The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 2 2020/21 Semester 2

Question 1.

[10 marks each] Calculate the following integrals.

(a) $\int_0^1 e^x \cos(1 - e^x)\, dx$

(b) $\int_1^0 (1 + x)\, e^x\, dx$

(c) $\int (x + x^2) \sin(x)\, dx$

(d) $\int x \sin(x^2 + 1) \cos(x^2 - 1)\, dx$

(e) $\int \frac{x}{(x^2 - 2x + 1)(x^2 + 1)}\, dx$

My work :

(a) $\int_0^1 e^x \cos(1 - e^x)\, dx = -\int_0^1 \cos(1 - e^x)\, d(1 - e^x) = -\sin(1 - e^x)\Big|_0^1 = \sin(e - 1)$

(b) $\int_1^0 (1 + x)\, e^x\, dx = \Big[(1 + x)\, e^x - e^x \Big]_1^0 = -e$

(c) $\int (x + x^2) \sin(x)\, dx = \cdots = -(x + x^2) \cos x + (1 + 2x) \sin x + 2 \cos x + C$

(d) $\int x \sin(x^2 + 1) \cos(x^2 - 1)\, dx = \frac{1}{4} \int (\sin 2x^2 + \sin 2)\, d(x^2) = -\frac{1}{8} \cos 2x^2 + \frac{\sin 2}{4} x^2 + C$

(e) $\int \frac{x}{(x^2 - 2x + 1)(x^2 + 1)}\, dx = \int \left(\frac{0}{x - 1} + \frac{\frac{1}{2}}{(x - 1)^2} + \frac{-\frac{1}{2}}{x^2 + 1} \right) dx = -\frac{1}{2(x - 1)} - \frac{1}{2} \tan^{-1} x + C$

(f) $\int x^2 \sqrt{1 - x^2}\, dx$

(g) $\int_1^2 \frac{dx}{(x^2 + 2x)^{3/2}}$

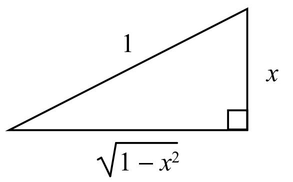
My work :

(f) Let $I = \int x^2 \sqrt{1 - x^2}\, dx$. Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1 - x^2} = \cos\theta$.

$$I = \int \sin^2\theta \cos\theta\, (\cos\theta\, d\theta) = \frac{1}{4}\int \sin^2 2\theta\, d\theta = \frac{1}{32}\int (1 - \cos 4\theta)\, d\,(4\theta)$$

$$= \frac{\theta}{8} - \frac{1}{32}\sin 4\theta + C = \frac{\theta}{8} - \frac{1}{16}\sin\theta\cos\theta\,(1 - 2\sin^2\theta) + C$$

$$= \frac{1}{8}\sin^{-1} x - \frac{1}{16}x\,(1 - 2x^2)\sqrt{1 - x^2} + C$$



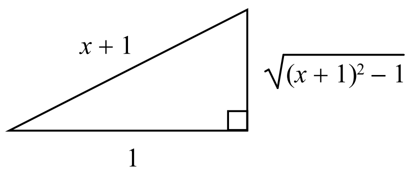
(g) Let $I = \int_1^2 \frac{dx}{(x^2 + 2x)^{3/2}} = \int_1^2 \frac{dx}{((x+1)^2 - 1)^{3/2}}$.

Let $x + 1 = \sec\theta$. When $x = 1$, $\theta = \frac{\pi}{3}$; when $x = 2$, $\theta = \cos^{-1}\frac{1}{3}$.

$dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{(x+1)^2 - 1} = \tan\theta$.

$$I = \int_1^2 \frac{dx}{((x+1)^2 - 1)^{3/2}} = \int_{\pi/3}^{\cos^{-1}(1/3)} \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int_{\pi/3}^{\cos^{-1}(1/3)} \cot\theta\csc\theta\, d\theta$$

$$= \Big[-\csc\theta\Big]_{\pi/3}^{\cos^{-1}(1/3)} = \frac{2}{\sqrt{3}} - \frac{3}{\sqrt{8}} \approx 0.09404$$



Question 2.

[10 marks] Let $f(t) = \int_0^{\sqrt{t}} \frac{1}{8 + 2s - s^2}\, ds$ and $F(x) = \int_0^{x^2} f(t)\, dt$. Find $F'(2)$.

My work :

$$F(x) = \int_0^{x^2} f(t)\, dt \quad \Rightarrow \quad F'(x) = f(x^2)\,(2x)$$

$$\Rightarrow \quad F'(2) = 4f(4) = 4\int_0^2 \frac{1}{8 + 2s - s^2}\, ds = 4\int_0^2 \frac{-1}{(s-4)(s+2)}\, ds$$

$$= \int_0^2 \left(\frac{-\frac{2}{3}}{s-4} + \frac{\frac{2}{3}}{s+2} \right) ds = -\frac{2}{3} \ln \frac{1}{2} + \frac{2}{3} \ln 2 = \frac{4}{3} \ln 2$$

Question 3.

(a) [10 marks] Find $T_2(x)$ (the Taylor polynomial of degree 2) for function $f(x) = xe^{-x}$ at $x_0 = 1$.

(b) [10 marks] Show that the error in approximating $f(x)$ by $T_2(x)$ for $x \in [1, \frac{3}{2}]$ is less than $\frac{1}{60}$. Note the fact that $e > \frac{5}{2}$. Do NOT use a calculator for question (b).

My work :

(a) $f'(x) = -xe^{-x} + e^{-x}$, $f''(x) = xe^{-x} - 2e^{-x}$, $f(1) = \frac{1}{e}$, $f'(1) = 0$, $f''(1) = -\frac{1}{e}$

The Taylor polynomial of degree 2 for $f(x)$ about $x_0 = 1$ is given by

$$T_2(x) = \frac{1}{e} - \frac{1}{2e}(x-1)^2$$

(b) $f^{(3)}(x) = -xe^{-x} + 3e^{-x}$

The remainder term is given by

$$R_2(x) := \frac{f^{(3)}(\xi)}{3!}(x-1)^3 = \frac{3-\xi}{6e^{\xi}}(x-1)^3$$

for some $\xi \in (1, x) \subseteq [1, \frac{3}{2}]$

$$|f(x) - T_2(x)| = |R_2(x)| = \frac{|3-\xi|}{6e^{\xi}}|x-1|^3 \le \frac{|3-1|}{6e^1}\left(\frac{1}{2}\right)^3 < \frac{1}{24} \cdot \frac{2}{5} = \frac{1}{60}$$