Extended Project - Chaos in a Double Pendulum

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This experiment investigates chaos in the planar double pendulum. Both a physical and computational version of the pendulum were analysed. No results were able to be recorded for the physical version - due to limitations in the camera used. The computational version produced linear results for small angle approximation and chaotic results for larger oscillations.

1 Introduction

A planar double pendulum (Figure 1)is a simple non-linear mechanical system that can exhibit chaotic motion. Chaos is used to describe motion that depends very sensitively on initial conditions. An analysis of this sensitivity to initial conditions, as well as a look into the phase plots and poincare sections, will be conducted both physically and computationally.

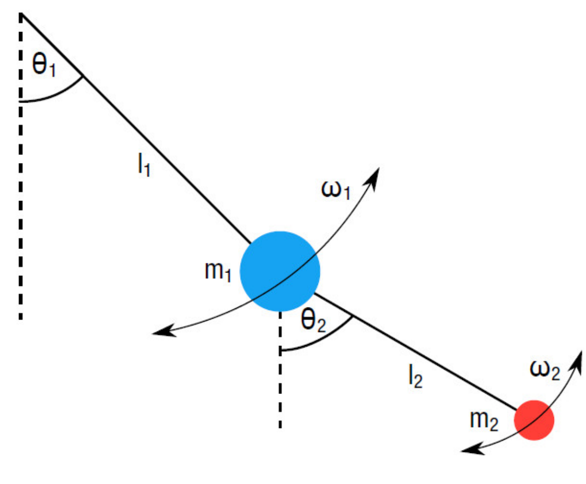

Figure 1: The Planar Double Pendulum [3]

2 Theory

The double pendulum is difficult to solve using Newtonian mechanics, however, using Lagrangian mechanics the equations of motion are more trivial to formulate. These equations are shown in Figure 2.

$$\ddot{\theta}_1 = \frac{-m_2 L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g m_2 \sin(\theta_2)\cos(\theta_1 - \theta_2) - m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin(\theta_1)}{L_1(m_1 + m_2) - m_2 L_1 \cos^2(\theta_1 - \theta_2)}$$

$$\ddot{\theta}_2 = \frac{m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g \sin(\theta_1)\cos(\theta_1 - \theta_2)(m_1 + m_2) + L_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)(m_1 + m_2) - g \sin(\theta_2)(m_1 + m_2)}{L_2(m_1 + m_2) - m_2 L_2 \cos^2(\theta_1 - \theta_2)}$$

Figure 2: The Equations of Motion of the double pendulum.[4]

Where θ_1 and θ_2 are the angle of each rod to the vertical, $\dot{\theta}_1$ and $\dot{\theta}_2$ are the angular velocities, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are the angular accelerations, m_1 and m_2 are the masses of the upper and lower bobs and l_1 and l_2 are the lengths of the upper and lower pendula.

These equations are non-linear, however, for small angle oscillations, the equations of motion become linear and quasi-periodic motion occurs. To analyse the double pendulum, three methods will be used.

Firstly, phase space plots can show important trends in a systems behaviour. In the case for a double pendulum, the phase space is 4 dimensional. To get tangible plots, the phase space must be confined to 3D then projected onto 2D. This means there are four phase space plots: θ_1 vs $\dot{\theta_1}$, θ_2 vs $\dot{\theta_2}$, θ_1 vs θ_2 and $\dot{\theta_1}$ vs $\dot{\theta_2}$.

The second method is a Poincare map. A Poincare map is a recurrence plot that allows fast and informative insight into the dynamics of the double pendulum. The different types of motion appear as finite number of points for periodic orbits, curve filling points for quasi periodic motion and area filling points for chaotic trajectories [2].

The final method is calculating the Lyapunov exponent. The Lyapunov exponent is defined as shown in (1).

$$|\delta Z(t)| \approx e^{\lambda t}|\delta Z_0|, \tag{1}$$

Where λ is the lyapunov exponent and δZ is an infinitesimal separation in trajectory. The Lyapunov exponent gives an excellent indication if a system is chaotic or not, however, it is defined by an infinitesimal initial separation which will be near impossible to accurately replicate using the physical system. If $\lambda > 0$, the system is chaotic. If $\lambda \leq 0$ the system is not chaotic. Additionally, if $\lambda < 0$, the trajectories converge to an attractor - either to a stable trajectory, called a limit cycle if its periodic, or to a stable equilibrium.

3 Experimental method

3.1 The Physical Double Pendulum

A physical double pendulum was created following the "blueprints" shown in Figure 3. The dimensions of L_1 is 3.5x20x0.7cm and the dimensions of L_2 is 3.5x12x0.7cm. The initial prototype created using this schematic was made out of wood and using skateboard wheel bearings. After concluding this set-up was too crude, a more professional double pendulum was made using transparent acrylic and smaller ball bearings. This set up is shown in Figure 4.

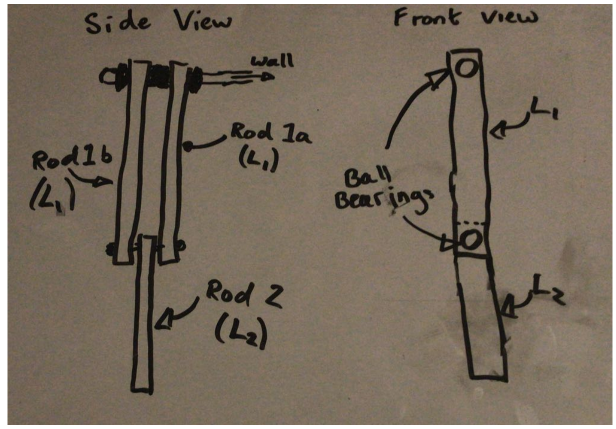

Figure 3: Original schematic drawn for the physical double pendulum.

Figure 4: Completed physical double pendulum.

To get data from the physical double pendulum, a video tracking software called "tracker" will be used. The black dots at the centre of each rod are to allow the tracker to autotrack the trajectory of the pendulum. Centering the axis on the pivot allows for x-y coordinates to be taken for every frame. These x-y coordinates can be converted to θ_1 and θ_2 trivially using simple trigonometry.

3.2 The Computational Simulation

The computational simulation of the double pendulum involves solving the equations of motion defined in Figure 2. This is done using the Runge Kutta 4th Order numerical analysis method in C. Using a step size h = 0.01 allows for an accurate approximation of the trajectory of the system. The lyapunov exponent can be calculated using a infinitesimal change in initial value of $\delta\theta_1 = 0.01$.

4 Results & Discussion

4.1 Physical Double Pendulum

Unfortunately, no results were obtained for the physical double pendulum. The tracker software couldn't follow the pendula with recordings at 60fps and unfortunately there was no access to a camera that could film at a higher frame rate. Coming to this limitation due to the lack of equipment was extremely disappointing, however, a few conclusions can be drawn from the visual analysis of the physical system.

Firstly, there was a lot of intrinsic friction in ball bearings and sometimes the lower rod would brush against/hit one of the upper rods when it rotated around the axis fully. This friction, coupled with air resistance, would lead to a negative lyapunov exponent - as the trajectory of the system would converge towards a stable equilibrium - stationary. This means that the double pendulum wouldn't exhibit true $\lambda > 0$ chaotic motion. However, for the short time that the pendulum is swinging, the motion can't be described as linear, periodic or even quasi-periodic.

This is speculative, and a deeper analysis into both the physical system (actually getting results and plotting) and a computer model involving friction would be needed to draw conclusions on how the friction of a mechanical system relates to the Lyapunov exponent.

4.2 The Computational Model

The analysis of the simulated double pendulum will take place in four parts - Basic plots, the phase space plots, the Poincare sections and the Lyapunov exponents.

4.3 Basic Plots

An initial look at the basic plots (θ vs t) for the small angle trajectory and for the large angle trajectory already shows a few signs of a change in behaviour of the system. Comparing Figure 5 and Figure 6 with Figure 7 and Figure 8 shows visual signs of nonlinearity and chaotic motion.

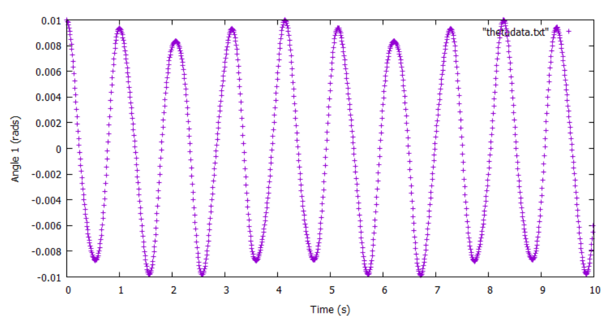

Figure 5: Plot of θ_1 vs time for the small angle approximation.

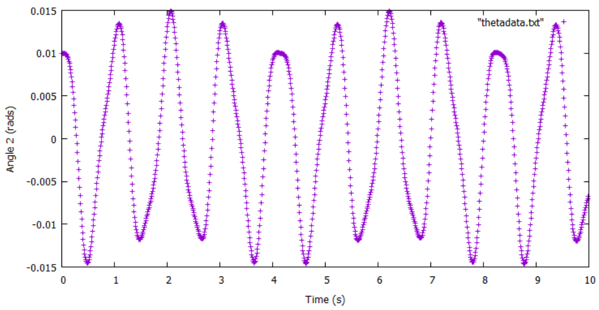

Figure 6: Plot of θ_2 vs time for the small angle approximation.

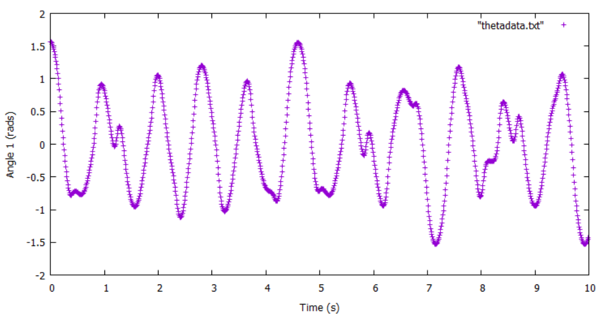

Figure 7: Plot of θ_1 vs time for initial conditions $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \dot{\theta_1} = \dot{\theta_2} = 0$.

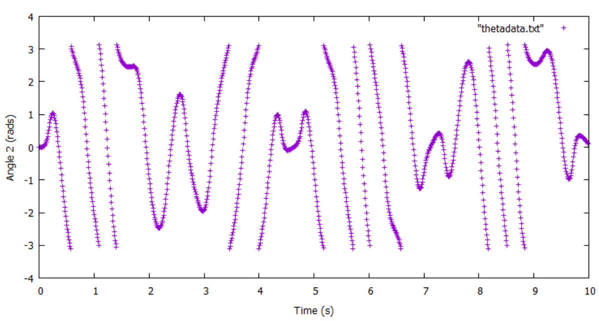

Figure 8: Plot of θ_2 vs time for initial conditions $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \dot{\theta_1} = \dot{\theta_2} = 0$.

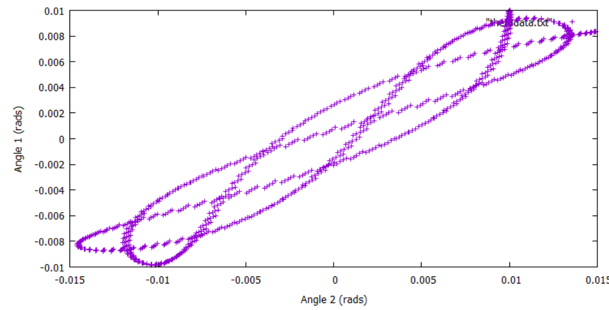

Figure 9: Plot of θ_1 vs θ_2 for the small angle approximation.

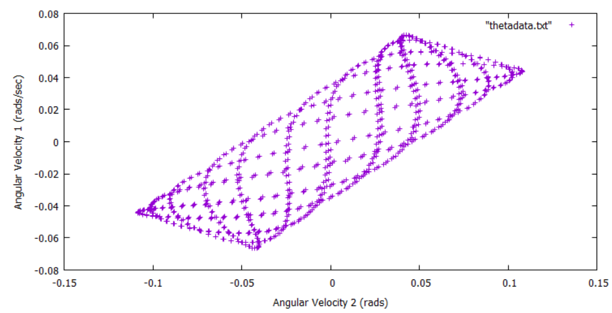

Figure 10: Plot of $\dot{\theta_1}$ vs $\dot{\theta_2}$ for the small angle approximation.

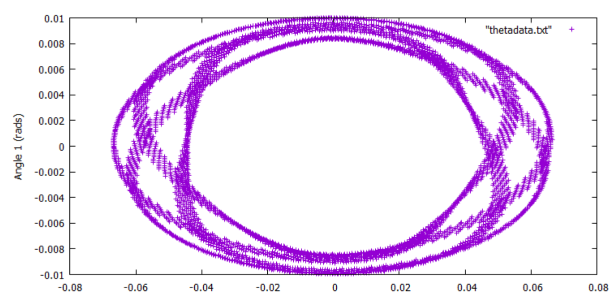

Figure 11: Plot of θ_1 vs $\dot{\theta_1}$ for the small angle approximation.

4.4 Phase Space Plots - Small Angle

Looking at the four phase space plots for the small angle approximation shows the linearity that is expected.

To analyse this further, the phase space plots for larger oscillations will be looked at for further signs of chaotic behaviour.

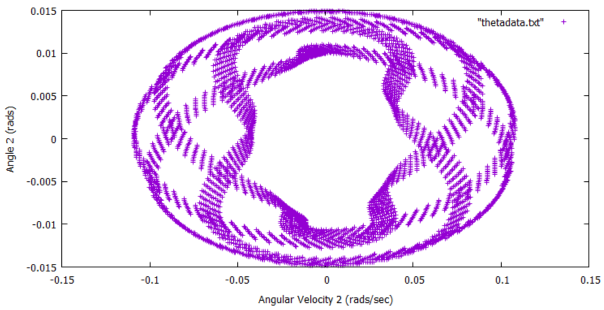

Figure 12: Plot of θ_2 vs $\dot{\theta_2}$ for the small angle approximation.

4.5 Phase Space Plots - Large Oscillations

These plots look much different to the phase space plots for the small angle approximation. The large oscillation plots look random with no obvious pattern in the trajectories in any phase space. These plots look non-linear and chaotic - to confirm this the Poincare maps can be looked at.

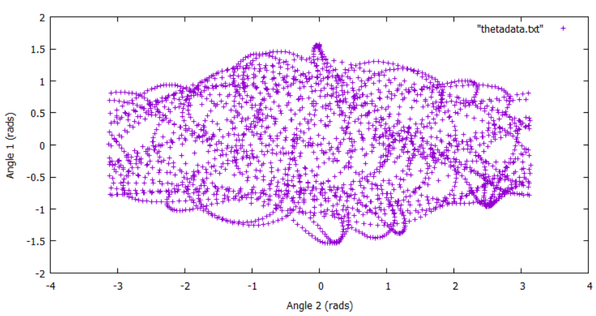

Figure 13: Plot of θ_1 vs θ_2 for large oscillation.

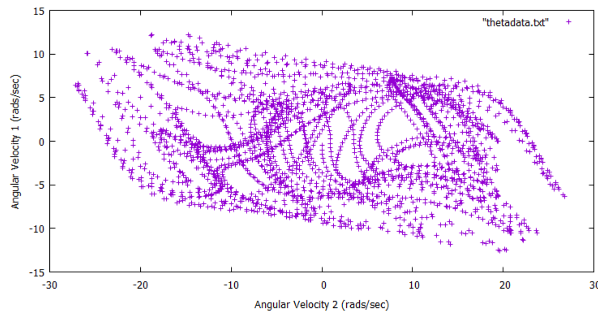

Figure 14: Plot of $\dot{\theta_1}$ vs $\dot{\theta_2}$ for large oscillation.

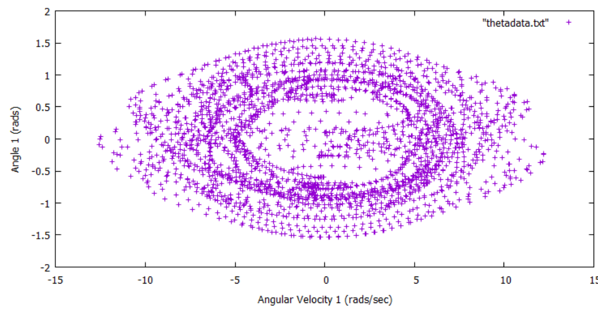

Figure 15: Plot of θ_1 vs $\dot{\theta_1}$ for large oscillation.

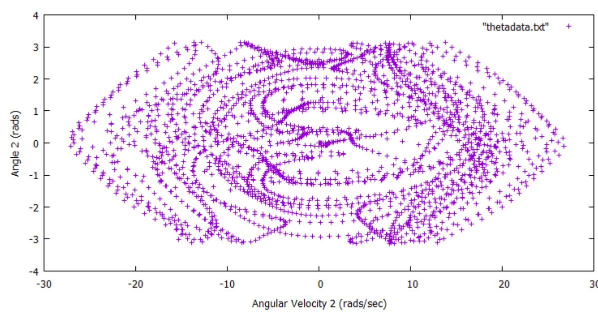

Figure 16: Plot of θ_2 vs $\dot{\theta_2}$ for large oscillation.

4.6 Poincare Plot

The Poincare maps are of when the lower pendulum passes through $\theta_2 = 0$ in the positive x direction ($\dot{\theta_2} > 0$). The map plots all such points over a 1000 second simulation.

In Figure 17, there are four distinct quasi-stable points. This is a representation of the linearity of

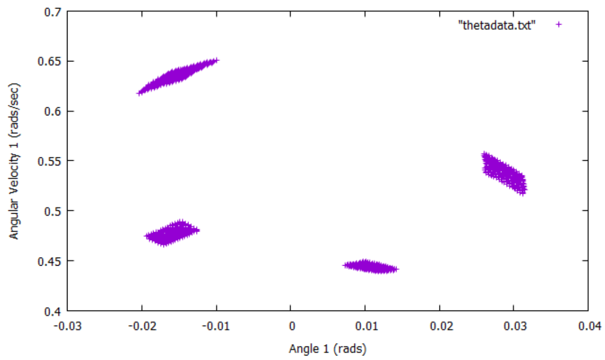

Figure 17: Poincare map plot of $\dot{\theta_1}$ vs θ_1 for when $\theta_2 = 0$ and $\dot{\theta_2} > 0$ for small angle approximation.

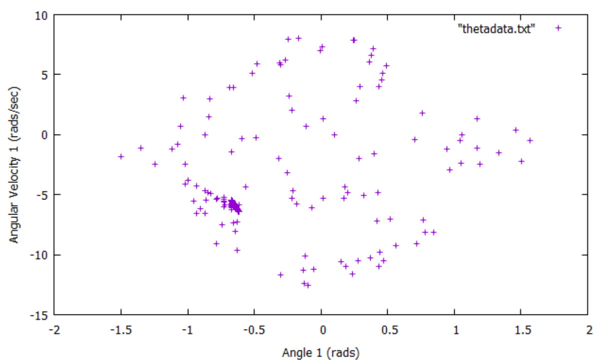

Figure 18: Poincare map plot of $\dot{\theta_1}$ vs θ_1 for when $\theta_2 = 0$ and $\dot{\theta_2} > 0$ for large oscillations.

the small angle approximation. When compared with Figure 18, it is apparent that the normal plot is behaving chaotically.

4.7 Lyapunov Exponent

Although it is already possible to conclude that the double pendulum acts chaotically for large oscillations, the Lyapunov exponent is an interesting way to quantify how chaos affects the system - and what a sensitivity to initial conditions means. The time period used is 60s and the infinitesimal change in initial θ_1 for each instance is 0.01 rads.

In Figure 19, the Lyapunov exponent is plotted as a function of the initial starting angle of θ_1. There is

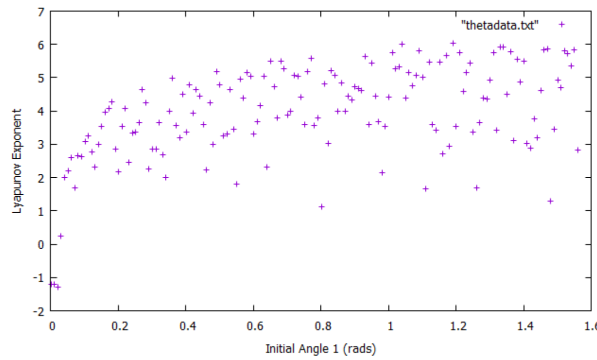

Figure 19: Plot of the Lyapunov exponent λ vs initial θ_1.

a varying range of the Lyapunov exponent, however, the actual numeric value of the exponent does not really mean anything - the important result is that $\lambda > 0$ in every single instance. This confirms chaotic motion.

Figure 20 shows the final θ_1 vs the initial θ_1. From this plot it is easy to visualise the systems descent into chaos as the initial energy of the system increases. The final plot Figure 21 limits the range of θ_1 to show a more visible representation of how fast the system descends into chaos.

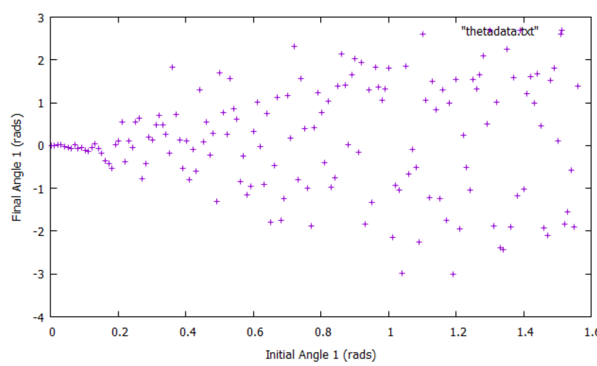

Figure 20: A plot of final θ_1 vs initial θ_1 over a range $0 < \theta_1 < \frac{\pi}{2}$.

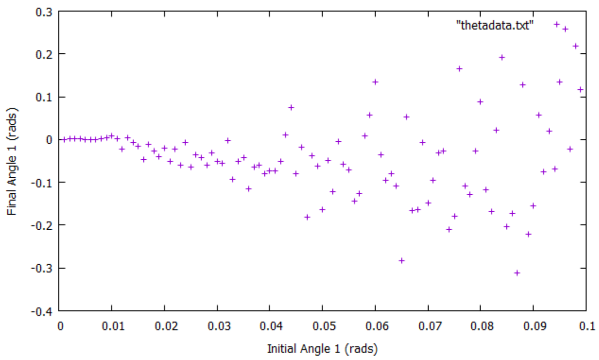

Figure 21: A plot of final θ_1 vs initial θ_1 over a range $0 < \theta_1 < 0.1$

5 Conclusions

This experiment attempted to use a physical double pendulum and a computational double pendulum to investigate non-linear dynamics and chaotic motion. The physical pendulum faced technical difficulties with the camera being used - and not being able to find a more suitable one. The computational model showed excellent results - showcasing the linearity for the small angle approximation of the system and chaotic behaviour for the large angle oscillation.

Developments for this experiment would be to firstly use a camera capable of recording at 100fps. This would allow the tracker software to produce accurate results. Also, creating the computational model with a damping factor to replicate the air resistance in the physical model would produce more realistic results.

References

[1] H. Goldstein, *Classical Mechanics*, (Addison-Wesley Press, INC).

[2] Roja Nunna, *Numerical Analysis of the Dynamics of a Double Pendulum*, Phys. Rev. Lett. (2009), 3.

[3] `http://herdingcats.typepad.com/my_weblog/2013/02/the-misunderstanding-of-chaos-again.html`, (accessed 9 March 2018).

[4] `http://sophia.dtp.fmph.uniba.sk/~kovacik/doublePendulum.pdf`, (accessed 28 March 2018).

[5] R. O. Nielsen, E. Have, B. T. Nielsen *The Double Pendulum*, Phys. Rev. Lett. (2013).

[6] `https://www.myphysicslab.com/pendulum/double-pendulum-en.html`, (accessed 28 March 2018).