© MMXXII All Rights Reserved, Aiden Rosenberg

Limits and Geometry

Aiden Rosenberg

December MMXXII A.D

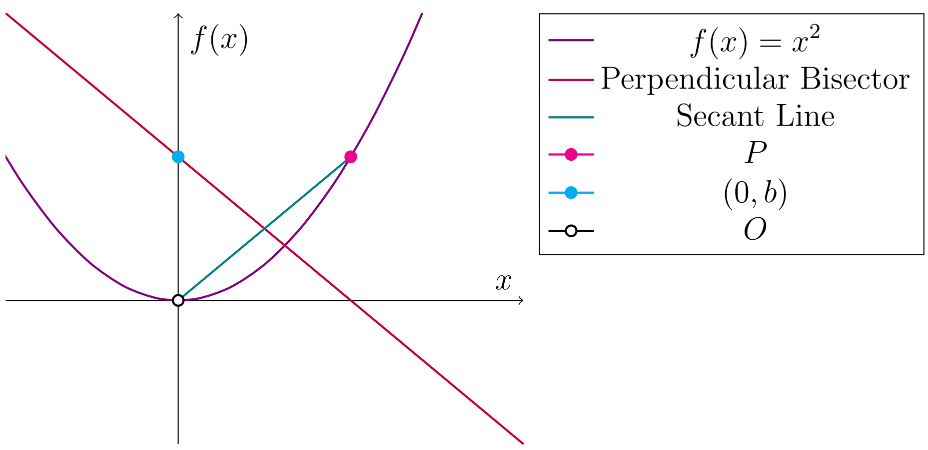
1 Problem

Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O denote the origin and $(0, b)$ the y-intercept of the perpendicular bisector of the line segment $\overline{OP}$. Find the $\lim\limits_{P \to O} b$.

2 Conjecture

Proposition 1. *The* $\lim\limits_{P \to O} b = k$ *such that* $k \in \mathbb{R}^+$.

3 Graph



4 Analysis

4.1 Numerical Analysis

a	0.1	0.01	0.001	0.0001	0.00001
b	0.505	0.50005	0.5000005	0.500000005	0.50000000005

The table above represents simulated values for b using graphical analysis via the Desmos engine. Link to the interactive graph: https://www.desmos.com/calculator/kwepdntlyx

4.2 Algebraic Analysis

Let m denote the slope of the secant line of the point $(0, 0)$ and $(0, a)$ $\therefore m = \dfrac{f(a)}{a} = \dfrac{a^2}{a} = a$. The equation for the perpendicular bisector of the line can be written as $y_\perp = \dfrac{-1}{a}(x - x_m) + y_m$ where (x_m, y_m) is the midpoint of the secant line. The point (x_m, y_m) can be expressed as $\left(\dfrac{a}{2}, \dfrac{f(a)}{2}\right) \Longrightarrow$ $y_\perp = \dfrac{-1}{a}\left(x - \dfrac{a}{2}\right) + \dfrac{f(a)}{2} \xRightarrow{\text{Simplifying}} y_\perp = \dfrac{-x}{a} + \underbrace{\dfrac{1}{2} + \dfrac{a^2}{2}}_{\text{Real Number}}$. When $x = 0 \Longrightarrow y = \underbrace{\dfrac{1 + a^2}{2}}_{y\text{-intercept}} = b$.

The $\lim\limits_{P \to O} b = \lim\limits_{a \to 0} \dfrac{1 + a^2}{2} = \boxed{\dfrac{1}{2}}$.

Q.E.D.