The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #06

Question 1 (*Basic Level*)

Evaluate the following:

(a) $\begin{pmatrix} 1 & 2 \\ 5 & -3 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 2 & 9 \end{pmatrix}$

(b) $5\begin{pmatrix} 2 & -3 \\ 6 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$

Question 2 (*Basic Level*)

Reduce the following matrices into reduced row echelon form:

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & -7 \\ 1 & -2 & -8 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & -1 & 4 \\ -1 & 1 & 3 & -5 \\ -2 & -1 & 3 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 1 & -8 \\ 1 & -3 & -18 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 7 \\ -4 & 1 & 1 \\ 1 & -2 & 5 \end{pmatrix}$

Question 3 (*Basic Level*)

(a) $\begin{aligned} 3x - 2y + z &= 15 \\ -3x + y + z &= 9 \\ x - 5y + 2z &= 24 \end{aligned}$

(b) $\begin{aligned} x - 5y + 2z &= 15 \\ 3x - 2y + z &= 9 \\ -3x + y + z &= -3 \end{aligned}$

(c) $\begin{aligned} x - y + 5z &= -3 \\ -2x + 5y + z &= -19 \\ 5x + y - 2z &= 33 \end{aligned}$

(d) $\begin{aligned} x + y - 5z &= 12 \\ -3x - 6y + 9z &= 7 \\ -4x - 14y &= -2 \end{aligned}$

Question 4 (*Standard Level*)

(a) Determine the value of h such that the system is consistent:

$$\begin{aligned} x + 2y - \ \ 3z &= -2 \\ 3x - 4y + 11z &= \ \ 4 \\ 4x + \ \ y + \ \ 2z &= \ \ h \end{aligned}$$

(b) For what values of k, the following system will have (i) no solution; (ii) many solutions?

$$\begin{aligned} x + 4y - 3z &= -10 \\ 3x + \ \ y + 2z &= \ \ 14 \\ -4x - 5y + \ \ z &= \ \ k \end{aligned}$$

(c) Find a relationship of the unknown constant a, b, c if the homogeneous system below has non-trivial solution.

$$\begin{aligned} x + 3y + az &= 0 \\ 2x - \ \ y + bz &= 0 \\ 4x + 5y + cz &= 0 \end{aligned}$$

Question 5 (*Standard Level*)

Find a general solution to the following linear systems:

(a) $$\begin{aligned} -x + \ \ y + \ \ 3z &= 10 \\ x + 2y - 15z &= -7 \end{aligned}$$

(b) $$\begin{aligned} x + 2y - 16z - \ \ 6w &= 19 \\ -3x + \ \ y + 20z - 17w &= -1 \end{aligned}$$

(c) $$\begin{aligned} x - \ \ y - \ \ 2z &= \ \ 1 \\ x + 2y + 10z &= -2 \\ 3x + \ \ y + 10z &= -1 \end{aligned}$$

(d) $$\begin{aligned} x + \ \ y - 13z &= 13 \\ x + 2y - 20z &= 18 \\ -x + 8y - 50z &= 32 \end{aligned}$$

Question 6 (*Standard Level*)

Find the inverse of the following matrices, or show that the inverse does not exist.

(a) $A = \begin{pmatrix} 1 & -1 & -6 \\ 2 & -1 & -9 \\ -2 & 1 & 10 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 1 & -3 \\ -2 & -1 & 3 \\ 3 & 2 & -5 \end{pmatrix}$

(c) $C = \begin{pmatrix} -6 & 1 & 20 \\ 1 & -6 & 20 \\ -1 & -4 & 20 \end{pmatrix}$

(d) $D = \begin{pmatrix} 3 & 1 & -9 \\ 2 & 3 & -9 \\ 1 & 1 & -4 \end{pmatrix}$

Using the results above, solve the linear systems

(e) $$\begin{aligned} x + \ \ y - 3z &= 1 \\ -2x - \ \ y + 3z &= 1 \\ 3x + 2y - 5z &= 1 \end{aligned}$$

(f) $$\begin{aligned} x + \ \ y - 3z &= 1 \\ -2x - \ \ y + 3z &= 2 \\ 3x + 2y - 5z &= 3 \end{aligned}$$

Question 7 (*Standard Level*)

Compute the following determinants by direct expansion

(a) $\begin{vmatrix} 4 & 0 & 2 \\ -1 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & 0 \\ 3 & 3 & 0 \\ 7 & 6 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 5 & 0 & -2 \\ 4 & -3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -1 & 1 & 3 & 4 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 3 & -1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 5 & 8 & -2 & -3 \end{vmatrix}$

Compute the following determinants by row or column operations

(e) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & -1 & 1 & -1 \\ 5 & 0 & 0 & 5 \end{vmatrix}$ (f) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Question 8 (*Intermediate Level*)

Factorize the following determinants.

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ (b) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Question 9 (*Intermediate Level*)

Solve the following linear system by using Cramer's rule.

$$\begin{aligned} x - 7y - z &= 25 \\ 8x + 4y + z &= -25 \\ 2x + y + 2z &= -15 \end{aligned}$$

Question 10 (*Revision*)

(a) Evaluate $\int x \cos nx \, dx$

(b) Evaluate $\int (x-2)^2 \sin nx \, dx$

(c) Evaluate $\int (3x-2)^2 \sin nx \, dx$

Question 11 (*Exam Level*)

For $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}$, use elementary row operations, or otherwise, to find A^{-1}. Hence solve the system of linear equations

$$\begin{cases} x + 2y \phantom{{}-2z} = 6b \\ 2x + 3y - 2z = 0 \\ y - z = -3b \end{cases},$$

where b is any real number.

Question 12 (*Exam Level*)

Let $A = \begin{pmatrix} 1 & -4 \\ a & -2 \end{pmatrix}$ and $B = \begin{pmatrix} b & 3b \\ 2b & 4b \end{pmatrix}$, where a and b are given real numbers, and

$X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$.

(a) If $AXA^T = B$, show that x_1, x_2, x_3 and x_4 satisfy the system of linear equations

$$\begin{pmatrix} 1 & -4 & -4 & 16 \\ a & -2 & -4a & 8 \\ a & -4a & -2 & 8 \\ a^2 & -2a & -2a & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b \\ 2b \\ 3b \\ 4b \end{pmatrix} \qquad (*)$$

(b) Determine the value(s) of a and b such that the system of linear equations $(*)$ is consistent.

(c) When $a = 1$ and $b = 2$, find a matrix X such that $AXA^T = B$.

Basic Properties of Determinants

1. $\det A = \det A^T$

2. Negative if two rows (or columns) are swapped

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

3. Take out common factor in each row (or column)

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix}$$

4. Zero if a row (or a column) consists of zero only

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & 0 & c_3 \end{vmatrix}$$

5. Zero if a row (or a column) is a multiple of another

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6. Invariant under $R_i + kR_j \to R_i$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 + ka_1 & c_2 + ka_2 & c_3 + ka_3 \end{vmatrix}$$

7. Linear in each row (or column) (*Multilinearity*)

$$\begin{vmatrix} 2x+3p & 2y+3q & 2z+3r \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 2\begin{vmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 3\begin{vmatrix} p & q & r \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$