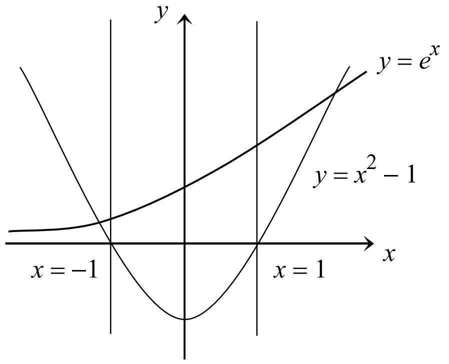
The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #05

Question 1. (*Standard Level*)

(a)



Note that $e^x > x^2 - 1$ for all $x \in [-1, 1]$

$$\therefore \quad \text{Area} = \int_{-1}^{1} (e^x - x^2 + 1)\, dx = \left[e^x - \frac{x^3}{3} + x \right]_{-1}^{1} = e - \frac{1}{e} + \frac{4}{3}$$

(b) $(x-2)^2 = x \iff x^2 - 5x + 4 = 0 \iff x = 1, 4$. Note that $(x-2)^2 < x$ for all $x \in (1, 4)$

$$\therefore \quad \text{Area} = \int_{1}^{4} |\,(x-2)^2 - x\,|\, dx = -\left[\frac{(x-2)^3}{3} - \frac{x^2}{2} \right]_{1}^{4} = \frac{16}{3} - \frac{5}{6} = \frac{9}{2}$$

(c) $\dfrac{1}{x} = \dfrac{1}{x^2} \iff x = 1$, and is undefined at $x = 0$. Note that $\dfrac{1}{x} > \dfrac{1}{x^2}$ for all $x \in (1, 2)$

$$\therefore \quad \text{Area} = \int_{1}^{2} \left| \frac{1}{x} - \frac{1}{x^2} \right| dx = \int_{1}^{2} \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \left[\ln x + \frac{1}{x} \right]_{1}^{2} = \ln 2 - \frac{1}{2}$$

(d) $\sqrt{x} = \dfrac{x}{2} \iff 4x = x^2 \iff x = 0, 4.$

$$\therefore \quad \text{Area} = \int_{0}^{9} |\sqrt{x} - \frac{x}{2}|\, dx = \int_{0}^{4} (\sqrt{x} - \frac{x}{2})\, dx + \int_{4}^{9} (\frac{x}{2} - \sqrt{x})\, dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{4} \right]_{0}^{4} + \left[\frac{x^2}{4} - \frac{2}{3} x^{3/2} \right]_{4}^{9} = \frac{59}{12}$$

(e) $x = \dfrac{1}{x} \iff x = \pm 1$

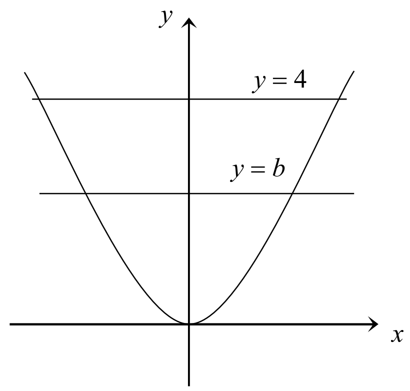
$$\therefore \quad \text{Area} = \int_{0}^{1} x\, dx + \int_{1}^{2} \frac{1}{x}\, dx = \left[\frac{x^2}{2} \right]_{0}^{1} + \Big[\ln x \Big]_{1}^{2} = \frac{1}{2} + \ln 2$$

(f) $2\sqrt{x} = 12 - 2x \iff x = (6-x)^2 \iff x^2 - 13x + 36 = 0 \iff x = 4, 9$ (rejected)

$\therefore$ Area $= \int_1^4 (12 - 2x - 2\sqrt{x})\, dx = \left[12x - x^2 - \frac{4}{3}x^{3/2} \right]_1^4$

$= \frac{64}{3} - \frac{29}{3} = \frac{35}{3}$

Question 2. (*Intermediate Level*)



$y = x^2 \iff x = \sqrt{y}$ if $x \geq 0$

$\frac{1}{2}$(Area of upper region)

$= \int_b^4 x\, dy = \int_b^4 \sqrt{y}\, dy = \frac{2}{3}y^{3/2}\Big|_b^4 = \frac{16}{3} - \frac{2}{3}b^{3/2}$

$\frac{1}{2}$(Area of lower region)

$= \int_0^b x\, dy = \int_0^b \sqrt{y}\, dy = \frac{2}{3}y^{3/2}\Big|_0^b = \frac{2}{3}b^{3/2}$

$\therefore \quad \frac{16}{3} - \frac{2}{3}b^{3/2} = \frac{2}{3}b^{3/2} \iff 4 = b^{3/2} \iff b = 4^{2/3}$

Question 3. (*Intermediate Level*)

$x^2 - c^2 = c^2 - x^2 \iff x = \pm c$

Note that $x^2 - c^2 < c^2 - x^2$ for all $x \in (-c, c)$

$\int_{-c}^{c} [\,(c^2 - x^2) - (x^2 - c^2)\,]\, dx = 2\int_{-c}^{c} (c^2 - x^2)\, dx = 2\left[c^2x - \frac{x^3}{3}\right]_{-c}^{c} = \frac{8}{3}c^3 = 576 \iff c = 6$

Question 4. (*Intermediate Level*)

(a) Volume $= \int_1^2 \pi y^2\, dx = \int_1^2 \pi (x^2 - x)^2\, dx = \pi \int_1^2 (x^4 - 2x^3 + x^2)\, dx$

$= \pi\left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}\right]_1^2 = \pi\left(\frac{16}{15} - \frac{1}{30}\right) = \frac{31}{30}\pi$

(b) Volume $= \int_0^1 2\pi xy\,dx = \int_0^1 2\pi xy\,dx = \int_0^1 2\pi x e^{-x}\,dx = -2\pi \int_0^1 x\,d(e^{-x})$

$$= -2\pi x e^{-x}\Big|_0^1 + 2\pi \int_0^1 e^{-x}\,dx = -\frac{2\pi}{e} - \frac{2\pi}{e} + 2\pi = 2\pi\left(1 - \frac{2}{e}\right)$$

(c) (i) $\frac{1}{2}x^2 + 3 = 12 - \frac{1}{2}x^2 \iff x^2 = 9 \iff x = \pm 3$

Volume $= \int_{-3}^{3} \pi\,[\,(12 - \frac{1}{2}x^2)^2 - (\frac{1}{2}x^2 + 3)^2\,]\,dx$

$$= \int_{-3}^{3} \pi\,(135 - 15x^2)\,dx = \pi\left[\,135x - \frac{15x^3}{3}\right]_{-3}^{3} = 540\pi$$

(ii) $\sec^2 x = \tan^2 x + 1$

Volume $= \int_0^1 \pi\,[\,\sec^2 x - \tan^2 x\,]\,dx = \pi \int_0^1 dx = \pi$

Question 5. (*Intermediate Level*)

(a) Note that $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^3 x = 1 \iff \tan x = 1 \iff x = \frac{\pi}{4}$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Volume $= \int_0^{\pi/4} \pi\,(y - 1)^2\,dx = \int_0^{\pi/4} \pi\,(\tan^3 x - 1)^2\,dx$

(b) Volume $= \pi\,(1)^2\,(\pi - 0) - \pi \int_0^{\pi} (\sin x - 1)^2\,dx = \pi^2 - \pi \int_0^{\pi} (\sin x - 1)^2\,dx$

Question 6. (*Intermediate Level*)

(a) $\int \ln y\,dy = y \ln y - \int dy = y \ln y - y + C$

(b) $y = 2^{x^2} \iff x^2 = \frac{\ln y}{\ln 2}$

Volume $= \pi \int_1^2 x^2\,dy = \pi \int_1^2 \frac{\ln y}{\ln 2}\,dy = \frac{\pi}{\ln 2}\Big[\,y \ln y - y\,\Big]_1^2 = \frac{\pi}{\ln 2}(2\ln 2 - 2) + \frac{\pi}{\ln 2}$

$$= \left(2 - \frac{1}{\ln 2}\right)\pi$$

Question 7. (*Intermediate Level*)

(a) $\dfrac{dy}{dx} = -\sin x$. Arc-length $= \displaystyle\int_0^{2\pi} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{2\pi} \sqrt{1+\sin^2 x}\,dx$

(b) $\dfrac{dx}{dy} = 1 + 3y^2$. Arc-length $= \displaystyle\int_1^4 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\,dy$

Question 8. (*Standard Level*)

(a) $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \dfrac{9}{4}x$

Arc-length $= \displaystyle\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_0^1 \sqrt{1+\frac{9}{4}x}\,d\left(1+\frac{9}{4}x\right)$

$= \displaystyle\frac{4}{9}\left[\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^1 = \frac{13\sqrt{13}-8}{27}$

(b) $\dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} = -\tan x \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1$

Arc-length $= \displaystyle\int_0^{\pi/4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|\,\Big|_0^{\pi/4} = \ln(\sqrt{2}+1$

(c) $\dfrac{dx}{dy} = \dfrac{1}{2}y - \dfrac{1}{2y} \Rightarrow \left(\dfrac{dx}{dy}\right)^2 = \dfrac{1}{4}(y^2 - 2 + \dfrac{1}{y^2}) \Rightarrow \left(\dfrac{dx}{dy}\right)^2 + 1 = \dfrac{1}{4}\left(y+\dfrac{1}{y}\right)^2$

Arc-length $= \displaystyle\int_1^e \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \frac{1}{2}\int_1^e (y+\frac{1}{y})\,dy = \frac{1}{2}\left[\frac{y^2}{2} + \ln y\right]_1^e = \frac{e^2+1}{4}$

(d) $\dfrac{dy}{dx} = \sqrt{x^3 - 1} \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = x^3 - 1$

Arc-length $= \displaystyle\int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big|_1^4 = \frac{62}{5}$

Question 9. (*Intermediate Level*)

(a) (i) $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

(ii) $\displaystyle\int \frac{\ln x}{x}\,dx = \int (\ln x)\,d(\ln x) = \frac{1}{2}(\ln x)^2 + C$

(b) $\dfrac{dy}{dx} = x - \dfrac{1}{4x} \;\Rightarrow\; \left(\dfrac{dy}{dx}\right)^2 = x^2 - \dfrac{1}{2} + \dfrac{1}{16x^2} \;\Rightarrow\; \left(\dfrac{dy}{dx}\right)^2 + 1 = \left(x + \dfrac{1}{4x}\right)^2$

$$\text{Surface area} = 2\pi\int_1^e y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^e \left(\frac{x^2}{2} - \frac{\ln x}{4}\right)\left(x + \frac{1}{4x}\right)dx$$

$$= 2\pi\int_1^e \left(\frac{1}{2}x^3 + \frac{1}{8}x - \frac{1}{4}x\ln x - \frac{1}{16}\frac{\ln x}{x}\right)dx$$

$$= 2\pi\left[\frac{x^4}{8} + \frac{x^2}{16} - \frac{1}{8}x^2\ln x + \frac{x^2}{16} - \frac{1}{32}(\ln x)^2\right]_1^e = \left(\frac{e^4}{4} - \frac{9}{16}\right)\pi$$

Question 10. (*Standard Level*)

$\dfrac{dy}{dx} = \dfrac{1}{2}(2x)^{-1/2}\cdot 2 = \dfrac{1}{\sqrt{2x}} \;\Rightarrow\; \left(\dfrac{dy}{dx}\right)^2 = \dfrac{1}{2x} \;\Rightarrow\; \left(\dfrac{dy}{dx}\right)^2 + 1 = \dfrac{1}{2x} + 1$

$$\text{Surface area} = 2\pi\int_0^{9/4} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{9/4}\sqrt{2x}\sqrt{\frac{1}{2x}+1}\,dx$$

$$= 2\pi\int_0^{9/4}\sqrt{2x+1}\,dx = \pi\left[\frac{2}{3}(2x+1)^{3/2}\right]_0^{9/4} = \left(\frac{11\sqrt{11}}{3\sqrt{2}} - \frac{2}{3}\right)\pi$$

Question 11. (*Revision*)

$$I_n = \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} = \lim_{k\to\infty}\int_0^k \frac{(x^2+1)\,dx}{(x^2+1)^{n+1}} - \lim_{k\to\infty}\int_0^k \frac{x^2\,dx}{(x^2+1)^{n+1}}$$

$$= I_{n-1} + \lim_{k\to\infty}\frac{1}{2n}x(x^2+1)^{-n}\Big|_0^k - \frac{1}{2n}\int_0^\infty \frac{dx}{(x^2+1)^n} = \frac{2n-1}{2n}I_{n-1}$$

$$\therefore\quad I_n = \frac{2n-1}{2n}I_{n-1} = \cdots = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}I_0 = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\int_0^\infty \frac{dx}{x^2+1}$$

$$= \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\lim_{k\to\infty}\left[\tan^{-1}x\right]_0^k = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\,\frac{\pi}{2}$$