AP Classroom Problems Unit 6

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6.01

1. A curve in the xy-plane is is defined by the parametric equations $x(t) = \cos(2t)$ and $y(t) = \sin(2t)$ for $t \geq 0$. Which of the following is equal to $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$?

   (a) $\frac{dx}{dt} = x'(t) = -2\sin(2t)$

   (b) $\frac{dy}{dt} = y'(t) = 2\cos(2t)$

$$\implies \sqrt{\big(-2\sin(2t)\big)^2 + \big(2\cos(2t)\big)^2} = 2\sqrt{\underbrace{\sin^2(2t) + \cos^2(2t)}_{0}} = \boxed{2}$$

2. A curve is defined by the parametric functions $x(t)$ and $y(t)$, where $\frac{dx}{dt} \neq 0$ and $2x(t) - y(t) = 4$ for all t. Which of the following equals $\frac{dy}{dx}$?

$$2x'(t) - y'(t) = 0$$
$$\frac{y'(t)}{x'(t)} = \frac{dy}{dx} = \boxed{2}$$

3. A curve is defined by the parametric equations $x(t) = at + b$ and $y(t) = ct + d$, where a, b, c and d are nonzero constants. Which of the following gives the slope of the line tangent to the curve at the point $(x(t), y(t))$?

   (a) $\frac{dx}{dt} = x'(t) = a$

   (b) $\frac{dy}{dt} = y'(t) = c$

$$\frac{y'(t)}{x'(t)} = \frac{dy}{dx} = \boxed{\frac{c}{a}}$$

4. An object moves in the xy-plane so that its position at any time t is given by the parametric equations $x(t) = t^3 - 3t^2 + 2$ and $y(t) = \sqrt{t^2 + 16}$. What is the rate of change of y with respect to x when $t = 3$?

   (a) $x'(t) = 3t^2 - 6t$

   (b) $y'(t) = \frac{t}{\sqrt{t^2+16}}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{t}{\sqrt{t^2+16}}}{3t^2 - 6t} = \frac{t}{3t(t-2)\sqrt{t^2+16}}\bigg|_{t=3} = \frac{3}{45} = \boxed{\frac{1}{15}}$$

5. If $x = e^{2t}$ and $y = \sin(2t)$, then $\frac{dy}{dx} =$

   (a) $x'(t) = 2e^{2t}$

   (b) $y'(t) = 2\cos(2t)$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \boxed{\frac{\cos(2t)}{e^{2t}}}$$

6. A curve C is defined by the parametric equations $x(t) = 3 + t^2$ and $y(t) = t^3 + 5t$. Which of the following is an equation of the line tangent to the graph of C at the point where $t = 1$?

   (a) $x(1) = 4$ and $y(1) = 6$

   (b) $x'(t) = 2t \implies x'(1) = 2$

   (c) $y'(t) = 3t^2 + 5 \implies y'(1) = 8$

   (d) $\frac{dy}{dx} = \frac{8}{2} = 4$

$$y = 4(x-4) + 6 = \boxed{4x - 10}$$

7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dy}{dt}$?

$$\begin{aligned} xy &= 10 \\ y\frac{dx}{dt} + x\frac{dy}{dt} &= 0 \\ \frac{-x}{y} \cdot \frac{dy}{dt} &= \frac{dy}{dt} \\ \frac{dy}{dt}\bigg|_{(2,5)} &= \boxed{\frac{-6}{5}} \end{aligned}$$

8. The position of a particle moving in the xy-plane is given by the parametric functions $x(t)$ and $y(t)$ for which $x'(t) = t\sin t$ and $y'(t) = 5e^{-3t} + 2$. What is the slope of the line tangent to the path of the particle at the point at which $t = 2$?

$$\frac{dy}{dx}\bigg|_{t=2} = \frac{y'(2)}{x'(2)} = \frac{(2e^6 + 5)\cdot e^{-6}}{2\sin 2} \approx \boxed{1.107}$$

9. Consider the curve in the xy-plane represented by $x = e^t$ and $y = te^{-t}$ for $t \geq 0$. The slope of the line tangent to the curve at the point where $x = 3$ is

   (a) $t = \ln x$

   (b) $x'(t) = e^t$

   (c) $y'(t) = (1-t)e^{-t}$

$$\frac{dy}{dx}\bigg|_{t=\ln 3} = \frac{y'(\ln 3)}{x'(\ln 3)} = \frac{1 - \ln 3}{9} \approx \boxed{-0.011}$$

10. For $0 \leq t \leq 13$, an object travels along an elliptical path given by the parametric equations $x = 3\cos t$ and $y = 4\sin t$. At the point where $t = 13$, the object leaves the path and travels along the line tangent to the path at that point. What is the slope of the line on which the object travels?

    (a) $x'(t) = -3\sin t$

    (b) $y'(t) = 4\cos t$

$$\frac{dy}{dx} = \frac{y'(13)}{x'(13)} = \frac{4\cos 13}{-3\sin 13} = \boxed{-\frac{4}{3\tan 13}}$$

11. A curve is described by the parametric equations $x = t^2 + 2t$ and $y = t^3 + t^2$. An equation of the line tangent to the curve at the point determined by $t = 1$ is

    (a) $x(1) = 3$ and $y(1) = 2$

    (b) $x'(t) = 2t + 2 \Longrightarrow x'(1) = 4$

    (c) $y'(t) = 3t^2 + 2t \Longrightarrow y'(1) = 5$

$$y = \frac{5}{4}(x-3) + 2$$
$$\Longrightarrow \boxed{7 = 5x - 4y}$$

12. For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

    (a) Vertical tangent occurs when $\frac{dx}{dt} = 0$

    (b) $x'(t) = t(3t - 2)$

    (c) $x'(t) = 0$ when $t = 0$ and $t = \frac{2}{3}$

$$\boxed{t = 0 \text{ and } t = \frac{2}{3} \text{ only}}$$

6.02

1. For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

$$\boxed{\text{See \# 12 above}}$$

2. If $x = t^2 + 1$ and $y = t^3$, then $\frac{d^2y}{dx^2} =$

    (a) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dy}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

    (b) $x'(t) = 2t$ and $y'(t) = 3t^2$

    (c) $\frac{dy}{dx} = \frac{3}{2}x \Longrightarrow \frac{d^2y}{dx^2} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$

$$\boxed{\frac{d^2y}{dx^2} = \frac{3}{4t}}$$

3. If $x = e^{2t}$ and $y = \sin(2t)$, then $\frac{dy}{dx} =$

See # 5 above

4. An object moves in the xy-plane so that its position at any time t is given by the parametric equations $x(t) = t^3 - 3t^2 + 2$ and $y(t) = \sqrt{t^2 + 16}$. What is the rate of change of y with respect to x when $t = 3$?

   (a) $x'(t) = 3t^2 - 6t$ and $y'(t) = \frac{t}{\sqrt{t^2+16}}$

   (b) $\frac{dy}{dx} = \frac{t}{(3t^2-6t)(\sqrt{t^2+16})}$

$$\frac{dy}{dx}\left(\frac{t}{(3t^2-6t)(\sqrt{t^2+16})}\right)\Bigg|_{t=3} = \frac{3}{45} = \boxed{\frac{1}{15}}$$

5. A curve C is defined by the parametric equations $x(t) = 3 + t^2$ and $y(t) = t^3 + 5t$. Which of the following is an equation of the line tangent to the graph of C at the point where $t = 1$?

   (a) $(x, y) : (4, 6)$

   (b) $x'(t) = 2t$ and $y'(t) = 3t^2 + 5$.

   (c) $\frac{dy}{dx} = \frac{3t^2+5}{2t}$

   (d) $\frac{dy}{dx}\Big|_{t=1} = 4$

$$y = 4(x - 4) + 6 = \boxed{4x - 10}$$

6. The position of a particle moving in the xy-plane is given by the parametric functions $x(t)$ and $y(t)$ for which $x'(t) = t\sin t$ and $y'(t) = 5e^{-3t} + 2$. What is the slope of the line tangent to the path of the particle at the point at which $t = 2$?

$$\frac{dy}{dx} = \frac{5e^{-3t} + 2}{t\sin t}\Bigg|_{t=2} = \frac{5e^{-6} + 2}{2\sin 2} \approx \boxed{1.107}$$

7. Which of the following gives the length of the path described by the parametric equations $x = \sin(t^3)$ and $y = e^{5t}$ from $t = 0$ to $t = \pi$?

   (a) $x'(t) = 3t^2\cos(t^3)$ and $y'(t) = 5e^{5t}$

$$\int_0^\pi \sqrt{\left(3t^2\cos^2(t^3)\right)^2 + \left(5e^{5t}\right)^2}\,dt = \boxed{\int_0^\pi \sqrt{9t^4\cos^2(t^3) + 25e^{10t}}\,dt}$$

8. Which of the following gives the length of the path described by the parametric equations $x(t) = 2 + 3t$ and $y(t) = 1 + t^2$ from $t = 0$ to $t = 1$?

   (a) $x'(t) = 3$ and $y'(t) = 2t$

$$\boxed{\int_0^1 \sqrt{9+4t^2}\,dt}$$

9. Consider the curve in the xy-plane represented by $x = e^t$ and $y = te^{-t}$ for $t \geq 0$. The slope of the line tangent to the curve at the point where $x = 3$ is

$\boxed{\text{See \# 9 above}}$

10. A curve is defined by the parametric equations $x(t) = at$ and $y(t) = bt$, where a and b are constants. What is the length of the curve from t =0 to $t = 1$?

    (a) $x'(t) = a$ and $y'(t) = b$

$$\int_0^1 \sqrt{a^2+b^2}\,dt = \boxed{\sqrt{a^2+b^2}}$$

11. What is the length of the curve defined by the parametric equations $x(t) = t^2 - 2t$ and $y(t) = t^3 - 4t$ for $0 \leq t \leq 2$?

$$\int_0^2 \sqrt{\left(2t-2\right)^2 + \left(3t^2-4\right)^2}\,dt \approx \boxed{6.511}$$

12. A curve is defined by the parametric equations $x(t) = at^2$ and $y(t) = bt$, where a and b are positive constants. What is $\frac{d^2y}{dx^2}$ in terms of t?

    (a) $x'(t) = 2at$ and $y'(t) = b$
    (b) $\frac{dy}{dx} = \frac{b}{2at}$

$$\frac{d^2y}{dx^2} = \frac{\frac{-b}{2at^2}}{2at} = \boxed{\frac{-b}{4a^2t^3}}$$

13. A curve is defined by the parametric equations $x(t) = e^{-3t}$ and $y(t) = e^{3t}$. What is $\frac{d^2y}{dx^2}$ in terms of t?

    (a) $x'(t) = -3e^{-3t}$ and $y'(t) = 3e^{3t}$
    (b) $\frac{dy}{dx} = -e^{6t}$

$$\frac{d^2y}{dx^2} = \frac{-6e^{6t}}{-3e^{-3t}} = \boxed{2e^{9t}}$$

14. A curve is defined by the parametric functions $x(t)$ and $y(t)$, where $\frac{dx}{dt} \neq 0$ and $2x(t) - y(t) = 4$ for all t. Which of the following equals $\frac{dy}{dx}$?

$\boxed{\text{See \#2 above}}$

15. A curve is defined by the parametric equations $x(t) = at + b$ and $y(t) = ct + d$, where a, b, c, and d are nonzero constants. Which of the following gives the slope of the line tangent to the curve at the point $(x(t), y(t))$?

$\boxed{\text{See \#3 above}}$

16. If $x = t^2 - 1$ and $y = \ln t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

    (a) $x'(t) = 2t$ and $y'(t) = \frac{1}{x}$

    (b) $\frac{dy}{dx} = \frac{1}{2t^2}$

$$\frac{d^2y}{dx^2} = \frac{-t^{-3}}{2t} = \boxed{\frac{-1}{2t^4}}$$

17. For $0 \leq t \leq 13$, an object travels along an elliptical path given by the parametric equations $x = 3\cos t$ and $y = 4\sin t$. At the point where $t = 13$, the object leaves the path and travels along the line tangent to the path at that point. What is the slope of the line on which the object travels?

See #10 above

6.03

1. For time $t > 0$, the position of a particle moving in the xy-plane is given by the parametric equations $x = 4t + t^2$ and $y = \frac{1}{3t+1}$. What is the acceleration vector of the particle at time $t = 1$?

    (a) $x'(t) = 4 + 2t \implies x''(t) = 2$

    (b) $y'(t) = \frac{-3}{(3t+1)^2} \implies y''(t) = \frac{18}{(3x+1)^3}$

$$\langle x''(1), y''(1) \rangle = \boxed{\left\langle 2, \frac{9}{32} \right\rangle}$$

2. A particle moves on a plane curve so that at any time $t > 0$ its x-coordinate is $t^3 - t$ and its y-coordinate is $(2t - 1)^3$. The acceleration vector of the particle at $t = 1$ is

    (a) $x(t) = t^3 - t \implies x'(t) = 3t^2 - 1 \implies x''(t) = 6t$

    (b) $y(t) = (2t - 1)^3 \implies y'(t) = 6(2t - 1)^2 \implies y''(t) = 24(2t - 1)$

$$\langle x''(1), y''(1) \rangle = \boxed{\left\langle 6, 24 \right\rangle}$$

3. For time $t > 0$, the position of a particle moving in the xy-plane is given by the vector $\langle \frac{1}{t}, e^{3t} \rangle$. What is the velocity vector of the particle at time $t = 2$?

    (a) $x(t) = t^-1 \implies x'(t) = \frac{-1}{t^2}$

    (b) $y(t) = e^{3t} \implies y'(t) = 3e^{3t}$

$$\langle x''(2), y''(2) \rangle = \boxed{\left\langle \frac{-1}{4}, 3e^6 \right\rangle}$$

4. A particle moves in the xy-plane with position given by $(x(t), y(t)) = (5 - 2t, t^2 - 3)$ at time t. In which direction is the particle moving as it passes through the point $(3, -2)$?

(a) $x(t)5 - 2t \implies x'(t) = -2$

(b) $y(t) = t^2 - 3 \implies y'(t) = 2t$

(c) The point $(3, -2)$ occores at $t = 1$

(d) $\langle x'(1), y'(1)\rangle = \Big\langle -2, 2\Big\rangle$

$$\boxed{\text{Up and to the left}}$$

5. A particle moves on the curve $y = \ln x$ so that the $x-$component has velocity $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(1, 0)$. At time $t = 1$, the particle is at the point

   (a) $x(1) = x(0) + \int_0^1 (t + 1)\, dt = \frac{5}{2}$

   (b) $y = \ln\left(\frac{5}{2}\right)$

$$\boxed{\left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right)}$$

6. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - 3t^2$ and $y = 2t^3 - 3t^2 - 12t$. For what values of t is the particle at rest?

   (a) $x'(t) = 3t^2 - 6t = 3t(t - 2)$

   (b) $y'(t) = 6t^2 - 6t = 6(t + 1)(t - 2)$

   (c) When $t = 2$ $x'(t) = 0$ and $y'(t) = 0$

$$\boxed{\text{2 only}}$$

7. The position of a particle moving in the xy-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = 12t - 3t^2$ At which of the following points (x, y) is the particle at rest?

   (a) $x'(t) = 3t(t - 2)$ and $y'(t) = -6(t - 2)$

   (b) The partical is at rest at $t = 2$

   (c) $x(2) = -4$ and $y(2) = 12$

$$\boxed{(-4, 12)}$$

8. For time $t \geq 0$, a particle moves with velocity vector given by $v(t) = \langle f(t), g(t)\rangle$, where f and g are continuous functions of t. At time $t = 3$, the particle is at position $(-4, 5)$. Which of the following expressions gives the particle's position at time $t = 1$?

$$\boxed{\left(-4 - \int_1^3 f(t)\, dt, 5 - \int_1^3 g(t), dt\right)}$$

9. The vector-valued function h is defined by $h(t) = \langle te^t, t^2e^t \rangle$. Which of the following is $h'(1)$?

$$h'(t) = \langle e^t + 4e^t, 2e^t + t^2e^t \rangle$$

$$\boxed{h'(1) = \langle 2e, 3e \rangle}$$

10. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $r(t) = \langle 3 + 30t - 8t^3, 9t^2 + 4t \rangle$. If $f(1) = \langle 4, -3 \rangle$, what is $f(-1)$?

$$\begin{aligned} f(-1) &= f(1) - \int_{-1}^{1} r(t)\, dt \\ &= \left\langle 4 - \int_{-1}^{1} \left(3 + 30t - 8t^3\right) dt, -3 - \int_{-1}^{1} \left(t^2 + 4t\right) dt \right\rangle \\ &= \left\langle 4 - \left[3t + 15t^2 - 2t^3\right]_{-1}^{1}, -3 - \left[3t^3 + 2t^2\right]_{-1}^{1} \right\rangle \\ &= \left\langle 4 - \left((3 + 15 - 2) - (3 + 15 - 2)\right), -4 - \left((3 + 2) - (-3 + 2)\right) \right\rangle \\ &= \boxed{\left\langle -2, -9 \right\rangle} \end{aligned}$$

6.04

1. If f is a vector-valued function defined by $f(t) = \langle e^{-t}, \cos t\$ \rangle$, then $f''(t) =$

$$f'(t) = \langle -e^{-t}, -\sin t \rangle$$

$$\boxed{f''(t) = \langle e^{-t}, -\cos t \rangle}$$

2. If a particle moves in the xy-plane so that at time $t > 0$ its position vector is $\langle \ln(t^2 + 2t), 2t^2 \rangle$, then at time $t = 2$, its velocity vector is

$$f(t) = \langle \ln(t^2 + 2t), 2t^2 \rangle$$

$$f'(t) = \left\langle \frac{2t}{t^2 + 2t}, 4t \right\rangle$$

$$\boxed{f'(2) = \left\langle \frac{3}{4}, 8 \right\rangle}$$

3. A curve in the xy-plane is given by parametric functions $x(t)$ and $y(t)$, where $\frac{dx}{dt} = 2 - \ln(t^3 + t + 1)$ and $\frac{dy}{dt} = 4\arctan(t^2 + 2) - 5$ for $t \geq 0$. The coordinates of the point on the curve where $t = 5$ are (-1.219,4.532). What is the y-coordinate of the point on the curve where $t = 0$?

$$y(0) = y(5) - \int_{0}^{5} \left(4\arctan(t^2 + 2) - 5\right) dt \approx \boxed{1.654}$$

4. For time $t \geq 0$, a particle moves in the xy-plane with velocity vector given by $v(t) = \langle \ln(t^3 + 3t + 1), 1 - \ln(t + 3) \rangle$. At time $t = 0$, the particle is at position $(1, -3)$. What is the particle's acceleration vector at time $t = 2.5$?

$$v'(t) = a(t) = \left\langle \frac{3(x^2+1)}{x^3+3x+1}, \frac{-1}{x+3} \right\rangle$$

$$a(2.5) = \left\langle \frac{174}{193}, \frac{-2}{11} \right\rangle$$

$$\boxed{a(2.5) \approx \langle 0.902, -0.182 \rangle}$$

5. The instantaneous rate of change of the vector-valued function $g(t)$ is given by $r(t) = \left\langle \sqrt{t^2+1}, \sin(t^2) \right\rangle$. If $g(7) = \langle \sqrt{2}, \pi \rangle$, which of the following is $g(0)$?

$$g'(t) = r(t)$$

$$g(0) = g(7) - \int_0^7 r(t)\, dt$$

$$g(0) = \left\langle \sqrt{2} - \int_0^7 \sqrt{t^2+1}\, dt, \pi - \int_0^7 \sin(t^2)\, dt \right\rangle$$

$$\boxed{g(0) \approx \langle -24.657, 2.536 \rangle}$$

6. For $t \geq 0$, the components of the velocity of a particle moving in the xy-plane are given by the parametric equations $x'(t) = \frac{1}{1+t}$and $y'(t) = ke^{kt}$ where k is a positive constant. The line $y = 4x + 3$ is parallel to the line tangent to the path of the particle at the point where $t = 2$. What is the value of k?

$$\frac{y'(2)}{x'(2)} = 4$$

$$\implies 4 = \frac{ke^{2k}}{\frac{1}{2+1}} = 3ke^{kt}$$

Solving via a graphical approach yields that $\boxed{k \approx 0.495}$

7. The position of a particle moving in the xy-plane is given by the parametric equations $x(t) = \cos(2t)$ and $y(t) = \sin(2t)$ for time $t \geq 0$. What is the speed of the particle when $t = 2.3$?

   (a) $s(t) = \sqrt{\big(y'(t)\big)^2 + \big(x'(t)\big)^2}$

   (b) $x'(t) = -\ln(2) \cdot \sin(e^t) \cdot 2^t$

   (c) $y'(t) = \ln(2) \cdot \cos(e^t) \cdot 2^t$

$$s(2.3) = \sqrt{\Big(\ln(2) \cdot \cos(e^t) \cdot 2^t\Big)^2 + \big(-\ln(2) \cdot \sin(e^t) \cdot 2^t\big)^2} \approx \boxed{3.14345}$$