The Hong Kong Polytechnic University
Department of Applied Mathematics
AMA1120 Tutorial Set #04

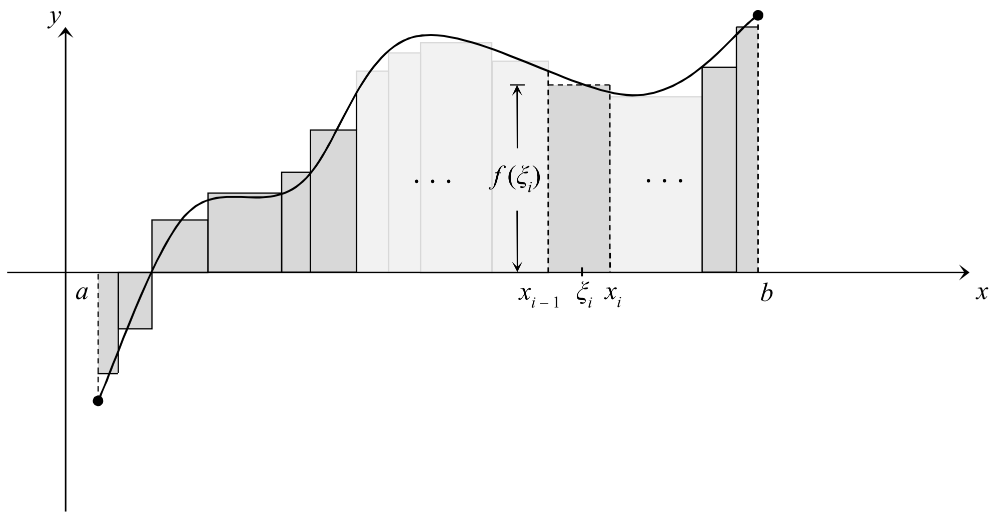
Definite Integral as Riemann Sum

A function f is called **Riemann integrable** on $[a, b]$ if $\exists\, L \in \mathbb{R}$ such that, for any partition $P: a = x_0 < x_1 < \cdots < x_n < b$ and any $\xi_i \in [\, x_{i-1}, x_i \,]$ in which $\| P \| := \max | x_i - x_{i-1} | \to 0$, the **Riemann sum**

$$\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}) \to L\,.$$

If such L exists in $\mathbb{R}$, we denote it as $\int_a^b f(x)\,dx$. In short, we mean

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}) = \int_a^b f(x)\,dx$$



Question 1. (*Intermediate Level*)

Use integration to evaluate the following limits:

(a) $\displaystyle \lim_{n \to \infty} \frac{1^s + 2^s + \cdots + n^s}{n^{s+1}}$, where $s > -1$

(b) $\displaystyle \lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$

(c) $\displaystyle \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{a+n}} + \frac{1}{\sqrt{2a+n}} + \cdots + \frac{1}{\sqrt{na+n}} \right)$, where $a \neq 0$

Question 2. (*Beginner's Level*)

Evaluate the following definite integrals:

(a) $\int_0^4 (3x - \frac{x^3}{4} + 2)\, dx$

(b) $\int_0^1 (14x^{\frac{4}{3}} - 7x^{\frac{3}{4}})\, dx$

(c) $\int_{-1}^1 \frac{1}{1+x^2}\, dx$

(d) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$

(e) $\int_0^{\pi/3} \frac{2}{\cos^2 x}\, dx$

(f) $\int_{\pi/4}^{\pi/2} \csc x \cot x\, dx$

(g) $\int_0^1 (e^x - x^e)\, dx$

(h) $\int_{-1}^2 |x|\, dx$

Question 3. (*Intermediate Level*)

Find the derivative $F'(x)$ wherever the function $F(x)$ is differentiable.

(a) $F(x) = \int_{-1}^x |t|\, dt$

(b) $F(x) = \int_1^{\cos x} \frac{1}{t}\, dt$

(c) $F(x) = \int_{\cos x}^1 \frac{1}{1+t^2}\, dt$

(d) $F(x) = \int_{x^2}^{x^3} \sin t\, dt$

(e) $F(x) = \int_x^1 e^{t^3}\, dt$

(f) $F(x) = \int_1^{x^2} \frac{t}{t^6+1}\, dt$

(g) $F(x) = \int_{\sqrt{x}}^{\sin x} \sqrt{t^2+3}\, dt$

(h) $F(x) = \int_{x^2}^{x^3} \frac{e^t}{t^2+4}\, dt$

Question 4. (*Intermediate Level*)

Evaluate the following definite integrals:

(a) $\int_0^{\pi} \cos x \cos 2x \cos 3x\, dx$

(b) $\int_0^1 \frac{dx}{\sqrt{8-4x-x^2}}$

(c) $\int_0^1 \frac{dx}{\sqrt{x^2+4x+8}}$

(d) $\int_0^1 x^5 \sqrt{1+x^2}\, dx$

(e) $\int_1^2 \frac{4x+6}{x^2+3x+1}\, dx$

(f) $\int_0^{\pi/2} \frac{1-\cos x}{1+\cos x}\, dx$

(g) $\int_1^{\sqrt{3}} \frac{dx}{(1+x^2)\tan^{-1} x}$

(h) $\int_0^1 \frac{1}{x^2\sqrt{x^2+4}}\, dx$

(i) $\int_{-1}^1 (6x^5 + |5x-1|)\, dx$

Question 5. (*Concept*)

(a) $$f(x) = \begin{cases} x^2, & \text{if } 0 \le x \le 3a, \\ 9a^2, & \text{if } 3a \le x \le 4a, \\ 25a^2 - x^2, & \text{if } 4a \le x \le 5a. \end{cases}$$

Evaluate $\int_0^{5a} f(x)\, dx$.

(b) Find the average value of $f(x) = x^2 + \sqrt{x}$ on [1, 4].

Question 6. (*Exam Level*)

Given a constant $a > 0$, show that $\int_{-a}^{a} f(x)\, dx = \int_0^a [f(-x) + f(x)]\, dx$ and hence, evaluate $\int_{-1}^{1} \ln(x + \sqrt{1 + x^2})\, dx$.

Question 7. (*Exam Level*)

(a) Consider the integral $I_n = \int_0^1 (1 - \sqrt{x})^n\, dx$, where n is a positive integer. Show that $I_n = \dfrac{n}{n+2} I_{n-1}$ and hence, evaluate $I_4 = \int_0^1 (1 - \sqrt{x})^4\, dx$.

(b) For any nonnegative integers m, n, prove that

$$\int_0^1 x^m (1 - x)^n\, dx = \frac{m!\, n!}{(m + n + 1)!}$$

Improper Integral

Let f be a real valued function on $[a, \infty)$ such that f is Riemann integrable on every finite subintervals $[b, c] \subseteq [a, \infty)$. Then we define the improper integral

$$\int_a^{\infty} f(x)\, dx = \lim_{A \to \infty} \int_a^A f(x)\, dx$$

If the limit exists and is finite, we say the integral is **convergent**; otherwise we say the integral is **divergent**.

Question 8. (*Standard Level*)

(a) $\int_2^{\infty} \dfrac{1}{x (\ln x)^3}\, dx$ (b) $\int_1^{\infty} \dfrac{(\ln x)^3}{x}\, dx$

Question 9. (*Intermediate Level*)

(a) Find out whether the improper integral $\int_1^{\infty} \frac{2\cos x + 2^x e^{-x}}{x^5+1}\,dx$ is convergent or not. Explain your answer.

(b) Suppose $f(x) = \int_{x^x}^{10} \sin\sqrt{t}\,dt$. Determine the value of x in $[1, 2]$ such that $f(x)$ attains its minimum.

Question 10. (*Exam Level*)

(a) Let f and g be continuous functions on $[a, b]$. Moreover $g(x) > 0$ for all $x \in [a, b]$. Show that there is a number $c \in [a, b]$ such that

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx\,.$$

(b) Use (a) to evaluate $\lim_{\delta\to 0^+} \frac{1}{\delta^4}\int_0^{\delta} \cos(x^2)\,x^3\,dx$.

Question 11.* (*Gamma Function – for fun only!*)

(a) Define $\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t}\,dt$ for $x > 0$. It is known that $\Gamma(x)$ is differentiable in $\mathbb{R}$. Using integration by parts, show that $\Gamma(x+1) = x\,\Gamma(x)$ for all $x > 0$.

(b) By (a), show that $\Gamma(n+1) = n!$ for all integers $n \geq 0$.

(c) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (*Hint* : You may use the fact that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz = 1$.)

† *Remark* It is also known that $\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin \pi z}$ for $z \notin \mathbb{Z}$.

Question 12. (*Exam Level*)

Define $I_n = \int_0^1 x^n\sqrt{1-x^2}\,dx$ for integer $n \geq 0$.

(a) Compute $I_0 = \int_0^1 \sqrt{1-x^2}\,dx$ and $I_1 = \int_0^1 x\sqrt{1-x^2}\,dx$.

(b) Using integration by parts, express I_{n+2} in terms of I_n.

(c) Compute I_5 and I_6.

Question 13. (*Standard Level*)
Discuss the convergence of the following improper integrals and evaluate the integrals if they are convergent.

(a) $\int_{1}^{\infty} \frac{1}{(3x+1)^2}\,dx$ (b) $\int_{0}^{\infty} \frac{x}{1+x^2}\,dx$ (c) $\int_{1}^{\infty} \frac{1}{x^p}\,dx,\ p>0$

(d) $\int_{1}^{\infty} \frac{\ln x}{x^p}\,dx,\ p>0$ (e) $\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx$

Question 14. (*Intermediate Level*)
Discuss the convergence of the following improper.

(a) $\int_{0}^{\infty} e^{-x^3}\,dx$ (b) $\int_{1}^{\infty} \frac{2\sin x + xe^{-x}}{x^4+x}\,dx$ (c) $\int_{2}^{\infty} \frac{1}{(\ln x)^2}\,dx$

Taylor Theorem, Linear and Quadratic Approximations, Remainder Term
Suppose f is continuous on $[a, x]$ and $f', f'', \ldots, f^{(n)}, f^{(n+1)}$ are continuous in (a, x), then $\exists\, \xi \in (a, x)$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

If $n = 1$, $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(\xi)(x-a)^2 = L(x) + R_1(x)$, where

$$L(x) = f(a) + f'(a)(b-a)$$

is the **linear approximation** of f around $x = a$, and $R_1(x) := \frac{f''(\xi)}{2}(x-a)^2$ is called the **remainder term** which can help us to estimate the error $|\, f(x) - L(x)\,|$.
Similarly, if $n = 2$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(x)(x-a)^2 + \frac{f^{(3)}(\xi)}{3!}(x-a)^3 = Q(x) + R_2(x),$$

where $$Q(x) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(a)(x-a)^2$$

is the **quadratic approximation** of f around $x = a$, and $R_2(x) := \frac{f^{(3)}(\xi)}{3!}(x-a)^3$ is called the **remainder term** which can help us to estimate the error $|\, f(x) - Q(x)\,|$.

Question 15. (*Theory*)

Derive the Taylor's Theorem.

Question 16. (*Intermediate Level*)

Let $f(x) = x^{3/4}$ and $x_0 = 16$.

(a) Use the linear approximation of $f(x)$ at x_0 to estimate $\sqrt[4]{17^3}$ and also estimate the error.

(b) Find the Taylor's polynomial of degree 2 of $f(x)$ at x_0, use it to estimate $\sqrt[4]{17^3}$ and also estimate the error.