The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #02

Question 1

(a) $f'(x) = 4x^3 - 12x + 8$

$f''(x) = 12x^2 - 12 = 0 \quad \Leftrightarrow \quad x = \pm 1$

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f''(x)$	$+$	$-$	$+$

The intervals where f is concave upwards are: $(-\infty, -1)$, $(1, +\infty)$

The interval where f is concave downwards is: $(-1, 1)$

(b) $f'(x) = \dfrac{2}{3}x^{-1/3}(1-x)^{1/3} + x^{2/3} \cdot \dfrac{1}{3}(1-x)^{-2/3}(-1) = \dfrac{\frac{2}{3} - x}{x^{1/3}(1-x)^{2/3}}$

$$f''(x) = (-1)\,x^{-1/3}(1-x)^{-2/3} + (\tfrac{2}{3} - x)(-\tfrac{1}{3}x^{-4/3})(1-x)^{-2/3} + (\tfrac{2}{3} - x)\,x^{-1/3} \cdot \tfrac{2}{3}(1-x)^{-5/3}$$

$= \dfrac{-\frac{2}{9}}{x^{4/3}(1-x)^{5/3}}$ is not continuous at $x = 0, 1$

x	$(-\infty, 0)$	$(0, 1)$	$(1, +\infty)$
$f''(x)$	$-$	$-$	$+$

The interval where f is concave upwards is: $(1, +\infty)$

The intervals where f is concave downwards are: $(-\infty, 0)$, $(0, 1)$

(c) $f'(x) = 1 - \dfrac{4}{(x-1)^2}$

$f''(x) = \dfrac{8}{(x-1)^3}$ is not continuous at $x = 1$

x	$(-\infty, 1)$	$(1, +\infty)$
$f''(x)$	$-$	$+$

The interval where f is concave upwards is: $(1, +\infty)$

The interval where f is concave downwards is: $(-\infty, 1)$

(d) $f(x) = \dfrac{x^3}{x^2 - 3x + 2} = x + 3 + \dfrac{7x - 6}{(x-1)(x-2)} = x + 3 - \dfrac{1}{x-1} + \dfrac{8}{x-2}$

$$f'(x) = 1 + \frac{1}{(x-1)^2} - \frac{8}{(x-2)^2}$$

$$f''(x) = \frac{16}{(x-2)^3} - \frac{2}{(x-1)^3} = 0 \;\Rightarrow\; 8\,(x-1)^3 = (x-2)^3 \;\Rightarrow\; 2x - 2 = x - 2 \;\Rightarrow\; x = 0$$

and f'' is not continuous at $x = 1, 2$

x	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, +\infty)$
$f''(x)$	$-$	$+$	$-$	$+$

The intervals where f is concave upwards are: $(0, 1)$, $(2, +\infty)$

The intervals where f is concave downwards are: $(-\infty, 0)$, $(1, 2)$

(e) $f'(x) = -2x\, e^{-x^2}$

$$f''(x) = 2\,(2x^2 - 1)\, e^{-x^2} = 0 \;\Leftrightarrow\; x = \pm\frac{1}{\sqrt{2}}$$

x	$(-\infty, -\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}}, +\infty)$
$f''(x)$	$+$	$-$	$+$

The intervals where f is concave upwards are: $(-\infty, -\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, +\infty)$

The interval where f is concave downwards is: $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Question 2

$\Leftarrow$) Trivial.

$\Rightarrow$) Suppose $\lim\limits_{x\to\infty} [\, f(x) - (mx + b)\,] = 0$ for some $m, b \in \mathbb{R}$. Then

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left(\frac{f(x) - mx}{x} + m \right) = \lim_{x\to\infty} \frac{f(x) - mx}{x} + m = \lim_{x\to\infty} \frac{f(x) - mx}{x} - \lim_{x\to\infty} \frac{b}{x} + m$$

$$= \lim_{x\to\infty} \frac{f(x) - (mx + b)}{x} + m = m$$

and trivially $b = \lim\limits_{x\to\infty} (\, f(x) - mx\,)$.

Question 3

(a) $\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \left(\frac{1}{x} - 1\right)^{1/3} = -1$

$$\lim_{x \to \pm\infty} [\, f(x) + x \,] = \lim_{x \to \pm\infty} [\, x^{2/3}(1-x)^{1/3} + x \,] = \lim_{x \to \pm\infty} x\left[\left(\frac{1}{x} - 1\right)^{1/3} + 1\right]$$

$$= \lim_{u \to -1^{\pm}} \frac{u+1}{u^3+1} = \lim_{u \to -1^{\pm}} \frac{1}{3u^2} = \frac{1}{3}$$

$\therefore \quad y = -x + \frac{1}{3}$ is an inclined asymptote of $y = f(x)$

(b) *Method 1*

$$\lim_{x \to 1^{\pm}} f(x) = \lim_{x \to 1^{\pm}} \frac{(x+1)^2}{x-1} = \pm\infty$$

$\therefore \quad x = 1$ is a vertical asymptote of $y = f(x)$.

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{(x+1)^2}{x(x-1)} = \lim_{x \to \pm\infty} \frac{(1+\frac{1}{x})^2}{1-\frac{1}{x}} = 1$$

$$\lim_{x \to \pm\infty} [\, f(x) - x \,] = \lim_{x \to \pm\infty} \left[\frac{(x+1)^2}{x-1} - x\right] = \lim_{x \to \pm\infty} \frac{3x+1}{x-1} = 3$$

$\therefore \quad y = x + 3$ is an inclined asymptote of $y = f(x)$.

Method 2

$$\frac{(x+1)^2}{x-1} = x + 3 + \frac{4}{x-1}$$

$\therefore \quad x = 1$ is a vertical asymptote of $y = f(x)$.

$\therefore \quad y = x + 3$ is an inclined asymptote of $y = f(x)$.

(c) $\frac{x^3}{x^2 - 3x + 2} = x + 3 + \frac{7x - 6}{x^2 - 3x + 2} = x + 3 + \frac{7x - 6}{(x-1)(x-2)}$

$\therefore \quad x = 1, \quad x = 2$ are vertical asymptotes of $y = f(x)$

$\therefore \quad y = x + 3$ is an inclined asymptote of $y = f(x)$.

(d) $\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{e^{-x^2}}{x} = 0, \quad \lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} e^{-x^2} = 0$

$\therefore \quad y = 0$ is a horizontal asymptote of $y = f(x)$.

Question 4

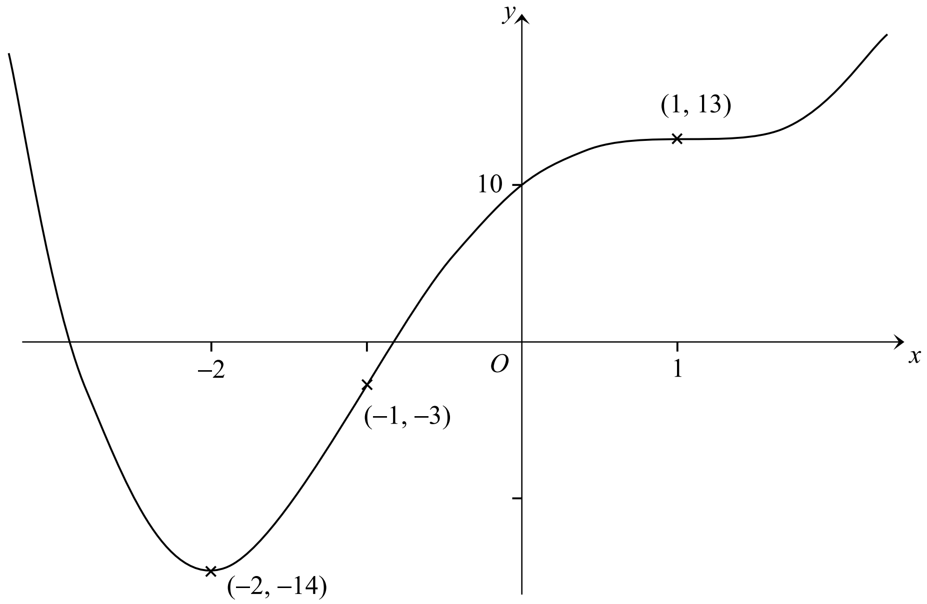
(a) Note that $f(x) = x^4 - 6x^2 + 8x + 10$ is defined on $\mathbb{R}$.

$f(0) = 10$

$f'(x) = 4x^3 - 12x + 8 = 0 \iff x = 1, -2$

$f''(x) = 12x^2 - 12 = 0 \iff x = \pm 1$

	$x <$	-2	$< x <$	-1	$< x <$	1	$< x$
f	↘	-14	↗	-3	↗	13	↗
f'	$-$	0	$+$	$+$	$+$	0	$+$
f''	$+$	$+$	$+$	0	$-$	0	$+$



(b) Note that $f(x) = x^{2/3}(1-x)^{1/3}$ is defined on $\mathbb{R}$.

$f(x) = 0 \iff x = 0, 1.$

$$f'(x) = -\frac{x - \frac{2}{3}}{x^{1/3}(1-x)^{2/3}} = 0 \quad \text{for } x \neq 0, 1 \iff x = \frac{2}{3}$$

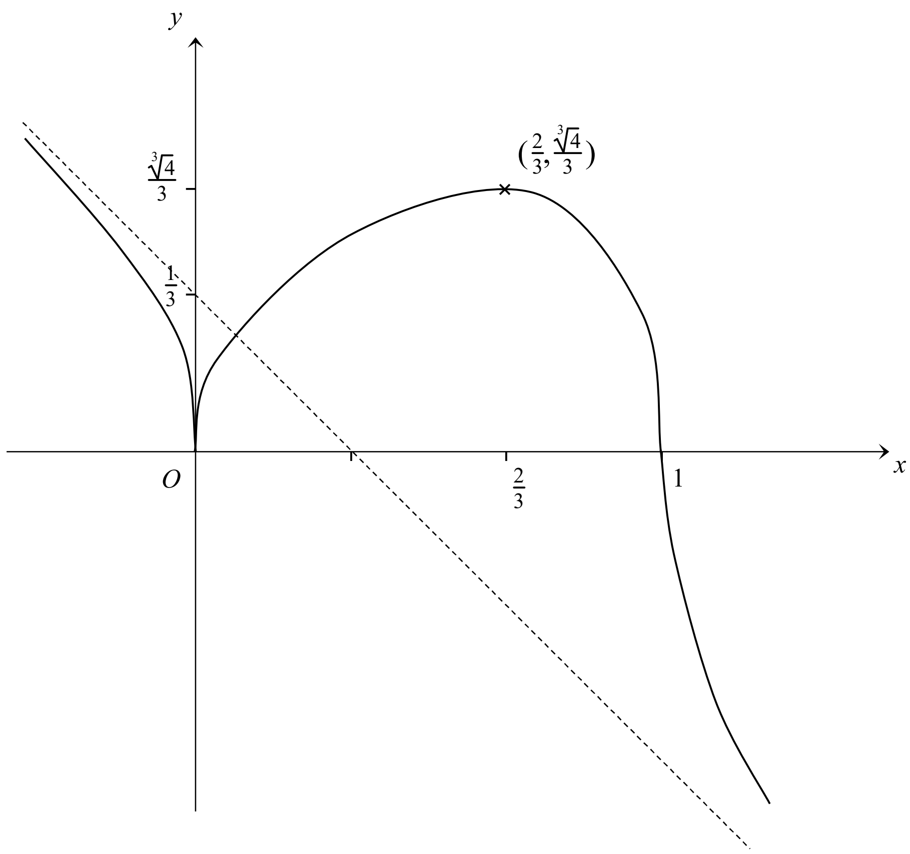
$$f'(0) = \lim_{x \to 0} \frac{x^{2/3}(1-x)^{1/3} - 0}{x - 0} = \lim_{x \to 0} \left(\frac{1-x}{x}\right)^{1/3} \text{ does not exist}$$

$$f'(1) = \lim_{x \to 1} \frac{x^{2/3}(1-x)^{1/3} - 0}{x - 1} = -\lim_{x \to 1} \left(\frac{x}{1-x}\right)^{2/3} \text{ does not exist}$$

$$f''(x) = -\frac{2}{9x^{4/3}(1-x)^{5/3}} \neq 0 \quad \text{for } x \neq 0, 1, \quad f''(0) \text{ and } f''(1) \text{ do not exist}$$

	$x <$	0	$< x <$	$\frac{2}{3}$	$< x <$	1	$< x$
f	↘	0	↗	$\frac{\sqrt[3]{4}}{3}$	↘	0	↘
f'	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$-$	$\nexists$	$-$	$-$	$-$	$\nexists$	$+$

By 3(a), $y = -x + \frac{1}{3}$ is an inclined asymptote of $y = f(x)$ when $x \to \pm\infty$.



(c) Note that $f(x) = \dfrac{(x+1)^2}{x-1} = x + 3 + \dfrac{4}{x-1}$ is defined on $\mathbb{R} \setminus \{1\}$.

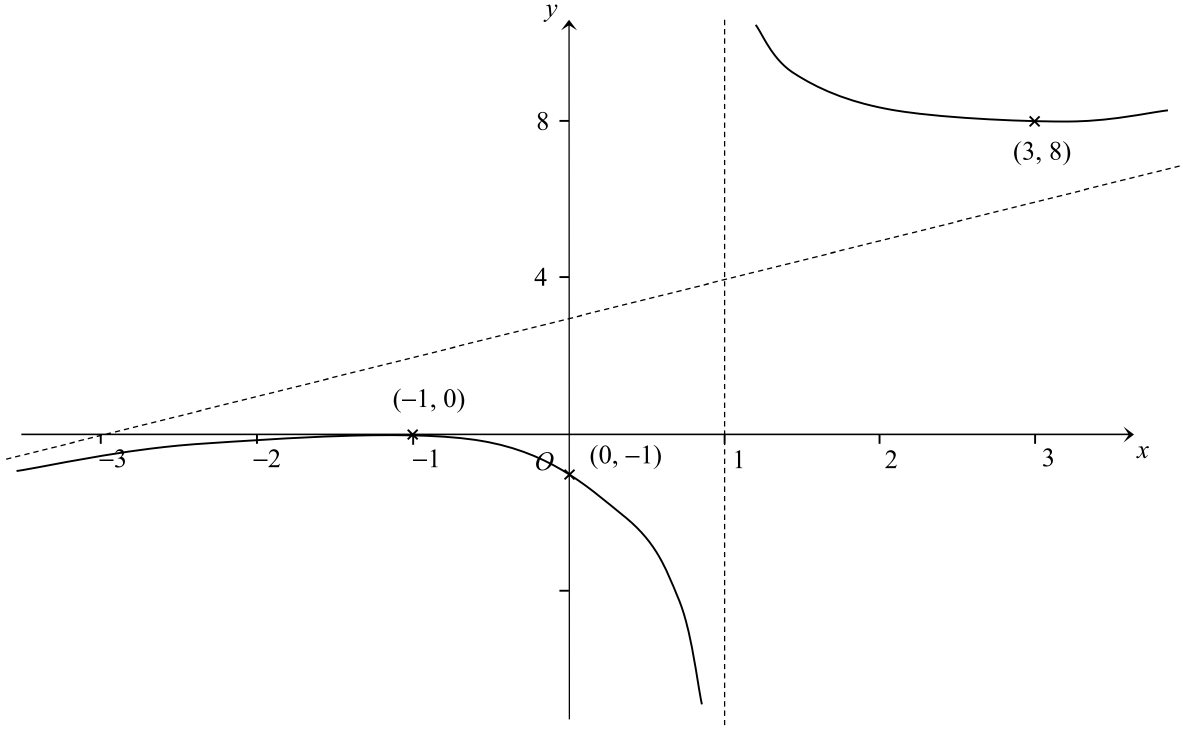
$f(x) = 0 \iff x = -1$, and $f(0) = -1$

$f'(x) = 1 - \dfrac{4}{(x-1)^2} = 0$ for $x \neq 1 \iff x = -1, 3$

$f''(x) = \dfrac{8}{(x-1)^3}$ for $x \neq 1$

$x = 1$ is vertical asymptote, $y = x + 3$ is an inclined asymptote.

	$x<$	-1	$<x<$	1	$<x<$	3	$<x$
f	$\nearrow$	0	$\searrow$	$\nexists$	$\searrow$	8	$\nearrow$
f'	$+$	0	$-$	$\nexists$	$-$	0	$+$
f''	$-$	$-$	$-$	$\nexists$	$+$	$+$	$+$



(d) Note that

$$f(x)=\frac{x^3}{x^2-3x+2}=x+3+\frac{7x-6}{(x-1)(x-2)}=x+3-\frac{1}{x-1}+\frac{8}{x-2}$$

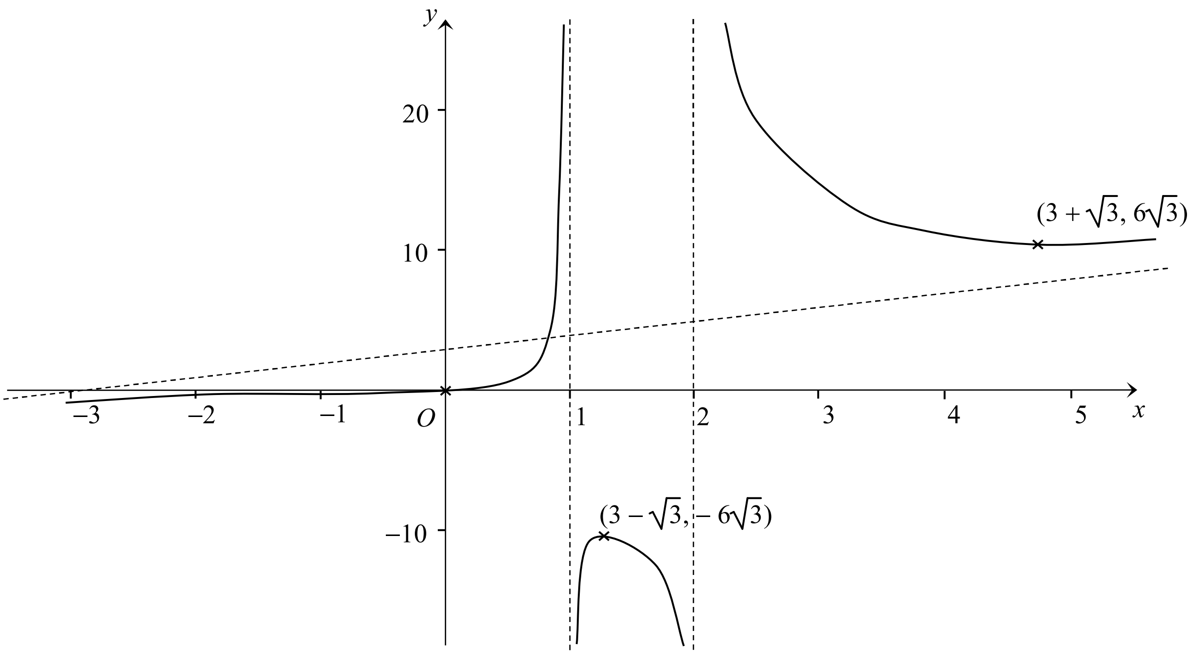
is defined on $\mathbb{R}\setminus\{1,2\}$.

$$f'(x)=1+\frac{1}{(x-1)^2}-\frac{8}{(x-2)^2}=0 \iff \frac{x^2(x^2-6x+6)}{(x-1)^2(x-2)^2}=0 \iff x=0,\ 3\pm\sqrt{3}$$

$$f''(x)=-\frac{2}{(x-1)^3}+\frac{16}{(x-2)^3}=0 \Rightarrow 8(x-1)^3=(x-2)^3 \Rightarrow 2x-2=x-2 \Rightarrow x=0$$

$x=1$, $x=2$ are vertical asymptotes, and $y=x+3$ is an inclined asymptote

	$x<$	0	$<x<$	1	$<x<$	$3-\sqrt{3}$	$<x<$	2	$<x<$	$3+\sqrt{3}$	$<x$
f	$\nearrow$	0	$\nearrow$	$\nexists$	$\nearrow$	$-6\sqrt{3}$	$\searrow$	$\nexists$	$\searrow$	$6\sqrt{3}$	$\nearrow$
f'	$+$	0	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$	0	$+$
f''	$-$	0	$+$	$\nexists$	$-$	$-$	$-$	$\nexists$	$+$	$+$	$+$



(e) Note that $f(x) = e^{-x^2} > 0$ is defined on $\mathbb{R}$.

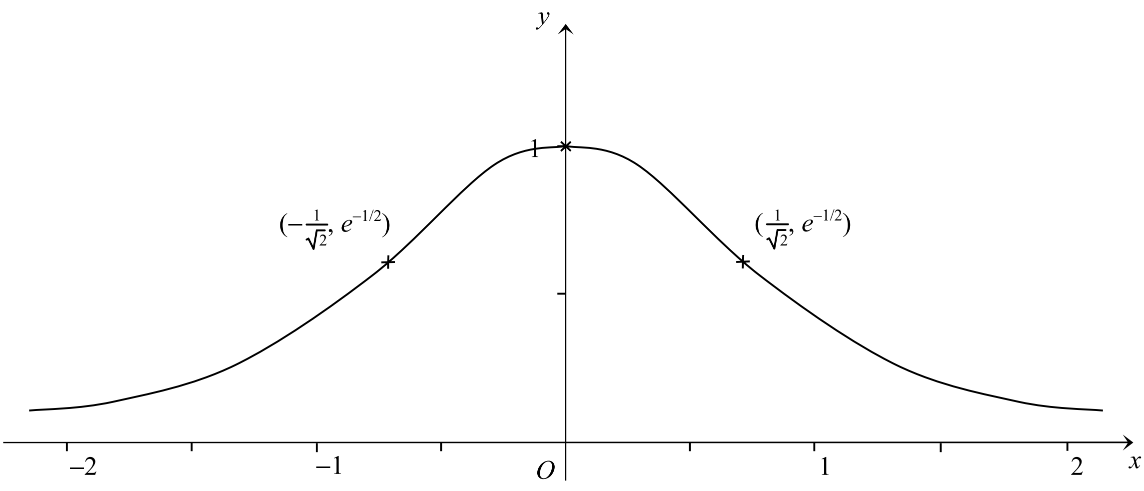
$f(0) = 1$

$f'(x) = -2x\, e^{-x^2} = 0 \iff x = 0$

$f''(x) = 2\,(2x^2 - 1)\, e^{-x^2} = 0 \iff x = \pm\dfrac{1}{\sqrt{2}}$

Since $\lim\limits_{x \to \pm\infty} [\, f(x) - 0\,] = \lim\limits_{x \to \pm\infty} e^{-x^2} = 0$, $y = 0$ is a horizontal asymptote

	$x <$	$-\frac{1}{\sqrt{2}}$	$< x <$	0	$< x <$	$\frac{1}{\sqrt{2}}$	$< x$
f	$\nearrow$	$e^{-1/2}$	$\nearrow$	1	$\searrow$	$e^{-1/2}$	$\searrow$
f'	$+$	$+$	$+$	0	$-$	$-$	$-$
f''	$+$	0	$-$	$-$	$-$	0	$+$



Question 5

(a) Let $g(x) = \dfrac{2x^2 + x - 1}{x - 1} = 0 \iff x = \frac{1}{2}, -1$, and g is undefined at $x = 1$.

	$x <$	-1	$< x <$	$\frac{1}{2}$	$< x <$	1	$< x$
g	$-$	0	$+$	0	$-$	$\nexists$	$+$

For $x \in (-1, \frac{1}{2}) \cup (1, +\infty)$, $f(x) = g(x) = 2x + 3 + \dfrac{2}{x-1} \quad \Rightarrow f'(x) = 2 - \dfrac{2}{(x-1)^2}$

For $x \in (-\infty, -1) \cup (\frac{1}{2}, 1)$, $f(x) = -g(x) \quad \Rightarrow f'(x) = -2 + \dfrac{2}{(x-1)^2}$

(b) $$f_-'(-1) = \lim_{x \to -1^-} \frac{-\dfrac{2x^2 + x - 1}{x - 1} - 0}{x - (-1)} = \lim_{x \to -1^-} \left(-\frac{2x-1}{x-1}\right) = -\frac{3}{2}$$

$$f_+'(-1) = \lim_{x \to -1^+} \frac{\dfrac{2x^2 + x - 1}{x - 1} - 0}{x - (-1)} = \lim_{x \to -1^+} \frac{2x-1}{x-1} = \frac{3}{2}$$

$\therefore \quad f_-'(-1) \neq f_+'(-1) \quad \Rightarrow \quad f'(-1)$ does not exist.

$$f_-'(\tfrac{1}{2}) = \lim_{x \to 1/2^-} \frac{\dfrac{2x^2 + x - 1}{x - 1} - 0}{x - \frac{1}{2}} = \lim_{x \to 1/2^-} \frac{2x+2}{x-1} = -6$$

$$f_+'(\tfrac{1}{2}) = \lim_{x \to 1/2^+} \frac{-\dfrac{2x^2 + x - 1}{x - 1} - 0}{x - \frac{1}{2}} = \lim_{x \to 1/2^+} \left(-\frac{2x+2}{x-1}\right) = 6$$

$\therefore \quad f_-'(\frac{1}{2}) \neq f_+'(\frac{1}{2}) \quad \Rightarrow \quad f'(\frac{1}{2})$ does not exist.

(c) $f'(x) = 0 \iff 2 = \dfrac{2}{(x-1)^2} \iff (x-1)^2 = 1 \iff x = 0, 2$

	$x <$	-1	$< x <$	0	$< x <$	$\frac{1}{2}$	$< x <$	1	$< x <$	2	$< x$
f	↘	0	↗	1	↘	0	↗	$\nexists$	↘	9	↗
f'	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$+$	$\nexists$	$-$	0	$+$

$\therefore \quad f$ has a local maximum at $x = 0$ and local minima at $x = -1, \frac{1}{2}, 2$.

(d) For $x > 1$, $f(x) = 2x + 3 + \dfrac{2}{x-1}$.

$\therefore$ $y = 2x + 3$ is an inclined asymptote, and $x = 1$ is a vertical asymptote.

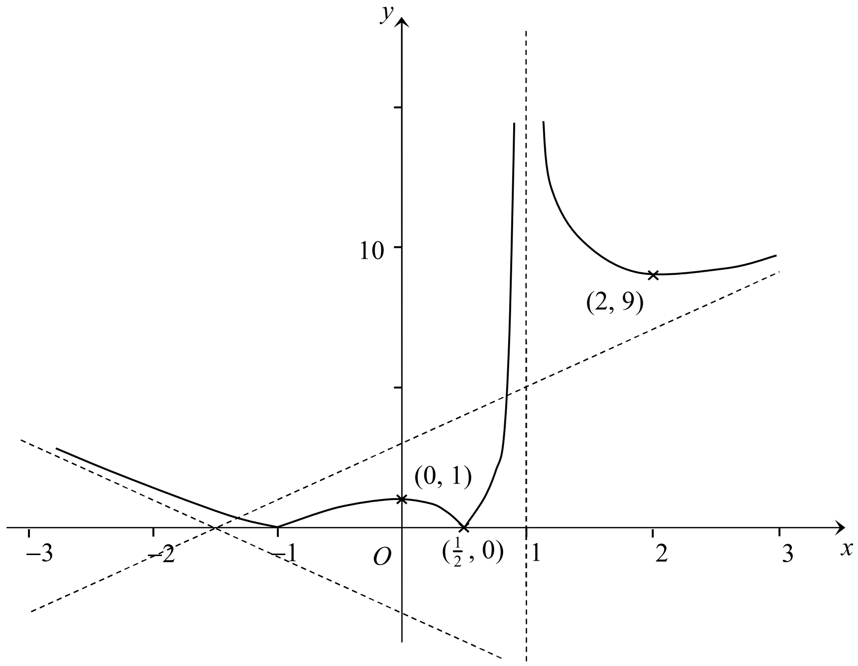
For $x < -1$, $f(x) = -2x - 3 - \dfrac{2}{x-1}$

$\therefore$ $y = -2x - 3$ is an inclined asymptote

(e) For $x \in (-1, \frac{1}{2}) \cup (1, +\infty)$, $f''(x) = \dfrac{4}{(x-1)^3}$

For $x \in (-\infty, -1) \cup (\frac{1}{2}, 1)$, $f''(x) = -\dfrac{4}{(x-1)^3}$

	$x <$	-1	$< x <$	0	$< x <$	$\frac{1}{2}$	$< x <$	1	$< x <$	2	$< x$
f	$\searrow$	0	$\nearrow$	1	$\searrow$	0	$\nearrow$	$\nexists$	$\searrow$	9	$\nearrow$
f'	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$+$	$\nexists$	$-$	0	$+$
f''	$+$	$\nexists$	$-$	$-$	$-$	$\nexists$	$+$	$\nexists$	$+$	$+$	$+$



Question 6

(a) $f(x) = x^2(2-x)\,e^{-x} = (-x^3 + 2x^2)\,e^{-x}$ is defined on $\mathbb{R}$.

$f'(x) = (x^3 - 5x^2 + 4x)\,e^{-x} = x\,(x-1)\,(x-4)\,e^{-x}$

$f''(x) = (-x^3 + 8x^2 - 14x + 4)\,e^{-x} = -(x-2)\,(x^2 - 6x + 2)\,e^{-x}$

(b) $f'(x) = 0 \iff x = 0, 1, 4$

$f''(x) = 0 \iff x = 2, 3 \pm \sqrt{7}$

	$x<$	0	$<x<$	$3-\sqrt{7}$	$<x<$	1	$<x<$	2	$<x<$	4	$<x<$	$3+\sqrt{7}$	$<x$
f	$\searrow$	0	$\nearrow$	0.1449	$\nearrow$	e^{-1}	$\searrow$	0	$\searrow$	$-32e^{-4}$	$\nearrow$	-0.4104	$\nearrow$
f'	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
f''	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$

$\therefore$ f has local minima at $x = 0, 4$, and local maximum at $x = 1$

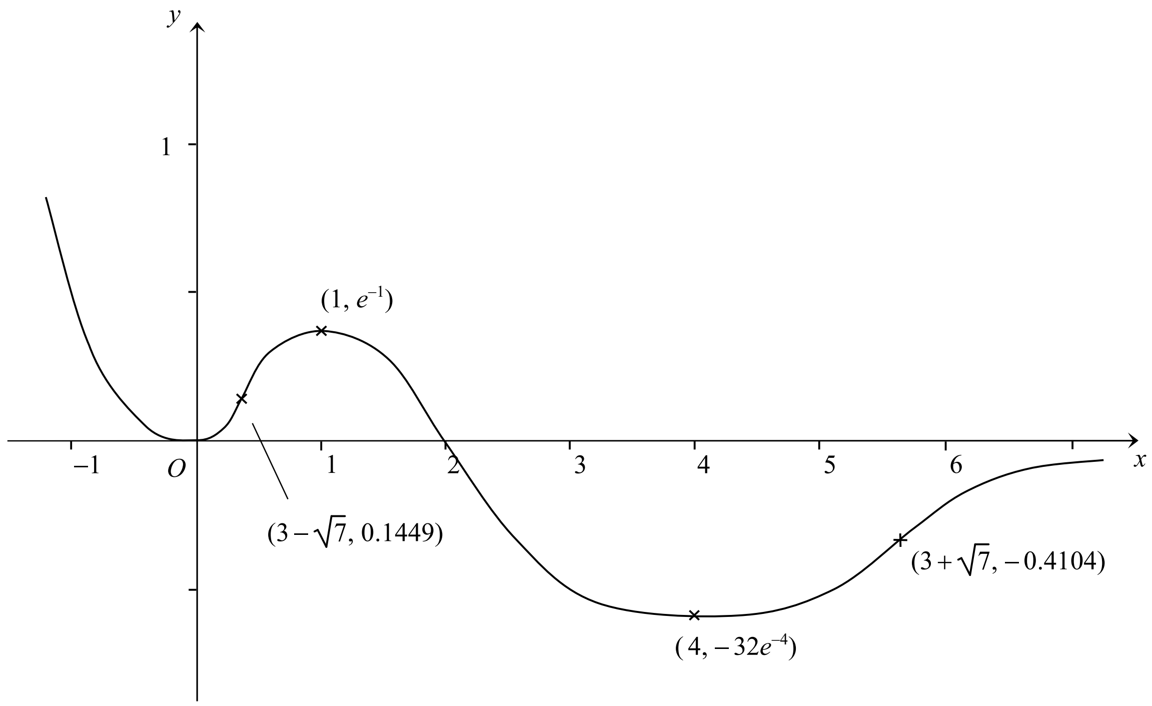
Change of convexity occurs at $x = 2, 3 \pm \sqrt{7}$

$\therefore$ f has points of inflexion at $x = 2, 3 \pm \sqrt{7}$.

(c) $\lim\limits_{x \to \infty} [f(x) - 0] = \lim\limits_{x \to \infty} \dfrac{-x^3 + 2x^2}{e^x} = 0$ by l'Hôpital's rule

$\therefore$ $y = 0$ is a horizontal asymptote.

(d)



Question 7

Note that $f(x) = \dfrac{36\,|\,x\,|}{(x-1)^2}$ is defined on $\mathbb{R} \setminus \{1\}$.

Let $g(x) = \dfrac{36x}{(x-1)^2} = \dfrac{36}{x-1} + \dfrac{36}{(x-1)^2}$

$$f'(x) = \begin{cases} -\dfrac{36}{(x-1)^2} - \dfrac{72}{(x-1)^3} & \text{if } x > 0 \\ \dfrac{36}{(x-1)^2} + \dfrac{72}{(x-1)^3} & \text{if } x < 0 \end{cases}$$

$f_-'(0) = \lim\limits_{x \to 0^-} \dfrac{-36x}{x(x-1)^2} = -36$ but $f_+'(0) = \lim\limits_{x \to 0^+} \dfrac{36x}{x(x-1)^2} = 36$

$\therefore$ $f'(0)$ does not exist.

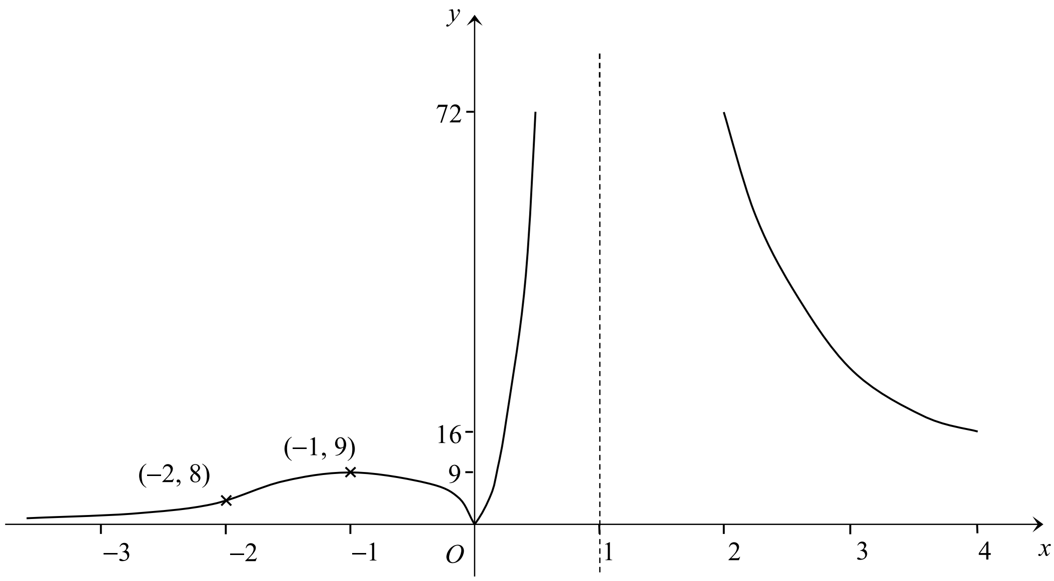
$\therefore$ $f'(x) = 0 \iff \dfrac{36}{(x-1)^2} = -\dfrac{72}{(x-1)^3} \iff x-1 = -2 \iff x = -1$

$$f''(x) = \begin{cases} \dfrac{72}{(x-1)^3} + \dfrac{216}{(x-1)^4} & \text{if } x > 0 \\ -\dfrac{72}{(x-1)^3} - \dfrac{216}{(x-1)^4} & \text{if } x < 0 \end{cases}$$

$\therefore$ $f''(x) = 0 \iff \dfrac{72}{(x-1)^3} = -\dfrac{216}{(x-1)^4} \iff x-1 = -3 \iff x = -2$

$x = 1$ is a vertical asymptote, $y = 0$ is a horizontal asymptote

	$x <$	-2	$< x <$	-1	$< x <$	0	$< x <$	1	$< x$
f	$\nearrow$	8	$\nearrow$	9	$\searrow$	0	$\nearrow$	$\nexists$	$\searrow$
f'	$+$	$+$	$+$	0	$-$	$\nexists$	$+$	$\nexists$	$-$
f''	$+$	0	$-$	$-$	$-$	$\nexists$	$+$	$\nexists$	$+$



Question 8

(a) $f(x) = x e^{-x^2}$ is defined on $\mathbb{R}$.

$f'(x) = (1 - 2x^2)\, e^{-x^2}$, $f''(x) = 2x\,(2x^2 - 3)\, e^{-x^2}$

(b) $f'(x) = 0 \iff x = \pm\frac{1}{\sqrt{2}}$, $f''(x) = 0 \iff x = 0, \pm\sqrt{\frac{3}{2}}$

	$x <$	$-\sqrt{\frac{3}{2}}$	$< x <$	$-\frac{1}{\sqrt{2}}$	$< x <$	0	$< x <$	$\frac{1}{\sqrt{2}}$	$< x <$	$\sqrt{\frac{3}{2}}$	$< x$
f	↘	$-\sqrt{\frac{3}{2}}\, e^{-\frac{3}{2}}$	↘	$-\frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$	↗	0	↗	$\frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$	↘	$\sqrt{\frac{3}{2}} e^{-\frac{3}{2}}$	↘
f'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
f''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

$\therefore$ f has local minimum at $x = -\frac{1}{\sqrt{2}}$, local maximum at $x = \frac{1}{\sqrt{2}}$.

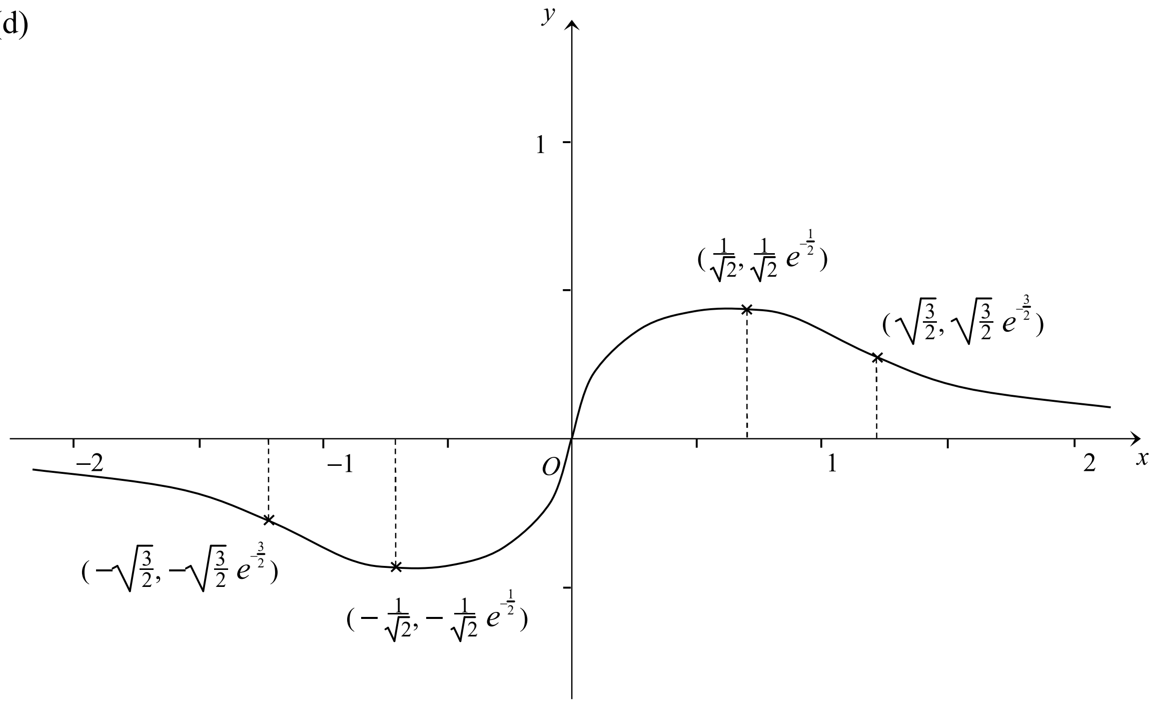
Change of convexity occurs at $x = 0, \pm\sqrt{\frac{3}{2}}$

$\therefore$ f has points of inflexion at $x = 0, \pm\sqrt{\frac{3}{2}}$.

(c) $\lim\limits_{x \to \pm\infty} [\, f(x) - 0\,] = \lim\limits_{x \to \pm\infty} \dfrac{x}{e^{x^2}} = \lim\limits_{x \to \pm\infty} \dfrac{1}{2x e^{x^2}} = 0$ by l'Hôpital's rule

$\therefore$ $y = 0$ is a horizontal asymptote.

(d)



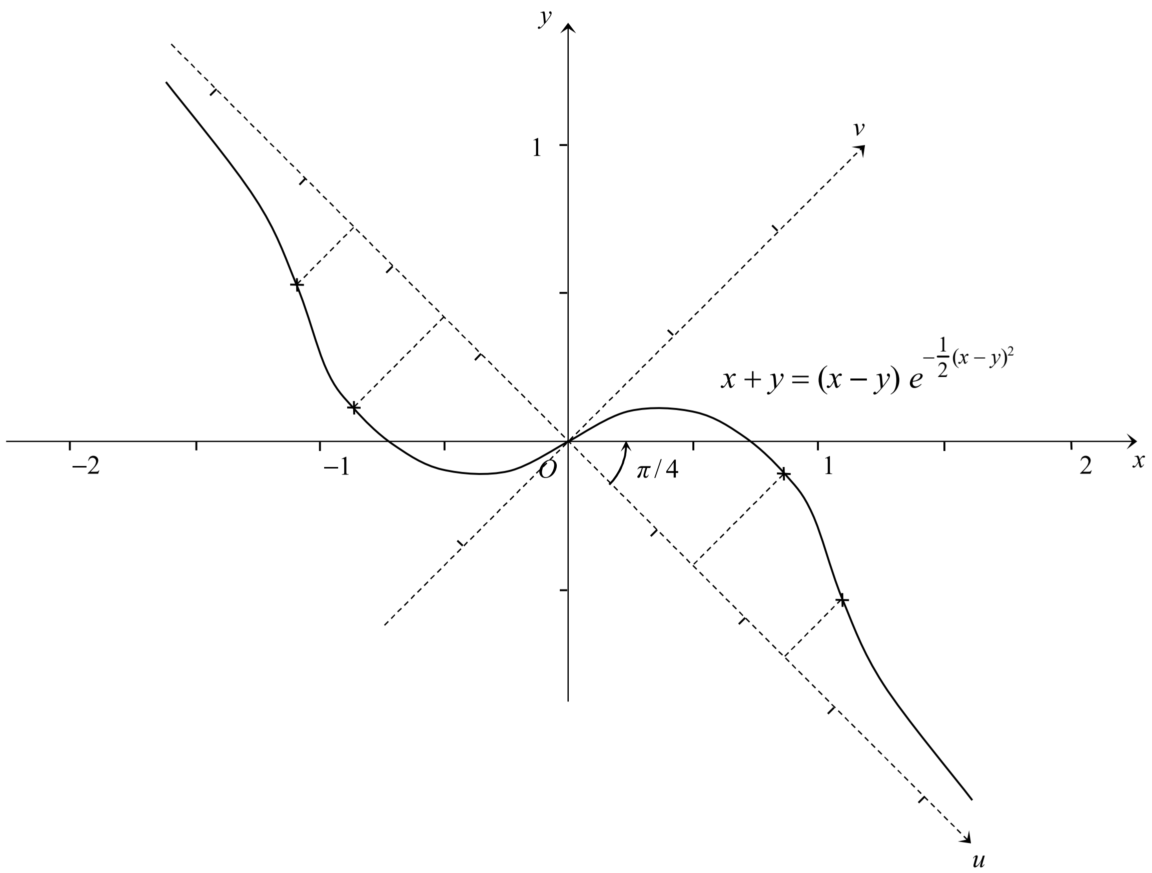
(e) Consider the change of variable

$$u = \frac{x-y}{\sqrt{2}} = x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4}$$

$$v = \frac{x+y}{\sqrt{2}} = x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4}$$

Hence

$$x + y = (x-y)\,e^{-\frac{1}{2}(x-y)^2} \quad \Leftrightarrow \quad v = ue^{-u^2}$$



Question 9

(a) $g(x) = \dfrac{x^2 - 16}{x - 5} = x + 5 + \dfrac{9}{x - 5}$

$g'(x) = 1 - \dfrac{9}{(x-5)^2} = 0 \quad \Leftrightarrow \quad x = 2, 8$

and g' is not defined at $x = 5$

x	$(-\infty, 2)$	$(2, 5)$	$(5, 8)$	$(8, +\infty)$
$g'(x)$	$+$	$-$	$-$	$+$

The intervals where $g' > 0$: $(-\infty, 2), (8, +\infty)$

The intervals where $g' < 0$: $(2, 5), (5, 8)$

$\therefore$ g has local minimum at $x = 8$ and local maximum at $x = 2$.

(b) $g''(x) = \dfrac{18}{(x-5)^3}$ which is discontinuous at $x = 5$

x	$(-\infty, 5)$	$(5, +\infty)$
$g''(x)$	$-$	$+$

The interval where $g'' > 0$: $(5, +\infty)$

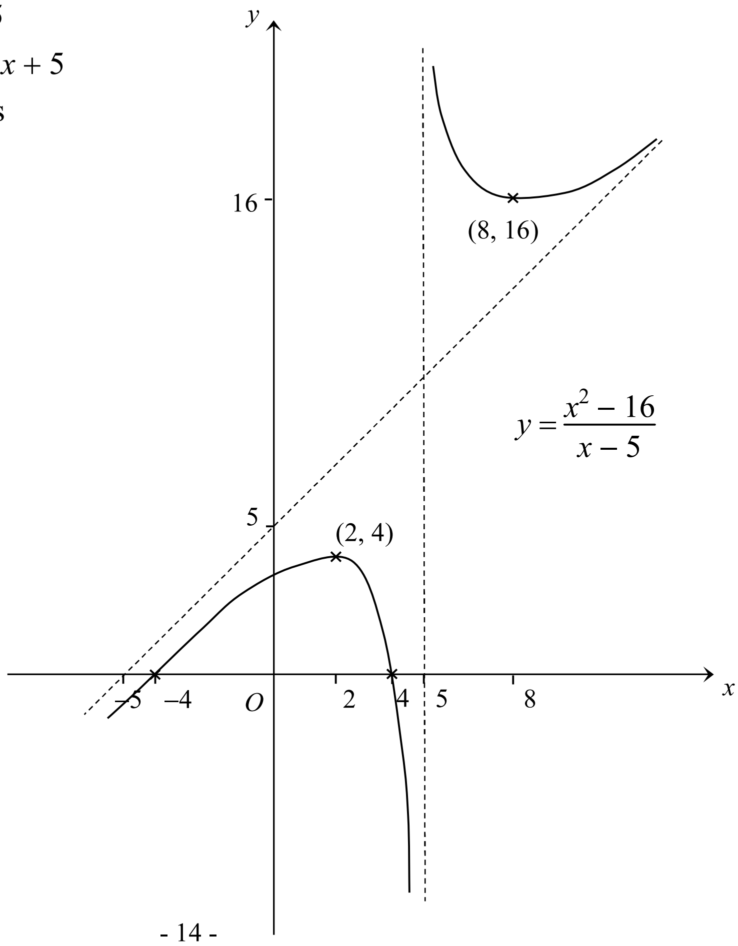
The interval where $g'' < 0$: $(-\infty, 5)$

$\therefore$ g has no points of inflection.

(c) Vertical asymptote: $x = 5$

Inclined asymptote: $y = x + 5$

No horizontal asymptotes



Question 10

(a) $f'(x) = 2x\,e^{-x} - x^2\,e^{-x} = -\,x\,(x-2)\,e^{-x} = 0 \iff x = 0, 2$

$\therefore$ $x = 0$ and $x = 2$ are stationary points

$f''(x) = 2\,e^{-x} - 4x\,e^{-x} + x^2\,e^{-x} = [\,(x-2)^2 - 2\,]\,e^{-x} = 0 \iff x = 2 \pm \sqrt{2}$

$\therefore$ $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$ are inflection points at which change of convexity occurs (see table in (b)).

(b) Note that

	$x <$	0	$< x <$	$2-\sqrt{2}$	$< x <$	2	$< x <$	$2+\sqrt{2}$	$< x$
f	↘	0	↗	0.1910	↗	$4e^{-2}$	↘	0.3835	↘
f'	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
f''	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

$\therefore$ f has local minimum at $x = 0$ with value 0

f has local maximum at $x = 2$ with value $4e^{-2}$.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2\,e^{-x} = \infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2\,e^{-x} = \lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0$$

$\therefore$ f has global minimum at $x = 0$ with value 0 and has no global maximum.

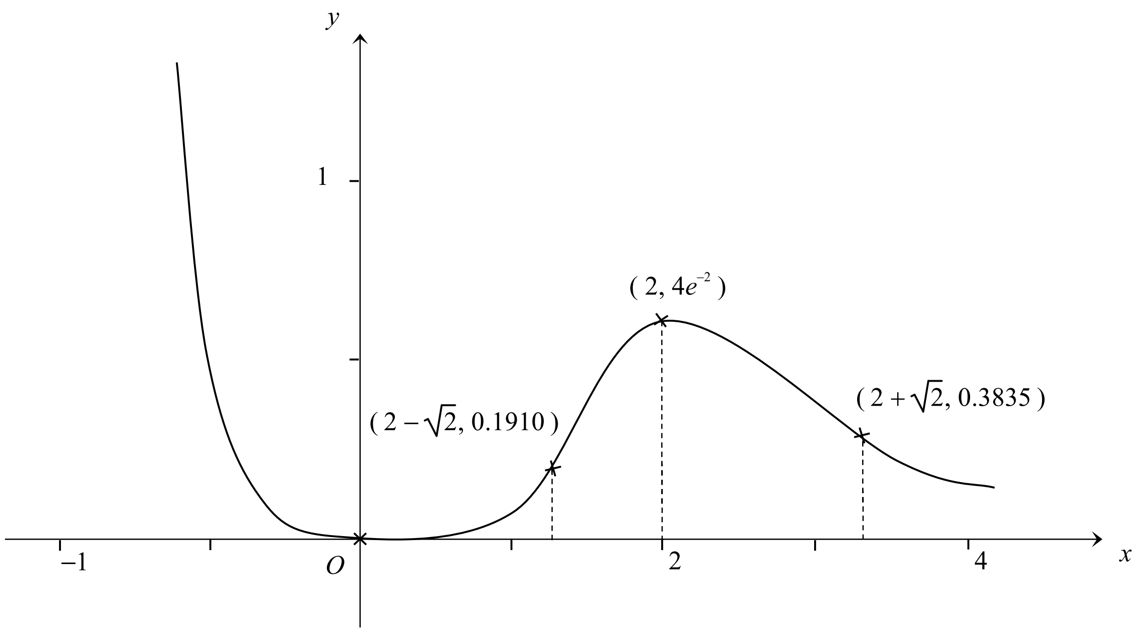
(c) The interval of increasing is $(0, 2)$

The intervals of decreasing are $(-\infty, 0)$, $(2, +\infty)$

The intervals of concave-up are $(-\infty, 2-\sqrt{2})$, $(2+\sqrt{2}, +\infty)$

The interval of concave-down is $(2-\sqrt{2}, 2+\sqrt{2})$

(d) No vertical and inclined asymptotes. Horizontal asymptotes: $y = 0$.



Question 11* (*Pure Global Optimization – Must-do!*)

(a) Consider $f(x) = x^{1/3}\, e^{-x/3}$ on $[-1, 27]$

$f'(x) = \dfrac{1}{3}x^{-2/3}\, e^{-x/3} - \dfrac{1}{3}x^{1/3}\, e^{-x/3} = \dfrac{1}{3}x^{-2/3}\,(1-x)\, e^{-x/3} = 0 \iff x = 1$

x	-1	0	1	27
$f(x)$	$-e^{1/3}$	0	$e^{-1/3}$	$3e^{-9}$

Thus f attains its absolute maximum at $x = 1$ with absolute maximum value $e^{-1/3}$, and f attains its absolute minimum at $x = -1$ with absolute minimum value $-e^{1/3}$.

(b) Consider $f(x) = \dfrac{x}{x^3+2}$ on $[0, 2]$

$f'(x) = \dfrac{x^3 + 2 - x\,(3x^2)}{(x^3+2)^2} = 0 \iff 2 = 2x^3 \iff x = 1$

x	0	1	2
$f(x)$	0	$\frac{1}{3}$	$\frac{1}{5}$

Thus f attains its absolute maximum at $x = 1$ with absolute maximum value $\dfrac{1}{3}$,

and f attains its absolute minimum at $x = 0$ with absolute minimum value 0.

If the domain is changed to $(0, \infty)$,

x	0^+	1	∞
$f(x)$	0	$\frac{1}{3}$	0

Thus f attains its absolute maximum at $x = 1$ with absolute maximum value $\dfrac{1}{3}$,

and f has no absolute minimum on the domain.

(c) Consider $f(x) = 2\sin x + \sin 2x$ on $[0, \frac{3\pi}{2}]$

$f'(x) = 2\cos x + 2\cos 2x = 0 \iff \cos x + 2\cos^2 x - 1 = 0 \iff \cos x = -1, \frac{1}{2}$

$\iff x = \pi, \frac{\pi}{3}$

x	0	$\frac{\pi}{3}$	π	$\frac{3\pi}{2}$
$f(x)$	0	$\frac{3\sqrt{3}}{2}$	0	-2

Thus f attains its absolute maximum at $x = \dfrac{\pi}{3}$ with absolute maximum value $\dfrac{3\sqrt{3}}{2}$,

and f attains its absolute minimum at $x = \frac{3\pi}{2}$ with absolute minimum value -2.

Question 12 (*Application Questions*)

(a) Let x and $30 - x$ be the two positive numbers. Thus $0 < x < 30$.

Let $f(x) = x^2 + 2(30 - x)^2$, which we want to minimize.

$f'(x) = 2x - 4(30 - x) = 6x - 120 = 0 \Rightarrow x = 20$

x	0^+	20	30^-
$f(x)$	1800	600	900

Alternative Method: (first derivative test) †

x	(0, 20)	20	(20, 30)
$f(x)$	↘	600	↗
$f'(x)$	−	0	+

Thus, the two positive numbers are 20 and 10, and the minimum value is 600.

(b) Let $r > 0$ be the radius and $h > 0$ be the height (in m) of the container respectively.

The volume $= \pi r^2 h = 10\pi \Rightarrow h = \dfrac{10}{r^2}$

Let $C(r) = 8\pi r^2 + 8 \cdot 2\pi r h + 2\pi r^2 = 10\pi r^2 + \dfrac{160\pi}{r}$ be the cost that we want to minimize.

$C'(r) = 20\pi r - \dfrac{160\pi}{r^2} = 0 \Rightarrow r^3 = 8 \Rightarrow r = 2$

r	0^+	2	$+\infty$
$C(r)$	$+\infty$	120π	$+\infty$

Alternative Method: (first derivative test) †

r	(0, 2)	2	$(2, +\infty)$
$C(r)$	↘	120π	↗
$C'(r)$	−	0	+

Thus, the radius and the height of the cylindrical cylinder are 2 m and $\dfrac{5}{2}$ m respectively, and the minimum cost is $\$120\pi \approx \$376.99$.

† **Remark** In the first derivative test, one must make sure the table data sufficiently imply the stationary point is the global optimal.

(c) Let $f(x) = x^2 + \left(x + \dfrac{20}{x}\right)$ be the squared distance from the curve to the origin, which we want to minimize. Here, $x + \dfrac{20}{x} > 0$, i.e. $x > 0$.

$f'(x) = 2x + 1 - \dfrac{20}{x^2} = 0 \Rightarrow 2x^3 + x^2 - 20 = (x - 2)(2x^2 + 5x + 10) = 0 \Rightarrow x = 2$

x	$(0, 2)$	2	$(2, +\infty)$
$f(x)$	$\searrow$	16	$\nearrow$
$f'(x)$	$-$	0	$+$

By first derivative test, the point that is closest to the origin on the curve is $(2, \sqrt{12})$.

(d) Note that $2A = 3B \Rightarrow B = \dfrac{2}{3}A$, $A + B + C = 20 \Rightarrow C = 20 - \dfrac{5}{3}A$

$A, B, C > 0 \Rightarrow 0 < A < 12$

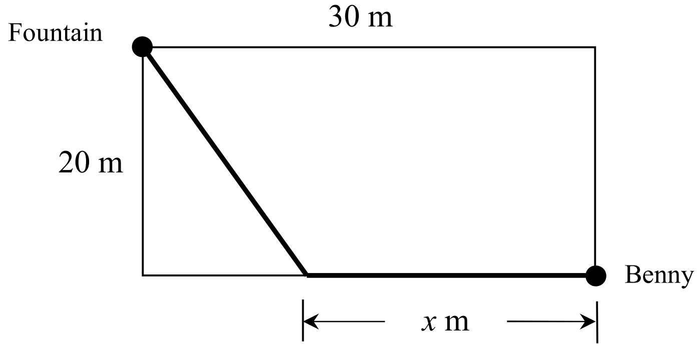
Let $f(A) = ABC = \dfrac{2}{3}A^2\left(20 - \dfrac{5}{3}A\right)$ be the product we want to maximize.

$f'(A) = \dfrac{4}{3}A\left(20 - \dfrac{5}{3}A\right) - \dfrac{10}{9}A^2 = \dfrac{80}{3}A - \dfrac{10}{3}A^2 = 0 \Rightarrow A = 0$ or 8

A	$(0, 8)$	8	$(8, 12)$
$f(A)$	$\nearrow$	$\dfrac{2560}{9}$	$\searrow$
$f'(A)$	$+$	0	$-$

By first derivative test, the product is maximized when $A = 8$, $B = \dfrac{16}{3}$ and $C = \dfrac{20}{3}$ and the value is $\dfrac{2560}{9}$.

(e) Let x (in m) be the distance of sidewalk Benny will walk, $0 < x < 30$.



The distance across the grass = $\sqrt{20^2 + (30 - x)^2} = \sqrt{(x - 30)^2 + 400}$

Let $T(x) = \dfrac{x}{1.0} + \dfrac{\sqrt{(x-30)^2 + 400}}{0.8} = x + \dfrac{5}{4}\sqrt{(x-30)^2 + 400}$ be the time we minimize.

$T'(x) = 1 + \dfrac{5}{4}\dfrac{x - 30}{\sqrt{(x-30)^2 + 400}} = 0 \Rightarrow \dfrac{\sqrt{(x-30)^2 + 400}}{5} = \dfrac{30 - x}{4} = \dfrac{20}{3}$

$\Rightarrow x = \dfrac{10}{3}, \dfrac{170}{3}$ (rejected)

x	$(0, \frac{10}{3})$	$\frac{10}{3}$	$(\frac{10}{3}, 30)$
$T(x)$	↘	45	↗
$T'(x)$	–	0	+

$\therefore$ Benny must walk $\dfrac{10}{3}$ m along sidewalk to reach the fountain in the smallest time.

(f) (i) For each fare adjustment $\$x$,

the ridership = $250000 - 23500x$, and the fare = $\$(11.3 + x)$

$\therefore R(x) = (11.3 + x)(250000 - 23500x)$

(ii) $R'(x) = 250000 - 23500x - 23500(11.3 + x) = -15550 - 47000x = 0$

$\Rightarrow x = -0.33$

$R''(x) = -47000 < 0 \ \forall x$, so $R(x)$ is always concave downwards (concave function).

$\therefore R(x)$ is maximized if and only if $R'(x) = 0 \Leftrightarrow x = -0.33$

Thus the company should decrease the fare by \$0.33 to maximize the revenue, where the fare is \$10.97.

(g) Let r and h be the base radius and height of the cylinder.

Surface area = $2\pi r^2 + 2\pi r h = 2\pi \Rightarrow r \neq 0$ and $h = \dfrac{1 - r^2}{r}$

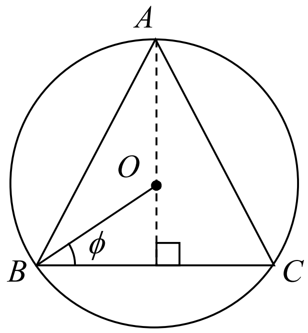
$\therefore$ Volume = $\pi r^2 h = \pi r(1 - r^2) =: V(r)$

$V'(r) = \pi(1 - r^2) + \pi r(-2r) = \pi(1 - 3r^2) = 0 \Rightarrow r = \dfrac{1}{\sqrt{3}}$

r	$(0, \frac{1}{\sqrt{3}})$	$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}}, \infty)$
$V(r)$	↗	$\frac{2\pi}{3\sqrt{3}}$	↘
$V'(r)$	+	0	–

Hence the largest possible volume is $\dfrac{2\pi}{3\sqrt{3}}$ and the radius is $\dfrac{1}{\sqrt{3}}$.

(h) Since ΔABC is isosceles, we can extend AO to D on BC such that $AD \perp BC$.



$BC = 2r \cos \phi$ and $AD = r + r \sin \phi$.

Area of $\Delta ABC = \dfrac{1}{2}(2r\cos\phi)(r + r\sin\phi) = r^2 (\cos\phi + \dfrac{1}{2}\sin 2\phi) =: A(r)$

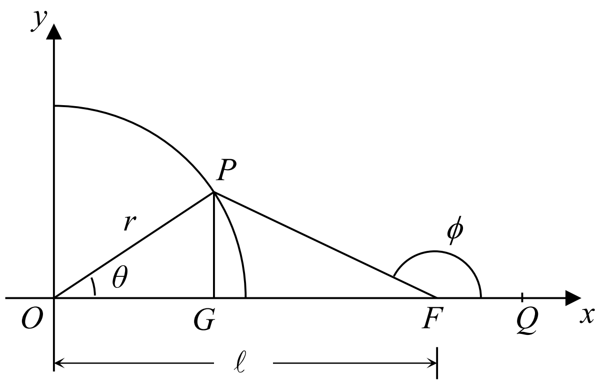
$A'(\phi) = r^2(-\sin\phi + \cos 2\phi) = 0 \iff 1 - \sin\phi - 2\sin^2\phi = 0$

$\iff \sin\phi = \dfrac{1}{2}, -1$ (rejected) $\iff \phi = \dfrac{\pi}{6}$

ϕ	$(0, \frac{\pi}{6})$	$\frac{\pi}{6}$	$(\frac{\pi}{6}, \frac{\pi}{2})$
$A(\phi)$	↗	max	↘
$A'(\phi)$	+	0	−

Hence, the area of ΔABC is the greatest when $\phi = \dfrac{\pi}{6}$, i.e. ΔABC is equilateral.

(i) (a) As depicted, first suppose $r < \ell$



Add PG so that $PG \perp OF$, $\therefore$ $PG = r\sin\theta$ and $OG = r\cos\theta$

$\tan\phi = -\tan(\pi - \phi) = -\dfrac{PG}{GF} = -\dfrac{r\sin\theta}{\ell - r\cos\theta} = \dfrac{r\sin\theta}{r\cos\theta - \ell}$.

(b) Since $\phi \mapsto \tan\phi$ is a strictly increasing function on $(\pi/2, \pi]$, maximizing or minimizing ϕ is equivalent to maximizing or minimizing $\tan\phi$ over θ.

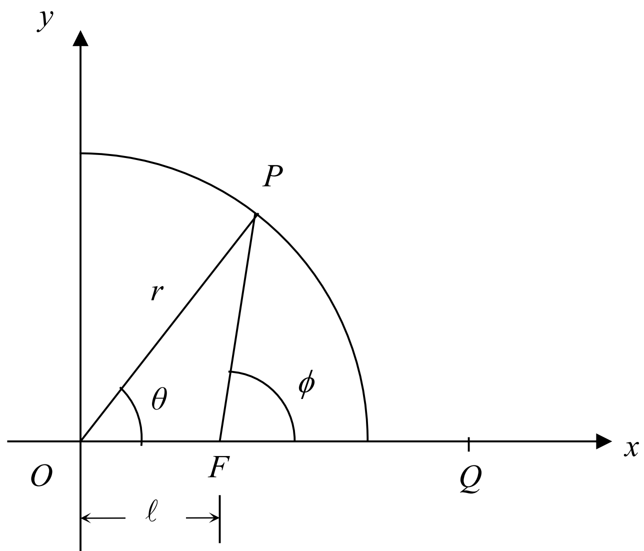
Consider $g(\theta) := \dfrac{r\sin\theta}{r\cos\theta - \ell} = \dfrac{\sin\theta}{\cos\theta - 2}$ on $[0, \pi/2]$ if $\ell = 2r$.

$$g'(\theta) = \frac{(\cos\theta - 2)\cos\theta - (\sin\theta)(-\sin\theta)}{(\cos\theta - 2)^2} = 0 \Leftrightarrow \cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$$

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$g(\theta)$	0	$-\frac{\sqrt{3}}{3}$	$-\frac{1}{2}$

Thus the least value of ϕ is when $\tan\phi = -\dfrac{\sqrt{3}}{3} \Leftrightarrow \phi = \dfrac{5\pi}{6}$.

(c) When $r = 2\ell > \ell$, the picture becomes



$\therefore\ 0 \le \phi \le \pi - \tan^{-1}\dfrac{r}{\ell} = \pi - \tan^{-1} 2$ which can be attained when $\theta = \dfrac{\pi}{2}$

That is to say, the greatest value of ϕ is $\pi - \tan^{-1} 2$.

Remark For part (c), one should not use $\tan\phi$ as a strategy of finding the maximum or minimum ϕ. (Why?)

Question 13*** (*Concept Level*)

(a) If $x = c$ is a global minimizer, then $f\,'(c) = 0$ since f is differentiable at $x = c$. Now suppose $f\,'(c) = 0$. Let $y \in \mathbb{R}$. By Taylor Theorem, $\exists\ \xi$ in between y and c such that

$$f(y) = f(c) + f\,'(c)\,(y - c) + \frac{1}{2}\,f\,''(\xi)\,(y - c)^2 = f(c) + \frac{1}{2}f\,''(\xi)\,(y - c)^2 \geq f(c).$$

Since y is arbitrary, this shows $x = c$ is a global minimizer.

(b) Since $f(x)$ is concave downwards on all x, $g(x) = -f(x)$ is concave upwards for all x. Then $f\,'(c) = 0 \Leftrightarrow g'(c) = 0 \Leftrightarrow x = c$ is a global minimizer of $g \Leftrightarrow x = c$ is a global maximizer of f.