AP Classroom Problems Unit 4

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5.01

1. If a and b are positive constants, then $\lim_{x\to\infty} \dfrac{\ln(bx+1)}{\ln(ax^2+3)}$

   (a) $\dfrac{d}{dx}\left(\ln(bx+1)\right) = \dfrac{b}{bx+1}$

   (b) $\dfrac{d}{dx}\left(\ln(ax^2+3)\right) = \dfrac{2ax}{ax^2+3}$

$$\lim_{x\to\infty} \frac{\ln(bx+1)}{\ln(ax^2+3)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b(ax^2+3)}{2ax(bx+1)} = \lim_{x\to\infty} \frac{abx^2+3b}{2abx^2+2ax} \overset{\text{H}}{=}$$

$$\lim_{x\to\infty} \frac{2abx}{4abx+2a} = \lim_{x\to\infty} = \frac{bx}{2bx+1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b}{2b} = \boxed{\frac{1}{2}}$$

OR

   (a) $\ln(bx+1) \sim \ln bx - \ln x + \ln b \sim \ln x$

   (b) $\ln(ax^2+3) \sim \ln(ax^2) = (\ln x^2 + \ln a) \sim \ln x^2 = 2\ln x$

$$\Longrightarrow \lim_{x\to\infty} \frac{\ln(bx+1)}{\ln(ax^2+3)} = \lim_{x\to\infty} \frac{\ln x}{2\ln x} = \boxed{\frac{1}{2}}$$

2. What is the $\lim_{x\to 0} \dfrac{x^2}{1-\cos x}$?

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2}{\cos x} = \boxed{2}$$

3. What is the $\lim_{x\to\infty} \dfrac{x^3}{e^{3x}}$?

$$\lim_{x\to\infty} \frac{x^3}{e^{3x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3x^2}{3e^{3x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6x}{9e^{3x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6}{27e^{3x}} = \frac{2}{9} \lim_{x\to\infty} \frac{1}{e^{3x}} \overset{\text{H}}{=} \frac{2}{9} \cdot \frac{1}{\infty} = \boxed{0}$$

4. What is the $\lim_{x\to\frac{\pi}{2}} \dfrac{3\cos x}{2x-\pi}$?

$$\lim_{x\to\frac{\pi}{2}} \frac{3\cos x}{2x-\pi} \overset{\text{H}}{=} \frac{-3\sin x}{2} = \boxed{\frac{-3}{2}}$$

5. What is the $\lim_{x\to 0} \dfrac{6e^{4x}-2e^{3x}-4}{\sin(2x)}$?

$$\lim_{x\to 0} \frac{6e^{4x}-2e^{3x}-4}{\sin(2x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{24e^{4x}-6e^{3x}}{2\cos 2x} = \boxed{9}$$

6. Let f be the function defined by $f(x) = 2x + 3e^{-5x}$, and let g be a differentiable function with derivative given by $g'(x) = \dfrac{1}{x} + 4\cos(5x)$. It is known that $\lim_{x\to\infty} g(x) = \infty$. What is the value of $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$?

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{2-\frac{15}{e^{5x}}}{\frac{1}{x}+4\cos(5x)} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

7. What is the $\lim_{t\to 0} \dfrac{\sin t}{\ln(2e^t-1)}$?

$$\lim_{t\to 0} \frac{\sin t}{\ln(2e^t-1)} \overset{\text{H}}{=} \lim_{t\to 0} \frac{\cos(t)\cdot(2e^t-1)}{2e^t} = \boxed{\frac{1}{2}}$$

8. What is the $\lim_{x\to 1} \dfrac{x^2-1}{\sin(\pi x)}$?

$$\lim_{x\to 1} \frac{x^2-1}{\sin(\pi x)} \overset{\text{H}}{=} \lim_{x\to 1} \frac{2x}{\pi\cos(\pi x)} = \boxed{-\frac{2}{\pi}}$$

9. What is the $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{2\sin^2\theta}$?

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{2\sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{4\sin\theta\cdot\cos\theta} \overset{\text{H}}{=} \lim_{\theta\to 0} \frac{\cos\theta}{4\big(\cos^2\theta-\sin^2\theta\big)} = \boxed{\frac{1}{4}}$$

10. Let f and g be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$ exists. What then is the $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$?

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \boxed{\lim_{x\to 0} \frac{f'(x)}{g'(x)}}$$

11. What is the $\lim_{x\to 0} \dfrac{\sin x\cos x}{x}$?

$$\lim_{x\to 0} \frac{\sin x\cos x}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos^2 x-\sin^2 x}{1} = \boxed{1}$$

12. Let g be a continuously differentiable function with $g(1) = 6$ and $g'(1) = 3$. What is the $\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6}$?

$$\lim_{x\to 1} \frac{\int_1^x g(t)\,dt}{g(x)-6} \overset{\text{H}}{=} \lim_{x\to 1} \frac{\frac{d}{dx}\left(\int_1^x g(t)\,dt\right)}{\frac{d}{dx}\left(g(x)\right)} = \lim_{x\to 1} \frac{g(x)}{g'(x)} = \frac{6}{3} = \boxed{2}$$

13. If f is the function defined by $f(x) = \frac{x^2-4}{\sqrt{x}-\sqrt{2}}$, then $\lim_{x\to 2} f(x)$ is equivalent to which of the following?

$$f(x) = \frac{x^2-4}{\sqrt{x}-\sqrt{2}} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} = (x+2)(\sqrt{x}+\sqrt{2})$$

$$\boxed{\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2)(\sqrt{x}+\sqrt{2})}$$

14. Let g and h be the functions defined by $g(x) = -2x^2 + 4x + 1$ and $h(x) = \frac{x^2}{2} - x + \frac{11}{2}$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for all x, what is $\lim_{x\to 1} f(x)$?

(a) $\lim_{x\to 1} g(x) = 3$

(b) $\lim_{x\to 1} h(x) = 5$

$$3 \le \lim_{x\to 1} f(x) \le 5$$

$$\boxed{\text{The limit cannot be determined from the information given.}}$$

15. Let f, g, and h be the functions defined by $f(x) = \frac{\sin x}{2x}$, $g(x) = x^4 \cos\left(\frac{1}{x^2}\right)$, and $h(x) = \frac{x^2}{\tan x}$ for $x \ne 0$. All of the following inequalities are true on the interval $[-1, 1]$ for $x \ne 0$. Which of the inequalities can be used with the squeeze theorem to find the limit of the function as x approaches 0?

I. $\frac{1}{4} \le f(x) \le x^2 + \frac{1}{2}$: **False** $\because \frac{1}{4} \ne \underbrace{\lim_{x\to 0}(x^2+0.5)}_{\frac{1}{2}}$

II. $-x^4 \le g(x) \le x^4$: **True** $\because \lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$

III. $\frac{-1}{x^2} \le h(x) \le \frac{1}{x^2}$: **False** $\because \underbrace{\lim_{x\to 0}\frac{-1}{x^2}}_{-\infty} \ne \underbrace{\lim_{x\to 0}\frac{1}{x^2}}_{\infty}$

$$\boxed{\text{II only}}$$

5.02

1. Let R be the region between the graph of $y = e^{-2x}$ and the x-axis for $x \geq 3$. What is the area of R?

$$R = \int_3^\infty e^{-2x}\,dx = \lim_{a\to\infty}\int_3^a e^{-2x}\,dx = \lim_{a\to\infty}\left.\frac{-1}{2e^{2x}}\right|_3^a = \frac{1}{2e^6} + \underbrace{\lim_{a\to\infty}\frac{1}{2e^{2a}}}_{0} = \boxed{\frac{1}{2e^6}}$$

2. What is the $\displaystyle\int_1^\infty \frac{x}{\left(1+x^2\right)^2}\,dx$?

$$\int_1^\infty \frac{x}{\left(1+x^2\right)^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{x}{\left(1+x^2\right)^2}\,dx = \lim_{a\to\infty}\left.\frac{-1}{2(1+x^2)}\right|_1^a = \frac{1}{4} - \underbrace{\lim_{a\to\infty}\frac{1}{2(1+a^2)}}_{0} = \boxed{\frac{1}{4}}$$

3. What is the $\displaystyle\int_1^\infty xe^{-x^2}\,dx$?

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{a\to\infty}\int_1^a xe^{-x^2}\,dx = \lim_{a\to\infty}\left.\frac{-1}{2e^{x^2}}\right|_1^a = \frac{1}{2e} + \underbrace{\lim_{a\to\infty}\frac{1}{2e^{a^2}}}_{0} = \boxed{\frac{1}{2e}}$$

4. What is the $\displaystyle\int_1^\infty \frac{x^2}{\left(x^3+2\right)}\,dx$?

$$\int_1^\infty \frac{x^2}{\left(x^3+2\right)}\,dx = \lim_{a\to\infty}\int_1^a \frac{x^2}{\left(x^3+2\right)}\,dx = \lim_{a\to\infty}\left.\frac{-1}{3(x^3+2)^2}\right|_1^a = \frac{1}{9} - \underbrace{\lim_{a\to\infty}\frac{1}{3(a^3+2)}}_{0} = \boxed{\frac{1}{9}}$$

5. If $\displaystyle\int_1^x f(t)\,dt = \frac{20x}{\sqrt{4x^2+21}} - 4$, then $\displaystyle\int_1^\infty f(t)\,dt$ is

$$\int_1^\infty f(t)\,dt = \lim_{a\to\infty}\left.\frac{20x}{\sqrt{4x^2+21}}\right|_1^a = \lim_{a\to\infty}\left(\frac{20a}{\sqrt{4a^2+21}}\right) - \underbrace{\left(\frac{20}{\sqrt{21+4}}\right)}_{4}$$

(a) $$\lim_{a\to\infty}\frac{20a}{\sqrt{4a^2+21}} = \lim_{a\to\infty}\frac{20}{\sqrt{4+\underbrace{\frac{21}{a^2}}_{0}}} = \frac{20}{\sqrt{4}} = 10$$

$$\underbrace{\lim_{a\to\infty}\frac{20a}{\sqrt{4a^2+21}}}_{10} - 4 = \boxed{6}$$

6. If R is the unbounded region between the graph of $y = \dfrac{1}{x\ln^2 x}$ and the x-axis for $x \geq 3$, then the area of R is

$$R = \int_3^\infty \frac{1}{x\ln^2 x}\,dx = \lim_{a\to\infty} \frac{1}{x\ln^2 x}\,dx = \lim_{a\to\infty} \frac{-1}{\ln x}\bigg|_3^a = \frac{1}{\ln 3} - \underbrace{\lim_{a\to\infty} \frac{1}{\ln a}}_{0} = \boxed{\frac{1}{\ln 3}}$$

7. What is the $\displaystyle\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx$?

$$\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx = \lim_{a\to\infty}\int_0^a \frac{x}{\left(1+x^2\right)^2}\,dx = \lim_{a\to\infty} \frac{-1}{2(1+x^2)}\bigg|_0^a = \frac{1}{2} - \underbrace{\lim_{a\to\infty} \frac{1}{2(1+a^2)}}_{0} = \boxed{\frac{1}{2}}$$

8. If g is a twice-differentiable function, where $g(1) = 0.5$ and $\lim\limits_{x\to\infty} g(x) = 4$, then $\displaystyle\int_1^\infty g'(x)$ is

$$\int_1^\infty g'(x) = \lim_{x\to a} g(x)\bigg|_0^a = \lim_{x\to a} g(a) - g(1) = 4 - 0.5 = \boxed{3.5}$$

5.03

n	100	200	300	400	500
$\displaystyle\sum_{k=1}^{n}\left(\frac{1}{x_k}\right)\cdot\frac{1}{n}$	5.19	5.88	6.28	6.57	6.79

1. The table above shows several Riemann sum approximations to $\displaystyle\int_0^1 \frac{1}{x}\,dx$ using right-hand endpoints of n subintervals of equal length of the interval $[0,1]$. Which of the following statements best describes the limit of the Riemann sums as n approaches infinity?

$$\sum_{k=1}^{n}\left(\frac{1}{x_k}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\left(\frac{1}{x_k}\right)\left(\frac{k+1}{n} - \frac{k}{n}\right)$$

(a) $\displaystyle\lim_{a\to\infty}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to\infty} \ln|x|\bigg|_b^1 = \infty$

$$\boxed{\text{The limit of the Riemann sums does not exist because } \int_0^1 \frac{1}{x}\,dx \text{ does not exist.}}$$

2. What is the $\displaystyle\int_0^3 \frac{dx}{(1-x)^2}$

(a) Let $u = 1 - x \implies du = -dx$

$$\int_0^3 \frac{dx}{(1-x)^2} = \int_{-2}^{1} \frac{1}{u^2}\,du = \lim_{b\to 0}\left[\int_{-2}^{b} \frac{1}{u^2}\,du + \int_{b}^{1} \frac{1}{u^2}\,du\right]$$

$$\lim_{b\to 0}\left[\frac{-1}{u}\Bigg|_{-2}^{b} + \frac{-1}{u}\Bigg|_{0}^{1}\right] = \lim_{b\to 0^-}\left(\underbrace{\frac{-1}{b}}_{\infty} - \frac{1}{2}\right) + \lim_{b\to 0^+}\left(-1 + \underbrace{\frac{1}{b}}_{\infty}\right) = \infty$$

$$\boxed{\text{Divergent}}$$

3. $\displaystyle\int_1^\infty \frac{1}{x^P}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^P}\,dx$ both diverge when $\boxed{P=1}$

$$\int \frac{1}{x^P}\,dx = \begin{cases} \dfrac{-1}{(P-1)x^{P-1}} & \text{if } P \neq 1 \\ \ln|P|, & \text{if } P = 0 \end{cases} \tag{1}$$

4. Which of the following statements about the integral $\displaystyle\int_0^\pi \sec^2 x\,dx$ is true?

$$\int_0^\pi \sec^2 x\,dx = \lim_{b\to\frac{\pi}{2}^-} \tan x\Big|_0^b + \lim_{a\to\frac{\pi}{2}^+} \tan x\Big|_a^\pi = \lim_{b\to\frac{\pi}{2}^-} \tan b - \lim_{a\to\frac{\pi}{2}^+} \tan a = \infty$$

$$\boxed{\text{The integral diverges because } \lim_{x\to\frac{\pi}{2}^-} \tan x \text{ does not exist.}}$$

5. An antiderivative of $\dfrac{e^x}{e^x-1}$ is $\ln|e^x-1|$. Which of the following statements about the integral $\displaystyle\int_{-2}^{2} \frac{e^x}{e^x-1}\,dx$ is true?

$$\int_{-2}^{2} \frac{e^x}{e^x-1}\,dx = \lim_{b\to 0^-} \ln|e^x-1|\Big|_{-2}^{b} + \lim_{a\to 0^+} \ln|e^x-1|\Big|_{a}^{2}$$

$$\boxed{\text{The integral diverges because } \lim_{x\to 0^-} \ln|e^x-1| \text{ does not exist.}}$$

6. What is the $\displaystyle\int_0^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$?

(a) Let $u = 1+\sqrt{x} \implies du = \dfrac{1}{2\sqrt{x}}$

$$\int_0^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx = 2\int_1^3 \frac{1}{u}\,du = 2\ln|u|\Big|_1^3 = 2\ln 3 - 2\ln 1 = \boxed{2\ln 3}$$