AP Classroom Problems Unit 1

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1 Thinking Backwards & Substitution

1.
$$\text{If } \int_0^k \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln 4, \text{ where } k>0, \text{ then } k =$$

(a) $u = x^2+4 \Longrightarrow \frac{du}{2x} = dx$

(b) $\int_0^k \frac{x}{x^2+4}\,dx$

$$\Longrightarrow \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+4| + C$$

$$\Longrightarrow \frac{1}{2}\ln|x^2+4|\Big|_0^k = \frac{1}{2}\ln|k^2+4| - \frac{1}{2}\ln(4) = \frac{1}{2}\ln 4$$

$$\Longrightarrow \ln(\frac{k^2+4}{4}) = \ln 4$$

$$\Longrightarrow k^2+4 = 16$$

$$\Longrightarrow k = \sqrt{12}$$

2.
$$\int (e^x + e)\,dx$$

$$= e^x + ex + C$$

3.
$$\int (\sec(x)\cdot\tan(x)\,dx$$

$$= \sec(x) + C$$

4.
$$\int_0^3 (x+1)^{\frac{1}{2}}\,dx$$

(a) $u = x+1 \Longrightarrow du = dx$

$$\Longrightarrow \int u^{\frac{1}{2}}\,du$$

$$\Longrightarrow \frac{2x^{\frac{3}{2}}}{3}\Big|_3^0 = \frac{2(4)^{3/2}-2}{3} = \frac{14}{3}$$

5.

$$\int \frac{x}{x^2-4}\,dx$$

(a) $u = x^2 - 4 \Longrightarrow \frac{du}{2x} = dx$

$$\Longrightarrow \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$\Longrightarrow \frac{1}{2}|x^2-4| + C$$

6.

$$\int_2^4 \frac{1}{5-3x}\,dx$$

$$= \frac{\ln|-3x+5|}{-3}$$

(a) $\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{b}$

7.

$$\int_0^1 (x^3+x)(x^4+2x^2+9)^{1/2}\,dx$$

(a) $u = x^4 + 2x^2 + 9 \Longrightarrow du = (4x^3+4x)dx = 4(x^3+x)dx$

$$\Longrightarrow \frac{1}{4}\int u^{1/2}\,du = \frac{u^{3/2}}{6}$$

$$\Longrightarrow \left.\frac{(x^4+2x^2+9)^{3/2}}{6}\right|_0^1$$

$$\Longrightarrow \frac{(12)^{3/2}-9^{3/2}}{6} = \frac{24\sqrt{3}-27}{6} = 4\sqrt{3} - \frac{9}{2}$$

8.

$$\int_{\pi/6}^{\pi/2} (\sin^5(2x)\cdot\cos(2x))\,dx$$

(a) $u = \sin(2x) \Longrightarrow du = 2\cos(x)\,dx$

(b) $\sin(\pi) = 0$ & $\sin(\pi/3) = \frac{\sqrt{3}}{2}$

$$\Longrightarrow \frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^{0} u^5\,du = \frac{-9}{256}$$

9.

$$\text{If } \int_1^2 f(x-c)\,dx = 5 \text{ where } c \text{ is a constant, then } \int_{1-c}^{2-c} f(x)\,dx = 5$$

(a) $u = x - c \Longrightarrow dx = du$

(b) $u_a = 1 - c$ and $u_b = 2 - c$ *Changing the bounds*

10.
$$\int \cos(3x)\,dx$$

(a) $u = 3x \Longrightarrow \frac{du}{3} = dx$

$$\Longrightarrow \frac{1}{3}\int \cos(u)\,du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(3x) + C$$

2 Integration by Parts, Substitution, and Anti-derivatives

1.
Let g be a differentiable function such that $\displaystyle\int g(x)e^{\frac{x}{4}}\,dx = 4g(x)e^{\frac{x}{4}} - \int 8x^2 e^{\frac{x}{4}}\,dx$.

Which of the following could be an expression for $g(x)$?

(a)
$$\int g(x)\cdot h'(x)\,dx = g(x)\cdot h(x) - \int g'(x)\cdot h(x)\,dx$$

(b) $h'(x) = e^{\frac{x}{4}} \Longrightarrow h(x) = 4e^{\frac{x}{4}}$

(c) $g'(x) = 2x^2 \Longrightarrow g(x) = \frac{2x^3}{3}$

2.
$$\int_1^2 (9x^2 - 4x + 1)\cdot \ln x\,dx$$

(a) Let $u = \ln(x) \Longrightarrow du = \frac{1}{x}dx$

(b) $dv = (9x^2 - 4x + 1)dx \Longrightarrow v = 3x^3 - 2x^2 + x$

$$\Longrightarrow \ln(x)(3x^3 - 2x^2 + x) - \int (3x^2 - 2x + 1)\,dx$$

$$\Longrightarrow \ln(x)(3x^3 - 2x^2 + x) - (x^3 - x^2 + x)$$

$$\Longrightarrow \ln(x)(3x^3 - 2x^2 + x) - x^3 + x^2 - x)\Big|_1^2$$

$$\Longrightarrow \Big(\ln(2)(3(8) - 8 + 2) - 8 + 4 - 2)\Big) - \Big(\ln(1)(3 - 2 + 1) - 1 + 1 - 1\Big)$$

$$\Longrightarrow (18\ln(2) - 6) - (2\ln(1) - 1)$$

$$\Longrightarrow 18\ln(2) - 5$$

x	1	2	3	4
$f(x)$	-2	1	6	3
$f'(x)$	2	4	-1	-4

3. The function f has a continuous second derivative. The table above gives values of f and its derivative, f', at selected values of x. What is the value of $\int x \cdot f''(x)\,dx$?

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(a) Let $u = x \implies dx = du$

(b) $dv = f''(x)dx \implies v = f'(x)$

$$\implies xf'(x) - \int f'(x)\,dx$$

$$\implies xf'(x) - f(x)\,dx\Big|_1^2$$

$$\implies \Big(2f'(2) - f(2)\Big) - \Big(f'(1) - f(1)\Big)$$

$$\implies ((2)(4) - 1) - (2 - -2) = 7 - 4 = 3$$

4.

$$\int \frac{4x^4 + 3}{4x^5 + 15x + 2}\,dx$$

(a) $u = 4x^5 + 15x + 2 \implies du = (20x^4 + 15x)dx = 5(4x^4 + 3)dx$

$$\implies \frac{1}{5}\int \frac{1}{u}\,du$$

$$\implies \frac{1}{5}\ln|u| + C$$

$$\implies \frac{1}{5}\ln|4x^5 + 15x + 2| + C$$

5.

$$\int_0^1 (x+2)(3x^2 + 12x + 1)^{1/2}\,dx$$

(a) Let $u = 3x^2 + 12x + 1 \implies du = 6(x+2)dx$

(b) $b = 3(1)^2 + 12 + 12 = 16$ & $a = 3(0) + 12(0) + 1 = 1$

$$\implies \frac{1}{6}\int_1^{16} u^{1/2}\,du$$

$$\implies \frac{1}{6} \cdot \frac{2u^{3/2}}{3}\Big|_1^{16}$$

$$\implies \frac{u^{3/2}}{9}\Big|_1^{16} = \frac{64 - 1}{9} = 7$$

6.

$$\int \frac{1}{x^2 + 4}\,dx$$

$$\implies \frac{1}{4}\int \frac{1}{(\frac{x^2}{4} + 1)}\,dx$$

(a) Let $u^2 = \frac{x^2}{4} \implies u = \frac{x}{2} \implies 2du = dx$

$$\implies \frac{1}{2}\int \frac{1}{u^2+1}\,du$$
$$\implies \frac{1}{2}\arctan(u) + C$$
$$\implies \frac{1}{2}\arctan(\frac{x}{2}) + C$$

7.
$$\int \frac{3x^2+4x+1}{3x^3+6x^2+3x+5}\,dx$$

(a) Let $u = 3x^3+6x^2+3x+5 \implies du = 3(3x^2+4x+1)dx$

$$\implies \frac{1}{3}\int \frac{1}{u}\,du$$
$$\implies \frac{1}{3}\ln|u| + C$$
$$\implies \frac{1}{3}\ln|3x^3+6x^2+3x+5| + C$$

8.
$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$
$$\implies \int \frac{1}{\sqrt{9(1-\frac{x^2}{9})}}$$
$$\implies \frac{1}{3}\int \frac{1}{\sqrt{(1-\frac{x^2}{9})}}$$

(a) Let $u^2 = \frac{x^2}{9} \implies u = \frac{x}{3} \implies 3du = dx$

$$\implies \int \frac{1}{\sqrt{1-u^2}}\,du$$
$$\implies \arcsin(u) + C$$
$$\implies \arcsin(\frac{x}{3}) + C$$

9. See #7 in section 1

3 All anti-differentiation strategies in one assignment.

1.
$$\int_0^1 x\sqrt{1+8x^2}\,dx$$

(a) Let $u = 1+8x^2 \implies \frac{du}{16} = x\cdot dx$

(b) $b = 1+8(1)^2 = 9$

(c) $a = 1 + 8(0)^2 = 1$

$$\Longrightarrow \frac{1}{16}\int_1^9 u^{1/2}\,du = \frac{1}{16}\cdot\frac{2u^{3/2}}{3}\bigg|_9^1$$

$$\Longrightarrow \frac{u^{3/2}}{24}\bigg|_9^1 = \frac{27}{24} - \frac{1}{24} = \frac{26}{24} = \frac{13}{12}$$

2.

$$\int_0^1 \frac{5x+8}{x^2+3x+2}\,dx$$

$$\Longrightarrow \frac{5x+8}{(x+2)(x+1)}$$

$$\Longrightarrow \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Longrightarrow \frac{A(x+2)+B(x+1)}{(x+2)(x+1)} = \frac{5x+8}{(x+2)(x+1)}$$

$$\Longrightarrow A + B = 5 \quad \& \quad 2A + B = 8$$

$$\Longrightarrow A = 3 \quad \& \quad B = 2$$

$$\Longrightarrow \int \frac{3}{x+1} + \frac{2}{x+2}\,dx$$

$$\Longrightarrow \Big[3\ln(x+1) + 2\ln(x+2)\Big]_0^1$$

$$\Longrightarrow \Big(3\ln(2) + 2\ln(3)\Big) - \Big(3\ln(1) + 2\ln(2)\Big)$$

$$\Longrightarrow 18$$

3. If f is a function such that $f'(x) = -f(x)$, then $\int xf(x)\,dx =$

(a) Let $u = x \Longrightarrow du = dx$

(b) $dv = f(x)\,dx \Longrightarrow v = \int f(x)\,dx = -\int f'(x)\,dx = -f(x)$

$$-x\cdot f(x) - \int -f'(x)\,dx$$

$$-x\cdot f(x) + \int f'(x)\,dx$$

$$-x\cdot f(x) + f(x) + C$$

$$(1-x)\cdot f(x) + C$$

$$-f(x)\cdot(x+1) + C$$

4. See #9 in section 1

5.

$$\int \frac{-2x^2+7x-8}{(x+2)(2x-1)(1-x)}\,dx$$

$$\implies \frac{A}{x+2}+\frac{B}{2x-1}+\frac{C}{1-x}$$

$$\implies \frac{A(2x-1)(1-x)+B(x+2)(1-x)+C(x+2)(2x-1)}{(x+2)(2x-1)(1-x)}=\frac{-2x^2+7x-8}{(x+2)(2x-1)(1-x)}$$

$$\begin{cases} -2A-B+2C & =-2 \\ 3A-B+3C & =7 \\ -A+2B-2C & =-8 \end{cases}$$

$$\implies A=2 \quad B=-4 \quad C=-1$$

$$\implies \int \left(\frac{2}{x+2}+\frac{-4}{2x-1}+\frac{-1}{1-x}\right)dx$$

$$\implies 2\ln|x+2|-2\ln|2x-1|+\ln|1-x|+C$$

6.

$$\int (x^3+1)^2\,dx$$

(a) $(x^3+1)^2 = x^6+2x^3+1$

$$\implies \int (x^6+2x^3+1)\,dx$$

$$\implies \frac{1}{7}x^7+\frac{1}{2}x^4+x+C$$

7.

$$\int \sqrt{x}\left(9x^3-2\sqrt{x}+\frac{8}{x}\right)dx$$

$$\implies \int (9x^{7/2}-2x+\frac{8}{x^{1/2}})$$

$$\implies 2x^{9/2}-x^2+16x^{1/2}+C$$

8.

$$\int_0^5 \frac{3x+11}{x+2}\,dx$$

(a)

$$\begin{array}{r|l} & \quad\quad\;\; 3 \\ \cline{2-2} x+2\,) & \;\; 3x+11 \\ & -3x\;\;-6 \\ \cline{2-2} & \quad\quad\;\; 5 \end{array}$$

$$\implies \int_0^5 3+\frac{5}{x+2}\,dx$$

$$\implies 3x+5\ln(x+2)\Big|_0^5 = 5\ln(\frac{7}{2})+15$$

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(b) Let $u = x + 2 \implies du = dx$

$$\int_2^7 3 + \frac{5}{u}\, du = 3u + 5\ln(u)\Big|_2^7 = 5\ln(\frac{7}{2}) + 15$$

9.

$$\int \frac{4}{x^2 + 4x + 8}\, dx$$
$$\implies 4\int \frac{1}{x^2 + 4x + 8}\, dx$$
$$\implies 4\int \frac{1}{(x+2)^2 + 4}\, dx$$
$$\implies \int \frac{1}{\frac{(x+2)^2}{4} + 1}\, dx$$

(a) Let $u^2 = \frac{(x+2)^2}{4} \implies u = \frac{x+2}{2} = \frac{x}{2} + 1$

(b) $\implies 2du = dx$

$$\implies 2\int \frac{1}{u^2 + 1}\, du$$
$$\implies 2\arctan(u) + C$$
$$\implies 2\arctan(\frac{x+2}{2}) + C$$

10.

$$\int \frac{1}{x^2 - 2x + 2}\, dx$$
$$\implies \int \frac{1}{(x-1)^2 + 1}\, dx$$

(a) Let $u = x - 1 \implies du = dx$

$$\implies \int \frac{1}{u^2 + 1}\, dx$$
$$\implies \arctan(u) + C$$
$$\implies \arctan(x - 1) + C$$

11.

$$\int \left(5e^{2x} + \frac{1}{x}\right) dx$$

(a) Let $u = 2x \implies \frac{du}{2} = dx$

$$\implies \frac{5}{2}\int e^u\, du + \int \frac{1}{x}\, dx$$
$$\implies \frac{5}{2}e^{2x} + \ln|x| + C$$

12. See #2 in section 2

13.

$$\text{Using the substitution } u = \sqrt{x}, \text{ find an equivalent form of } \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

(a) If $u = \sqrt{x} \Longrightarrow 2du = \frac{1}{\sqrt{x}}$

(b) $b = \sqrt{4} = 2$ & $a = \sqrt{1} = 1$

$$\Longrightarrow 2\int_1^2 e^u\, du = 2e(e-1)$$

14.

x	2	4
$f(x)$	7	13
$g(x)$	2	9
$g'(x)$	1	7
$g''(x)$	5	8

The table above gives selected values of twice-differentiable functions f and g, as well as the first two derivatives of g. If $f'(x) = 3$ for all values of x, what is the value of $\int_2^4 f(x)\cdot g''(x)\, dx$?

(a) Let $u = f(x) \Longrightarrow f'(x)dx$

(b) $dv = g''(x)dx \Longrightarrow v = g'(x)$

$$\Longrightarrow f(x)\cdot g'(x) - \int g'(x)\cdot f'(x)\, dx$$

$$\Longrightarrow f(x)\cdot g'(x) - 3\int g'(x)$$

$$\Longrightarrow f(x)\cdot g'(x) - 3g(x)\Big|_2^4$$

$$\Big(f(4)\cdot g'(4) - 3g(4)\Big) - \Big(f(2)\cdot g'(2)\cdot 3g(2)\Big)$$

$$\Big(13\cdot 7 - 3(9)\Big) - \Big(7\cdot -1\cdot 6\Big) = 63$$

15.

$$\text{The function } f \text{ is continuous and } \int_0^8 f(u)\, du = 6. \text{ What is the value of } \int_1^3 x\cdot f(x^2-1)\, dx?$$

(a) Let $u = x^2 - 1 \Longrightarrow du = 2xdx$

$$\Longrightarrow \frac{1}{2}\int_0^8 f(u)\, du = 3$$