The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #04

Question 1. (*Intermediate Level*)

(a) $\displaystyle \lim_{n\to\infty} \frac{1^s + 2^s + \cdots + n^s}{n^{s+1}} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^s \frac{1}{n} = \int_0^1 x^s \, dx = \frac{1}{s+1}$

(b) $\displaystyle \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \frac{1}{n} = \int_0^1 \frac{1}{1+x} \, dx = \ln 2$

(c) $\displaystyle \lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{a+n}} + \frac{1}{\sqrt{2a+n}} + \cdots + \frac{1}{\sqrt{na+n}} \right)$

$\displaystyle = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{\frac{ak}{n}+1}} \frac{1}{n} = \int_0^1 \frac{dx}{\sqrt{ax+1}} = \frac{2}{a}\sqrt{ax+1}\,\Big|_0^1$

$\displaystyle = \frac{2}{a}(\sqrt{a+1} - 1)$

Question 2. (*Beginner's Level*)

(a) $\displaystyle \int_0^4 (3x - \frac{x^3}{4} + 2) \, dx = \left[\frac{3x^2}{2} - \frac{x^4}{16} + 2x \right]_0^4 = 16$

(b) $\displaystyle \int_0^1 (14x^{\frac{4}{3}} - 7x^{\frac{3}{4}}) \, dx = \left[6x^{\frac{7}{3}} - 4x^{\frac{7}{4}} \right]_0^1 = 2$

(c) $\displaystyle \int_{-1}^1 \frac{1}{1+x^2} \, dx = \tan^{-1} x \,\Big|_{-1}^1 = \frac{\pi}{2}$

(d) Let $\displaystyle I = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx$. Let $x = \sin\theta$, $dx = \cos\theta \, d\theta$.

When $x = -\frac{1}{2}$, $\theta = -\frac{\pi}{6}$; when $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$. $\therefore\ \displaystyle I = \int_{-\pi/6}^{\pi/6} d\theta = \frac{\pi}{3}$.

(e) $\displaystyle\int_0^{\pi/3} \frac{2}{\cos^2 x}\, dx = 2\int_0^{\pi/3} \sec^2 x\, dx = 2\tan x \Big|_0^{\pi/3} = 2\sqrt{3}$

(f) $\displaystyle\int_{\pi/4}^{\pi/2} \csc x \cot x\, dx = -\csc x \Big|_{\pi/4}^{\pi/2} = \sqrt{2} - 1$

(g) $\displaystyle\int_0^1 (e^x - x^e)\, dx = \left[e^x - \frac{x^{e+1}}{e+1} \right]_0^1 = e - \frac{1}{e+1} - 1$

(h) $\displaystyle\int_{-1}^2 |x|\, dx = \int_{-1}^0 (-x)\, dx + \int_0^2 x\, dx = -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^2}{2}\Big|_0^2 = \frac{5}{2}$

Question 3. (*Intermediate Level*)

(a) $F'(x) = |x|$

(b) $F'(x) = \dfrac{1}{\cos x} \cdot (-\sin x) = -\tan x$

(c) $F'(x) = -\dfrac{1}{1+\cos^2 x}(-\sin x) = \dfrac{\sin x}{1+\cos^2 x}$

(d) $F'(x) = 3x^2 \sin x^3 - 2x \sin x^2$

(e) $F'(x) = -e^{x^3}$

(f) $F'(x) = \dfrac{x^2}{x^{12}+1} \cdot 2x = \dfrac{2x^3}{x^{12}+1}$

(g) $F'(x) = \cos x\sqrt{\sin^2 x + 3} - \sqrt{x+3} \cdot \dfrac{1}{2} x^{-\frac{1}{2}} = \cos x\sqrt{\sin^2 x + 3} - \dfrac{1}{2}\sqrt{1 + \dfrac{3}{x}}$

(h) $F'(x) = \dfrac{e^{x^3}}{x^6+4} \cdot 3x^2 - \dfrac{e^{x^2}}{x^4+4} \cdot 2x = \dfrac{3x^2 e^{x^3}}{x^6+4} - \dfrac{2x\, e^{x^2}}{x^4+4}$

Question 4. (*Intermediate Level*)

(a) $\int_0^{\pi} \cos x \cos 2x \cos 3x \, dx$

$= \frac{1}{2}\int_0^{\pi} (\cos 3x + \cos x) \cos 3x \, dx = \frac{1}{4}\int_0^{\pi} (\cos 6x + 1 + \cos 4x + \cos 2x) \, dx$

$= \frac{1}{4}\left[\frac{1}{6}\sin 6x + x + \frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}$

(b) Let $I = \int_0^1 \frac{dx}{\sqrt{8 - 4x - x^2}} = \int_0^1 \frac{dx}{\sqrt{12 - (x+2)^2}}$

Let $x + 2 = 2\sqrt{3} \sin \theta, \quad dx = 2\sqrt{3} \cos \theta \, d\theta$

When $x = 0$, $\theta = \sin^{-1}\frac{1}{\sqrt{3}}$. When $x = 1$, $\theta = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\therefore \ I = \int_{\sin^{-1}\frac{1}{\sqrt{3}}}^{\frac{\pi}{3}} \frac{2\sqrt{3}\cos\theta \, d\theta}{2\sqrt{3}\cos\theta} = \frac{\pi}{3} - \sin^{-1}\frac{1}{\sqrt{3}}$

(c) Let $I = \int_0^1 \frac{dx}{\sqrt{x^2 + 4x + 8}} = \int_0^1 \frac{dx}{\sqrt{(x+2)^2 + 4}}$

Let $x + 2 = 2 \tan \theta, \ dx = 2 \sec^2 \theta \, d\theta$

When $x = 0$, $\theta = \frac{\pi}{4}$. When $x = 1$, $\theta = \tan^{-1}\frac{3}{2}$

$\therefore \ I = \int_{\frac{\pi}{4}}^{\tan^{-1}\frac{3}{2}} \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int_{\frac{\pi}{4}}^{\tan^{-1}\frac{3}{2}} \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| \Big|_{\frac{\pi}{4}}^{\tan^{-1}\frac{3}{2}}$

$= \ln\left|\sqrt{\frac{9}{4} + 1} + \frac{3}{2}\right| - \ln|\sqrt{2} + 1| = \ln\frac{\sqrt{13} + 3}{2(\sqrt{2} + 1)} = \ln\frac{\sqrt{26} - \sqrt{13} + 3\sqrt{2} - 3}{2}$

Method 2

Let $I = \int_0^1 \frac{dx}{\sqrt{x^2 + 4x + 8}} = \int_0^1 \frac{dx}{\sqrt{(x+2)^2 + 4}}$

Let $x + 2 = 2 \sinh \theta, \ dx = 2 \cosh \theta \, d\theta$

When $x = 0$, $\theta = \sinh^{-1} 1$. When $x = 1$, $\theta = \sinh^{-1}\frac{3}{2}$

$\therefore \ I = \int_0^1 \frac{2\cosh\theta \, d\theta}{2\cosh\theta} = \int_0^1 d\theta = \sinh^{-1}\frac{3}{2} - \sinh^{-1} 1$

(d) Let $I = \int_0^1 x^5 \sqrt{1+x^2}\, dx$

Let $u = 1 + x^2$, $du = 2x\, dx$. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$

$\therefore\ I = \int_1^2 (u-1)^2 u^{1/2} \cdot \frac{1}{2} du = \frac{1}{2}\int_1^2 (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$

$= \frac{1}{2}\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} \right]_1^2 = \frac{1}{2}\left(\frac{2}{7}(8) - \frac{4}{5}(4) + \frac{2}{3}(2) \right)\sqrt{2} - \frac{1}{2}\left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3} \right)$

$= \frac{22}{105}\sqrt{2} - \frac{8}{105}$

(e) Let $I = \int_1^2 \frac{4x+6}{x^2+3x+1}\, dx$

Let $u = x^2 + 3x + 1$, $du = (2x+3)\, dx$. When $x = 1$, $u = 5$. When $x = 2$, $u = 11$

$\therefore\ I = 2\int_5^{11} \frac{du}{u} = 2 \ln \frac{11}{5} = \ln \frac{121}{25}$

(f) $\int_0^{\pi/2} \frac{1-\cos x}{1+\cos x}\, dx = \int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx = \int_0^{\pi/2} \left(\sec^2 \frac{x}{2} - 1 \right) dx = \left[2\tan\frac{x}{2} - x \right]_0^{\pi/2} = 2 - \frac{\pi}{2}$

(g) Let $I = \int_1^{\sqrt{3}} \frac{dx}{(1+x^2)\tan^{-1} x}$

Let $u = \tan^{-1} x$, $du = \frac{dx}{1+x^2}$. When $x = 1$, $u = \frac{\pi}{4}$. When $x = \sqrt{3}$, $u = \frac{\pi}{3}$

$\therefore\ I = \int_{\pi/4}^{\pi/3} \frac{du}{u} = \ln |u| \Big|_{\pi/4}^{\pi/3} = \ln \frac{4}{3}$

(h) $\int_0^1 \frac{1}{x^2\sqrt{x^2+4}}\, dx \geq \int_0^1 \frac{dx}{x^2\sqrt{4+4}} = -\frac{1}{2\sqrt{2}\, x}\Big|_0^1 = -\frac{1}{2\sqrt{2}} + \lim_{x\to 0} \frac{1}{2\sqrt{2}\, x} = +\infty$

(*Technique for Indefinite Integral*)

Let $I = \int \frac{1}{x^2\sqrt{x^2+4}}\, dx$. Let $x = 2\sinh\theta$, $dx = 2\cosh\theta\, d\theta$

$\therefore\ I = \int \frac{1}{4\sinh^2\theta}\, d\theta = \frac{1}{4}\int \operatorname{csch}^2\theta\, d\theta = -\frac{1}{4}\coth\theta + C = -\frac{1}{4} \cdot \frac{\sqrt{(\frac{x}{2})^2+1}}{\frac{x}{2}} + C$

$= -\frac{\sqrt{x^2+4}}{4x} + C$

(i) $\displaystyle\int_2^{\infty} \frac{1}{x\,(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\, d\,(\ln x) = -\frac{1}{2}\,(\ln x)^{-2}\Big|_2^{\infty} = \frac{1}{2\,(\ln 2)^2}$

(j) $\displaystyle\int_{-1}^{1} (6x^5 + |\,5x - 1\,|)\,dx$

$$= \int_{-1}^{1/5} (6x^5 - 5x + 1)\,dx + \int_{1/5}^{1} (6x^5 + 5x - 1)\,dx$$

$$= \left[\, x^6 - \frac{5x^2}{2} + x \,\right]_{-1}^{1/5} + \left[\, x^6 + \frac{5x^2}{2} - x \,\right]_{1/5}^{1} = \frac{3127}{31250} + \frac{5}{2} + \frac{5}{2} + \frac{3123}{31250} = \frac{26}{5}$$

Question 5. (*Concept*)

(a) $\displaystyle\int_0^{5a} f(x)\,dx = \int_0^{3a} x^2\,dx + \int_{3a}^{4a} 9a^2\;dx + \int_{4a}^{5a} (25a^2 - x^2)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^{3a} + 9a^2 \cdot a \;+ 25a^2 \cdot a - \left[\frac{x^3}{3}\right]_{4a}^{5a} = 9a^3 + 9a^3 + 25a^3 - \frac{61}{3}\,a^3 = \frac{68}{3}\,a^3$$

(b) The average value of $f(x)$

$$= \frac{1}{4-1}\int_1^4 (x^2 + \sqrt{x})\,dx = \frac{1}{3}\left[\frac{x^3}{3} + \frac{2}{3}\,x^{3/2}\right]_1^4 = \frac{77}{9}$$

Question 6. (*Exam Level*)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^{a} f(x)\,dx = -\int_a^{0} f(-u)\,du + \int_0^{a} f(x)\,dx$$

$$= \int_0^{a} [\,f(-x) + f(x)\,]\,dx$$

$$\int_{-1}^{1} \ln\,(x + \sqrt{1+x^2})\,dx = \int_0^{1} [\,\ln\,(-x + \sqrt{1+x^2}) + \ln\,(x + \sqrt{1+x^2})\,]\,dx$$

$$= \int_0^{1} \ln\,(1 + x^2 - x^2)\,dx = 0$$

Question 7. (*Exam Level*)

(a) $\displaystyle I_n = \int_0^1 (1-\sqrt{x})^n\,dx = \int_0^1 (1-\sqrt{x})^{n-1}\,(1-\sqrt{x})\,dx$

$$= I_{n-1} - \int_0^1 (1-\sqrt{x})^{n-1}\sqrt{x}\,dx = I_{n-1} + \frac{2}{n}\,x\,(1-\sqrt{x})^n\Big|_0^1 - \frac{2}{n}\int_0^1 (1-\sqrt{x})^n\,dx$$

$$= I_{n-1} - \frac{2}{n}\,I_n \quad\Rightarrow\quad I_n = \frac{n}{n+2}\,I_{n-1}$$

Hence $I_4 = \frac{4}{6} I_3 = \frac{4}{6}\cdot\frac{3}{5} I_2 = \frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot I_1 = \frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot I_0 = \frac{1}{15}$

(b) For all nonnegative integers m, n ,

$$I_{m,n} = \int_0^1 (1-x)^n \, d\left(\frac{x^{m+1}}{m+1}\right) = \frac{x^{m+1}(1-x)^n}{m+1}\bigg|_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}\,dx$$

$$= \frac{n}{m+1} I_{m+1,\,n-1}$$

$$\therefore \quad I_{m,n} = \frac{n!}{(m+1)(m+2)\cdots(m+n)} I_{m+n+1,\,0} = \frac{n!m!}{(m+n)!} I_{m+n+1,\,0}$$

$$= \frac{n!m!}{(m+n)!}\int_0^1 x^{m+n+1}\,dx = \frac{m!\,n!}{(m+n+1)!}$$

Question 8. (*Intermediate Level*)

(a) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to+\infty}\int_2^b (\ln x)^{-3}\,d(\ln x) = \lim_{b\to+\infty}\left[\frac{(\ln x)^{-2}}{-2}\right]_2^b = \frac{1}{2(\ln 2)^2}$

(b) $\displaystyle\int_1^\infty \frac{(\ln x)^3}{x}\,dx = \lim_{b\to+\infty}\int_1^b (\ln x)^3\,d(\ln x) = \lim_{b\to+\infty}\left[\frac{(\ln x)^4}{4}\right]_1^b = +\infty$

Question 9. (*Intermediate Level*)

(a) $\displaystyle\int_1^\infty \left|\frac{2\cos x + 2^x e^{-x}}{x^5+1}\right| dx \le \int_1^\infty (2x^{-5} + x^{-5})\,dx = \lim_{b\to+\infty} -\frac{3}{4}x^{-4}\bigg|_1^b = \frac{3}{4} < \infty$

$\therefore$ $\displaystyle\int_1^\infty \frac{2\cos x + 2^x e^{-x}}{x^5+1}\,dx$ converges absolutely hence is convergent.

(b) $\displaystyle f(x) = \int_{x^x}^{10} \sin\sqrt{t}\,dt$

$\Rightarrow$ $f'(x) = -\sin x^{x/2}\cdot e^{x\ln x}(\ln x + 1) < 0$ for all $x \in [1, 2]$

$\therefore$ Hence f is strictly decreasing, i.e. f attains its minimum at $x = 2$.

Question 10. (*Exam Level*)

(a) Let $f(x_{\min}) = \min f(x)$, $f(x_{\max}) = \max f(x)$. Hence

$$f(x_{\min}) \le f(x) \le f(x_{\max}) \;\Rightarrow\; f(x_{\min}) \le \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le f(x_{\max})$$

By Intermediate Value Theorem, $\exists\, c \in [a, b]$ in between $x_{\min}$ and $x_{\max}$ such that

$$f(c) = \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \;\Rightarrow\; \int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

(b) $\displaystyle \lim_{\delta \to 0^+} \frac{1}{\delta^4}\int_0^\delta \cos(x^2)\,x^3\,dx = \lim_{\delta \to 0^+} \frac{\cos(c^2)}{\delta^4}\int_0^\delta x^3\,dx = \lim_{\delta \to 0^+} \frac{\cos(c^2)}{4} = \frac{1}{4}$ for some $c \in (0, \delta)$

Question 11.** (*Gamma Function – for fun only!*)

(a) $$\Gamma(x+1) = \lim_{b \to +\infty} \int_0^b t^x e^{-t}\,dt = \lim_{b \to +\infty} \left(-t^x e^{-t}\Big|_0^b + x\int_0^b t^{x-1} e^{-t}\,dt \right)$$

$$= \lim_{b \to +\infty} -\frac{b^x}{e^b} + x\,\Gamma(x) = x\,\Gamma(x)$$

for all $x \ge 0$, since $\displaystyle \lim_{b \to +\infty} \frac{b^x}{e^b} = 0$ by l'Hôpital's rule.

(b) Note that $\displaystyle \Gamma(0) = \lim_{b \to +\infty} \int_0^b e^{-t}\,dt = \lim_{b \to +\infty} -e^{-t}\Big|_0^b = 1.$

Hence for integer $n \ge 0$, $\Gamma(n+1) = n\,\Gamma(n) = \cdots = n!\,\Gamma(0) = n!$.

(c) $\displaystyle \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt$. Let $t = \dfrac{z^2}{2}$, $z \ge 0 \Rightarrow z = \sqrt{2t} \Rightarrow dz = \dfrac{\sqrt{2}}{2} t^{-1/2}\,dt$.

When $t = 0$, $z = 0$; when $t \to \infty$, $z \to \infty$. Since $e^{-\frac{z^2}{2}}$ is an even function,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int_0^\infty e^{-\frac{z^2}{2}}\,dz = \sqrt{\pi}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{z^2}{2}}\,dz = \sqrt{\pi}$$

Question 12. (*Exam Level*)

(a) Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$. When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/2$.

$$I_0 = \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4}$$

Let $u = 1 - x^2$, $du = -2x\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$.

$$I_1 = \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 u^{1/2}\,du = \frac{1}{3}u^{1/2}\Big|_0^1 = \frac{1}{3}$$

(b) $$I_{n+2} = \int_0^1 x^{n+2}\sqrt{1-x^2}\,dx = -\frac{1}{3}x^{n+1}(1-x^2)^{3/2}\Big|_0^1 + \frac{n+1}{3}\int_0^1 x^n\sqrt{1-x^2}\,(1-x^2)\,dx$$

$$= \frac{n+1}{3}I_n - \frac{n+1}{3}I_{n+2}$$

$$\Rightarrow \quad \frac{n+4}{3}I_{n+2} = \frac{n+1}{3}I_n \quad \Rightarrow \quad I_{n+2} = \frac{n+1}{n+4}I_n$$

Idea:

$$\begin{tikzcd}
-\frac{1}{2}x^{n+1} \arrow[dr, "+"] & -2x\sqrt{1-x^2} \\
-\frac{1}{2}(n+1)x^n \arrow[r, "-"] & \frac{2}{3}(1-x^2)^{3/2}
\end{tikzcd}$$

(c) $$I_5 = \frac{4}{7}I_3 = \frac{4}{7}\cdot\frac{2}{5}I_1 = \frac{4}{7}\cdot\frac{2}{5}\cdot\frac{1}{3} = \frac{8}{105}$$

$$I_6 = \frac{5}{8}I_4 = \frac{5}{8}\cdot\frac{3}{6}I_2 = \frac{5}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{\pi}{4} = \frac{5}{256}\pi$$

Question 13. (*Standard Level*)

(a) $$\int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{(3x+1)^2}\,dx = \lim_{b\to+\infty}\frac{1}{3}\int_1^b \frac{d(3x+1)}{(3x+1)^2} = \lim_{b\to+\infty}\left[-\frac{1}{3(3x+1)}\right]_1^b$$

$$= \frac{1}{12}$$

(b) $$\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{b\to+\infty}\frac{1}{2}\int_0^b \frac{d(1+x^2)}{1+x^2} = \lim_{b\to+\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_0^b = +\infty$$

(c) If $p = 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to+\infty}\Big[\ln x\Big]_1^b = +\infty$. Now suppose $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to+\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \begin{cases} +\infty, & \text{if } 0 < p < 1 \\ \frac{1}{p-1}, & \text{if } p > 1 \end{cases}$$

(d) If $p = 1$, $\int_1^\infty \frac{\ln x}{x^p}\,dx = \int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to+\infty}\int_1^b \ln x\, d(\ln x) = \lim_{b\to+\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b = +\infty$.

Now suppose $p \neq 1$.

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b \frac{\ln x}{x^p}\,dx = \lim_{b\to+\infty}\left(\frac{1}{1-p}x^{-p+1}\ln x\Big|_1^b + \frac{1}{p-1}\int_1^b \frac{1}{x^p}\,dx\right)$$

$$= \lim_{b\to+\infty}\frac{1}{1-p}\,b^{-p+1}\ln b + \frac{1}{p-1}\int_1^b \frac{1}{x^p}\,dx$$

$$= \lim_{b\to+\infty}\frac{b^{-p+1}\,[(1-p)\ln b - 1] + 1}{(1-p)^2} = \begin{cases} +\infty, & \text{if } 0 < p < 1 \\ \frac{1}{(p-1)^2}, & \text{if } p > 1 \end{cases}$$

since when $p > 1$, $\lim_{b\to+\infty}\frac{\ln b}{b^{p-1}} = \lim_{b\to+\infty}\frac{1/b}{(p-1)\,b^{p-2}} = \lim_{b\to+\infty}\frac{1}{(p-1)\,b^{p-1}} = 0$.

(e) $\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \int_e^\infty \frac{d(\ln x)}{(\ln x)^p}$

Let $u = \ln x$. When $x = e$, $u = 1$; when $x \to +\infty$, $u \to +\infty$

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \int_1^\infty \frac{du}{u^p} = \begin{cases} +\infty, & \text{if } 0 < p \leq 1 \\ \frac{1}{p-1}, & \text{if } p > 1 \end{cases}$$

Question 14. (*Intermediate Level*)

(a) Since $e^{-x^3} > 0$, $\int_0^\infty e^{-x^3}\,dx$ converges if it is bounded above.

For $x \geq 1$, $x^3 \geq x \Rightarrow e^{-x^3} \leq e^{-x}$. Thus

$$\int_0^\infty e^{-x^3}\,dx = \int_0^1 e^{-x^3}\,dx + \int_1^\infty e^{-x^3}\,dx \leq \int_0^1 e^{-x^3}\,dx + \int_1^\infty e^{-x}\,dx$$

$$= \int_0^1 e^{-x^3}\,dx + \lim_{b\to+\infty}\Big[-e^{-x}\Big]_1^b = \int_0^1 e^{-x^3}\,dx + \frac{1}{e} < +\infty$$

where $\int_0^1 e^{-x^3}\,dx < +\infty$ since e^{-x^3} is continuous on [0, 1].

(b) (See the reference: Improper Integral - Absolute convergence implies convergence)

$$\int_1^{\infty} \left| \frac{2\sin x + xe^{-x}}{x^4 + x} \right| dx \le \int_1^{\infty} \left(\frac{2\,|\sin x|}{x^4 + x} + \frac{xe^{-x}}{x^4 + x} \right) dx$$

$$\le \int_1^{\infty} \left(\frac{2}{x^4} + e^{-x} \right) dx = \lim_{b \to +\infty} \left[-\frac{2}{3}x^{-3} - e^{-x} \right]_1^b = \frac{2}{3} + \frac{1}{e} < +\infty.$$

Thus $\int_1^{\infty} \frac{2\sin x + xe^{-x}}{x^4 + x}\, dx$ converges absolutely.

(c) $\int_2^{\infty} \frac{1}{(\ln x)^2}\, dx \ge \int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_2^{\infty} \frac{d(\ln x)}{\ln x} = \lim_{b \to +\infty} \left[\ln(\ln x) \right]_2^b = +\infty$

Question 15. (*Theory*)

Suppose f is continuous on $[a, x]$ and $f', f'', \ldots, f^{(n)}, f^{(n+1)}$ are continuous in (a, x).

$f(x) - f(a)$

$= \int_a^x f'(u)\, du = \int_a^x f'(u)\, d(u - x)$

$= \left[f'(u)(u - x) - \frac{1}{2} f''(u)(u - x)^2 - \cdots - (-1)^n \frac{1}{n!} f^{(n)}(u)(u - x)^n \right]_a^x$

$\quad - (-1)^{n+1} \int_a^x f^{(n+1)}(u) \frac{(u - x)^n}{n!}\, du$

$= f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x - a)^n$

$\quad + f^{(n+1)}(\xi) \cdot \frac{1}{n!} \int_a^x (x - u)^n\, du$ for some $\xi \in (a, x)$ (cf. Question 10 (a))

$= f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}$

$\therefore f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}$

i.e. Taylor's Theorem is proved.

Question 16. (*Intermediate Level*)

(a) $f(x) = x^{\frac{3}{4}} \Rightarrow f'(x) = \frac{3}{4} x^{-\frac{1}{4}} \Rightarrow f'(16) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

Thus the linear approximation at $x_0 = 16$ is given by

$$L(x) = f(16) + f'(16)(x - 16) = 8 + \frac{3}{8}(x - 16)$$

$$\sqrt[4]{17^3} = f(17) \approx L(17) = 8 + \frac{3}{8}(1) = 8.375$$

$$f''(x) = -\frac{3}{16} x^{-\frac{5}{4}}$$

By Taylor Theorem, since f is C^2 near $x_0 = 16$, $\exists\, \xi \in (16, 17)$ such that

$$\sqrt[4]{17^3} = f(17) = f(16) + f'(16)(17 - 16) + \frac{1}{2} f''(\xi)(17 - 16)^2 = L(17) - \frac{3}{32}\xi^{-\frac{5}{4}}$$

$$\Rightarrow \quad |\sqrt[4]{17^3} - L(17)| \le \frac{3}{32}\xi^{-\frac{5}{4}} \le \frac{3}{32}(16)^{-\frac{5}{4}} = \frac{3}{1024} \approx 0.00293$$

(b) $f''(x) = -\frac{3}{16} x^{-\frac{5}{4}} \Rightarrow f''(16) = -\frac{3}{512}$

Thus the Taylor's polynomial of degree 2 at $x_0 = 16$ is given by

$$T(x) = f(16) + f'(16)(x - 16) + \frac{1}{2} f''(16)(x - 16)^2 = 8 + \frac{3}{8}(x - 16) - \frac{3}{1024}(x - 16)^2$$

$$\sqrt[4]{17^3} = f(17) \approx T(17) = 8 + \frac{3}{8}(1) - \frac{3}{1024}(1)^2 = \frac{8573}{1024} \approx 8.37207$$

$$f'''(x) = \frac{15}{64} x^{-\frac{9}{4}}$$

By Taylor Theorem, since f is C^3 near $x_0 = 16$, $\exists\, \xi \in (16, 17)$ such that

$$\sqrt[4]{17^3} = f(17) = f(16) + f'(16)(17 - 16) + \frac{1}{2} f''(16)(17 - 16)^2 + \frac{1}{3!} f'''(\xi)(17 - 16)^3$$

$$= T(17) + \frac{1}{6} \cdot \frac{15}{64}\xi^{-\frac{9}{4}} = T(17) + \frac{5}{128}\xi^{-\frac{9}{4}}$$

$$\Rightarrow \quad |\sqrt[4]{17^3} - T(17)| \le \frac{5}{128}\xi^{-\frac{9}{4}} \le \frac{5}{128}(16)^{-\frac{9}{4}} = \frac{5}{65536} \approx 0.0000763$$