AP calc Practice FRQs

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1 FRQ #1

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \leq t \leq 40$ are shown in the table above.

a.) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to aproximate $\int_0^{40} v(t)\,dt$ Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

$$\int_0^{40} v(t)\,dt \approx 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229\,\text{miles}$$

The plane travels 229 miles during the 40 minutes.

b.) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

$$v'(c) = \frac{v(b) - v(a)}{a - b} = 0\,,\text{MVT}$$

The average rate of change when $t \in [25, 30]$ & $t \in [0, 15]$ is zero.

$a \leq c \leq b \therefore$ There are two possible values where $v'(c) = 0$.

c.) The function f, defined by $f(t) = 6 + \cos(\frac{t}{10}) + 3\sin(\frac{7t}{40})$ is used to model the velocity of the plane, in miles per minute, for $0 \leq t \leq 40$ According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

$$f'(23) \approx -0.407\,\text{miles per minute}^2$$

d.) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval$0 \leq t \leq 40$?

$$\frac{1}{40}\int_0^{40} f(t)\,dt \approx 5.915\,\text{miles per minute}$$

FRQ #2

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ equation for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

a.) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$$\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$$

$$y = 44t + 1400\Big|_{t=\frac{1}{4}} = 1411\,\text{tons}$$

b.) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

$$\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)\Big|_{W=1400} = 1.76.$$

$$\text{Since}\frac{d^2W}{dt^2} > 0\,\text{when}\,t \in [0, \frac{1}{4}] \therefore \text{ part (a) is an underestimate}$$

c.) Find the particular solution $W = W()$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

$$\int \frac{1}{W - 300}\,dW = \int \frac{1}{25}\,dt$$

$$\ln|W - 300| = 125t + C$$

$$\ln|1400 - 300| = \frac{1}{25}(0) + C \Rightarrow C = \ln|1100|$$

$$W - 300 = 1100e^{\frac{1}{25}t}$$

$$W(t) = 1100e^{\frac{1}{25}t} + 300\,\text{when}\,t \in [0, 20]$$

FRQ #3

For $0 \leq t \leq 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos(\frac{t}{5})$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

a.) Show that the number of mosquitoes is increasing at time $t = 6$.

$$R(6) \approx 4.437 \because R(6) > 0\,, \text{the number of mosquitoes is increasing at}\, t = 6$$

b.) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

$$R'(6) \approx -1.913 \Rightarrow R'(6) < 0$$
$$\therefore \text{the number of mosquitoes is increasing at a decreasing rate.}$$

c.) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

$$1000 + \int_0^{31} R(t)\,dt \approx 964\,\text{mosquitoes}$$

d.) To the nearest whole number, what is the maximum number of mosquitoes for $0 \leq t \leq$ 31 Show the analysis that leads to your conclusion.

$$R(t) = 0\,\text{when}\, t = \{0, 2.5\pi, 7.5\pi\}$$
$$R(t) > 0\,\text{when}\, t \in [0, 2.5\pi]\,\&\, t \in [7.5\pi, 31]$$
$$R(t) < 0\,\text{when}\, t \in [2.5\pi, 7.5\pi]$$
$$\text{Absolute max is at the } 2.5\pi \text{ or at the end points}$$
$$1000 + \int_0^{2.5\pi} R(t)\,dt \approx 1039\,\text{mosquitoes}$$

FRQ #4

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimetres from the centre of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimetre. Values of $f(r)$ for selected values of r are given in the table above.

a.) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

$$f'(2.25) \approx \frac{f(2.5) - f(2)}{2.5 - 2} = 8$$

At distance $r = 2.25$ from the centre of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimetre per centimetre.

b.) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 rf(r)\,dr$. Approximate the value of $2\pi \int_0^4 rf(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$$RHS_4 = 2\pi(1 \cdot f(1) \cdot (1-0) + 2 \cdot f(2) \cdot (2-1) + 2.5 \cdot f(2.5) \cdot (2.5-2)$$
$$+4 \cdot f(4) \cdot (4 - 2.5))$$
$$= 269\pi$$

c.) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$$\frac{d}{dr}(r \cdot f(r)) = f(r) + r \cdot f'(r)$$

Since $f'(r) > 0$ & $r > 0$ when $t \in [0,4] \implies \frac{d}{dr}(r \cdot f(r) > 0$ when $t \in [0,4]$ $\therefore$ the right Riemann sum of $2\pi \int_0^4 rf(r)\,dr$ is an overestimate

d.) The density of bacteria in the petri dish, for $1 \leq t \leq 14$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k , $1 < k < 14$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \leq k \leq 4$?

$$g_{avg} = \frac{1}{4-1} \cdot \int_1^4 g(r)\,dr$$
$$g_{avg} \approx 9.875$$
$$g(k) = g_{avg} \implies k \approx 2.497$$

FRQ # 5

A particle, P, is moving along the x-axis. The velocity of particle P at time t is given by $v_P(t) = \sin(t^{1.5})$ for $0 \leq t \leq \pi$. At time $t = 0$, particle P is at position $x = 5$. A second particle, Q, also moves along the x-axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \leq t \leq \pi$. At time $t = 0$, particle Q is at position $x = 10$.

a.) Find the positions of particles P and Q at time $t = 1$.

$$x_P(1) = 5 + \int_0^1 v_P(t)\,dt \approx 5.3706$$
$$x_Q(1) = 10 + \int_0^1 v_Q(t)\,dt \approx 8.5643$$

b.) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

$$v_P(1) \approx 0.841471 \because v_P(1) > 0 \,\text{particle } P \text{ is moving to the right.}$$
$$v_Q(1) = -1 \because v_Q(1) < 0 \,\text{particle } Q \text{ is moving to the left.}$$

At time $t = 1$, $x_P(1) < x_q(1)$ $\therefore$ particle P is to the left of particle Q. Hence at time $t = 1$, particles P and Q are moving toward each other.

c.) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

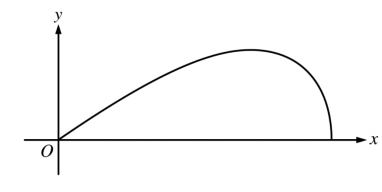
$$v'_Q(1) \approx 1.0268$$
$$v_Q(1) = -1$$

$\because$ the velocity of the particle is less than zero and acceleration is greater than zero,at $t = 1$; the speed of particle Q is decreasing.

d.) Find the total distance travelled by particle P over the time interval $0 \leq t \leq \pi$.

$$\int_0^\pi |v_P(t)|\,dt = 1.93148$$

FRQ 5



A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$ where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

a.) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

$$\begin{aligned} 0 = 6x\sqrt{4-x^2} &\Rightarrow x = 0,\ x = 2 \\ \int_0^2 6x\sqrt{4-x^2}\,dx \\ u = 4 - x^2 &\Rightarrow du = 2x\,dx \\ -3\int_4^0 u^{1/2}\,du &\Longrightarrow 3\int_0^4 u^{1/2}\,du \\ = 2u^{3/2}\Big|_0^4 &= 2(4^{3/2}) = 16 \end{aligned}$$

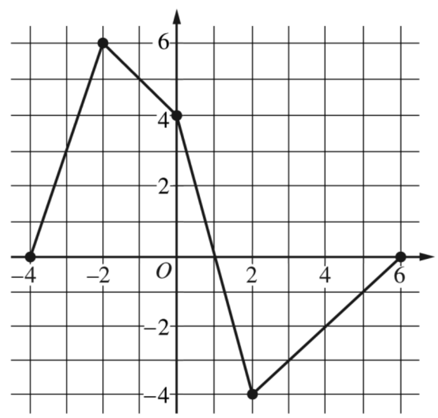
b.) It is known that, for $y = cx\sqrt{4-x^2}$, $\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

$$\begin{aligned} 0 = \frac{c(4-2x^2)}{\sqrt{4-x^2}} &\Rightarrow x = \sqrt{2} \\ y = c\sqrt{2}\sqrt{4-(\sqrt{2})^2} = c\sqrt{2}\sqrt{2} &= 2c \\ 1.6 = 2c &\Rightarrow c = 0.6 \end{aligned}$$

c.) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$$
\begin{aligned}
2\pi &= \pi \int_0^2 (cx\sqrt{4-x^2})^2 \, dx \\
2 &= c^2 \int_0^2 x^2(4-x^2))^2 \\
2 &= c^2 \int_0^2 (4x^2 - x^4) \\
2 &= c^2 \left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 \\
2 &= c^2 \cdot \frac{64}{15} \\
c &= \sqrt{\frac{15}{32}}
\end{aligned}
$$

FRQ #6



Graph of f

Let f be a continuous function defined on the closed interval $4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

1. On what open intervals is the graph of G concave up? Give a reason for your answer.

$$G'(x) = f(x) \implies G''(x) = f'(x)$$
$$f'(x) > 0 \text{ when } x \in [-4, -2] \;\&\; x \in [2, 6]$$
$$\therefore \text{ G is CCU when } x \in [-4, -2] \;\&\; x \in [2, 6]$$

2. Let P be the function defined by $P(x) = g(x) \cdot f(x)$. Find $P'(3)$.

$$P'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$$
$$G'(x) = f(x) \;\&\; G(x) = \int_0^x f(t)\,dt$$
$$\Rightarrow P'(3) = f(3) \cdot f(3) + f'(3) \cdot \int_0^3 f(t)\,dt$$
$$\Rightarrow (-3)(-3) + (1)(\frac{-7}{2}) = \frac{11}{2}$$

3. Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$

$$G(2) = \int_0^2 f(t)\,dt = 0$$
$$(2)^2 - 2(2) = 0$$
$$\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} \overset{\text{H}}{=}$$
$$\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{G'(x)}{2x - 2} = \frac{-4}{2} = -2$$

4. Find the average rate of change of G on the interval $[4, 2]$. Does the Mean Value Theorem guarantee a value c, $4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$$G(2) = \int_0^2 f(t)\,dt = 0 \;\&\; G(-4) = \int_0^{-4} f(t)\,dt = -16$$
$$\frac{G(2) - G(-4)}{2 - (-4)} = \frac{16}{6} = \frac{8}{3}$$

Yes, $G'(x) = f(x) \because G$ is differentiable when $x \in (-4, 2) \therefore G$ continuous when $x \in [-4, 2]$ ergo MVT applies. This guarantees for $-4 < c < 2$ $f'(c) = \frac{8}{3}$.

FRQ # 7

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

a.) Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$

$$4y\frac{dy}{dx} = \frac{dy}{dx} \cdot \sin x + y \cos x$$
$$4y\frac{dy}{dx} - \frac{dy}{dx} \cdot \sin x = y \cos x$$
$$\frac{dy}{dx}(4y - \sin x) = y \cos x$$
$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

b.) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

$$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}\bigg|_{(0,\sqrt{3})} = \frac{1}{4}$$
$$y = \frac{1}{4}x + \sqrt{3}$$

c.) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

$$\frac{y\cos x}{4y - \sin x} = 0$$
$$0 = y\cos x \Rightarrow x = \frac{\pi}{2}$$
$$2y^2 - 6 = y\sin(\frac{\pi}{2})$$
$$2y^2 - 6 = y$$
$$0 = 2y^2 - 6 - y$$
$$0 = (2y+3)(y-2)$$

Since $4(2) - \sin(\frac{\pi}{2}) \neq 0 \therefore$ at $(\frac{\pi}{2}, 2)$ the line tangent to the curve is horizontal.

d.) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

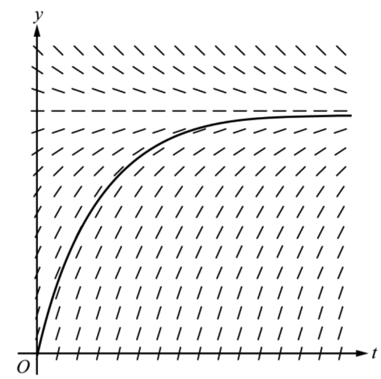
$$\frac{d^2y}{dx^2} = \frac{(\frac{dy}{dx}\cos x - y\sin x)(4y - \sin x) - (4x\frac{dy}{dx} - \cos x)(y\cos x)}{(4y - \sin x)^2}$$
$$\frac{d^2y}{dx^2}\bigg|_{(\frac{\pi}{2},2)} = \frac{(-2)(8-1) - 0}{(8-1)^2} = \frac{-14}{49} = \frac{-2}{7}$$

f has a relative maximum at $(\frac{\pi}{2}, 2) \because \frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} < 0$.

FRQ #8

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

a.) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ is given below. Sketch the solution curve through the point (0, 0).



b.) Using correct units, interpret the statement $\lim_{t\to\infty} A(t) = 12$ in the context of this problem.

The amount of medication in the patients blood stream stabilise at 12 milligrams.

c.) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ with initial condition $A(0) = 0$.

$$\begin{aligned}
\frac{dy}{dt} = \frac{12-y}{3} \implies \frac{1}{3} \cdot dy &= \frac{1}{(12-y)} \cdot dt \\
\int \frac{1}{3}\, dt &= \int \frac{1}{(12-y)}\, dy \\
\frac{1}{3}t + C &= -\ln(12-y) \\
C &= -\ln(12) \\
-\frac{1}{3}t + \ln(12) &= \ln(12-y) \\
e^{-\frac{1}{3}t+\ln(12)} &= 12-y \\
12e^{-\frac{1}{3}t} &= 12-y \\
-12e^{-\frac{t}{3}} &= y
\end{aligned}$$

d.) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modelled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

$$\left.\frac{dy}{dt}\right|_{(1,2.5)} = 3 - \frac{2.5}{3} = \frac{6.5}{3}$$

$$\frac{d^2y}{dt^2} = \left.\frac{-\frac{dy}{dt}(t+2) - y}{(t+2)^2}\right|_{(1,2.5)} = \frac{-6.5 - 2.5}{9} = \frac{-4}{9} < 0$$

Since at $(1, 2.5)$ $\frac{d^2y}{dt^2} < 0$ and $\frac{dy}{dt} > 0$, the amount of medication in the persons blood is increasing at a decreasing rate.