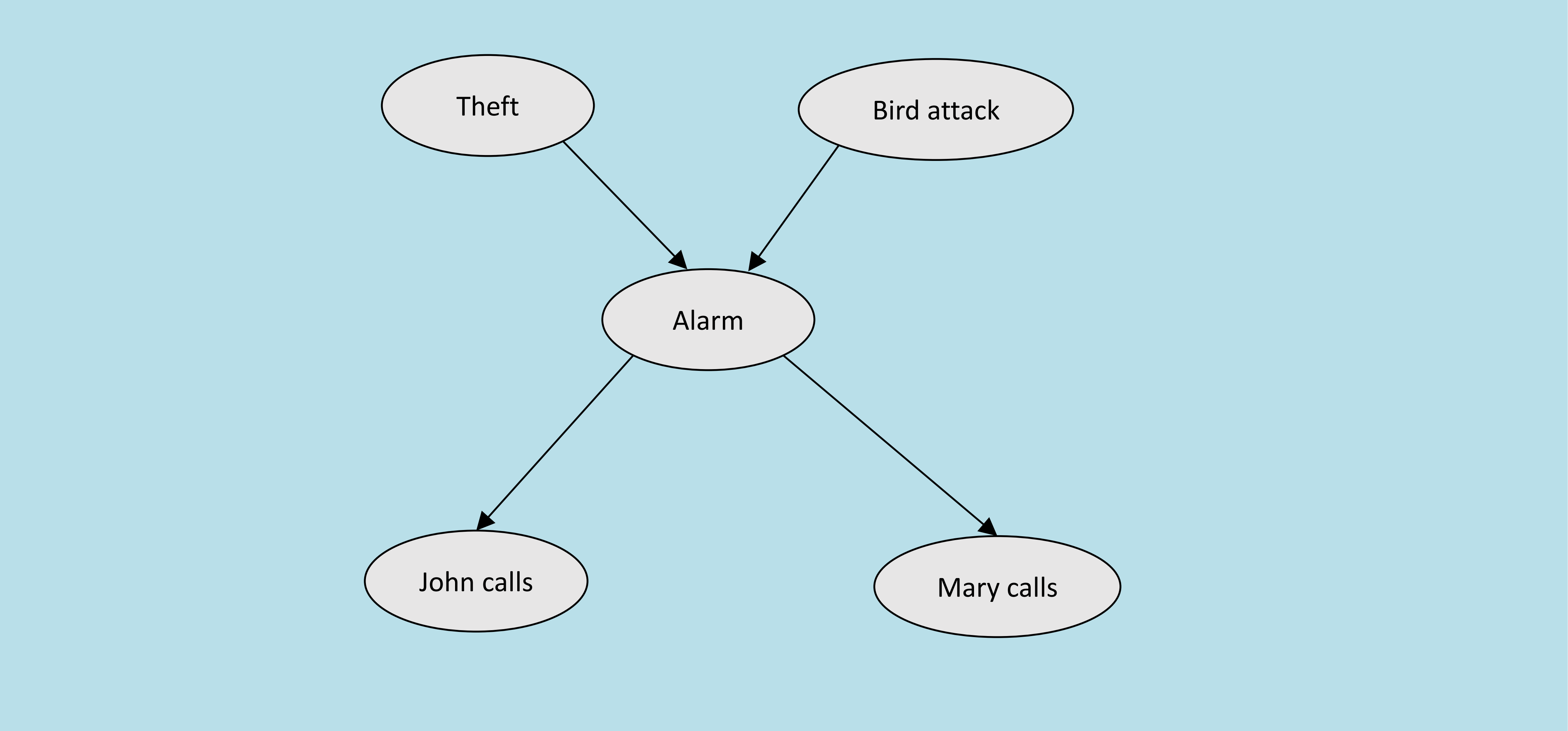
- Network topology reflects causal knowledge:
  - A thief can set the alarm off
  - A bird shitting on my car’s roof can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

?

How would the model look like? Why?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020) | Image source: ↗ Pixabay (2019) / ↗ CC0 | The following example is adapted from Judea Pearl who made a similar example with earthquakes and burglaries.

8.2 Example: Porsche Theft Protection



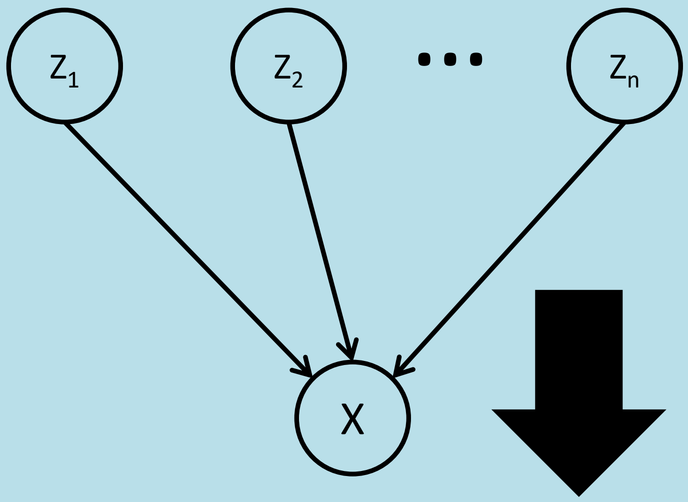
Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Conditional Probability Distributions in Bayesian Networks

- In addition to the structure, we need a conditional probability distribution for each node given the random variables of its parents

- Specify a conditional distribution for each node given its parents:

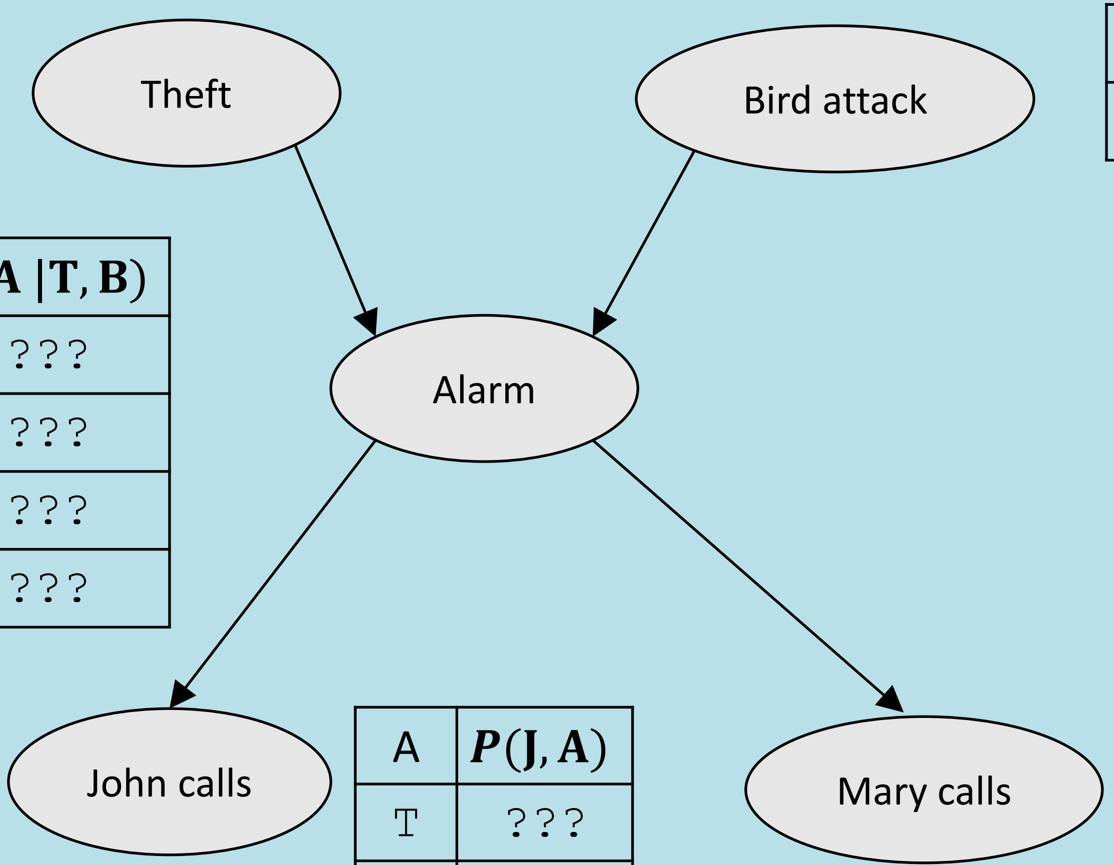
$$P\ (X \mid Parents(X))$$



P (X | Z_1, …, Z_n)

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Example 1: Porsche Theft Protection I



$\boldsymbol{P}(\mathbf{T})$
???

$\boldsymbol{P}(\boldsymbol{B})$
???

T	B	$\boldsymbol{P}(\mathbf{A}\ \vert \mathbf{T},\mathbf{B})$
T	T	???
T	F	???
F	T	???
F	F	???

A	$\boldsymbol{P}(\mathbf{J},\mathbf{A})$
T	???
F	???

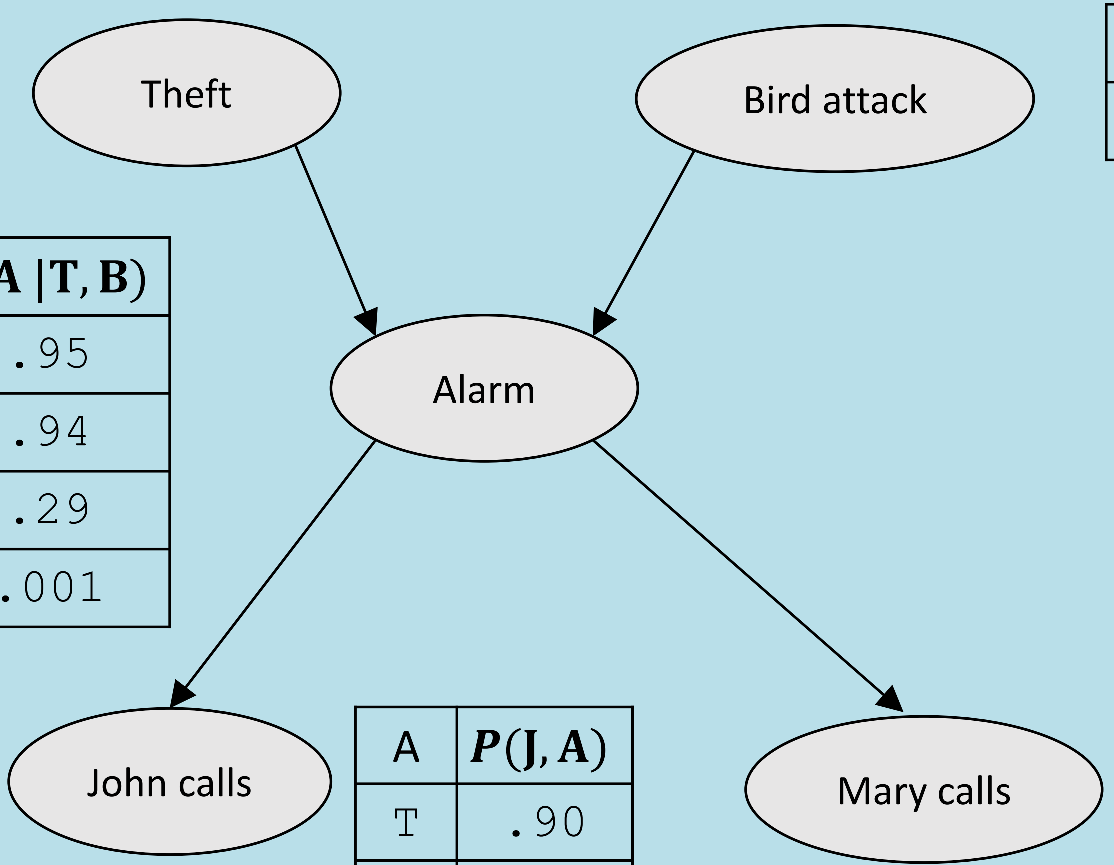
A	$\boldsymbol{P}(\mathbf{M},\mathbf{A})$
T	???
F	???

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Example 1: Porsche Theft Protection II

$P(\mathbf{T})$
.001

T	B	$P(\mathbf{A} \mid \mathbf{T}, \mathbf{B})$
T	T	.95
T	F	.94
F	T	.29
F	F	.001



$P(B)$
0.002

A	$P(\mathbf{J}, \mathbf{A})$
T	.90
F	.05

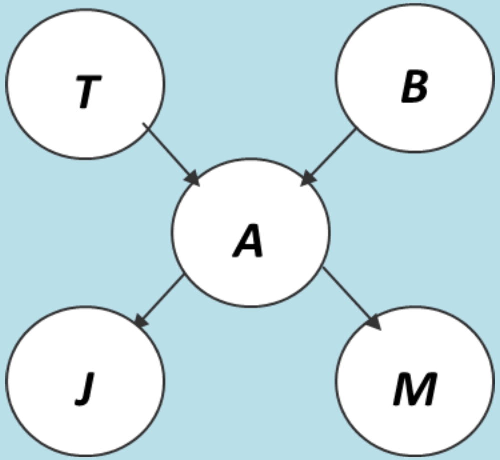
A	$P(\mathbf{M}, \mathbf{A})$
T	.70
F	.01

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 The Joint Probability Distribution

- For each node X_i, we know $P(X_i \mid Parents(X_i))$
- How do we get the full joint distribution $P(X_1, \ldots, X_n)$?
- Using chain rule:

$$P(X_1,\ldots,X_n) = \prod_{i=1}^{n} P\left(X_i \mid X_1,\ldots,X_{i-1}\right) = \prod_{i=1}^{n} P\left(X_i \mid Parents(X_i)\right)$$

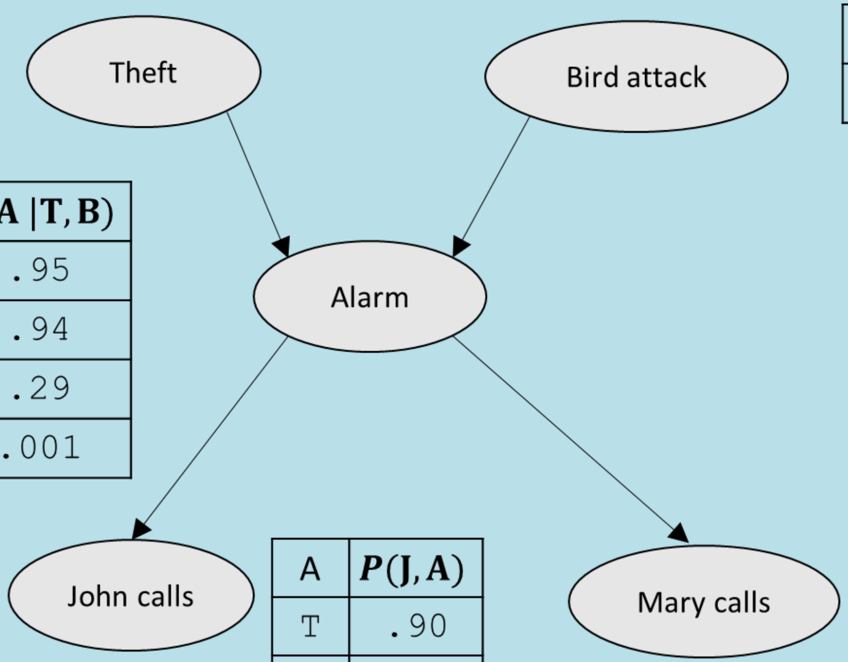


?
What is the probability that the alarm starts and both John and Mary call, but there is neither a bird attack nor a thief stealing the car?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Calculation of the Joint Probability Distribution

What is the probability that the alarm starts and both John and Mary call, but there is neither a bird attack nor a thief stealing the car?



$P(T)$
.001

$P(B)$
0.002

T	B	$P(A \mid T,B)$
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	$P(J,A)$
T	.90
F	.05

A	$P(M,A)$
T	.70
F	.01

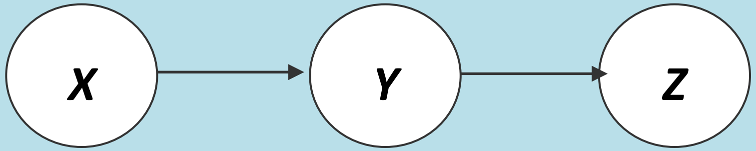
- $P(J, M, A, \neg T, \neg B)$

$$= P(\neg T)\, P(\neg B)\, P(A \mid \neg T, \neg B)\, P(J \mid A) P(M \mid A)$$

$$= 0.999 \cdot 0.998 \cdot 0.001 \cdot 0.9 \cdot 0.7 \approx 0.00063$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

- Key assumption: X is conditionally independent of every non-descendant node given its parents
- Example: causal chain

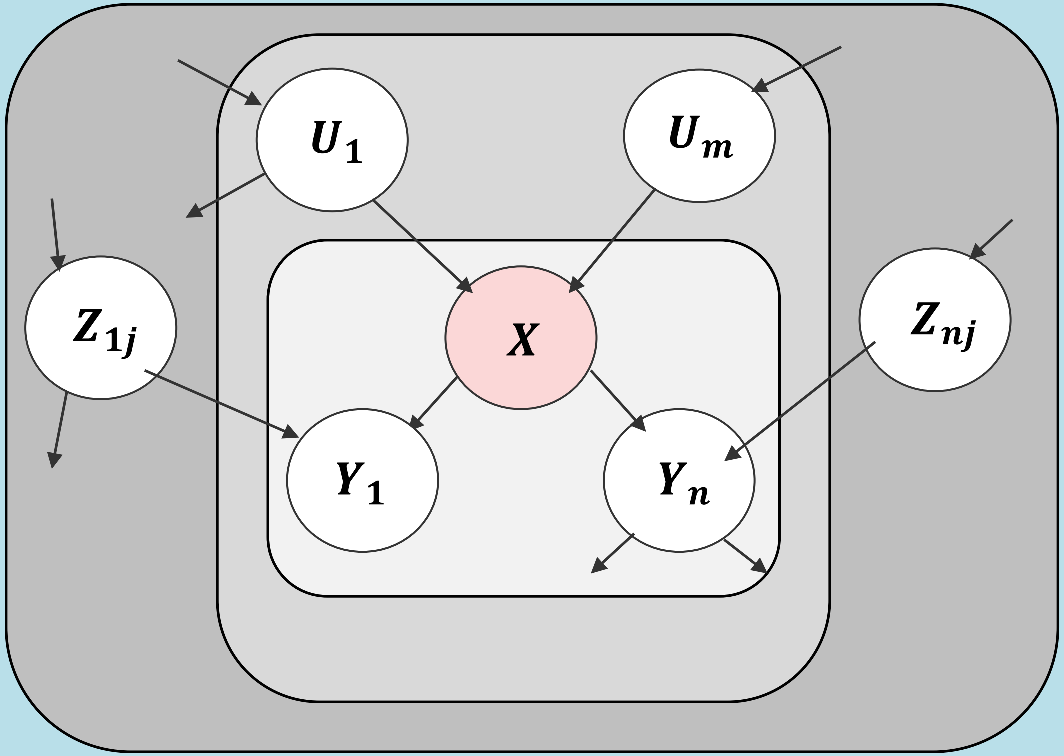


- Are X and Z independent?
- Is Z independent of X given Y?

$$P(Z \mid X, Y) = \frac{P(X, Y, Z)}{P(X, Y)} = \frac{P(X)P(Y \mid X)P(Z \mid Y)}{P(X)P(Y \mid X)} = P(Z \mid Y)$$

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

- Each node is conditionally independent of its nondescendants given its parents



$$P(X \mid U_1, \ldots, U_m, Z_{1j}, \ldots, Z_{nj}) = P(X \mid U_1, \ldots, U_m)$$

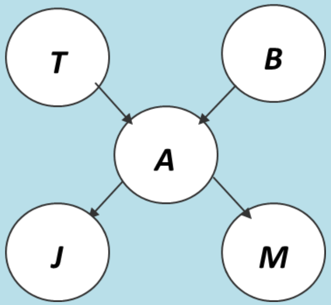
D **Markov Blanket**

Each node is conditionally independent of all other nodes given its Markov blanket (Parents + children + children's parents)

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Compactness

- Suppose we have a Boolean variable X_i with k Boolean parents. How many rows does its conditional probability table have?
  - 2^k rows for all the combinations of parent values
  - Each row requires one number p for $X_i = true$



- If each variable has no more than k parents, how many numbers does the complete network require?
  - $O(n \cdot 2^k)$ numbers – vs. $O(2^n)$ for the full joint distribution

- How many nodes for the car thief network?
  - $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

? Why do we need only 31 and not 32 nodes for the network?

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Constructing Bayesian Networks

- Identify the direct relationships of your network
  - $P(M \mid J, A, T, B) = P(M \mid A)$

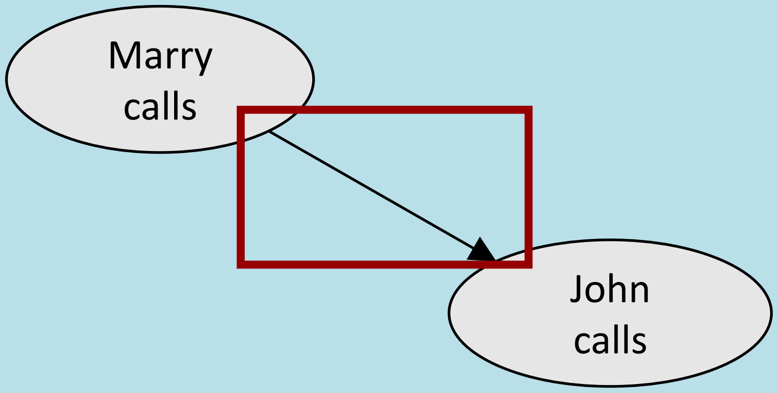
- Choose an ordering of variables $X_1, \dots, X_n$

- While you have remaining variables
  - add X_i to the network
  - select parents from $X_1, \dots, X_{i-1}$ such that

    $P(X_i \mid Parents(X_i)) = P(X_i \mid X_1, \dots X_{i-1})$ and define the probability table

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

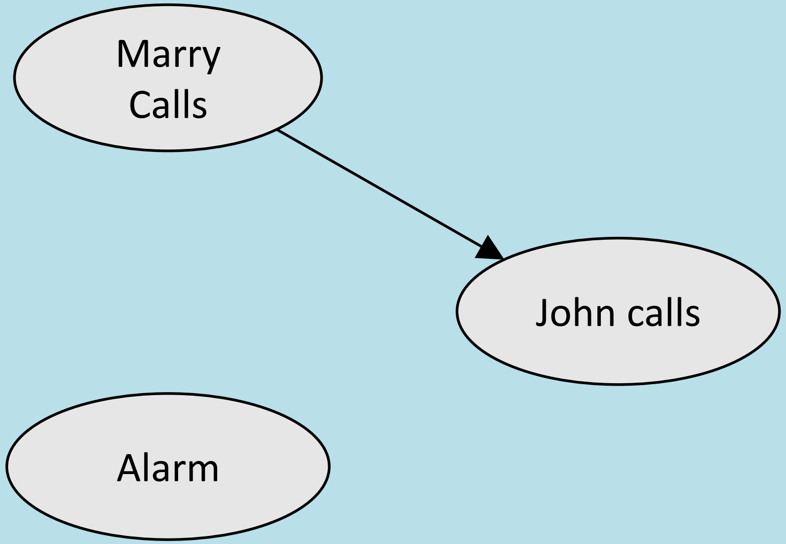
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(J | M) = P(J)?
- If Marry calls, it is more likely that John calls as well



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 2

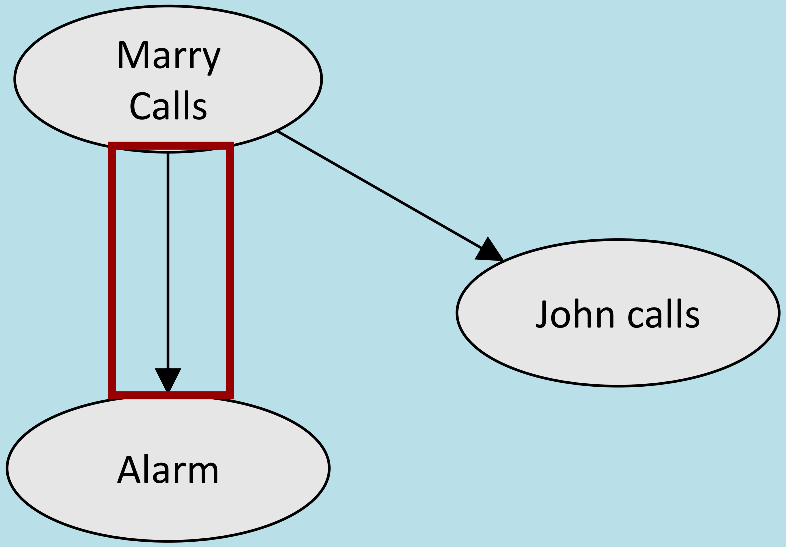
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(A |J, M) = P(A)? X
- If Marry and John call, the probability that the alarm has gone off is larger than if they don‘t call
- Node A needs parents J or M



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 3

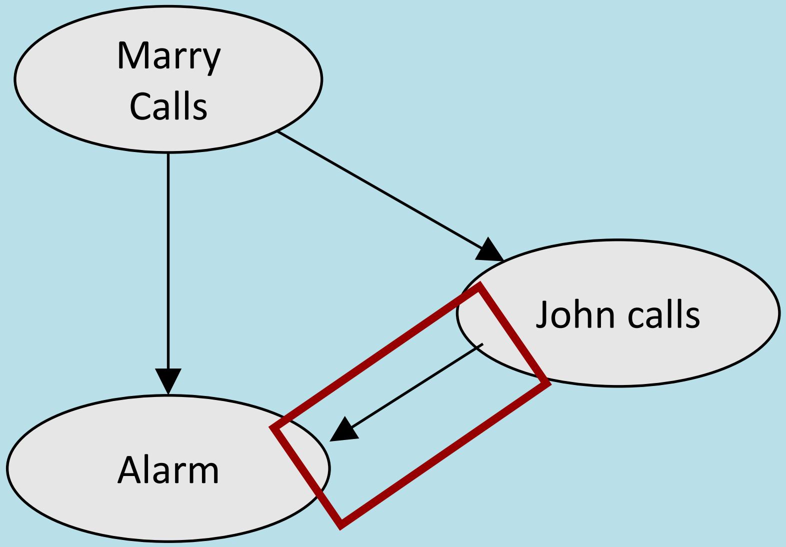
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(A |J, M) = P(A | J)? X
- If Marry and John call, the probability that the alarm has gone off is larger than if only John calls



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 4

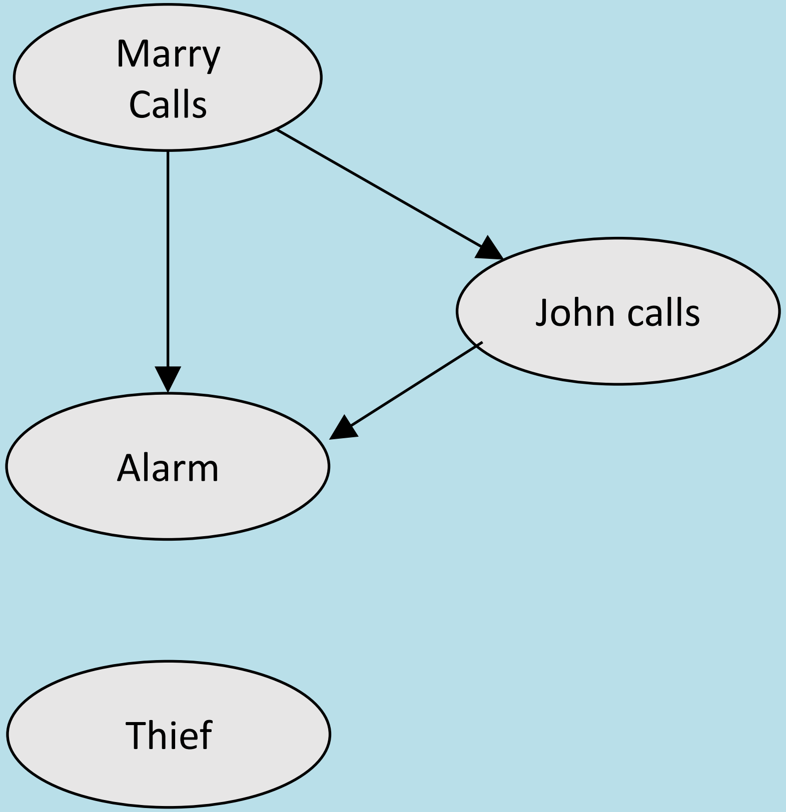
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(A |J, M) = P(A | M)? X
- If Marry and John call, the probability that the alarm has gone off is larger than if only Marry calls



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 5

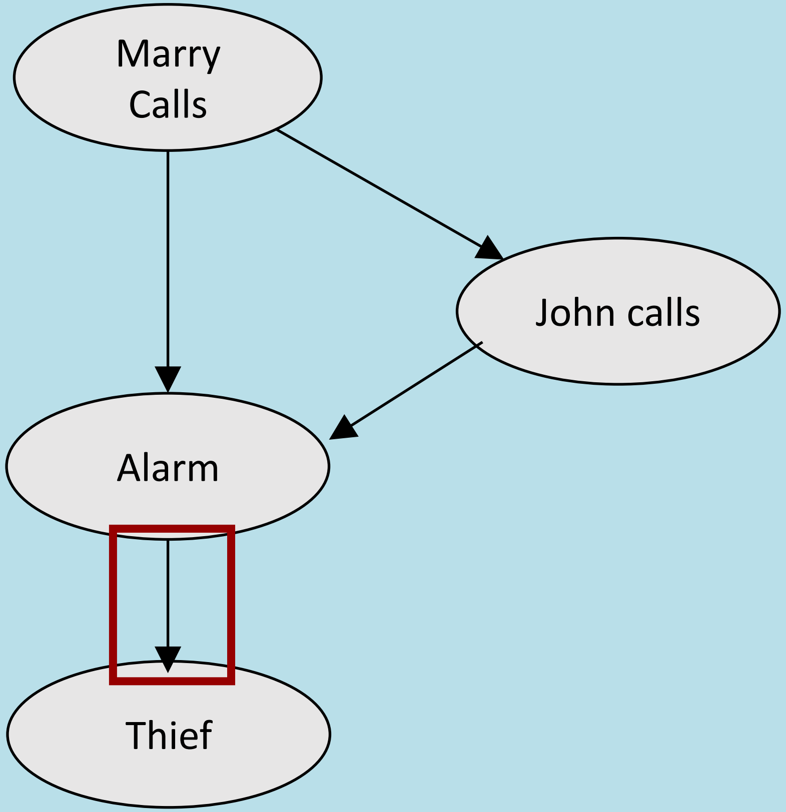
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(T |A, J, M) = P(T)? X
- Knowing whether Mary or John called and whether the alarm went off influences my knowledge about whether the car has been stolen
- Node T needs parents A, J, M



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 6

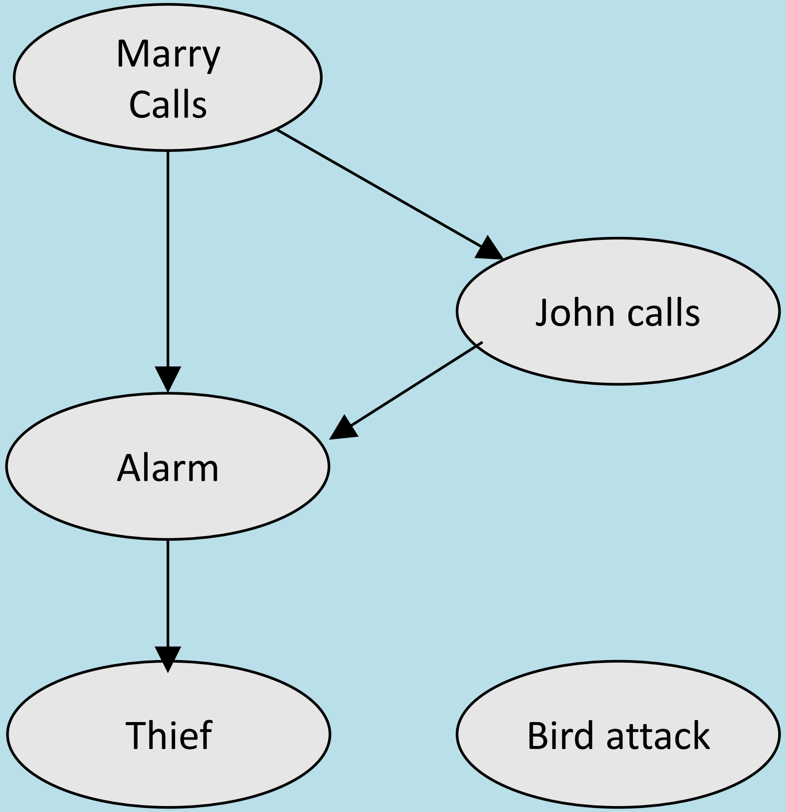
- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(T |A, J, M) = P(T | A)? ✓
- If I know that the alarm has gone off, knowing that John or Mary have called does not add to my knowledge of whether there has been a thief or not
- Thus, no edges from M and J, only from T



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 7

- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack
- P(B |T, A, J, M) = P(B | A)? X
- Knowing whether there has been an alarm does not suffice to determine the probabiliy of a bird shitting on my car, we have to know if my car is stolen or not



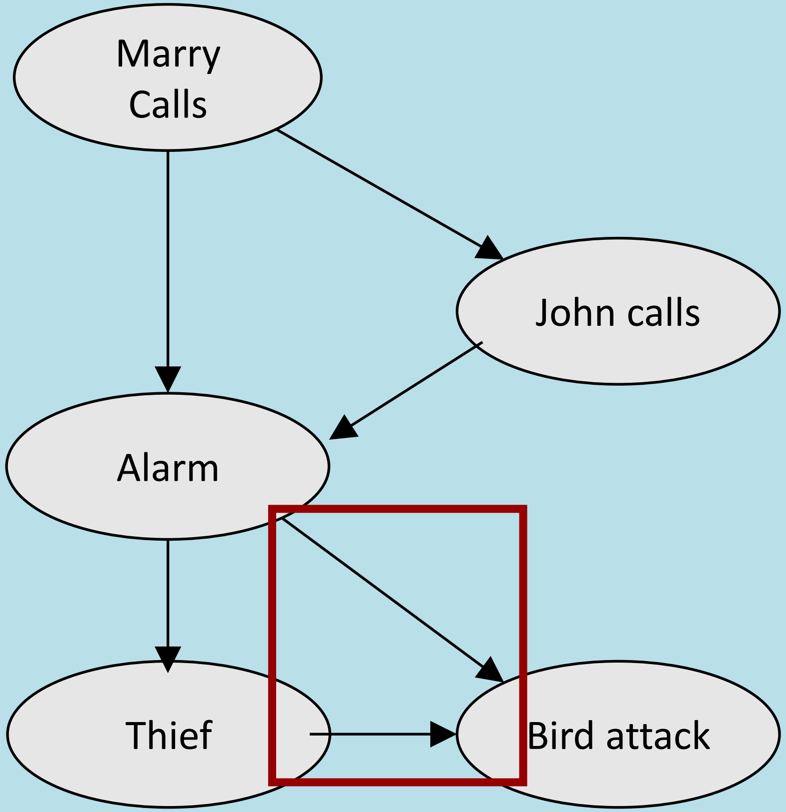
Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Step 8

- Suppose we choose the ordering MarryCalls, JohnCalls, Alarm, Thief, BirdAttack

- P(B |T, A, J, M) = P(B | A)?
- P(B |T, A, J, M) = P(B | A, T)?

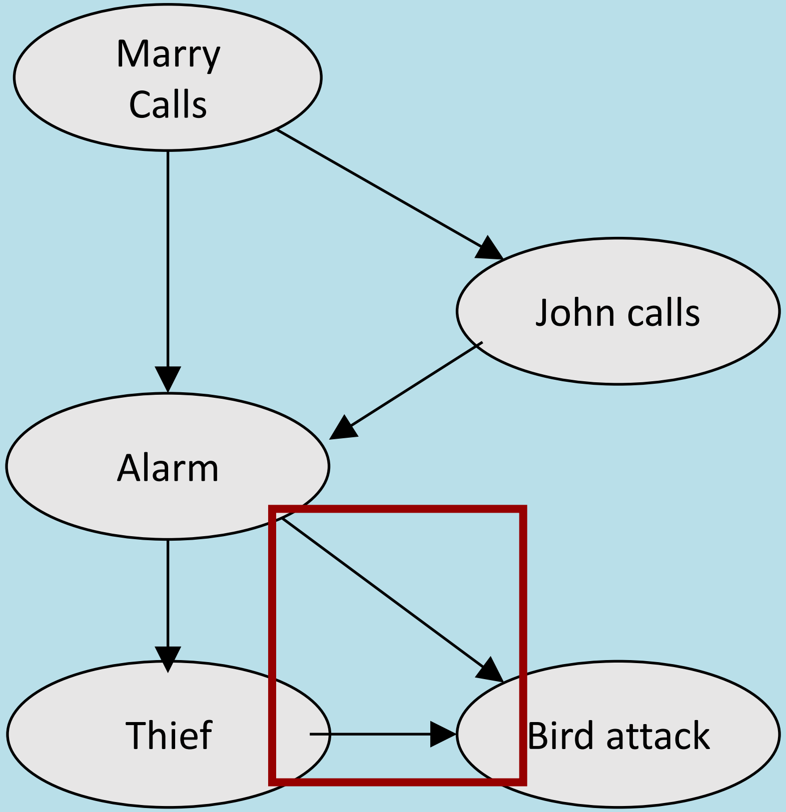
- Knowing whether there has been an alarm and whether my car has been stolen, no other factors will determine our knowledge about whether there has been a bird with digestive problems



Adapted from Rusell, S., & Norvig, P. (2016)

8.2 Example 1 ▶ Discussion

- Deciding conditional independence is hard in noncausal directions
  - The causal direction seems much more natural
- Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed
- Worst possible ordering (fully connected network): Marry calls, John calls, Birth attack, Thief, Alarm



Adapted from Rusell, S., & Norvig, P. (2016)

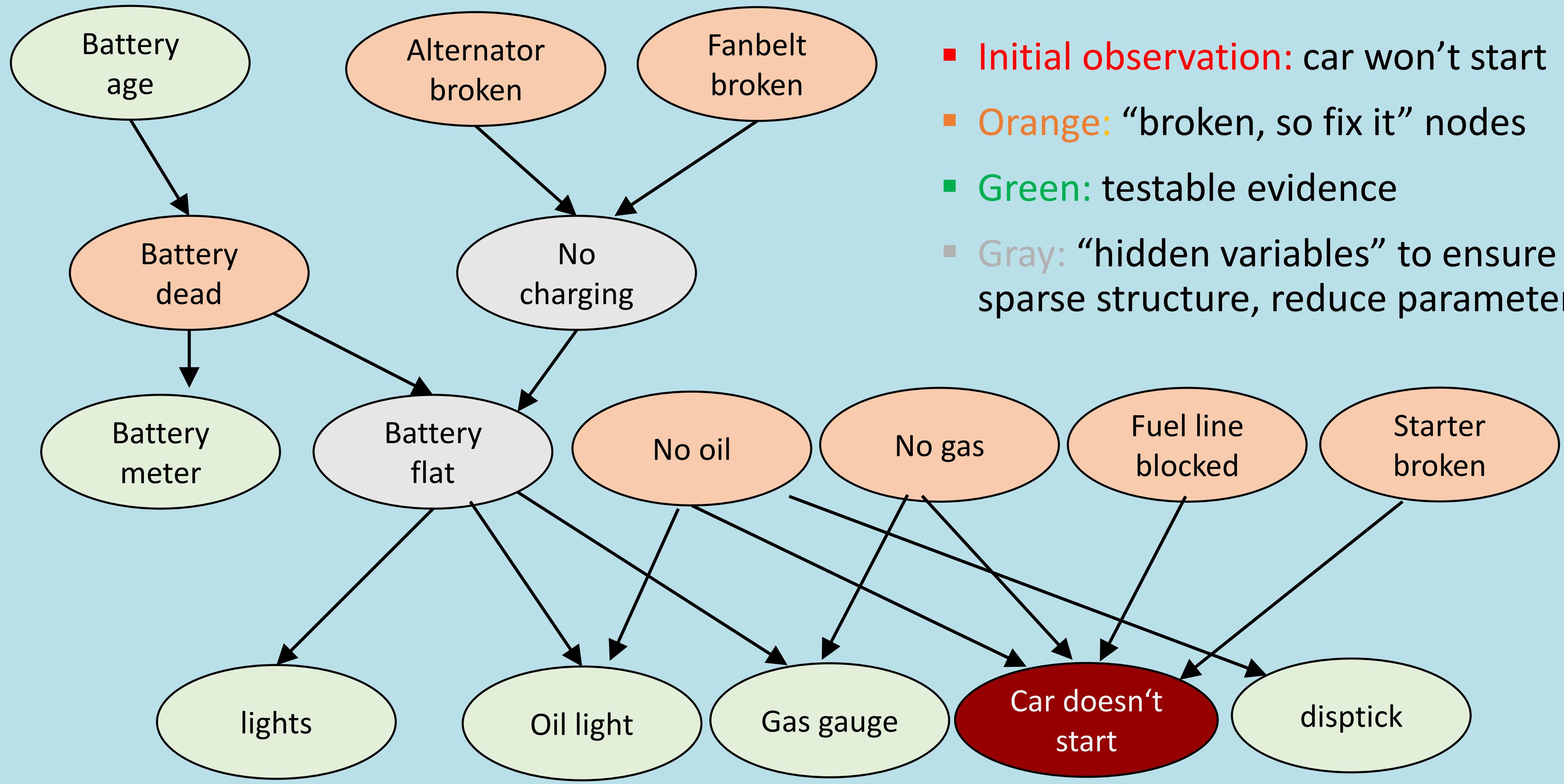
8.2 Bayesian Networks and Continous Variables

- Problems:
  - Conditional probability tables require discrete values
  - Many real-live scenarios have continous variables

- Solution:
  - Discretization (see 📚 “Chapter 4 – Data and Feature Engineering“)
  - Define probability density functions (require specific methods for combining discrete and continous variables)

Adapted from Rusell, S., & Norvig, P. (2016)

8.2 A more Realistic Car Diagnosis Network

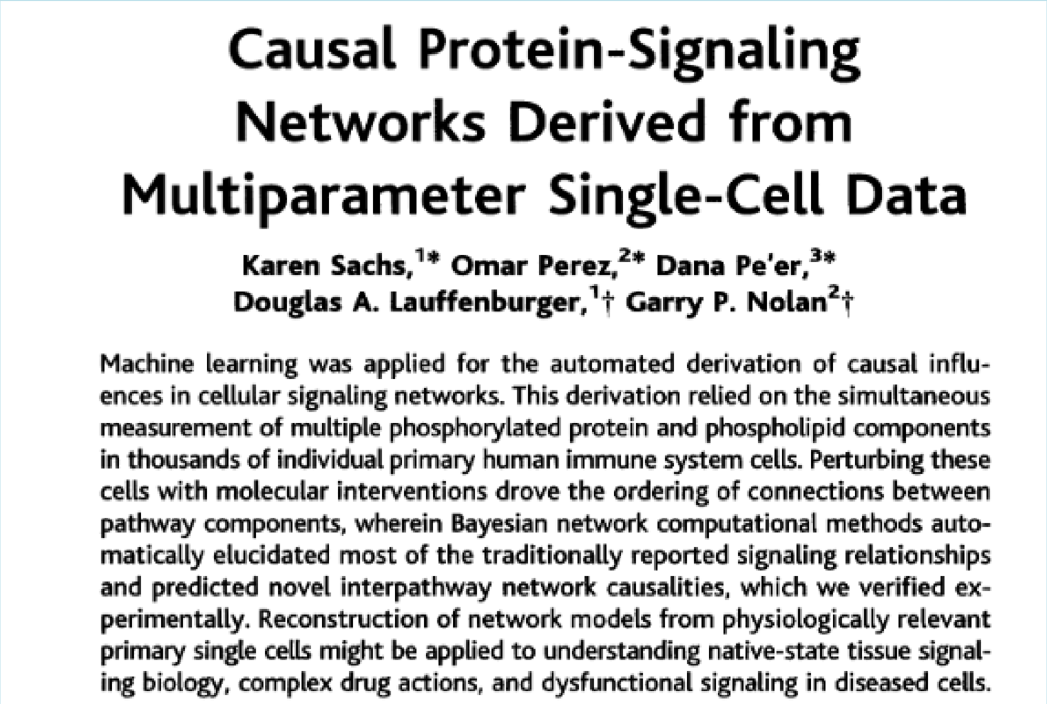


- Initial observation: car won't start
- Orange: "broken, so fix it" nodes
- Green: testable evidence
- Gray: "hidden variables" to ensure sparse structure, reduce parameters

Adapted from Rusell, S., & Norvig, P. (2016); Frochte, J. (2020);

8.2 Application in Bioinformatics

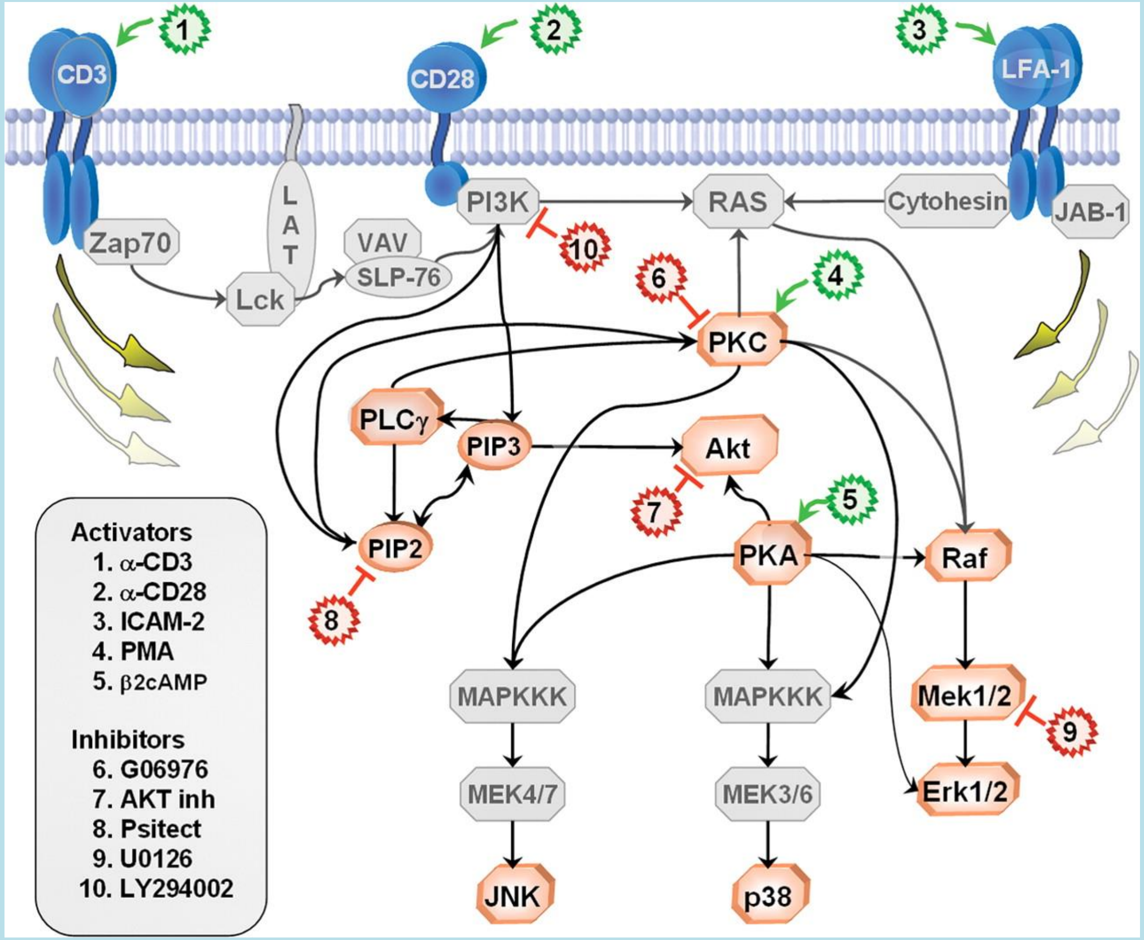
- Probabilistic Modelling for the automated derivation of causal influences in cellular signaling network



Causal Protein-Signaling Networks Derived from Multiparameter Single-Cell Data

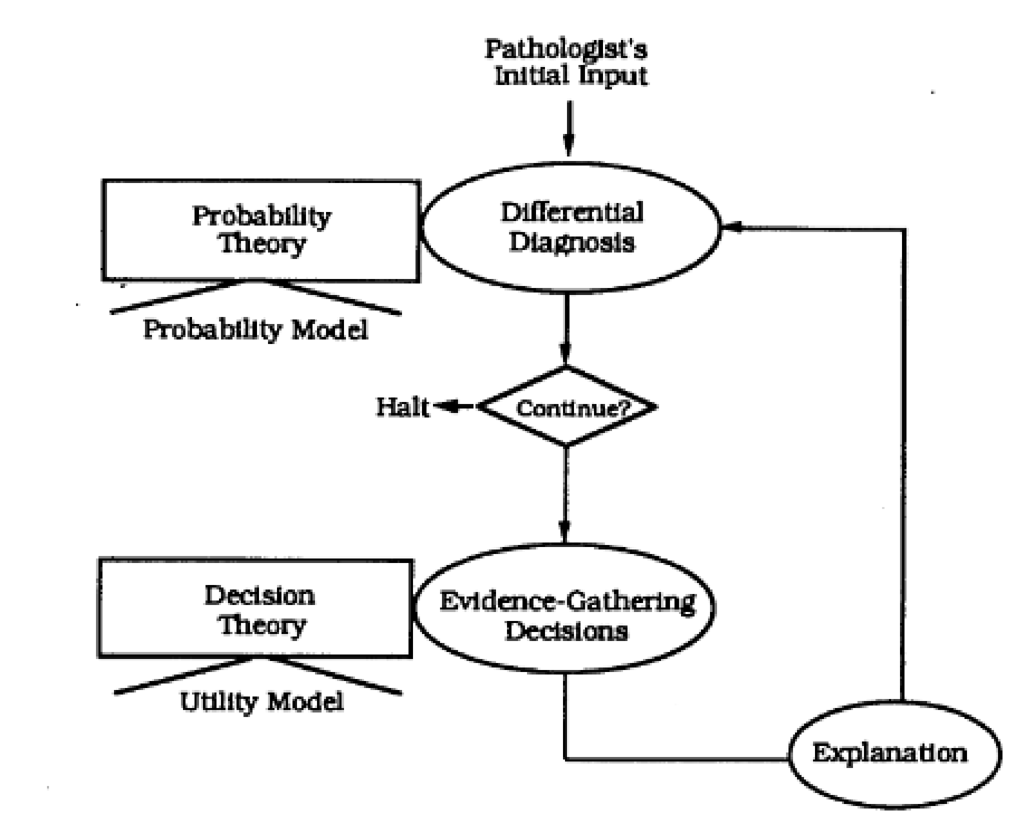
Karen Sachs,[1*] Omar Perez,[2*] Dana Pe'er,[3*] Douglas A. Lauffenburger,[1†] Garry P. Nolan[2†]

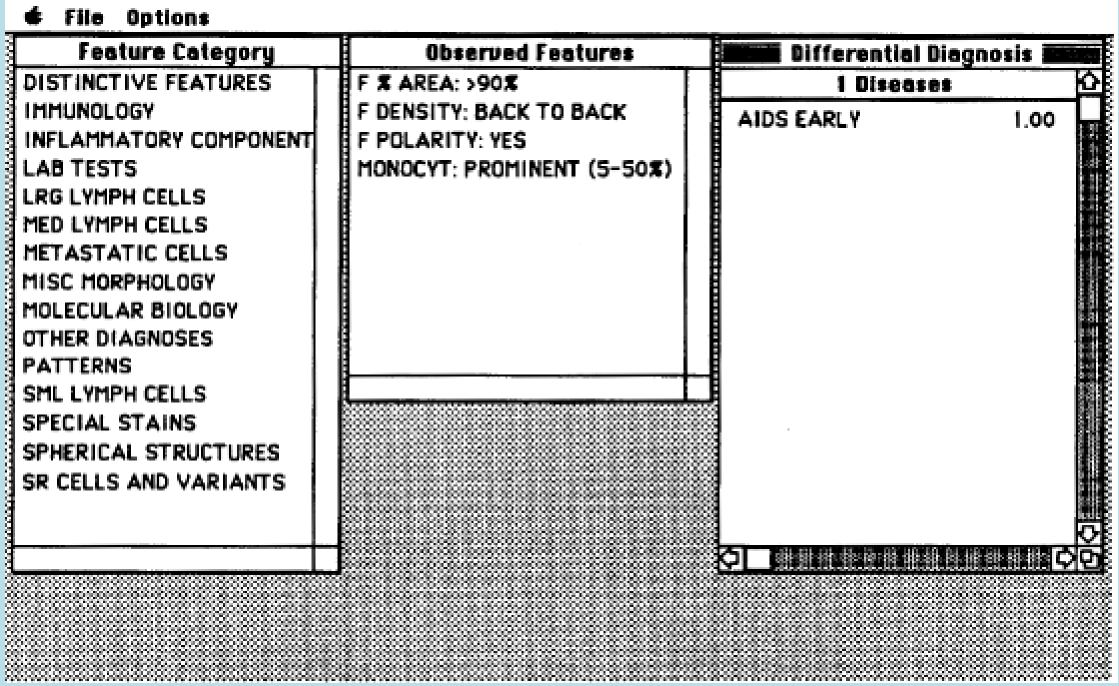
Machine learning was applied for the automated derivation of causal influences in cellular signaling networks. This derivation relied on the simultaneous measurement of multiple phosphorylated protein and phospholipid components in thousands of individual primary human immune system cells. Perturbing these cells with molecular interventions drove the ordering of connections between pathway components, wherein Bayesian network computational methods automatically elucidated most of the traditionally reported signaling relationships and predicted novel interpathway network causalities, which we verified experimentally. Reconstruction of network models from physiologically relevant primary single cells might be applied to understanding native-state tissue signaling biology, complex drug actions, and dysfunctional signaling in diseased cells.



Adapted from Sachs K et al. (2005) | Image Source: Sachs K et al. (2005)

8.2 Expert System and Probabilistic Modelling





- Normativ Expert Systems: Pathfinder IV System
- Research project (Stanford & Sth. California), expert systems based on decision theory for medical knowledge reasoning

Adapted from Heckerman, D. E. et al. (1992) | Image Source: Heckerman, D. E. et al. (1992)

Your turn!

Task

Given a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes $X1$, $X2$, and $X3$.

- Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
- Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

Outline

Image source: ↗ Pixabay (2019) / ↗ CC0

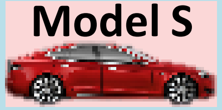
- **What you will learn:**
  - Concepts of statistics and probability theory to model agents that can act under uncertainty
  - Foundations of Markov theory, Bayes theory and sequence analysis for to reason under uncertainty according to the laws of probability theory
  - How to analyze and build optimal agent decision-making that can handle uncertainty

- **Duration:**
  - 215 min

- **Relevant for Exam:**
  - 10.1 – 10.4

8.3 Example: Electric Vehicle Market in 2019

- You startet your career as an AI-Specialist at the Porsche AG. At the moment we assume the following three EVs in the European market:



- Let us suppose that your newest electric car, the Taycan controls now 20 % of the EV market. The competitors Model S controlls 30 %, and the Renault Zoe 50%.
- The sales manager (best friend from your studytime at university) ask you to analyze the impact of a new marketing programm on the leasing market of EVs.

8.3 Example: Electric Vehicle Market

Suppose they find evidence that for the next buy:

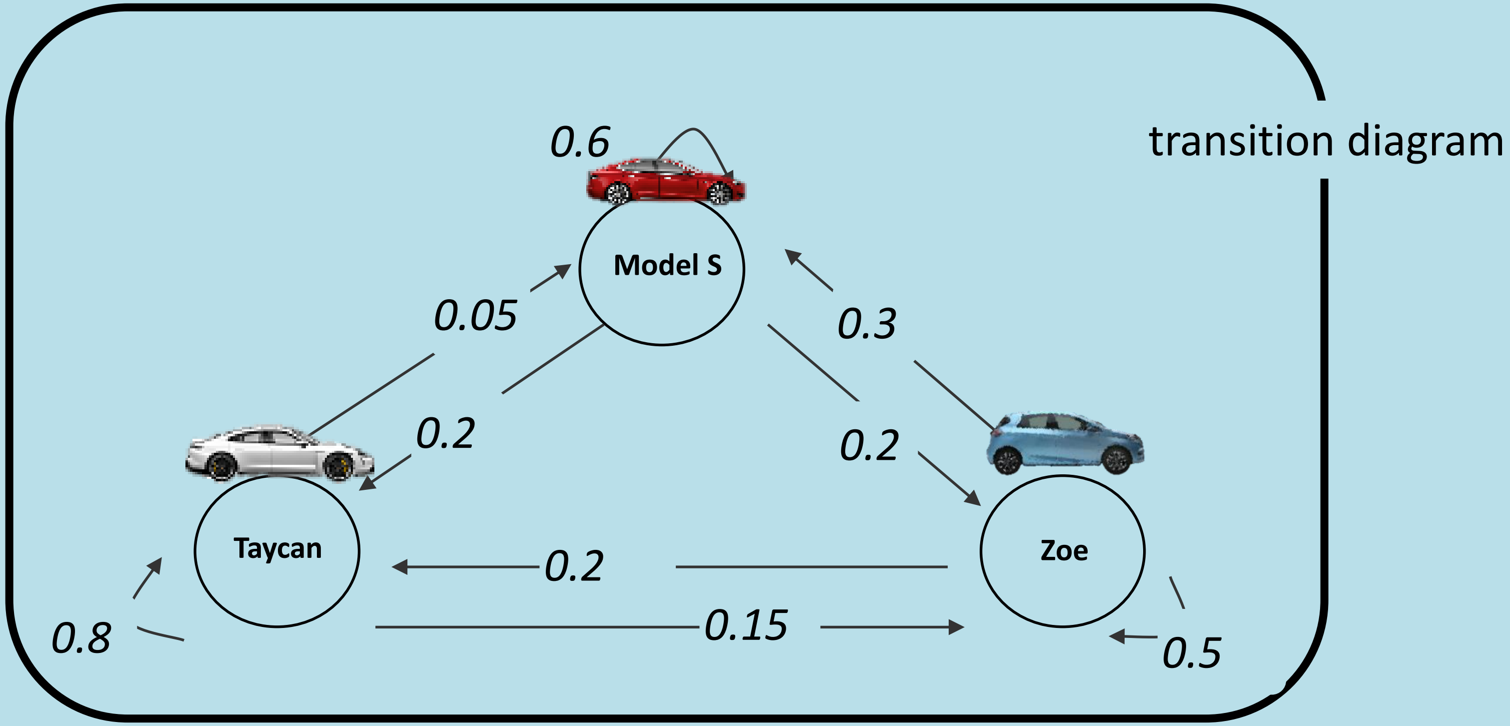
- Someone driving Taycan will be so impressed that he continues to drive Taycan with 0.8 probability. The others will switch to Tesla with 0.05 and to the Renault Zoe with 0.15 probability.

- Someone driving Tesla will begin to drive Taycan with 0.2 probability. The others will continue to drive Tesla with 0.6 probabilty or start to drive Zoe with 0.2 probability.

- Someone driving Zoe will like the marketing campaign with 0.2 probabilty, so he will also start to drive Taycan. The others will switch to Tesla with 0.3, or stay to Renault with 0.5 probability.

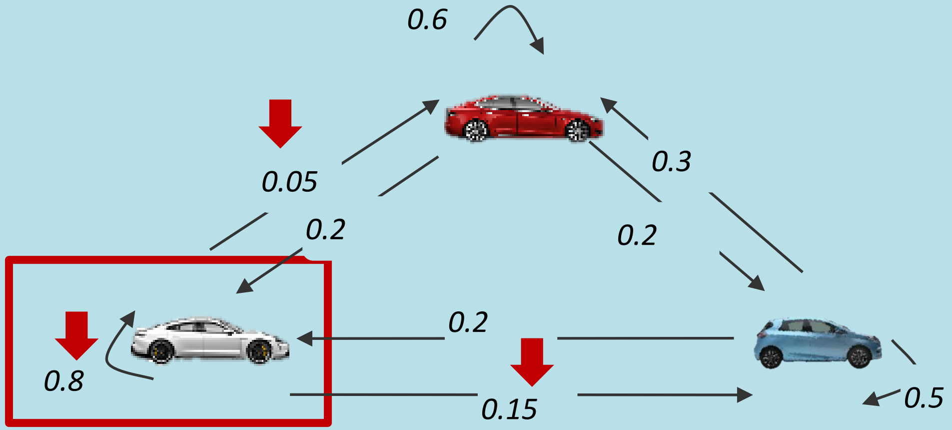
? How could you model these different dynamics in this case?

8.3 Markov Model of the Car Market



initial market segmentation with 3 states $\pi_0 = [0.2, 0.3, 0.5]$

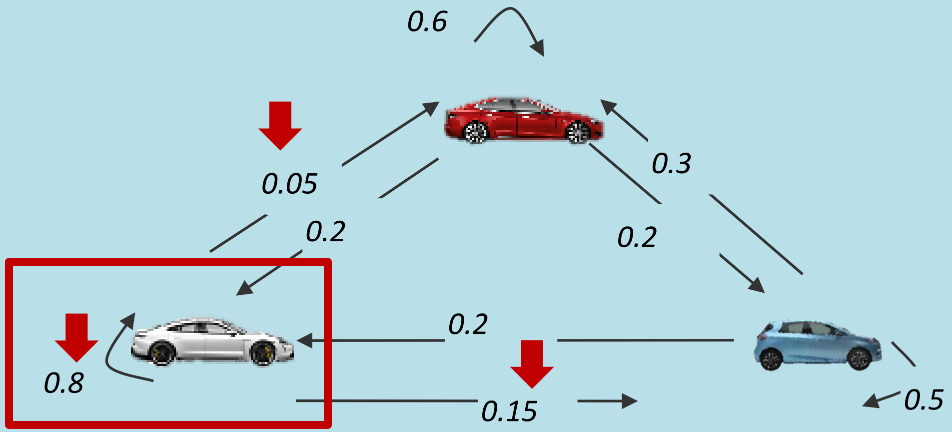
- Now we write this transition diagram in a more formal way, so that we can do some math with it…



$$P = (p_{ij}) = \begin{array}{cc} & \textit{to } j \\ \textit{from } i & \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ & & \\ & & \end{bmatrix} \end{array}$$

- The row of the matrix indicates, that there is a 0.8 probability that you stay at Taycan ($p_{TT} = 0.8$) and 0.05 that you switch to Tesla ($p_{TM} = 0.05$) and 0.15 that you switch to Renault ($p_{TR} = 0.15$).

- And finally we get the transition probability matrix



$$P = (p_{ij}) = \begin{array}{c c} & \textit{to } j \\ \textit{from } i & \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ & & \\ & & \end{bmatrix} \end{array}$$

Always check the row sums. They must be 1 (probabilities always sum up to 1)

8.3 Calculate market segmentation in 1 month

- The final transition probability matrix allows us, together with the initial state distribution, to calculate some probabilities...

- For instance: What is the probability that someone drives Taycan after 1 month or 1 timestep of the marketing campaign?

$$P = \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$\pi_0 = [0.2, 0.3, 0.5]$$

8.3 Calculate market segmentation in 1 month

$$P = \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$\pi_0 = [0.2, 0.3, 0.5]$$

$$P(\text{Taycan}) = P(\text{drives Taycan } \textit{and stays}) + P(\text{drives Tesla } \textit{and switches to}) + P(\text{drives Renault Zoe } \textit{and switches to})$$

$$P(\text{Taycan}) = 0.2 \cdot 0.8 + 0.3 \cdot 0.2 + 0.5 \cdot 0.2 = 0.32$$

8.3 Classroom Task

Previous Exam Task!

Your turn!

Task

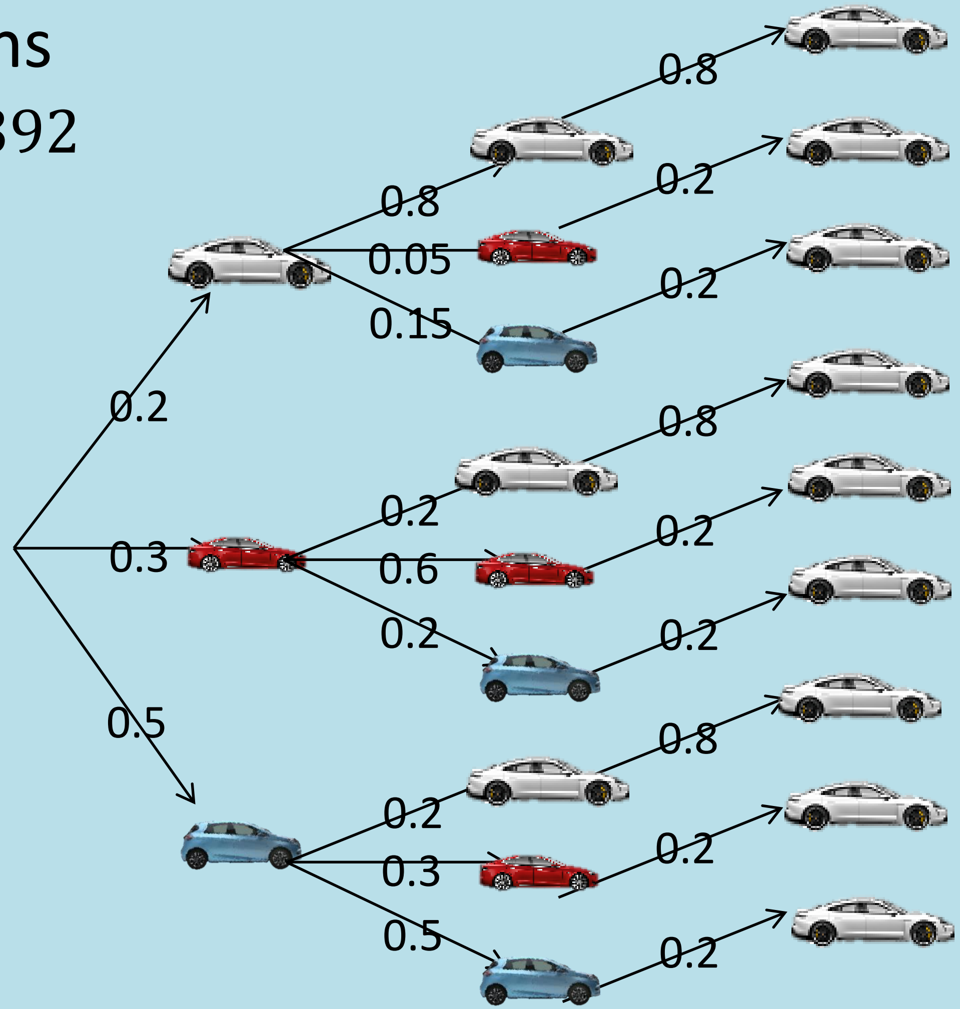
Please discuss with your neighbor:

- How do you calculate the market segmentation?
- What is the market segmentation after 2 timesteps later?

$$P = \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \qquad \pi_0 = [0.2, 0.3, 0.5]$$

8.3 Calculate Market Segmentation in t Months/Time Steps

- After 2 months
  - $P(J)_2 = 0.392$



$$P(Taycan)_2 = 0.2 \cdot 0.8 \cdot 0.8 + 0.2 \cdot 0.05 \cdot 0.2 + 0.2 \cdot 0.15 \cdot 0.2 + 0.3 \cdot 0.2 \cdot 0.8 + 0.3 \cdot 0.6 \cdot 0.2 + 0.3 \cdot 0.2 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.8 + 0.5 \cdot 0.3 \cdot 0.2 + 0.5 \cdot 0.5 \cdot 0.2$$

- If you take a look at our calculation, you can notice that we can rewrite it as:

$$\pi_1 = \pi_0 \cdot P$$

$$\pi_1 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$P(Taycan)_1 = 0.2 \cdot 0.8 + 0.3 \cdot 0.2 + 0.5 \cdot 0.2$$

$$P(Taycan)_1 = 0.32$$

$$\pi_1 = [P(Taycan), P(Tesla), P(Zoe)] = [0.32, 0.34, 0.34]$$

! Remember: Matrix multiplication is rows by columns

8.3 State Probability Distribution

- And then we can write π_2 as

$$\pi_2 = \pi_1 \cdot P = \pi_0 \cdot P \cdot P$$

$$\pi_2 = \overbrace{\begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}}^{\pi_1} \cdot \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

- And

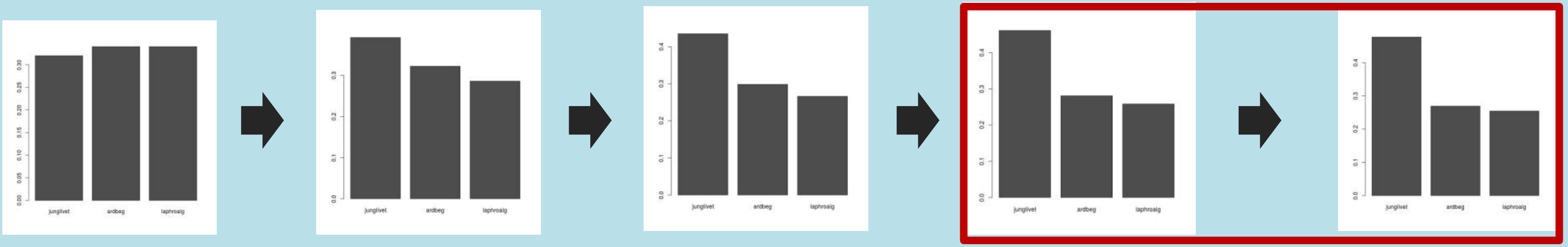
$$\pi_3 = \pi_2 \cdot P = \pi_1 \cdot P \cdot P = \pi_0 \cdot P \cdot P \cdot P$$

- Therefore

D **State probability distribution:**

$$\pi_n = \pi_0 \cdot P^n$$
$$\pi_n = \pi_{n-1} \cdot P$$

- If you calculate and plot the market segmentation



- It looks like that the segmentation is getting closer and closer to a stable distribution, $\pi = [0.5, \ 0.25, 0.25]$

- In Markov chain terms, the matrix π is called **stationary** matrix
- And the system is in steadystate, if the steadystate is

$$\pi_\infty = \pi_\infty \cdot P$$

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Identify the Unique Stationary Matrix

- A regular Markov chain is a unique stationary matrix, that can be found by solving the equation:

$$\pi = \pi \cdot P$$

- Example: $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$

$$u_1 \cdot 0.8 + u_2 \cdot 0.2 + u_3 \cdot 0.2 = u_1$$
$$u_1 \cdot 0.05 + u_2 \cdot 0.6 + u_3 \cdot 0.3 = u_2$$
$$u_1 \cdot 0.15 + u_2 \cdot 0.2 + u_3 \cdot 0.5 = u_3$$

$$s.t.\ \ u_i \geq 0; \sum u_i = 1$$

8.3 Identify the Unique Stationary Matrix II

$$u_1 \cdot 0.8 + u_2 \cdot 0.2 + u_3 \cdot 0.2 = u_1$$
$$u_1 \cdot 0.05 + u_2 \cdot 0.6 + u_3 \cdot 0.3 = u_2$$
$$u_1 \cdot 0.15 + u_2 \cdot 0.2 + u_3 \cdot 0.5 = u_3$$

$$\boxed{\begin{aligned} u_1 &= 2t \\ u_2 &= t \\ u_3 &= t \end{aligned}}$$

Problem: dependent system

$$s.t.\ \ u_i \geq 0$$
$$\sum u_i = 1$$

Solution: norm the probabilities

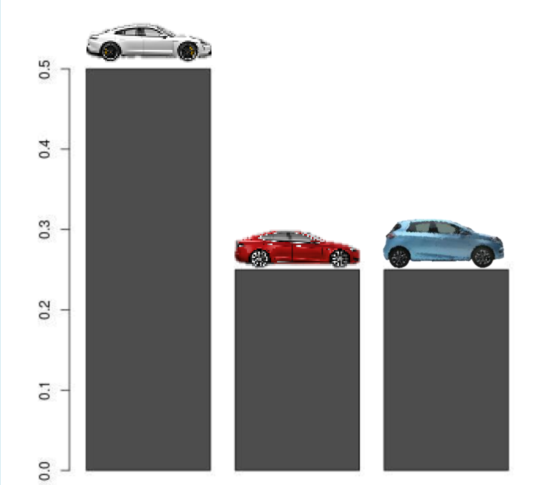
8.3 Identify the Unique Stationary Matrix III

$$u_1 + u_2 + u_3 = 1$$
$$4t = 1 \ \rightarrow t = 0.25$$

$$u_1 = 2t$$
$$u_2 = t$$
$$u_3 = t$$

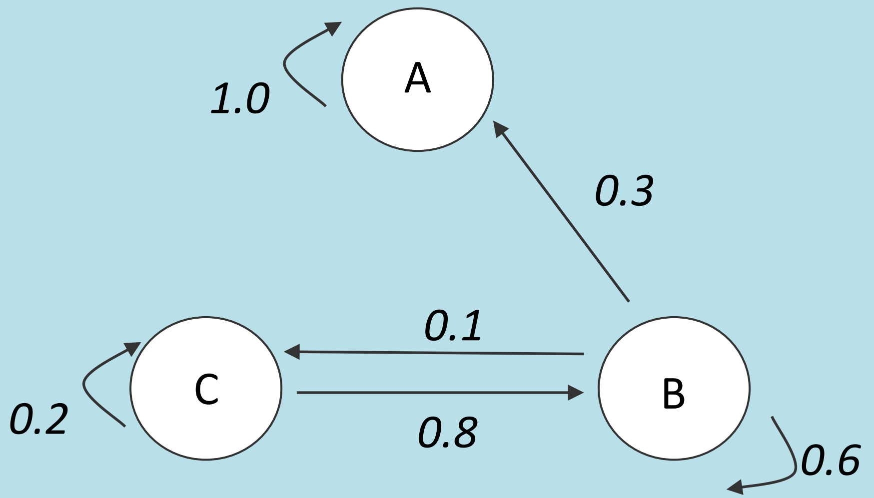
$$u_1 = 0.5$$
$$u_2 = 0.25$$
$$u_3 = 0.25$$

Barplot of stationary matrix



8.3 Markov Chain

▶ Represent data as consequtive decisions.



- Random walk in a state space.
- A Markov chain must satisfy the Markov property, which says that if you want to predict where the chain will be at a future time, and if we know the present state, then the entire past history is irrelevant

Example
- User behaviour (user analytics), supply chain flow

+ Parameters can be easily calculated, speed of modelling, decision process can be represented graphically

- You have to model the states(need system knowledge), and the states must be observable, very simple method

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Definition: Markov Property

D

- Random walk in a state space S, which we will assume is finite, say $n \in \; N_0 \; and \; s_0, \dots, s_{n-1}, s_n, s_{n+1} \in S$.
- We let X_t denote which element of S the walk is visiting at the time t.
- A Markov chain must satisfy the Markov property, which says that if you want to predict where the chain will be at a future time, and if we know the present state, then the entire past history is irrelevant

$$P(X_{n+1} = s_{n+1} \,|X_0 = s_0, X_1 = s_1, \dots, X_n = s_n) = P(X_{n+1} = s_{n+1}|X_n = s_n)$$

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Definition: Markov Property

D

- Random walk in a state space S, which we will assume is finite, say $n \in\ N_0\ and\ s_0, \dots, s_{n-1}, s_n, s_{n+1} \in S$.

- We let X_t denote which element of S the walk is visiting at the time t.

-

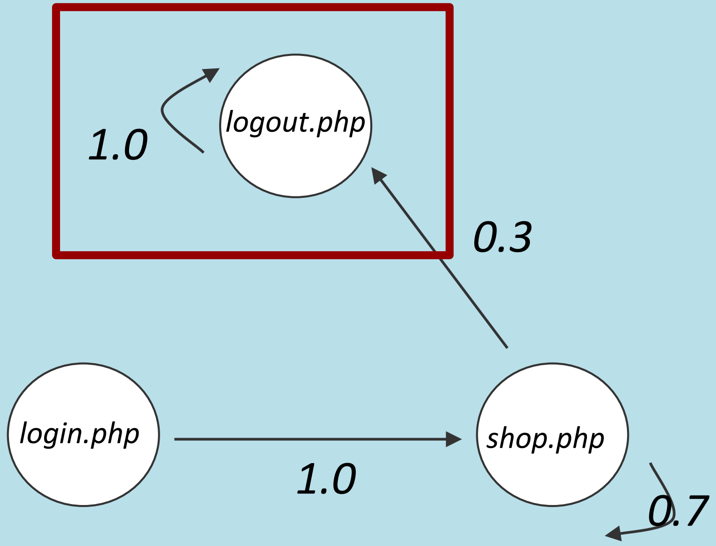
Mathematical way of saying:

„Today is the first day of the rest of your life“

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Absorbing Markov Chains I

- A state in a Markov chain is called **absorbing**, if once the state is entered it is impossible to leave it
- In general absorbing markov chains can be treated like regular markov chains



Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Absorbing Markov Chains II

- The presence of an absorbing state does not guarantee that the chain approach a stationary matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \quad \pi = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix} \, or \, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

- To proof that, just check if the power of P starts oscillating

- A Markov chain is absorbing, if: there is at least one absorbing state and it is possible to go from each non-absorbing state to at least one absorbing state in a finite number of steps

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Technical Definitions: Transition Matrix

- Let the state space be $I = \{1, 2, \ldots, M\}$. The transition matrix P is the $M \times M$ matrix where element p_{ij} is the probability that the chain goes from state i to state j in one step

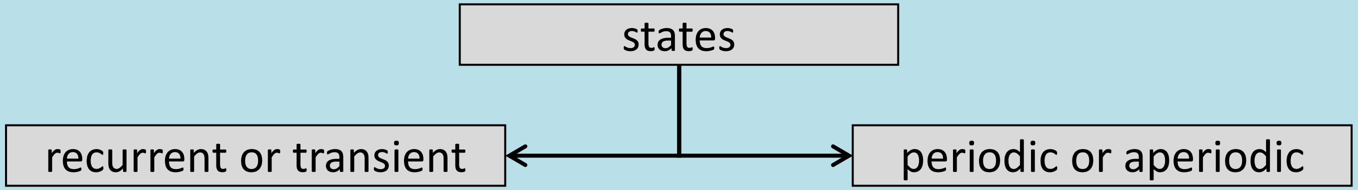
$$p_{ij} = P(X_{n+1} = s_j \,|X_n = s_i)$$

- To find the probability that the chain goes from state i to state j in exactly m steps, take the (i, j) element of P^m

$$p_{ij}{}^{(m)} = P(X_{n+m} = s_j \,|X_n = s_i)$$

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

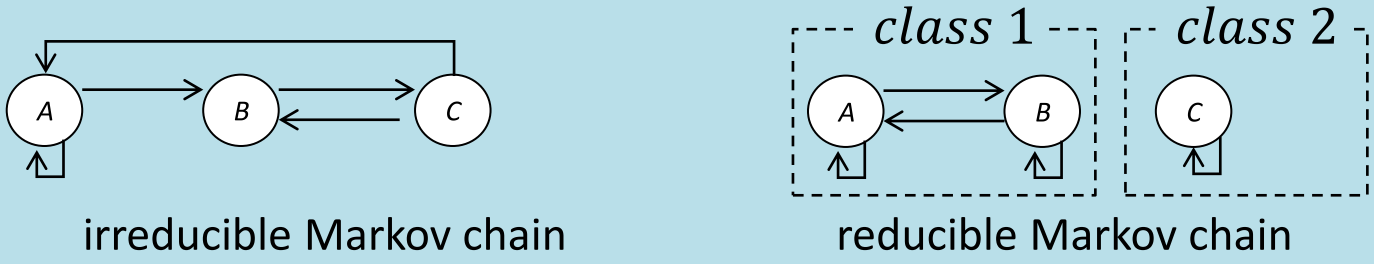
8.3 Technical Definitions: State Properties



- A state is either **recurrent** or **transient**.
  - if you start at a recurrent state, then you will always return back to that state at some point in the future.
  - otherwise you are at a transient state. There is some positive probability that once you leave you will never return.
- A state is either **periodic** or **aperiodic**.
  - if you start at a periodic state of period k, then the greatest common divisor (GCD) of the possible numbers of steps it would take to return back is $k > 1$.
  - otherwise you are at an aperiodic state. The GCD of the possible numbers of steps it would take to return back is 1.

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

- A chain is **irreducible** if you can get from anywhere to anywhere.
- If a chain (on a finite state space) is irreducible, then all of its states are recurrent.

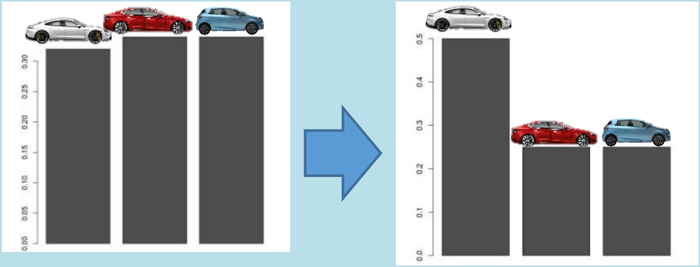


- A chain is periodic if any of its states are periodic, and is aperiodic if none of its states are periodic. In an irreducible chain, all states have the same period.

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Technical Definitions: Stationary Distribution

- Let us say that the vector π be a probability mass function of the different states
- We call the the vector π a stationary distribution if $\pi = \pi \cdot P$
- As a consequence if X_t has the stationary distribution π, then all future $X_{t+1},\ X_{t+2}, \ldots$ also have the stationary distribution
- For irreducible, aperiodic chains, the stationary distribution π exists, is unique, and is the long-run probability of a chain being at state i.



Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Classroom Task

Previous Exam Task!

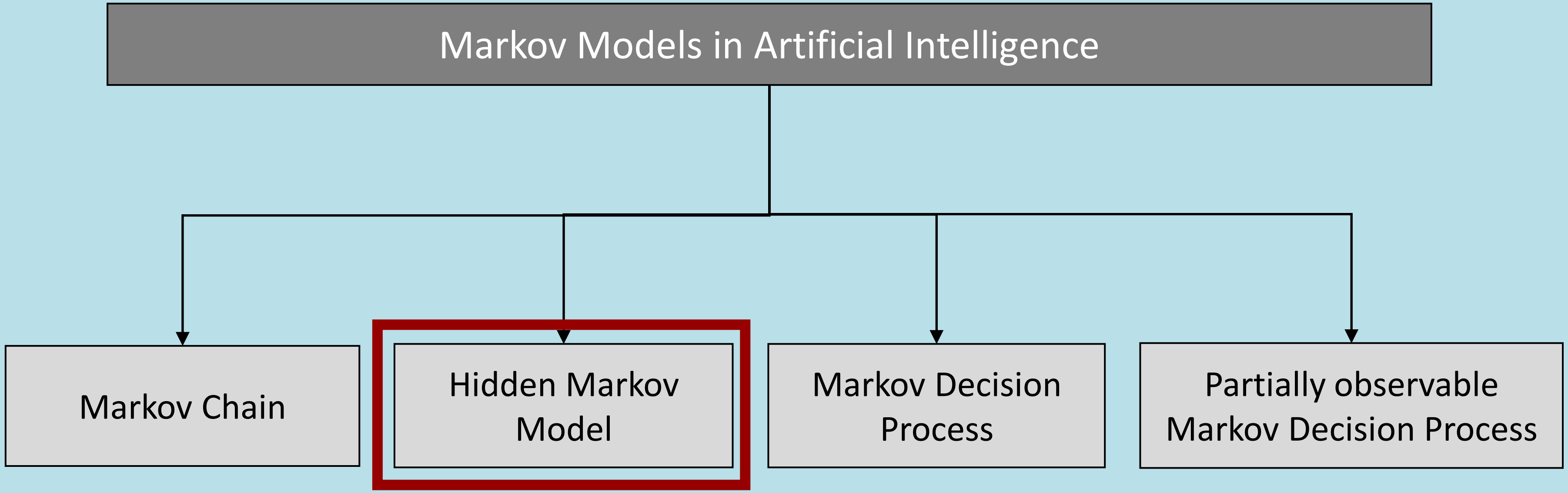
Your turn!

Task

Please discuss with your neighbor:

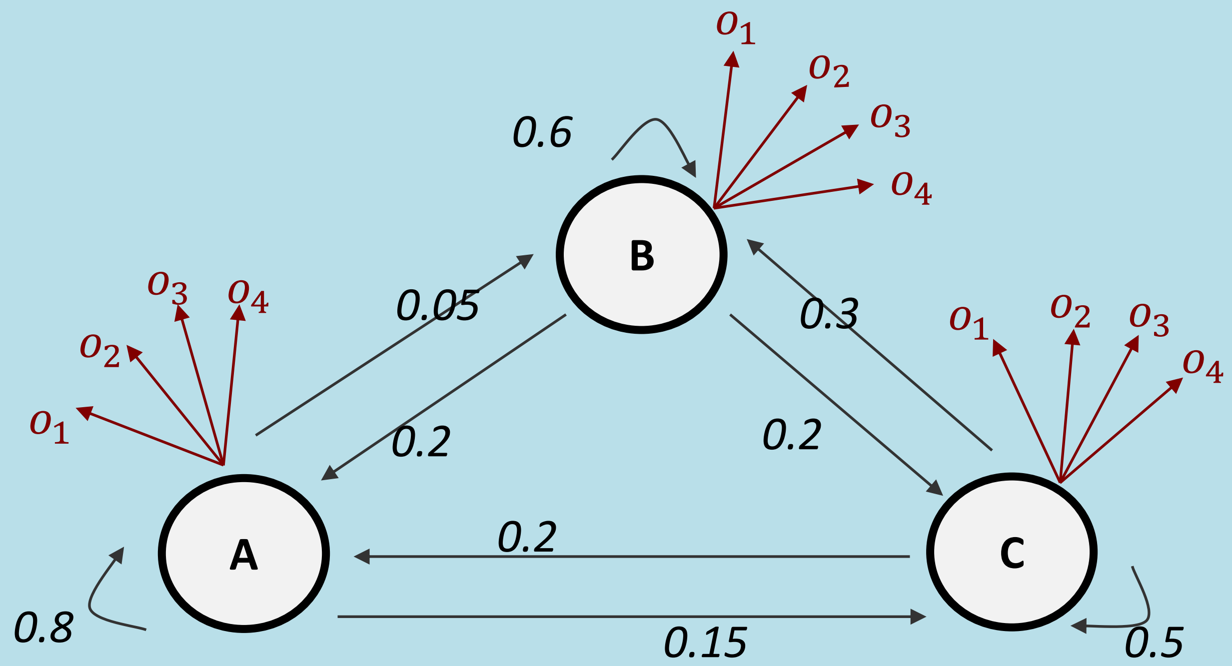
- How do you identify the percentage of consumer that use a product after 1, 2 or 3 timesteps?
- What is the inital state matrix?
- How do you compute the percentage of consumer that use a product in the long run?

8.3 Types of Markov Models



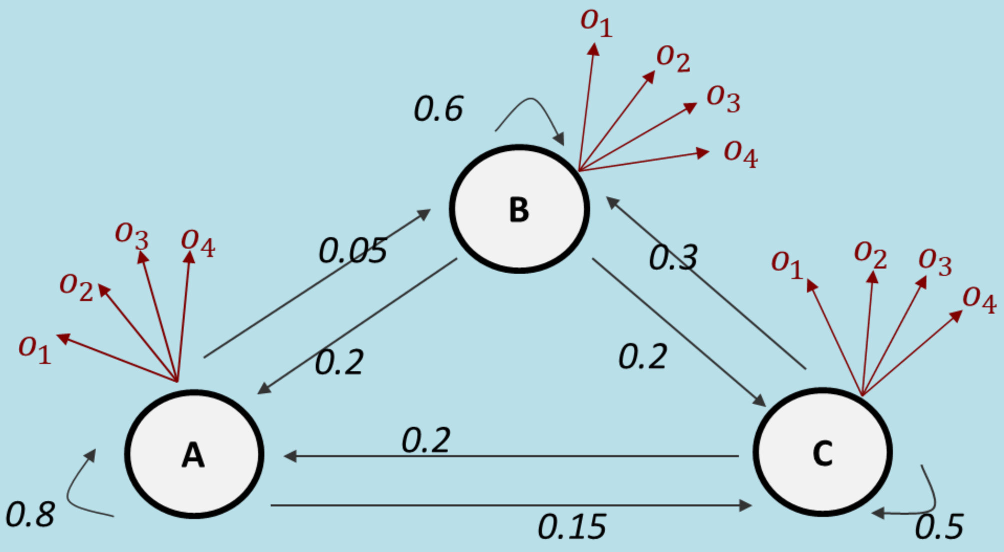
Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Conceptual Structure of a HMM



Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

▶ Model an unobservable process that generates outcomes by observing these outcomes



- Observed or future states Y (e.g. purchase) are outcomes of an unobserved markov model (e.g. preferences)
- States space S, which we will assume is finite, say $n \in N_0$ and $s_0, \dots, s_{n-1}, s_n, s_{n+1} \in S$, and state transition matrix P where element p_{ij} is the probability that the chain goes from state s_i to state s_j in one step t
- Emission probability matrix B with probabilities bij and observations space O, which is finite, say $n \in N_0$ and $o_0, \dots, o_{n-1}, o_n, o_{n+1} \in O$

Example

- Bioinformatics, spech recognition, useranalytics

\+ well studied understandable AI, algorithms learn fast, HMM are able to capture hidden states

\- HMM can only model problems with a finite number of static distributions, time spent in a state is not captured explicitly

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Example 2: Plan your National Park Roadtrip

- You and your girl/boyfriend plan to make a **spontaneous kayak tour** in the Everglades National Park .

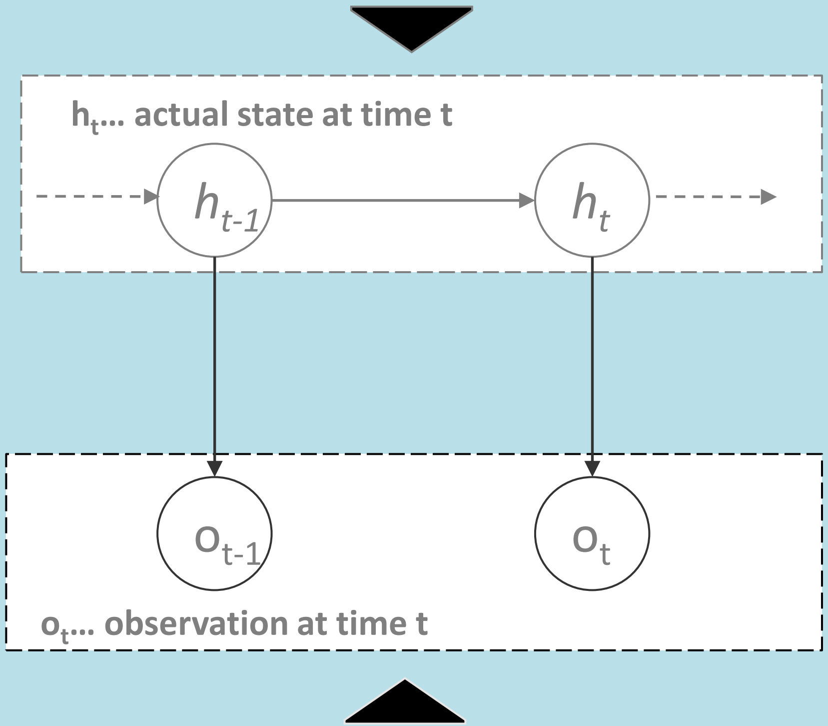
- However **the weather is extremely changeable in spring**. So there are no accurate weather reports and **you have to rely on the ancient knowledge from your native grandmother**.

- She says, that **after three sunny days** the spring is coming and the changes are good for the kayak weekend. So you have to **check out the streets on the national parks webcam** (if they are dry, rather dry, rather wet, wet) to decide when to start your tour.

8.3 Use a Hidden Markov Model (HMM) for that!

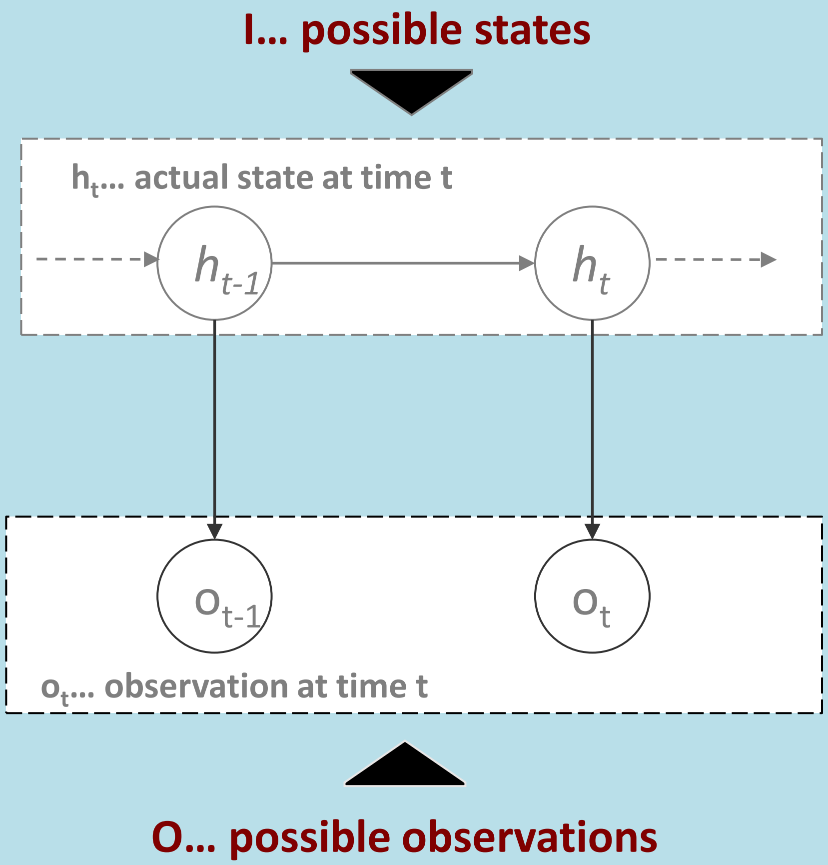
After three sunny days the street is dry

I… possible states



O … possible observations

8.3 State and Observation



Kayak Tour: After three sunny days the weather is good and dry

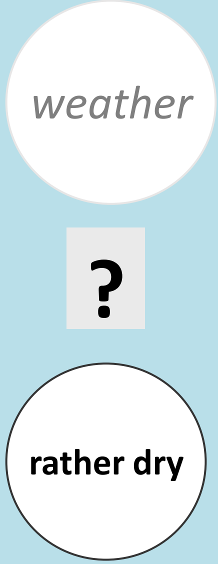
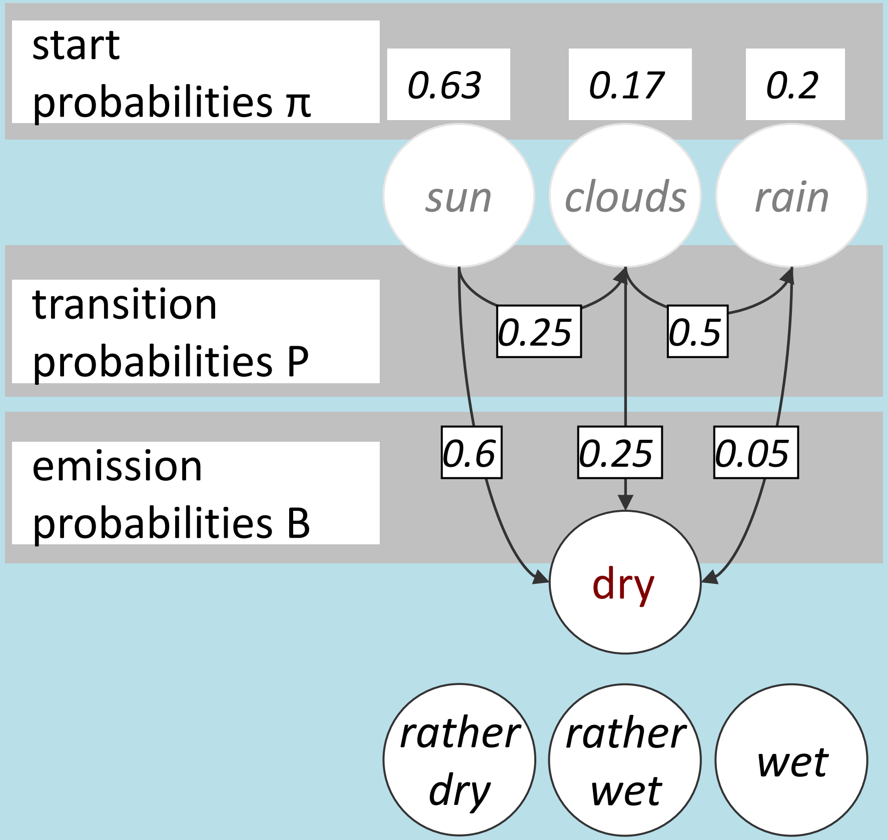


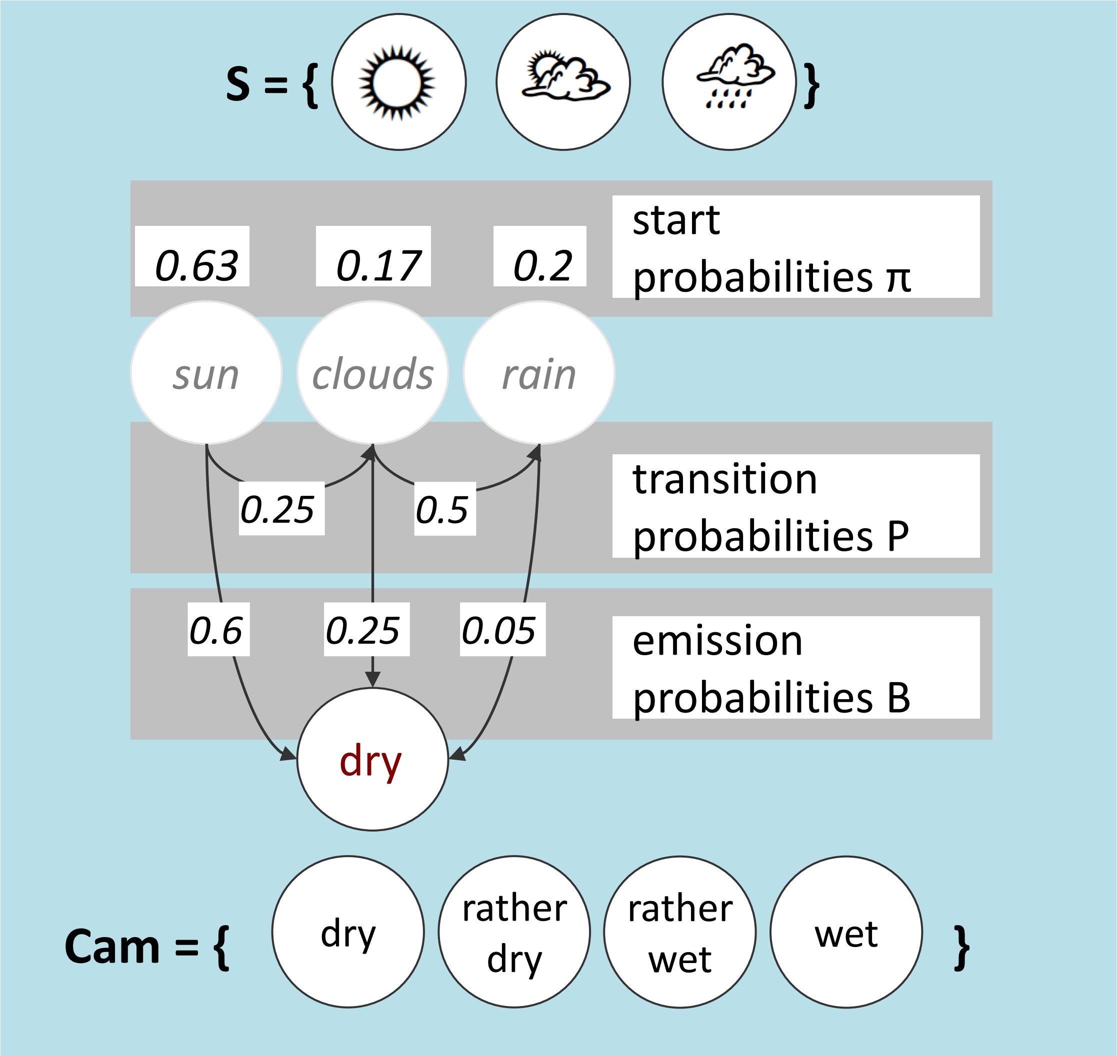
Image source: ↗ Pixabay (2019) / ↗ CC0

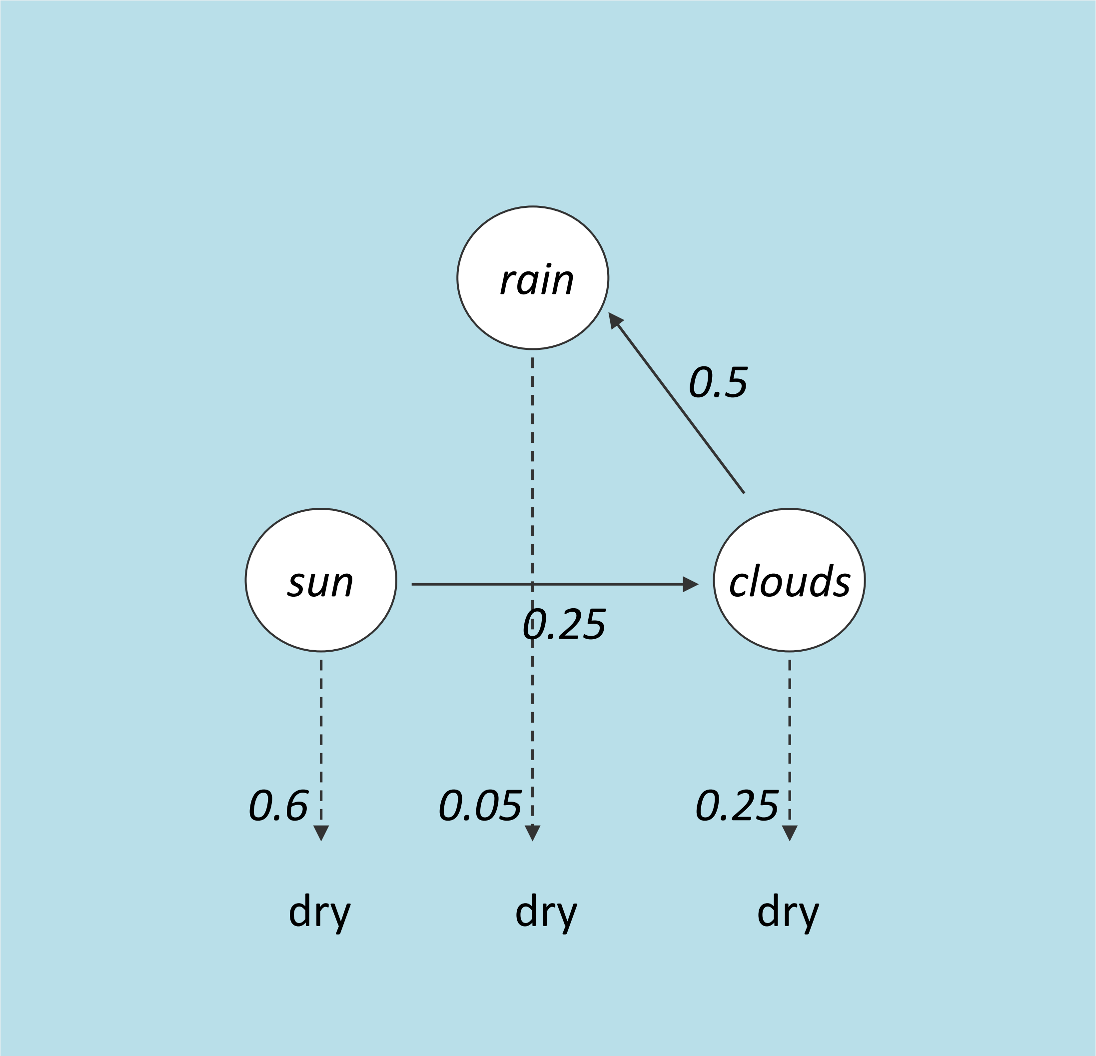


You know

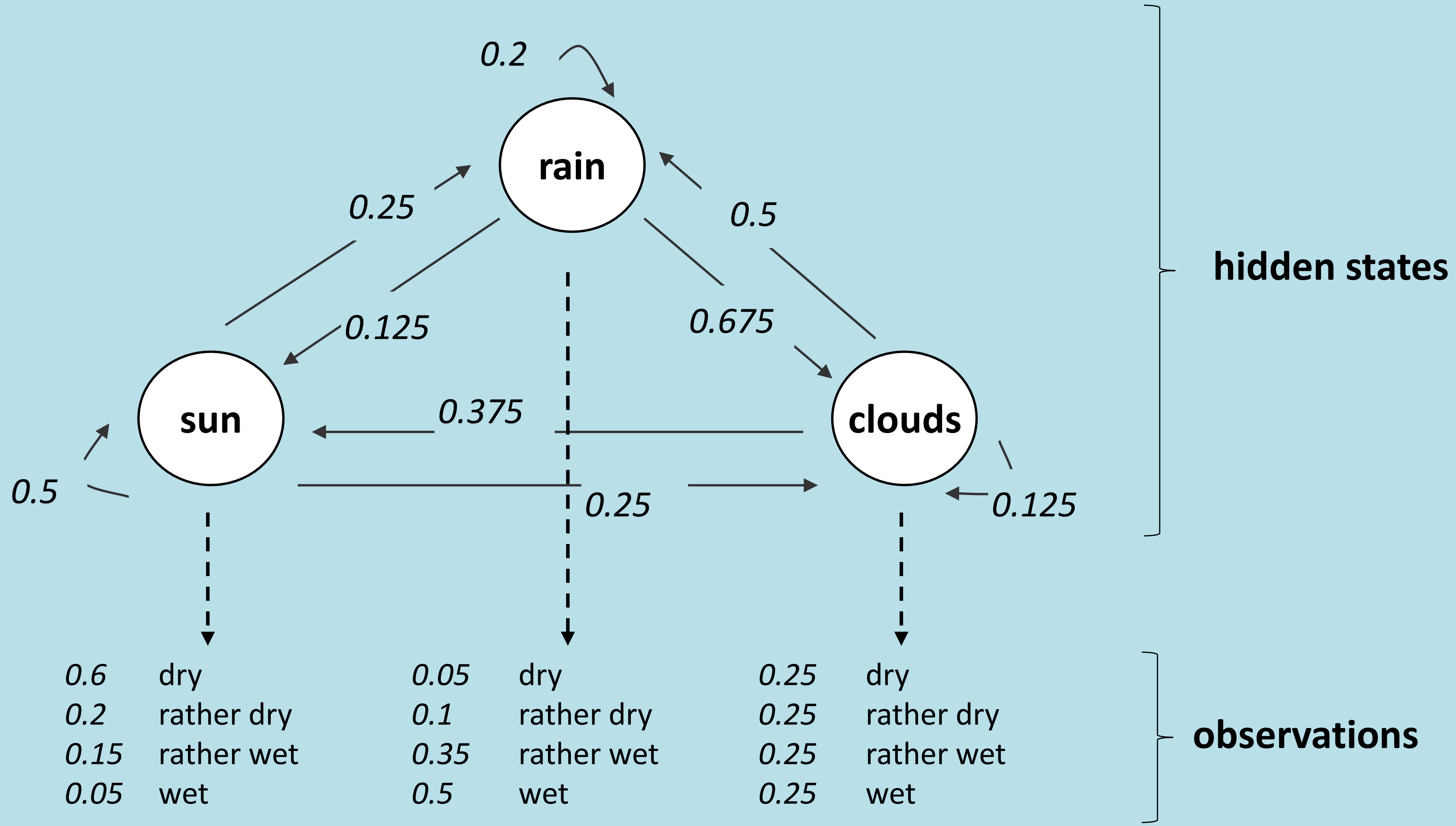
▪ which states exist	**S**
▪ how frequently they occur	**π**
▪ which observations may occur	**O**
▪ how probable it is that a sunny day is followed by a rainy day etc.	**P**
▪ how probable it is that the cloth is dry on a sunny day, on a rainy day, etc.	**B**

8.3 Markov-Model Completion





8.3 Modelling the Final Hidden Markov Process



Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Overview: Values of the Markov Model

	Acronym	Levels
States	$S = \{s_1 \ldots s_M\}$ or $I = \{i_1 \ldots i_M\}$	Sun, Clouds, Rain
Transition probability matrix	$P = \{p_{ij}\}$	0.5 0.25 0.25 0.375 0.125 0.5 0.125 0.675 0.2
Emission probability matrix	$B = \{b_{ij}\}$	0.6 0.2 0.15 0.05 0.25 0.25 0.25 0.25 0.05 0.1 0.35 0.5
Start probabilities	π_0	0.63 0.17 0.2
Observations	O	Dry, Rather dry, Rather wet, Wet
Actual observations	Y	Dry, dry, dry, wet, ratherdray, dry, …

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Common Problems in Sequence-Analysis

- A popular paper by Rabiner (1989) introducted the idea that HMM should be characterized by three fundamental problems:

HMM Analysis Problems

Decoding Problem	Given a HMM and a sequence of observations V, what is the **most likely state sequence** in the model that produced the observations?
Evaluation Problem	Given an HMM and a sequence of observations V, what is the **probability that the observations are generated by the mode**l?
Learning Problem	Given a model and a sequence of observations, how should we **adjust the model parameters** in order to maximize

Adapted from Rabiner LR (1989);Ferguson J (1960)

- **Decoding Problem**
  - Given a HMM and a sequence of observations O, what is the most likely state sequence in the model that produced the observations?

- Given a sequence of observations
  - $O = Dry, Rather\ dry, Rather\ wet, Wet$
- What portion of the sequence was generated from the sunny weather, what portion from the cloudy day, and what portion from the rainy day?



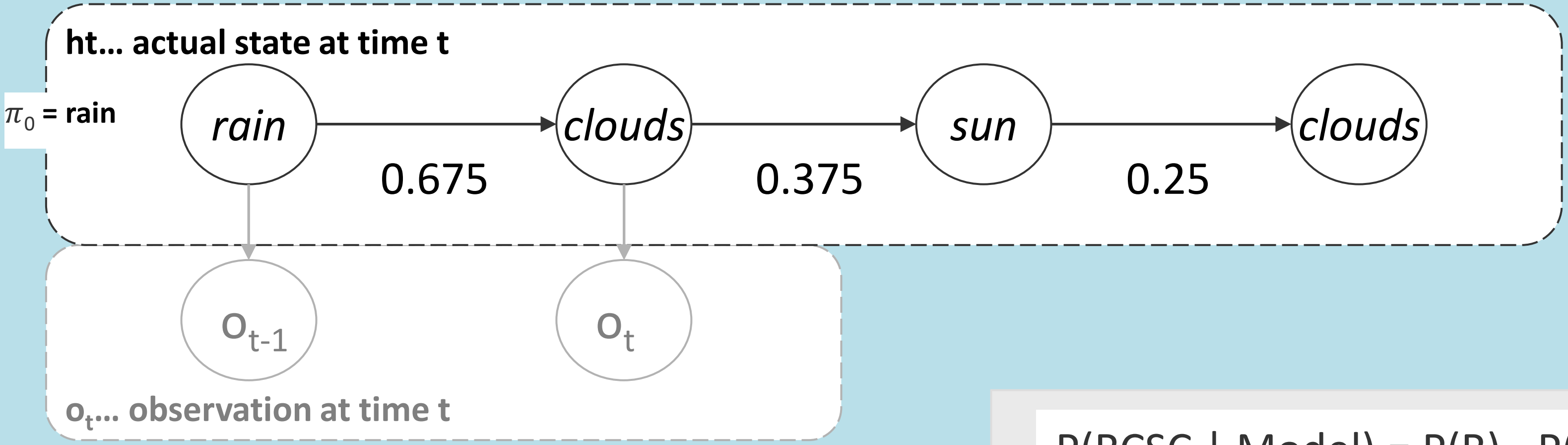
- $O = Dry, rather\ dry, rather\ wet, wet, \ldots$

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 HMM: How to read it

How likely is it that on the three days following a rainy day the sequence "clouds, sun, clouds" will occur?

	Sun	*Clouds*	*Rain*
Sun	0.5	**0.25**	0.25
Clouds	**0.375**	0.125	0.5
Rain	0.125	**0.675**	0.2

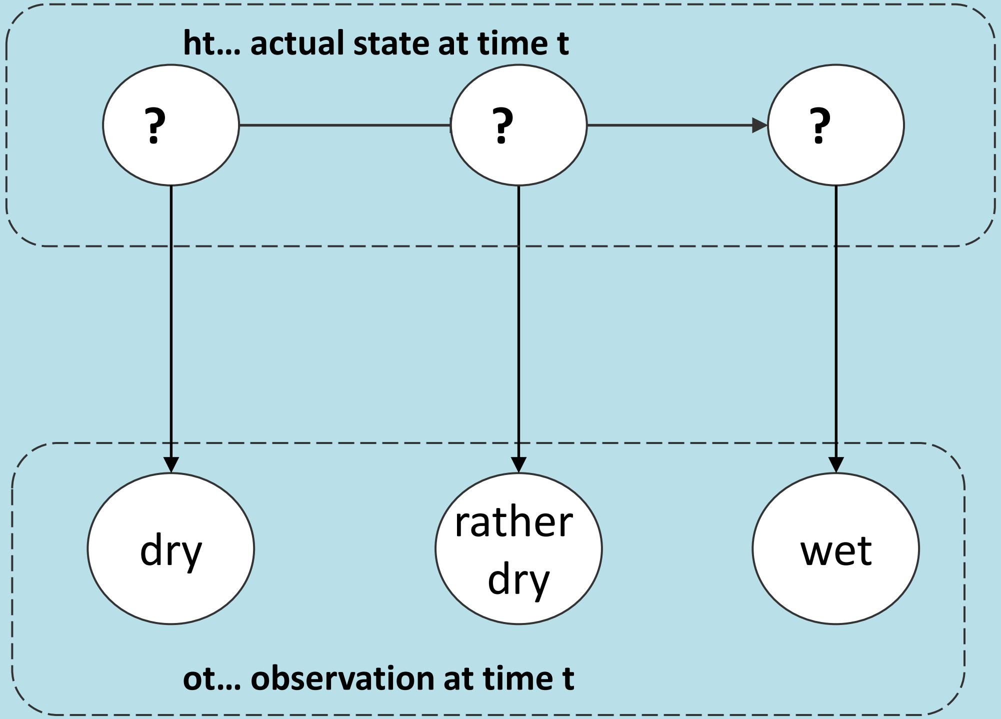


$$P(RCSC \mid Model) = P(R) \cdot P(C|R) \cdot P(S|C) \cdot P(C|S)$$
$$P(RCSC \mid Model) = 1 \cdot 0.675 \cdot 0.375 \cdot 0.25$$
$$P(RCSC \mid Model) = 0.06328125$$

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Example 2: A Common Decoding Problem

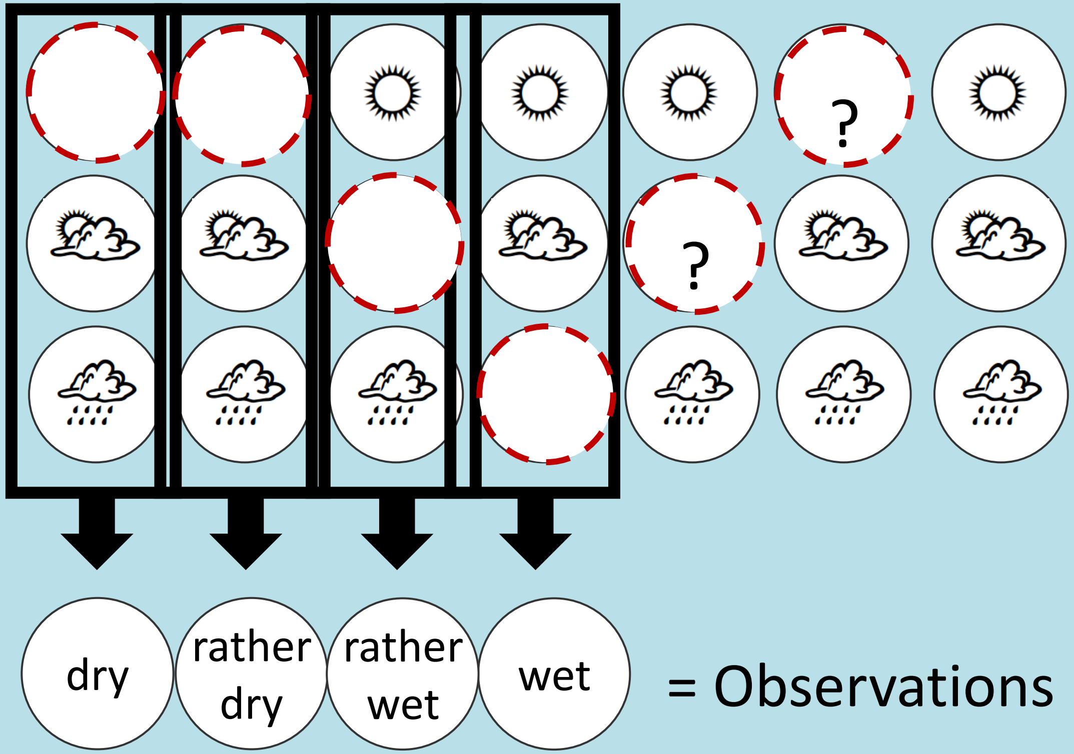
Which is the most probable hidden state sequence if the observed sequence is "dry, rather dry, wet"?



Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 The Viterbi Algorithm: Main idea

General principle: Dynamic programming



Algorithm: Viterbi

for *each state* $i = 1,2,\dots N$

$\quad T_1[i,1] \leftarrow \pi_i \cdot B_{iy_1}$

$\quad T_2[i,1] \leftarrow 0$

end

for ***each*** *observation* $j = 2,3,\dots T$

$\quad for$ ***each*** *state* $i = 1,2,\dots N$

$\quad\quad T_1[i,1] \leftarrow \max_k(T_1[k,j-1] \cdot P \cdot B$

$\quad\quad T_2[i,1] \leftarrow \operatorname{argmax}_k(T_1[k,j-1] \cdot P \cdot B$

$\quad$ ***end***

end

$z_T \leftarrow \operatorname{argmax}_k(T_1[k,T])$

$x_T \leftarrow s_{zT}$

$for\ j = T, T-1, \dots, 2$

$\quad z_{j-1} \leftarrow T_2[z_j, j]$

$\quad x_j - 1 \leftarrow s_{z_{j-1}}$

end

Adapted from Rusell, S., & Norvig, P. (2016); Waldmann, K. H. & Stocker, U. M. (2012)

8.3 Example 2 ▶ State 1

- **Start probabilities π**

sun	0.63
clouds	0.17
rain	0.20

- **Sequence**: „dry, rather dry, wet“

- **Emission probabilities B:**

	dry	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5

	dry	
sun		$0.63 \cdot 0.6 = 0.378$
clouds		$0.17 \cdot 0.25 = 0.0425$
rain		$0.2 \cdot 0.05 = 0.01$

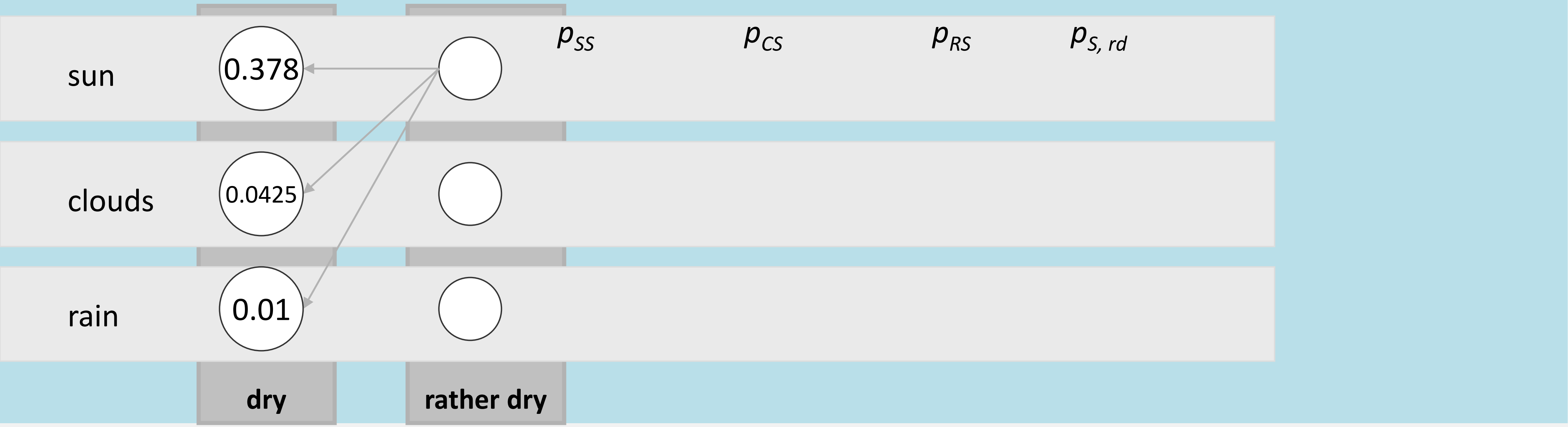
8.3 Example 2 ▶ State 2 = "sun"

- **transition probabilities P**

after ▶ *follows* ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ *then* ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



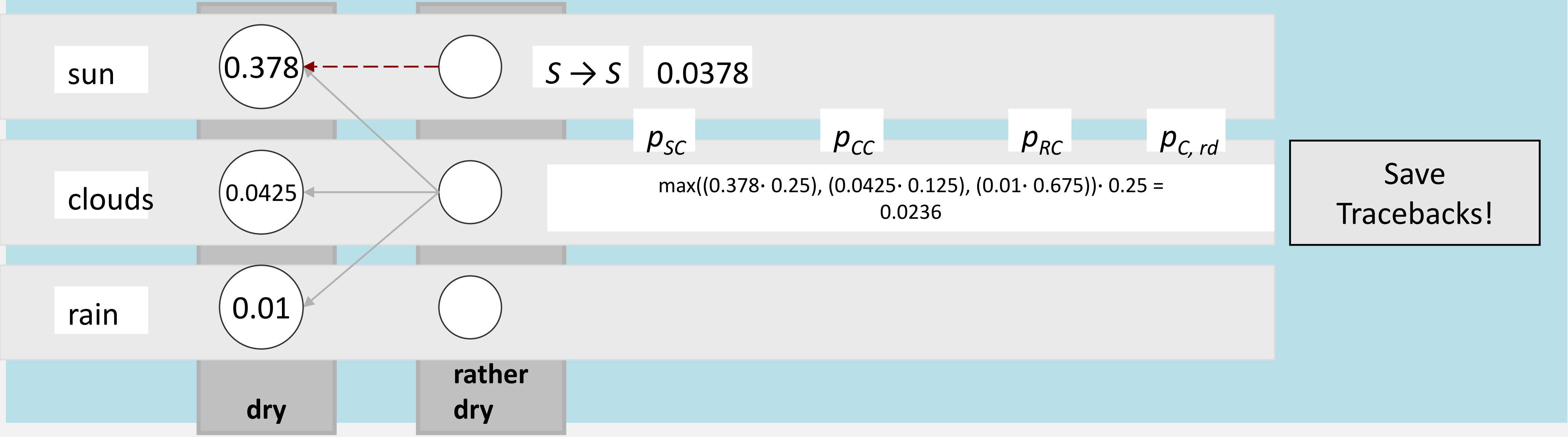
8.3 Example 2 ▶ State 2 = "clouds"

- **transition probabilities P**

after ▶ follows ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ then ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



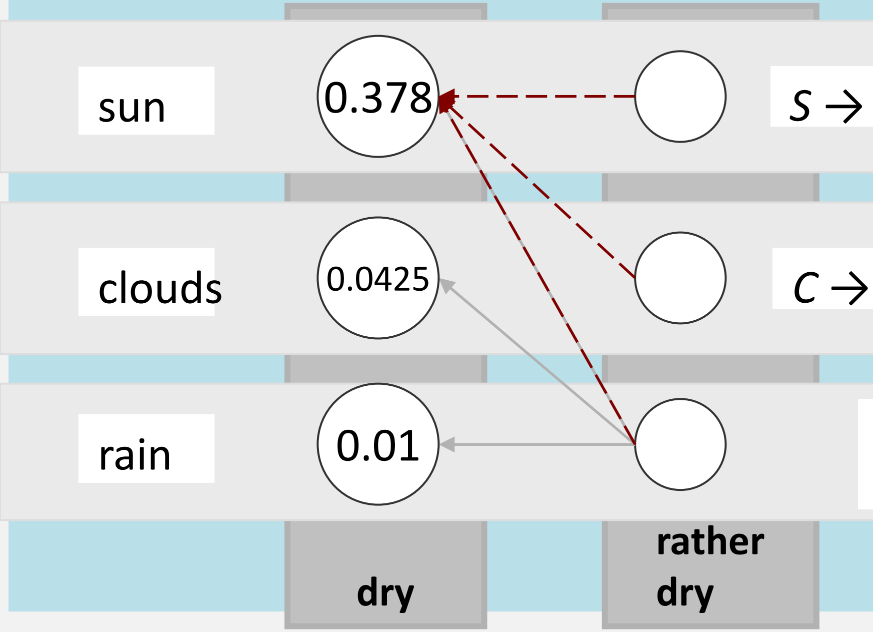
8.3 Example 2 ▶ State 2 = "rain"

- **transition probabilities P**

after ▶ follows ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ then ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



$S \rightarrow S$ 0.0378

$C \rightarrow S$ 0.0236

p_{SR} p_{CR} p_{RR} $p_{R,\,rd}$

max ((0.378· 0.25), (0.0425· 0.5), (0.01 · 0.2)) × 0.1 =
= 0.00945

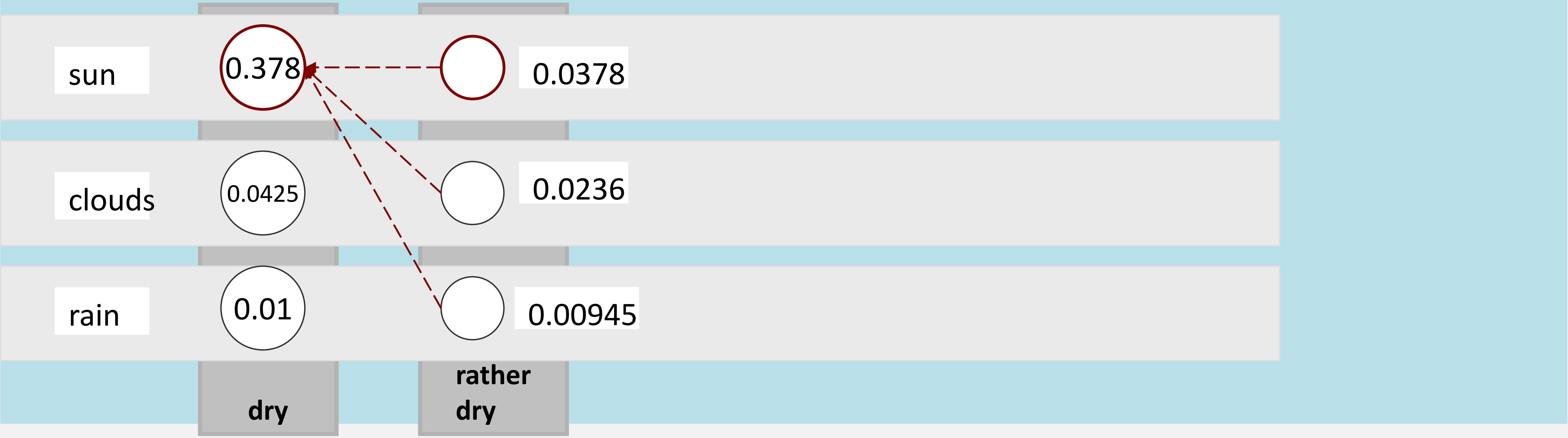
8.3 Example 2 ▶ path($state_1$, $state_2$)

- **transition probabilities P**

after ▶ follows ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ then ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



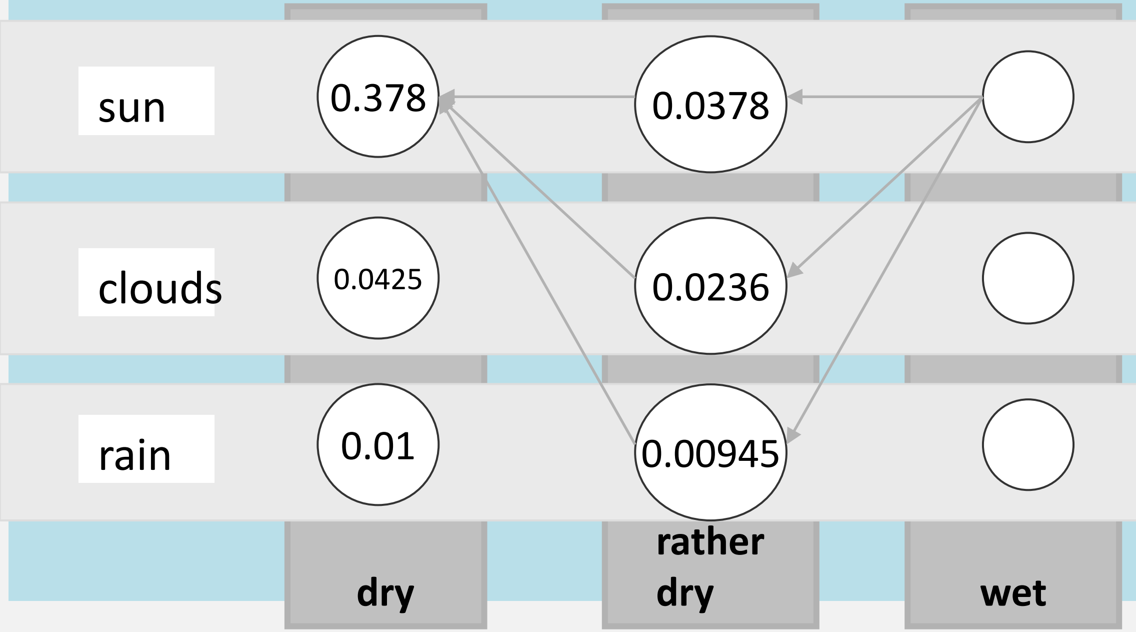
8.3 Example 2 ▶ State 3 = "sun"

- **transition probabilities P**

after ▶ follows ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ then ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



max ((0.0378 · 0.5), (0.0236 · 0.375),
(0.00945· 0.125))· 0.05 =
= 0.000945

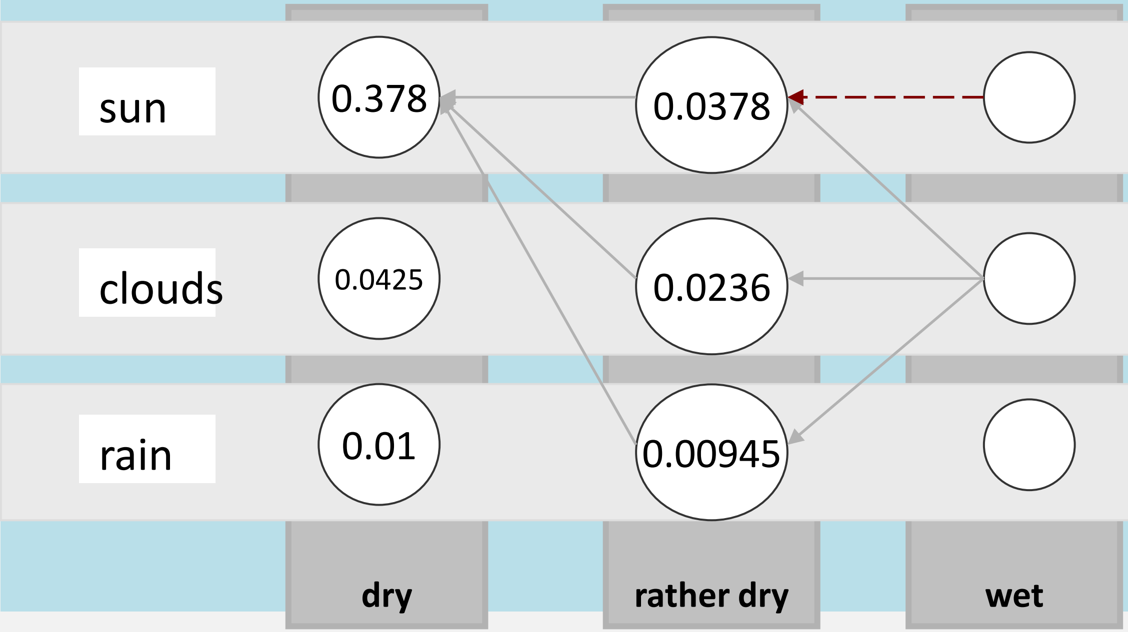
8.3 Example 2 ▶ State 3 = "clouds"

- **transition probabilities P**

after ▶ *follows* ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ *then* ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



S → *S* 0.000945

max ((0.0378 · 0.25), (0.0236 · 0.125), (0.00945· 0.675))· 0.25 =
= 0.00236

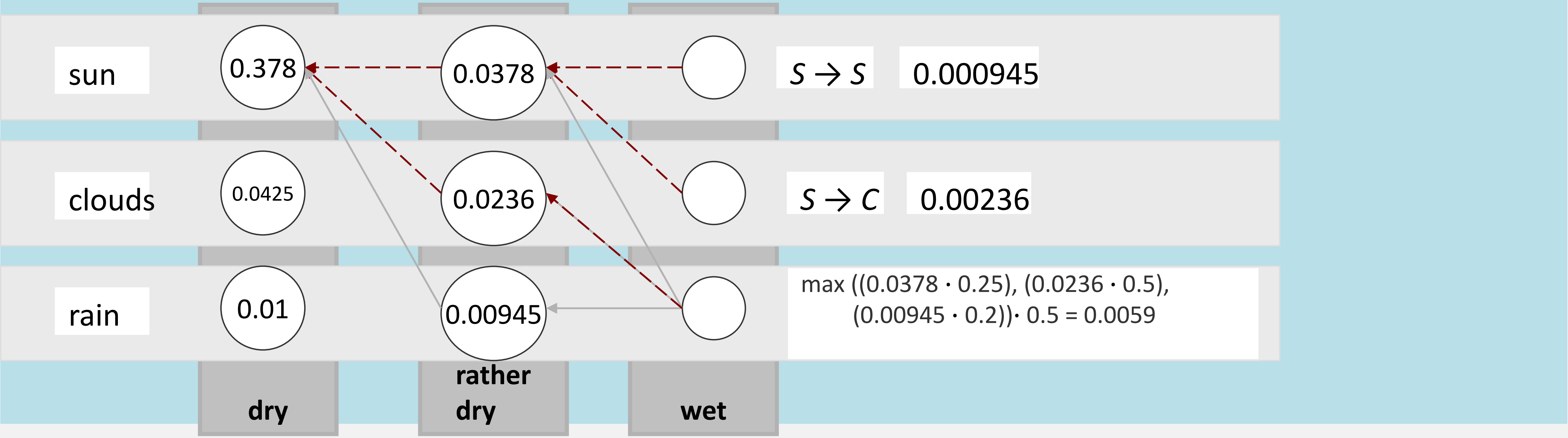
8.3 Example 2 ▶ State 3 = "rain"

- **transition probabilities P**

after ▶ *follows* ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ *then* ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



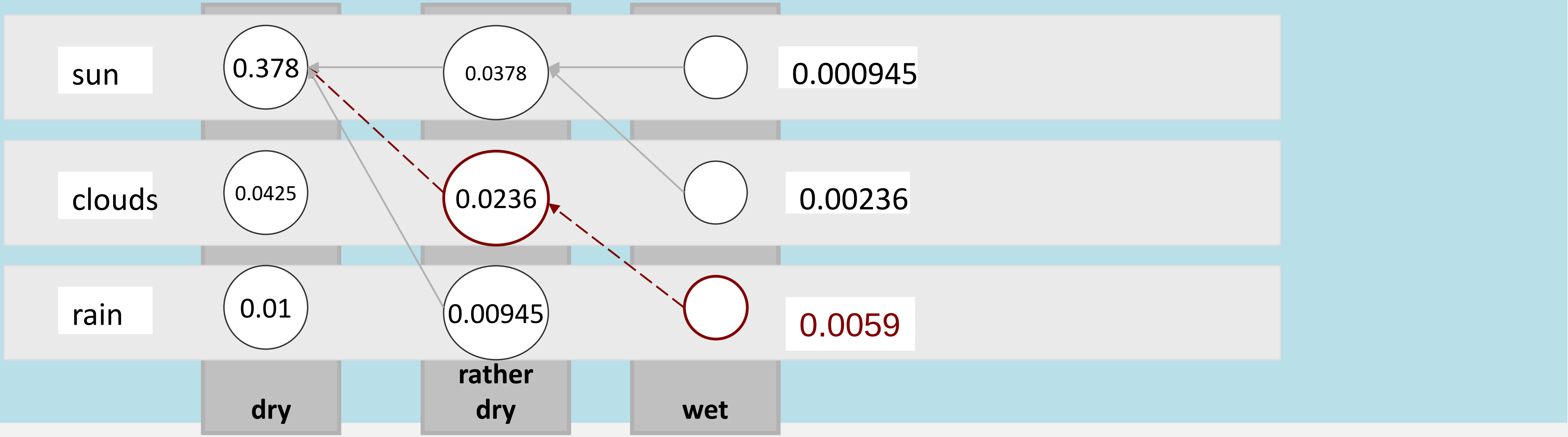
8.3 Example 2 ▶ path($state_2$, $state_3$)

- **transition probabilities P**

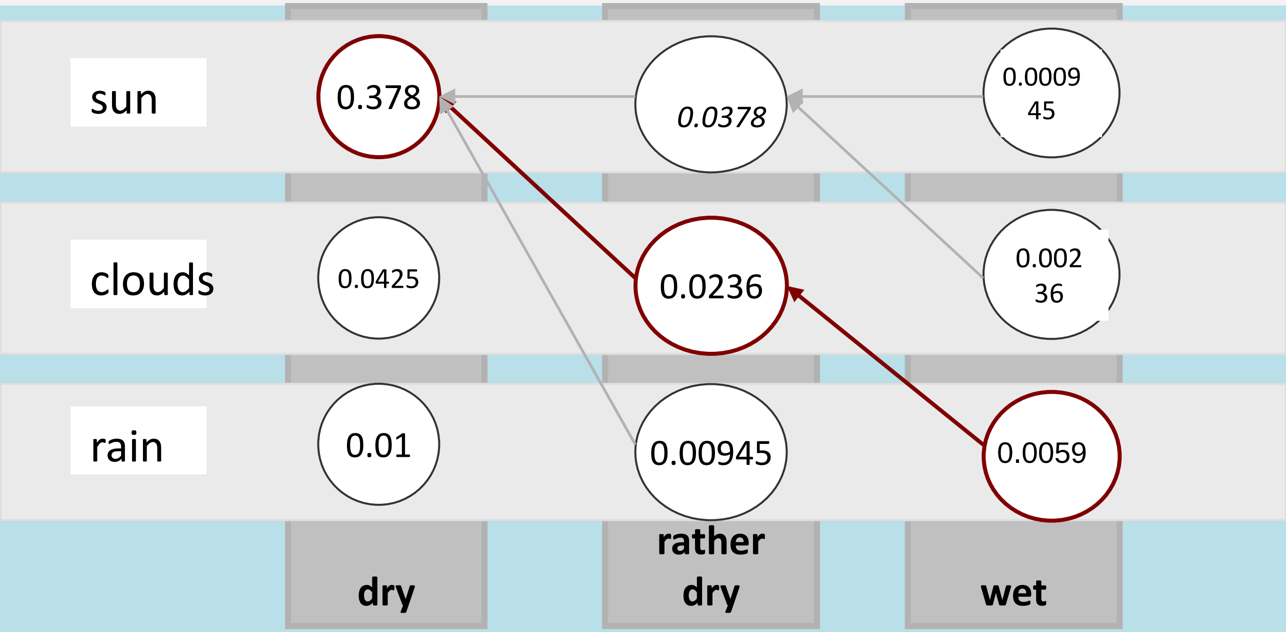
after ▶ *follows* ▼	*sun*	*clouds*	*rain*
sun	0.5	0.25	0.25
clouds	0.375	0.125	0.5
rain	0.125	0.675	0.2

- **emission probabilities B**

If ▶ *then* ▼	*dry*	*rather dry*	*rather wet*	*wet*
sun	0.6	0.2	0.15	0.05
clouds	0.25	0.25	0.25	0.25
rain	0.05	0.1	0.35	0.5



8.3 Example 2 ▶ path($state_1$, $state_2$, $state_3$)



Reconstruction of the path using the tracebacks!

most probable sequence:
sun -> clouds -> rain

Note:
There are several algorithms for computing Markov chains. We used ***VITERBI****.*

HMM Analysis Problems

Decoding Problem	Given a HMM and a sequence of observations O, what is the **most likely state sequence** in the model that produced the observations?
Evaluation Problem	Given an HMM and a sequence of observations O, what is the **probability that the observations are generated by the mode**l?
Learning Problem	Given a model and a sequence of observations, how should we **adjust the model parameters** in order to maximize

Adapted from Rabiner LR (1989);Ferguson J (1960)

- **Evaluation Problem**
  - Given an HMM and a sequence of observations V, what is the probability that the observations are generated by the model?

- Given a sequence of observations
  - $V = \{Dry, Rather\ dry, Rather\ wet, Wet\}$
- How likely is this sequence, given our model of how the weather works?

Adapted from Rabiner LR (1989);Ferguson J (1960)

8.3 Common Problems in Sequence-Analysis

HMM Analysis Problems

Decoding Problem

Given a HMM and a sequence of observations O, what is the **most likely state sequence** in the model that produced the observations?

Evaluation Problem

Given an HMM and a sequence of observations O, what is the **probability that the observations are generated by the mode**l?

Learning Problem

Given a model and a sequence of observations, how should we **adjust the model parameters** in order to maximize

Adapted from Rabiner LR (1989);Ferguson J (1960)

8.3 Learning Problem

- **Learning Problem**
  - Given a model and a sequence of observations, how should we adjust the model parameters in order to maximize
- Given a sequence of observations
  - $V = \{Dry, Rather\ dry, Rather\ wet, Wet\}$
- How “rainy” is the rainy day? How “sunny” is the sunny day? How often does the weather change from sunny to rainy, and back?
- Note: We need to know how many states exist

Adapted from Rabiner LR (1989);Ferguson J (1960)

Your turn!

Task

What is the basic idea behind the VITERBI algorithm? Which general programming principle does it rely on?

Outline

8	Probabilistic Reasoning and Modelling
8.1	From Uncertainty to Probability
8.2	Probabilistic Reasoning
8.3	Probabilistic Reasoning over Time
8.4	Decision Theory and Decision-Making
8.5	Game Theory and Sequential Decision-Making
8.6	Generative Modelling

Lectorial 6: Intelligent Agents in Action

Image source: ↗ Pixabay (2019) / ↗ CC0

▶ **What you will learn:**
- Concepts of statistics and probability theory to model agents that can act under uncertainty
- Foundations of Markov theory, Bayes theory and sequence analysis for to reason under uncertainty according to the laws of probability theory
- How to analyze and build optimal agent decision-making that can handle uncertainty

▶ **Duration:**
- 215 min

▶ **Relevant for Exam:**
- 10.1 – 10.4

8.4 Decision-Making under Uncertainty to Decision Theory

- In chapter 8.1 we defined decision theory as

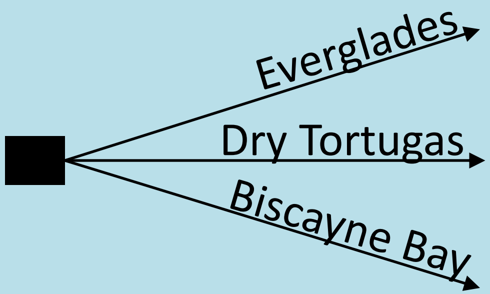
$$Decision\ theory\ =\ probability\ theory\ +\ utility\ theory$$

- **Key question**: What is utility, and how can we use it in Artificial Intelligence

Adapted from Rusell, S., & Norvig, P. (2016)

8.4 Non-Deterministic vs. Probabilistic Uncertainty

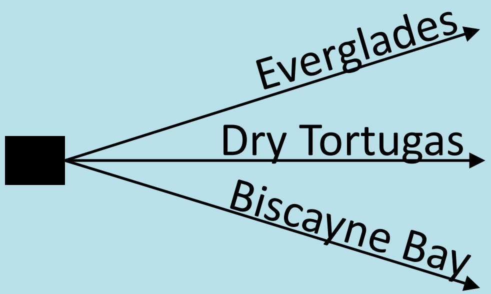
Non-deterministic model



$\{\, Everglades,\; Dry\ Tortugas,\; Biscayne\ Bay \,\}$

- Decision that is best for worst case

Probabilistic model



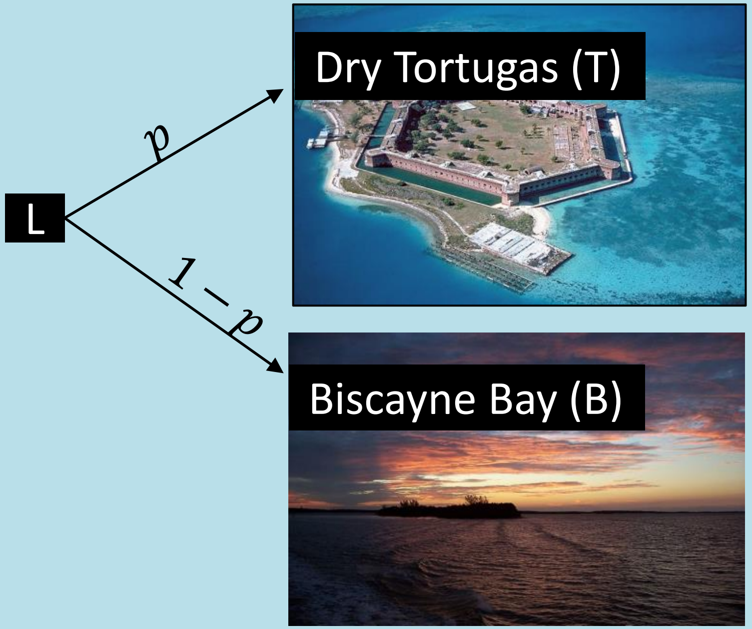
$\{\, P(E), Everglades;\; P(T), Dry\ Tortugas;\; P(B), Biscayne\ Bay \,\}$

- Decision that maximizes expected utility value (MEU)

- Similar concept than adversarial search

Adapted from Rusell, S., & Norvig, P. (2016)

- In the following, we call such probabilistic events lotteries
- A lottery consists of a set of events (prizes) and probabilities



Lottery $L = [p, T; (1-p), B]$

- Preferences
  - An agent likes certain prizes better than others
  - An agent therefore also likes certain lotteries better than others
- Notations: $T \succ B$ vs. $T \sim B$ vs. $T \succcurlyeq B$

Adapted from Rusell, S., & Norvig, P. (2016) | Image source: ↗Fort-Jefferson Dry-Tortugas(2005) by U.S. National Park Service from Wikimedia / public domain; ↗Fort-Jefferson Dry-Tortugas(2005) by U.S. National Park Service from Wikimedia / ↗CC BY-SA 3.0

8.4 Constraints for Preferences in Rational Behavior I

The following six rules define constrains on preferences

- Orderability

$$(A \succ B) \lor (B \succ A) \lor (B \sim A)$$

- Transitivity

$$(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$$

- Continuity

$$A \succ B \succ C \;\Rightarrow\; \exists p\,[p, A; 1-p, C] \sim B$$

- Substitutability

$$A \sim B \Rightarrow [p, A; 1-p, C] \sim [p, B;\ 1-p, C]$$

- Monotonicity

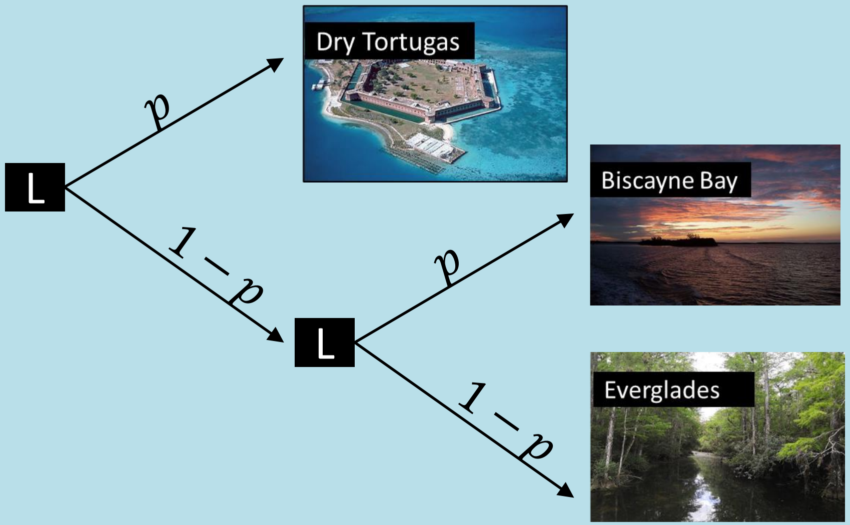
$$A \succ B \Rightarrow (p \geq q \;\Leftrightarrow\; [p, A; 1-p, B] \succcurlyeq [q, A; 1-q, B]$$

Adapted from Rusell, S., & Norvig, P. (2016)

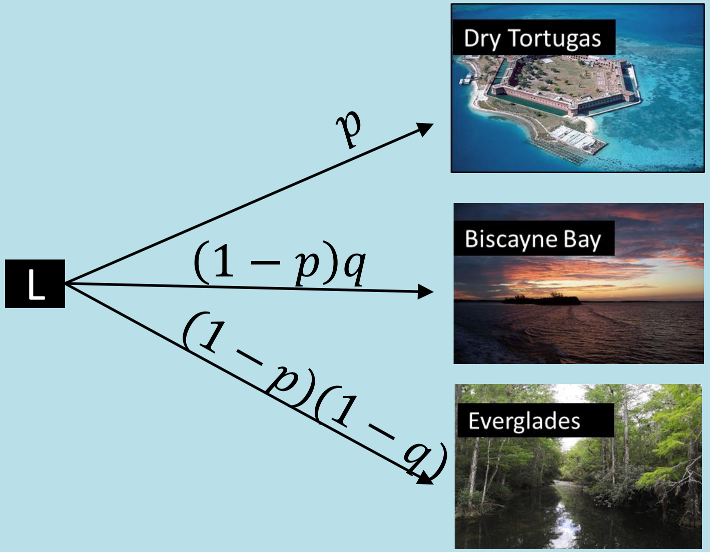
8.4 Constraints for Preferences in Rational Behavior II

- Another property that should be obeyed is that lotteries are decomposable
- Therefore, no rational agent should have a perference between the two equivalent formulations

$$[p, T; 1-p, [q, B; 1-q, C]] \sim [p, T; (1-p)q, B;\ (1-p)(1-q), E]$$



is equal to

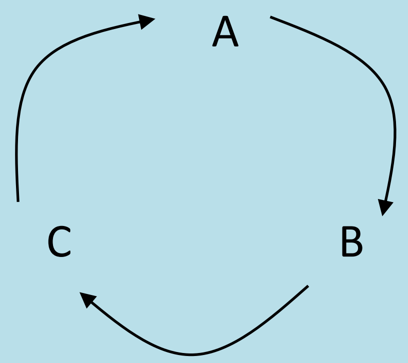


Adapted from Rusell, S., & Norvig, P. (2016) | Image source: ↗Fort-Jefferson Dry-Tortugas(2005) by U.S. National Park Service from Wikimedia / public domain; ↗Fort-Jefferson Dry-Tortugas(2005) by U.S. National Park Service from Wikimedia / ↗CC BY-SA 3.0

8.4 Preferences and Irrational Behavior

- Preferences between events (prizes) may be arbitrary

- However, violating the axioms results in irrational behavior
  - E.g. Preferences may be cyclic, $A \succ B, B \succ C, C \succ A$



- Agents with cyclic preferences will lose all its money in trading games

Adapted from Rusell, S., & Norvig, P. (2016)

8.4 Utility Functions

- Natural way for measuring how desirable certain prizes are is using utility functions. An utility function assigns a numerical value to each prize
- Utility function naturally lead to preferences

$$A \succ B \Leftrightarrow U(A) > U(B)$$

- The expected utility of an event is the expected value of the utility function in a lottery

$$EU(X) = \sum_{x \in X} P(x) \cdot U(x)$$

- A utility function in a deterministic environment (no lotteries) is also called a value function

Adapted from Rusell, S., & Norvig, P. (2016)

- **Problem**: How to measure Utility
- **Economic workaround**: Measure utility with money

- E.g. If you win 1.000.000€, are you willing to bet them on a double-or-nothing coin flip? What is with a triple or nothing?

$$U(1\ Mio.) > EU([0.5, 0; 0.5, 3\ Mio.])$$

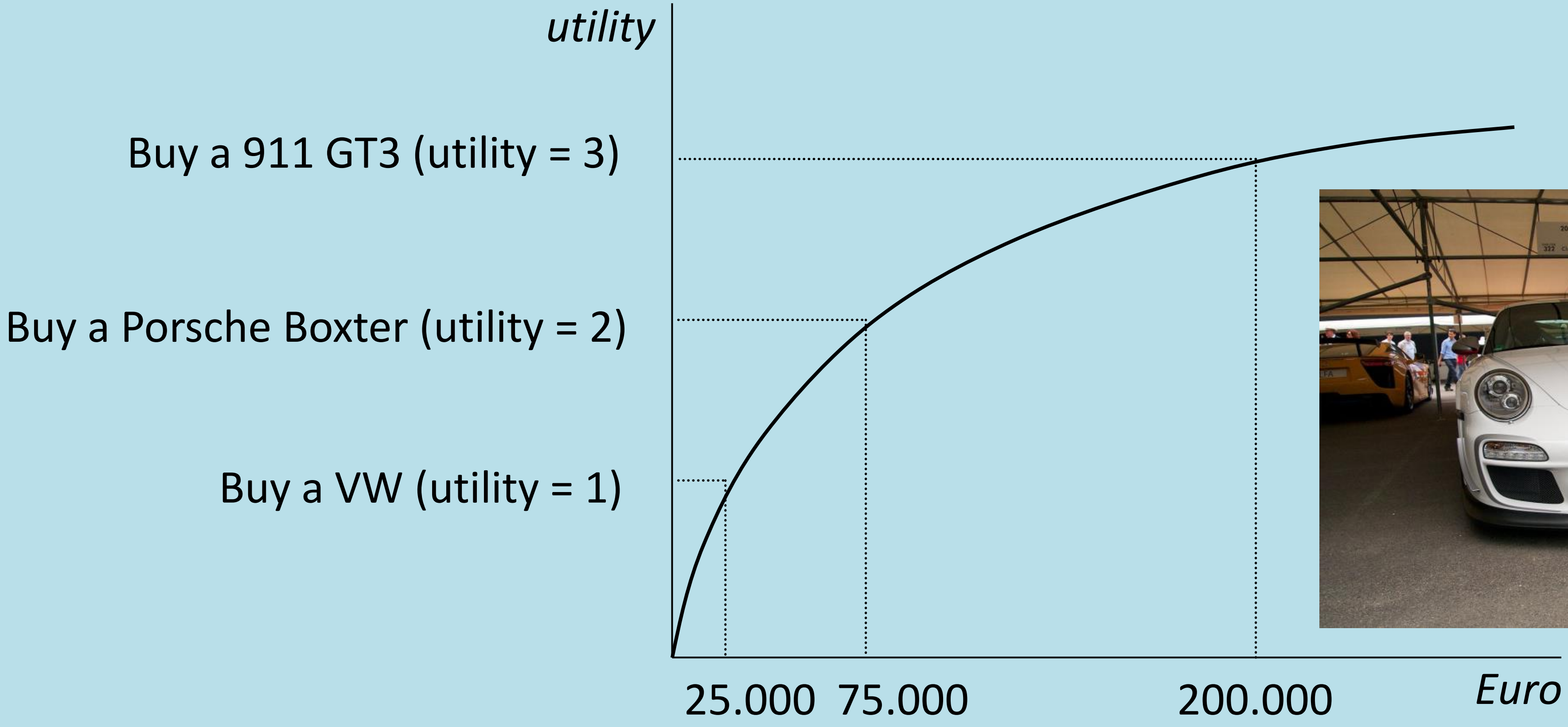
$$U(1\ Mio.) > 0.5\ U(0) + 0.5\ U(3\ Mio.)$$

! This does not mean that money behaves as a utility function, because it says nothing about preferences between lotteries involving money.

Adapted from Rusell, S., & Norvig, P. (2016);

8.4 Decreasing Marginal Utility

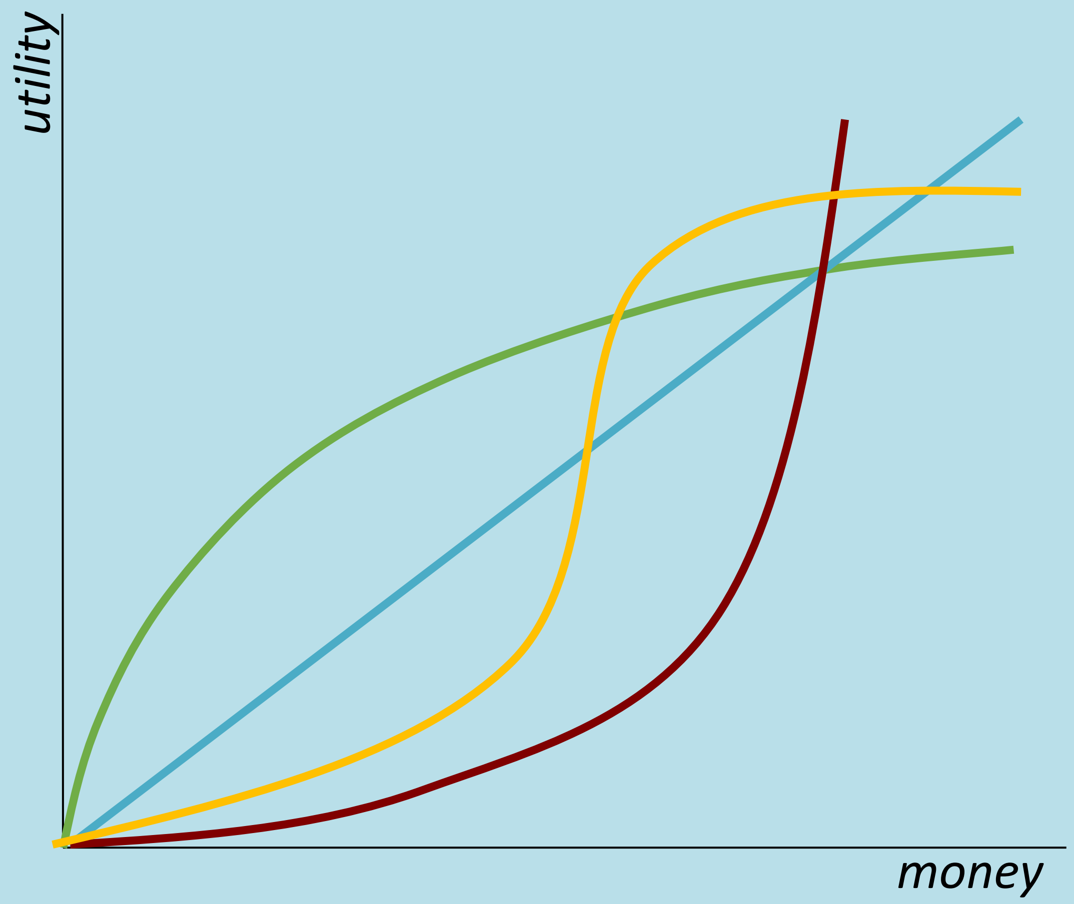
- Typically, at some point, having an extra dollar does not make people much happier (decreasing marginal utility)



Adapted from Rusell, S., & Norvig, P. (2016) Image source: ↗White Porsche 911 GT3 RS 4.0 (2019) by Adam Russell on Flickr from Wikimedia / ↗CC BY-SA 2.0

Examples of Risk Attitudes

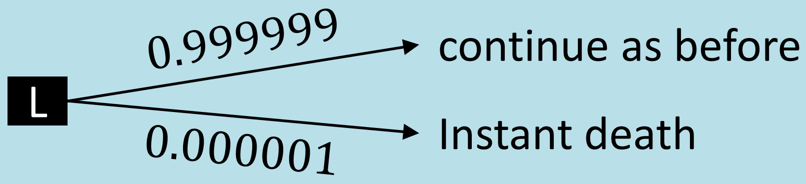
- Green has decreasing marginal utility
  → risk-averse
- Blue has constant marginal utility
  → risk-neutral
- Red has increasing marginal utility
  → risk-seeking
- Yellow's marginal utility is sometimes increasing, sometimes decreasing
  → neither risk-averse (everywhere) nor risk-seeking (everywhere)



Adapted from Rusell, S., & Norvig, P. (2016)

8.4 Other Units of Measurements for Utility

- In other domains money is no adequate measurement for utility, e.g. medicine and safety-critical environments
- Quality-Adjusted Life Year (QALY)
- Micromort is the lottery of dying with a probability of one in a million



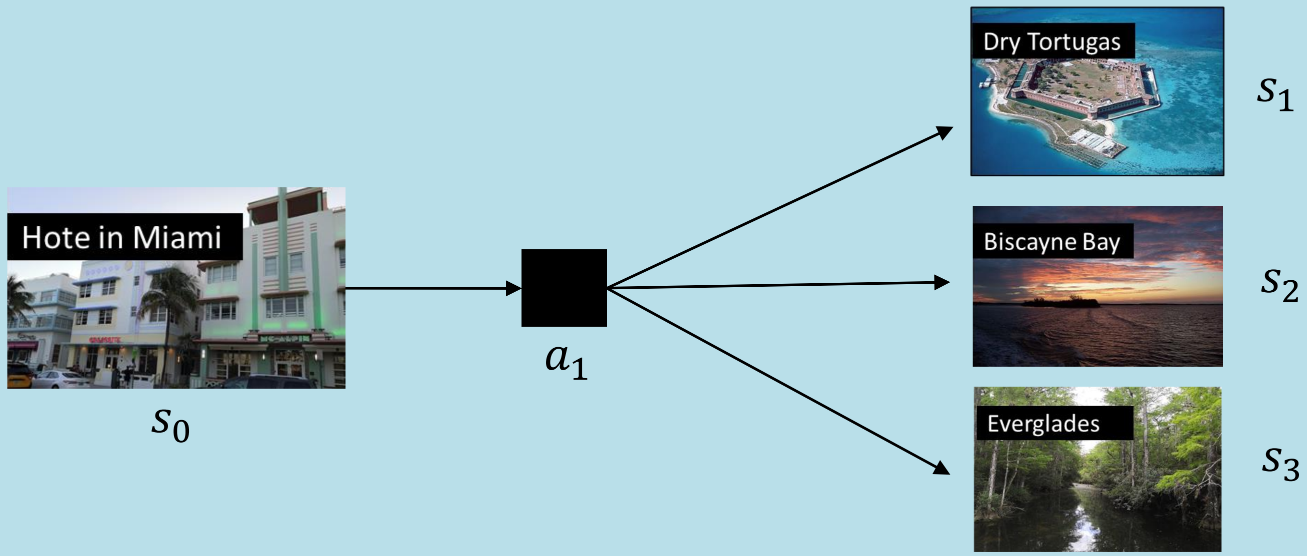
- Research suggest that a micromort is worth about 50 €

Adapted from Rusell, S., & Norvig, P. (2016);

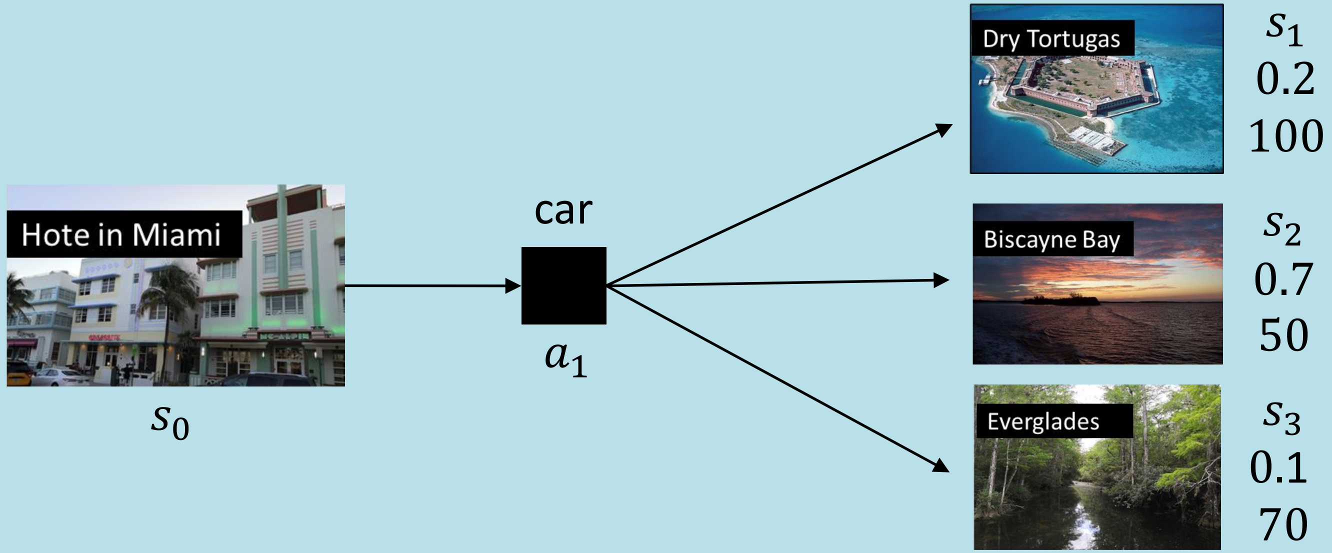
8.4 Example 1: Maximizing Expected Roadtrip Utility

D **Maximizing Expected Utility principle (MEU):**

A rational agent should choose the action that maximizes agent’s expected utility: $a^* = \arg\max_a EU(a \mid e)$



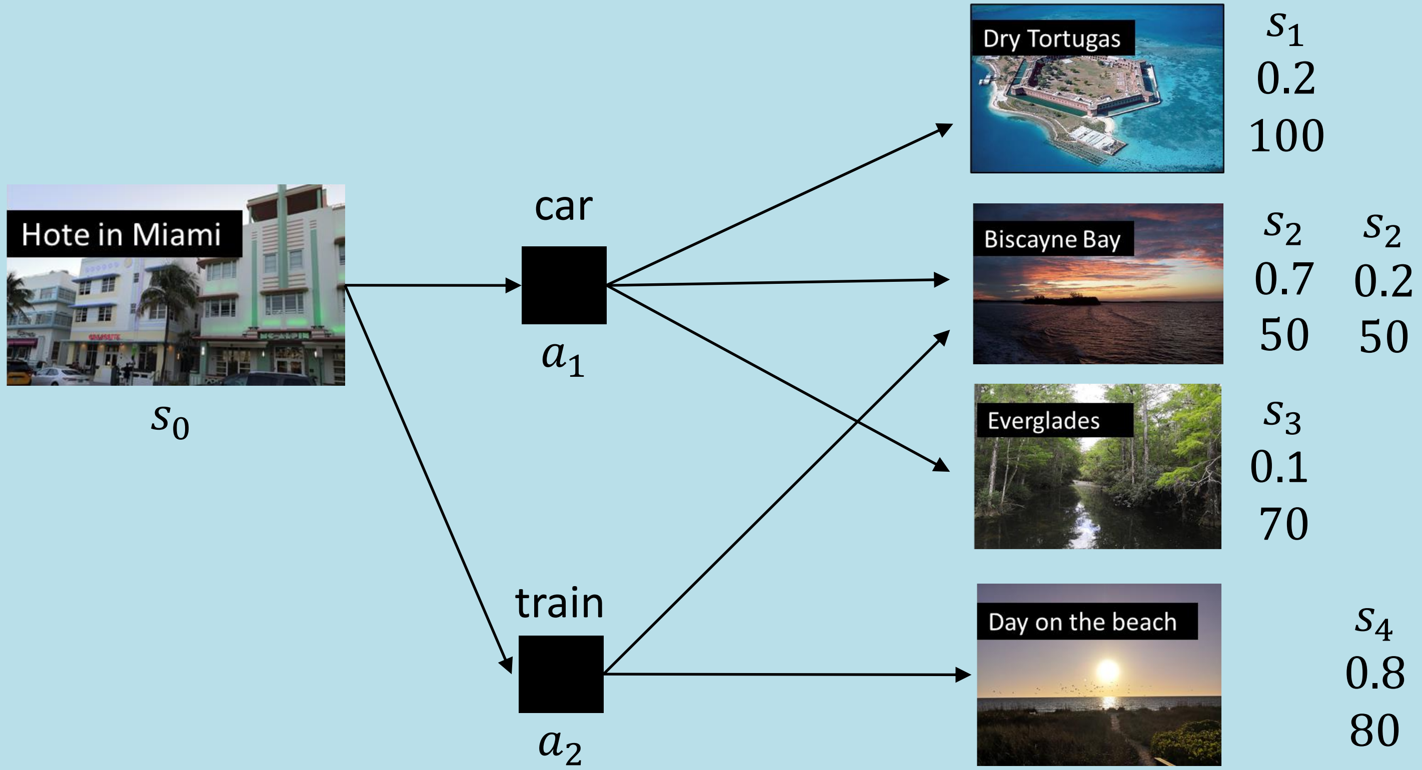
Adapted from Rusell, S., & Norvig, P. (2016);



$$U(Hotel) = 100 \cdot 0.2 + 50 \cdot 0.7 + 70 \cdot 0.1$$
$$= 20 + 35 + 7$$
$$= 62$$

Adapted from Rusell, S., & Norvig, P. (2016);

8.4 Example 1 ▶ Choice Between 2 Actions

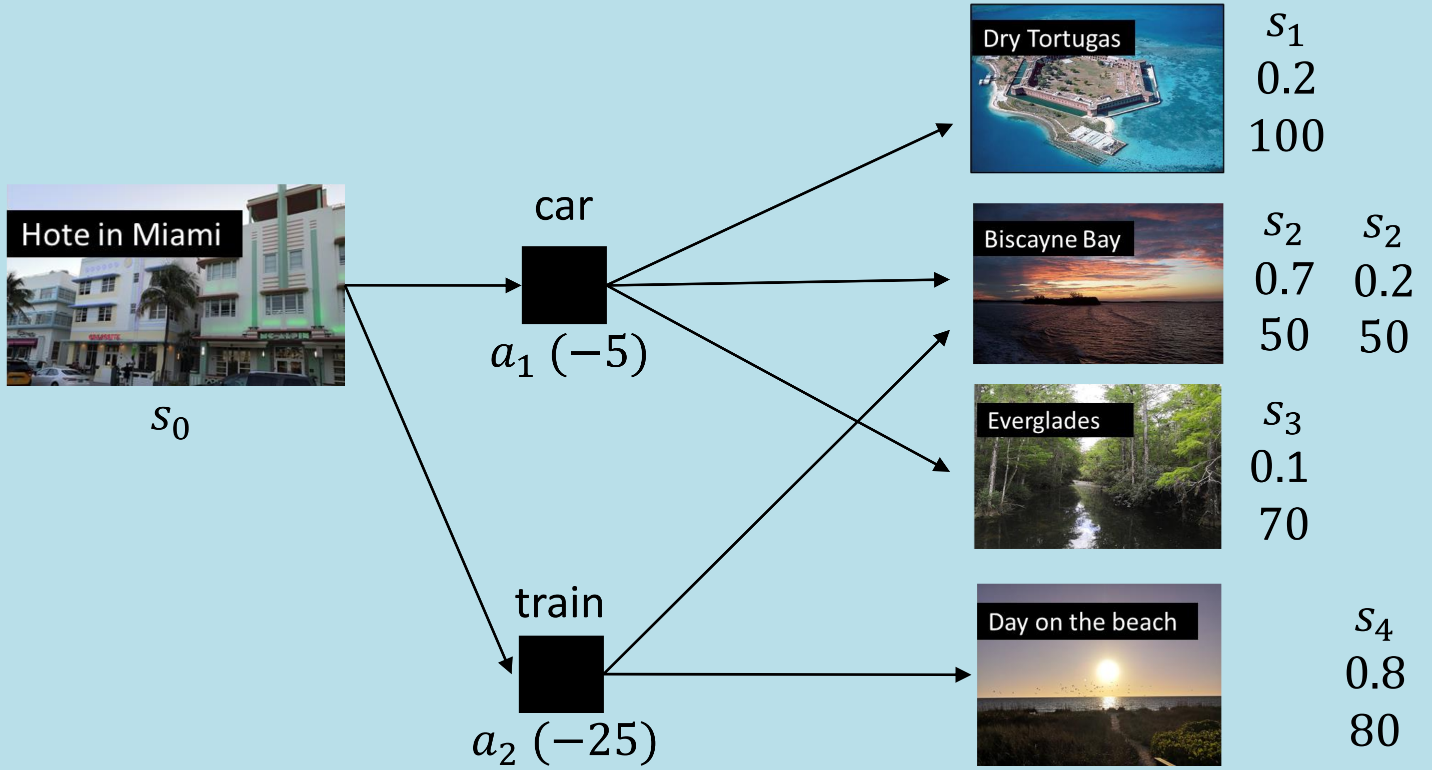


$$U_1(Hotel) = 62 \qquad U(Hotel) = \max\{U_1(Hotel), U_2(Hotel)\}$$

$$U_2(Hotel) = 74 \qquad U(Hotel) = 74 \ \rightarrow go\ by\ train\ (a_2)$$

Adapted from Rusell, S., & Norvig, P. (2016);

8.4 Example 1 ▶ Adding Action Costs



$$U_1(Hotel) = 62 - 5 = 57 \qquad U(Hotel) = 57\ \rightarrow pick\ driving\ by\ car\ (a_1)$$

$$U_2(Hotel) = 74 - 25 = 49$$

Adapted from Rusell, S., & Norvig, P. (2016);

8.4 MEU and Artificial Intelligence

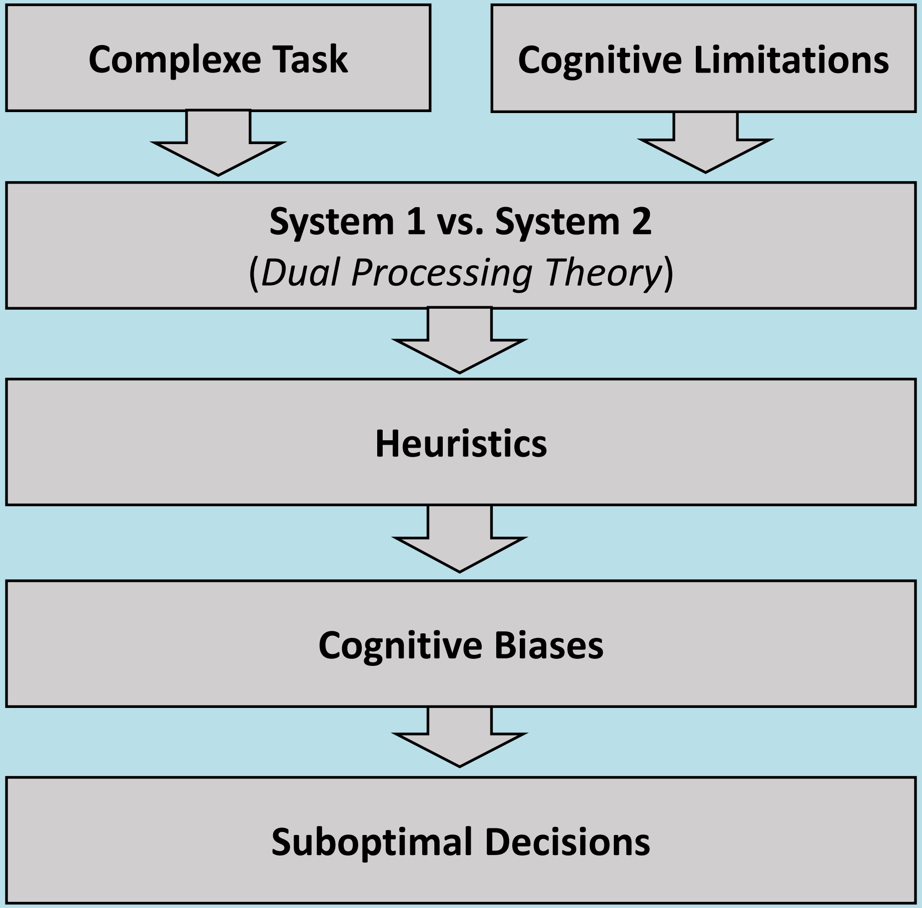
- A rational agent should choose the action that maximizes the expected utility
- The MEU principle provides a normative criterion for rational choice of action
- Do we now have a working definition of rational behavior, and therefore solved/generated true artificial intelligence?

- Not quite, we have to
  - Build a complete model (actions, utilities, states)
  - Even if you have a complete model, it will be computationally intractable

- In fact, a truly rational agent takes into account the utility of reasoning as well – bounded rationality

Adapted from Rusell, S., & Norvig, P. (2016);

8.4 Bounded Rationality



"I distinguish between **two cognitive systems**, a fast system 1 and a slow system 2"

Daniel Kahneman,

"maximizing with the goal of satisficing, of finding a course of action that is **good enough**"

Herbert Simon

"In general, these heuristics are quite useful, but sometimes they **lead to severe and systematic errors**."


Amos Tversky,
Daniel Kahneman,

Adapted from Jung D. (2019); Kahneman D. (2012); Gigerenzer G. & Selten R.(2002)

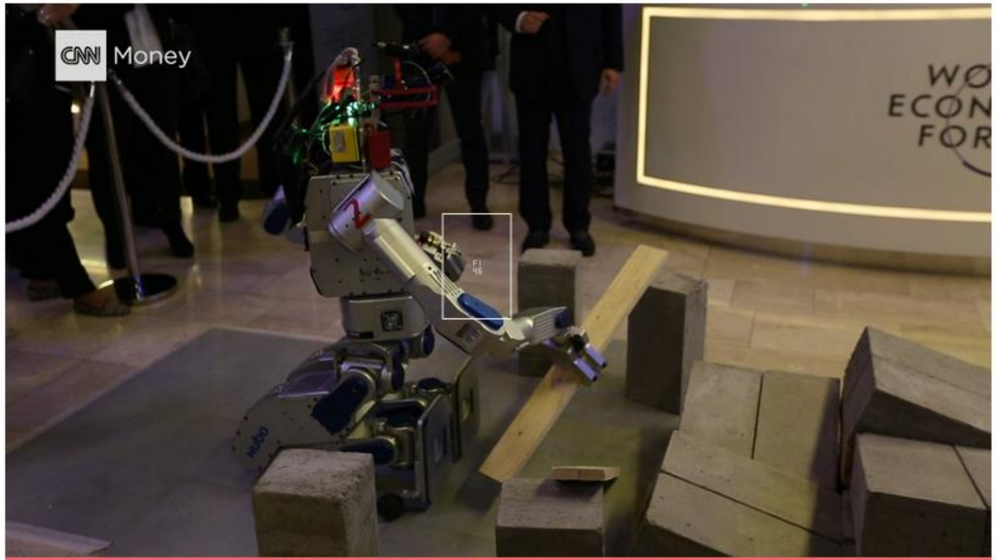
Business Case: Robo Advisors in Financial Consulting (2016)

CNN Money International + Markets Economy Companies Tech Autos India

30% of bank jobs are under threat

by Matt Egan @MattEganCNN

April 4, 2016: 11:20 AM ET



Message from Davos: The robots are coming

Robo-Advisor

Der Roboter verliert seine Unschuld

Robo-advisors: The future of investing or the latest financial craze?

Robo-Advisor - vertrauen Sie einem Computer?

Internet und Geldanlage (Teil 2)

Robo-Advisor: Reich mit System?

KOLUMNE **Verkehrte (Finanz)welt**

Keine Angst vorm Robo-Advisor

Online-Vermögensverwaltung

Robo-Advisors sammeln Millionen

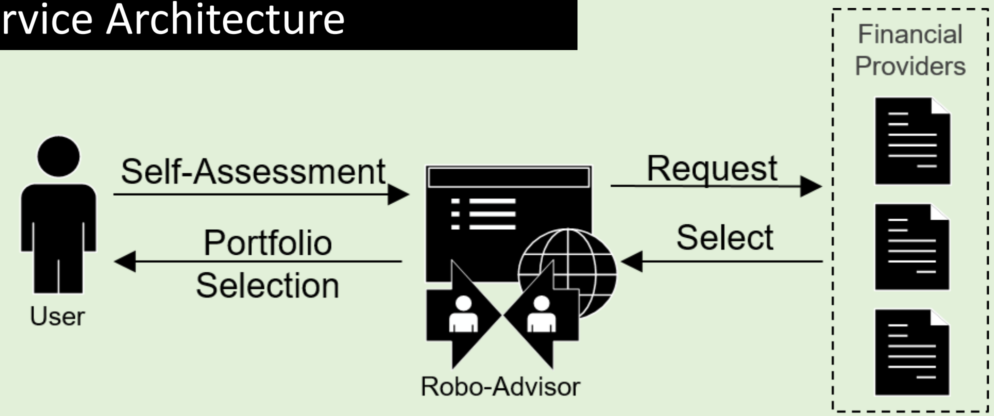
Robo-advisers are trying to tackle the gender-investing gap

Business Case: Robo Advisors in Financial Consulting

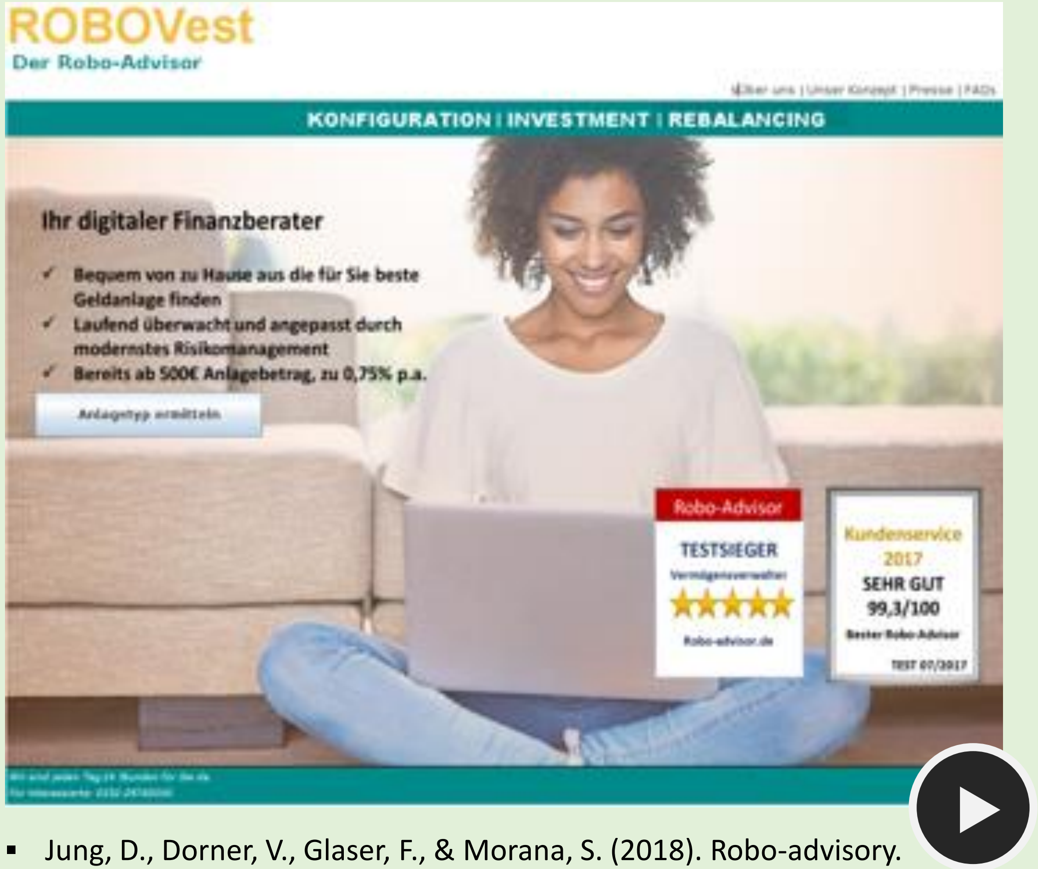
Robo-Advisors

- Guide users through a self-assessment process and give financial investment advisory
- Robo-advisor computes different investment recommendations and supports individual portfolio selection based on State-of-the-Art artificial intelligence

Service Architecture



Adapted from Jung D. (2019); Jung D. et al (2018)

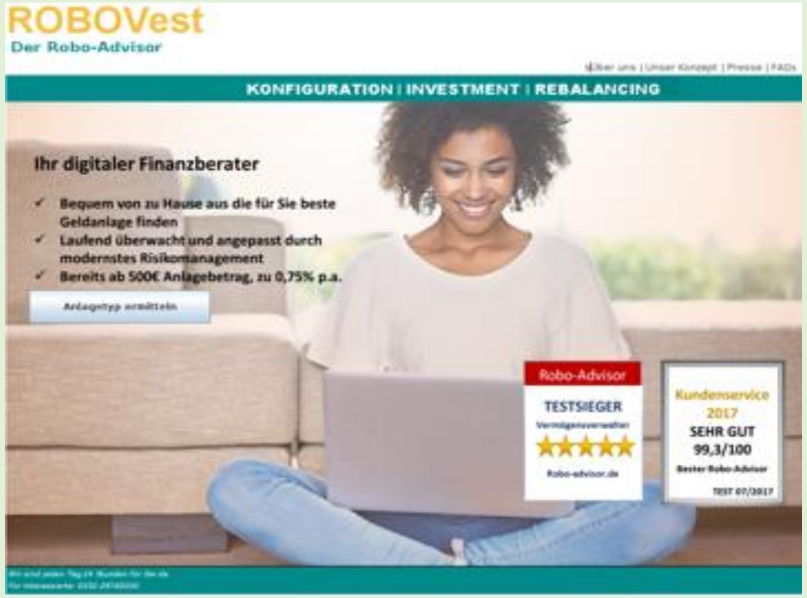


- Jung, D., Dorner, V., Glaser, F., & Morana, S. (2018). Robo-advisory. *Business & Information Systems Engineering*, *60*(1), 81-86, Best Paper of the Year 2019, online available at Springer Link: ↗https://link.springer.com/article/10.1007/s12599-018-0521-9

Business Case: Robo Advisors in Financial Consulting

01 | Executive Summary

Information systems researchers design and build financial decision support systems based on statistics, artificial intelligence and quantitative finance for private investors. Robo-advisors are such decision support systems aiming to provide independent advice, and support private households in investment decisions and wealth management. Their design and use and thus their ability to support financial decision-making is a challenge for researchers and practioneers.



02 | Solution

- Build information systems that use artificial intelligence and findings from finance to support individual portfolio selection
- In particularly, in situations where private investors are likely to make wrong investment decisions.

03 | References

- https://link.springer.com/article/10.1007/s12599-018-0521-9

Take-Aways

- Intelligent systems can support financial decision-making
- They provide a challenge for individuals to get high qualitative, low-budget advice

Adapted from Jung D. (2019); Jung D. et al (2018)

Your turn!

Task

Chris considers four used cars before buying the one with maximum expected utility. Pat considers ten cars and does the same.

- All other things being equal, which one is more likely to have the better car?
- Which is more likely to be disappointed with their car's quality?

8. Exercises

Workbook Exercises

- Please read the chapters 13 to 17 from Rusell, S., & Norvig, P. (2016) to understand the theory behind the concepts of this lecture. Then work through the exercises of each chapter.

Coding Exercises

The following coding exercise is an old bonus task for students.

☑ Previous Bonus Task!

- Implement a markov chain in python to solve the example we solved by hand
- Generalize your markov chain code in a module you can use for other types of problems

8. References

Literature

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Images

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Further reading

- Probabilistic reasoning and decision theory and cognitive science has huge influence on artificial intelligence. From the past to the present cognitive scientists and decision theories like Herbert Simon in the beginning or Geoffrey Hinton today are responsible for pathbreaking findings in AI. Take a look at their work to get a non-computer-science perspective on many AI topics and problems.
- Bounded rationality and decision theory influences your everydays decision-making more than you think (see Kahnemann‘s ↗Thinking fast and slow, and Richard Thaler‘s ↗Nudging, both Nobel-Price-Lauratees).
- Robo-advisors are a very new type of information systems that target to help humans to overcome their congitive limitations with decision theory and artificial intelligence and decision-making. If you are interesting you can take a look at my PHD research (↗Literature Overview)

8. Glossary

Bayes Rule	*Describes the probability of an event, based on prior knowledge of conditions that might be related to the event.*
Bayesian Networks	*Represents a set of variables and their conditional dependencies via a directed acyclic graph*
De Finetti's theorem	*De Finetti's theorem implies that no rational agent can have beliefs that violate the axioms of probability.*
Decision Theory	*Decision Theory is a combination of probability and utility theory to analyse and model decision-making. Utility theory is used to represent and infer preferences, while probability theory allows us to analyze chance events in a logically sound manner.*
Decision-Theoretic Agent	*A decision-theoretic agent that selects rational actions based on decision and utility theory.*
Random Variables	*A Random Variable is a variable whose possible values are numerical outcomes of a random phenomenon*
Markov Property	*if you want to predict where the chain will be at a future time, and if we know the present state, then the entire past history is irrelevant*
Maximizing Expected Utility principle (MEU)	*A rational agent should choose the action that maximizes agent's expected utility*
Statistical Independence	*Two events are independent, statistically independent if the occurrence of one does not affect the probability of occurrence of the other*
Robo-advisor	*Robo-advisors are information systems that computes different investment recommendations and supports individual portfolio selection based on State-of-the-Art artificial intelligence and decision theory*