The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 2 2023/24 Semester 2

Question 1.

(10 marks each) Calculate the following indefinite integrals.

(a) $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\, dx$

(b) $\displaystyle\int \frac{2x+1}{x^3-1}\, dx$

(c) $\displaystyle\int (x^2+3) \sin(2x)\, dx$

(d) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-4}}\, dx$

My work :

(a) $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\, dx = \int \tan^3 x \sec x\, dx = \int (\sec^2 x - 1)\, d(\sec x) = \frac{\sec^3 x}{3} - \sec x + C$

(b) $\displaystyle\int \frac{2x+1}{x^3-1}\, dx = \int \frac{(2x+1)\, dx}{(x-1)(x^2+x+1)} = \int \left(\frac{1}{x-1} + \frac{-(x+\frac{1}{2})+\frac{1}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}} \right) dx$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

(c) $\displaystyle\int (x^2+3)\sin(2x)\, dx = \cdots = -\frac{1}{2}(x^2+3)\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

$$= \frac{x}{2}\sin 2x - \frac{2x^2+5}{4}\cos 2x + C$$

(d) Let $\displaystyle I = \int \frac{1}{x^3\sqrt{x^2-4}}\, dx$. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-4} = 2\tan\theta$

$$\therefore\ I = \int \frac{2\sec\theta\tan\theta\, d\theta}{(2\sec\theta)^3(2\tan\theta)} = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C = \frac{1}{16}\operatorname{sgn}(x)\cos^{-1}\frac{2}{x} + \frac{\sqrt{x^2-4}}{8x^2} + C$$

Remark: $\displaystyle I = \frac{1}{16}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + \frac{\sqrt{x^2-4}}{8x^2} + C$ is also acceptable, but not $\displaystyle\frac{1}{16}\operatorname{sgn}(x)\sin^{-1}\frac{\sqrt{x^2-4}}{x} + \frac{\sqrt{x^2-4}}{8x^2} + C$ nor $\displaystyle\frac{1}{16}\sin^{-1}\frac{\sqrt{x^2-4}}{x} + \frac{\sqrt{x^2-4}}{8x^2} + C$

Question 2.

(10 marks each) Calculate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^7}{\sqrt{x^4+1}}\,dx$

(b) $\displaystyle\int_0^1 x^2 \sin^{-1}(x)\,dx$

(c) $\displaystyle\int_1^6 \frac{1}{x+2\sqrt{x+3}}\,dx$

My work :

(a) Let $I = \displaystyle\int_0^1 \frac{x^7}{\sqrt{x^4+1}}\,dx$. Let $x^2 = \tan\theta$, $2x\,dx = \sec^2\theta\,d\theta$, $\sqrt{x^4+1} = \sec\theta$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/4$.

$$\therefore\ I = \frac{1}{2}\int_0^{\pi/4} \frac{(\tan\theta)^3}{\sec\theta}\cdot\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} (\sec^2\theta - 1)\,d\,(\sec\theta)$$

$$= \frac{1}{2}\left[\frac{\sec^3 x}{3} - \sec x\right]_0^{\pi/4} = \frac{2-\sqrt{2}}{6}\ \boxed{\text{or } \frac{1}{3\,(2+\sqrt{2})}}$$

(b) Let $I = \displaystyle\int_0^1 x^2 \sin^{-1}(x)\,dx$. Let $x = \sin u$, $dx = d\,(\sin u)$.

When $x = 0$, $u = 0$; when $x = 1$, $u = \pi/2$.

$$\therefore\ I = \int_0^{\pi/2} (\sin u)^2\,u\,d\,(\sin u) = \int_0^{\pi/2} u\,d\left(\frac{\sin^3 u}{3}\right)$$

$$= u\left(\frac{\sin^3 u}{3}\right)\Big|_0^{\pi/2} - \frac{1}{3}\int_0^{\pi/2} \sin^3 u\,du = \frac{\pi}{6} + \frac{1}{3}\int_0^{\pi/2} (1-\cos^2 u)\,d\,(\cos u)$$

$$= \frac{\pi}{6} + \frac{1}{3}\left[\cos u - \frac{\cos^3 u}{3}\right]_0^{\pi/2} = \frac{\pi}{6} - \frac{2}{9}$$

(c) Let $I = \displaystyle\int_1^6 \frac{1}{x+2\sqrt{x+3}}\,dx$. Let $x+3 = u^2$, $dx = 2u\,du$, $x + 2\sqrt{x+3} = u^2 + 2u - 3$.

When $x = 1$, $u = 2$; when $x = 6$, $u = 3$.

$$\therefore\ I = \int_2^3 \frac{2u\,du}{u^2+2u-3} = \int_2^3 \frac{2u\,du}{(u+3)\,(u-1)} = \int_2^3 \left(\frac{\frac{3}{2}}{u+3} + \frac{\frac{1}{2}}{u-1}\right)du$$

$$= \left[\frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1|\right]_2^3 = \frac{3}{2}\ln\frac{6}{5} + \frac{1}{2}\ln 2$$

Question 3.

(10 marks) Let $F(x) = \int_{x^2}^{x^2+1} e^{-t^2}\, dt.$

Compute $F'(x)$ and find all local extrema of $F(x)$.

My work :

$F'(x) = e^{-(x^2+1)^2}(2x) - e^{-(x^2)^2}(2x) = 2x\, e^{-x^4}(e^{-2x^2-1} - 1) = 0 \;\Rightarrow\; x = 0$

x	$(-\infty, 0)$	0	$(0, +\infty)$
$F(x)$	↗	max	↘
$F'(x)$	$+$	0	$-$

$\therefore$ F attains a local maximum at $x = 0$.

Question 4.

Let $f(x) = x \ln x$

(a) (10 marks) Find the degree 2 Taylor polynomial of $f(x)$ at $x_0 = 1$.

(b) (10 marks) Use the Taylor polynomial in part (a) to estimate the value of $f(1.1)$ and show that the error of this estimation is at most $\frac{1}{6000}$.

My work :

(a) $f'(x) = \ln x + 1,\; f''(x) = \dfrac{1}{x} \;\Rightarrow\; f(1) = 0,\; f'(1) = 1,\; f''(1) = 1$

The degree 2 Taylor polynomial of $f(x)$ at $x_0 = 1$ is given by

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) + \frac{1}{2}(x-1)^2$$

(b) $f(1.1) \approx T_2(1.1) = 0.1 + \dfrac{1}{2}(0.1)^2 = 0.105,\; f'''(x) = -\dfrac{1}{x^2}$

The remainder term is $R_2(x) = \dfrac{f'''(\xi)}{3!}(x-1)^3 = -\dfrac{(x-1)^3}{6\xi^2}$ for some ξ between 1 and x.

$$|f(1.1) - T_2(1.1)| = |R_2(1.1)| = \frac{(0.1)^3}{6\xi^2} \text{ for some } \xi \in (1, 1.1)$$

$$\le \frac{(0.1)^3}{6(1)^2} = \frac{1}{6000}$$