ES 1: Dato un pto materiale di massa $m=1$ è vincolato a muoversi su una superficie liscia di equazione $z^2-x^2-y^2=1$ $(z\geq 0)$ (z verticale)

(i) Scrivere l'hamiltoniana
(ii) Scrivere le eq. di Hamilton
(iii) Mostrare che il sistema è completamente integrabile
(iv) Portare alle quadrature l'equazione di HJ.

(i) La superficie si può parametrizzare con

$$\begin{cases} x = \mathrm{sh}\theta \sin\varphi \\ y = \mathrm{sh}\theta\cos\varphi \\ z = \mathrm{ch}\theta \end{cases} \qquad \begin{cases} \dot x = \dot\theta\,\mathrm{ch}\theta\sin\varphi + \dot\varphi\,\mathrm{sh}\theta\cos\varphi \\ \dot y = \dot\theta\,\mathrm{ch}\theta\cos\varphi - \dot\varphi\,\mathrm{sh}\theta\sin\varphi \\ \dot z = \dot\theta\,\mathrm{sh}\theta \end{cases}$$

$$L = \frac{1}{2}\left((\mathrm{ch}^2\theta + \mathrm{sh}^2\theta)\dot\theta^2 + \mathrm{sh}^2\theta\,\dot\varphi^2\right) - g\,\mathrm{ch}\theta$$

$$g_{\mu\nu} = \begin{pmatrix} \mathrm{ch}^2\theta+\mathrm{sh}^2\theta & 0 \\ 0 & \mathrm{sh}^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} \frac{1}{\mathrm{ch}^2\theta+\mathrm{sh}^2\theta} & 0 \\ 0 & \frac{1}{\mathrm{sh}^2\theta} \end{pmatrix}$$

$$\boxed{H = \frac{1}{2}\left[\frac{p_\theta^2}{\mathrm{ch}^2\theta+\mathrm{sh}^2\theta} + \frac{p_\varphi^2}{\mathrm{sh}^2\theta}\right] + g\,\mathrm{ch}\theta}$$

NB: $\mathrm{ch}^2\theta + \mathrm{sh}^2\theta = 1 + 2\,\mathrm{sh}^2\theta = 2\,\mathrm{ch}^2\theta - 1 = \mathrm{ch}(2\theta)$

(ii)

$$\begin{cases} \dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mathrm{ch}^2\theta+\mathrm{sh}^2\theta} \\ \dot p_\theta = -\frac{\partial H}{\partial\theta} = -\left(g\,\mathrm{sh}\theta - \frac{p_\varphi^2}{\mathrm{sh}^3\theta}\mathrm{ch}\theta - \frac{p_\theta^2}{\mathrm{ch}^2(2\theta)}\mathrm{sh}(2\theta)\right) \\ \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{\mathrm{sh}^2\theta} \\ \dot p_\varphi = -\frac{\partial H}{\partial\varphi} = 0 \end{cases}$$

(iii) Integrali primi: H p_φ; ovviamente $\{H, p_\varphi\} = 0$ (altrimenti p_φ non sarebbe integrale primo).
Due gradi di libertà e due integrali primi in involuzione $\Rightarrow$ il sistema è completam. integrabile

(iv) $S = -Et + J\varphi + W(\theta; E, J)$

$$\frac{1}{2}\left[\frac{1}{\mathrm{ch}2\theta}(W')^2 + \frac{J^2}{\mathrm{sh}^2\theta}\right] + g\,\mathrm{ch}\theta = E$$

$$p_\theta = \frac{\partial S}{\partial\theta} = \sqrt{\phi} \qquad \alpha = -\frac{\partial S}{\partial E}$$

$$p_\varphi = \frac{\partial S}{\partial\varphi} = J \qquad \beta = -\frac{\partial S}{\partial J}$$

$$(W')^2 = \left[2(E - g\,\mathrm{ch}\theta) - \frac{J^2}{\mathrm{sh}^2\theta}\right]\mathrm{ch}2\theta = \phi(\theta)$$

$$W = \pm\int d\theta\,\sqrt{\phi(\theta)}$$

$$\alpha = t \mp \int d\theta\,\frac{\partial\sqrt{\phi}}{\partial E}$$

che possiamo invertire per $\theta(t; E, J, \alpha)$

$$\beta = -\varphi \mp \int d\theta\,\frac{\partial\sqrt{\phi}}{\partial J}$$

che noto $\theta(t)$ fornisce $\varphi(t; E, J, \alpha, \beta)$

ES2: Data

$$H = \frac{(q^2)^2(p_2)^4 - 2(q^2)^2(p_2)^2 + (q^2)^2 + (q^1)^2 + 2g(p_2)^3 - 2g\,p_2}{2(p_2^2-1)} =$$

$$= \frac{(q^2)^2(p_2^2-1)^2 + (q^1)^2 + 2g\,p_2(p_2^2-1)}{2(p_2^2-1)} = \frac{(q^2)^2}{2}(p_2^2-1) + \frac{(q^1)^2}{2(p_2^2-1)} + g\,p_2$$

e la funzione generatrice

$$F = q^1 y + a\, q^2\, \mathrm{ch}\, x$$

determinare a in modo che la nuova Hamiltoniana sia $K = \frac{1}{2}\left(p_x^2 + \frac{p_y^2}{\mathrm{sh}^2 x}\right) + g\,\mathrm{ch}\, x$

$$\begin{cases} p_x = \frac{\partial F}{\partial x} = a\, q^2\, \mathrm{sh}\, x \\ p_y = \frac{\partial F}{\partial y} = q^1 \end{cases} \qquad \begin{cases} p_1 = -\frac{\partial F}{\partial q^1} = -y \\ p_2 = -\frac{\partial F}{\partial q^2} = -a\,\mathrm{ch}\, x \end{cases}$$

$$p_2^2 - 1 = a^2\mathrm{ch}^2 x - 1 = (a^2-1) + a^2\mathrm{sh}^2 x$$

$$q^1 = p_y$$

$$q^2 = \frac{p_x}{a\,\mathrm{sh}\, x}$$

$$-a g\,\mathrm{ch}\, x$$

$$K = \frac{1}{2}\frac{p_x^2}{a^2\mathrm{sh}^2 x}\left[(a^2-1) + a^2\mathrm{sh}^2 x\right] + \frac{1}{2}\frac{p_y^2}{(a^2-1) + a^2\mathrm{sh}^2 x}$$

$$\boxed{a = -1}$$

$$K = \frac{1}{2}\left[p_x^2 + \frac{p_y^2}{\mathrm{sh}^2 x}\right] + g\,\mathrm{ch}\, x$$