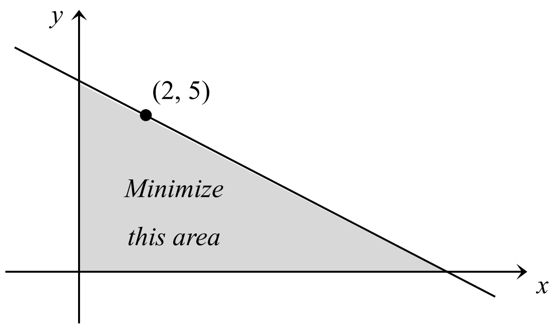
The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Final Exam 2020/21 Semester 2

Question 1.

(a) In the picture below, a straight line with a negative slope passes through point (2, 5), and a triangle is formed by this line with the x-axis and y-axis. Find the equation of the line such that the area of the triangle is minimized. [5 marks]



(b) Calculate the following integrals. [5 marks each]

(i) $\displaystyle\int \frac{x^2+8x-3}{x\,(x^2+3x)}\,dx$

(ii) $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx$

(iii) $\displaystyle\int_0^{+\infty} x^3 e^{-x^2}\,dx$

(iv) $\displaystyle\int_1^{2} \sqrt{x^{-1}+1}\,dx$

My work :

(a) Let $y = m\,(x-2)+5$ be the equation of the line passing through (2, 5), and $m < 0$.

y-intercept $= 5-2m > 0$, and x-intercept $= 2-\frac{5}{m} > 0$

Area of the triangle $= \dfrac{1}{2}\left(2-\dfrac{5}{m}\right)(5-2m) = \dfrac{(2m-5)^2}{-2m} = A\,(m)$

$A'(m) = \dfrac{(2m+5)(2m-5)}{-2m^2} = 0 \iff m = -\dfrac{5}{2}$

m	$(-\infty, -\frac{5}{2})$	$-\frac{5}{2}$	$(-\frac{5}{2}, 0)$
$A\,(m)$	↘	min	↗
$A'(m)$	–	0	+

The equation of the line is $y = -\frac{5}{2}x + 10$.

(b) (i) $\displaystyle\int \frac{x^2+8x-3}{x\,(x^2+3x)}\,dx = \cdots = \int\left(\frac{3}{x}+\frac{-1}{x^2}+\frac{-2}{x+3}\right)dx = 3\ln|x|+\frac{1}{x}-2\ln|x+3|+C$

(ii) $$\int_0^{\pi/4}\frac{x\sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x\tan x\sec^2 x\,dx = \int_0^{\pi/4} x\,d\left(\frac{\tan^2 x}{2}\right)$$
$$= \left[\frac{x\tan^2 x}{2}\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}(\sec^2 x - 1)\,dx = \frac{\pi}{8}-\frac{1}{2}\left[\tan x - x\right]_0^{\pi/4}$$
$$= \frac{\pi}{8}-\frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{\pi}{4}-\frac{1}{2}$$

(iii) $$\int_0^{+\infty} x^3 e^{-x^2}\,dx = \lim_{b\to+\infty}\int_0^b x^3 e^{-x^2}\,dx = \lim_{b\to+\infty}\frac{1}{2}\int_0^b (-x^2)\,e^{-x^2}\,d(-x^2)$$
$$= \lim_{b'\to+\infty}\frac{1}{2}\int_0^{-b'} u\,e^u\,du = \cdots = \lim_{b'\to+\infty}\frac{1}{2}\left[u\,e^u - e^u\right]_0^{-b'}$$
$$= \lim_{b'\to-\infty}\frac{1}{2}\left(\frac{-b'}{e^{b'}}-\frac{1}{e^{b'}}+1\right) = \frac{1}{2}\text{, by l'Hôpital's rule}$$

(iv) Let $\displaystyle I = \int_1^2\sqrt{x^{-1}+1}\,dx = 2\int_1^2\sqrt{1+x}\,d(\sqrt{x}) \boxed{= 2\int_1^2\sqrt{1+u^2}\,du}$

Let $\sqrt{x}=\tan\theta$, $d(\sqrt{x}) = \sec^2\theta\,d\theta$, $\sqrt{1+x}=\sec\theta$

When $x=1$, $\theta=\pi/4$; when $x=2$, $\theta=\tan^{-1}\sqrt{2}$

$$I = 2\int_{\pi/4}^{\tan^{-1}\sqrt{2}}\sec^3\theta\,d\theta = 2\left[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_{\pi/4}^{\tan^{-1}\sqrt{2}}$$
$$= \sqrt{6}+\ln(\sqrt{3}+\sqrt{2})-\sqrt{2}-\ln(\sqrt{2}+1)\approx 1.3001$$

Question 2.

(a) Let $f(x) = \int_0^{\sqrt{x}} \sin(t^2)\, dt$, where $0 < x < 2\pi$.

(i) Find all the open subintervals of $(0, 2\pi)$ where f is increasing or decreasing. [5 marks]

(ii) Find the degree 2 Taylor polynomial of f at π. [5 marks]

(iii) Demonstrate that there is a $\xi \in [0, x]$ such that $f(x) = \sqrt{x}\sin(x) - \frac{2\sqrt{x^3}}{3}\cos(\xi)$.

[5 marks]

(b) Let $I(x) = \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt$, where $x > 0$.

(i) By the comparison test or otherwise, demonstrate that the improper integral $I(x)$ is convergent for any given $x > 0$. [5 marks]

(ii) Demonstrate that $I(x) \le x^{-1}e^{-\frac{x^2}{2}}$ [5 marks]

My work :

(a) (i) $f'(x) = \frac{1}{2\sqrt{x}}\sin x = 0 \Leftrightarrow x = \pi$

X	$(0, \pi)$	π	$(\pi, 2\pi)$
$f(x)$	$\nearrow$	max	$\searrow$
$f'(x)$	$+$	0	$-$

The open subinterval where f is increasing is $(0, \pi)$.

The open subinterval where f is decreasing is $(\pi, 2\pi)$.

(ii) $f''(x) = \dfrac{x\cos x - \frac{1}{2}\sin x}{2x^{3/2}} \Rightarrow f''(\pi) = -\frac{1}{2\sqrt{\pi}}$. $f(\pi) = \int_0^{\sqrt{\pi}} \sin(t^2)\, dt$

The degree 2 Taylor polynomial of f at π is given by

$$T_2(x) = f(\pi) + f'(\pi)(x - \pi) + \frac{f''(\pi)}{2}(x - \pi)^2 = \int_0^{\sqrt{\pi}} \sin(t^2)\, dt - \frac{1}{4\sqrt{\pi}}(x - \pi)^2$$

Remark : $S(x) := \int_0^x \sin(t^2)\, dt$ cannot be computed in closed-form, and is called **Fresnel sine integral function**.

(iii) $f(x) = \int_0^{\sqrt{x}} \sin(t^2)\, dt = t \sin(t^2) \Big|_0^{\sqrt{x}} - \int_0^{\sqrt{x}} 2t^2 \cos(t^2)\, dt$

$= \sqrt{x} \sin x - \cos \xi \int_0^{\sqrt{x}} 2t^2\, dt$ for some $\xi \in (0, x)$,

by Mean-Value Theorem for Integral

$= \sqrt{x} \sin x - \dfrac{2\sqrt{x^3}}{3} \cos \xi$

(b) (i) For $t \ge 1$, $t \le t^2 \Rightarrow e^{-\frac{t^2}{2}} \le e^{-\frac{t}{2}}$. Hence, for all $x > 0$,

$$I(x) = \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt \le \int_0^1 e^{-\frac{t^2}{2}}\, dt + \int_1^{+\infty} e^{-\frac{t^2}{2}}\, dt \le \int_0^1 e^{-\frac{0^2}{2}}\, dt + \lim_{b \to +\infty} \int_1^b e^{-\frac{t}{2}}\, dt$$

$$= (1 - 0) + \lim_{b \to +\infty} 2\left(e^{-\frac{1}{2}} - e^{-\frac{b}{2}}\right) = 1 + \frac{2}{\sqrt{e}} < +\infty.$$

Since $e^{-t^2/2} > 0$ for all t, $I(x)$ is convergent.

Alternative Solution

$$I(x) = \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt \le \int_0^1 e^{-\frac{t^2}{2}}\, dt + \int_1^{+\infty} e^{-\frac{t^2}{2}}\, dt \le \int_0^1 e^{-\frac{0^2}{2}}\, dt + \lim_{b \to +\infty} \int_0^b e^{-\frac{t}{2}}\, dt$$

$$= (1 - 0) + \lim_{b \to +\infty} 2\left(1 - e^{-\frac{b}{2}}\right) = 1 + 2 = 3 < +\infty.$$

(ii) For all $x > 0$, we have

$$I(x) = \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt = \int_x^{+\infty} (-t^{-1})\, d\left(e^{-\frac{t^2}{2}}\right) = \lim_{b \to +\infty} -t^{-1} e^{-\frac{t^2}{2}} \Big|_x^b - \int_x^{+\infty} e^{-\frac{t^2}{2}} t^{-2}\, dt$$

$$= x^{-1} e^{-\frac{x^2}{2}} - \int_x^{+\infty} e^{-\frac{t^2}{2}} t^{-2}\, dt \le x^{-1} e^{-\frac{x^2}{2}}$$

Question 3.

Consider the curve given by the equation $4e^{2x} - 4e^x y + 1 = 0$, where $0 \le x \le 1$.

(a) Find the arc-length of this curve. [5 marks]

(b) Find the area bounded by this curve together with the lines $y = 0$, $x = 0$, and $x = 1$. [5 marks]

(c) Find the area of the surface obtained by rotating this curve about the x-axis. [5 marks]

(d) Find the volume of the solid obtained by rotating the area in 3(b) about the x-axis. [5 marks]

(e) Find the volume of the solid obtained by rotating the area in 3(b) about the y-axis. [5 marks]

My work :

(a) $4e^{2x} - 4e^x y + 1 = 0 \;\Rightarrow\; y = \frac{1}{4}e^{-x} + e^x \;\Rightarrow\; y' = -\frac{1}{4}e^{-x} + e^x$

$$\Rightarrow \quad (y')^2 + 1 = \left(\frac{1}{4}e^{-x} + e^x\right)^2 \;\Rightarrow\; \sqrt{(y')^2 + 1} = \frac{1}{4}e^{-x} + e^x$$

$$\text{Arc-length} = \int_0^1 \sqrt{(y')^2 + 1}\,dx = \int_0^1 \left(\frac{1}{4}e^{-x} + e^x\right) dx = \left[-\frac{1}{4}e^{-x} + e^x\right]_0^1 = e - \frac{1}{4e} - \frac{3}{4}$$

(b) Note that $\frac{1}{4}e^{-x} + e^x > 0$ for all $0 \le x \le 1$

$$\text{Area bounded} = \int_0^1 \left(\frac{1}{4}e^{-x} + e^x\right) dx = e - \frac{1}{4e} - \frac{3}{4}$$

(c) $$\text{Surface area} = 2\pi \int_0^1 y\sqrt{(y')^2 + 1}\,dx = 2\pi \int_0^1 \left(\frac{1}{4}e^{-x} + e^x\right)^2 dx$$

$$= \pi \int_0^1 \left(\frac{1}{8}e^{-2x} + 1 + 2e^{2x}\right) dx = \pi\left[-\frac{1}{16}e^{-2x} + x + e^{2x}\right]_0^1 = \pi\left(e^2 - \frac{1}{16e^2} + \frac{1}{16}\right)$$

(d) $$\text{Volume about } x\text{-axis} = \pi \int_0^1 y^2\,dx = \pi \int_0^1 \left(\frac{1}{4}e^{-x} + e^x\right)^2 dx = \pi\left(\frac{e^2}{2} - \frac{1}{32e^2} + \frac{1}{32}\right)$$

(e) $$\text{Volume about } y\text{-axis} = 2\pi \int_0^1 xy\,dx = 2\pi \int_0^1 x\left(\frac{1}{4}e^{-x} + e^x\right) dx$$

$$= \cdots = 2\pi\left[-\frac{1}{4}xe^{-x} - \frac{1}{4}e^{-x} + xe^x - e^x\right]_0^1 = \left(\frac{5}{2} - \frac{1}{e}\right)\pi$$

Question 4.

(a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find the adjoint matrix of $2A^{\mathsf{T}}$ and also find A^{-1}. [5 marks]

(b) Let $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find $A^{1120}B^{1121} + A^{1121}B^{1120}$. [5 marks]

(c) Consider the system of linear equations

$$\begin{cases} ax_1 + x_2 + 2x_3 = a + 1 \\ x_1 + x_2 + x_3 = -a + 1 \\ 3x_1 + x_2 + (a^2 - 1)x_3 = 1 \end{cases}.$$

By Gaussian elimination or otherwise, find all possible values of a such that the system is

(i) inconsistent;

(ii) consistent with infinitely many solutions; or

(iii) consistent with a unique solution.

Also solve the system when it is consistent. [15 marks]

My work :

(a) $2A^{\mathsf{T}} = \begin{bmatrix} 2 & -2 & 6 \\ 6 & 0 & 2 \\ 4 & 4 & -2 \end{bmatrix} \Rightarrow \text{adj}\,(2A^{\mathsf{T}}) = \begin{bmatrix} -8 & 20 & -4 \\ 20 & -28 & 32 \\ 24 & -16 & 12 \end{bmatrix}$; $\text{adj}\,A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & -7 & -4 \\ -1 & 8 & 3 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -8 & 0 & 5 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{vmatrix} = -(-8)(2) + (-1)(5) = 11$$

$$A^{-1} = \frac{\text{adj}\,A}{\det A} = \begin{bmatrix} -\frac{2}{11} & \frac{5}{11} & \frac{6}{11} \\ \frac{5}{11} & -\frac{7}{11} & -\frac{4}{11} \\ -\frac{1}{11} & \frac{8}{11} & \frac{3}{11} \end{bmatrix}$$

(b) $B^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, and

$$AB = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix} = -B^2, \text{ so } A^2B = A\,(-B^2) = B^3$$

We claim $A^kB = (-1)^k B^{k+1}$ for $k \in \mathbb{N}$. We use induction on k. The base case $k = 1$ has been shown. Then $A^{k+1}B = A\,[A^kB] = A\,[\,(-1)^k B^{k+1}] = (-1)^{k+1} B^{k+2}$. Thus the claim is proved.

$\therefore\quad A^{1120}B = (-1)^{1120} B^{1121} = B^{1121}$ and $A^{1121}B = (-1)^{1121} B^{1122} = -B^{1122}$

$\Rightarrow\quad A^{1120}B^{1121} + A^{1121}B^{1120} = (B^{1121})\,B^{1120} - (B^{1122})\,B^{1119} = 0$

(c) $$\left[\begin{array}{ccc|c} a & 1 & 2 & a+1 \\ 1 & 1 & 1 & 1-a \\ 3 & 1 & a^2-1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ a & 1 & 2 & a+1 \\ 3 & 1 & a^2-1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 1-a & 2-a & a^2+1 \\ 0 & -2 & a^2-4 & 3a-2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 2 & 4-a^2 & 2-3a \\ 0 & 1-a & 2-a & a^2+1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 2 & 4-a^2 & 2-3a \\ 0 & 0 & -a(a+1)(a-2) & -a(a-5) \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow$ $a\,(a+1)\,(a-2) = 0$ and $a\,(a-5) \neq 0$ $\Leftrightarrow$ $a = -1, 2$.

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow$ $a = 0$. In this case,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ x_1 = t,\ x_2 = 1-2t,\ x_3 = t$ for all $t \in \mathbb{R}$

(iii) The system is consistent with unique solution $\Leftrightarrow$ $a \neq 0, -1, 2$. In this case,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 2 & 4-a^2 & 2-3a \\ 0 & 0 & 1 & \frac{a-5}{(a+1)(a-2)} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 2 & 0 & -\frac{2(a^2+2a+4)}{a+1} \\ 0 & 0 & 1 & \frac{a-5}{(a+1)(a-2)} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1-a \\ 0 & 1 & 0 & -\frac{a^2+2a+4}{a+1} \\ 0 & 0 & 1 & \frac{a-5}{(a+1)(a-2)} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2a^2-5}{(a+1)(a-2)} \\ 0 & 1 & 0 & -\frac{a^2+2a+4}{a+1} \\ 0 & 0 & 1 & \frac{a-5}{(a+1)(a-2)} \end{array}\right]$$

$$\therefore\ x_1 = \frac{2a^2-5}{(a+1)(a-2)},\ x_2 = -\frac{a^2+2a+4}{a+1},\ x_3 = \frac{a-5}{(a+1)(a-2)}$$

Alternative Solution

(iii) The system is consistent with unique solution

$$\Leftrightarrow \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & a^2-1 \end{vmatrix} = a^3 - a^2 - 2a = a\,(a+1)\,(a-2) \neq 0 \quad \Leftrightarrow \quad a \neq 0, -1, 2.$$

In this case, by Cramer's rule,

$$x_1 = \frac{\begin{vmatrix} a+1 & 1 & 2 \\ 1-a & 1 & 1 \\ 1 & 1 & a^2-1 \end{vmatrix}}{a\,(a+1)\,(a-2)} = \frac{2a^3-5a}{a\,(a+1)\,(a-2)} = \frac{2a^2-5}{(a+1)(a-2)}$$

$$x_2 = \frac{\begin{vmatrix} a & a+1 & 2 \\ 1 & 1-a & 1 \\ 3 & 1 & a^2-1 \end{vmatrix}}{a\,(a+1)\,(a-2)} = \frac{-a^4+8a}{a\,(a+1)\,(a-2)} = -\frac{a^2+2a+4}{a+1}$$

$$x_3 = \frac{\begin{vmatrix} a & 1 & a+1 \\ 1 & 1 & 1-a \\ 3 & 1 & 1 \end{vmatrix}}{a\,(a+1)\,(a-2)} = \frac{a^2-5a}{a\,(a+1)\,(a-2)} = \frac{a-5}{(a+1)(a-2)}$$

If $a = 0$, the system becomes

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If $a = -1$, the system becomes

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 2 \\ 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 1 & 2 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 2 \\ 0 & -2 & -3 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

If $a = 2$, the system becomes

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & 3 \\ 3 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 5 \\ 0 & -2 & 0 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow$ $a = -1, 2$.

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow$ $a = 0$. In this case, $x_1 = t,\ x_2 = 1 - 2t,\ x_3 = t$ for all $t \in \mathbb{R}$