The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #05

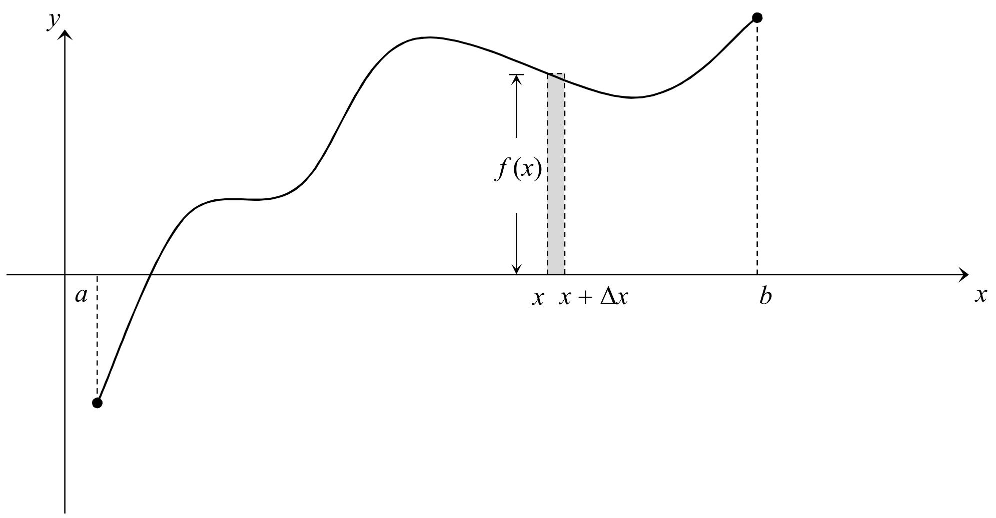
Area under Graph

Given a Riemann integrable function $f(x)$ on $[a, b]$. For a small change from x to $x + \Delta x$ within $[a, b]$, we can approximate **the change of area under graph** by

$$\Delta A = f(x)\,\Delta x$$

Thus by means of integration,

$$\text{total area under graph} = \int dA = \int_a^b f(x)\,dx$$



Question 1. (*Standard Level*)

Find the area of the region enclosed by the given curves.

(a) $y = e^x,\ y = x^2 - 1,\ \ x = -1,\ \ x = 1$

(b) $y = (x-2)^2,\ \ y = x$

(c) $y = \dfrac{1}{x},\ \ y = \dfrac{1}{x^2},\ \ x = 2$

(d) $y = \sqrt{x},\ y = \dfrac{x}{2},\ \ x = 9$

(e) $y = x,\ x = 2,\ y = \dfrac{1}{x},\ \ y = 0$

(f) $y = 2\sqrt{x},\ y = 12 - 2x,\ \ x = 1$

Question 2. (*Intermediate Level*)

Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Question 3. (*Intermediate Level*)

Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

Volume of the Revolution

Disk Method

Let $f(x)$ be a Riemann integrable function on $[a, b]$. The small change of the volume of revolution bounded by $y = f(x)$ and the x-axis # about x-axis † can be approximated by

$$\Delta V = \pi\, [f(x)]^2\, \Delta x$$

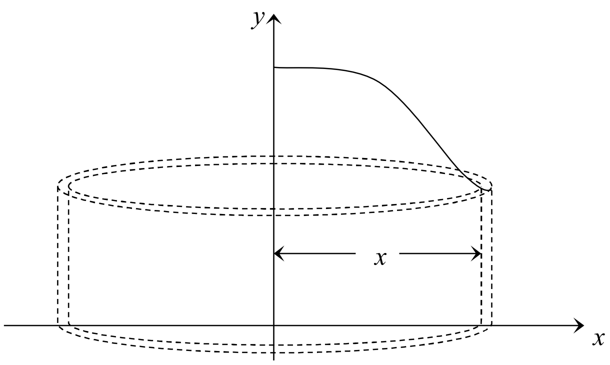
$$\text{the volume of the solid} = \pi \int_a^b [f(x)]^2\, dx$$

Shell Method

Let $f(x) \geq 0$ be a Riemann integrable function on $[a, b]$. The small change of the volume of revolution bounded by $y = f(x)$ and the x-axis # about y-axis † can be approximated by

$$\Delta V = 2\pi x f(x)\, \Delta x$$

$$\text{the volume of the solid} = 2\pi \int_a^b x f(x)\, dx$$



Question 4. (*Intermediate Level*)

(a) Find the volume of the solid generated by revolving the region bounded by the curve of $y = x^2 - x$, x-axis and the lines of $x = 1$, $x = 2$ about the x-axis.

(b) Find the volume of the solid generated by rotating the region bounded by the curve $y = e^{-x}$, x-axis and the lines of $x = 0$, $x = 1$ about the y-axis.

(c) Find the volume of the solid generated by revolving the region bounded by the given curves about the x-axis.

(i) $y = \frac{1}{2}x^2 + 3, \;\; y = 12 - \frac{1}{2}x^2$

(ii) $y = \sec x, \;\; y = \tan x, \;\; x = 0, \;\; x = 1$

Question 5. (*Intermediate Level*)

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line:

(a) $y = \tan^3 x, \;\; y = 1, x = 0$ about $y = 1$

(b) $y = 0, \;\; y = \sin x$ for $0 \le x \le \pi$ about $y = 1$

Question 6. (*Intermediate Level*)

(a) Find $\int \ln y \, dy$

(b) Find the volume of the solid of revolution generated by revolving the region bounded by the curve $y = 2^{x^2}$ and the straight line $y = 2$ about the y-axis.

Arc Length

Let $x(t)$ and $y(t)$ be functions such that $x'(t)$ and $y'(t)$ Riemann integrable functions on t. The small change of arc-length can be approximated by

$$\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \, \Delta t$$

provided that $\Delta t > 0, a < t < b$

$$\text{the total arc-length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

Question 7. (*Intermediate Level*)

Set up, but do not evaluate, an integral for the length of the curve

(a) $y = \cos x$ for $0 \le x \le 2\pi$

(b) $x = y + y^3$ for $1 \le y \le 4$

Question 8. (*Standard Level*)

Find the arc length of the curve.

(a) $y = x^{3/2}$, $0 \le x \le 1$

(b) $y = \ln \cos x$, $0 \le x \le \dfrac{\pi}{4}$

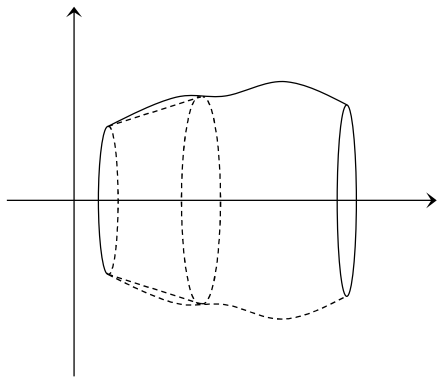
(c) $x = \dfrac{1}{4} y^2 - \dfrac{1}{2} \ln y$, $1 \le y \le e$

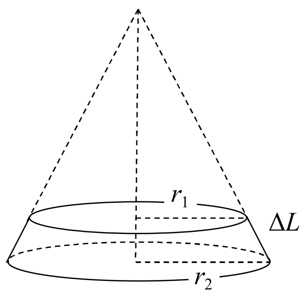
(d) $y = \displaystyle\int_1^x \sqrt{t^3 - 1}\, dt$, $1 \le x \le 4$

Surface Area of Revolution

The lateral surface area of a frustum is equal to

$$S = \pi\,(r_1 + r_2)\,\Delta L\,.$$





Given a continuous function $f(x) \ge 0$ on $[a, b]$, the lateral surface area element of revolution is

$$\Delta S = \pi\,[\,f(x) + f(x + \Delta x)\,]\,\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\text{the surface area of revolution} = \int_a^b 2\pi f(x)\,\sqrt{1 + (f'(x))^2}\,dx$$

Question 9. (*Intermediate Level*)

(a) Evaluate

(i) $\int x \ln x \, dx$ (ii) $\int \frac{\ln x}{x} \, dx$

(b) Consider the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$ where $1 \le x \le e$.

Find the area of the surface generated by rotating the curve about the x-axis.

Question 10. (*Standard Level*)

Find the area of the surface of revolution generated by revolving $y = \sqrt{2x}$, $0 \le x \le \frac{9}{4}$ about the x-axis.

Question 11. (*Revision*)

For $n = 0, 1, 2, \ldots$, it is known that $\lim_{k \to \infty} \int_0^k \frac{dx}{(x^2 + 1)^{n+1}}$ exists.

Denote this limit by I_n.

For $n \ge 1$, express I_n in terms of I_{n-1} and hence show that

$$I_n = \frac{(2n-1)(2n-3) \cdots 1}{(2n)(2n-2) \cdots 2} \frac{\pi}{2}$$