# Improper Integral Practice

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## Question 1

A. $\displaystyle\int \frac{dx}{x^2+1} = \arctan x + C$

B. The $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$ is improper because of an indefinite integral i.e for $x \in [0, \infty)$.

C. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \underbrace{\int_0^1 \frac{dx}{1+x^2}}_{\text{Definite Integral}} + \underbrace{\int_1^\infty \frac{dx}{1+x^2}}_{\text{Improper Integral}}$. Since $\dfrac{1}{1+x^2} < \dfrac{1}{x^2}$ on $(1, \infty)$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges. Therefore by the comparison test $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges.

D. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \int_0^1 \frac{dx}{1+x^2} + \int_1^\infty \frac{dx}{1+x^2} = \arctan x\Big|_0^1 + \lim_{b\to\infty} \arctan x\Big|_1^b = \boxed{\frac{\pi}{2}}$

## Question 2

1. $\displaystyle\int x^{-3/5}\,dx = \frac{5x^{2/5}}{2} + C$

2. The $\displaystyle\int_0^\infty x^{-3/5}\,dx$ is improper because of an indefinite integral i.e for $x \in (0, \infty)$.

3. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \underbrace{\int_0^1 x^{-3/5}\,dx}_{\text{Converges}} + \underbrace{\int_1^\infty x^{-3/5}\,dx}_{\text{Diverges}}$. Since $\displaystyle\int_1^\infty \frac{1}{x^P}\,dx$ converges when $P > 1$ and diverges when $P \leq 1$, and $\dfrac{3}{5} < 1$ therefore the $\displaystyle\int_1^\infty \frac{dx}{x^{3/5}}$ diverges.

4. $\displaystyle\int_0^1 x^{-3/5} = \lim_{a\to 0} \frac{5x^{2/5}}{2}\Big|_a^1 = \boxed{\frac{5}{2}}$

# Question 3

1. $\displaystyle\int x^{-1/3}\,dx = \frac{3x^{2/3}}{2} + C$

2. The $\displaystyle\int_{-8}^{1} x^{-1/3}\,dx$ is improper because the $\displaystyle\lim_{x\to0^+} x^{-1/3} = \infty$ and the $\displaystyle\lim_{x\to0^-} x^{-1/3} = -\infty$ therefore there is a vertical asymptote at $x = 0$.

3. $\displaystyle\int_{-8}^{1} x^{-1/3}\,dx = \underbrace{\int_{-8}^{0} x^{-1/3}\,dx}_{\text{Converges}} + \underbrace{\int_{0}^{1} x^{-1/3}\,dx}_{\text{Converges}}$. Since $\displaystyle\int_0^1 \frac{1}{x^P}\,dx$ converges when $P < 1$ and diverges when $P \geq 1$, and $\frac{1}{3} < 1$ therefore the $\displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ and $\displaystyle\int_{-1}^{0} x^{-1/3}\,dx$ converges.

4. $\displaystyle\int_{-8}^{1} x^{-1/3}\,dx = \lim_{a\to0^-} \frac{3x^{2/3}}{2}\bigg|_{-8}^{a} + \lim_{b\to0^+} \frac{3x^{2/3}}{2}\bigg|_{b}^{1} = \boxed{-\frac{9}{2}}$

# Question 4

1. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -e^{1/x} + C$

2. The $\displaystyle\int_0^{\ln 2} \frac{e^{1/x}}{x^2}\,dx$ is improper because the $\displaystyle\lim_{x\to0^+} \frac{e^{1/x}}{x^2} = \infty$ therefore there is a vertical asymptote at $x = 0$.

3. The $\displaystyle\underbrace{\int_0^{\ln 2} \frac{e^{1/x}}{x^2}\,dx}_{\text{Converges}}$. Since $\displaystyle\frac{e^{1/x}}{x^2} > \frac{1}{x^2}$ on $(0, \ln 2)$ and $\displaystyle\int_0^{\ln 2} \frac{dx}{x^2}$ diverges. Therefore by the comparison test $\displaystyle\int_0^{\ln 2} \frac{e^{1/x}}{x^2}$ diverges.

# Question 5

1. $\displaystyle\int \frac{dx}{x^2+5x+6} = \int \frac{dx}{(x+2)(x+3)} = \ln\left|\frac{x+2}{x+3}\right| + C$

2. The $\displaystyle\int_{-1}^{\infty} \frac{dx}{x^2+5x+6}$ is improper because of an indefinite integral i.e for $x \in [-1, \infty)$.

3. The $\displaystyle\int_{-1}^{\infty} \underbrace{\int_{-1}^{1} \frac{dx}{x^2+5x+6}}_{\text{Definite integral}} + \underbrace{\int_{1}^{0} \frac{dx}{x^2+5x+6}}_{\text{Improper integral}}$. Since $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+5x+6}$ converges and $\displaystyle\frac{1}{x^2+5x+6} < \frac{1}{x^2}$ for $x \in (1, \infty)$ and $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ converges, therefore by the comparison test $\displaystyle\int_1^{\infty} \frac{dx}{x^2+5x+6}$ converges.

4. $\displaystyle\int_{-1}^{\infty} \frac{dx}{x^2+5x+6} = \lim_{b\to\infty} \ln\left|\frac{x+2}{x+3}\right|\Bigg|_{-1}^{b} = \lim_{b\to\infty} \underbrace{\ln\left|\frac{b+2}{b+3}\right|}_{\ln(1)} - \ln\frac{1}{2} = \ln 2$

## Question 6

1. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

2. The $\displaystyle\int_0^{\frac{\pi}{2}} \tan x\,dx$ is improper because the $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$, therefore there is a vertical asymptote at $x = \dfrac{\pi}{2}$.

3. $\underbrace{\displaystyle\int_0^{\frac{\pi}{2}} \tan x\,dx}_{\text{Diverges}}$. Since $\displaystyle\lim_{x\to\frac{\pi}{2}} \tan x = \frac{1}{\frac{\pi}{2}-x}$ and since the $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}$ diverges then $\displaystyle\int_0^{\frac{\pi}{2}} \tan x\,dx$ diverges.

4. $\displaystyle\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{b\to\frac{\pi}{2}} \ln|\sec x|\Big|_0^b = \lim_{b\to\frac{\pi}{2}} \ln|\sec b| - \ln(1) = \infty$

## Question 7

1. $\displaystyle\int \frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1} + C$

2. The $\displaystyle\int_5^{\infty} \frac{dx}{\sqrt{x-1}}$ is improper because of an indefinite integral i.e for $x \in [5,\infty)$.

3. $\underbrace{\displaystyle\int_5^{\infty} \frac{dx}{\sqrt{x-1}}}_{\text{Diverges}}$. Since $\dfrac{1}{\sqrt{x-1}} > \dfrac{1}{x^{1/2}}$ on $(5,\infty)$ and $\displaystyle\int_5^{\infty} x^{-1/2}\,dx$ diverges, therefore by the comparison test $\displaystyle\int_5^{\infty} \frac{dx}{\sqrt{x-1}}$ diverges.

4. $\displaystyle\int_5^{\infty} \frac{dx}{\sqrt{x-1}} = \lim_{b\to\infty} 2\sqrt{x-1}\Big|_5^b = \infty$

## Question 8

1. $\displaystyle\int \frac{dx}{\sqrt{4-x}} = -2\sqrt{4-x} + C$

2. The $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$ is improper because the $\displaystyle\lim_{x\to4^-} \frac{dx}{\sqrt{4-x}} = \infty$, therefore there is a vertical asymptote at $x = 4$.

3. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}} = \int_0^4 \frac{dx}{\sqrt{x}} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x}}}_{\text{Converges}} + \underbrace{\int_1^4 \frac{dx}{\sqrt{x}}}_{\text{Definite integral}}$. Since $\displaystyle\int_0^1 \frac{1}{x^P}\,dx$ converges when $P<1$ and diverges when $P \geq 1$, and $\frac{1}{2} < 1$ therefore the $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ diverges.

4. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to\infty} -2\sqrt{x-1}\Big|_0^4 = \infty$

# Question 9

1. $\displaystyle\int \frac{dx}{x^2+5x+6} = \int \frac{dx}{(x+2)(x+3)} = \ln\left|\frac{x+2}{x+3}\right| + C$

2. The $\displaystyle\int_{-5}^{0} \frac{dx}{x^2+5x+6}$ is improper because the $\displaystyle\underbrace{\lim_{x\to-2^-} \frac{1}{x^2+5x+6}}_{-\infty} \neq \underbrace{\lim_{x\to-2^+} \frac{1}{x^2+5x+6}}_{\infty}$ and $\displaystyle\underbrace{\lim_{x\to-3^-} \frac{1}{x^2+5x+6}}_{\infty} \neq \underbrace{\lim_{x\to-3^+} \frac{1}{x^2+5x+6}}_{-\infty}$, therefore there is a vertical asymptotes at $x=-2$ and $x=-3$.

3. $\displaystyle\int_{-5}^{0} \frac{dx}{x^2+5x+6} = \underbrace{\int_{-5}^{-3} \frac{dx}{x^2+5x+6}}_{\text{Diverges}} + \underbrace{\int_{-3}^{2} \frac{dx}{x^2+5x+6}}_{\text{Diverges}} + \underbrace{\int_{-2}^{0} \frac{dx}{x^2+5x+6}}_{\text{Diverges}}$. Since $\displaystyle\int_{-2}^{0} \frac{1}{x^2+5x+6} = \lim_{a\to-2} \ln\left|\frac{x+2}{x+3}\right|\Bigg|_a^0 = 2\ln 3 - \underbrace{\ln(0)}_{\infty} = \infty$ then $\displaystyle\int_{-5}^{0} \frac{dx}{x^2+5x+6}$ diverges.

# Question 10

1. $\displaystyle\int \frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1} + C$

2. The $\displaystyle\int_1^5 \frac{dx}{\sqrt{x-1}}$ is improper because $\displaystyle\lim_{x\to1^+} \frac{1}{\sqrt{x-1}} = \infty$ and therefore there is a vertical asymptote at $x=1$.

3. $\displaystyle\int_1^5 \frac{dx}{\sqrt{x}} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x}}}_{\text{Converges}} + \underbrace{\int_1^5 \frac{dx}{\sqrt{x}}}_{\text{Definite integral}}$. Since $\displaystyle\int_0^1 \frac{1}{x^P}\,dx$ converges when $P<1$ and diverges when $P \geq 1$ and $\frac{1}{2} < 1$, therefore the $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ converges.

4. $\displaystyle\int_1^5 \frac{dx}{\sqrt{x}} = \lim_{a\to0} 2\sqrt{x-1}\Big|_a^1 = 4$