AP Classroom Problems Unit 7

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Notes

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} \tag{2}$$

7.01

1. What is the slope of the line tangent to the polar curve $r = 1 + 2\sin\theta$ at $\theta = 0$?

   (a) $\frac{dr}{d\theta} = 2\cos\theta$

$$\frac{dy}{dx} = \left.\frac{2\cos\theta \cdot \sin\theta + (1 + 2\sin\theta)\cdot\cos\theta}{2\cos^2\theta - (1 + 2\sin\theta)\cdot\sin\theta}\right|_{\theta=0} = \boxed{\frac{1}{2}}$$

2. A polar curve is given by the equation $r = \frac{10\theta}{\theta^2+1}$ for $\theta \geq 0$. What is the instantaneous rate of change of r with respect to θ when $\theta = 2$?

$$\frac{dr}{d\theta} = \left.\frac{-10(\theta^2 - 1)}{(\theta^2+1)^2}\right|_{\theta=2} = \boxed{\frac{-6}{5}}$$

3. A polar curve is given by the differentiable function $r = f(\theta)$ for $0 \leq \theta \leq 2\pi$. If the line tangent to the polar curve at $\theta = \frac{\pi}{3}$ is horizontal, which of the following must be true?

$$0 = \left.\frac{dy}{d\theta}\right|_{\frac{\pi}{3}} = \boxed{\frac{\sqrt{3}}{2}f'\left(\frac{\pi}{3}\right) + \frac{1}{2}f\left(\frac{\pi}{3}\right)}$$

4. For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 4$?

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left.\frac{\theta^2 + 2}{(\cos\theta - \theta\sin\theta)^3}\right|_{\theta=4} \approx \boxed{1.34607}$$

5. What is the slope of the line tangent to the polar curve $r = 2\theta$ at the point $\theta = \frac{\pi}{2}$?

(a) $\frac{dr}{d\theta} = 2$

$$\frac{dy}{dx} = \frac{2\sin\theta + (2\theta)\cdot\cos\theta}{2\cos\theta - (2\theta)\cdot\sin\theta}\bigg|_{\theta=\frac{\pi}{2}} = \boxed{\frac{-2}{\pi}}$$

6. What is the slope of the line tangent to the polar curve $r = 2\cos\theta - 1$ at the point where $\theta = \pi$?

   (a) $\frac{dr}{d\theta} = -2\sin\theta$

$$\frac{dy}{dx} = \frac{-2\sin^2\theta + (2\cos\theta - 1)\cos\theta}{-2\sin\theta\cos\theta - (2\cos\theta - 1)\sin\theta}\bigg|_{\theta=\pi} = \boxed{\frac{1}{0}}$$

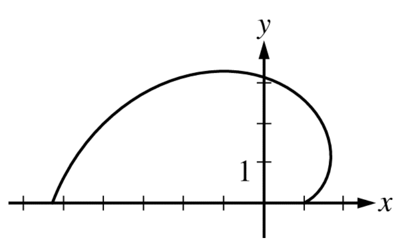
$$\boxed{\text{Undefined}}$$

7. What is the slope of the line tangent to the polar curve $r = \cos\theta$ at the point where $\theta = \frac{\pi}{6}$?

   (a) $\frac{dr}{d\theta} = -\sin\theta$
   (b) $\frac{dy}{d\theta} = -\sin^2\theta + \cos^2\theta$
   (c) $\frac{dx}{d\theta} = -\sin\theta\cos\theta - \cos\theta\sin\theta$

$$\frac{dy}{dx} = \frac{-\sin^2\theta + \cos^2\theta}{-\sin\theta\cos\theta - \cos\theta\sin\theta}\bigg|_{\theta=\frac{\pi}{6}} = \boxed{\frac{-1}{\sqrt{3}}}$$

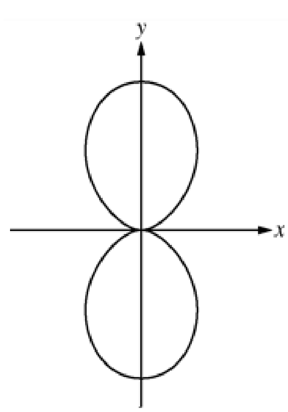
7.02



1.

The graph above shows the polar curve $r = 2\theta + \cos\theta$ for $0 \le \theta \le \pi$. What is the area of the region bounded by the curve and the x-axis?

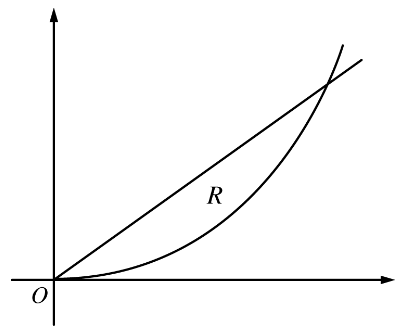
$$A = \frac{1}{2}\int_0^{\pi} \left(2\theta + \cos\theta\right)^2 d\theta = \frac{8\pi^3 + 3\pi - 48}{12} \approx \boxed{17.456}$$



2.

Which of the following expressions gives the total area enclosed by the polar curve $r = \sin^2\theta$ shown in the figure above?

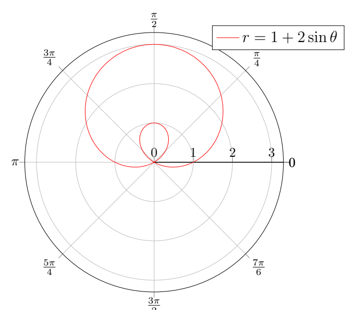
$$\boxed{\int_0^{\pi} \sin^4\theta \, d\theta}$$



3\. Let R be the region in the first quadrant that is bounded by the polar curves $r = \theta$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k?

$$R = \frac{1}{2}\int_0^k \theta^2 \, d\theta = \left.\frac{\theta^3}{6}\right|_0^k = \boxed{\frac{k^3}{6}}$$

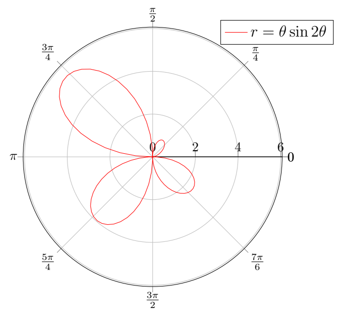
4. Which of the following integrals represents the area enclosed by the smaller loop of the graph of $r = 1 + 2\sin\theta$?



(a) $0 = 1 + 2\sin\theta$ when $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$

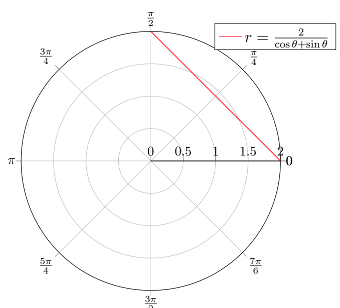
$$\boxed{A = \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left(1 + 2\sin\theta\right)^2 d\theta}$$

5. Which of the following gives the total area enclosed by the graph of the polar curve $r = \theta \sin 2\theta$ for $0 \le \theta \le 2\pi$?



$$A = \boxed{\frac{1}{2} \int_0^{2\pi} |\theta \sin 2\theta|^2 \, d\theta}$$

6. Which of the following integrals gives the area of the region that is bounded by the graphs of the polar equations $\theta = 0$, $\theta = \frac{\pi}{4}$, and, $r = \frac{2}{\cos\theta + \sin\theta}$?

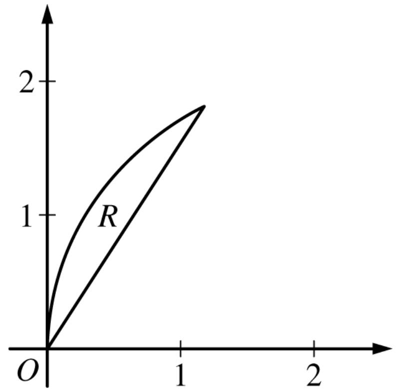


$$A = \boxed{\int_0^{\frac{\pi}{4}} \frac{2}{\left(\cos\theta + \sin\theta\right)^2} \, d\theta}$$

7.03

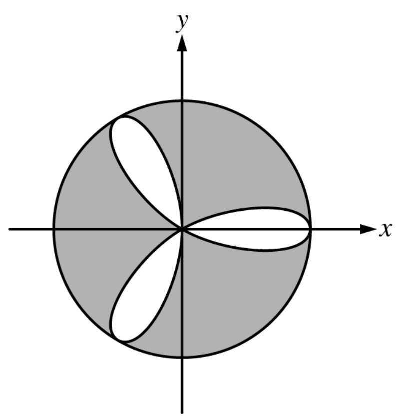
1. What is the total area between the polar curves $r = 5\sin(3\theta)$ and $r = 8\sin(3\theta)$?

$$A = \frac{1}{2} \int_0^{\pi} \left((8\sin(3\theta))^2 - \left(5\sin(3\theta)\right)^2 \right) d\theta \approx \boxed{30.631}$$

2. 

Let R be the region in the first quadrant that is bounded above by the polar curve $r = 4\cos\theta$ and below by the line $\theta = 1$, as shown in the figure above. What is the area of R?

$$R = \frac{1}{2}\int_{1}^{\frac{\pi}{2}} \left(4\cos\theta\right)^2 d\theta = -2(\sin(2) - \pi + 2) \approx \boxed{0.465}$$

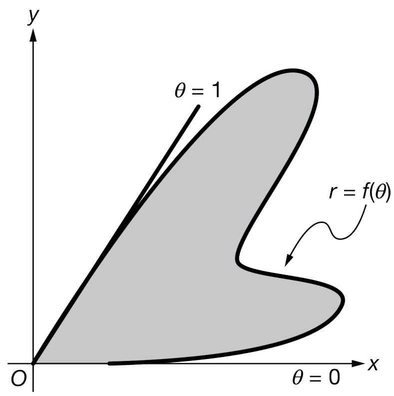
3. 

The figure above shows the graphs of the polar curves $r = 2\cos(3\theta)$and $r = 2$. What is the sum of the areas of the shaded regions?

$$A = \underbrace{4\pi}_{\text{Area of circle}} - \underbrace{\frac{1}{2}\int_{0}^{\pi} (2\cos(3\theta))^2}_{\text{Area of rose}} = 3\pi \approx \boxed{9.425}$$

4. What is the area of the region R bounded by the graph of the polar curve $r = \sqrt{1 + \frac{3\theta}{\pi}}$ and the x-axis for $0 \le \theta \le \pi$?

$$R = \frac{1}{2}\int_{0}^{\pi} \left(1 + \frac{3\theta}{\pi}\right) d\theta = \frac{1}{2}\left[\theta + \frac{3\theta^2}{2\pi}\right]_0^{\pi} = \boxed{\frac{5\pi}{4}}$$

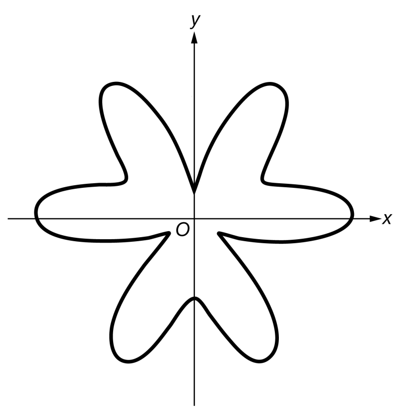
5. 

θ	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
r	1	4	3	5	2

Let R be the region bounded by the graph of the polar curve $r = f(\theta)$ and the lines $\theta = 0$ and $\theta = 1$, as shaded in the figure above. The table above gives values of the polar function $r = f(\theta)$ at selected values of θ. What is the approximation for the area of region R using a right Riemann sum with the four subintervals indicated by the data in the table?

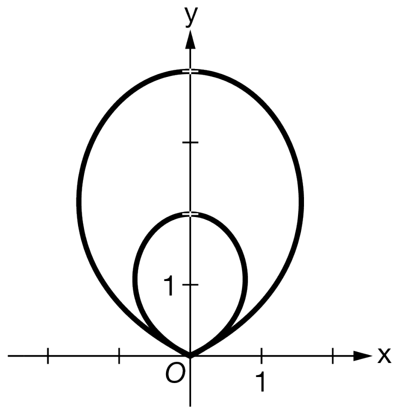
(a) Note that Area $\approx \sum_{i=1}^{n} \frac{1}{2} r(\theta_i)^2 \, \Delta\theta$

$$A = \frac{1}{2} \cdot \underbrace{\frac{1}{4}}_{\Delta\theta \cdot} \left(f\left(\frac{1}{4}\right)^2 + f\left(\frac{1}{2}\right)^2 + f\left(\frac{3}{4}\right)^2 \right) \approx \boxed{\frac{1}{8}(16 + 9 + 25 + 4)}$$

6. 

What is the area of the region bounded by the graph of the polar curve $r = 1 + \frac{1}{2}\cos(6\theta) + \frac{1}{4}\sin(3\theta)$, shown in the figure above?

$$A = \frac{1}{2} \cdot \int_0^{2\pi} \left(1 + \frac{1}{2}\cos(6\theta) + \frac{1}{4}\sin(3\theta)\right)^2 d\theta = \frac{37\pi}{32} \approx \boxed{3.632}$$



7.

The figure above shows the graphs of the polar curves $r = 2\sin^2\theta$ and $r = 4\sin^2\theta$ for $0 \leq \theta \leq \pi$. Which of the following integrals gives the area of the region bounded between the two polar curves?

$$A = \frac{1}{2}\int_0^{\pi}\left(\left(4\sin^2\theta\right)^2 - \left(2\sin^2\theta\right)^2\right)d\theta = \boxed{\int_0^{\pi} 6\sin^4\theta\,d\theta}$$

Extra Polar Practice

1. Which of the following intgrals represents the area enclosed by the smaller loop of the graph of $r = 1 + 2\sin\theta$?

   (a) $r = 0$ when $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$

$$\boxed{A = \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\left(1 + 2\sin\theta\right)^2 d\theta}$$

2. What is the area of the region enclosed by the lemniscate $r^2 = 18\cos(2\theta)$?

$$A = 4\left[\frac{1}{2}\int_0^{\frac{\pi}{4}} 18\cos(2\theta)\,d\theta\right] = 2\int_0^{\frac{\pi}{4}} 18\cos(2\theta)\,d\theta$$

   (a) Let $u = 2\theta \Longrightarrow \frac{du}{2} = d\theta$

$$A = 18\sin(2\theta)\Big|_0^{\frac{\pi}{4}} = \boxed{18}$$

3. The area of one loop of the graph of the polar equation $r = 2\sin(3\theta)$ is given by which of the following expresions?

   (a) $r = 0$ when $\theta = \pi 3$ and $\theta = 0$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} 4\sin^2(3\theta)\,d\theta = \boxed{3\int_0^{\frac{\pi}{3}}\sin^2(3\theta)\,d\theta}$$

4. The area of the region inside the polar curve $r = 4\sin\theta$ and outside the polar curve $r = 2$ is given by

(a) The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

$$\boxed{A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(16\sin^2\theta - 4\right) d\theta}$$

5. Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$

$$A = \frac{1}{2}\int_0^{\pi} \left(4\cos^2\theta - \cos^2\theta\right) d\theta = \frac{3}{2}\int_0^{\pi} \cos^2\theta\, d\theta = \boxed{3\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta}$$

6. Which of the following represents the graph of the polar curve $r = 2\sec\theta$?

$$r = 2\sec\theta \implies r\cos\theta = 2 \implies \boxed{x = 2}$$

7. Which of the following represents the area of the region enclosed by the loop of the graph of the polar curve $r = 4\cos(3\theta)$?

$$\boxed{A = 8\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\,\mathrm{d}\theta}$$

8. What is the area of the region enclosed by the polar curve $r = \sin(2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$?

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\,\mathrm{d}\theta$$

$$2A = \int_0^{\frac{\pi}{2}} \sin^2(2\theta)\,\mathrm{d}\theta$$

(a) $u = \sin(2\theta) \implies du = 2\cos(2\theta)\, d\theta$

(b) $v = \frac{-1}{2}\cos(2\theta) \Leftrightarrow dv = \sin(2\theta)\, d\theta$

(c) Note: $\cos^2(\theta) = 1 - \sin^2(\theta)$

$$2A = \frac{-1}{2}\cos(2\theta)\sin(2\theta) + \int \cos^2(2\theta)\,\mathrm{d}\theta$$

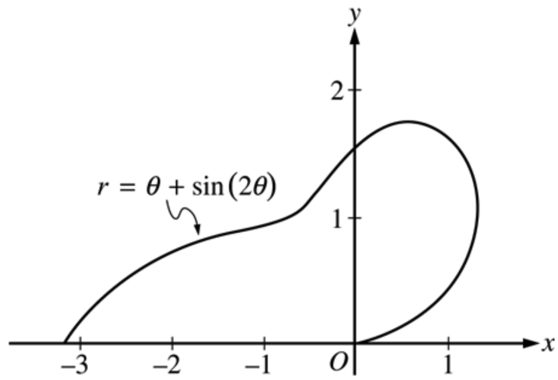
$$2A = \frac{-1}{2}\cos(2\theta)\sin(2\theta) + \int \left(1 - \sin^2(2\theta)\right)\mathrm{d}\theta$$

$$2A = \frac{-1}{2}\cos(2\theta)\sin(2\theta) + \theta - 2A$$

$$A = \frac{1}{4}\left[\frac{-1}{2}\cos(2\theta)\sin(2\theta) + \theta\right]_0^{\frac{\pi}{2}}$$

$$A = \boxed{\frac{\pi}{8}}$$

FRQ 1



The curve above is drawn in the xy-plane and is discribed by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivitive of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$

(a) Find the area bounded by the curve and the x-axis.

$$A = \frac{1}{2}\int_0^{\pi} \left(\theta + \sin(2\theta)\right)^2 d\theta = \frac{\pi\left(2\pi^2 - 3\right)}{12} \approx 4.382$$

(b) Find the angle θ that corresponds to the point on the curve with the x-coordinate -2.

$$-2 = r\cos\theta = (\theta + \sin(2\theta)) \cdot \cos\theta$$
$$\theta \approx 2.786$$

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negitive. What does this say about r? What does this fact say about the curve?

Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decresing on this interval. This means the curve is getting closer to the orgin.

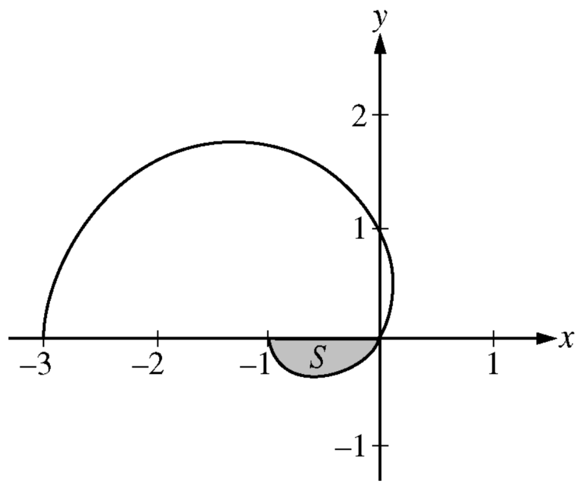
(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that cooresponds to the point on the curve in the first quadrant with the greatest distance from the orgin. Justify your answer.

The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\frac{\pi}{3}$

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs at $\frac{\pi}{3}$.

FRQ 2



The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

$$r(0) = -1; r(\theta) = 0 \text{ when } \theta = \frac{\pi}{3}.$$

$$S = \frac{1}{2}\int_0^{\frac{\pi}{3}} \big(1 - 2\cos(\theta)\big)^2 \, d\theta$$

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = 2\sin^2\theta + (1 - 2\cos\theta)\cdot\cos\theta$$

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

When $\theta = \dfrac{\pi}{2}$, we have $x = 0$ and $y = 1$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.