

AP Classroom Problems Unit 7

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Notes

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \quad (1)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta} \left(\frac{dy}{dx} \right)}{\frac{dx}{d\theta}} \quad (2)$$

7.01

1. What is the slope of the line tangent to the polar curve $r = 1 + 2 \sin \theta$ at $\theta = 0$?

(a) $\frac{dr}{d\theta} = 2 \cos \theta$

$$\frac{dy}{dx} = \frac{2 \cos \theta \cdot \sin \theta + (1 + 2 \sin \theta) \cdot \cos \theta}{2 \cos^2 \theta - (1 + 2 \sin \theta) \cdot \sin \theta} \Big|_{\theta=0} = \boxed{\frac{1}{2}}$$

2. A polar curve is given by the equation $r = \frac{10\theta}{\theta^2+1}$ for $\theta \geq 0$. What is the instantaneous rate of change of r with respect to θ when $\theta = 2$?

$$\frac{dr}{d\theta} = \frac{-10(\theta^2 - 1)}{(\theta^2 + 1)^2} \Big|_{\theta=2} = \boxed{\frac{-6}{5}}$$

3. A polar curve is given by the differentiable function $r = f(\theta)$ for $0 \leq \theta \leq 2\pi$. If the line tangent to the polar curve at $\theta = \frac{\pi}{3}$ is horizontal, which of the following must be true?

$$0 = \frac{dy}{d\theta} \Big|_{\frac{\pi}{3}} = \boxed{\frac{\sqrt{3}}{2} f' \left(\frac{\pi}{3} \right) + \frac{1}{2} f \left(\frac{\pi}{3} \right)}$$

4. For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos \theta - \theta \sin \theta$ and $\frac{dy}{d\theta} = \sin \theta + \theta \cos \theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 4$?

$$\frac{d^2y}{d\theta^2} = \frac{\theta^2 + 2}{(\cos \theta - \theta \sin \theta)^3} \Big|_{\theta=4} \approx \boxed{1.34607}$$

5. What is the slope of the line tangent to the polar curve $r = 2\theta$ at the point $\theta = \frac{\pi}{2}$?

(a) $\frac{dr}{d\theta} = 2$

$$\frac{dy}{dx} = \frac{2 \sin \theta + (2\theta) \cdot \cos \theta}{2 \cos \theta - (2\theta) \cdot \sin \theta} \bigg|_{\theta=\frac{\pi}{2}} = \boxed{\frac{-2}{\pi}}$$

6. What is the slope of the line tangent to the polar curve $r = 2 \cos \theta - 1$ at the point where $\theta = \pi$?

(a) $\frac{dr}{d\theta} = -2 \sin \theta$

$$\frac{dy}{dx} = \frac{-2 \sin^2 \theta + (2 \cos \theta - 1) \cos \theta}{-2 \sin \theta \cos \theta - (2 \cos \theta - 1) \sin \theta} \bigg|_{\theta=\pi} = \boxed{\frac{1}{0}}$$

Undefined

7. What is the slope of the line tangent to the polar curve $r = \cos \theta$ at the point where $\theta = \frac{\pi}{6}$?

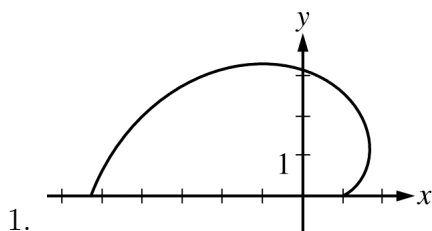
(a) $\frac{dr}{d\theta} = -\sin \theta$

(b) $\frac{dy}{d\theta} = -\sin^2 \theta + \cos^2 \theta$

(c) $\frac{dx}{d\theta} = -\sin \theta \cos \theta - \cos \theta \sin \theta$

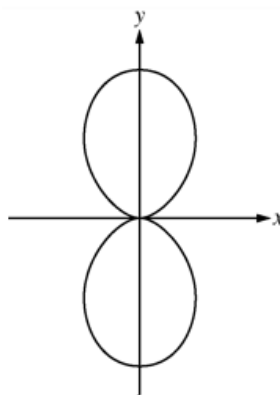
$$\frac{dy}{dx} = \frac{-\sin^2 \theta + \cos^2 \theta}{-\sin \theta \cos \theta - \cos \theta \sin \theta} \bigg|_{\theta=\frac{\pi}{6}} = \boxed{\frac{-1}{\sqrt{3}}}$$

7.02



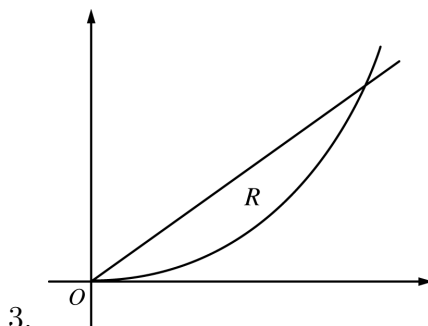
The graph above shows the polar curve $r = 2\theta + \cos \theta$ for $0 \leq \theta \leq \pi$. What is the area of the region bounded by the curve and the x -axis?

$$A = \frac{1}{2} \int_0^\pi (2\theta + \cos \theta)^2 d\theta = \frac{8\pi^3 + 3\pi - 48}{12} \approx \boxed{17.456}$$



Which of the following expressions gives the total area enclosed by the polar curve $r = \sin^2 \theta$ shown in the figure above?

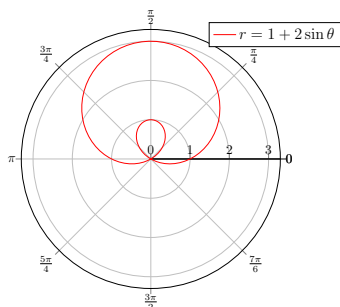
$$\int_0^\pi \sin^4 \theta \, d\theta$$



Let R be the region in the first quadrant that is bounded by the polar curves $r = \theta$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k ?

$$R = \frac{1}{2} \int_0^k \theta^2 \, d\theta = \frac{\theta^3}{6} \Big|_0^k = \frac{k^3}{6}$$

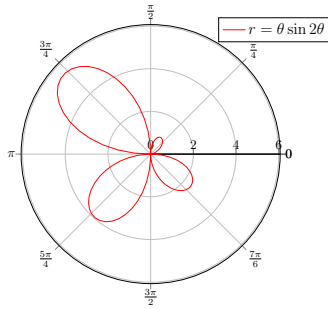
4. Which of the following integrals represents the area enclosed by the smaller loop of the graph of $r = 1 + 2 \sin \theta$?



- (a) $0 = 1 + 2 \sin \theta$ when $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$

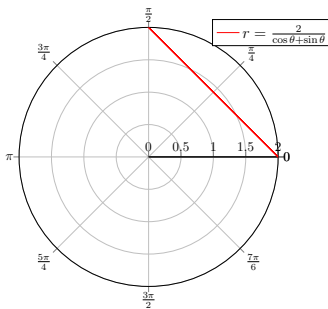
$$A = \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 2 \sin \theta)^2 \, d\theta$$

5. Which of the following gives the total area enclosed by the graph of the polar curve $r = \theta \sin 2\theta$ for $0 \leq \theta \leq 2\pi$?



$$A = \frac{1}{2} \int_0^{2\pi} |\theta \sin 2\theta|^2 d\theta$$

6. Which of the following integrals gives the area of the region that is bounded by the graphs of the polar equations $\theta = 0$, $\theta = \frac{\pi}{4}$, and, $r = \frac{2}{\cos \theta + \sin \theta}$?

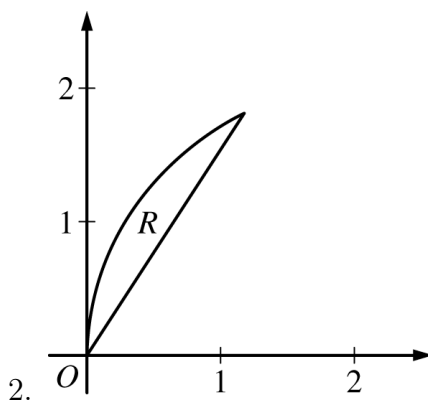


$$A = \int_0^{\pi/4} \frac{2}{(\cos \theta + \sin \theta)^2} d\theta$$

7.03

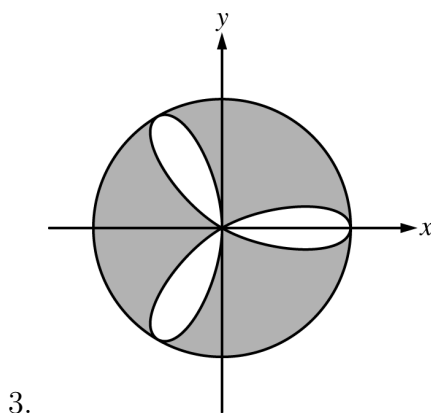
1. What is the total area between the polar curves $r = 5 \sin(3\theta)$ and $r = 8 \sin(3\theta)$?

$$A = \frac{1}{2} \int_0^{\pi} \left((8 \sin(3\theta))^2 - (5 \sin(3\theta))^2 \right) d\theta \approx \boxed{30.631}$$



Let R be the region in the first quadrant that is bounded above by the polar curve $r = 4 \cos \theta$ and below by the line $\theta = 1$, as shown in the figure above. What is the area of R ?

$$R = \frac{1}{2} \int_1^{\frac{\pi}{2}} (4 \cos \theta)^2 d\theta = -2(\sin(2) - \pi + 2) \approx \boxed{0.465}$$

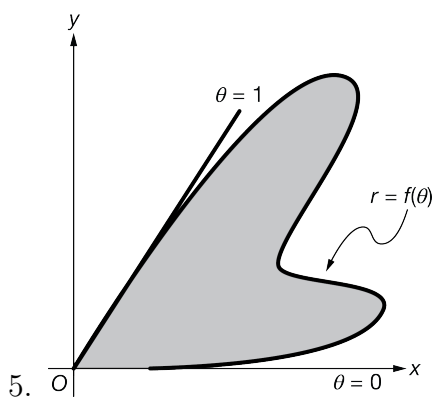


The figure above shows the graphs of the polar curves $r = 2 \cos(3\theta)$ and $r = 2$. What is the sum of the areas of the shaded regions?

$$A = \underbrace{4\pi}_{\text{Area of circle}} - \underbrace{\frac{1}{2} \int_0^\pi (2 \cos(3\theta))^2 d\theta}_{\text{Area of rose}} = 3\pi \approx \boxed{9.425}$$

4. What is the area of the region R bounded by the graph of the polar curve $r = \sqrt{1 + \frac{3\theta}{\pi}}$ and the x -axis for $0 \leq \theta \leq \pi$?

$$R = \frac{1}{2} \int_0^\pi \left(1 + \frac{3\theta}{\pi}\right) d\theta = \frac{1}{2} \left[\theta + \frac{3\theta^2}{2\pi} \right]_0^\pi = \boxed{\frac{5\pi}{4}}$$

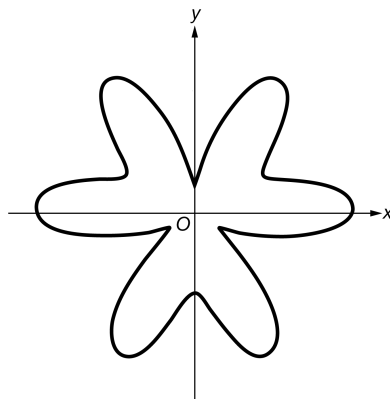


θ	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
r	1	4	3	5	2

Let R be the region bounded by the graph of the polar curve $r = f(\theta)$ and the lines $\theta = 0$ and $\theta = 1$, as shaded in the figure above. The table above gives values of the polar function $r = f(\theta)$ at selected values of θ . What is the approximation for the area of region R using a right Riemann sum with the four subintervals indicated by the data in the table?

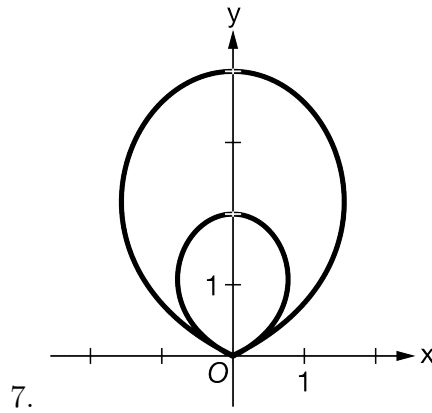
(a) Note that $\text{Area} \approx \sum_{i=1}^n \frac{1}{2} r(\theta_i)^2 \Delta\theta$

$$A = \frac{1}{2} \cdot \underbrace{\frac{1}{4}}_{\Delta\theta} \left(f\left(\frac{1}{4}\right)^2 + f\left(\frac{1}{2}\right)^2 + f\left(\frac{3}{4}\right)^2 \right) \approx \boxed{\frac{1}{8}(16 + 9 + 25 + 4)}$$



What is the area of the region bounded by the graph of the polar curve $r = 1 + \frac{1}{2} \cos(6\theta) + \frac{1}{4} \sin(3\theta)$, shown in the figure above?

$$A = \frac{1}{2} \cdot \int_0^{2\pi} \left(1 + \frac{1}{2} \cos(6\theta) + \frac{1}{4} \sin(3\theta) \right) d\theta = \frac{37\pi}{32} \approx \boxed{3.632}$$



The figure above shows the graphs of the polar curves $r = 2 \sin^2 \theta$ and $r = 4 \sin^2 \theta$ for $0 \leq \theta \leq \pi$. Which of the following integrals gives the area of the region bounded between the two polar curves?

$$A = \frac{1}{2} \int_0^\pi \left((4 \sin^2 \theta)^2 - (2 \sin^2 \theta) \right) d\theta = \boxed{\int_0^\pi 6 \sin^2 \theta d\theta}$$