The Making of the Atomic Bomb

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Introduction

The study of the diffusion of neutrons in a fissile material has a central role (and had) in the understanding and the development of nuclear weapons.
These studies started in 1939, as a consequence of the Germany's threat, at Los Alamos (New Mexico, USA) under the name of *Manhattan Project* and culminated with 3 major events:

- The *Trinity* test, on 16th July 1945, at Los Alamos, with Plutonium as the used fissile material;
- The *Little Boy* bomb, on 6th August 1945, dropped on Hiroshima, with Uranium as the used fissile material;
- The *Fat Man* bomb, on 9th August 1945, dropped on Nagasaki, with Plutonium as the used fissile material.

Indeed, the diffusion of neutrons in fissile materials gives origin to a release of secondary neutrons, generated by collisions between the free neutrons and the medium nuclei; this process causes the raise in size of the fissile material and, if a critical value of neutron density is exceeded, the result is an exponential growth which leads to an intense explosion. So, our aim is to calculate this critical value, study the behaviour of the neutron density and try to figure out how much a bomb is efficient, in relationship with the problem of how save money, since the fissile materials are very expensive.

1 The Critical Mass of Nuclear Weapons

The first question is: how do we represent the diffusion of the neutrons, taking into account the fact that, being in fissile materials, neutrons generate secondary neutrons?
If the material is non-fissile, neutron density would be simply described by a diffusion equation, which is a parabolic partial differential equation, written as

$$\frac{\partial n}{\partial t} = \mu \nabla^2 n$$

where

$$n = n\left(t, x\right), t \in \mathbb{R}^+, x \in \mathbb{R}^s, s = [1, 2, 3]$$

and μ is the diffusion constant.
But since the material is fissile, we have to consider the production of secondary neutrons, which can be accounted at the equation level as a source term:

$$\frac{\partial n}{\partial t} = \mu \nabla^2 n + \eta n \tag{1.1}$$

where η is the rate of secondary neutron formation.
At this point, we need a set of boundary conditions, i. e. we have to specify what happens at the edges of the material. In a first approximation, we can assume that neutrons at boundaries simply escape and disappear; so, we can set Dirichlet boundary conditions:

$$n\left(t, 0\right) = n\left(t, L\right) = 0 \tag{1.2}$$

where L is the lenght size of the fissile material.
Of course, a more realistic model cannot allow that neutrons reach the edges and magically disappear: to this aim, we will set, in a second moment, different and more accurate boundary conditions, Neumann ones.
Moreover, we need also an initial condition; we simply consider that, at initial time, the neutron density is well determined by a known distribution function $f\left(x\right)$:

$$n\left(0, x\right) = f\left(x\right) \tag{1.3}$$

So, summarizing:

$$\begin{cases} \frac{\partial n}{\partial t} = \mu \nabla^2 n + \eta n, \mu, \eta > 0 \\ n\left(t, 0\right) = n\left(t, L\right) = 0 \\ n\left(0, x\right) = f\left(x\right) \end{cases} \tag{1.4}$$

One can write the two constants in terms of

$$\mu = \frac{1}{3}\lambda_t v_{neut} \tag{1.5}$$

$$\eta = \frac{1}{\lambda_f} v_{neut} \left(\nu - 1\right) \tag{1.6}$$

where v_{neut} is the neutron speed, λ_t is the transport free path, λ_f is the fission free path and ν is the number of the secondary neutron generated (the $(\nu - 1)$ is because one neutron is used to generate ν new neutrons). A better understanding of the origin and the meaning of these constants can be found in the Appendix A.

Thus, our aim is to solve the differential equation within its boundary and initial conditions and we'll make so step by step, proceeding first with Cartesian coordinates, in one, two and three dimension(s), then with Cylindrical and Spherical coordinates. Then, once obtained a formal solution in the various cases, we'll try to understand what we expect that a bomb should do in relationship to the diffusion of the neutrons, calculating the critical value above which we have the unstoppable reaction chain that lead to the explosion; we use these result and compare them to understand which is the "best shape" for an efficient bomb. Finally, we'll study a more realistic model, making use of the more accurate Neumann boundary conditions.

1.1 Cartesian coordinates

We'll first use the Cartesian coordinates, which simplicity stays in the definition of the Laplacian of the differential equation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.7}$$

However, we'll proceed step by step, starting from the "unrealistic" one-dimensional and two-dimensional models and then we'll move to the real three dimensional problem.

1.1.1 One-dimensional case

In this case, the system (1.4) simply reduces to

$$\begin{cases} \frac{\partial n}{\partial t} = \mu \frac{\partial^2 n}{\partial x^2} + \eta n \\ n\left(t, 0\right) = n\left(t, L\right) = 0 \\ n\left(0, x\right) = f\left(x\right) \end{cases} \tag{1.8}$$

In order to solve the partial differential equation, we can use the method of the variable separation: it consists in assuming that the unknown double-variable function (in this case $n\left(t, x\right)$) can be written as a product of one-variable functions, as follows

$$n\left(t, x\right) = T(t)X(x) \tag{1.9}$$

Substituting in the equation and dividing by $T(t)X(x)$, we find that:

$$\frac{1}{T}\frac{\partial T}{\partial t} - \eta = \mu \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{1.10}$$

Since the two sides depend on different variables but are, at the same time, equal, then they must be equal to a constant, so we can write

$$\frac{1}{T}\frac{\partial T}{\partial t} - \eta = \mu \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\alpha \tag{1.11}$$

Soon we'll understand the minus sign; so, we obtain

$$\begin{cases} \frac{dT}{dt} = \left(\eta - \alpha\right) T \\ \frac{d^2 X}{dx^2} = -\frac{\alpha}{\mu} X \end{cases} \tag{1.12}$$

Thus, from a single partial differential equation, we have obtained two ordinary differential equations, which we know how to solve; the solutions, indeed are:

$$\begin{cases} T(t) = Ae^{(\eta-\alpha)t} \\ X(x) = B_1 cos\left(\sqrt{\frac{\alpha}{\mu}}x\right) + B_2 sin\left(\sqrt{\frac{\alpha}{\mu}}x\right) \end{cases} \tag{1.13}$$

So, our former unknown function is

$$n\left(t,x\right) = e^{(\eta-\alpha)t}\left[Ccos\left(\sqrt{\frac{\alpha}{\mu}}x\right) + Dsin\left(\sqrt{\frac{\alpha}{\mu}}x\right)\right] \tag{1.14}$$

where we have appropriately scaled the constants.
Now, to determine these constants, we can use boundary and initial conditions; let's start with boundary ones. The first boundary condition, $n\left(t,0\right) = 0$, implies that $C = 0$ and so

$$n\left(t,x\right) = De^{(\eta-\alpha)t} sin\left(\sqrt{\frac{\alpha}{\mu}}x\right) \tag{1.15}$$

From the second boundary condition, $n\left(t,L\right) = 0$, we derive that

$$\sqrt{\frac{\alpha}{\mu}} = \frac{p\pi}{L}, p \in \mathbb{Z} \tag{1.16}$$

Since the diffusion equation, even with the source term, is linear, we can apply the superposition principle and write

$$n\left(t,x\right) = \sum_{p=1}^{\infty} a_p exp\left[\left(\eta-\alpha\right)t\right] sin\left(\frac{p\pi}{L}x\right) \tag{1.17}$$

The reason for which we have used the sum for the only positive values of p, can be understood calculating the latter constant a_p. Of course, we are left over with the initial condition, $n\left(0,x\right) = f\left(x\right)$, where $f\left(x\right)$ is a certain distribution function, representing the initial distribution of the neutrons (neutron density) at an initial time. From (1.17), we therefore have

$$f\left(x\right) = \sum_{p=1}^{\infty} a_p sin\left(\frac{p\pi}{L}x\right) \tag{1.18}$$

Since the following integral

$$\int_0^L dx f\left(x\right) sin\left(\frac{p\pi}{L}x\right) = \frac{L}{2}a_p \tag{1.19}$$

then, we can determine the constant a_p as

$$a_p = \frac{2}{L}\int_0^L dx f\left(x\right) sin\left(\frac{p\pi}{L}x\right) \tag{1.20}$$

Now, if we had used the sum over all integers of p, we would have obtained only half of the values for a_p, but because of the additivity of sin function with opposite values, we would have obtained the same value of (1.20).
From the eigenvalue equation (1.16), we derive that

$$\alpha = \mu\left(\frac{p\pi}{L}\right)^2, p \in \mathbb{N} \tag{1.21}$$

and so, our final solution is

$$n\left(t,x\right) = \sum_{p=1}^{\infty} a_p exp\left[\left(\eta - \mu\frac{p^2\pi^2}{L^2}\right)t\right] sin\left(\frac{p\pi}{L}x\right) \tag{1.22}$$

At this point, let us come back to the physics.
What we want for our bomb is that the number of neutrons, i. e. the neutron density, could raise

uncontrollably; so, we want that the neutron density, expressed in the (1.22), to be exponentially unbounded, which means to require that

$$\eta - \mu \frac{p^2 \pi^2}{L^2} \geq 0 \tag{1.23}$$

Thus, the critical condition is obtained when (1.23) is satisfied at minimum and it is

$$L_{crit} = p\pi \sqrt{\frac{\mu}{\eta}} \tag{1.24}$$

The worst case, in which we have the shortest lenght, is $p = 1$, and gives the simple critical condition

$$L_{crit} = \pi \sqrt{\frac{\mu}{\eta}} \tag{1.25}$$

Thus, for every cores which exceed this value of size, the result will be an unstoppable reaction chain which will lead to an intense explosion.

Numerics

After having calculated the critical lenght through (1.25), resulting $L_{crit} \sim 11.05\ cm$, we have considered a lenght of $L = 11.1\ cm$; as initial distribution function, we have first considered a Gaussian

$$n\left(0, x\right) = f(x) = Aexp\left[\frac{-4\lambda\left(x - \frac{1}{2}L\right)^2}{L^2}\right]$$

Since this kind of distribution function trivially does not satisfy the Dirichlet boundary conditions, we have chosen $A = 1$ and $\lambda = 100$, in order to have at the origin $x = 0$ and at the edge $x = L$

$$n\left(0, 0\right) = n\left(0, L\right) = exp\left(-\lambda\right) \sim 0$$

Then, we have calculated the a_p coefficients through (1.20), making use of trapezoidal rule for numerical integration. The results are shown in the following table

Table 1: Values for a_p coefficients

p	a_p	p	a_p
1	0.176155	17	0.0298098
2	-4.15417E-11	18	6.80615E-11
3	-0.167673	19	-0.0191194
4	6.96101E-11	20	-7.09234E-11
5	0.151915	21	0.0116724
6	-8.27296E-11	22	6.37905E-11
7	-0.131011	23	-0.00678285
8	7.97565E-11	24	-3.83573E-11
9	0.107543	25	0.00375174
10	-5.37758E-11	26	1.04918E-11
11	-0.0840277	27	-0.00197525
12	1.61804E-11	28	1.07378E-11
13	0.0624932	29	0.000989876
14	2.41976E-11	30	-3.41061E-11
15	-0.0442397	31	-0.00047218
16	-5.51074E-11	32	4.82731E-11

Notice that a_p coefficient with even p are zero. Moreover, we see how coefficients over those calculated are tending to zero, so we have stopped our calculation at $p = 32$.
Then, we have calculated the neutron density function, as (1.22) tells us to do, and plotting it we have obtained the following graph

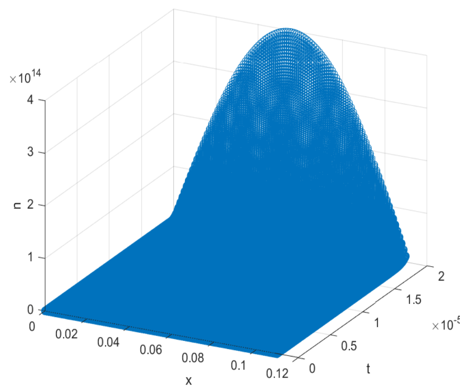


Figure 1: Plot of $n\left(t,x\right)$

Notice how abruptly the number density of neutrons raise without control; this would be not possible if we had considered a lenght smaller than the critical lenght. Moreover, we see how this happens in a so small time of the order of 100 μs.
As final calculation, since $f(x)$ does not satisfy the boundary conditions, we have compared it with $n\left(0,x\right)$ and plotted the difference between these two functions; the result is the following graph

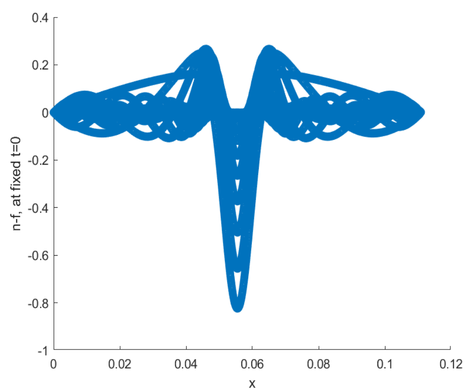


Figure 2: Plot of $n\left(0,x\right)-f(x)$

We can observe that the difference is, in practice, zero near the points subject to the Dirichlet boundary conditions, as we have set appropriately the constants of $f(x)$, but the it increases in the middle of the core; however, this difference does not overcome the unity.

1.1.2 Two-dimensional case

With two coordinates, the system (1.4) reduces to

$$\begin{cases} \frac{\partial n}{\partial t}=\mu\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+\eta n \\ n\left(t,0,y\right)=n\left(t,x,0\right)=n\left(t,L_x,y\right)=n\left(t,x,L_y\right)=0 \\ n\left(0,x,y\right)=f\left(x,y\right) \end{cases} \tag{1.26}$$

Again, we use the method of the variable separation as follows

$$n\left(t,x,y\right)=T(t)X(x)Y(y) \tag{1.27}$$

Substituting in the equation and dividing by $T(t)X(x)Y(y)$ and rearranging, we find that:

$$\frac{1}{T}\frac{\partial T}{\partial t}-\eta=\mu\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\mu\frac{1}{X}\frac{\partial^2 Y}{\partial y^2}=-\alpha \tag{1.28}$$

Applying another time the variable separation method, we obtain

$$\begin{cases} \frac{dT}{dt} = (\eta - \alpha) T \\ \frac{d^2X}{dx^2} = -\frac{\alpha_1}{\mu} X \\ \frac{d^2Y}{dy^2} = -\frac{\alpha_2}{\mu} Y \end{cases} \tag{1.29}$$

where $\alpha_1 + \alpha_2 = \alpha$. Repeating the same steps made for the one-dimensional case, we would find that

$$n(t, x, y) = \sum_{p,q=1}^{\infty} a_{p,q} exp \left[(\eta - \alpha) t \right] sin \left(\frac{p\pi}{L_x} x \right) sin \left(\frac{q\pi}{L_y} y \right) \tag{1.30}$$

and that

$$\sqrt{\frac{\alpha_1}{\mu}} = \frac{p\pi}{L_x} \Rightarrow \alpha_1 = \mu \left(\frac{p\pi}{L_x} \right)^2 \tag{1.31}$$

$$\sqrt{\frac{\alpha_2}{\mu}} = \frac{q\pi}{L_y} \Rightarrow \alpha_2 = \mu \left(\frac{q\pi}{L_y} \right)^2 \tag{1.32}$$

Thus, we finally have

$$n(t, x, y) = \sum_{p,q=1}^{\infty} a_{p,q} exp \left\{ \left[\eta - \mu\pi^2 \left(\frac{p^2}{L_x^2} + \frac{q^2}{L_y^2} \right) \right] t \right\} sin \left(\frac{p\pi}{L_x} x \right) sin \left(\frac{q\pi}{L_y} y \right) \tag{1.33}$$

So, the critical condition is

$$\mu\pi^2 \left(\frac{p^2}{L_x^2} + \frac{q^2}{L_y^2} \right) = \eta \tag{1.34}$$

The worst case, corresponding to $p = q = 1$, gives

$$\sqrt{\frac{1}{L_x^2} + \frac{1}{L_y^2}} = \frac{1}{\pi} \sqrt{\frac{\eta}{\mu}} \tag{1.35}$$

If the fissile material is symmetric in size, that is $L_x = L_y$, then

$$L_{crit} = \pi \sqrt{\frac{2\mu}{\eta}} \tag{1.36}$$

which is exactly $\sqrt{2}$ times bigger than what we found in the one-dimensional case. As in the one dimensional case, the coefficients can be calculated, through the initial condition, by

$$a_{p,q} = \frac{4}{L^2} \int_0^L \int_0^L dxdy f(x, y) \, sin \left(\frac{p\pi}{L} x \right) sin \left(\frac{q\pi}{L} y \right) \tag{1.37}$$

Numerics

Through (1.36), we have calculated the critical lenght, resulting about $L_{crit} \sim 15.62\ cm$, so we have chosen a lenght of $L = 15.7\ cm$. For the initial distribution, we have taken a suitable function which satisfies the Dirichlet boundary condition, as follows

$$f(x, y) = \frac{16xy}{L^2} \left(1 - \frac{x}{L} \right) \left(1 - \frac{y}{L} \right)$$

Indeed, the numerical integration is not simple as in the one-dimensional case, because we should perform a multidimensional integration; so we have chosen a distribution function which gives, through (1.36), an exact result:

$$a_{p,q} = 256 \frac{1 - (-1)^q - (-1)^p + (-1)^{p+q}}{p^3 q^3 \pi^6}$$

Thus, we have calculated the coefficients and they are illustrated in the following table

Table 2: Values for $a_{p,q}$ coefficients

p	q	$a_{p,q}$	p	q	$a_{p,q}$
1	1	1.06513E+00	3	4	0
1	2	0	3	5	3.15593E-04
1	3	3.94491E-02	4	1	0
1	4	0	4	2	0
1	5	8.52100E-03	4	3	0
2	1	0	4	4	0
2	2	0	4	5	0
2	3	0	5	1	8.52100E-03
2	4	0	5	2	0
2	5	0	5	3	3.15593E-04
3	1	3.94491E-02	5	4	0
3	2	0	5	5	6.81680E-05
3	3	1.46108E-03			

As we see, increasing the number of coefficients, we have negligible effects.
We have so calculated the neutron density function through the (1.33) and, in particular, we have plotted it at fixed time $t = 5 \cdot 10^{-5}$ s, when the explosion is occurred, as shown in the following figure

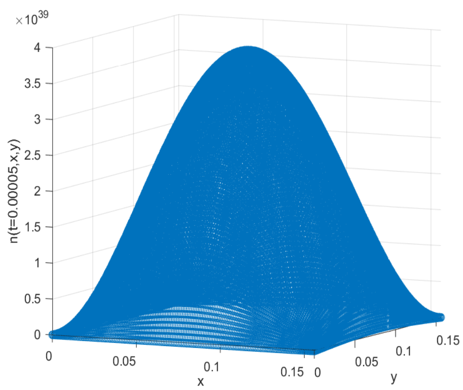


Figure 3: Plot of $n\left(x,y\right)$ at fixed time

Notice how the explosion has its centre in the middle of the core.

1.1.3 Three-dimensional case

Let us come to the first realistic model, since our world is 3-dimensional (at least we see it so), the 3D Cartesian model: in this case, the bomb assumes the shape of a parallelepiped. With three coordinates, the system (1.4) reduces to

$$\begin{cases} \frac{\partial n}{\partial t} = \mu\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right) + \eta n \\ n\left(t,0,y,z\right) = n\left(t,x,0,z\right) = n\left(t,x,y,0\right) = n\left(t,L_x,y,z\right) = n\left(t,x,L_y,z\right) = n\left(t,x,y,L_z\right) = 0 \\ n\left(0,x,y,z\right) = f\left(x,y,z\right) \end{cases} \tag{1.38}$$

Again, we use the method of the variable separation as follows

$$n\left(t,x,y,z\right) = T(t)X(x)Y(y)Z(z) \tag{1.39}$$

Repeating the same steps made for the previous cases, we would find that

$$n(t,x,y,z)=\sum_{p,q,k=1}^{\infty} a_{p,q,k} exp\left\{\left[\eta-\mu\pi^2\left(\frac{p^2}{L_x^2}+\frac{q^2}{L_y^2}+\frac{k^2}{L_z^2}\right)\right]t\right\} sin\left(\frac{p\pi}{L_x}x\right) sin\left(\frac{q\pi}{L_y}y\right) sin\left(\frac{k\pi}{L_z}z\right) \tag{1.40}$$

So, the critical condition in the worst case, corresponding to $p=q=k=1$, is

$$\sqrt{\frac{1}{L_x^2}+\frac{1}{L_y^2}+\frac{1}{L_z^2}}=\frac{1}{\pi}\sqrt{\frac{\eta}{\mu}} \tag{1.41}$$

If the fissile material is symmetric in size, that is $L_x=L_y=L_z$, then

$$L_{crit}=\pi\sqrt{\frac{3\mu}{\eta}} \tag{1.42}$$

which is exactly $\sqrt{3}$ times bigger than what we found in the one-dimensional case. Thus, we have found the critical value that a cubic bomb must exceed to generate an explosion.
We can also ask, instead of the size, what mass a bomb should exceed: in other words, we can find the so called **Critical Mass**. Calculating it is a simple job:

$$M_{crit}=\rho_0 V_{crit} \tag{1.43}$$

where ρ_0 is the density of the used fissile material (kg/m^3). In the case of a cubic bomb, we certainly have

$$M_{crit}=\rho_0 V_{crit}=\rho_0 L_{crit}^3=\rho_0\pi^3\left(\frac{3\mu}{\eta}\right)^{\frac{3}{2}} \tag{1.44}$$

Considering the U^{235} as the fissile material, the critical mass comes out to be about $131kg$, which is a very large mass, considering the high cost of the fissile material. Thus, in making a nuclear bomb, one has to proceed further in order to discover cheaper and more stable solutions.
Finally, as in the other two previous cases, the coefficients can be calculated, through the initial condition, by

$$a_{p,q,k}=\frac{8}{L^3}\int_0^L\int_0^L\int_0^L dxdydz f(x,y,z)\, sin\left(\frac{p\pi}{L}x\right) sin\left(\frac{q\pi}{L}y\right) sin\left(\frac{k\pi}{L}z\right) \tag{1.45}$$

Numerics

Using (1.42), we have first calculated the critical lenght, resulting $L_{crit}\sim 19.14\ cm$, and then, via (1.44) we have calculated the value of the critical mass, resulting $M_{crit}\sim 131\ kg$. We have decided to use a lenght of $L=19.2\ cm$.
We have considered, for the same reasons of the two-dimensional case, a distribution function like

$$f(x,y,z)=\frac{8xyz}{L^3}\left(1-\frac{x}{L}\right)\left(1-\frac{y}{L}\right)\left(1-\frac{z}{L}\right)$$

Indeed, this satisfies the Dirichlet boundary conditions and the integral (1.45) can be done exactly, avoid a problematic triple numerical integration. The result is

$$a_{p,q,k}=-512\frac{(-1)^q-(-1)^{q+k}-1+(-1)^k+(-1)^p-(-1)^{p+k}-(-1)^{p+q}+(-1)^{p+q+q}}{p^3q^3k^3\pi^9}$$

Thus, we have calculated the first 64 coefficients and they are illustrated in the following table

Table 3: Values for $a_{p,q,k}$ coefficients

p	q	k	$a_{p,q,k}$	p	q	k	$a_{p,q,k}$
1	1	1	1.37408E-01	3	1	1	5.08917E-03
1	1	2	0	3	1	2	0
1	1	3	5.08917E-03	3	1	3	1.88488E-04
1	1	4	0	3	1	4	0
1	2	1	0	3	2	1	0
1	2	2	0	3	2	2	0
1	2	3	0	3	2	3	0
1	2	4	0	3	2	4	0
1	3	1	5.08917E-03	3	3	1	1.88488E-04
1	3	2	0	3	3	2	0
1	3	3	1.88488E-04	3	3	3	6.98103E-06
1	3	4	0	3	3	4	0
1	4	1	0	3	4	1	0
1	4	2	0	3	4	2	0
1	4	3	0	3	4	3	0
1	4	4	0	3	4	4	0
2	1	1	0	4	1	1	0
2	1	2	0	4	1	2	0
2	1	3	0	4	1	3	0
2	1	4	0	4	1	4	0
2	2	1	0	4	2	1	0
2	2	2	0	4	2	2	0
2	2	3	0	4	2	3	0
2	2	4	0	4	2	4	0
2	3	1	0	4	3	1	0
2	3	2	0	4	3	2	0
2	3	3	0	4	3	3	0
2	3	4	0	4	3	4	0
2	4	1	0	4	4	1	0
2	4	2	0	4	4	2	0
2	4	3	0	4	4	3	0
2	4	4	0	4	4	4	0

Notice how many coefficients are zero.
Finally, we have computed the neutron density and, in particular, the neutron density at fixed time $t = 2 \cdot 10^{-5}$ s and fixed height $z = \frac{L}{2}$ and we have performed a contour-plot, as shown in Figure 4. Again, the growth of the neutron density number is concentrated in the middle of the fissile material.

1.2 Spherical coordinates

Instead of a cubic one, suppose to build up a spherical bomb. Then, instead of using Cartesian coordinates, we attempt to solve the system (1.4) through spherical ones.
First, we have to remember that, in spherical coordinates, the Laplacian assumes the following form

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \tag{1.46}$$

For the sake of simplicity, we consider a spherical symmetry, that is we consider that the neutron density function does not depend on angular variables but only depends on radial changes; in other words

$$n\left(t, r, \theta, \varphi\right) = n\left(t, r\right) \tag{1.47}$$

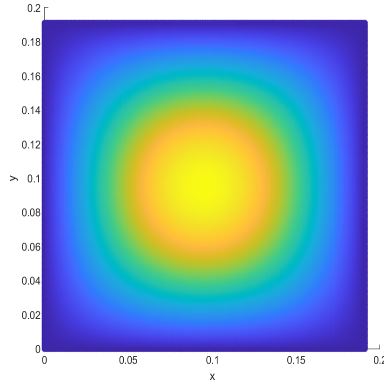


Figure 4: Plot of $n\,(x,y)$ at fixed time and fixed height

So, the Laplacian is oversimplified and the system (1.4) reduces to

$$\begin{cases} \frac{\partial n}{\partial t} = \mu \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) n + \eta n \\ n\,(t, r_1) = 0 \\ n\,(0, r) = f\,(r) \end{cases} \tag{1.48}$$

where r_1 is the ball radius. Notice that this problem is analogous to 1D Cartesian case; however, we'll have a different solution and require continuity at the origin.
As always, we attempt the problem via variable separation method, writing

$$n\,(t, r) = T(t)R(r) \tag{1.49}$$

Substituting, dividing by TR and rearranging, we obtain

$$\frac{1}{T}\frac{\partial T}{\partial t} - \eta = \mu \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \mu \frac{2}{rR}\frac{\partial R}{\partial r} = -\alpha \tag{1.50}$$

from which we get the two following ODEs

$$\begin{cases} \frac{dT}{dt} = (\eta - \alpha)\,T \\ \frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{\alpha}{\mu} R = 0 \end{cases} \tag{1.51}$$

The first has already been solved previously

$$T(t) = Aexp\,[(\eta - \alpha)\,t] \tag{1.52}$$

the second has a well-known solution too, which is

$$R(r) = \frac{1}{r}\left[Bsin\left(\sqrt{\frac{\alpha}{\mu}} r \right) + Ccos\left(\sqrt{\frac{\alpha}{\mu}} r \right) \right] \tag{1.53}$$

Beyond the boundary and initial conditions, we have also to require that the solution is finite at $r = 0$, so we need to impose that $C = 0$ in order to have

$$R(r) = \frac{B}{r} sin\left(\sqrt{\frac{\alpha}{\mu}} r \right) \tag{1.54}$$

After rescaling the constants, the overall solution is

$$n\,(t, r) = \frac{A}{r} exp\,[(\eta - \alpha)\,t]\, sin\left(\sqrt{\frac{\alpha}{\mu}} r \right) \tag{1.55}$$

From the boundary condition, $n\,(t, r_1) = 0$, we find that

$$\sqrt{\frac{\alpha}{\mu}} = \frac{p\pi}{r_1}, p \in \mathbb{Z} \tag{1.56}$$

which is, indeed, in a narrow analogy with the 1D Cartesian case. Applying the superposition principle, we can write

$$n\left(t,r\right)=\sum_{p=1}^{\infty}\frac{a_p}{r}exp\left[\left(\eta-\mu\frac{p^2\pi^2}{r_1^2}\right)t\right]sin\left(\frac{p\pi}{r_1}r\right) \tag{1.57}$$

Now, the critical condition is simply read as

$$r_{1_{crit}}=p\pi\sqrt{\frac{\mu}{\eta}} \tag{1.58}$$

In the worst case of smallest critical radius, namely $p=1$, we have

$$r_{1_{crit}}=\pi\sqrt{\frac{\mu}{\eta}} \tag{1.59}$$

which is the exact same value we found for L_{crit} in the one-dimensional Cartesian case. However, since the volumes are calculated in a different way (and L_{crit} is $\sqrt{3}$ times bigger in three-dimensional case), we have a significant difference on critical volumes (and so critical masses). Indeed, the critical volumes of a cubic bomb and a spherical bomb respectively are

$$V_{cube_{crit}}=L^3_{cube_{crit}}=\pi^3\left(\frac{3\mu}{\eta}\right)^{\frac{3}{2}} \tag{1.60}$$

$$V_{sphere_{crit}}=\frac{4}{3}\pi r^3_{1cube_{crit}}=\frac{4}{3}\pi^4\left(\frac{3\mu}{\eta}\right)^{\frac{3}{2}} \tag{1.61}$$

Thus, the ratio between the critical volumes and so between the critical masses, is

$$\frac{M_{cube_{crit}}}{M_{sphere_{crit}}}=\frac{V_{cube_{crit}}}{V_{sphere_{crit}}}=\frac{3^{\frac{5}{2}}}{4\pi}\sim 1.24>1 \tag{1.62}$$

which means that a spherical bomb is more efficient than a cubic one, in the sense that we require a less quantity of fissile material to reach the critical condition. Thus, we have understood that the shape of the bomb can play a central role in the making of the atomic bomb.
As final remark, like the previous cases, we can calculate the coefficients a_p through the initial condition; similarly to what we have seen for the 1D Cartesian case, we can calculate them by the following

$$a_p=\frac{2}{r_1}\int_0^{r_1}rn\left(0,r\right)sin\left(\frac{p\pi}{r_1}r\right)dr \tag{1.63}$$

Numerics

We have calculated the critical radius and the critical mass, resulting $r_{1_{crit}}\sim 11.05\ cm$ and $M_{crit}\sim 106\ kg$, which is smaller than 3D-Cartesian case; moreover, we have calculated the ratio between the cubic critical volume and the spherical critical volume, resulting, as shown in (1.62), 1.24. We have decided to use a radius of $r_1=11.5\ cm$.
In the spirit of the previous cases, we have chosen the following initial distribution function, which satisfies the Dirichlet boundary condition

$$f\left(r\right)=\left[1-\left(\frac{r}{r_1}\right)^2\right]$$

We have then calculated the a_p coefficients in two ways, through (1.63): numerically and exactly, since one can show that, performing the integral analytically

$$a_p=12\frac{r_1\left(-1\right)^{1+p}}{p^3\pi^3}$$

The coefficients are illustrated in the following table

Table 4: Values for a_p coefficients

p	a_p	p	a_p
1	0.0445071	16	-1.0866e-05
2	-0.00556339	17	9.05905e-06
3	0.00164841	18	-7.63154e-06
4	-0.000695424	19	6.48886e-06
5	0.000356057	20	-5.56339e-06
6	-0.000206051	21	4.80587e-06
7	0.000129758	22	-4.17986e-06
8	-8.6928e-05	23	3.65802e-06
9	6.10523e-05	24	-3.21955e-06
10	-4.45071e-05	25	2.84846e-06
11	3.34389e-05	26	-2.53227e-06
12	-2.57564e-05	27	2.2612e-06
13	2.02581e-05	28	-2.02747e-06
14	-1.62198e-05	29	1.82488e-06
15	1.31873e-05	30	-1.64841e-06

Again, the coefficient become negligible after a certain value of p. The results are the same for both ways and errors do not come until the eleventh decimal digit, as we have shown in the Table 5, making the difference between the exact and the approximated value. We have then calculated the

Table 5: Error values

p	$a_{p_{numeric}} - a_{p_{exact}}$	p	$a_{p_{numeric}} - a_{p_{exact}}$
1	4.77444e-13	16	-7.60342e-12
2	-9.61217e-13	17	8.07702e-12
3	1.43155e-12	18	-8.55166e-12
4	-1.89512e-12	19	9.02882e-12
5	2.37365e-12	20	-9.50401e-12
6	-2.85387e-12	21	9.97786e-12
7	3.32757e-12	22	-1.04529e-11
8	-3.80086e-12	23	1.09284e-11
9	4.27434e-12	24	-1.14045e-11
10	-4.75155e-12	25	1.18793e-11
11	5.22912e-12	26	-1.23531e-11
12	-5.70172e-12	27	1.28289e-11
13	6.17551e-12	28	-1.3305e-11
14	-6.65215e-12	29	1.37798e-11
15	7.12813e-12	30	-1.42545e-11

neutron density function and plotted it, as shown in the figure 5. Of course, the origin of the explosion coincides with the $r = 0$ point; to make an complete overview, we have extended the plot to negative radius values, as shown in figure 6.

1.3 Cylindrical coordinates

Let us consider a cylindrical object as our fissile material. First of all, the Laplacian in cylindrical coordinates is

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \tag{1.64}$$

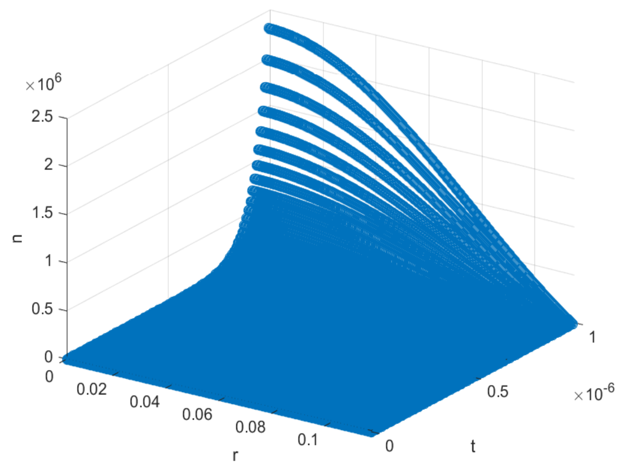


Figure 5: Plot of $n\,(t, r)$

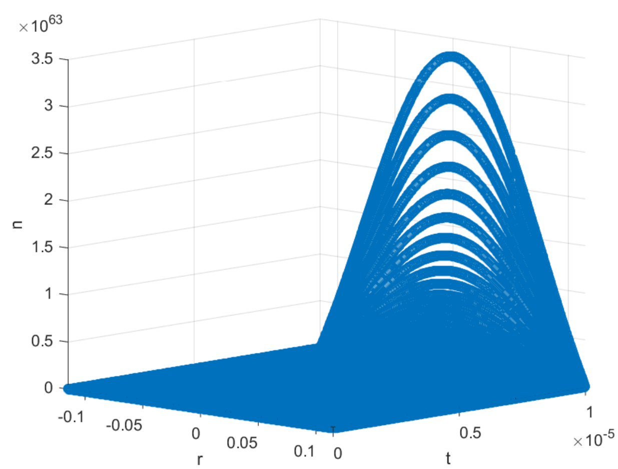


Figure 6: Plot of $n\,(t, r)$ extended

Supposing again that there is no angular dependence, the system (1.4) reduces to

$$\begin{cases} \frac{\partial n}{\partial t} = \mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) n + \mu \frac{\partial^2 n}{\partial z^2} + \eta n \\ n\left(t, r_1, z\right) = n\left(t, r, 0\right) = n\left(t, r, L\right) = 0 \\ n\left(0, r, z\right) = f\left(r, z\right) \end{cases} \tag{1.65}$$

where the domain is $r \in [0, r_1], \varphi \in [0, 2\pi], z \in [0, L]$. Setting

$$n\left(t, r, z\right) = T(t)R(r)Z(z) \tag{1.66}$$

one can find that

$$\frac{1}{T}\frac{\partial T}{\partial t} - \eta = \mu \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \mu \frac{1}{rR}\frac{\partial R}{\partial r} + \mu \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -l \tag{1.67}$$

So, the first ODE to solve is the well known

$$\frac{dT}{dt} = \left(\eta - l\right) T \Rightarrow T = Aexp\left[\left(\eta - l\right) t\right] \tag{1.68}$$

From the other side of equation (1.67), we have

$$\mu \frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \mu \frac{1}{rR}\frac{\partial R}{\partial r} + l = -\mu \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k \tag{1.69}$$

from which we deduce the other two ODEs to solve

$$\begin{cases} \frac{d^2 Z}{dz^2} = -\frac{k}{\mu} Z \\ \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \frac{l-k}{\mu} R = 0 \end{cases} \tag{1.70}$$

The solution of the first equation is very simple

$$Z = Aexp\left(i\sqrt{\frac{k}{\mu}} z\right) + B\left(-i\sqrt{\frac{k}{\mu}} z\right) \tag{1.71}$$

From the boundary conditions, $Z(0) = Z(L) = 0$, we discover first that $A = -B$ and so

$$Z = A\left[exp\left(i\sqrt{\frac{k}{\mu}} z\right) - \left(-i\sqrt{\frac{k}{\mu}} z\right)\right] = 2iAsin\left(\sqrt{\frac{k}{\mu}} z\right) \tag{1.72}$$

and, second, that

$$\sqrt{\frac{k}{\mu}} L = p\pi, p \in \mathbb{Z} \tag{1.73}$$

Thus

$$Z = 2iAsin\left(\frac{p\pi}{L} z\right) \tag{1.74}$$

The solution of the R equation is more difficult; however, there is a way to reconduce it to a well-known differential equation. Indeed, multiplying both sides by r^2 and substituting for the value of k coming from the equation (1.73), $k = \mu \left(\frac{p\pi}{L}\right)^2$, we find

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + r^2\left(\frac{l}{\mu} - \frac{p^2\pi^2}{L^2}\right) R = 0 \tag{1.75}$$

Making the substitution $r = \frac{x}{\sqrt{\frac{l}{\mu} - \frac{p^2\pi^2}{L^2}}}$, we obtain

$$x^2 \frac{d^2 R}{dx^2} + x\frac{dR}{dx} + x^2 R = 0 \tag{1.76}$$

which is the standard form of the Bessel's equation, whose solution is

$$R = AJ_0(x) + BY_0(x) \tag{1.77}$$

where $J_0(x)$ is the zero-order first-type Bessel function and $Y_0(x)$ is the zero-order second-type Bessel function.
Since we want our solution to be finite at the origin $r = 0$, which corresponds to $x = 0$, and because $Y_0(x)$ is singular in this point, we have to require $B = 0$; thus

$$R = AJ_0 \left(r\sqrt{\frac{l}{\mu} - \frac{p^2\pi^2}{L^2}} \right) \tag{1.78}$$

From the boundary condition, $R(r_1) = 0$, we find that

$$r_1\sqrt{\frac{l}{\mu} - \frac{p^2\pi^2}{L^2}} = \alpha_q \tag{1.79}$$

where α_q is the q-th zero of the J_0- Bessel function. Inverting

$$l = \left(\frac{\alpha_q^2}{r_1^2} + \frac{p^2\pi^2}{L^2} \right) \mu \tag{1.80}$$

Therefore, the final solution results, making use of the superposition principle

$$n\left(t,r,z\right) = \sum_{p,q=1}^{\infty} a_{p,q} J\left(\frac{\alpha_q r}{r_1}\right) sin\left(\frac{p\pi}{L}z\right) exp\left[\frac{\eta r_1^2 L^2 - \mu\left(\alpha_q^2 L^2 + p^2\pi^2 r_1^2\right)}{r_1^2 L^2} t\right] \tag{1.81}$$

However, due to the orthogonality condition of the Bessel function, one can find that $a_{p,q} = 0$ for every $p \neq 1$ and so we get the oversemplification

$$n\left(t,r,z\right) = \sum_{q=1}^{\infty} a_{1,q} J\left(\frac{\alpha_q r}{r_1}\right) sin\left(\frac{\pi}{L}z\right) exp\left[\frac{\eta r_1^2 L^2 - \mu\left(\alpha_q^2 L^2 + \pi^2 r_1^2\right)}{r_1^2 L^2} t\right] \tag{1.82}$$

So, we can derive the critical condition for the cylindrical case, which results to be

$$\eta r_{1_{crit}}^2 L_{crit}^2 = \mu\alpha_q^2 L_{crit}^2 + \mu\pi^2 r_{1_{crit}}^2 \tag{1.83}$$

The worst case correspond to $q = 1$, which correspond to

$$\eta r_{1_{crit}}^2 L_{crit}^2 = \mu\alpha_1^2 L_{crit}^2 + \mu\pi^2 r_{1_{crit}}^2 \tag{1.84}$$

Solving for the radius $r_{1_{crit}}$

$$r_{1_{crit}} = \frac{\sqrt{\left(\eta r_{1_{crit}}^2 - \pi^2\mu\right)\mu} L_{crit}\alpha_1}{\eta L_{crit}^2 - \pi^2\mu} \tag{1.85}$$

So, we can express the volume of the critical cylinder as a function of L_{crit}

$$V_{crit} = \frac{\pi\mu L_{crit}^3 \alpha_1}{\eta L_{crit}^2 - \pi^2\mu} \tag{1.86}$$

Since we want to find the minimum lenght above which we have the unstoppable explosion, we can differentiate this volume with respect to the critical lenght and equal to zero, $\frac{dV_{crit}}{dL_{crit}}$, finding that

$$L_{crit} = \pi\sqrt{\frac{3\mu}{\eta}} = L_{crit_{Cartesian3D}} \tag{1.87}$$

and

$$r_{1_{crit}} = \alpha_1\sqrt{\frac{3\mu}{2\eta}} \tag{1.88}$$

Comparing the critical volume of the cylinder

$$V_{crit} = \pi r_{1_{crit}}^2 L_{crit} \tag{1.89}$$

with that of the sphere, we find that

$$\frac{M_{cylinder_{crit}}}{M_{sphere_{crit}}} = \frac{V_{cylinder_{crit}}}{V_{sphere_{crit}}} \sim 1.14 \tag{1.90}$$

Thus, the cylindrical bomb requires less fissile material than a cubic one but more than a spherical one. Notice, however, that for some pratical matters, the *Little Boy* bomb was cylindrical, so not always the choice falls on the "best shape" bomb.

Numerics

The critical lenght is the same of the 3D Cartesian case, $L_{crit} \sim 19.14\ cm$; in order to calculate the critical radius, through the (1.88), we have first calculated, by the bisection method, the first 10 zeros of the zero-order Bessel function of the first kind, shown in the Table 6. Then, we have calculated the

Table 6: First 10 zeros of J_0

q	α_q
1	2.40483
2	5.52008
3	8.65373
4	11.7915
5	14.9309
6	18.0711
7	21.2116
8	24.3525
9	27.4934
10	30.6346

critical radius and so, the critical mass, resulting respectively $r_{1_{crit}} \sim 10.36\ cm$ and $M_{crit} \sim 121\ kg$. We have also tested the (1.90), founding that the ratio between the cylindrical critical volume and the spherical critical volume is effectively around 1.14. We have then chosen the following values for the radius and the lenght: $r_1 = 10.4\ cm$ and $L = 19.2\ cm$.
As the initial distribution function, we have taken in consideration the following

$$f\left(r,z\right) = \left(1 - \frac{r^2}{r_1^2}\right) sin\left(\frac{\pi z}{L}\right)$$

which satisfies the Dirichlet boundary conditions. One can show, considering the orthogonality properties of the Bessel functions, that the coefficients can be calculated as

$$a_{1,q} = \frac{4\left[\alpha_q J_0\left(\alpha_q\right) + 2J_1\left(\alpha_q\right)\right]}{J_1\left(\alpha_q\right)^2 \alpha_q^3}$$

where J_1 is the first-order Bessel function of the first kind. The result of this calculation are found in the Table 7.

Table 7: Values for $a_{1,q}$ coefficients

q	$a_{1,q}$
1	1.10802
2	-0.139778
3	0.0454766
4	-0.0209909
5	0.0116363
6	-0.00722119
7	0.0048379
8	-0.00342569
9	0.00252985
10	-0.00193141

In the end, we have, as usual, calculated the neutron density at fixed time and plotted it, both for only positive radius values and for negative and positive radius values. The graphs are shown in Figure 7 and Figure 8, respectively.

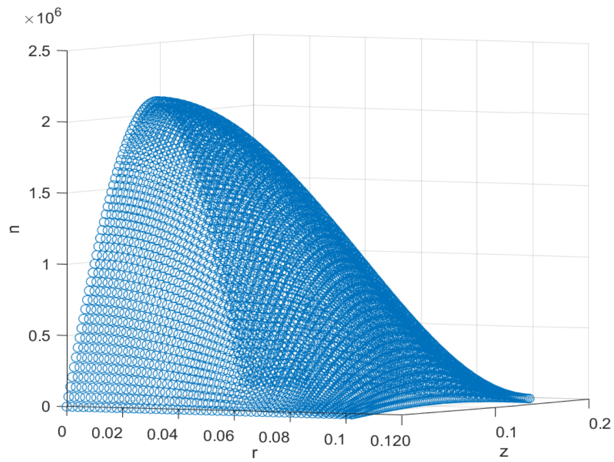


Figure 7: Plot of $n\,(r, z)$

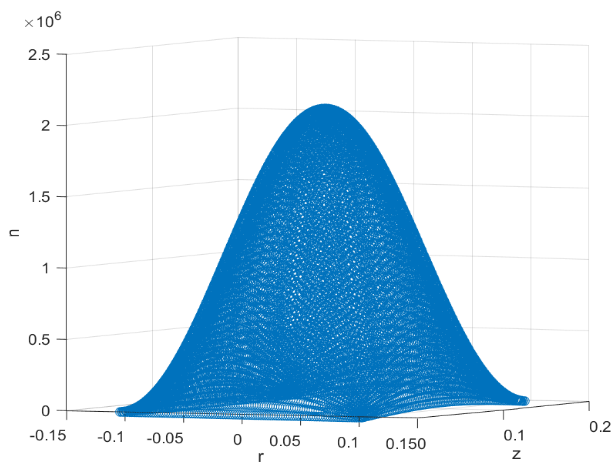


Figure 8: Plot of $n\,(r, z)$ with radius negative values

1.4 Neumann boundary condition

Hitherto, we have developed models of neutron diffusion based on the fact that, once arrived at the edges of the fissile material, neutrons escape simply disappearing; to this aim, we have set the so called Dirichlet boundary conditions.
A more realistic model, however, cannot allow that neutrons disappearing but we expect a "smoother" behaviour at the bounds: as the neutrons approach the borders, the number of neutrons, in these border points, goes through decreasing values but does not simply vanish. In other words, the neutrons escaping will not contribute anymore to the generation of secondary neutrons by fission but their presence will be a very important factor in the reaching of the critical mass and so, of the explosion. Thus, what we expect, setting a new set of boundary conditions, the so called Neumann boundary conditions, that the critical mass that a bomb has to reach is less than the value calculated using Dirichlet conditions.

From now on, we'll use spherical coordinates; the system (1.48) has to be modified into the following

$$\begin{cases} \frac{\partial n}{\partial t} = \mu\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)n + \eta n \\ \frac{dn(t,r_1)}{dr} = -\frac{3}{2}\frac{n(t,r_1)}{\lambda_t} \\ n\left(0,r\right) = f\left(r\right) \end{cases} \tag{1.91}$$

where λ_t is the transport free path. Notice how the new boundary conditions allow the escaping neutrons to not contribute anymore to fission but at the same time they cannot magically disappear. Of course, the solution of the PDE is the same of before; however, for some useful reasons, we put the η constant to the R equation. From (1.50)

$$\frac{1}{T}\frac{\partial T}{\partial t} = \mu\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \mu\frac{2}{rR}\frac{\partial R}{\partial r} + \eta = -\alpha \tag{1.92}$$

So, we have the following two ODEs to solve

$$\begin{cases} \frac{dT}{dt} = -\alpha T \\ \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{\eta+\alpha}{\mu}R = 0 \end{cases} \tag{1.93}$$

The solutions are similar to (1.52) and (1.53), with the proper constants

$$\begin{cases} T(t) = Aexp\left(-\alpha t\right) \\ R(r) = \frac{1}{r}\left[Bsin\left(kr\right) + Ccos\left(kr\right)\right] \end{cases} \tag{1.94}$$

where $k = \sqrt{\frac{\eta+\alpha}{\mu}}$. Requiring continuity at the origin, we have $C = 0$, so we can write

$$R(r) = \frac{B}{r}sin\left(kr\right) \tag{1.95}$$

Let us now apply the boundary condition, $\frac{dR(r_1)}{dr} = -\frac{3}{2}\frac{R(r_1)}{\lambda_t}$; derivating with respect to r and substituting $r = r_1$, we obtain that

$$-\frac{B}{r_1^2}sin\left(kr_1\right) + \frac{Bk}{r_1}cos\left(kr_1\right) = -\frac{3}{2}\frac{B}{r_1\lambda_t}sin\left(kr_1\right) \tag{1.96}$$

Rearranging, we obtain the following equation

$$-1 + kr_1cot\left(kr_1\right) + \frac{3}{2}\frac{r_1}{\lambda_t} = 0 \tag{1.97}$$

The critical value, as we have learnt, is that value of the radius for which the T function becomes unbounded for $t \to +\infty$; in this case, the critical condition simply reads $\alpha = 0$ which allows us to determine the critical radius directly by equation (1.97), where $k = \sqrt{\frac{\eta+\alpha}{\mu}}|_{\alpha=0} = \sqrt{\frac{\eta}{\mu}}$

$$-1 + \sqrt{\frac{\eta}{\mu}}r_1cot\left(\sqrt{\frac{\eta}{\mu}}r_1\right) + \frac{3}{2}\frac{r_1}{\lambda_t} = 0 \tag{1.98}$$

The last equation, solved numerically (since it is a transcendental equation), gives us the value of the critical radius. In terms of the critical mass one can show that, while for the spherical Dirichlet case we have obtained a critical mass of around $106kg$, here we obtain a value much smaller, around $46kg$.This means that our expectations were right and the more realistic model which makes use of Neumann boundary conditions predicts a critical mass much smaller than that predicted by a model which uses Dirichlet boundary conditions.

Finally, concerning the value of the constant, we can set a different initial condition; as we have noticed, if we use a radius which is bigger than the critical radius, no matter what particular form $f(r)$ has, since the $n(t,r)$ is destinated to increase very rapidly. So, we can choose a particular initial condition at the boundary in which the neutron density is a well-known constant, like $n(0,r_1) = 1$, from which we derive that

$$n\left(t,r\right) = \frac{r_1}{sin\left(kr_1\right)}exp\left(-\alpha t\right)\frac{sin\left(kr\right)}{r} \tag{1.99}$$

Numerics

In order to find the critical radius, we have solved numerically the equation (1.98); the method used is the bisection one (a trial with a different method, Newton-Raphson method, has been done but it was unsuccessful because of the high value of the derivatives in the extrema) and it gives the following critical radius and a critical mass: $r_{1_{crit}} \sim 8.36\ cm$ and $M_{crit} \sim 45.9\ kg$. Notice how these values are much smaller than the analogous values with the Dirichlet boundary conditions. We decided to consider as value for the radius $r_1 = 8.5\ cm$.
Then, to calculate and plot the neutron density function, we have first determined, again with bisection method, the values of k and α, inserting the used value of r_1 in the (1.97); therefore, we have plotted both neutron density functions, with only positive and with any values of radius, as we show in the Figure 9 and the Figure 10, respectively. Notice the principal difference with the Dirichlet case: as we can see, even at the boundaries the neutron density (and so the number of neutrons) increases with time, unlike the Dirichlet case, in which, at the boundaries, the neutrons are always zero (they disappear when escaping). So, we can consider the Neumann case as a more realistic model. Moreover,

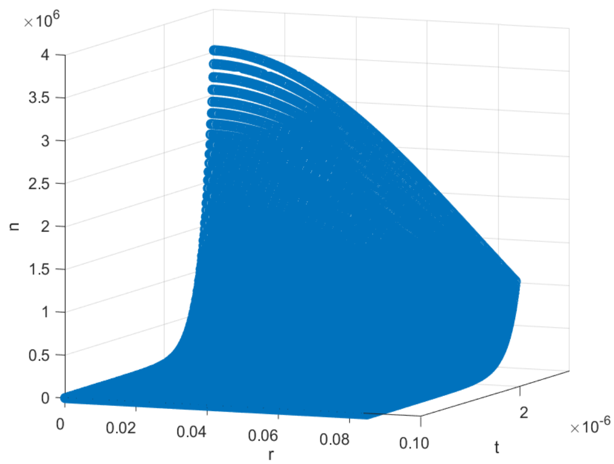


Figure 9: Plot of $n\,(t, r)$

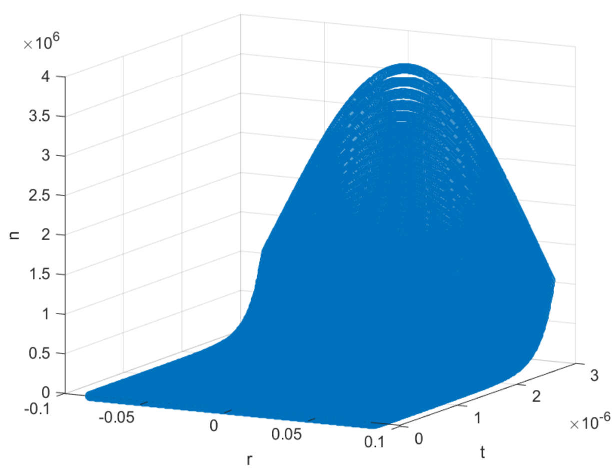


Figure 10: Plot of $n\,(t, r)$ extended

we have also plotted, for any value of the radius, the initial neutron density function, as shown in the Figure 11.

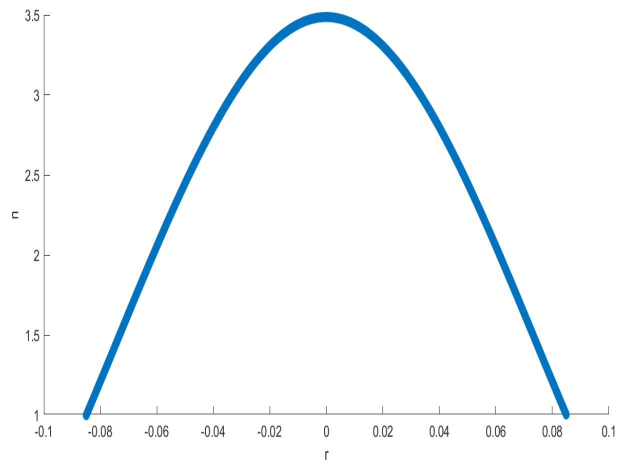


Figure 11: Plot of $n\left(r\right)$ at $t=0$

2 The effect of a Tamper

The fissile material has a huge cost; as we have seen, even in the best model we have analyzed, the spherical bomb with Neumann boundary conditions, the critical mass is around $46kg$, which is a very big amount of fissile material. The solution to this problem can be the addition of a **Tamper**, that is a non-fissile material surrounding the fissile material (that, from now on, we will call **Core**). The effect of a such material, which has a very smaller cost than fissile one, is to reduce the value of the critical mass needed to make the explosion: this occurs because the tamper reflect back some of those escaping neutrons, causing a minor loss of neutrons (for this reason, a modern name of the tamper is "reflector").

Thus, while the core satisfies the seen diffusion equation with the source term, the tamper satisfies the simple diffusion equation, without the source term, since in non-fissile materials fission does not occur and so no secondary neutrons are generated. So, we will distinguish between the neutron density of the core and the neutron density of the tamper, n_{core} and n_{tamper} respectively, which satisfy two slightly different partial differential equations; considering again spherical coordinates, we have already solved the equation for n_{core}; about n_{tamper}, the associated equation is

$$\frac{\partial n_{tamper}}{\partial t}=\mu_{tamper}\nabla^2 n_{tamper} \tag{2.1}$$

Notice that we are assuming that the tamper does not capture neutrons, but these can simply move through the material.

In order to solve the last equation, we have to proceed like in the other cases, through the variable separation method, but now it is convenient to make explicit the constants defined through the equations (1.5) and (1.6); moreover, the constant α which in the previous cases had the dimension of the reciprocal of time, is redefined here as $-\alpha\rightarrow\frac{\alpha}{\tau}$, where τ is the mean time that a neutron can travel in the core before causing a fission, defined as

$$\tau=\frac{\lambda_f^{core}}{v_{neut}} \tag{2.2}$$

so the new constant α is dimensionless. Separating variables, the equation for the tamper neutron density is

$$\frac{1}{T_{tamper}}\frac{\partial T_{tamper}}{\partial t}=\mu_{tamper}\frac{1}{R_{tamper}}\frac{\partial^2 R_{tamper}}{\partial r^2}+\mu_{tamper}\frac{2}{rR_{tamper}}\frac{\partial R_{tamper}}{\partial r}=-\frac{\delta}{\tau} \tag{2.3}$$

where remind that $\mu_{tamper}=\frac{\lambda_t^{tamper}v_{neut}}{3}$. Solving this equation is even simpler because we have not a source term; so, one can show that the solution is

$$n_{tamper}=\begin{cases}\frac{A}{r}+B,\delta=0\\ e^{\frac{\delta}{\tau}t}\left(A\frac{e^{\frac{r}{d_{tamper}}}}{r}+B\frac{e^{-\frac{r}{d_{tamper}}}}{r}\right),\delta>0\end{cases} \tag{2.4}$$

where $d_{tamper} = \sqrt{\frac{\lambda_t^{tamper}\lambda_f^{core}}{3\delta}}$. On the other hand, the neutron density of the core, making explicit all constants, is

$$n_{core} = A_{core} e^{\frac{\alpha}{\tau}t} \frac{sin\left(\frac{r}{d_{core}}\right)}{r} \tag{2.5}$$

where $d_{core} = \sqrt{\frac{\lambda_t^{core}\lambda_f^{core}}{3(\nu-1-\alpha)}}$.
Now, about boundary condition, let us name the radius of the core r_{core} and the total radius with the tamper r_{tamper}, as shown in the following figure

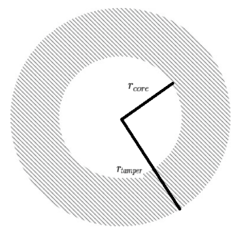


Figure 12: Schematic illustration of a tamped bomb core

First of all, on the core/tamper interface, we can expect continuity, if no neutrons are created or lost on it, so we can set

$$n_{core}(r_{core}) = n_{tamper}(r_{core}) \tag{2.6}$$

The second condition to apply is on the edges of the bomb; we can set at these points the Neumann boundary conditions as follows

$$n_{tamper}(r_{tamper}) = -\frac{2}{3}\lambda_t^{tamper} \frac{\partial n_{tamper}(r_{tamper})}{\partial r} \tag{2.7}$$

Applying (2.6) and (2.7), first of all we deduce that, in the end, $\alpha = \delta$; moreover, we get the critical condition. One can show that, if we restrict ourselves to $\delta = \alpha = 0$ case, the critical condition can be written as

$$\left(1 + \frac{2r_{crit}\lambda_t^{tamper}}{3r_{tamper}^2} - \frac{r_{crit}}{r_{tamper}}\right)\left[\frac{r_{crit}}{d_{core}} cot\left(\frac{r_{crit}}{d_{core}}\right) - 1\right] + \frac{\lambda_t^{tamper}}{\lambda_t^{core}} = 0 \tag{2.8}$$

We can use this equation in two different ways:

1. if we want to know the critical radius (and so the critical mass of the core) given the size of the tamper or of the entire bomb r_{tamper}, we can solve it for r_{crit};

2. if we want to know how much non-fissile material is needed to have a desired critical radius r_{crit}, we can solve for r_{tamper}.

Despite the difficulty in treating this critical condition, which has to be solved numerically, we can see what happens in some special cases, for understanding how the presence of the tamper can reduce the critical radius value. The special case we consider is supposing that the tamper is very thick, so that $r_{tamper} >> r_{crit}$; in this case, equation (2.8) is simplified to

$$\frac{r_{crit}}{d_{core}} cot\left(\frac{r_{crit}}{d_{core}}\right) = 1 - \frac{\lambda_t^{tamper}}{\lambda_t^{core}} \tag{2.9}$$

Let's consider two sub-cases

- the tamper is the vacuum itself: in this case, we have a total cross section of $\sigma_{tamper} = 0$ because there is no reaction of neutron scattering and so $\lambda_t^{tamper} = \frac{1}{n\sigma_{tamper}} \to \infty$ (see Appendix A), since $\lambda_t^{tamper} \to \infty$, the critical condition becomes

$$\frac{r_{crit}}{d_{core}} cot\left(\frac{r_{crit}}{d_{core}}\right) = -\infty \tag{2.10}$$

whose solution is

$$\frac{r_{crit}}{d_{core}} = \pi \Rightarrow r_{crit} = \pi d_{core} \tag{2.11}$$

- a more realistic case with $\lambda_t^{tamper} \neq 0$ but finite, assuming $\lambda_t^{tamper} \sim \lambda_t^{core}$: in this case, we are assuming that the neutron scattering properties of the tamper are similar to those of the core; the critical condition becomes

$$\frac{r_{crit}}{d_{core}} cot\left(\frac{r_{crit}}{d_{core}}\right) = 0 \tag{2.12}$$

whose solution is

$$\frac{r_{crit}}{d_{core}} = \frac{\pi}{2} \Rightarrow r_{crit} = \frac{\pi}{2} d_{core} \tag{2.13}$$

The first sub-case is analogous to the previous problems in which there wasn't the tamper and for which we have already calculated the critical radius (and critical mass). Thus, what we notice is that

$$(r_{crit})_{infinitelythicktamper} = \frac{1}{2}(r_{crit})_{notamper} \tag{2.14}$$

So, in a first approximation, the tamper has the effect of reducing the critical radius by a factor of 2, with respect to the case in which we have no tamper: this translates in the fact that, the critical volume and so, the critical mass, is reduced by a factor of $2^3 = 8$ with respect to the case of no tamper. Thus, the tamper, which is a non-fissile material and so it is cheaper than fissile material, has the effect of contribute positively to the lowering of the critical mass in the making of the nuclear bomb, making so possible to save money and resources.

Numerics

As the Tamper material, we have considered what it was used in the Hiroshima *Little Boy*, the tungsten-carbide (WC). The tungsten has five isotopes W^{180}, W^{182}, W^{183}, W^{184} and W^{186}, with abundances 0.0012, 0.265, 0.1431, 0.3064, and 0.2843, respectively. The reference [3] gives elastic-scattering cross sections for the four most abundant of these as (in order of increasing weight) 4.369, 3.914, 4.253, and 4.253 barns (the W^{180} is not taken in consideration because of its small abundance). The abundance-weighted average of these is 4.235 barns. Adding the 2.352 barns elastic-scattering cross-section for C^{12} gives a total of 6.587 barns. The total atomic weight results 183.84 + 12.0=195.84 $\frac{g}{mol}$ and the mass density is 14800 $\frac{kg}{m^3}$.
Thus, we have calculated first the value of the new critical mass and tamper mass (and its thickness) with the adding of this kind of tamper, with a total radius of the bomb $r_{tamper} = 11.7\ cm$, resulting that: $M_{crit} \sim 25.1\ kg$, $M_{tamper} \sim 79.5\ kg$ and $L_{tamper} \sim 4.9\ cm$. Notice how much less is this critical mass with respect to the no-tamper case: indeed, tungsten-carbide material is much cheaper than uranium, so one can save a lot of money considering tampers.
As final computation, we have made a plot in which one can analyze the dependence between the tamper mass and the critical mass, as shown in Figure 13. To do this, we have explicitated, in the equation (2.8), the r_{tamper} as a function of r_{crit}, finding that

$$r_{tamper} = \frac{-3 - \sqrt{9 + 24j\lambda_t^{tamper}}}{6j}$$

where

$$j = \frac{tan\left(\frac{r_{crit}}{d_{core}}\right)}{tan\left(\frac{r_{crit}}{d_{core}}\right) - \frac{r_{crit}}{d_{core}}} \frac{\lambda_t^{tamper}}{\lambda_t^{core} r_{crit}} - \frac{1}{r_{crit}}$$

We first note that, for M_{tamper} approaching to zero, we obtain the same critical mass we have obtained in the case of no tamper (about 46 kg); moreover, we must note also that the adding of tamper has its limits, in the sense we cannot reduce the critical mass as we want but there is a certain point in which, even adding more and more tamper material, the critical mass remains nearly the same: this behaviour is induced by the asymptote belonging to the cotangent function in equation (2.8). Furthermore, if we go before the value of this asymptote, the resulting plot is a negative function, as one can see taking the minus sign of the square root in the r_{tamper} equation, which means that adding tamper material is even worse and not physically feasible. Thus, we can certainly save money but surely we cannot build a bomb for free; of course, one can try to find other tamper materials (and core ones) in order to lower further the needed critical mass.

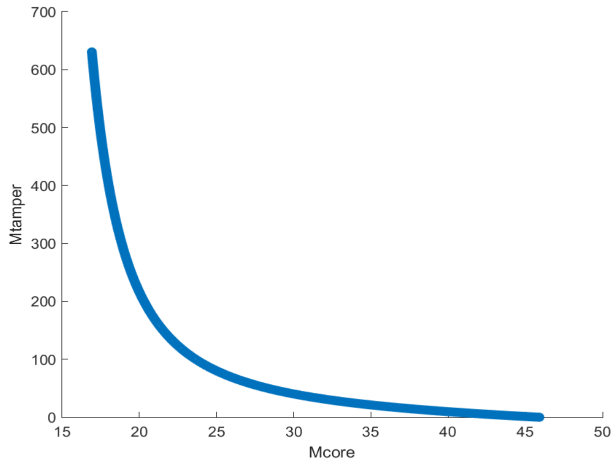


Figure 13: Plot of $M_{tamper}\,(M_{core})$

3 Energy released and Efficiency of a bomb

A complete fission of 1 kg of U^{235} liberates an energy of 17 kt (kilotons), equivalent to the explosion of 17000 *tons* of TNT; however, *Little Boy* was about 64 kg but liberated "only" 13 kt, so the bomb was, in a certain sense, inefficient. Thus, our final goal, is to estimate the efficiency of a bomb; for the sake of simplicity, we consider an untamped core.
The efficiency of a nuclear weapon involves three different time scales. The first is mechanical in nature: the time required to assemble the sub-critical components into a critical assembly before fission is initiated. Estimate this kind of time is not too easy because of the occurrence of the spontaneous fission, which can give some troubles like pre-detonation. For example, the *Little Boy* bomb was cylindrical in order to control it like a gun: a projectile of fissile material is fired inside a barrel toward a target of other fissile material, just before the release. So, if we would make an esteem of the assembly time, considering a projectile of 10 cm fired at the speed of 10^3 m/s, then the time required from the bomb to be fully engaged is 100 μs.
The spontaneous fission was not a real problem for *Little Boy* bomb, because it was made of Uranium, which has a low probability of spontaneous fission, but it was for *Trinity* and *Fat Man*, that were made of Plutonium, which is known to have a very high probability to have a spontaneous fission.
For a first treatment, let us ignore this kind of time and concentrate on the other two.
The first of these is nuclear in nature: it is the time required for all of the fissile material to be consumed; we call it $t_{fission}$.
The second is again mechanical: when the fission begins, the core starts to expand itself due to the gas pressure of the fission fragments and this expansion will lead to a loss of criticality after a time which we can call $t_{criticality}$, in which the reaction rate will diminish. In other words, we have not considered, in our previous treatments, that the overcoming of critical radius will lead to an exponentially increasing number of neutrons, leading to the explosion, but, at the same time, this growth change the size of the core and so it will arrive at a certain point in which there is no more overcoming of critical radius (and so critical mass), stopping the exponential reaction chain.
Thus, in evaluating the efficiency of a nuclear weapon, we must compare these two times; of course, if $t_{criticality} > t_{fission}$, the core material will be totally "burnt" before of the loss of criticality and so the efficiency would be 100%. However, it is usual that the loss of criticality occurs so much sooner than the whole core undergoes fission, so we have to estimate these two times and compare them.
Now, first estimate $t_{fission}$; on average, we know, by definition, that a neutron will cause another fission after a time $\tau = \frac{\lambda_f}{v_{neut}}$; so, the rate of fission per neutron is naturally defined as $\frac{1}{\tau}$. The density of the neutrons at a certain time t, as a function of only time, is of course given by (1.94) with $\alpha \to -\frac{\alpha}{\tau}$

$$N(t) = N_0 e^{\frac{\alpha}{\tau}t} \tag{3.1}$$

The number of neutrons in a certain volume is of course the neutron density multiplied by the volume

$$\#(t) = N_0 V e^{\frac{\alpha}{\tau}t} \tag{3.2}$$

Thus, the total number of fission per second is nothing else than

$$fissions/s = \frac{\#(t)}{\tau} = \frac{N_0 V}{\tau} e^{\frac{\alpha}{\tau}t} \tag{3.3}$$

N_0 is of course the initial value of the neutron density. It is now important to remind now, however, that since the core is expanding, α and τ are now functions of time; for the sake of simplicity, we continue to consider them constant, even if this would result in an overesteem of the efficiency.
The total number of nuclei within the core can be so calculated by integrating the (3.3) between 0 and $t_{fission}$

$$nV = \frac{N_0 V}{\tau} \int_0^{t_{fission}} e^{\frac{\alpha}{\tau}t} dt = \frac{N_0 V}{\alpha} \left(e^{\frac{\alpha}{\tau}t_{fission}} - 1 \right) \tag{3.4}$$

Since $e^{\frac{\alpha}{\tau}t_{fission}} >> 1$, we can invert the last equation to obtain the fission time

$$t_{fission} = \frac{\tau}{\alpha} log \left(\frac{\alpha n}{N_0} \right) \tag{3.5}$$

Now, let us occupy of the critical time; this time is due to the expansion of the core which lead the criticality condition to shutdown at a certain point: this point is also called **second criticality**.
Assume we have C untamped and critical masses; to satisfy the critical condition we need of course $C > 1$ critical masses. The initial radius of the core is so

$$r_i = C^{\frac{1}{3}} r_{crit} \tag{3.6}$$

This is the initial value of the core before its expansion.
Let us now define the energy released by fission; if each fission liberates an energy E_f, then the rate of energy liberation throughout the entire volume of the core will be

$$\frac{dE}{dt} = \frac{N_0 V E_f}{\tau} e^{\frac{\alpha}{\tau} t} \tag{3.7}$$

Integrating from 0 to some time t, we get the energy released at time t

$$E(t) = \frac{N_0 V E_f}{\tau} \int_0^t dt e^{\frac{\alpha}{\tau} t} \sim \frac{N_0 V E_f}{\alpha} e^{\frac{\alpha}{\tau} t} \tag{3.8}$$

In order to determine the pressure within the core, we use thermodynamics and, in particular the equation of state $P(t) = \gamma U(t)$, where $U(t) = \frac{E(t)}{V}$ and γ can be either $\frac{2}{3}$, if we have a classical gas pressure, or $\frac{1}{3}$, if we have a radiation (ultra-relativistic) pressure. Thus, we can write

$$P(t) = \frac{\gamma N_0 E_f}{\alpha} e^{\frac{\alpha}{\tau} t} \equiv P_0 e^{\frac{\alpha}{\tau} t} \tag{3.9}$$

The question of whether of the two possible values of γ we have to choose is easily answered: indeed, since the expansion is in reality an intense and fast explosion, which gives raise to γ-rays and X-rays, we are clearly in presence of a gas of photons/radiation and so we must use $\gamma = \frac{1}{3}$.
Now, in order to model the expanding core, we assume a radial expansion, so the radius of the core is a function of time $r(t)$ with r_i as its initial values; the velocity expansion, $v(t)$, which is not v_{neut}, is the speed at which every atom is moved outwards. Moreover, whereas the mass of the core remains always the same and it is equal to CM_{crit}, the mass density is a function of time $\rho(t)$, since the volume of the core is in expansion. Then, the total kinetic energy is

$$K_{core} = \frac{1}{2} M v^2 = \frac{1}{2}\frac{4}{3}\pi r^3 \rho v^2 = \frac{2\pi}{3}\rho v^2 r^3 \tag{3.10}$$

By the Work-Energy Theorem, we know that $P(t)dV = W = dK$ and so

$$P(t)\frac{dV}{dt} = \frac{dK_{core}}{dt} \tag{3.11}$$

Since

$$\frac{dK_{core}}{dt} = \frac{2\pi}{3}\rho r^3 2v \frac{dv}{dt} \tag{3.12}$$

and since

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \tag{3.13}$$

then, using the equation (3.9)

$$\frac{dv}{dt} = \frac{3P_0}{\rho r} e^{\frac{\alpha}{\tau} t} \tag{3.14}$$

The solution of this equation is not trivial, since ρ and r are actually functions of time, so we have to face the ρr product in that denominator and use conveniently an approximation.
First of all, as the core expands, its (mass) density in terms of a general radius r will be

$$\rho(r) = C\rho_0 \left(\frac{r_{crit}}{r}\right)^3 \tag{3.15}$$

in order that it is a decreasing function for values of $r(t) > r_{crit}$.
The critical condition holds until $\rho r(t) = \rho_0 r_{crit}$, which is the condition of the reaching of the second criticality; substituting the value of ρ in the equation (3.15), we get the radius of the second criticality

$$r_{secondcriticality} = C^{\frac{1}{2}} r_{crit} \tag{3.16}$$

Therefore, we can define the range of radius in which the criticality condition holds, as

$$\Delta r = r_{secondcriticality} - r_i = \left(C^{\frac{1}{2}} - C^{\frac{1}{3}}\right) r_{crit} \tag{3.17}$$

Since the initial and the second criticality values of ρr do not differ very greatly, we can consider this product of time function as a constant and precisely as the mean value between these two values

$$\rho r \to \langle \rho r \rangle = \frac{1}{2}\left(1 + C^{\frac{1}{3}}\right) \rho_0 r_{crit} \tag{3.18}$$

Now, the equation (3.14) is easily solved

$$v(t) = \frac{3P_0}{\langle \rho r \rangle} \int_0^t dt e^{\frac{\alpha}{\tau}t} \sim \frac{3P_0 \tau}{\langle \rho r \rangle \alpha} e^{\frac{\alpha}{\tau}t} \tag{3.19}$$

Now, we have all the elements to compute $t_{criticality}$; since $v(t) = \frac{dr(t)}{dt}$, we can integrate again, obtaining $r(t)$ and then, since $r(t_{criticality}) = \Delta r$, we have just to invert, getting

$$t_{criticality} \sim \frac{\tau}{\alpha} log\left(\frac{\Delta r \alpha^2 \langle \rho r \rangle}{3P_0 \tau^2}\right) = \frac{\tau}{\alpha} log\left(\frac{\Delta r \alpha^3 \langle \rho r \rangle}{3\gamma \tau^2 N_0 E_f}\right) \tag{3.20}$$

Firstly, we notice that both $t_{fission}$ and $t_{criticality}$ depend on N_0, even if they are not highly sensitive to its value.
In order to define the efficiency, we can think at the ratio of the energy released after $t_{criticality}$ and the energy released if the whole fissile material is consumed, namely after $t_{fission}$. The first of these two energies is also called the Yield and through the equation(3.8) and various substitutions, we can write it as

$$Y = \frac{N_0 V E_f}{\alpha} e^{\frac{\alpha}{\tau} t_{criticality}} = \frac{\Delta r \alpha^2 \langle \rho r \rangle V}{3\gamma \tau^2} \tag{3.21}$$

Since the energy released by a complete fission is by definition $E_f nV$, then the efficiency of a nuclear weapon can be calculated as

$$Efficiency = \frac{Y}{E_f nV} = \frac{\Delta r \alpha^2 \langle \rho r \rangle}{3\gamma n \tau^2 E_f} \tag{3.22}$$

Notice that both the Yield and the Efficiency do not depend on the initial neutron density.
Another justification of the $\gamma = \frac{1}{3}$ choice comes from the efficiency. Consider the energy per nucleus at time $t_{criticality}$; it clearly is equal to $\frac{Y}{nV} = Efficiency \cdot E_f$. So, even if we had a bad bomb, say with an efficiency of 0.1%, because $E_f \sim 180\ MeV$, then the energy per nucleus at time $t_{criticality}$ is about 180 keV, that is much more than 2 keV, over which the radiation pressure dominates gas pressure. As final useful formulae, one can obtain, by simple substitution, the speed and the pressure of the expansion at time $t_{criticality}$

$$v\left(t_{criticality}\right) = \frac{\alpha \Delta r}{\tau} \tag{3.23}$$

$$P\left(t_{criticality}\right) = \frac{\alpha^2 \Delta r \langle \rho r \rangle}{3\tau^2} \tag{3.24}$$

Curiously, the pressure at this time does not depend on the value of γ.
As a latter discussion, we can repeat the same calculations that Fermi did on 16th July 1945, when at 5:30 a.m. *Trinity* test was started. As we can read from an unclassified document (Appendix B), while other scientist and collaborators of Manhattan project were being shocked by the incredible blast generated by the first ever artificial nuclear explosion, Enrico Fermi tried to calculate the effective energy blast by measuring the displacement of some small pieces of paper dropped, caused by the blast wave, ensuring that there was no wind. He was positioned in a tower far about 10 miles (16 km) from the explosion and dropped the pieces of paper from about 6 feet (about 1.83 m) and, since the measured displacement was of about 2.5 m, he estimated a blast of about 10 kt. Trying to repeat the possible calculations that Fermi maybe did, we can suppose that the desired energy blast, corresponding to the Yield, since criticality is lost after $t_{criticality}$, can be simply determined by the energy conservation; in other words, we can suppose that the energy blast is transformed in the kinetic energy of the moving air, as follow

$$Y_F = \frac{1}{2} M_{air} v_{air}^2 \tag{3.25}$$

Of course, it will arrive a certain point in which, because M_{air} depends on the distance from the explosion, the kinetic energy has completely "consumed" the Yield. Since the pieces of paper perform the projectile motion and assuming a pure horizontal velocity (which can be good at so far distances), we can relate the displacement d to v_{air} through

$$d = \sqrt{\frac{2h}{g}} v_{air} \tag{3.26}$$

where $h \sim 1.83\ m$ and $g = 9.8\ m/s^2$. About M_{air}, we can write

$$M_{air} = \frac{1}{2}\rho_{air} V_{air} = \frac{2}{3}\pi L^3 \rho_{air} \tag{3.27}$$

where $L \sim 16\ km$ and ρ_{air} is the density of the air and the one half factor indicates the fact that the blast propagates in an emispherical way (we neglect also the energy loss with obstacles like the ground). However, the value of ρ_{air} could be low than the standard value because of the very high temperature induced by the explosion, so I arbitrarily assume that it can be between 0 kg/m^3 and the characteristic value at ambient temperature, i.e. 1.2 kg/m^3, so I choose the value in the middle $\rho_{air} \sim 0.6\ kg/m^3$, which correspond to the order of some hundreds of Kelvin.
Entering these values, we derive an energy blast of the order of ten thousands of kilotons, which is in agreement with Fermi's calculation.
However, we must alert that this value is far from the real value, which is bigger; indeed, we have neglected in the equation (3.25) the energy loss by thermal heating, so we would have had

$$Y_F = \frac{1}{2} M_{air} v_{air}^2 + T.H. \tag{3.28}$$

so, in reality, the Yield results more than that calculated through equation (3.25).

Numerics

We have copied the result obtained in Neumann spherical case, in particular, the critical radius of the core. Then, we have calculated the Efficiency in two different ways: first, we have expressed it as a function of the parameter C and, secondly, we have restricted ourselves to a particular value of it. In the first case, so, we have obtained a plot of how efficiency varies with the parameter C, as shown in the Figure 14.

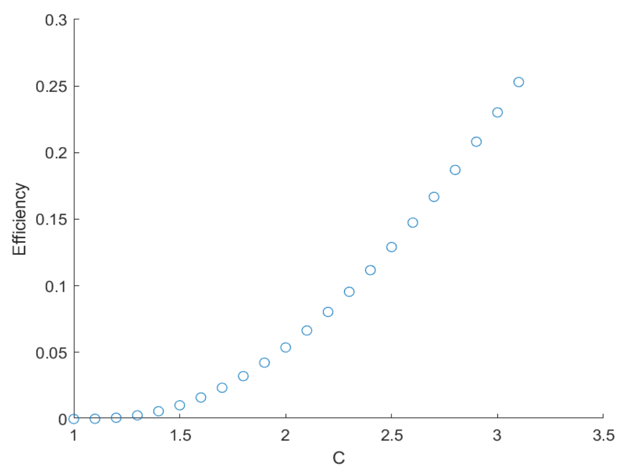


Figure 14: Plot of Efficiency(C)

We can note how the efficiency increases if we add more and more mass over the critical value; however, this have its own limits of course: indeed, the value of C influences the value of k and so α, determined with a transcendental equation, involving the cotangent function; the result is that C cannot be anything and so the efficiency cannot be increased at will.
Then, we have chosen a particular value of C, $C = 1.5$, and calculated all the relevant quantities, as shown in the Table 8.

Table 8: Relevant values for $C = 1.5$

Quantity	Physical meaning	Value	Unit
r_{crit}	Critical radius	8.36311	cm
r_i	Initial radius	9.57338	cm
$r_{secondcriticality}$	Radius at crit. shutdown	10.2427	cm
Δr	Expansion radius	0.669305	cm
Efficiency	Efficiency	1.02404	%
$P\left(t_{criticality}\right)$	Pressure at crit. shutdown	4.71901e+15	Pa
Y	Yield (Energy released at crit. shutdown)	15.2304	kt
$t_{fission}$	Time needed to fission all nuclei	1.66884e-06	s
$t_{criticality}$	Time to crit. shutdown	1.5398e-06	s
Y_F	Hypothetical Yield (Fermi)	10.4858	kt

Conclusions

We have outlined a first insight of the physics behind the making of a nuclear weapon.
We have seen how the fission, exceeded a certain value of fissile material mass, can generate an intense explosion, by a simple accretion of neutron density. We have analyzed ways to reduce this value, due to the high cost of fissile materials, seeing how the shape and the tamper can optimize it. We have also seen its limits; first of all, the expansion of the core, due to the continuous growth of the neutron density, set a limit in which there is no longer criticality, giving an explosion, in terms of energy, less powerful than its maximum and, mostly, leaving the unfissioned material free, causing more damages to the life of that place (and for many kilometers around), due to radioactivity. Moreover, we have to consider another process, the spontaneous fission, which can detonate the nuclear weapon before than expected, causing uncountable damages. We have also seen the limits of our model(s): passing from Dirichlet boundary conditions to Neumann ones, gives us a better understanding of what it is really happening; however, another trouble emerges in the model we used and it is a very former assumption: the movement of the neutrons is regulated by a diffusion equation. A diffusion approach is appropriate if neutron scattering is isotropic and even if this is not so, a diffusion approach will still be reasonable if neutrons suffer enough scatterings so as to effectively erase non-isotropic angular effects. Unfortunately, neither of these conditions are fulfilled in the case of a uranium core: fast neutrons elastically scattering against uranium show a strong forward-peaked effect and, moreover, the mean free path of a fast neutron in U^{235} is about 3.6 cm, which is only about half of the 8.4 cm bare critical radius. So, we have used a diffusion equation but we have to be careful in doing it; however, the lack of enough unclassified material and, mostly, the fact that results of our diffusion theory are comparable to what is experimentally measured, gives us a reasonable motivation to effectively use diffusion theory.
If the making of nuclear weapons is one of the worst applications of the physics and the entire science from an obvious ethical point of view, on the other hand it may represent how far the human race has gone, at least in terms of the huge amount of the energy generated by such an explosion. Of course, there are better ways to use the fissile materials and one is certainly the production of energy for peaceful scopes: in a certain sense, the requirement for a stable and controlled fission in the reactors is opposite to what we have set for the explosions, because to maintain stable the generation of secondary neutrons and to avoid that the neutron density may grow exponentially, we need a fissile material mass to be much **less** than the critical mass value. Surely, in terms of costs, the requiring of less mass than critical value is advantageous and this and the other reasons should be enough to show how human being can be so evil in choosing deliberately a value of mass bigger than a limit which is so carefully respected in reactors, paying not only huge costs but also a lot of human lives.
However, if one is able to remain indifferent to ethical and war purposes, almost as Fermi during *Trinity* test, one can face the physics behind a nuclear weapon like any other branch of science, obtaining very interesting results. Moreover, one can aim to use these and other results to more noble goals in an undetermined future.

Appendix: A Origin of the used constants

We have seen that

$$\mu = \frac{1}{3}\lambda_t v_{neut} \tag{A.1}$$

$$\eta = \frac{1}{\lambda_f} v_{neut}\left(\nu - 1\right) \tag{A.2}$$

where λ_t is the transport free path and λ_f is the fission free path; but what do these constants stay for?

The mean free path is the average distance that a neutron can travel without any interaction. To understand on what it depends on, consider a thin slab, of thickness s and a cross-sectional area of Σ, bombarded by neutrons at a rate R_0; the number density of nuclei in the material is

$$n = \frac{\rho N_A}{A} \tag{A.3}$$

where ρ is the mass density of the material, N_A is Avogadro's number and A is the atomic weight of the material.

First of all, we ask how many reactions there will be because of the collisions? To answer this, we know that the volume of the slab is, by definition, Σs, so the number of nuclei in the slab is

$$\# = \Sigma s n \tag{A.4}$$

Calling σ the reaction cross-sectional area of each nucleus to the incoming neutrons, the total area is $\Sigma s n \sigma$ while the fraction of area available for reaction clearly is

$$\frac{\Sigma s n \sigma}{\Sigma} = s n \sigma \tag{A.5}$$

Therefore, the probability that a neutron precipitates in a reaction is

$$P_{reaction} = s n \sigma \tag{A.6}$$

Conversely, the probability of a neutron passing through, without interaction, and escaping, is

$$P_{escape} = 1 - P_{reaction} = 1 - s n \sigma \tag{A.7}$$

Consider now a block of thickness x made up of m thin slabs like the previous one, of thickness s, such that $x = ms$; calling N_0 the neutrons incident on the face of the first block, the number of neutrons that pass through the first slab is $N_0 P_{escape}$, the number of neutrons that pass through the second slab is $N_0 P^2_{escape}$, the number of neutrons that pass through the third slab is $N_0 P^3_{escape}$, and so on. After m slabs, the number of neutrons escaped is so

$$N_{escape} = N_0 P^m_{escape} = N_0 P^{\frac{x}{s}}_{escape} = N_0 \left(1 - s n \sigma\right)^{\frac{x}{s}} \tag{A.8}$$

Introducing $z = -sn\sigma$, we can write

$$N_{escape} = N_0 \left(1 + z\right)^{\frac{-xn\sigma}{z}} = N_0 \left[\left(1 + z\right)^{\frac{1}{z}}\right]^{-\sigma n x} \tag{A.9}$$

Performing the continuum limit, since s is very small, $s \to 0 \Rightarrow z \to 0$, we obtain

$$N_{escape} = N_0 e^{-\sigma n x} \tag{A.10}$$

Conversely, the number of reacting neutrons is

$$N_{reaction} = N_0 - N_{escape} = N_0 \left(1 - e^{-\sigma n x}\right) \tag{A.11}$$

Thus, the probability of reaction is

$$P_{reaction} = 1 - e^{-\sigma n x} \tag{A.12}$$

The probability density function is

$$\rho_{reaction} = \frac{dP_{reaction}}{dx} = \sigma n e^{-\sigma n x} \tag{A.13}$$

Now, what we are looking for is the mean free path, that is the average distance that a neutron can travel without any interaction; thus, by definition it correspond to

$$\lambda = \int_0^L x\rho_{reaction}dx = \int_0^L \sigma n x e^{-\sigma n x}dx = \frac{1 - e^{-\sigma n L} - \sigma n L e^{-\sigma n L}}{\sigma n} \tag{A.14}$$

For large L, this simplifies to

$$\lambda = \frac{1}{\sigma n} \tag{A.15}$$

This answer to our former question, because now we have understood that

$$\lambda_t = \frac{1}{\sigma_t n} \tag{A.16}$$

and

$$\lambda_f = \frac{1}{\sigma_f n} \tag{A.17}$$

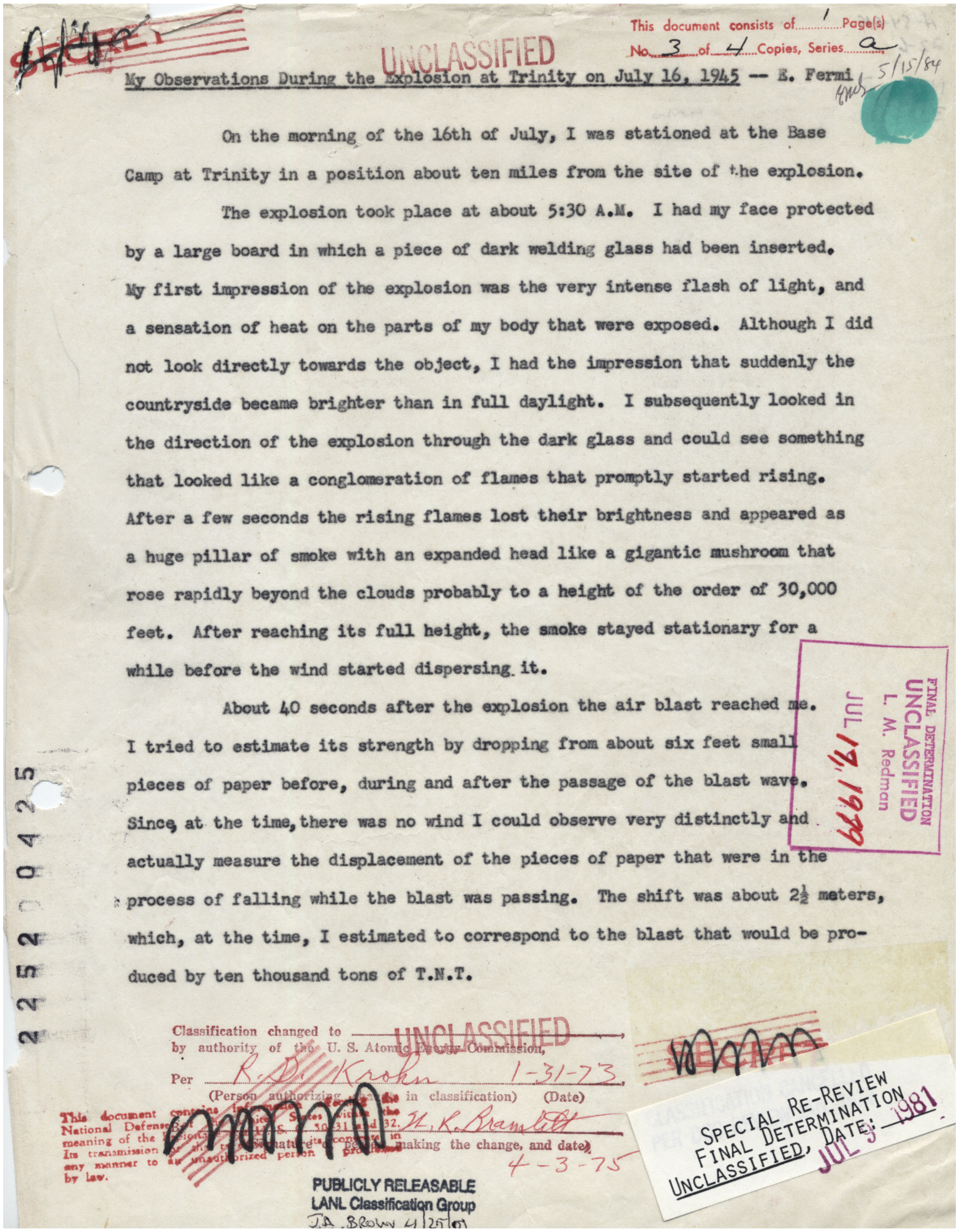
We end this discussion by illustrating a table of the used constants for the principal fissile materials, U^{235} and Pu^{239}, adapted from *Reed, B. C. (2015), The Physics of the Manhattan project, 3rd Ed., Springer.*

Some parameters forU^{235} and Pu^{239}

symbol	U^{235}	Pu^{239}	units	description
σ_f	1.2350E-28	1.8000E-28	m^2	fission cross-section
σ_{el}	4.5660E-28	4.3940E-28	m^2	elastic scattering cross-section
σ_t	5.8010E-28	6.1940E-28	m^2	total/transport cross-section, $(\sigma_f + \sigma_{el})$
n	4.7940E+28	3.9300E+28	$1/m^3$	neutron number density
λ_f	0.1689	0.1414	m	fission free path, $1/(n\sigma_f)$
λ_t	0.0360	0.0411	m	transport free path, $1/(n\sigma_t)$
ν	2.6370	3.1720	-	secondary neutrons created by fission
ρ	18.7100	15.6000	g/cm^3	density
A_t	235.04	239.05	-	atomic weight
E_N	2	2	MeV	neutron energy
e_p	1.6022E-13	1.6022E-13	J/MeV	conversion factor
m_N	1.6749E-27	1.6749E-27	kg	neutron mass
v	1.9561E+07	1.9561E+07	m/s	neutron speed, $\sqrt{2\,e_p\,E_N/m_N}$
μ	2.3446E+5	2.6786E+5	m^2/s	diffusion constant, $\lambda_t v/3$
η	1.8958E+8	3.0055E+8	1/s	neutron generation rate, $v(\nu-1)/\lambda_f$
τ	8.6347E-9	7.2268E-9	s	mean travel time to neutron fission, λ_f/v

Appendix B: Fermi's unclassified document

A copy of the unclassified document written by Enrico Fermi at the dawn of the first artificial nuclear explosion is shown below



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My Observations During the Explosion at Trinity on July 16, 1945 -- E. Fermi

On the morning of the 16th of July, I was stationed at the Base Camp at Trinity in a position about ten miles from the site of the explosion.

The explosion took place at about 5:30 A.M. I had my face protected by a large board in which a piece of dark welding glass had been inserted. My first impression of the explosion was the very intense flash of light, and a sensation of heat on the parts of my body that were exposed. Although I did not look directly towards the object, I had the impression that suddenly the countryside became brighter than in full daylight. I subsequently looked in the direction of the explosion through the dark glass and could see something that looked like a conglomeration of flames that promptly started rising. After a few seconds the rising flames lost their brightness and appeared as a huge pillar of smoke with an expanded head like a gigantic mushroom that rose rapidly beyond the clouds probably to a height of the order of 30,000 feet. After reaching its full height, the smoke stayed stationary for a while before the wind started dispersing it.

About 40 seconds after the explosion the air blast reached me. I tried to estimate its strength by dropping from about six feet small pieces of paper before, during and after the passage of the blast wave. Since, at the time, there was no wind I could observe very distinctly and actually measure the displacement of the pieces of paper that were in the process of falling while the blast was passing. The shift was about $2\frac{1}{2}$ meters, which, at the time, I estimated to correspond to the blast that would be produced by ten thousand tons of T.N.T.

FINAL DETERMINATION
UNCLASSIFIED
L. M. Redman
JUL 17, 1979

Classification changed to UNCLASSIFIED
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Per R. D. Krohn 1-31-73
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4-3-75

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FINAL DETERMINATION
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