The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 2 2022/23 Semester 2

Question 1.

[10 marks each] Calculate the following integrals.

(a) $\int \cos(7 - 4x)\, dx$

(b) $\int \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx$

(c) $\int \sqrt{x} \ln(2x)\, dx$

(d) $\int \csc^6 x \cot^4 x\, dx$

My work :

(a) $\int \cos(7 - 4x)\, dx = -\frac{1}{4}\int \cos(7 - 4x)\, d\,(7 - 4x) = -\frac{1}{4}\sin(7 - 4x) + C$

(b) $\int \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx = -2\int e^{-\sqrt{x}}\, d\,(-\sqrt{x}) = -2e^{-\sqrt{x}} + C$

(c) Let $I = \int \sqrt{x} \ln(2x)\, dx$. Let $x = \frac{1}{2}e^{2u}$, $dx = e^{2u}\, du$, $u = \frac{1}{2}\ln(2x)$,

$$I = \int \frac{1}{\sqrt{2}} e^{u}\,(2u) \cdot e^{2u}\, du = \int \sqrt{2}u\, e^{3u}\, du = \frac{\sqrt{2}}{3} u\, e^{3u} - \frac{\sqrt{2}}{9} e^{3u} + C$$

$$= \frac{\sqrt{2}}{3} \cdot \frac{1}{2}\ln(2x) \cdot 2\sqrt{2}\, x^{3/2} - \frac{\sqrt{2}}{9} \cdot 2\sqrt{2}\, x^{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\ln(2x) - \frac{4}{9}x^{3/2} + C$$

(d) $\int \csc^6 x \cot^4 x\, dx = -\int \csc^4 x \cot^4 x\, d\,(\cot x) = -\int (\cot^8 x + 2\cot^6 x + \cot^4 x)\, d\,(\cot x)$

$$= -\frac{\cot^9 x}{9} - \frac{2\cot^7 x}{7} - \frac{\cot^5 x}{5} + C$$

(e) $\displaystyle\int_0^1 \frac{1}{e^x\,(1+e^x)}\,dx$

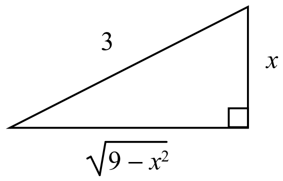
(f) $\displaystyle\int \frac{1}{x^4\sqrt{9-x^2}}\,dx$

My work :

(e) $$\int_0^1 \frac{1}{e^x\,(1+e^x)}\,dx = \int_0^1 \frac{e^{-2x}}{e^{-x}+1}\,dx = -\int_0^1 \frac{e^{-x}\,d\,(e^{-x})}{e^{-x}+1} = \int_0^1 \left(\frac{1}{e^{-x}+1} - 1\right) d\,(e^{-x})$$

$$= \Big[\ \ln\,(e^{-x}+1) - e^{-x}\,\Big]_0^1 = \ln\left(\frac{1}{e}+1\right) - \frac{1}{e} - \ln 2 + 1 = \ln\frac{1+e}{2e} - \frac{1}{e} + 1$$

(f) Let $I = \displaystyle\int \frac{1}{x^4\sqrt{9-x^2}}\,dx$.



Let $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta,\ \sqrt{9-x^2} = 3\cos\theta$.

$$I = \int \frac{3\cos\theta\,d\theta}{(3\sin\theta)^4\,(3\cos\theta)} = \frac{1}{81}\int \csc^4\theta\,d\theta = -\frac{1}{81}\int (\cot^2\theta + 1)\,d\,(\cot\theta)$$

$$= -\frac{1}{81}\left(\frac{\cot^3\theta}{3} + \cot\theta\right) + C = -\frac{(9-x^2)^{3/2}}{243x^3} - \frac{\sqrt{9-x^2}}{81x} + C$$

Question 2.

Let $f(x) = e^{-x^2}\sin x$ and $F(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) [10 marks] Calculate $F''(x)$

(b) [10 marks] Find the local maximizers of F in the interval (0, 10).

Note: You need to find value(s) of x where F attains local / relative maximum; no need to find the corresponding value(s) of F for this particular question.

My work :

(a) $F''(x) = f'(x) = e^{-x^2}\cos x - 2xe^{-x^2}\sin x$

(b) $F'(x) = f(x) = e^{-x^2}\sin x = 0 \iff x = n\pi$ for $n \in \mathbb{Z}$

Since $x \in (0, 10)$, $x = \pi, 2\pi, 3\pi$.

x	π	2π	3π
$F''(x)$	−	+	−

The local maximizers of F in (0, 10) are: $x = \pi$ and $x = 3\pi$.

Question 3.

(a) [10 marks] Let $f(x) = (1-x)^{1/3}$. Find the degree 2 Taylor polynomial of f at at $x_0 = 0$.

(b) [10 marks] By the Taylor polynomial in 3 (a), estimate $(\frac{1}{9})^{1/3}$. Formulate your answer as a fraction, and prove that the error is at most 5×10^{-4}.

My work :

(a) $f(x) = (1-x)^{1/3},\ f'(x) = -\dfrac{1}{3}(1-x)^{-2/3},\ f''(x) = -\dfrac{2}{9}(1-x)^{-5/3}$

$f(0) = 1,\ \ f'(0) = -\dfrac{1}{3},\ \ f''(0) = -\dfrac{2}{9}$

The Taylor polynomial of degree 2 for $f(x)$ about $x_0 = 0$ is given by

$T_2(x) = 1 - \dfrac{1}{3}x - \dfrac{1}{9}x^2$

(b) By (a), $(\frac{8}{9})^{1/3} \approx T_2(\frac{1}{9}) = 1 - \dfrac{1}{3}\cdot\dfrac{1}{9} - \dfrac{1}{9}\left(\dfrac{1}{9}\right)^2 = \dfrac{701}{729}$, $\quad\therefore\ (\frac{1}{9})^{1/3} = \frac{1}{2}(\frac{8}{9})^{1/3} \approx \dfrac{701}{1458}$

$f^{(3)}(x) = -\dfrac{10}{27}(1-x)^{-8/3}$

The remainder term is given by $R_2(x) := \dfrac{f^{(3)}(\xi)}{3!}x^3 = -\dfrac{5}{81}(1-\xi)^{-8/3}x^3$

for some $\xi \in (0, x)$ if $x > 0$.

$$\left|(\tfrac{1}{9})^{1/3} - \tfrac{1}{2}T_2(\tfrac{1}{9})\right| = \tfrac{1}{2}\left|(\tfrac{8}{9})^{1/3} - T_2(\tfrac{1}{9})\right| = \tfrac{1}{2}\,|R_2(\tfrac{1}{9})| \le \frac{1}{2}\cdot\frac{5}{81}\left(1-\frac{1}{9}\right)^{-8/3}\left(\frac{1}{9}\right)^3$$

$$= \frac{5}{81 \times 2 \times 2^8 \times 9^{3-8/3}} \le \frac{5}{81 \times 1024} < 5 \times 10^{-4}$$