The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Final Exam 2022/23 Semester 2

Question 1.

(a) Consider the function $f(x) = \dfrac{|x| + 1}{x^2 + 1}, x \in \mathbb{R}$.

(i) Find all intervals on which the function is increasing or decreasing. [5 marks]

(ii) Is f convex (i.e., concave up) on $(1, +\infty)$? Explain why. [5 marks]

(b) Let $F(x) = \displaystyle\int_{\ln x}^{\ln(x^2)} \sin(e^t)\, dt$, where $x > 0$.

(i) Find $F'(x)$. [5 marks]

(ii) Find $F''(x)$. [5 marks]

(iii) Find the degree-2 Taylor polynomial of F at $x_0 = 1$ (the remainder is not needed).

[5 marks]

My work :

(a) (i) $f(x) = \dfrac{\operatorname{sgn}(x)\, x + 1}{x^2 + 1}$

$$\Rightarrow f'(x) = \frac{(x^2 + 1)\operatorname{sgn}(x) - (|x| + 1)\, 2x}{(x^2 + 1)^2} = -\operatorname{sgn}(x)\frac{|x|^2 + 2|x| - 1}{(x^2 + 1)^2}.$$

$$f'_+(0) = \lim_{x \to 0^+} \frac{\frac{x+1}{x^2+1} - 1}{x - 0} = \lim_{x \to 0^+} 1 - x = 1, \quad f'_-(0) = \lim_{x \to 0^+} \frac{\frac{-x+1}{x^2+1} - 1}{x - 0} = \lim_{x \to 0^+} -1 - x = -1$$

$\therefore$ f is not differentiable at $x = 0$.

$$f'(x) = -\operatorname{sgn}(x)\frac{|x|^2 + 2|x| - 1}{(x^2 + 1)^2} = 0 \Leftrightarrow |x|^2 + 2|x| - 1 = 0 \Leftrightarrow |x| = \sqrt{2} - 1$$

$\therefore$ The critical points of f are: $x = \sqrt{2} - 1$ and $x = 1 - \sqrt{2}$.

x	$(-\infty, 1 - \sqrt{2})$	$(1 - \sqrt{2}, 0)$	$(0, \sqrt{2} - 1)$	$(\sqrt{2} - 1, +\infty)$
$f'(x)$	$+$	$-$	$+$	$-$

The open intervals where f is increasing are: $(-\infty, 1 - \sqrt{2})$, $(0, \sqrt{2} - 1)$

The open intervals where f is decreasing are: $(1 - \sqrt{2}, 0)$, $(\sqrt{2} - 1, +\infty)$

(ii) $$f''(x) = \frac{-(x^2+1)^2(2|x|+2) + \operatorname{sgn}(x)(|x|^2+2|x|-1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{-2(|x|^2+1)(|x|+1) + 4|x|(|x|^2+2|x|-1)}{(x^2+1)^3}$$

$$= \frac{2|x|^3+6x^2-6|x|-2}{(x^2+1)^3} = \frac{2(|x|-1)(x^2+4|x|+1)}{(x^2+1)^3} > 0$$

for all $x \in (1, +\infty)$. $\therefore$ f is convex on $(1, +\infty)$.

(b) (i) $$F(x) = \int_{\ln x}^{\ln (x^2)} \sin(e^t)\, dt \quad \Rightarrow \quad F'(x) = (\sin x^2)\frac{2}{x} - (\sin x)\frac{1}{x} = \frac{2\sin x^2 - \sin x}{x}$$

(ii) $$F''(x) = \frac{x(2(\cos x^2)(2x) - \cos x) - (2\sin x^2 - \sin x)}{x^2}$$

$$= \frac{4x^2\cos x^2 - x\cos x - 2\sin x^2 + \sin x}{x^2}$$

(iii) $F(1) = \int_0^0 \sin(e^t)\, dt = 0$, $F'(1) = \sin 1$, $F''(1) = 3\cos 1 - \sin 1$

The degree-2 Taylor polynomial of F around $x = 1$ is given by

$$T_2(x) = (\sin 1)(x-1) + \frac{1}{2}(3\cos 1 - \sin 1)(x-1)^2$$

Question 2.

Evaluate the following integrals:

(a) $\displaystyle\int \frac{dx}{x^{\frac{1}{3}} + x^{\frac{1}{5}}}$ [5 marks]

(b) $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2 + 4x + 3}}\, dx$ [5 marks]

(c) $\displaystyle\int_0^1 \ln(1 + x^2)\, dx$ [5 marks]

(d) $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^3 x\, dx$ [5 marks]

(e) $\displaystyle\int \frac{x^3 - x^2 - 3x + 2}{x(x^2 - 2x)(x^2 + 1)}\, dx$ [5 marks]

My work :

(a) Let $x = u^{15}$, $dx = 15u^{14}\, du$.

$$\int \frac{dx}{x^{\frac{1}{3}} + x^{\frac{1}{5}}} = \int \frac{15u^{14}\, du}{u^5 + u^3} = 15\int \frac{u^{11}\, du}{u^2 + 1} = 15\int \left(u^9 - u^7 + u^5 - u^3 + u - \frac{u}{u^2 + 1}\right) du$$

$$= 15\left(\frac{u^{10}}{10} - \frac{u^8}{8} + \frac{u^6}{6} - \frac{u^4}{4} + \frac{u^2}{2} - \frac{1}{2}\ln(u^2 + 1)\right) + C$$

$$= \frac{3}{2}x^{\frac{2}{3}} - \frac{15}{8}x^{\frac{8}{15}} + \frac{5}{2}x^{\frac{2}{5}} - \frac{15}{4}x^{\frac{4}{15}} + \frac{15}{2}x^{\frac{2}{15}} - \frac{15}{2}\ln(x^{\frac{2}{15}} + 1) + C$$

(b) Let $\displaystyle I = \int_0^1 \frac{x}{\sqrt{x^2 + 4x + 3}}\, dx = \int_0^1 \frac{x}{\sqrt{(x + 2)^2 - 1}}\, dx.$

Let $x + 2 = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2 + 4x + 3} = \tan\theta.$

When $x = 0$, $\theta = \dfrac{\pi}{3}$; when $x = 1$, $\theta = \cos^{-1}\dfrac{1}{3}$

$$I = \int_{\pi/3}^{\cos^{-1}(1/3)} \frac{\sec\theta - 2}{\tan\theta}(\sec\theta\tan\theta\, d\theta) = \int_{\pi/3}^{\cos^{-1}(1/3)} (\sec^2\theta - 2\sec\theta)\, d\theta$$

$$= \Big[\tan\theta - 2\ln|\sec\theta + \tan\theta|\Big]_{\pi/3}^{\cos^{-1}(1/3)} = \sqrt{8} - \sqrt{3} - 2\ln\frac{3 + \sqrt{8}}{2 + \sqrt{3}}$$

(c) $\displaystyle\int_0^1 \ln(1+x^2)\,dx = x\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2\,dx}{1+x^2} = \ln 2 - \int_0^1 \left(2 - \frac{2}{1+x^2}\right) dx$

$$= \ln 2 - \left[2x - 2\tan^{-1} x\right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

(d) $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^3 x\,dx = \int_0^{\pi/4} \tan^2 x \sec^2 x\,d(\sec x) = \int_0^{\pi/4} (\sec^4 x - \sec^2 x)\,d(\sec x)$

$$= \left[\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}\right]_0^{\pi/4} = \frac{2}{15}(\sqrt{2}+1)$$

(e) $\displaystyle\int \frac{x^3 - x^2 - 3x + 2}{x(x^2-2x)(x^2+1)}\,dx = \int \frac{x^3 - x^2 - 3x + 2}{x^2(x-2)(x^2+1)}\,dx = \cdots = \int \left(\frac{1}{x} + \frac{-1}{x^2} + \frac{0}{x-2} + \frac{-x+2}{x^2+1}\right)$

$$= \ln|x| + \frac{1}{x} - \frac{1}{2}\ln(x^2+1) + 2\tan^{-1} x + C$$

Question 3.

(a) Find the arc length of the curve defined by $4x^4 + 3 - 12xy = 0$ from $x = 2$ to $x = 4$.

[5 marks]

(b) Consider the region bounded by the curve $y = x^2$ $(x \geq 0)$, the x-axis, and the line $x = 2$. Find the volume of the solid bounded by revolving this region about the y-axis.

[5 marks]

(c) Prove that

$$\frac{1}{\sqrt{1121\pi}} \leq \int_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \sin(t^2)\, dt \leq \frac{1}{\sqrt{1120\pi}}.$$

[10 marks]

My work :

(a) $4x^4 + 3 - 12xy = 0 \Rightarrow y = \frac{1}{3}x^3 + \frac{1}{4x} \Rightarrow y' = x^2 - \frac{1}{4x^2}$

$$1 + (y')^2 = 1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = \left(x^2 + \frac{1}{4x^2}\right)^2$$

$$\text{Arc-length} = \int_2^4 \sqrt{1 + (y')^2}\, dx = \int_2^4 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_2^4 = \frac{899}{48}$$

(b) $\text{Volume} = 2\pi \int_0^2 xy\, dx = 2\pi \int_0^2 x^3\, dx = 2\pi \cdot \frac{2^4}{4} = 8\pi$

(c)
$$\int_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \sin(t^2)\, dt = \int_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \frac{\sin(t^2)}{2t}\, d(t^2)$$

$$= \int_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \sin(t^2)\, d(t^2) \cdot \frac{1}{2\xi} \quad \text{for some } \xi \in [\sqrt{1120\pi}, \sqrt{1121\pi}]$$

by Mean Value Theorem for Integral

$$= [-\cos t^2]_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \cdot \frac{1}{2\xi} = \frac{1}{\xi}$$

Thus, we have $\frac{1}{\sqrt{1121\pi}} \leq \int_{\sqrt{1120\pi}}^{\sqrt{1121\pi}} \sin(t^2)\, dt = \frac{1}{\xi} \leq \frac{1}{\sqrt{1120\pi}}$

Question 4.

(a) Define $X = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & 5 & 6 \end{bmatrix}$, $Y = \begin{bmatrix} -8 & 17 & -7 \\ 4 & -7 & 3 \\ -2 & 3 & -1 \end{bmatrix}$.

(i) Find X^{-1}. [5 marks]

(ii) Find $X^{1120}Y^{1120}$. [10 marks]

(b) Consider the system of linear equations

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & a \\ -4 & -\frac{1}{2}a & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix},$$

where a is a number. Determine the values of a such that the system is

(i) inconsistent; [5 marks]

(ii) consistent with infinitely many solutions and solve the system; [5 marks]

(iii) consistent with a unique solution and solve the system. [5 marks]

My work :

(a) (i) $X^{-1} = \frac{1}{-2}\begin{bmatrix} 8 & -17 & 7 \\ -4 & 7 & -3 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & \frac{17}{2} & -\frac{7}{2} \\ 2 & -\frac{7}{2} & \frac{3}{2} \\ -1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

(ii) By (i), we see that $XY = YX = 2I$. Thus $X^{1120}Y^{1120} = (XY)^{1120} = 2^{1120}I$.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 3 & 1 & a & 1 \\ -4 & -\frac{a}{2} & 5 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & a+3 & -8 \\ 0 & -\frac{a}{2} & 1 & 20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & a+3 & -8 \\ 0 & 0 & a^2+3a+2 & 40-8a \end{array}\right]$

(iii) The system is consistent with a unique solution $\Leftrightarrow$ $a \neq -1, -2$. In this case,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & a+3 & -8 \\ 0 & 0 & 1 & \frac{-8a+40}{a^2+3a+2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3a^2+a+46}{a^2+3a+2} \\ 0 & 1 & 0 & \frac{-8(5a+17)}{a^2+3a+2} \\ 0 & 0 & 1 & \frac{-8a+40}{a^2+3a+2} \end{array}\right], \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3a^2+a+46}{a^2+3a+2} \\ \frac{-8(5a+17)}{a^2+3a+2} \\ \frac{-8a+40}{a^2+3a+2} \end{bmatrix}$$

If $a=-1$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 0 & 48 \end{array}\right]$$

If $a=-2$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 0 & 56 \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow$ $a=-1,-2$.

(ii) There are no values of a such that the system is consistent with infinitely many solutions.

Alternative Method

(iii) The system is consistent with a unique solution

$$\Leftrightarrow \begin{vmatrix} 1 & 0 & -1 \\ 3 & 1 & a \\ -4 & -\frac{a}{2} & 5 \end{vmatrix} = \frac{1}{2}(a^2+3a+2) = \frac{1}{2}(a+2)(a+1) \neq 0 \Leftrightarrow a \neq -1, -2.$$

In this case, by Cramer's rule,

$$x = \frac{\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & a \\ 8 & -\frac{a}{2} & 5 \end{vmatrix}}{\frac{1}{2}(a+2)(a+1)} = \frac{3a^2+a+46}{(a+2)(a+1)}, \quad y = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 3 & 1 & a \\ -4 & 8 & 5 \end{vmatrix}}{\frac{1}{2}(a+2)(a+1)} = \frac{-8(5a+17)}{(a+2)(a+1)}$$

$$z = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & 1 \\ -4 & -\frac{a}{2} & 8 \end{vmatrix}}{\frac{1}{2}(a+2)(a+1)} = \frac{-8a+40}{(a+2)(a+1)}$$

If $a=-1$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 3 & 1 & -1 & 1 \\ -4 & \frac{1}{2} & 5 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -8 \\ 0 & \frac{1}{2} & 1 & 20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 0 & 24 \end{array}\right]$$

If $a=-2$, the system becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 3 & 1 & -2 & 1 \\ -4 & 1 & 5 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 1 & 1 & 20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 0 & 28 \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow$ $a=-1,-2$.

(ii) There are no values of a such that the system is consistent with infinitely many solutions.