The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #06

Question 1

(a) $\begin{pmatrix} 1 & 2 \\ 5 & -3 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 7 & 6 \end{pmatrix}$

(b) $5\begin{pmatrix} 2 & -3 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 10 & -15 \\ 30 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 2 \\ -13 & -4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 20 \\ -2 \end{pmatrix}$

Question 2

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & -7 \\ 1 & -2 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -12 \\ 0 & -3 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & -1 & 4 \\ -1 & 1 & 3 & -5 \\ -2 & -1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 2 & 2 & -1 \\ 0 & 1 & 1 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 9 \\ 0 & 2 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & -19 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 1 & -8 \\ 1 & -3 & -18 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -18 \\ -5 & 1 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -18 \\ 0 & -14 & -98 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -18 \\ 0 & 1 & 7 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 7 \\ -4 & 1 & 1 \\ 1 & -2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 7 \\ 0 & 9 & 29 \\ 0 & -4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & \frac{29}{9} \\ 0 & -4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & \frac{29}{9} \\ 0 & 0 & \frac{98}{9} \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & \frac{29}{9} \\ 0 & 0 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Question 3

(a) $\left(\begin{array}{ccc|c} 3 & -2 & 1 & 15 \\ -3 & 1 & 1 & 9 \\ 1 & -5 & 2 & 24 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 24 \\ -3 & 1 & 1 & 9 \\ 3 & -2 & 1 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 24 \\ 0 & -14 & 7 & 81 \\ 0 & 13 & -5 & -57 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 24 \\ 0 & 1 & -\frac{1}{2} & -\frac{81}{14} \\ 0 & 13 & -5 & -57 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 24 \\ 0 & 1 & -\frac{1}{2} & -\frac{81}{14} \\ 0 & 0 & \frac{3}{2} & \frac{255}{14} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 24 \\ 0 & 1 & -\frac{1}{2} & -\frac{81}{14} \\ 0 & 0 & 1 & \frac{85}{7} \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -5 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{85}{7} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{85}{7} \end{array}\right), \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{8}{7} \\ \frac{2}{7} \\ \frac{85}{7} \end{pmatrix}$

(b) $\left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 3 & -2 & 1 & 9 \\ -3 & 1 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 0 & 13 & -5 & -36 \\ 0 & -14 & 7 & 42 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 0 & -14 & 7 & 42 \\ 0 & 13 & -5 & -36 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 13 & -5 & -36 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & \frac{3}{2} & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & 2 & 15 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -5 & 0 & 11 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right), \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

(c) $\left(\begin{array}{ccc|c} 1 & -1 & 5 & -3 \\ -2 & 5 & 1 & -19 \\ 5 & 1 & -2 & 33 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 5 & -3 \\ 0 & 3 & 11 & -25 \\ 0 & 6 & -27 & 48 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 5 & -3 \\ 0 & 1 & \frac{11}{3} & -\frac{25}{3} \\ 0 & 6 & -27 & 48 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -1 & 5 & -3 \\ 0 & 1 & \frac{11}{3} & -\frac{25}{3} \\ 0 & 0 & -49 & 98 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 5 & -3 \\ 0 & 1 & \frac{11}{3} & -\frac{25}{3} \\ 0 & 0 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$, $\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}$

(d) $\left(\begin{array}{ccc|c} 1 & 1 & -5 & 12 \\ -3 & -6 & 9 & 7 \\ -4 & -14 & 0 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -5 & 12 \\ 0 & -3 & -6 & 43 \\ 0 & -10 & -20 & 46 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -5 & 12 \\ 0 & 1 & 2 & -\frac{43}{3} \\ 0 & -10 & -20 & 46 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 1 & -5 & 12 \\ 0 & 1 & 2 & -\frac{43}{3} \\ 0 & 0 & 0 & -\frac{292}{3} \end{array}\right)$, $\therefore$ the system has no solution.

Question 4

(a) $\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -4 & 11 & 4 \\ 4 & 1 & 2 & h \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -10 & 20 & 10 \\ 0 & -7 & 14 & h+8 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & -7 & 14 & h+8 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h+1 \end{array}\right)$

The system is consistent iff $h + 1 = 0 \Leftrightarrow h = -1$

(b) $\left(\begin{array}{ccc|c} 1 & 4 & -3 & -10 \\ 3 & 1 & 2 & 14 \\ -4 & -5 & 1 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & -3 & -10 \\ 0 & -11 & 11 & 44 \\ 0 & 11 & -11 & k-40 \end{array}\right)$

(i) The system has no solution iff $k - 40 \neq -44 \Leftrightarrow k \neq -4$

(ii) The system has many solution iff $k - 40 = -44 \Leftrightarrow k = -4$

(c) The system has non-trivial solution iff

$$\begin{vmatrix} 1 & 3 & a \\ 2 & -1 & b \\ 4 & 5 & c \end{vmatrix} = 14a + 7b - 7c = 0 \quad \Leftrightarrow \quad 2a + b - c = 0$$

Question 5

(a) $$\left(\begin{array}{ccc|c} -1 & 1 & 3 & 10 \\ 1 & 2 & -15 & -7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -15 & -7 \\ -1 & 1 & 3 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -15 & -7 \\ 0 & 3 & -12 & 3 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & 2 & -15 & -7 \\ 0 & 1 & -4 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -7 & -9 \\ 0 & 1 & -4 & 1 \end{array}\right)$$
$\therefore \quad (x, y, z)^T = (7t - 9, 4t + 1, t)^T$, where $t \in \mathbb{R}$

(b) $$\left(\begin{array}{cccc|c} 1 & 2 & -16 & -6 & 19 \\ -3 & 1 & 20 & -17 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -16 & -6 & 19 \\ 0 & 7 & -28 & -35 & 56 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -16 & -6 & 19 \\ 0 & 1 & -4 & -5 & 8 \end{array}\right)$$
$$\to \left(\begin{array}{cccc|c} 1 & 0 & -8 & 4 & 3 \\ 0 & 1 & -4 & -5 & 8 \end{array}\right)$$
$\therefore \quad (x, y, z, w)^T = (3 + 8s - 4t, 8 + 4s + 5t, s, t)^T$, where $s, t \in \mathbb{R}$

(c) $$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 1 & 2 & 10 & -2 \\ 3 & 1 & 10 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 3 & 12 & -3 \\ 0 & 4 & 16 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 4 & 16 & -4 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$\therefore \quad (x, y, z)^T = (-2t, -4t - 1, t)^T$, where $t \in \mathbb{R}$

(d) $$\left(\begin{array}{ccc|c} 1 & 1 & -13 & 13 \\ 1 & 2 & -20 & 18 \\ -1 & 8 & -50 & 32 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -13 & 13 \\ 0 & 1 & -7 & 5 \\ 0 & 9 & -63 & 45 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -13 & 13 \\ 0 & 1 & -7 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & 0 & -6 & 8 \\ 0 & 1 & -7 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$\therefore \quad (x, y, z)^T = (6t + 8, 7t + 5, t)^T$, where $t \in \mathbb{R}$

Question 6

(a) $$\left(\begin{array}{ccc|ccc} 1 & -1 & -6 & 1 & 0 & 0 \\ 2 & -1 & -9 & 0 & 1 & 0 \\ -2 & 1 & 10 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & -6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & -1 & -2 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & -6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 6 & 6 \\ 0 & 1 & 0 & -2 & -2 & -3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 3 \\ 0 & 1 & 0 & -2 & -2 & -3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore \quad \begin{pmatrix} 1 & -1 & -6 \\ 2 & -1 & -9 \\ -2 & 1 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 4 & 3 \\ -2 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

(b) $$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ -2 & -1 & 3 & 0 & 1 & 0 \\ 3 & 2 & -5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & -1 & 4 & -3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & 3 \\ 0 & 1 & 0 & -1 & 4 & 3 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 4 & 3 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\therefore \quad \begin{pmatrix} 1 & 1 & -3 \\ -2 & -1 & 3 \\ 3 & 2 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 4 & 3 \\ -1 & 1 & 1 \end{pmatrix}$$

(c) $$\begin{vmatrix} -6 & 1 & 20 \\ 1 & -6 & 20 \\ -1 & -4 & 20 \end{vmatrix} = 0, \quad \therefore \quad \begin{pmatrix} -6 & 1 & 20 \\ 1 & -6 & 20 \\ -1 & -4 & 20 \end{pmatrix} \text{ is not invertible}$$

(d) $$\left(\begin{array}{ccc|ccc} 3 & 1 & -9 & 1 & 0 & 0 \\ 2 & 3 & -9 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 2 & 3 & -9 & 0 & 1 & 0 \\ 3 & 1 & -9 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & -2 & 3 & 1 & 0 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 4 & 8 & -27 \\ 0 & 1 & 0 & 1 & 3 & -9 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 5 & -18 \\ 0 & 1 & 0 & 1 & 3 & -9 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{array}\right)$$

$$\therefore \quad \begin{pmatrix} 3 & 1 & -9 \\ 2 & 3 & -9 \\ 1 & 1 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 & -18 \\ 1 & 3 & -9 \\ 1 & 2 & -7 \end{pmatrix}$$

(e) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ -2 & -1 & 3 \\ 3 & 2 & -5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 4 & 3 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$$

(f) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ -2 & -1 & 3 \\ 3 & 2 & -5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 4 & 3 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 16 \\ 4 \end{pmatrix}$$

Question 7

(a) $$\begin{vmatrix} 4 & 0 & 2 \\ -1 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = -26$$

(b) $$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 3 & 0 \\ 7 & 6 & 5 \end{vmatrix} = 15$$

(c) $$\begin{vmatrix} 0 & 5 & 0 & -2 \\ 4 & -3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -1 & 1 & 3 & 4 \end{vmatrix} = -4 \times (-3)(10+2) - (-1) \times 0 = 144$$

(d) $$\begin{vmatrix} 1 & 3 & -1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 5 & 8 & -2 & -3 \end{vmatrix} = 1 \times 2 - 5 \times (-2) = 12$$

(e) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & -1 & 1 & -1 \\ 5 & 0 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -5 & -5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 5 & 15 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 5 & 15 \end{vmatrix} = 40$$

(f) $$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -6 & 1 \\ 0 & -6 & -8 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 20 \end{vmatrix} = 96$$

Question 8

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$

$$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a)$$

(b) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\,[\,-(b-c)^2-(a-b)(a-c)\,]$$

$$= (a+b+c)\,(ab+bc+ca-a^2-b^2-c^2)$$

Question 9

$\begin{vmatrix} 1 & -7 & -1 \\ 8 & 4 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 105,\ \begin{vmatrix} 25 & -7 & -1 \\ -25 & 4 & 1 \\ -15 & 1 & 2 \end{vmatrix} = -105,\ \begin{vmatrix} 1 & 25 & -1 \\ 8 & -25 & 1 \\ 2 & -15 & 2 \end{vmatrix} = -315,\ \begin{vmatrix} 1 & -7 & 25 \\ 8 & 4 & -25 \\ 2 & 1 & -15 \end{vmatrix} = -525$

By Cramer's rule,

$\therefore \quad x = \dfrac{-105}{105} = -1,\ \ y = \dfrac{-315}{105} = -3,\ \ z = \dfrac{-525}{105} = -5$

Question 10

(a) $\int x\cos nx\,dx = \dfrac{1}{n}x\sin nx + \dfrac{1}{n^2}\cos nx + C$

(b) $\int (x-2)^2\sin nx\,dx = -\dfrac{1}{n}(x-2)^2\cos nx + \dfrac{2}{n^2}(x-2)\sin nx + \dfrac{2}{n^3}\cos nx + C$

(c) $\int (3x-2)^2\sin nx\,dx = -\dfrac{1}{n}(3x-2)^2\cos nx + \dfrac{6}{n^2}(3x-2)\sin nx + \dfrac{18}{n^3}\cos nx + C$

Question 11

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & -2 & -2 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\therefore \quad \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 6b \\ 0 \\ -3b \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 6b \\ 0 \\ -3b \end{pmatrix} = \begin{pmatrix} 2b \\ 2b \\ 5b \end{pmatrix}$$

Question 12

(a) $$AXA^T = \begin{pmatrix} 1 & -4 \\ a & -2 \end{pmatrix} \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} 1 & a \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} x_1 - 4x_2 & x_3 - 4x_4 \\ ax_1 - 2x_2 & ax_3 - 2x_4 \end{pmatrix} \begin{pmatrix} 1 & a \\ -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - 4x_2 - 4x_3 + 16x_4 & ax_1 - 4ax_2 - 2x_3 + 8x_4 \\ ax_1 - 2x_2 - 4ax_3 + 8x_4 & a^2x_1 - 2ax_2 - 2ax_3 + 4x_4 \end{pmatrix} = B = \begin{pmatrix} b & 3b \\ 2b & 4b \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 1 & -4 & -4 & 16 \\ a & -2 & -4a & 8 \\ a & -4a & -2 & 8 \\ a^2 & -2a & -2a & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b \\ 2b \\ 3b \\ 4b \end{pmatrix}$$

Remark (a) involves vectorization of matrix. In general, $\mathrm{vec}(ABC) = (C^T \otimes A)\ \mathrm{vec}(B)$

(b) The system is consistent if and only if $b = 0$ or

$$\begin{vmatrix} 1 & -4 & -4 & 16 \\ a & -2 & -4a & 8 \\ a & -4a & -2 & 8 \\ a^2 & -2a & -2a & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & 4a-2 & 0 & 8-16a \\ a & 0 & 4a-2 & 8-16a \\ a^2 & 4a^2-2a & 4a^2-2a & 4-16a^2 \end{vmatrix}$$

$$= (4a-2)^3 \begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ a & a & -4a-2 \end{vmatrix} = (4a-2)^3 \begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 4a-2 \end{vmatrix}$$

$$= (4a-2)^4 \neq 0$$

$\Leftrightarrow\ b = 0$ or $a \neq \dfrac{1}{2}$

Remark If $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, then $\det(A \otimes B) = (\det A)^m (\det B)^n$.

(c) *Method 1*

When $a = 1$ and $b = 2$, the system becomes

$$\left(\begin{array}{cccc|c} 1 & -4 & -4 & 16 & 2 \\ 1 & -2 & -4 & 8 & 4 \\ 1 & -4 & -2 & 8 & 6 \\ 1 & -2 & -2 & 4 & 8 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -4 & 16 & 2 \\ 0 & 2 & 0 & -8 & 2 \\ 0 & 0 & 2 & -8 & 4 \\ 0 & 2 & 2 & -12 & 6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -4 & 16 & 2 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 1 & 1 & -6 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -4 & -4 & 16 & 2 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -4 & 16 & 2 \\ 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -4 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 14 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \quad \therefore\ X = \begin{pmatrix} 14 & 2 \\ 1 & 0 \end{pmatrix}$$

Method 2

$$A^{-1} = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} -2 & 4 \\ -1 & 1 \end{pmatrix}$$

$$X = A^{-1} B (A^T)^{-1} = \frac{1}{2}\begin{pmatrix} -2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}\frac{1}{2}\begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 2 \\ 1 & 0 \end{pmatrix}$$