The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Midterm Test 2021/22 Semester 3

Question 1.

Let $f(x) = \frac{1}{5}x^5 - x^4$.

(a) Find the open intervals on which f is increasing and the open intervals on which f is decreasing. Identify all the relative maxima and minima of f. [10 marks]

(b) Find the open intervals on which f is concave upwards and the open intervals on which f is concave downwards. Identify all the inflection points of f. [10 marks]

(c) Find the absolute maximum and absolute minimum of f on the closed interval $[-2, 2]$. [5 marks]

(d) Find all the intercepts and asymptotes of f (if any), and hence sketch the graph of f. [5 marks]

My work :

(a) $f'(x) = x^4 - 4x^3 = x^3(x-4) = 0 \iff x = 0, 4$

x	$(-\infty, 0)$	$(0, 4)$	$(4, +\infty)$
$f'(x)$	+	−	+

The open intervals where f is increasing are: $(-\infty, 0)$, $(4, +\infty)$

The open interval where f is decreasing is: $(0, 4)$

The relative maximum of f is $x = 0$ with value 0.

The relative minimum of f is $x = 4$ with value $-\frac{256}{5}$.

(b) $f''(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \iff x = 0, 3$

x	$(-\infty, 0)$	$(0, 3)$	$(3, +\infty)$
$f'(x)$	−	−	+

The open interval where f is concave upwards are: $(-\infty, 3)$

The open interval where f is concave downwards is: $(3, +\infty)$

Inflection points: $(0, 0)$, $(3, -\frac{162}{5})$. Change of concavity occurs at $(3, -\frac{162}{5})$.

(c)

x	-2	0	2
$f(x)$	$-\dfrac{112}{5}$	0	$-\dfrac{48}{5}$

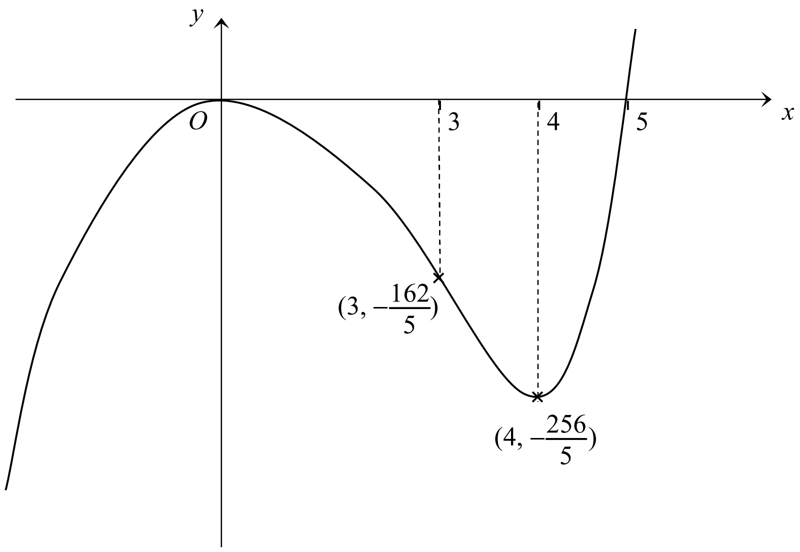
The absolute maximum of f on $[-2, 2]$ is $x = 0$ with value 0

The absolute minimum of f on $[-2, 2]$ is $x = -2$ with value $-\dfrac{112}{5}$.

(d) $f(x) = \dfrac{1}{5}x^5 - x^4 = 0 \iff x = 0, 5.$ $\therefore$ x-intercepts: $x = 0$ and $x = 5$, y-intercept: $y = 0$

$\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = +\infty$, $\therefore$ There are no horizontal and inclined asymptotes.

Since f is continuous everywhere, there is no vertical asymptotes.



Question 2.

If $g(x) = \displaystyle\int_{\sqrt{\sin x + 1}}^{10} \frac{(1 + t^2)^{1120}}{t^4}\, dt$, find $g'(0)$. [8 marks]

My work :

$g'(x) = -\dfrac{(1 + \sin x + 1)^{1120}}{(\sin x + 1)^2} \dfrac{1}{2} (\sin x + 1)^{-1/2} \cos x.$ $\therefore$ $g'(0) = -\dfrac{2^{1120}}{(1)^2} \dfrac{1}{2} (1)(1) = -2^{1119}$

Question 3.

Evaluate the following integrals (show steps).

(a) $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$ [10 marks]

(b) $\int \frac{2x}{x^2 + 2x + 10}\, dx$ [8 marks]

(c) $\int \sin^{-1}\left(\frac{x}{7}\right) dx$ [8 marks]

(d) $\int \tan x\, dx$ [8 marks]

(e) $\int_0^{\sqrt{27}} \frac{2}{x + \sqrt[3]{x}}\, dx$ [8 marks]

(f) $\int \frac{1 + x\cos x}{x + e^{-\sin x}}\, dx$ [5 marks]

My work :

(a) $\int_1^2 \frac{e^{1/x}}{x^2}\, dx = -\int_1^2 e^{1/x}\, d\left(\frac{1}{x}\right) = -e^{1/x}\Big|_1^2 = e - e^{1/2}$

(b) $\int \frac{2x}{x^2 + 2x + 10}\, dx = \int \frac{2(x+1) - 2}{(x+1)^2 + 3^2}\, dx = \ln(x^2 + 2x + 10) - \frac{2}{3}\tan^{-1}\frac{x+1}{3} + C$

(c) Let $u = \sin^{-1}\left(\frac{x}{7}\right) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $7\sin u = x \Rightarrow 7\cos u\, du = dx$

$$\int \sin^{-1}\left(\frac{x}{7}\right) dx = \int u\,(7\cos u)\, du = 7u\sin u + 7\cos u + C = x\sin^{-1}\left(\frac{x}{7}\right) + \sqrt{49 - x^2} + C$$

(d) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + C = \ln|\sec x| + C$

(e) Let $x = u^3$, $dx = 3u^2\, du$. When $x = 0$, $u = 0$; when $x = \sqrt{27}$, $u = \sqrt{3}$.

$$\int_0^{\sqrt{27}} \frac{2}{x + \sqrt[3]{x}}\, dx = \int_0^{\sqrt{3}} \frac{6u^2\, du}{u^3 + u} = \int_0^{\sqrt{3}} \frac{6u\, du}{u^2 + 1} = 3\ln(u^2 + 1)\Big|_0^{\sqrt{3}} = 3\ln 4$$

(f) Note that $d\,(x + e^{-\sin x}) = (x - e^{-\sin x}\cos x)\,dx$

$$\int \frac{1 + x\cos x}{x + e^{-\sin x}}\,dx = \int \frac{1 - e^{-\sin x}\cos x + e^{-\sin x}\cos x + x\cos x}{x + e^{-\sin x}}\,dx$$

$$= \int \frac{d\,(x + e^{-\sin x})}{x + e^{-\sin x}} + \int \frac{(x + e^{-\sin x})\cos x}{x + e^{-\sin x}}\,dx$$

$$= \ln\,(x + e^{-\sin x}) + \int \cos x\,dx \;= \ln\,(x + e^{-\sin x}) + \sin x + C$$

Question 4.

(a) Prove the reduction formula

$$\int \frac{(\ln x)^m}{x^n}\,dx \;= -\frac{(\ln x)^m}{(n-1)\,x^{n-1}} + \frac{m}{n-1}\int \frac{(\ln x)^{m-1}}{x^n}\,dx,$$

where $m \ge 1$ and $n \ge 2$. [8 marks]

(b) Use (a) to compute $\int_1^e \frac{(\ln x)^3}{x^2}\,dx$. [7 marks]

My work :

(a) $$\int \frac{(\ln x)^m}{x^n}\,dx = \int (\ln x)^m\, d\left(\frac{x^{-n+1}}{-n+1}\right) = \frac{x^{-n+1}\,(\ln x)^m}{-n+1} - \int \frac{x^{-n+1}}{-n+1}\,d\,[\,(\ln x)^m\,]$$

$$= -\frac{(\ln x)^m}{(n-1)\,x^{n-1}} + \frac{m}{n-1}\int \frac{(\ln x)^{m-1}}{x^n}\,dx$$

(b) Let $I_{m,\,n} = \int_1^e \frac{(\ln x)^m}{x^n}\,dx$. Then from (a), we have

$$I_{m,\,n} = -\frac{(\ln x)^m}{(n-1)\,x^{n-1}}\Bigg|_1^e + \frac{m}{n-1}\int_1^e \frac{(\ln x)^{m-1}}{x^n}\,dx = -\frac{1}{(n-1)\,e^{n-1}} + \frac{m}{n-1}\,I_{m-1,\,n}$$

$$\therefore\ \int_1^e \frac{(\ln x)^3}{x^2}\,dx \;=\; I_{3,\,2} \;= -\frac{1}{e} + 3\,I_{2,\,2} \;= -\frac{1}{e} + 3\left(-\frac{1}{e} + 2\,I_{1,\,2}\right) = -\frac{4}{e} + 6\,I_{1,\,2}$$

$$= -\frac{4}{e} + 6\left(-\frac{1}{e} + I_{0,\,2}\right) = -\frac{10}{e} + 6\int_1^e x^{-2}\,dx \;= -\frac{10}{e} + 6\left(-\frac{1}{e} + 1\right) = 6 - \frac{16}{e}$$