The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 2 2021/22 Semester 2

Question 1.

[10 marks each] Calculate the following integrals.

(a) $\int_0^{\sqrt{3}} x^5 \sqrt{x^2+1}\, dx$

(b) $\int_0^2 2x\, e^{2x}\, dx$

(c) $\int x^2 \cos(x)\, dx$

(d) $\int_0^{\pi/2} \sin(2x) \sin(3x)\, dx$

(e) $\int \dfrac{x^2+1}{(x-1)(x-2)(x+3)}\, dx$

My work :

(a) Let $I = \int_0^{\sqrt{3}} x^5 \sqrt{x^2+1}\, dx$. Let $u = x^2 + 1,\ du = 2x\, dx$.

When $x = 0$, $u = 1$; when $x = \sqrt{3}$, $u = 4$.

$$I = \frac{1}{2}\int_1^4 (u-1)^2 u^{1/2}\, du = \frac{1}{2}\int_1^4 (u^{5/2} - 2u^{3/2} + u^{1/2})\, du = \frac{1}{2}\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_1^4 = \frac{848}{105}$$

(b) $\int_0^2 2x\, e^{2x}\, dx = \left[\, xe^{2x} - \frac{1}{2}e^{2x}\,\right]_0^2 = \frac{3}{2}e^4 + \frac{1}{2}$

(c) $\int x^2 \cos(x)\, dx = \cdots = x^2 \sin x + 2x \cos x - 2\sin x + C$

(d) $\int_0^{\pi/2} \sin(2x)\sin(3x)\, dx = \frac{1}{2}\int_0^{\pi/2} (\cos x - \cos 5x)\, dx = \frac{1}{2}\left[\sin x - \frac{1}{5}\sin 5x\right]_0^{\pi/2} = \frac{2}{5}$

(e) $\int \dfrac{x^2+1}{(x-1)(x-2)(x+3)}\, dx$

$$= \int \left(\frac{-\frac{1}{2}}{x-1} + \frac{1}{x-2} + \frac{\frac{1}{2}}{x+3} \right) dx = -\frac{1}{2}\ln|x-1| + \ln|x-2| + \frac{1}{2}\ln|x+3| + C$$

(f) $\displaystyle\int \frac{1}{1-e^{2x}}\,dx$

(g) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx$ with $|x|>1$

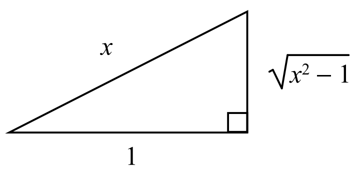
My work :

(f) $$\int \frac{1}{1-e^{2x}}\,dx = \int \frac{e^{-2x}}{e^{-2x}-1}\,dx = -\frac{1}{2}\int \frac{d\,(e^{-2x}-1)}{e^{-2x}-1} = -\frac{1}{2}\ln|\,e^{-2x}-1\,| + C$$

(g) Let $\displaystyle I = \int \frac{1}{\sqrt{x^2-1}}\,dx$.

Let $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-1} = \tan\theta$.

$$I = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\,x + \sqrt{x^2-1}\,| + C$$



Question 2.

[10 marks] Let $f(t) = \int_0^{e^t} (s+2)\cos(s)\,ds$ and $F(x) = \int_0^{\ln x} f(t)\,dt$. Find $F'(\pi)$.

My work :

$$F(x) = \int_0^{\ln x} f(t)\,dt \quad\Rightarrow\quad F'(x) = f(\ln x)\,\frac{1}{x}$$

$$\Rightarrow\quad F'(\pi) = \frac{1}{\pi}\,f(\ln\pi) = \frac{1}{\pi}\int_0^{\pi}(s+2)\cos(s)\,ds = \frac{1}{\pi}\Big[\,(s+2)\sin s + \cos s\,\Big]_0^{\pi} = -\frac{2}{\pi}$$

Question 3.

(a) [10 marks] Find the Taylor polynomial of degree 3 for function $f(x) = \sin(x)\cos(x)$ at $x_0 = 0$. In other words, approximate $f(x)$ by

$$T_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3,$$

where a_0, a_1, a_2, a_3 are the numbers that you should find out.

(b) [10 marks] Show that the error in approximating $f(x)$ by $T_3(x)$ for $x \in [0, \frac{\pi}{4}]$ is less than $\frac{1}{3}(\frac{\pi}{4})^4$. Do NOT use a calculator for question (b).

My work :

(a) $f(x) = \dfrac{1}{2}\sin 2x,\ f'(x) = \cos 2x,\ f''(x) = -2\sin 2x,\ f^{(3)}(x) = -4\cos 2x$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f^{(3)}(0) = -4$

The Taylor polynomial of degree 3 for $f(x)$ about $x_0 = 0$ is given by

$$T_3(x) = x - \frac{4}{3!}x^3 = x - \frac{2}{3}x^3$$

(b) $f^{(4)}(x) = 8\sin 2x$

The remainder term is given by $R_3(x) := \dfrac{f^{(4)}(\xi)}{4!}x^4 = \dfrac{\sin 2\xi}{3}x^4$

for some $\xi \in (0, x) \subseteq [0, \frac{\pi}{4}]$

$$|f(x) - T_3(x)| = |R_3(x)| = \frac{|\sin 2\xi|}{3}|x|^4 < \frac{1}{3}\left(\frac{\pi}{4}\right)^4$$