The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #01

Question 1

(a) (1) $f'(x) = 6x^2 - 18x - 24$ (2) $f'(x) = \frac{2}{3}x^{-1/3}$

(3) $f'(x) = 2x\, e^{x^2}$ (4) $f'(x) = 3\sec^2(3x - 1)$

(5) $f'(x) = \frac{1}{1 + x^2}$ (6) $f'(x) = 4x - \frac{1}{x}$

(b) (1) $f''(x) = 12x - 18$ (2) $f''(x) = -\frac{2}{9}x^{-4/3}$

(3) $f''(x) = 2e^{x^2} + 4x^2 e^{x^2} = 2e^{x^2}(1 + 2x^2)$

(4) $f''(x) = 18\sec^2(3x - 1)\tan(3x - 1)$

(5) $f''(x) = -\frac{2x}{(1 + x^2)^2}$ (6) $f''(x) = 4 + \frac{1}{x^2}$

Question 2

(a) $f'(x) = 6x^2 - 18x - 24 = 0 \iff x = -1, 4$

$\therefore$ f has stationary points at $x = -1, 4$.

(b) $f'(x) = 4x - \frac{1}{x} = 0 \Rightarrow 4x^2 - 1 = 0 \Rightarrow x = \frac{1}{2}, -\frac{1}{2}$ (rej.)

$\therefore$ f has a stationary point at $x = \frac{1}{2}$.

(c) $f'(x) = \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} = 0 \iff x = \pm 1$

$\therefore$ f has stationary points at $x = \pm 1$.

(d) $f'(x) = 2x\, e^{-x} - x^2 e^{-x} = e^{-x} x(2 - x) = 0 \iff x = 0, 2$

$\therefore$ f has stationary points at $x = 0, 2$.

Question 3

(a)

x	$(-\infty, -1)$	$(-1, 4)$	$(4, +\infty)$
$f'(x)$	$+$	$-$	$+$

The intervals where f is strictly increasing are: $(-\infty, -1)$, $(4, +\infty)$

The interval where f is strictly decreasing is: $(-1, 4)$

(b)

x	$(0, \frac{1}{2})$	$(\frac{1}{2}, +\infty)$
$f'(x)$	$-$	$+$

The interval where f is strictly increasing is: $(\frac{1}{2}, +\infty)$

The interval where f is strictly decreasing is: $(0, \frac{1}{2})$

(c)

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	$-$	$+$	$-$

The interval where f is strictly increasing is: $(-1, 1)$

The intervals where f is strictly decreasing are: $(-\infty, -1)$, $(1, +\infty)$

(d)

x	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	$-$	$+$	$-$

The interval where f is strictly increasing is: $(0, 2)$

The intervals where f is strictly decreasing are: $(-\infty, 0)$, $(2, +\infty)$

Question 4

(a)

x	$(-\infty, -1)$	$(-1, 4)$	$(4, +\infty)$
$f'(x)$	$+$	$-$	$+$

By first derivative test, f has a local maximum at $x = -1$ and a local minimum at $x = 4$.

By second derivative test,

$f''(-1) = -30 < 0 \Rightarrow f$ has a local maximum at $x = -1$

$f''(4) = 30 > 0 \Rightarrow f$ has a local minimum at $x = 4$

(b)

x	$(0, \frac{1}{2})$	$(\frac{1}{2}, +\infty)$
$f'(x)$	$-$	$+$

By first derivative test, f has a local minimum at $x = \frac{1}{2}$

By second derivative test, $f''(\frac{1}{2}) = 8 > 0 \;\Rightarrow\; f$ has a local minimum at $x = \frac{1}{2}$

(c)

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	$-$	$+$	$-$

By first derivative test, f has a local minimum at $x = -1$ and a local maximum at $x = 1$.

$$f''(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)\cdot 2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)\,(-2x) - (1-x^2)\,(4x)}{(1+x^2)^3} = \frac{2x^3 - 6x}{(1+x^2)^3} = \frac{2x\,(x^2-3)}{(1+x^2)^3}$$

By second derivative test, $f''(-1) > 0$ and $f''(1) < 0$, f has a local minimum at $x = -1$ and a local maximum at $x = 1$.

Question 5

(a) If f is constant, then the result is trivial. Hence WLOG assume f is nonconstant. Say suppose there is $x_0 \in (a, b)$ such that $f(x_0) > f(a) = f(b)$. Since f is continuous on $[a, b]$, this suggests there must be a local maximum attained at $x = \xi \in (a, b)$. Since f is differentiable in (a, b), $f'(\xi) = 0$. On the other hand, if no $x \in (a, b)$ satisfies $f(x) > f(a) = f(b)$, then since f is nonconstant, there must be $x_0 \in (a, b)$ such that $f(x_0) < f(a) = f(b)$. Since f is continuous on $[a, b]$, this suggests there must be a local minimum attained at $x = \xi \in (a, b)$. Since f is differentiable in (a, b), $f'(\xi) = 0$.

(b) Let $g(x) = f(x) - \left(f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a) \right)$.

If f is continuous on $[a, b]$ and differentiable in (a, b), so is g.

Observe that $g(a) = f(a) - f(a) = 0$ and $g(b) = f(b) - f(b) = 0$.

Hence by Rolle's Theorem, $\exists\, \xi \in (a, b)$ such that

$$0 = g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} \;\Leftrightarrow\; f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Question 6

(a) $f(x) = x^3 - x^2 - x + 1$, $f'(x) = 3x^2 - 2x - 1$, $f(2) = 3$, $f(1) = 0$, $\dfrac{f(2) - f(1)}{2 - 1} = 3$

$\therefore\ f'(\xi) = 3\xi^2 - 2\xi - 1 = 3 \Rightarrow \xi = \dfrac{2 \pm \sqrt{2^2 - 4(3)(-4)}}{2(3)} = \dfrac{1 \pm \sqrt{13}}{3}$

Note $\xi \in (1, 2)$, $\xi = \dfrac{1 + \sqrt{13}}{3}$

(b) $f(x) = x^{2/3}$, $f'(x) = \dfrac{2}{3}x^{-1/3}$, $f(8) = 4$, $f(-8) = 4$, $\dfrac{f(8) - f(-8)}{8 - (-8)} = 0$

$\therefore\ f'(\xi) = \dfrac{2}{3}\xi^{-1/3} = 0 \Rightarrow \dfrac{2}{3} = 0$, which is a contradiction.

Hence such ξ does not exist in $(-8, 8)$.

Question 7 (*Standard Level*)

Let $f(u) = e^u$, which is continuous on $[0, x]$ and differentiable in $(0, x)$, $0 < x \le 1$.

By Mean Value Theorem, $\exists\ \xi \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi) \Leftrightarrow \frac{e^x - 1}{x} = e^{\xi}$$

$$1 = e^0 < e^{\xi} < e^x \le e \Rightarrow 1 < \frac{e^x - 1}{x} < e \Rightarrow 1 + x < e^x < 1 + ex$$

Question 8 (*Standard Level*)

If $x = 0$, the inequality is trivial. WLOG suppose $x > 0$. Let $f(u) = \ln(1 + 2u) - 2u$ which is continuous on $[0, x)$ and differentiable in $(0, x)$. Note that $f'(u) = \dfrac{2}{1 + 2u} - 2$ for $u > 0$.

By Mean Value Theorem, $\exists\ \xi \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi) \Leftrightarrow \ln(1 + 2x) - 2x = \left(\frac{2}{1 + 2\xi} - 2\right)x < 0 \Leftrightarrow \ln(1 + 2x) < 2x.$$

Let $g(u) = \ln(1 + 2u) - 2u + 2u^2$ which is continuous on $[0, x)$ and differentiable in $(0, x)$. Note that $g'(u) = \dfrac{2}{1 + 2u} - 2 + 4u$ for $u > 0$.

By Mean Value Theorem, $\exists\ \eta \in (0, x)$ such that

$$\frac{g(x) - g(0)}{x - 0} = g'(\xi) \Leftrightarrow \ln(1 + 2x) - 2x + 2x^2 = \left(\frac{2}{1 + 2\eta} - 2 + 4\eta\right)x = \left(4\eta - \frac{4\eta}{1 + 4\eta}\right)x > 0.$$

Hence we have shown that $2x - 2x^2 \le \ln(1 + 2x) \le 2x$ for $x \ge 0$.

Question 9 (*Standard Level*)

Let $f(x) = \tan^{-1} x$, which is continuous on $[a, b]$ and differentiable in (a, b), $0 < a < b$.

By Mean Value Theorem, $\exists\ \xi \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \iff \frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+\xi^2}$$

$$a < \xi < b \Rightarrow \frac{1}{1+a^2} > \frac{1}{1+\xi^2} > \frac{1}{1+b^2} \Rightarrow \frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Put $a = 1$, $b = \frac{4}{3}$, we have $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Question 10 (*Standard Level*)

Since $f(x) = 2x + \sin 3x$, which is continuous on $[0, \pi]$ and differentiable in $(0, \pi)$.

Note that $f'(x) = 2 + 3\cos 3x$. By Mean Value Theorem, $\exists\ \xi \in (0, \pi)$ such that

$$\frac{f(\pi)-f(0)}{\pi-0} = f'(\xi) \iff 2\pi = (2 + 3\cos 3\xi)\,\pi.$$

For such $\xi \in (0, \pi)$,

$2\pi = (2 + 3\cos 3\xi)\,\pi \iff 2 = 2 + 3\cos 3\xi \iff \cos 3\xi = 0$

Hence if we pick $\xi = \frac{\pi}{6} \in (0, \pi)$, the statement of the Mean Value Theorem is verified.