The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #07

Question 1

(a) Suppose A is non-singular, then $\det A \neq 0$. Hence $A\,(\text{adj}\,A) = (\det A)\,I$, which implies $\dfrac{A}{\det A}\,(\text{adj}\,A) = I$, i.e. adj A is non-singular.

Suppose adj A is non-singular. Now we assume $\det A = 0$, which implies

$A\,(\text{adj}\,A) = 0I \;\Rightarrow\; A = A\,(\text{adj}\,A)\,(\text{adj}\,A)^{-1} = 0I \;\Rightarrow\; \text{adj}\,A = 0I$, which is a contradiction.

(b) WLOG assume $\det A \neq 0$. Now we have

$$\begin{aligned} A\,(\text{adj}\,A) &= (\det A)\,I \\ (\det A)\det(\text{adj}\,A) &= (\det A)^n \\ \therefore \quad \det(\text{adj}\,A) &= (\det A)^{n-1} \end{aligned}$$

(c) $A\,(\text{adj}\,A) = (\det A)\,I \;\Rightarrow\; \dfrac{A}{\det A}(\text{adj}\,A) = I \;\Rightarrow\; (\text{adj}\,A)^{-1} = \dfrac{A}{\det A}$

$(A^{-1})\,(\text{adj}\,A^{-1}) = (\det A^{-1})\,I \;\Rightarrow\; \text{adj}\,A^{-1} = \dfrac{A}{\det A}$

Hence $(\text{adj}\,A)^{-1} = \text{adj}\,A^{-1}$

Question 2

(a)
$$\begin{vmatrix} a & 1 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & a \end{vmatrix} = a\begin{vmatrix} a & 0 & 0 \\ 4 & 3 & 2 \\ -2 & -1 & a \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -2 & -1 & a \end{vmatrix} = (a^2-1)(3a+2)$$
$$= (a-1)(a+1)(3a+2)$$

(b) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions

$\Leftrightarrow \quad \det A = (a-1)(a+1)(3a+2) = 0$

$\Leftrightarrow \quad a = -1, -\dfrac{2}{3}, 1$

(c) $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -2 & -1 & 0 \end{vmatrix} = -2$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 0 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 0 & -2 & 4 & -2 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -3 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

(d) $(\det B)^3 = \det(\operatorname{adj} B) = \det A \;\Rightarrow\; \det B = (\det A)^{\frac{1}{3}} = -2^{\frac{1}{3}}$

$AB = (\operatorname{adj} B)\, B = (\det B)\, I$

$$\therefore\; B = (\det B)\, A^{-1} = -2^{\frac{1}{3}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 & -2^{1/3} & 0 & 0 \\ -2^{1/3} & 0 & 0 & 0 \\ 2^{4/3} & 0 & 0 & 2^{1/3} \\ -2^{1/3} & 0 & -2^{-2/3} & -3\cdot 2^{-2/3} \end{pmatrix}$$

Question 3

(a) $\begin{vmatrix} -1-k & -1 & 3 \\ 3 & 2-k & -6 \\ -1 & 0 & 1-k \end{vmatrix} = -k^3 + 2k^2 - 5k + 1$

$$A^{-1} = \frac{1}{1-5k+2k^2-k^3}\begin{pmatrix} (k-2)(k-1) & 3\,(k+1) & 2-k \\ 1-k & k^2+2 & 1 \\ 3k & 3(1-2k) & k^2-k+1 \end{pmatrix}^{\mathsf{T}}$$

$$= \frac{1}{1-5k+2k^2-k^3}\begin{pmatrix} (k-2)(k-1) & 1-k & 3k \\ 3\,(k+1) & k^2+2 & 3(1-2k) \\ 2-k & 1 & k^2-k+1 \end{pmatrix}$$

(b) $$\begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} + 2B = \begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} B$$

$$\begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 3 \\ 3 & 0 & -6 \\ -1 & 0 & -1 \end{pmatrix} B$$

$$\therefore\ B = \begin{pmatrix} -3 & -1 & 3 \\ 3 & 0 & -6 \\ -1 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{-9} \begin{pmatrix} 0 & 9 & 0 \\ -1 & 6 & 1 \\ 6 & -9 & 3 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{-9} \begin{pmatrix} 0 & -1 & 6 \\ 9 & 6 & -9 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{9} & -\frac{4}{3} \\ -2 & -\frac{1}{3} & 2 \\ 0 & -\frac{2}{9} & \frac{1}{3} \end{pmatrix}$$

Question 4

(a) By elementary row operations, we have

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & 3 & 0 \\ 3 & 3 & 1 & 1 & 1 \\ -4 & -5 & -4 & 3 & -1 \\ -1 & -5 & 10 & -4 & b \end{array}\right) \xrightarrow[\substack{R_3 + 4R_1 \to R_3 \\ R_4 + R_1 \to R_4}]{R_2 - 3R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & 2 & -4 & 3 & 0 \\ 0 & -3 & 13 & -8 & 1 \\ 0 & 3 & -20 & 15 & -1 \\ 0 & -3 & 6 & -1 & b \end{array}\right)$$

$$\xrightarrow[R_4 - R_2 \to R_4]{R_3 + R_2 \to R_3} \left(\begin{array}{cccc|c} 1 & 2 & -4 & 3 & 0 \\ 0 & -3 & 13 & -8 & 1 \\ 0 & 0 & -7 & 7 & 0 \\ 0 & 0 & -7 & 7 & b-1 \end{array}\right)$$

Hence the system is consistent $\Leftrightarrow$ $b = 1$.

When $b = 1$,

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & 3 & 0 \\ 0 & 1 & -\frac{13}{3} & \frac{8}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 + 4R_3 \to R_1]{R_2 + \frac{13}{3}R_3 \to R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{7}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore\ \mathbf{x} = \left(\frac{2}{3} - \frac{7}{3}t,\ -\frac{1}{3} + \frac{5}{3}t,\ t,\ t\right)^{\mathsf{T}}$, where $t \in \mathbb{R}$.

(b) (i) $\det(\mathbf{A}) = -p + 2 \;\Rightarrow\;$ $\mathbf{A}$ is singular iff $p = 2$.

(ii) $$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(iii) (α) $\mathbf{Ax} = \mathbf{b}$ has one and only one solution if and only if $p \neq 2$.

If $p = 2$, then

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & q \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & q \end{array}\right)$$

(β) $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions $\Leftrightarrow$ $q = 0$ and $p = 2$.

(γ) $\mathbf{Ax} = \mathbf{b}$ has no solution $\Leftrightarrow$ $q \neq 0$ and $p = 2$.

Question 5

Using elementary row operations, we have

$$\left(\begin{array}{cccc|c} 0 & 2 & 6 & 4 & 0 \\ 1 & -2 & -13 & -4 & p \\ -2 & 4 & 17 & 1 & q \\ -2 & 2 & 11 & -3 & -2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & -2 & -13 & -4 & p \\ 0 & 2 & 6 & 4 & 0 \\ -2 & 4 & 17 & 1 & q \\ -2 & 2 & 11 & -3 & -2 \end{array}\right)$$

$$\xrightarrow[R_4 + 2R_1 \to R_4]{R_3 + 2R_1 \to R_3} \left(\begin{array}{cccc|c} 1 & -2 & -13 & -4 & p \\ 0 & 2 & 6 & 4 & 0 \\ 0 & 0 & -9 & -7 & 2p+q \\ 0 & -2 & -15 & -11 & 2p-2 \end{array}\right) \xrightarrow{R_4 + R_2 \to R_4} \left(\begin{array}{cccc|c} 1 & -2 & -13 & -4 & p \\ 0 & 2 & 6 & 4 & 0 \\ 0 & 0 & -9 & -7 & 2p+q \\ 0 & 0 & -9 & -7 & 2p-2 \end{array}\right)$$

$$\xrightarrow{R_4 - R_3 \to R_4} \left(\begin{array}{cccc|c} 1 & -2 & -13 & -4 & p \\ 0 & 2 & 6 & 4 & 0 \\ 0 & 0 & -9 & -7 & 2p+q \\ 0 & 0 & 0 & 0 & -q-2 \end{array}\right)$$

(i) $\mathbf{Ax} = \mathbf{b}$ has no solution $\Leftrightarrow$ $q \neq -2$

(ii) $\mathbf{Ax} = \mathbf{b}$ has infinitely many solution $\Leftrightarrow$ $q = -2$

(iii) There is no value of p and q such that $\mathbf{Ax} = \mathbf{b}$ has one and only one solution.

Question 6

By elementary row operations, we have

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & 1 & 4a-1 & 2 \\ 2 & a & a+1 & 2 \end{array}\right) \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & 4a-4 & -4 \\ 0 & a+2 & a-1 & -2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & a-1 & -1 \\ 0 & a+2 & a-1 & -2 \end{array}\right) \xrightarrow{R_3 - (a+2)R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & a-1 & -1 \\ 0 & 0 & 1-a^2 & a \end{array}\right)$$

(b) If $1 - a^2 \neq 0$, i.e. $a \neq \pm 1$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & a-1 & -1 \\ 0 & 0 & 1-a^2 & a \end{array}\right) \xrightarrow{\frac{1}{1-a^2}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & a-1 & -1 \\ 0 & 0 & 1 & \frac{a}{1-a^2} \end{array}\right)$$

$$\xrightarrow[R_1 - R_3 \to R_1]{R_2 - (a-1)R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & \frac{2-a-2a^2}{1-a^2} \\ 0 & 1 & 0 & \frac{-1}{a+1} \\ 0 & 0 & 1 & \frac{a}{1-a^2} \end{array}\right) \xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1-2a^2}{1-a^2} \\ 0 & 1 & 0 & \frac{-1}{a+1} \\ 0 & 0 & 1 & \frac{a}{1-a^2} \end{array}\right)$$

$$\therefore \ \mathbf{x} = \left(\frac{1-2a^2}{1-a^2}, -\frac{1}{a+1}, \frac{a}{1-a^2}\right)^{\mathsf{T}}.$$

(c) If $a = \pm 1 \neq 0$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & a-1 & -1 \\ 0 & 0 & 0 & a \end{array}\right) \quad \therefore \text{ No solutions exist.}$$

(a) There is no value of a such that the system has infinitely many solutions.

Alternative Solution

(b) The system has unique solution if and only if

$$\begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 4a-1 \\ 2 & a & a+1 \end{vmatrix} = -4a^2 + 4 \neq 0 \iff a \neq \pm 1$$

By Cramer's rule,

$$\begin{vmatrix} 2 & -1 & 1 \\ 2 & 1 & 4a-1 \\ 2 & a & a+1 \end{vmatrix} = -8a^2 + 4, \quad \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 4a-1 \\ 2 & 2 & a+1 \end{vmatrix} = 4a - 4, \quad \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & a & 2 \end{vmatrix} = 4a$$

$$\therefore \ \mathbf{x} = \left(\frac{1-2a^2}{1-a^2}, -\frac{1}{a+1}, \frac{a}{1-a^2} \right)^T.$$

If $a = 1$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 2 \end{array}\right) \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & 0 & -4 \\ 0 & 3 & 0 & -2 \end{array}\right) \xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 - 3R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\therefore$ No solution exists.

If $a = -1$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & 1 & -5 & 2 \\ 2 & -1 & 0 & 2 \end{array}\right) \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -8 & -4 \\ 0 & 1 & -2 & -2 \end{array}\right) \xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$\therefore$ No solution exists.

(c) The system is inconsistent if and only if $a = \pm 1$.

(a) There is no value of a such that the system has infinitely many solutions.

Question 7

(a) The system is consistent for any b_1, b_2 and b_3 if and only if

$$\begin{vmatrix} 1 & 1 & -1 \\ -a & -1 & a \\ a^2 & 1 & -a \end{vmatrix} = a^3 - 2a + a = a(a-1)^2 \neq 0 \iff a \neq 0, 1$$

(b) If $a = 0$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -1 & 0 & b_2 \\ 0 & 1 & 0 & b_3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 0 & b_3 \\ 0 & -1 & 0 & b_2 \end{array}\right) \xrightarrow{R_3 + R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & b_2 + b_3 \end{array}\right)$$

The system is consistent if and only if $b_2 + b_3 = 0 \iff b_2 = -b_3$

(c) If $a = 1$ and $b_1 = b_2 = b_3 = 0$, then the system becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 + R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore\ \mathbf{x} = (-s + t, s, t)^T$, where $s, t \in \mathbb{R}$.

Question 8

$$\det A = \begin{vmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} a_n & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \end{vmatrix}$$

$$= (-1)^{n-1} a_1 a_2 \cdots a_{n-1} a_n`$$

A is invertible $\iff \det A = (-1)^{n-1} a_1 a_2 \cdots a_{n-1} a_n \neq 0 \iff a_1,\ a_2,\ \ldots,\ a_n \neq 0$

By Gauss-Jordan method, we have

$$\left(\begin{array}{ccccc|ccccc} 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & 0 & 0 & \cdots & 0 \\ a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|ccccc} a_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & a_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1/a_n \\ 0 & 1 & 0 & \cdots & 0 & 1/a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 1/a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1/a_{n-1} & 0 \end{array}\right)$$

$$\therefore \quad A^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1/a_n \\ 1/a_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/a_{n-1} & 0 \end{pmatrix}$$

Question 9

By row operations, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ \alpha & \beta & \gamma & 1 \\ \beta & \gamma & \alpha & 1 \\ \gamma & \alpha & \beta & 1 \end{array}\right) \xrightarrow{R_4 + R_3 + R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ \alpha & \beta & \gamma & 1 \\ \beta & \gamma & \alpha & 1 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & 3 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 - \beta R_1 \to R_3 \\ R_4 - (\alpha+\beta+\gamma)R_1 \to R_4}]{R_2 - \alpha R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & \beta-\alpha & \gamma-\alpha & 1-3\alpha \\ 0 & \gamma-\beta & \alpha-\beta & 1-3\beta \\ 0 & 0 & 0 & 3-3(\alpha+\beta+\gamma) \end{array}\right)$$

The system has a unique solution if and only if $\alpha+\beta+\gamma = 1$ and

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & \beta-\alpha & \gamma-\alpha \\ 0 & \gamma-\beta & \alpha-\beta \end{vmatrix} = -\alpha^2 - \beta^2 - \gamma^2 + \alpha\beta + \beta\gamma + \gamma\alpha$$

$$= -\frac{(\alpha-\beta)^2}{2} - \frac{(\beta-\gamma)^2}{2} - \frac{(\gamma-\alpha)^2}{2} \neq 0$$

$\Leftrightarrow$ $\alpha+\beta+\gamma = 1$ and not all α, β, γ are equal.

If $\alpha+\beta+\gamma = 1$ and $\alpha = \beta = \gamma = \frac{1}{3}$, the system now becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which implies the system has infinitely many solutions.

Conversely, we will have $\begin{cases} \alpha+\beta+\gamma = 1 \\ \beta-\alpha = \gamma-\beta = 0 \text{ or } \gamma-\alpha = \alpha-\beta = 0 \end{cases}$, i.e. $\alpha = \beta = \gamma = \frac{1}{3}$.

Hence the system is consistent if and only if $\alpha+\beta+\gamma = 1$. †

† *Remark* We need to check whether $\alpha+\beta+\gamma = 1$ is truly a necessary and sufficient condition for the system to be consistent.

Question 10

By row operations, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ (b+1) & 1 & (ab+a) & cb+c \\ (b+1)^2 & 1 & a^2(b+1)^2 & c^2(b+1)^2 \end{array}\right)$$

$$\xrightarrow{R_3-(b+1)R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ b+1 & 1 & a\,(b+1) & c\,(b+1) \\ 0 & -b & a\,(a-1)(b+1)^2 & c\,(c-1)(b+1)^2 \end{array}\right)$$

$$\xrightarrow{R_2-(b+1)R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -b & (a-1)(b+1) & (c-1)(b+1) \\ 0 & -b & a\,(a-1)(b+1)^2 & c\,(c-1)(b+1)^2 \end{array}\right)$$

$$\xrightarrow{R_3-R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -b & (a-1)(b+1) & (c-1)(b+1) \\ 0 & 0 & (b+1)(a-1)(ab+a-1) & (c-1)(b+1)(cb+c-1) \end{array}\right)$$

(a) The system is consistent with infinitely many solutions if and only if

$$b=-1 \text{ or } \begin{cases} b=0 \\ a=c \end{cases} \text{ or } \begin{cases} b=0 \\ c=1 \end{cases} \text{ or } \begin{cases} ab+a-1=0 \\ cb+c-1=0 \\ b\neq 0 \end{cases} \text{ or } \begin{cases} ab+a-1=0 \\ c=1 \end{cases}$$

$$\text{or } \begin{cases} a=1 \\ cb+c-1=0 \\ b\neq 0 \end{cases} \text{ or } \begin{cases} a=1 \\ c=1 \end{cases}$$

$$\Leftrightarrow \quad b=-1 \text{ or } \begin{cases} b=0 \\ a=c \end{cases} \text{ or } \begin{cases} b=0 \\ c=1 \end{cases} \text{ or } \begin{cases} ab+a-1=0 \\ cb+c-1=0 \end{cases} \text{ or } \begin{cases} ab+a-1=0 \\ c=1 \end{cases}$$

$$\text{or } \begin{cases} a=1 \\ cb+c-1=0 \end{cases}$$

(b) The system is consistent with one and only one solution if and only if

$b\neq 0$ and $(ab+a-1)(a-1)(b+1)\neq 0$

$\Leftrightarrow \quad a\neq 1$ and $b\neq 0,-1$ and $ab+a-1\neq 0$

(c) The system is inconsistent if and only if

$$\begin{cases} b=0 \\ a\neq c \\ c\neq 1 \end{cases} \text{ or } \begin{cases} ab+a-1=0 \\ cb+c-1\neq 0 \\ c\neq 1 \end{cases} \text{ or } \begin{cases} a=1 \\ cb+c-1\neq 0 \\ c\neq 1 \end{cases}$$