The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Final Exam 2023/24 Semester 2

Question 1.

(a) Consider the function $f(x) = \dfrac{2x}{x^2+1}$, $x \in \mathbb{R}$.

(i) Find the intervals on which f is increasing or decreasing, also find all the relative extrema of f in $\mathbb{R}$. [6 marks]

(ii) Find the intervals on which f is concave up or concave down and identify the inflection points. [6 marks]

(b) Let $F(x) = \displaystyle\int_x^{x^2} \cos(t^2)\,dt$, where $x > 0$.

(i) Compute $F'(x)$ and $F''(x)$. [6 marks]

(ii) Find the degree-2 Taylor polynomial of F at $x_0 = 0$ and use it to estimate $F(0.1)$. [7 marks]

My work :

(a) (i) $f(x) = \dfrac{2x}{x^2+1} \;\Rightarrow\; f'(x) = \dfrac{(x^2+1)(2) - (2x)\,2x}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

$$f'(x) = \frac{2(1-x^2)}{(x^2+1)^2} = 0 \;\Leftrightarrow\; 1 - x^2 = 0 \;\Leftrightarrow\; x = \pm 1$$

$\therefore$ The critical points of f are: $x = \pm 1$.

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	$-$	$+$	$-$

The open interval where f is increasing is: $(-1, 1)$

The open intervals where f is decreasing are: $(-\infty, -1)$, $(1, +\infty)$

(ii) $$f''(x) = \frac{(x^2+1)^2(-4x) - 2(1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(-4x) - 2(1-x^2)(2)(2x)}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3} = 0$$

$\Leftrightarrow\; x = 0, \pm\sqrt{3}$.

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, +\infty)$
$f''(x)$	$-$	$+$	$-$	$+$

The open intervals where f is concave up are: $(-\sqrt{3}, 0)$, $(\sqrt{3}, +\infty)$

The open intervals where f is concave down are: $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

(b) (i) $F(x) = \int_x^{x^2} \cos(t^2)\, dt \quad \Rightarrow \quad F'(x) = 2x\cos(x^4) - \cos(x^2)$

$F''(x) = 2\cos(x^4) - 8x^4\sin(x^4) - 2x\sin(x^2)$

(ii) $F(0) = \int_0^0 \cos(t^2)\, dt = 0,\ F'(0) = -1,\ F''(0) = 2$

The degree-2 Taylor polynomial of F around $x = 0$ is given by

$T_2(x) = -x + x^2$

$\therefore\ F(0.1) \approx T_2(0.1) = -0.09$

Question 2.

Evaluate the following integrals:

(a) $\int \frac{1}{x^2} \sec\left(\frac{1}{x}\right) dx$ [5 marks]

(b) $\int \frac{1}{\ln(x^x)} dx$ [5 marks]

(c) $\int_0^1 x^2 e^{-x} dx$ [5 marks]

(d) $\int \frac{x+2}{(x-1)(x+2)-10} dx$ [5 marks]

(e) $\int \frac{\sqrt{x+3}}{x+4} dx$ [5 marks]

My work :

(a) $\int \frac{1}{x^2} \sec\left(\frac{1}{x}\right) dx = -\int \sec\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right) = -\ln \left| \sec\left(\frac{1}{x}\right) + \tan\left(\frac{1}{x}\right) \right| + C$

(b) $\int \frac{1}{\ln(x^x)} dx = \int \frac{1}{x \ln x} dx = \int \frac{d(\ln x)}{\ln x} = \ln|\ln x| + C$

(c) $\int_0^1 x^2 e^{-x} dx = \cdots = \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^1 = 2 - \frac{5}{e}$

(d) $\int \frac{x+2}{(x-1)(x+2)-10} dx = \int \frac{x+2}{x^2+x-12} dx = \int \frac{x+2}{(x+4)(x-3)} dx$

$$= \int \left(\frac{\frac{2}{7}}{x+4} + \frac{\frac{5}{7}}{x-3} \right) dx = \frac{2}{7} \ln|x+4| + \frac{5}{7} \ln|x-3| + C$$

(e) Let $u^2 = x + 3$, $2u\, du = dx$.

$$\int \frac{\sqrt{x+3}}{x+4} dx = \int \frac{2u^2\, du}{u^2+1} = \int \left(2 - \frac{1}{u^2+1}\right) du = 2u - \tan^{-1} u + C$$

$$= 2\sqrt{x+3} - \tan^{-1}\sqrt{x+3} + C$$

Question 3.

(a) Find the arc length of the curve defined by $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$, $1 \le x \le 2$. [6 marks]

(b) Consider the region bounded by $y = -3x$ and $y = x^2 + 2$ in the xy-plane.

(i) Find the area of this region; [6 marks]

(ii) Find the volume of the solid obtained by revolving this region about the x-axis. [7 marks]

(c) Discuss the convergence of the improper integral $\displaystyle\int_1^\infty \frac{x^2 \sin x}{x^4 + e^{-x}}\, dx$. [6 marks]

My work :

(a) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2} \Rightarrow y' = x^3 - \dfrac{1}{4x^3}$

$$1 + (y')^2 = 1 + \left(x^3 - \frac{1}{4x^3}\right)^2 = \left(x^3 + \frac{1}{4x^3}\right)^2$$

$$\text{Arc-length} = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(x^3 + \frac{1}{4x^3}\right) dx = \left[\frac{x^4}{4} - \frac{1}{8x^2}\right]_1^2 = \frac{123}{32}$$

(b) (i) $\begin{cases} y = -3x \\ y = x^2 + 2 \end{cases} \Rightarrow x^2 + 3x + 2 = 0 \Rightarrow x = -2, -1$

$$\text{Area} = \int_{-2}^{-1} [\,(-3x) - (x^2 + 2)\,]\, dx = -\left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_{-2}^{-1} = \frac{5}{6} - \frac{2}{3} = \frac{1}{6}$$

(ii) $$\text{Volume} = \pi \int_{-2}^{-1} [\,(-3x)^2 - (x^2 + 2)^2\,]\, dx = \pi \int_{-2}^{-1} (5x^2 - x^4 - 4)\, dx$$

$$= \pi \left[\frac{5x^3}{3} - \frac{x^5}{5} - 4x\right]_{-2}^{-1} = \frac{38}{15} - \frac{16}{15} = \frac{22}{15}$$

(c) $$\int_1^\infty \left|\frac{x^2 \sin x}{x^4 + e^{-x}}\right| dx \le \int_1^\infty \frac{1}{x^2}\, dx = \lim_{b \to +\infty} -\frac{1}{x}\bigg|_1^b = 1 < \infty$$

$\therefore$ $\displaystyle\int_1^\infty \frac{x^2 \sin x}{x^4 + e^{-x}}\, dx$ converges absolutely hence is convergent.

Question 4.

(a) Let $A = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) Compute $AB - BA$. [5 marks]

(ii) Compute $A^{-1} B^{-1}$. [5 marks]

(b) Consider the following system of linear equations

$$\begin{bmatrix} 1 & 1 & p \\ 3 & 0 & 2 \\ 0 & 3 & p^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3p \\ 9 \end{bmatrix},$$

where p is a constant. Determine the possible values of p such that the system is

(i) consistent with a unique solution;

(ii) consistent with infinitely many solutions and solve the system;

(iii) inconsistent.

Also solve the system when it has infinitely many solutions. [15 marks]

My work :

(a) (i) $$AB - BA = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 \\ 0 & 10 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -3 \\ 0 & 10 & 10 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $$A^{-1} B^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & -1 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{10} & 2 \\ 0 & \frac{1}{10} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $$\left[\begin{array}{ccc|c} 1 & 1 & p & 5 \\ 3 & 0 & 2 & 3p \\ 0 & 3 & p^2 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & p & 5 \\ 0 & -3 & 2-3p & 3p-15 \\ 0 & 3 & p^2 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & p & 5 \\ 0 & -3 & 2-3p & 3p-15 \\ 0 & 0 & p^2-3p+2 & 3p-6 \end{array}\right]$$

(i) The system is consistent with a unique solution $\Leftrightarrow p^2 - 3p + 2 \neq 0 \Leftrightarrow p \neq 1, 2$

If $p = 1$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -3 & -1 & -12 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

If $p = 2$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -3 & -4 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 2 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow$ $p = 2$. In this case,

$(x, y, z) = (2 - \frac{2}{3}t,\ 3 - \frac{4}{3}t,\ t)$, where $t \in \mathbb{R}$

(iii) The system is inconsistent $\Leftrightarrow$ $p = 1$.

Alternative Method

(i) The system is consistent with a unique solution

$$\Leftrightarrow \begin{vmatrix} 1 & 1 & p \\ 3 & 0 & 2 \\ 0 & 3 & p^2 \end{vmatrix} = -3\,(p^2 + 3p + 2) \neq 0 \quad \Leftrightarrow \ p \neq 1, 2.$$

If $p = 1$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -3 & -1 & -12 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

If $p = 2$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -3 & -4 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 2 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow$ $p = 2$. In this case,

$(x, y, z) = (2 - \frac{2}{3}t,\ 3 - \frac{4}{3}t,\ t)$, where $t \in \mathbb{R}$

(iii) The system is inconsistent $\Leftrightarrow$ $p = 1$.