The Hong Kong Polytechnic University

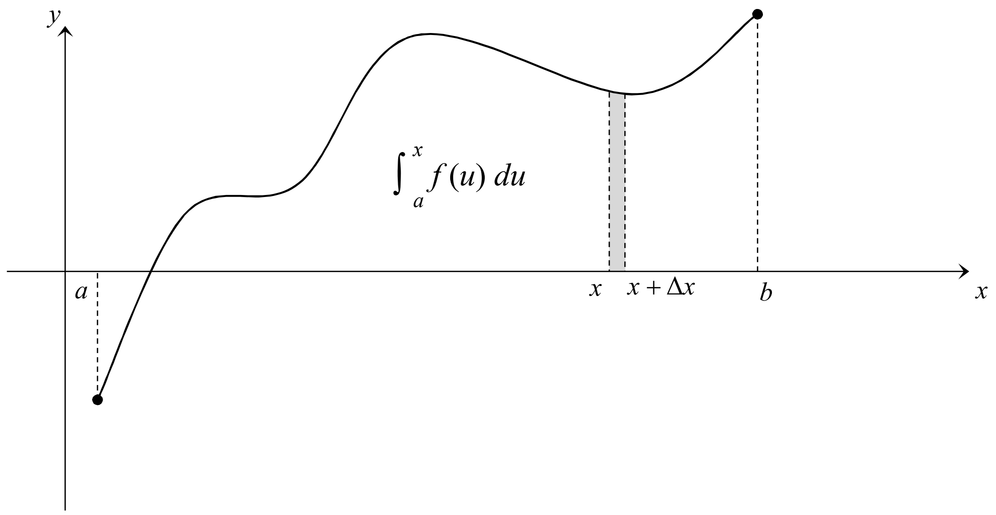
Department of Applied Mathematics

AMA1120 Tutorial Set #03

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(u)\, du$ is differentiable in (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(u)\, du = f(x)$$



This suggests we can solve integration problem by **anti-differentiation**, i.e. to find a function $F(x)$ such that $F'(x) = f(x)$. However, if $F'(x) = G'(x) = f(x)\ \forall\, x \in [a, b]$, then $(F - G)'(x) = 0\ \forall\, x \in [a, b] \Rightarrow F(x) = G(x) + C\ \forall\, x \in [a, b]$ for some constant $C \in \mathbb{R}$.

Question 1. (*Basic Level*)

Find

(a) (i) $\int (1 - x)(1 - 2x)(1 - 3x)\, dx$ (ii) $\int \sqrt{x}\, dx$

(b) (i) $\int \sin x\, dx$ (ii) $\int \cos x\, dx$ (iii) $\int \tan x\, dx$

(iv) $\int \cot x\, dx$ (v) $\int \sec x\, dx$ (vi) $\int \csc x\, dx$

(c) (i) $\int e^x\, dx$ (ii) $\int \frac{1}{x}\, dx$ (iii) $\int \frac{1}{x^2 + 1}\, dx$

Question 2. (*Beginner's Level*)
Find

(a) $\int (3 + 3\sqrt{x})\, dx$

(b) $\int (3 - \cos x + 3x^2)\, dx$

(c) $\int \left(2e^x + \dfrac{3}{x^2}\right) dx$

(d) $\int \dfrac{2}{9 + x^2}\, dx$

(e) $\int \dfrac{1}{5 - 3x}\, dx$

(f) $\int (2x - 3)^{10}\, dx$

(g) $\int (3e^x + \dfrac{2}{x} - \sin 2)\, dx$

(h) $\int (x^3 - 2)\,(\dfrac{1}{x} - 5)\, dx$

(i) $\int \dfrac{x^2 + 3x - 2}{\sqrt{x}}\, dx$

(j) $\int (2 + \tan^2 x)\, dx$

(k) $\int (2 \sin t - 2 \cos t + t^{\frac{5}{4}})\, dt$

(l) $\int \cos x\, (\tan x + \sec x)\, dx$

(m) $\int \dfrac{\sin x}{\sin 2x}\, dx$

(n)* $\int \sinh x\, dx$

(o)* $\int \tanh x\, dx$

(p)* $\int \operatorname{sech} x\, dx$

Change of Variable Formula

Let ϕ have continuous derivative on $[\, c, d\,]$. If f is continuous on $[\, a, b\,] \supseteq \phi(\, [\, c, d\,]\,)$, then

$$\int_{\phi(c)}^{\phi(d)} f(x)\, dx = \int_{c}^{d} f(\phi(u))\, \phi'(u)\, du$$

Question 3. (*Intermediate Level*)
Find

(a) $\int \sec(2x - 3) \tan(2x - 3)\, dx$

(b) $\int 2 \cos^3 x \sin x\, dx$

(c) $\int \dfrac{x}{(x^2 + 2)^2}\, dx$

(d) $\int \dfrac{\tan^{-1} x}{1 + x^2}\, dx$

(e) $\int x\sqrt{2x + 3}\, dx$

(f) $\int \dfrac{1}{\sqrt{x^2 + 1}}\, dx$

(g) $\int \dfrac{1}{x\,(\ln x)^3}\, dx$

(h) $\int x^3 \cos(x^4 + 2)\, dx$

(i) $\int \sin 4x \cos 5x\, dx$

(j) $\int x^5 \sqrt{x^2 + 1}\, dx$

(k) $\int \frac{1}{\sqrt{4x - x^2}}\, dx$

(l) $\int x \sin(\ln x^2)\, dx$

(m) $\int \frac{dx}{1 + \cos x}$

(n) $\int \sqrt{x^2 + 8x + 6}\, dx$

(o) $\int \frac{dx}{\sqrt{2x^2 + 3x + 5}}$

(p) $\int \sin^2 x \cos 2x\, dx$

(q) $\int \frac{dx}{2x\sqrt{1 + (\ln x)^2}}$

(r) $\int (3x + 1)\cos(3x^2 + 2x - 1)\, dx$

(s) $\int \sin 2x \cos^2 3x\, dx$

(t) $\int 3x\,[\,\sec(x^2 + 2)\,]^3 \tan(x^2 + 2)\, dx$

(u) $\int \frac{2x^2}{\sqrt{9 - x^2}}\, dx$

(v) $\int \frac{x^2}{\sqrt{9x - x^2}}\, dx$

(w) $\int x^5 \sqrt{x^3 - 1}\, dx$

(x) $\int \frac{\sqrt{1 + \sin x}}{\sec x}\, dx$

(y) $\int \sqrt{x^2 - 9}\, dx$

(z) $\int \frac{x^2}{\sqrt{x^2 - 25}}\, dx$

Integration by Parts

Let u, v be continuously differentiable function on $[a, b]$. Then

$$\int_a^b u(x)\, d\,(\,v(x)\,) = u(x)\,v(x)\Big|_a^b - \int_a^b v(x)\, d\,(\,u(x)\,)$$

Question 4. (*Intermediate Level*)

Find

(a) $\int x e^x\, dx$

(b) $\int \ln x\, dx$

(c) $\int x^2 e^{-x}\, dx$

(d) $\int x \sin 3x\, dx$

(e) $\int 2x \sec^2 3x\, dx$

(f) $\int \sec^3 x\, dx$

(g) $\int \sin^{-1} x\, dx$

(h) $\int \cos^{-1} x\, dx$

(i) $\int \tan^{-1} x\, dx$

(j) $\int \cot^{-1} x\, dx$

(k) $\int \sec^{-1} x\, dx$

(l) $\int \csc^{-1} x\, dx$

(m) $\int (x^2 + 3x + 1)\, e^x\, dx$

(n) $\int e^{2\sin x} \sin^2 x \cos x\, dx$

(o) $\int x^2 (\ln x)^3\, dx$

(p) $\int (x \sin x - x^2 \cos x)\, dx$

(q) $\int e^{2x} \sin x \cos x\, dx$

(r) $\int (3x^2 - 5x + 1) \ln x\, dx$

(s) $\int e^{\sqrt{t}}\, dt$

(t) $\int x\, e^{3x} \sin 5x\, dx$

(u) $\int \sin \sqrt{x}\, dx$

(v) $\int \sec^{-1} \sqrt{x}\, dx$

(w) $\int x^{\frac{3}{2}} \tan^{-1} x^{\frac{1}{2}}\, dx$

(x) $\int x^2 \exp x^{\frac{3}{2}}\, dx$

(y) $\int e^x \sin^{-1} (e^x)\, dx$

(z) $\int \ln (1 + \sqrt{x})\, dx$

Question 5. (*Exam Level*)

(a) Define $J_n = \int (\ln x)^n\, dx$ for $n \geq 0$. Express J_n in terms of J_{n-1} for $n \geq 1$. Hence find J_3.

(b) Suppose $I_n = \int x^2 (\ln x)^n\, dx$. Show that $I_n = \dfrac{x^3 (\ln x)^n}{3} - \dfrac{n}{3} I_{n-1}$. Hence find I_3.

(c) Establish the reduction formula for $I_n := \int (1 + x)^n \sin 2x\, dx$. Hence find I_4.

Question 6. (*Standard Level*)

(a) Establish the reduction formula for $I_n := \int \sin^n x\, dx$.

(b) Establish the reduction formula for $I_n := \int \cos^n x\, dx$.

(c) Establish the reduction formula for $I_n := \int \tan^n x\, dx$.

(d) Establish the reduction formula for $I_n := \int \cot^n x\, dx$.

(e) Establish the reduction formula for $I_n := \int \sec^n x\, dx$.

(f) Establish the reduction formula for $I_n := \int \csc^n x\, dx$.

Question 7. (*Exam Level*)

(a) Derive the reduction formula

$$\int x^m (\ln x)^n \, dx = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx$$

(b) Derive the reduction formula

$$\int \sin^m x \cos^n x \, dx = -\frac{1}{m+n} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$$

Partial Fractions

Suppose $f(x)$, $g(x)$ are polynomials in real coefficients, where $\deg f(x) < \deg g(x)$ and

$$g(x) = \prod_{k=1}^{M} (\alpha_k x - \beta_k)^{m_k} \prod_{k=1}^{N} (a_k x^2 + b_k x + c_k)^{n_k}.$$

Then

$$\begin{aligned} \frac{f(x)}{g(x)} &= \sum_{k=1}^{M} \left(\frac{C_{k1}}{\alpha_k x - \beta_k} + \cdots + \frac{C_{km_k}}{(\alpha_k x - \beta_k)^{m_k}} \right) \\ &\quad + \sum_{k=1}^{N} \left(\frac{A_{k1} x + B_{k1}}{a_k x^2 + b_k x + c_k} + \cdots + \frac{A_{kn_k} x + B_{kn_k}}{(a_k x^2 + b_k x + c_k)^{n_k}} \right) \end{aligned}$$

in which we call this process as to resolve $\frac{f(x)}{g(x)}$ into **partial fractions**.

Question 8. (*Standard Level*)

Find

(a) $\displaystyle\int \frac{3x+2}{x^2+1} \, dx$

(b) $\displaystyle\int \frac{x^2-2x-1}{(x-1)(x^2+1)} \, dx$

(c) $\displaystyle\int \frac{x-11}{x^2+3x-4} \, dx$

(d) $\displaystyle\int \frac{x^2}{(x-1)(x-2)^2} \, dx$

(e) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x} \, dx$

(f) $\displaystyle\int \frac{2x^2+11x+33}{(2x-3)(4x^2+9)} \, dx$

(g) $\displaystyle\int \frac{x+\sqrt{x}}{x+1} \, dx$

(h) $\displaystyle\int \frac{2x^3-3x^2+5x-9}{2x^2-3x-2} \, dx$

(i) $\displaystyle\int \frac{dx}{x^4+1}$

(j) $\displaystyle\int \frac{x^3}{x^4+1} \, dx$

(k) $\displaystyle\int \frac{x^2-2}{x^2+1} \, dx$

(l) $\displaystyle\int \frac{1}{x^5+2x^3+x} \, dx$

(m) $\displaystyle\int \frac{2x^3+3x^2+4}{(x+1)^4} \, dx$

(n) $\displaystyle\int \frac{x^3+x^2+2x+1}{x^4+2x^2+1} \, dx$

(o) $\int \frac{5e^{-x}}{e^{-2x} + 4e^{-x} + 3}\, dx$

(p) $\int \frac{e^{2x}}{e^{2x} + 3e^{x} + 2}\, dx$

(q) $\int \frac{2 \sin \theta}{\cos^2 \theta + \cos \theta - 2}\, d\theta$

(r) $\int \frac{9 \sec \theta}{1 + \sin \theta}\, d\theta$

(s) $\int \frac{dx}{1 + e^{x} + e^{-x}}$

(t) $\int \frac{dx}{1 + e^{x}}$

Question 9. (*AMA1500 Midterm Past Paper*)

Given $\frac{-3x^6 + 9x^5 - 12x^4 + 18x^3 - 22x^2 + 14x - 6}{(x-1)^2(x^2 - x + 1)} = \frac{-3x^6 + 9x^5 - 12x^4 + 18x^3 - 22x^2 + 14x - 6}{x^4 - 3x^3 + 4x^2 - 3x + 1}$,

resolve the above rational function into partial fractions.

Question 10. (*Concept Discussion*)

The integral

$$\int \frac{1 + 2x^2}{x^5 (1 + x^2)^3}\, dx = \int \frac{x + 2x^3}{(x^4 + x^2)^3}\, dx$$

would require solving eleven equations in eleven unknowns if the method of partial fractions were used. Indeed, there is a simpler way, use the substitution $u = x^4 + x^2$ to evaluate it.

Question 11. (*Challenging Level – Have Fun!*)

(a) $\int \frac{3 + \cos \theta}{2 - \cos \theta}\, d\theta$

(b) $\int x \left(\frac{1 - x^2}{1 + x^2} \right)^{1/2} dx$

(c) $\int \sqrt{1 + \sin t}\, dt$

(d) $\int \sqrt{1 + \cos t}\, dt$

(e) $\int \frac{d\theta}{2 + 2 \cos \theta + \sin \theta}$

(f) $\int \frac{d\theta}{2 + 2 \sin \theta + \cos \theta}$

(g) $\int \ln (x^2 + x + 1)\, dx$

(h) $\int \frac{\tan^{-1} x}{x^2}\, dx$

(i) $\int \frac{\tan^{-1} x}{(x-1)^3}\, dx$

(j) $\int \frac{\sin^{-1} x}{x^2}\, dx$

(k) $\int \frac{\sqrt{1 + \sin^2 x}}{\sec x \csc x}\, dx$

(l) $\int \frac{e^{\sqrt{\sin x}}}{(\sec x) \sqrt{\sin x}}\, dx$

(m) $\int \sqrt{\tan \theta}\; d\theta$

(n) $\int \frac{x^{1/3}}{x^{1/2} + x^{1/4}}\, dx$