The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 1 2021/22 Semester 2

Question 1.

(60 marks) Let $f(x) = 5x^3 - 3x^5, \; -\infty < x < \infty$

(a) Find all critical points, if any.

(b) Find all open intervals where the function is increasing or decreasing, if any.

(c) Find all local (i.e., relative) and global (i.e., absolute) maximum and minimum, if any.

(d) Find all open intervals where the function is concave-up (i.e., convex) or concave-down (i.e., concave), if any.

(e) Find all inflection points, if any.

(f) Sketch the curve of the function $f(x)$.

My work :

(a) $f'(x) = 15x^2 - 15x^4 = 0 \;\Leftrightarrow\; x = -1, 0, 1$

The critical points of f are: $-1, 0, 1$

(b)

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
f'	$-$	0	$+$	0	$+$	0	$-$
f	↘	-2	↗	0	↗	2	↘

The open interval where f is increasing is: $(-1, 1)$

The open intervals where f is decreasing are: $(-\infty, -1)$, $(1, +\infty)$

(c) f attains local minimum at $x = -1$ and $f(-1) = -2$

f attains local maximum at $x = 1$ and $f(1) = 2$

Since $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = -\infty$, f has no global maximum nor global minimum.

(d) $f''(x) = 30x - 60x^3 = 0 \;\Leftrightarrow\; x = 0, \pm\frac{1}{\sqrt{2}}$

x	$(-\infty, -\frac{1}{\sqrt{2}})$	$-\frac{1}{\sqrt{2}}$	$(-\frac{1}{\sqrt{2}}, 0)$	0	$(0, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}}, +\infty)$
f''	$+$	0	$-$	0	$+$	0	$-$
f	convex	$-\frac{7}{4\sqrt{2}}$	concave	0	convex	$\frac{7}{4\sqrt{2}}$	concave

The open intervals where f is convex are: $(-\infty, -\frac{1}{\sqrt{2}})$, $(0, \frac{1}{\sqrt{2}})$

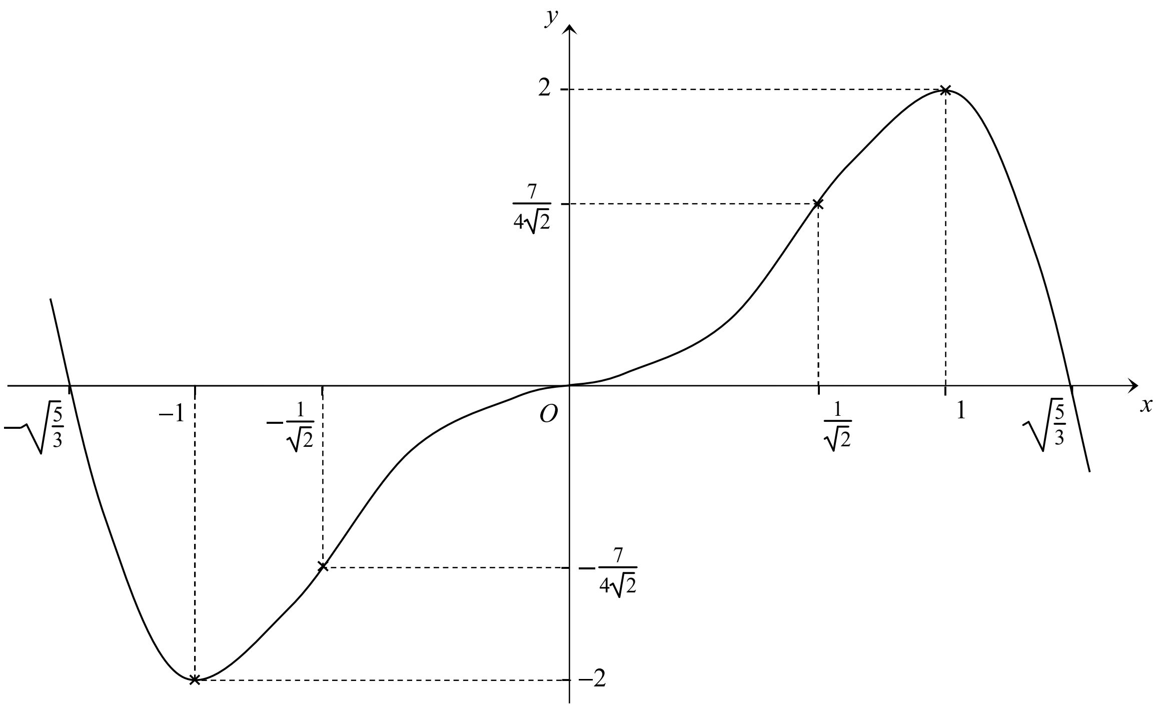
The open intervals where f is concave are: $(-\frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{\sqrt{2}}, +\infty)$

(e) Change of convexity / concavity occurs at $x = 0, \pm\frac{1}{\sqrt{2}}$

The inflection points are: $0, \pm\frac{1}{\sqrt{2}}$

(f) y-intercept: $f(0) = 0$

x-intercepts: $f(x) = 5x^3 - 3x^5 = 0 \Leftrightarrow x = 0, \pm\sqrt{\frac{5}{3}}$



Question 2.

(20 marks) Let a and b be two real numbers such that $a > b > 0$. Demonstrate that

$$e^{a^2} - e^{b^2} > 2\,(ab - b^2)\,e^{b^2}$$

My work :

Let $f(x) = e^{x^2}$, which is continuous on $[b, a]$ and differentiable in (b, a), and

$f'(x) = 2x\, e^{x^2}$,

$f''(x) = (2 + 4x^2)\, e^{x^2} > 0$ for all $x \Rightarrow f'$ is strictly increasing.

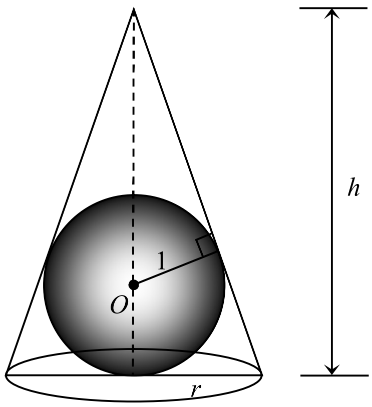
By Mean-Value Theorem, $\exists\, \xi \in (b, a)$ such that

$f(a) - f(b) = f'(\xi)(a - b) > f'(b)(a - b)$, since $a - b > 0$

i.e. $e^{a^2} - e^{b^2} > (2b\, e^{b^2})(a - b) = 2(ab - b^2)\, e^{b^2}$

Question 3.

(20 marks) A ball of radius 1 is contained in a right circular cone. Find the smallest possible surface area of the cone. [Hint: If the base radius of the cone is r and its height is h, then its surface area is $\pi r^2 + \pi r\sqrt{h^2 + r^2}$.]



My work :

By similar triangle, we have $\dfrac{r}{1} = \dfrac{h}{\sqrt{(h-1)^2 - 1^2}} = \sqrt{\dfrac{h}{h-2}}$ where $h > 2$

As hinted, the surface area $= \pi r^2 + \pi r\sqrt{h^2 + r^2} = \pi r^2\left(1 + \sqrt{1 + \dfrac{h^2}{r^2}}\right)$

$$= \frac{\pi h}{h-2}\left[1 + \sqrt{1 + h(h-2)}\,\right] = \frac{\pi h^2}{h-2} =: S(h) \text{ on } h > 2$$

$S'(h) = \pi \cdot \dfrac{(h-2)(2h) - h^2}{(h-2)^2} = 0 \quad \Leftrightarrow \quad h = 4$

h	$(2, 4)$	4	$(4, +\infty)$
$S(h)$	↘	8π	↗
$S'(h)$	$-$	0	$+$

The smallest possible surface area of the cone is 8π with $r = \sqrt{2}$ and $h = 4$.