AP Classroom Problems Unit 8

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8.01

1. If $a_k = (-1)^k$ for $k = 0, 1, 2, \ldots$, which of the following statements about the infinite series $\sum_{k=0}^{\infty} a_k$ is true?

   The series can be written as $1+(-1)+1+(-1)+1+\cdots$. The sequence of partial sums of the series is $1, 0, 1, 0, \cdots$, and this sequence of alternating 1s and 0s does not converge. Therefore, $\boxed{\text{the series diverges.}}$

2. If $a_n = 1$, for all positive integers n, what is the value of S_n, the nth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$?

   Each term a_n of the infinite sequence is equal to 1. It follows that S_n, the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$, is the sum of n 1s. Therefore,. $\boxed{S_n = n}$

3. The infinite series $\sum_{k=1}^{\infty}$ has nth partial sum $S_n = \frac{n}{3n+1}$ for $n \geq 1$. What is the sum of the series $\sum_{k=1}^{\infty} a_k$?

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \boxed{\frac{1}{3}}$$

8.02

1. Which of the following series converge to 2?

   I. $\sum_{n=1}^{\infty} \frac{2n}{n+3}$ *Diverges*

   II. $\sum_{n=1}^{\infty} \frac{-8}{(-3)^n} \implies S = \frac{8/3}{4/3} = 2$

   III. $\sum_{n=0}^{\infty} \frac{1}{2^n} \implies S = \frac{1}{1/2} = 2$

$$\boxed{\text{II and III only}}$$

2. If $f(x) = \sum_{k=1}^{\infty} \left(\sin^2 x\right)^k$, then $f(1)$ is

$$f(1) = \frac{\sin^2(1)}{1 - \sin^2(1)} = \tan^2(1) \approx \boxed{2.426}$$

3. Let x be a real number. Which of the following statements about the infinite series $\sum_{k=0}^{\infty} e^{kx}$ is true?

$$\boxed{\text{The sum of the series is } \frac{1}{1-e^x} \text{ if } x < 0.}$$

4. If x and y are positive real numbers, which of the following conditions guarantees that the infinite series $\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)\left(\frac{x}{y^2}\right)^n$ converges?

$$\boxed{x < y^2}$$

5. If b and t are real numbers such that $0 < |t| < |b|$, which of the following infinite series has sum $\frac{1}{b^2+t^2}$?

   (a) $a = -1$

   (b) $r = -\frac{t^2}{x^2}$

$$\boxed{\frac{1}{b^2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{t^2}{x^2}\right)^2}$$

6. What is the sum of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}}$?

   (a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}} = \frac{-2}{e^2} + \frac{4}{e^3} - \frac{8}{e^4} + \frac{16}{e^5} + \cdots + \frac{(-2)^n}{e^{n+1}}$

   (b) $a_1 = \frac{-2}{e^2}$

   (c) $r = \frac{-2}{e}$

$$S = \frac{-\frac{2}{e^2}}{1 - \frac{-2}{e}} = \boxed{\frac{-2}{e^2 + 2e}}$$

7. What is the value of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$?

   (a) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots + \frac{2^{n+1}}{3^n}$

   (b) $a_1 = \frac{4}{3}$

   (c) $r = \frac{2}{3}$

$$S = \frac{4/3}{1 - 2/3} = \boxed{4}$$

8. Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $an > 0$ for all n. The first term of the series is $a_1 = 48$, and the third term is $a_3 = 12$. Which of the following statements about $\sum_{n=1}^{\infty} a_n$ is true?

> Let r denote the common ratio of the geometric series. Then we have $a_2 = 48r$ and $a_3 = 48r^2$. We are also given that $a_3 = 12$, so we can solve for r as follows: $12 = 48r^2 \quad \Rightarrow \quad r^2 = \frac{1}{4} \quad \Rightarrow \quad r = \frac{1}{2}$. Applying formula for the sum of a geometric series: $\sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r} = \frac{48}{1-\frac{1}{2}} = 96$ So the statement that is true is: $\boxed{\displaystyle\sum_{n=1}^{\infty} a_n = 96}$

9. Consider the series $\sum_{n=1}^{\infty} a_n$. If $a_1 = 16$ and $\frac{a_{n+1}}{a_n} = \frac{1}{2}$ for all integers $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is

> We have $\frac{a_{n+1}}{a_n} = \frac{1}{2}$ for all $n \geq 1$. This means that $a_{n+k} = \frac{1}{2^k} a_n$. Thus, the series $\sum_{n=1}^{\infty} a_n$ can be written as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = a_1 + \frac{1}{2}a_1 + \frac{1}{2^2}a_1 + \cdots = a_1\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$ The sum of an infinite geometric series with first term a and common ratio r is $\frac{a}{1-r}$ (provided that $|r| < 1$), so $\sum_{n=1}^{\infty} a_n = a_1\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = 16\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = 16 \cdot \frac{1}{1-\frac{1}{2}} = \boxed{32}$

10. What is the value of $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$?

    (a) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n} = \frac{9}{5} - \frac{27}{25} + \frac{81}{125} + \cdots + \frac{(-3)^{n+1}}{5^n}$
    (b) $a_1 = \frac{9}{5}$
    (c) $r = \frac{-3}{5}$

$$S = \frac{9/5}{1 + \frac{3}{5}} = \boxed{\frac{9}{8}}$$

11. The sum of the infinite geometric series $\frac{3}{2} + \frac{9}{16} + \frac{27}{128} + \frac{81}{1{,}024} + \cdot$ is

    (a) $a_1 = \frac{3}{2}$
    (b) $r = \frac{3}{8}$

$$S = \frac{\frac{3}{2}}{1 - \frac{3}{8}} = \frac{3/2}{5/8} = \frac{12}{5} = \boxed{2.40}$$

12. What is the value of $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$?

    (a) $a_1 = 1$
    (b) $r = -\frac{2}{3}$

$$S = \frac{1}{5/3} = \boxed{\frac{3}{5}}$$

8.03

1. The n term test can be used to determine divergence for which of the following series?

   I. $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

   II. $\sum_{k=0}^{\infty} (-1)^k \left(\frac{k}{2k+1}\right)$

   III. $\sum_{k=1}^{\infty} \frac{3k^2-k^3}{5k^3}$

$$\boxed{\text{II and III only}}$$

2. Which of the following series diverge?

   I. $\sum_{n=1}^{\infty} \cos(2n)$

   II. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)$

   III. $\sum_{n=1}^{\infty} \left(n+\frac{1}{n^2}\right)$

$$\boxed{\text{I, II, and III}}$$

3. If $a_n = \cos\left(\frac{\pi}{n}\right)$ for $n = 1, 2, \cdots$, which of the following statements about $\sum_{n=0}^{\infty} a_n$ must be true?

$$\boxed{\text{The series diverges and } \lim_{n\to\infty} \neq 0.}$$

8.04

1. Let f be a positive, continuous, decreasing function. If $\int_1^{\infty} f(x)\,dx = 5$, which of the following statements about the series $\sum_{n=1}^{\infty} f(n)$ must be true?

$$\boxed{\sum_{n=1}^{\infty} f(n) \text{ converges, and } \sum_{n=1}^{\infty} f(n) > 5}$$

2. Let f be a positive, continuous, decreasing function such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n$ converges to k, which of the following must be true?

$$\boxed{\int_1^{\infty} f(x)\,dx \text{ converges}}$$

3. The integral test can be used to determine that which of the following statements about the infinite series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

$$\boxed{\text{The series converges because } \int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = -1+e.}$$

4. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The integral test can be used to verify convergence of the series because $f(x) = \frac{1}{x^2}$ is positive, continuous, and decreasing for $x \geq 1$. Which of the following inequalities is true?

$$\boxed{\underbrace{\int_2^{\infty} \frac{1}{x^2}\,dx}_{0.5} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\frac{\pi^2}{6}} < \underbrace{1 + \int_1^{\infty} \frac{1}{x^2}\,dx}_{2}}$$

5. The integral test can be used to conclude that which of the following statements about the infinite series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is true?

$$\boxed{\text{The series diverges, and the terms of the series have limit 0.}}$$

8.05

1. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Since $\int_1^{\infty} \frac{1}{x^{1/2}}\,dx = \infty \therefore$ the series diverges.

   II. $\sum_{n=1}^{\infty} \frac{3^n}{n!} = e^3 - 1$

   III. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n = \frac{-e}{e-\pi}$

$$\boxed{\text{II and III only}}$$

2. Which of the following series converge?

   I. $1 + (-1) + 1 + \cdots + (-1)^{n-1} + \cdots$. Since $\int_1^{\infty} (-1)^{x-1}\,dx = \infty \therefore$ the series diverges.

   II. $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$. Since $\int_1^{\infty} \frac{1}{2x-1}\,dx = \infty \therefore$ the series diverges.

   III. $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}} + \cdots$. Since $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \frac{3}{2} \therefore$ the series converges.

$$\boxed{\text{III only}}$$

3. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

   > We can use the comparison test with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to show that $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges. Note that $0 \leq |\sin n| \leq 1$, so $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$ for all n. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test with $p = 2 > 1$, we have that $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges by the comparison test.

   II. $\sum_{n=1}^{\infty} e^{-n}$

   > Since $\sum_{n=1}^{\infty} e^{-n}$ is a geometric series with first term e^{-1} and common ratio $r = e^{-1} < 1$, it converges by the geometric series test.

III. $\sum_{n=1}^{\infty} \frac{n+1}{n^2+n}$

We can use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ to show that $\sum_{n=1}^{\infty} \frac{n+1}{n^2+n}$ diverges. Note that $\frac{n+1}{n^2+n} = \frac{n+1}{n(n+1)} = \frac{1}{n}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test with $p = 1 < 2$, we have that $\sum_{n=1}^{\infty} \frac{n+1}{n^2+n}$ also diverges by the limit comparison test.

$$\boxed{\text{I and II only}}$$

4. What are all values of p for which the infinite series $\sum_{n=1}^{\infty} \frac{n}{n^p+1}$ converges?

We can use the limit comparison test to determine the values of p for which the series converges. First, note that for all $p > 0$, we have $0 \leq \frac{n}{n^p+1} \leq \frac{n}{n^p} = \frac{1}{n^{p-1}}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is a p-series, and converges if $p - 1 > 1$, i.e. if $p > 2$. Thus, for $p > 2$, we have $0 \leq \frac{n}{n^p+1} \leq \frac{1}{n^{p-1}}$ for all n, and by the comparison test, the series $\sum_{n=1}^{\infty} \frac{n}{n^p+1}$ converges. Therefore, the series converges if and only if $p > 2$.

$$\boxed{p > 2}$$

5. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \zeta\left(\frac{3}{2}\right)$. Since $\int_1^{\infty} \frac{1}{n\sqrt{n}}\, dn = 2 \therefore \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges.

II. $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$. Since the series is geometric and $r = \frac{1}{3} < 1 \therefore$ by the geometric series test $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

III. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$. Since $\int_1^{\infty} \frac{1}{n \ln n}\, dn = \ln(\ln n)\Big|_0^{\infty} = \infty \therefore \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges.

$$\boxed{\text{I and II only}}$$

6. What are all values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}+n}$ diverges?

Note that for any positive integer n, we have $n^{2p} + n \geq n^{2p}$. Therefore, we can write $\frac{1}{n^{2p}+n} \leq \frac{1}{n^{2p}} = \frac{1}{n^p \cdot n^p}$. Now, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ is a p-series, which converges if and only if $p > \frac{1}{2}$. Therefore, by the limit comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}+n}$ diverges if and only if $p \leq \frac{1}{2}$.

$$\boxed{p \leq \frac{1}{2}}$$

7. For what values of p will both series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ and $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converges?

Using the p-series test, we see that $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges if $2p > 1$, or equivalently, $p > \frac{1}{2}$. Using the geometric series test, we see that $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converges if $\left|\frac{p}{2}\right| < 1$, or equivalently, $|p| < 2$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ and $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converge for $\boxed{\frac{1}{2} < p < 2}$.

8. What are all values of p for which $\int_1^\infty \frac{1}{x^{2p}}\, dx$ converges?

$$\boxed{p > \frac{1}{2}}$$

9. Which of the following is a convergent p-series?

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^3 = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}$$

10. Which of the following is not a p-series?

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{e^n}}$$

11. Which of the following is the harmonic series?

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

12. Which of the following series converge?

    (a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Since the $\int_1^\infty \frac{1}{x^2}\, \mathrm{d}x = 1 \therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

    (b) $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the $\int_1^\infty \frac{1}{x}\, \mathrm{d}x = \infty \therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

    (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = (\sqrt{2} - 1) \cdot \zeta\left(\frac{1}{2}\right)$.

To prove the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, we can use the Alternating Series Test, which states that if a series $\sum_{n=1}^{\infty} (-1)^n b_n$ satisfies the following two conditions: The sequence b_n is positive and monotonically decreasing (i.e., $0 < b_{n+1} \leq b_n$) and the $\lim_{n\to\infty} b_n = 0$. Then the series converges. In our case, we have $b_n = \frac{1}{\sqrt{n}}$, which is a positive, monotonically decreasing sequence. To see that b_n is decreasing, note that $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ since $\sqrt{n+1} > \sqrt{n}$ for all $n \geq 1$. Furthermore, we have $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ since the denominator grows to infinity faster than the numerator. Thus, by the Alternating Series Test, we conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

13. Which of the following series diverge?

    I. $\sum_{k=3}^{\infty} \frac{2}{k^2+1} = \frac{1}{5}\left(5\pi \cosh(\pi) - 12\right)$.

    II. $\sum_{n=1}^{\infty} \left(\frac{6}{7}\right)^k = 6$

    III. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k} = 1 - \ln 2$

$$\boxed{\text{None}}$$

8.06

1. Which of the following series converges?

$$\boxed{\sum_{n=1}^{\infty} \frac{3n^2}{n^4 + 2n}}$$

2. Which of the following statements about the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is true?

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin n}{n} = 1$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ would also diverge by the limit comparison test.

$$\boxed{\text{The series diverges by limit comparison to the series } \sum_{n=1}^{\infty} \frac{1}{n}.}$$

3. Which of the following series can be used with the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n - n^2}$ converges or diverges?

The limit comparison test looks at the limit of the ratio of general terms of the two positive series. If this limit is finite and greater than 0, the two series either both converge or both diverge. For this series, $\lim_{n\to\infty} \frac{\frac{4^n}{5^n - n^2}}{\left(\frac{4}{5}\right)^n} = \lim_{n\to\infty} \left(\frac{4^n}{5^n - n^2} \cdot \frac{5^n}{4^n}\right) = \lim_{n\to\infty} \frac{1}{1 - \frac{n^2}{5^n}} = 1$. Since the limit is finite and nonzero and the geometric series $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ converges, the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n - n^2}$ will also converge by the limit comparison test.

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n}$$

4. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n + n}$ is true?

The series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series. Since $0 < \frac{2^n}{3^n + n} \leq \frac{2^n}{3^n}$ for all n, the comparison test shows that the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n + n}$ also converges

$$\boxed{\text{The series converges by comparison to the series } \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.}$$

5. Which of the following series can be used with the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ converges or diverges?

For this series, $\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$. Since the limit is finite and nonzero and the geometric series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ will also converge by the limit comparison test.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

6. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n$ converges, which of the following must be true?

$$\boxed{\text{If } b_n \leq a_n, \text{ then } \sum_{n=1}^{\infty} b_n \text{ converges.}}$$

7. If $0 < b_n < a_n$ for $n \geq 1$, which of the following must be true?

$$\boxed{\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} b_n = 0.}$$

8. If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$ for all n, which of the following statements must be true?

$$\boxed{\sum_{n=1}^{\infty} b_n \text{ diverges.}}$$

8.07

1. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Converges by p-seires test, i.e., $p = 2 > 1$.

   II. $\sum_{n=1}^{\infty} \frac{1}{n}$. Diverges by p-seires test, i.e., $p = 1$.

   III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Converges by the alternating series test, i.e. $\frac{(-1)^{n+1}}{\sqrt{n+1}} \leq \frac{(-1)^n}{\sqrt{n}}$,

$$\boxed{\text{I and III only}}$$

2. The Taylor series for a function f about $x = 0$ is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}$ and converges to f for all real numbers x. If the fourth-degree Taylor polynomial for f about $x = 0$ is used to

approximate $f\left(\frac{1}{2}\right)$ alternating series error bound?

Since $f(x) = \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \cdots$ the fourth-degree Taylor polynomial for f is $P(x) = \frac{x^2}{3!} - \frac{x^4}{5!}$. $P\left(\frac{1}{2}\right) = \frac{1}{3!}\left(\frac{1}{2}\right)^2 - \frac{1}{5!}\left(\frac{1}{2}\right)^4$. Using the Taylor series for f about $x = 0$, $\left(\frac{1}{2}\right) = \frac{1}{3!}\left(\frac{1}{2}\right)^2 - \frac{1}{5!}\left(\frac{1}{2}\right)^4 + \frac{1}{7!}\left(\frac{1}{2}\right)^6 - \frac{1}{9!}\left(\frac{1}{2}\right)^8 + \frac{1}{11!}\left(\frac{1}{2}\right)^{10} - \cdots$. This is an alternating series and converges by the alternating series test. Therefore the alternating series error bound can be used to approximate this value using the first two terms of the series, which is the same as $P\left(\frac{1}{2}\right)$. The alternating series error bound using the first two terms in the series for $f\left(\frac{1}{2}\right)$ is the absolute value of the third term, $\frac{1}{7!}\left(\frac{1}{2}\right)^6$, the first omitted term of the series, so $\left| f\left(\frac{1}{2}\right) - P\left(\frac{1}{2}\right) \le \frac{1}{7!}\left(\frac{1}{2}\right)^6 \right|$.

$$\boxed{\frac{1}{2^6 \cdot 7!}}$$

3. The power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n n^2}$ has radius of convergence 2. At which of the following values of x can the alternating series test be used with this series to verify convergence at x?

$$\boxed{4}$$

4. The alternating series test can be used to show convergence of which of the following alternating series?

   I. $4 - \frac{1}{9} + 1 - \frac{1}{81} + \frac{1}{4} - \frac{1}{729} + \frac{1}{16} - \cdots + a_n + \cdots$, where $a_n = \begin{cases} \frac{8}{2^n} & \text{if } n \text{ is odd} \\ -\frac{1}{3^n} & \text{if } n \text{ is even} \end{cases}$

   II. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots + a_n + \cdots$, where $a_n = \frac{(-1)^{n+1}}{n}$

   III. $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \frac{6}{11} - \frac{7}{13} + \frac{8}{15} - \cdots + a_n + \cdots$, where $a_n = (-1)^{n+1}\frac{n+1}{2n+1}$

$$\boxed{\text{II only}}$$

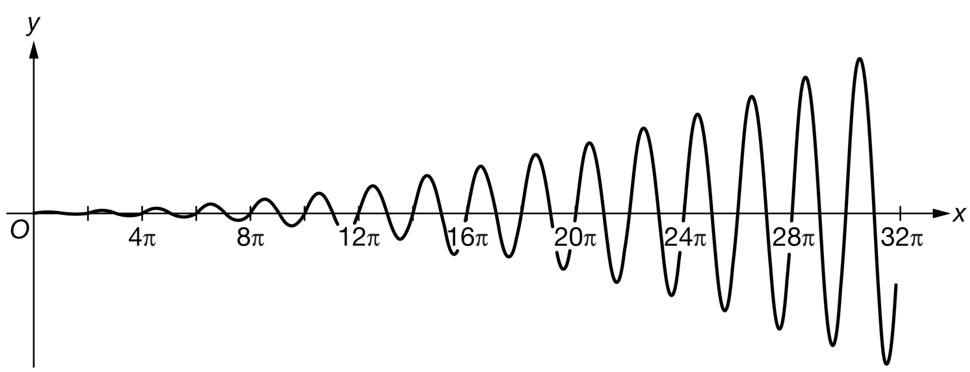
5. Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n}$?

   I. The series is alternating.

   II. $|a_n + 1| \le |a_n|$ for all $n \ge 2$

   III. $\lim_{n\to\infty} a_n = 0$

$$\boxed{\text{I and III only}}$$



6. Graph of g

Let f be the function defined by $f(x)=\frac{2+\cos x}{x^2}$. The derivative of f is $f'(x)=-\frac{x^2\sin x+2x(2+\cos x)}{x^4}$. The graph of the function g defined by $g(x)=x^2\sin x+2x(2+\cos x)$ is shown above for $0\le x\le 100$. Let $b_n=f(n)$ for all integers $n\ge 1$. Which of the following statements about the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ is true?

The alternating series test cannot be used to determine convergence because the terms b_n are not decr

7. The alternating series test can be used to show convergence for which of the following series?

   I. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots+a_n+\cdots$, where $a_n=(-1)^{n+1}\frac{1}{n^2}$

   II. $\sin 1-\frac{\sin 2}{4}+\frac{\sin 3}{9}-\frac{\sin 4}{16}+\frac{\sin 5}{25}-\frac{\sin 6}{36}+\cdots+b_n+\cdots$, where $b_n=(-1)^{n+1}\frac{\sin n}{n^2}$

   III. $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{2}-1}+\frac{1}{\sqrt{3}+1}-\frac{1}{\sqrt{3}-1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{4}-1}+\cdots+c_n+\cdots$, where $c_n=\begin{cases}\frac{1}{\sqrt{k+1}+1} & \text{if } n=2k-1\\ \frac{-1}{\sqrt{k+1}-1} & \text{if } n=2k\end{cases}$

$$\boxed{\text{I only}}$$

8. Which of the following statements is true?

$$\boxed{\text{The series } \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{9+n^2} \text{ converges by the alternating series test.}}$$

9. If the series $S=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is approximated by the partial sum $S_k=\sum_{n=1}^{k}(-1)^{n+1}\frac{1}{n^2}$, what is the least value of k for which the alternating series error bound guarantees that $|S-S_k|\le\frac{9}{10{,}000}$?

The alternating series error bound guarantees that $|S-S_k|\le\frac{1}{(k+1)^2}$. $\frac{1}{(k+1)^2}\le\frac{9}{10{,}000}\implies\frac{10{,}000}{9}\le(k+1)^2\implies\frac{100}{3}\le k+1$. Therefore, $|S-S_k|\le\frac{9}{10{,}000}$ is guaranteed by the alternating series error bound if $k\ge\frac{100}{3}-1\approx 33.333-1\approx 32.333$. The least k satisfying this inequality is $k=33$.

$$\boxed{33}$$

10. The series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges to S and $0<a_{k+1}<a_k$ for all k. If $S_n=\sum_{k=1}^{n}(-1)^{k+1}a_k$ is the nth partial sum of the series, which of the following statements must be true?

$$\boxed{|S-S_{15}|\le a_{16}}$$

11. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$ is approximated by the partial sum with 15 terms, what is the alternating series error bound?

$$\boxed{S_{16}=\frac{1}{33}}$$

8.08

1. Which of the following series are conditionally convergent i.e., if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges?

   I. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges based on the alternating series test but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges due to the p-series test.

   II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, based on the alternating series test and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges due to the p-series test.

   III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, converges based on the alternating series test but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges due to the p-series test.

   $$\boxed{\text{I and III only}}$$

2. Which of the following series converges for all real numbers x?

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{e^{n+1}x^{n+1}}{(n+1)n!}}{\frac{e^n x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{ex}{n+1}\right| = 0 < 1$$

   $$\boxed{\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}}$$

3. Which of the following series converge?

   I. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$

      i. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{8^{n+1}}{(n+1)n!}}{\frac{8^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{8}{n+1}\right| = 0 < 1$

   II. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$

      i. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)n!}{n^{101}}}{\frac{n!}{n^{100}}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1$

   III. $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$

      i. We can use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to show that $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$ converges. Let $L = \lim_{n\to\infty}\left(\frac{n+1}{(n)(n+2)(n+3)} \cdot \frac{n^2}{1}\right) = 1$. Since the limit converges to a finite value therefore the series $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$ converges by the limit comparison test.

   $$\boxed{\text{I and III only}}$$

4. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{n3^n}{x^n}$ converges?

   $$\boxed{|x| > 3}$$

5. For what values of p is the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p+2}$ conditionally convergent?

For $p-1>0$, or $p>1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p+2}$ is s an alternating series with individual terms that decrease in absolute value to 0. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p+2}$ converges for $p>1$ by the alternating series test. The series $\sum_{n=1}^{\infty} \frac{n}{n_p} = \sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is a p-series and therefore diverges for $p-1 \leq 1$, or $p \leq 2$. Since $\sum_{n=1}^{\infty} \frac{n}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ diverges for. $p \leq 2$, the series $\sum_{n=1}^{\infty} \frac{n}{n^p+2}$ diverges for $p \leq 2$ by the limit comparison test. Since the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p+2}$ converges for $p>1$ and the series of absolute values $\sum_{n=1}^{\infty} \frac{n}{n^p+2}$ diverges for $p \leq 2$, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p+2}$ is conditionally convergent for $1<p\leq 2$

$$\boxed{1 < p \leq 2 \text{ only}}$$

6. Which of the following series is conditionally convergent?

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{17+\sqrt{n}}{n}$ converges by the alternating series test: $\lim_{x\to\infty} \frac{17+\sqrt{n}}{n} = 0$ if $f(x) = \frac{17+\sqrt{x}}{x}$, then $f'(x) = \frac{-17}{x^2} - \frac{1}{2^{3/2}} < 0$ for $x>0$, showing that the terms $\frac{17+\sqrt{x}}{x}$ decrease as n increases. The series $\sum_{n=1}^{\infty} \frac{17+\sqrt{n}}{n}$ diverges, however, by the comparison with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, since $\frac{17+\sqrt{n}}{n} > \frac{\sqrt{n}n > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}}{}$ for all n. Therefore the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{17+\sqrt{n}}{n}$ is conditionally convergent.

$$\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{17+\sqrt{n}}{n}}$$

7. Which of the following statements is true about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$?

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is an alternating series with individual terms that decreases in absolute value to 0. Therefore, it converges by the alternating series test. The series of absolute values $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges, as it is a p-series with $p=\frac{1}{3}<1$. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is conditionally convergent.

$$\boxed{\text{The series converges conditionally.}}$$

8. If the ratio test is applied to the series $\sum_{n=0}^{\infty} \frac{n\pi^{2n}}{17^n}$, which of the following inequalities results, implying that the series converges?

Let $a_n = \frac{n\pi^{2n}}{17^n}$. The ratio test would look at the limit $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)\pi^{2(n+1)}}{17^{n+1}}}{\frac{n\pi^{2n}}{17^n}} = \lim_{n\to\infty} \frac{17^n(n+1)\pi^{2n+2}}{17^{n+1}n\pi^{2n}} = \lim_{n\to\infty} \frac{(n+1)\pi^2}{17n}$. Since $L = \frac{\pi^2}{17} < 1$ the series converges.

$$\boxed{\lim_{n\to\infty} \frac{\pi^2(n+1)}{17n} < 1}$$

9. If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$, which of the following series converges?

Since $\lim_{n\to\infty} \frac{\frac{a_{n+1}}{5^{n+1}}}{\frac{a_n}{5^n}} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n} \cdot \frac{5^n}{5^{n+1}}\right) = \frac{1}{5} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3}{5} < 1$, this series would converge by the ratio test.

$$\boxed{\sum_{n=1}^{\infty} \frac{a_n}{5^n}}$$

10. What are all positive values of p for which the series $\sum_{n=1}^{\infty} \frac{n^p}{4^n}$ will converge?

Let $a_n = \frac{n^p}{4^n}$. Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^p}{4^{n+1}} \cdot \frac{4^n}{n^p}\right) = \lim_{n\to\infty} \left(\frac{1}{4} \cdot \left(\frac{n+1}{n}\right)^p\right)$ for all positive values of p. By the ratio test, the series will converge for all $p > 0$.

$$\boxed{p > 0}$$

11. Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

$$\boxed{\lim_{n\to\infty} \frac{e}{n+1} < 1}$$