The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #01

Review on Limits and Differentiations

A function $f : I \to \mathbb{R}$ is called *continuous* at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

A function $f : I \to \mathbb{R}$ is called *differentiable* at x if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R}$.

Some Properties on Limits

1. $\lim_{x \to c} f(x)$ exists in $\mathbb{R}$ $\Leftrightarrow$ $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$ exist in $\mathbb{R}$ and $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$.

2. $f(x) \le g(x) \le h(x)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L \Rightarrow \lim_{x \to c} g(x) = L$

3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$, $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

4. In case of **indeterminate forms** $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ or $\pm\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \qquad \text{(}\textit{l'Hôpital's rule}\text{)}$$

Some Properties on Derivatives

1. $(f \pm g)' = f' \pm g'$, $(fg)' = fg' + f'g$, $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

2. If $f = g \circ h$, then $f'(x) = g'(\,\boldsymbol{h(x)}\,)\, h'(x)$ (*Chain rule*)

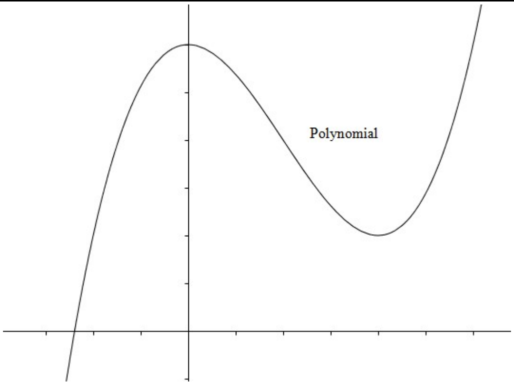
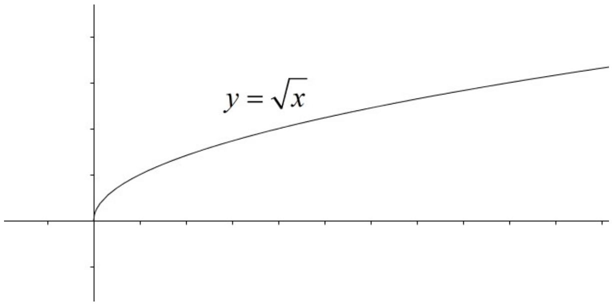
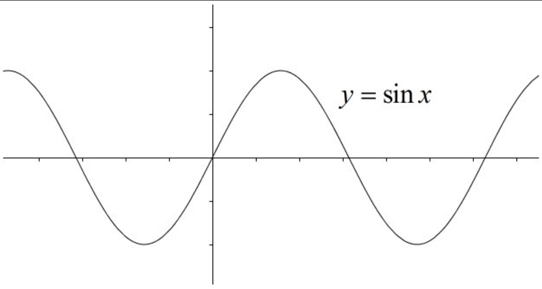
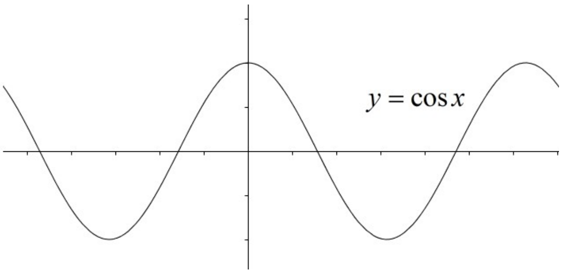
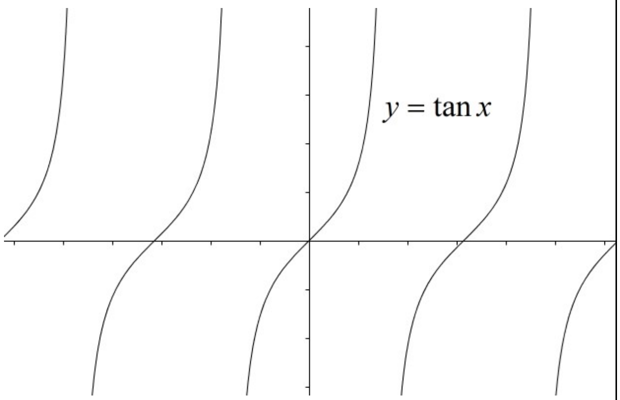
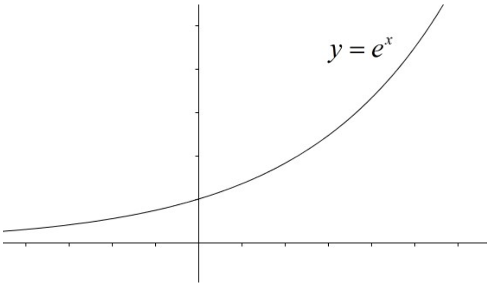
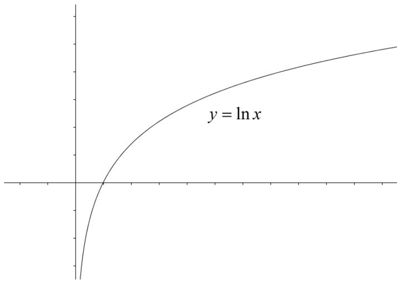
Some Special Derivatives

1. $\frac{d}{dx}(\text{constant}) = 0$, $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

2. $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$, $\frac{d}{dx}(\cot x) = -\csc^2 x$,

   $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Elementary Functions

 Polynomials continuous on the whole real lines	 Fractional powers of x continuous on $[0,+\infty)$
  Sine, Cosine function continuous on the whole real lines	 Tangent function continuous except $x=(2n+1)\,\pi/2$
 Exponential function continuous on the whole real lines	 Logarithm function continuous on $(0,+\infty)$

Question 1 (*Basic Level*)

(a) Find the derivatives of each of the following functions.

(1) $f(x) = 2x^3 - 9x^2 - 24x + 7$ (2) $f(x) = x^{2/3}$

(3) $f(x) = e^{x^2}$ (4) $f(x) = \tan(3x - 1)$

(5) $f(x) = \tan^{-1} x$ (6) $f(x) = 2x^2 - \ln x$

(b) Find the second derivatives of each of the functions in (a).

Stationary Points

A real number $c \in \mathbb{R}$ is called a stationary point of f if $f'(c) = 0$.

Question 2 (*Standard Level*)

(a) Find all stationary points of $f(x) = 2x^3 - 9x^2 - 24x + 7$.

(b) Find all stationary points of $f(x) = 2x^2 - \ln x$.

(c) Find all stationary points of $f(x) = \dfrac{x}{1 + x^2}$.

(d) Find all stationary points of $f(x) = x^2 e^{-x}$.

Intervals of Monotonicity

A function f is called **(strictly) monotone** if either f is **(strictly) increasing**, i.e. $f' \geq 0$ ($f' > 0$), or f is (strictly) decreasing, i.e. $f' \leq 0$ ($f' < 0$).

In general, f is not monotone everywhere, however we can determine the **intervals of monotonicity** where f is strictly increasing ($f' > 0$) or strictly decreasing ($f' < 0$).

Question 3 (*Standard Level*)

(a) Determine the intervals of strict monotonicity of $f(x) = 2x^3 - 9x^2 - 24x + 7$.

(b) Determine the intervals of monotonicity of $f(x) = 2x^2 - \ln x$.

(c) Determine the intervals of monotonicity of $f(x) = \dfrac{x}{1 + x^2}$.

(d) Determine the intervals of monotonicity of $f(x) = x^2 e^{-x}$.

Classification of Stationary Points

We can classify a stationary point c ($f'(c) = 0$) by *first derivative test*.

1. If $f'(x) \begin{cases} < 0 & \text{if } c - \delta < x < c \\ > 0 & \text{if } c + \delta > x > c \end{cases}$, then $x = c$ is called a **local minimum**.
2. If $f'(x) \begin{cases} > 0 & \text{if } c - \delta < x < c \\ < 0 & \text{if } c + \delta > x > c \end{cases}$, then $x = c$ is called a **local maximum**.

If f is twice differentiable, we can also classify a stationary point by *second derivative test*.

3. If $f''(c) > 0$, then $x = c$ is called a **local minimum**.
4. If $f''(x) < 0$, then $x = c$ is called a **local maximum**.

Question 4 (*Standard Level*)

(a) Classify all stationary points of $f(x) = 2x^3 - 9x^2 - 24x + 7$.

(b) Classify all stationary points of $f(x) = 2x^2 - \ln x$.

(c) Classify all stationary points of $f(x) = \dfrac{x}{1 + x^2}$.

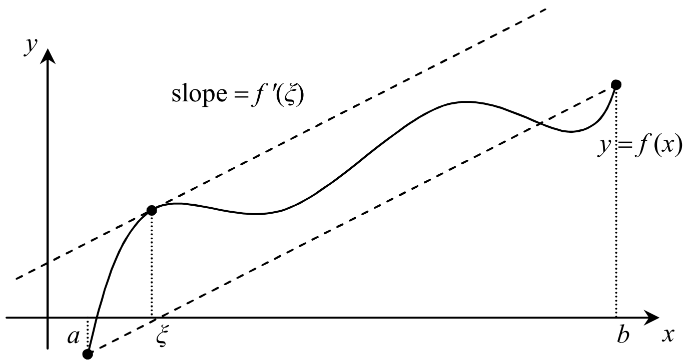
Rolle's Theorem

Let f be a real-valued function which is continuous on $[a, b]$, differentiable in (a, b) and satisfies $f(a) = f(b)$. Then there is a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Mean Value Theorem

Let f be a real-valued function which is continuous on $[a, b]$, differentiable in (a, b). Then there is a point $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



Question 5* (*Concept*)

(a) Explain the Rolle's Theorem.

(b) By Rolle's Theorem, prove the Mean Value Theorem.

Question 6 (*Intermediate Level*)

Find a number ξ, if exists, that is described by the Mean Value Theorem for

(a) $f(x) = x^3 - x^2 - x + 1$ on $[1, 2]$

(b) $f(x) = x^{2/3}$ on $[-8, 8]$

Question 7 (*Standard Level*)

By making use of Mean Value Theorem, prove that for $0 < x \le 1$, we have

$$1 + x < e^x < 1 + ex.$$

Question 8 (*Standard Level*)

Prove that $2x - 2x^2 \le \ln(1 + 2x) \le 2x$ for $x \ge 0$.

Question 9 (*Standard Level*)

By making use of Mean Value Theorem, prove that for $0 < a < b$,

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

and hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

Question 10 (*Standard Level*)

What does the Mean Value Theorem say about the function $f(x) = 2x + \sin 3x$ in the interval $[a, b] = [0, \pi]$? Find a point inside the interval $(0, \pi)$ that satisfies the conclusion of the Mean Value Theorem.