The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Final Exam 2021/22 Semester 2

Question 1.

Consider the function $f(x) = \dfrac{|x|}{x^2+1}$, $x \in \mathbb{R}$.

(a) Find $f'(x)$. [5 marks]

(b) Find all the critical points of f. [5 marks]

(c) Find all open intervals on which the function is increasing or decreasing. [5 marks]

(d) Find the global (i.e., absolute) maximum of f on $\mathbb{R}$. [5 marks]

(e) Is f concave (i.e., concave down) on $(-\infty, -2)$? Explain why. [5 marks]

My work :

(a) $f(x) = \begin{cases} \dfrac{x}{x^2+1}, & \text{if } x \geq 0 \\ -\dfrac{x}{x^2+1}, & \text{if } x < 0 \end{cases}$

For $x > 0$, $f'(x) = \dfrac{(x^2+1) - x\,(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$. For $x < 0$, $f'(x) = \dfrac{x^2-1}{(x^2+1)^2}$.

$f'_+(0) = \lim\limits_{x \to 0^+} \dfrac{\dfrac{x}{x^2+1} - 0}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{1}{x^2+1} = 1$, $f'_-(0) = \lim\limits_{x \to 0^+} \dfrac{\dfrac{-x}{x^2+1} - 0}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{-1}{x^2+1} = -1$

$\therefore$ f is not differentiable at $x = 0$.

(b) $f'(x) = 0 \iff |f'(x)| = \dfrac{|1-x^2|}{(x^2+1)^2} = 0 \iff x = \pm 1$

$\therefore$ The critical points of f are: $x = 1$ and $x = -1$.

(c)

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
$f'(x)$	$+$	$-$	$+$	$-$

The open intervals where f is increasing are: $(-\infty, -1)$, $(0, 1)$

The open intervals where f is decreasing are: $(-1, 0)$, $(1, +\infty)$

(d)

x	$-\infty$	-1	0	1	$+\infty$
$f(x)$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0

The absolute maximum of f on $\mathbb{R}$ is $x = \pm 1$ with value $\frac{1}{2}$

The absolute minimum of f on $\mathbb{R}$ is $x = 0$ with value 0.

(e) For $x < 0$, $f''(x) = \dfrac{(x^2+1)^2(2x) - (x^2-1)\cdot 2(x^2+1)(2x)}{(x^2+1)^4} = \dfrac{2x(3-x^2)}{(x^2+1)^3}$

or $f''(x) = \dfrac{x^2-1}{(x^2+1)^2}\left(\dfrac{1}{x^2-1} - \dfrac{2}{x^2+1}\right)(2x) = \dfrac{2x(3-x^2)}{(x^2+1)^3}$

For $x \in (-\infty, -2)$, we have $x < 0$ and $x^2 > 4 > 3 \Rightarrow f''(x) = \dfrac{2x(3-x^2)}{(x^2+1)^3} > 0$

$\therefore$ f is concave up on $(-\infty, -2)$, in particular f is not concave down on $(-\infty, -2)$.

Question 2.

Find the following integrals (5 marks each).

(a) $\displaystyle\int_0^1 |\, x + a \,|\, dx$ (a is a parameter)

(b) $\displaystyle\int_0^1 \frac{x^2}{1 + x^6}\, dx$

(c) $\displaystyle\int \frac{2x\, dx}{(x^2 + 1)(x^4 + x^2 + 1)}$

(d) $\displaystyle\int_0^{+\infty} e^{-x} \cos x\, dx$

(e) $\displaystyle\int_0^1 \frac{x \arctan x}{(1 + x^2)^2}\, dx$

My work :

(a) Let $u = x + a$, $du = dx$. When $x = 0$, $u = a$; when $x = 1$, $u = a + 1$

$$\int_0^1 |\, x + a \,|\, dx = \int_a^{a+1} |\, u \,|\, du = \begin{cases} \displaystyle\int_a^{a+1} u\, du, & \text{if } a \ge 0 \\ \displaystyle\int_a^{a+1} (-u)\, du, & \text{if } a + 1 \le 0 \\ \displaystyle\int_a^{0} (-u)\, du + \int_0^{a+1} u\, du, & \text{if } a < 0 \text{ and } a + 1 > 0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}[\, (a+1)^2 - a^2 \,], & \text{if } a \ge 0 \\ \dfrac{1}{2}[\, a^2 - (a+1)^2 \,], & \text{if } a \le -1 \\ \dfrac{1}{2}[\, a^2 + (a+1)^2 \,], & \text{if } -1 \le a \le 0 \end{cases}$$

Alternative Solution

$$\int_0^1 |\, x + a \,|\, dx = \int_a^{a+1} |\, u \,|\, du = \int_a^{0} |\, u \,|\, du + \int_0^{a+1} |\, u \,|\, du$$

$$= \int_a^{0} \operatorname{sgn}(u)\, u\, du + \int_0^{a+1} \operatorname{sgn}(u)\, u\, du$$

$$= \int_a^{0} \operatorname{sgn}(a)\, u\, du + \int_0^{a+1} \operatorname{sgn}(a+1)\, u\, du$$

$$= \frac{1}{2}[\, \operatorname{sgn}(a+1)\, (a+1)^2 - \operatorname{sgn}(a)\, a^2 \,]$$

(b) $\displaystyle\int_0^1 \frac{x^2}{1+x^6}\,dx = \frac{1}{3}\int_0^1 \frac{d\,(x^3)}{1+(x^3)^2} = \left[\frac{1}{3}\tan^{-1}(x^3)\right]_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12}$

(c) $\displaystyle\int \frac{2x\,dx}{(x^2+1)(x^4+x^2+1)}$

$$= \int \frac{d\,(x^2)}{(x^2+1)(x^4+x^2+1)} = \int \left(\frac{1}{x^2+1} + \frac{-(x^2+\frac{1}{2})+\frac{1}{2}}{(x^2+\frac{1}{2})^2+\frac{3}{4}}\right) d\,(x^2)$$

$$= \ln(x^2+1) - \frac{1}{2}\ln(x^4+x^2+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}} + C$$

(d) Let $\displaystyle I = \int_0^{+\infty} e^{-x}\cos x\,dx = \lim_{b\to+\infty}\int_0^b e^{-x}\cos x\,dx$

$$\therefore\ I = \lim_{b\to+\infty}\int_0^b e^{-x}\,d\,(\sin x) = \lim_{b\to+\infty}\left[\,e^{-x}\sin x\,\right]_0^b + \int_0^b e^{-x}\sin x\,dx = \lim_{b\to+\infty}\int_0^b e^{-x}\sin x\,dx$$

$$= \lim_{b\to+\infty}\int_0^b e^{-x}\,d\,(-\cos x) = \lim_{b\to+\infty}\left[\,e^{-x}(-\cos x)\,\right]_0^b - \lim_{b\to+\infty}\int_0^b e^{-x}\cos x\,dx$$

$$= 1 - I$$

$$\Rightarrow\ \int_0^{+\infty} e^{-x}\cos x\,dx = I = \frac{1}{2}$$

or $\displaystyle I = \lim_{b\to+\infty}\left[-\frac{1}{2}e^{-x}\cos x + \frac{1}{2}e^{-x}\sin x\right]_0^b = \frac{1}{2}$

(e) Let $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $dx = \sec^2\theta\,d\theta$. When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{4}$.

$$\int_0^1 \frac{x\arctan x}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{(\tan\theta)\,\theta}{\sec^4\theta}(\sec^2\theta\,d\theta) = \int_0^{\pi/4} \theta\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \theta\sin 2\theta\,d\theta = \frac{1}{2}\left[-\frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{8}$$

Question 3.

(a) Let $F(x) = \int_x^{x^2} \sin(t^2)\, dt$. Find the degree-2 Taylor polynomial of F at $x_0 = 0$ (the remainder is not needed). [5 marks]

(b) Consider the improper integral

$$\int_1^{+\infty} \left(\frac{1}{\sqrt{x^2+1}} + \frac{p}{x} \right) dx,$$

where p is a parameter.

(i) Is this improper integral convergent when $p < -1$? If yes, find the value, otherwise, explain why. [5 marks]

(ii) Is this improper integral convergent when $p = -1$? If yes, find the value, otherwise, explain why. [5 marks]

My work :

(a) $F'(x) = 2x \sin x^4 - \sin x^2 \;\Rightarrow\; F''(x) = 2 \sin x^4 + 2x (\cos x^4)(4x^3) - (\cos x^2)(2x)$

$\therefore \quad F(0) = 0,\ F'(0) = 0$ and $F''(0) = 0$

The degree-2 Taylor polynomial of F at $x_0 = 0$ is 0.

(b) Let $x = \tan\theta$, $\sqrt{x^2+1} = \sec\theta$, $dx = \sec^2\theta\, d\theta$.

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|x + \sqrt{x^2+1}| + C$$

$$\int_1^b \left(\frac{1}{\sqrt{x^2+1}} + \frac{p}{x} \right) dx = \Big[\ln|x + \sqrt{x^2+1}| + p \ln|x| \Big]_1^b = \ln \frac{b + \sqrt{b^2+1}}{b^{-p}(1+\sqrt{2})}$$

(i) When $p < -1$,

$$\int_1^{+\infty} \left(\frac{1}{\sqrt{x^2+1}} + \frac{p}{x} \right) dx = \lim_{b \to +\infty} \ln \frac{b + \sqrt{b^2+1}}{b^{-p}(1+\sqrt{2})}$$

$$= \ln \left(\lim_{b \to +\infty} \frac{\dfrac{1}{b^{-p-1}} + \sqrt{\dfrac{1}{b^{-2p-2}} + \dfrac{1}{b^{-2p}}}}{1+\sqrt{2}} \right) = -\infty$$

which is divergent.

(ii) When $p = -1$,

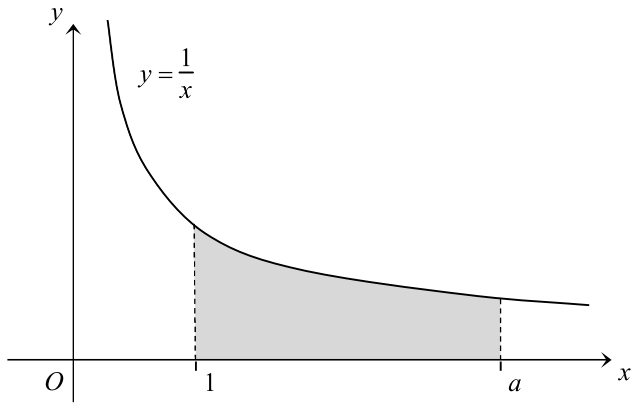
$$\int_1^{+\infty} \left(\frac{1}{\sqrt{x^2+1}} + \frac{p}{x} \right) dx = \lim_{b \to +\infty} \ln \frac{b + \sqrt{b^2+1}}{b(1+\sqrt{2})} = \ln \left(\lim_{b \to +\infty} \frac{1 + \sqrt{1 + 1/b^2}}{1+\sqrt{2}} \right)$$

$$= \ln \frac{2}{1+\sqrt{2}} = \ln(2(\sqrt{2}-1))$$

which is convergent.

Question 4.

Consider the graph of the function $y = \dfrac{1}{x}$ with $x \in [1, a]$, where a is a number greater than 1.



(a) On this graph, find the coordinate of the point that is the closest to the origin. [5 marks]

(b) Consider the area bounded by this graph with the straight lines $x = 1$, $x = a$, and $y = 0$. Find the volume of the solid obtained by rotating this area about the x-axis. [5 marks]

(c) Let $L(a)$ be the curve length of this graph, and $S(a)$ be the area of the surface obtained by rotating the graph about the x-axis. Does $\dfrac{S(a)}{L(a)}$ converge to a finite value when $a \to +\infty$? If yes, find the value; otherwise, explain why. [5 marks]

My work :

(a) The squared distance from the graph to the origin is given by

$$f(x) = x^2 + y^2 = x^2 + \frac{1}{x^2} \text{ with } x \in [1, a]$$

which is to be minimized.

$f'(x) = 2x - \dfrac{2}{x^3} = \dfrac{2(x^4 - 1)}{x^3} > 0$ for $x \in (1, a]$, i.e., f is strictly increasing on $(1, a]$.

$\therefore$ f attains its global minimum at $x = 1$.

The closest point on the graph to the origin is (1, 1).

(b) Volume of the solid $= \displaystyle\int_1^a \pi \left(\frac{1}{x}\right)^2 dx = \left[\frac{\pi x^{-1}}{-1}\right]_1^a = \pi\left(1 - \frac{1}{a}\right)$

(c) $\dfrac{dy}{dx} = -\dfrac{1}{x^2} \Rightarrow \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\dfrac{1}{x^4}}$

$L(a) = \int_1^a \sqrt{1+\dfrac{1}{x^4}}\,dx \geq \int_1^a dx \to +\infty$

and $S(a) = \int_1^a 2\pi\left(\dfrac{1}{x}\right)\sqrt{1+\dfrac{1}{x^4}}\,dx \geq 2\pi\int_1^a \dfrac{dx}{x} \to +\infty$ as $a \to +\infty$.

Thus, by l'Hôpital's rule and Fundamental Theorem of Calculus,

$$\lim_{a\to+\infty}\frac{S(a)}{L(a)} = \lim_{a\to+\infty}\frac{2\pi\left(\dfrac{1}{a}\right)\sqrt{1+\dfrac{1}{a^4}}}{\sqrt{1+\dfrac{1}{a^4}}} = \lim_{a\to+\infty}\frac{2\pi}{a} = 0$$

which is a finite value.

Question 5.

(a) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 0 \end{bmatrix}$$

Find $A^{-1}B$. Show the details of your calculation. [5 marks]

(b) Consider the system of linear equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & a \\ 1 & a & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix},$$

where a is a number. Determine the values of a such that the system is

(i) inconsistent; [5 marks]

(ii) consistent with infinitely many solutions and solve the system; [5 marks]

(iii) consistent with a unique solution and solve the system. [5 marks]

My work :

(a) *Method 1*:

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & -1 & 4 \\ 1 & -1 & 1 \end{pmatrix} \Rightarrow A^{-1}B = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & -1 & 4 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \\ 1 & -1 \end{pmatrix}$$

Method 2:

$$(A\mid B)=\left(\begin{array}{ccc|cc}1&0&2&1&0\\2&1&0&0&3\\1&1&1&2&0\end{array}\right)\to\left(\begin{array}{ccc|cc}1&0&2&1&0\\0&1&-4&-2&3\\0&1&-1&1&0\end{array}\right)\to\left(\begin{array}{ccc|cc}1&0&2&1&0\\0&1&-4&-2&3\\0&0&3&3&-3\end{array}\right)$$

$$\to\left(\begin{array}{ccc|cc}1&0&2&1&0\\0&1&-4&-2&3\\0&0&1&1&-1\end{array}\right)\to\left(\begin{array}{ccc|cc}1&0&0&-1&2\\0&1&0&2&-1\\0&0&1&1&-1\end{array}\right)\quad\therefore\ A^{-1}B=\begin{pmatrix}-1&2\\2&-1\\1&-1\end{pmatrix}$$

(b) *Method 1*:

$$\left(\begin{array}{ccc|c}1&-1&1&4\\1&1&a&1\\1&a&2&2\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1&1&4\\0&2&a-1&-3\\0&a+1&1&-2\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1&1&4\\0&2&a-1&-3\\0&0&3-a^2&3a-1\end{array}\right)$$

(i) The system is inconsistent $\Leftrightarrow$ $3-a^2=0$ and $3a-1\neq 0$ $\Leftrightarrow$ $a=\pm\sqrt{3}$.

(ii) The system is consistent with infinitely many solutions
$\Leftrightarrow$ $3-a^2=0$ and $3a-1=0$, which is impossible.

(iii) The system is consistent with unique solution $\Leftrightarrow$ $3-a^2\neq 0$ $\Leftrightarrow$ $a\neq\pm\sqrt{3}$.

$$\left(\begin{array}{ccc|c}1&-1&1&4\\0&2&a-1&-3\\0&0&3-a^2&3a-1\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1&1&4\\0&2&a-1&-3\\0&0&1&\frac{-3a+1}{a^2-3}\end{array}\right)$$

$$\to\left(\begin{array}{ccc|c}1&-1&0&\frac{4a^2+3a-13}{a^2-3}\\0&2&0&\frac{-4a+10}{a^2-3}\\0&0&1&\frac{-3a+1}{a^2-3}\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1&0&\frac{4a^2+3a-13}{a^2-3}\\0&1&0&\frac{-2a+5}{a^2-3}\\0&0&1&\frac{-3a+1}{a^2-3}\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&\frac{4a^2+a-8}{a^2-3}\\0&1&0&\frac{-2a+5}{a^2-3}\\0&0&1&\frac{-3a+1}{a^2-3}\end{array}\right)$$

$$\therefore\ x=\frac{4a^2+a-8}{a^2-3},\ y=\frac{-2a+5}{a^2-3},\ z=\frac{-3a+1}{a^2-3}$$

Method 2 :

(iii) The system is consistent with unique solution

$$\Leftrightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & a \\ 1 & a & 2 \end{vmatrix} = 3 - a^2 \neq 0 \quad \Leftrightarrow \quad a \neq \pm\sqrt{3}$$

By Cramer's rule, we have

$$x = \frac{\begin{vmatrix} 4 & -1 & 1 \\ 1 & 1 & a \\ 2 & a & 2 \end{vmatrix}}{3 - a^2} = \frac{8 - a - 4a^2}{3 - a^2}, \quad y = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & a \\ 1 & 2 & 2 \end{vmatrix}}{3 - a^2} = \frac{2a - 5}{3 - a^2}, \quad z = \frac{\begin{vmatrix} 1 & -1 & 4 \\ 1 & 1 & 1 \\ 1 & a & 2 \end{vmatrix}}{3 - a^2} = \frac{3a - 1}{3 - a^2}$$

When $a = \pm\sqrt{3}$, the system becomes

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & 1 & \pm\sqrt{3} & 1 \\ 1 & \pm\sqrt{3} & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 2 & \pm\sqrt{3} - 1 & -3 \\ 0 & \pm\sqrt{3} + 1 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 2 & \pm\sqrt{3} - 1 & -3 \\ 0 & 0 & 0 & \pm 3\sqrt{3} - 1 \end{array}\right)$$

(i) The system is inconsistent $\Leftrightarrow$ $a = \pm\sqrt{3}$.

(ii) There is no values of a such that the system has infinitely many solutions