The Hong Kong Polytechnic University

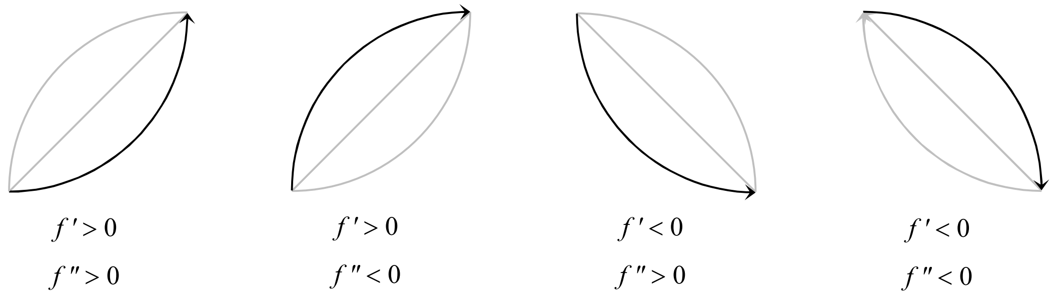
Department of Applied Mathematics

AMA1120 Tutorial Set #02

Concave Upwards and Concave Downwards

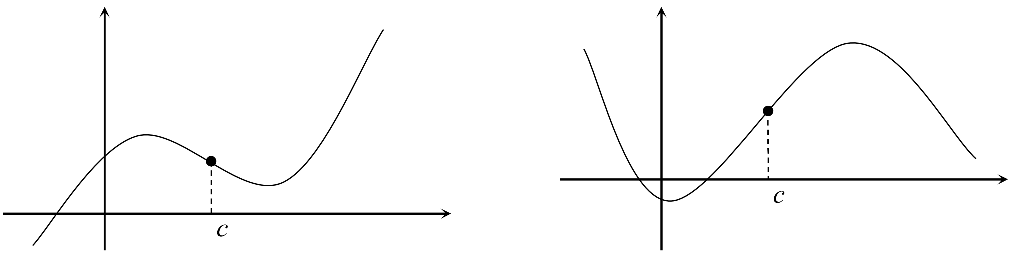
The graph $y = f(x)$ is **concave upwards on** I if and only if $f''(x) > 0$ for all $x \in I$.

The graph $y = f(x)$ is **concave downwards on** I if and only if $f''(x) < 0$ for all $x \in I$.



Second-order Stationary Points versus Points of Inflection (Inflexion)

A real number $c \in \mathbb{R}$ is called a second-order stationary point of f if $f''(c) = 0$. A point of inflection (inflexion) is a point c at which change of concavity / convexity occurs.



Question 1 (*Standard Level*)

Find the intervals in which $y = f(x)$ is concave upwards and concave downwards.

(a) $f(x) = x^4 - 6x^2 + 8x + 10$

(b) $f(x) = x^{2/3}(1 - x)^{1/3}$

(c) $f(x) = \dfrac{(x+1)^2}{x-1}$

(d) $f(x) = \dfrac{x^3}{x^2 - 3x + 2}$

(e) $f(x) = e^{-x^2}$

Asymptotes

If either $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty$, then

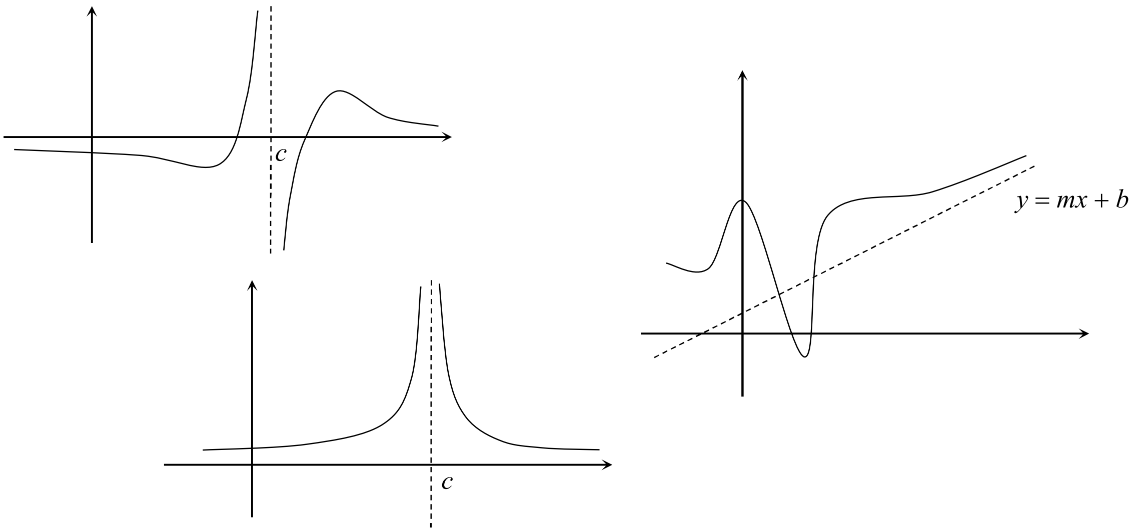
$$\boxed{x = c}$$

is called a **vertical asymptote** of the graph $y = f(x)$.

If $\lim_{x \to +\infty} [\, f(x) - (mx + b)\,] = 0$ or $\lim_{x \to -\infty} [\, f(x) - (mx + b)\,] = 0$, then

$$\boxed{y = mx + b}$$

is called an **inclined asymptote** (if $m \neq 0$) or a **horizontal asymptote** (if $m = 0$) of the graph $y = f(x)$.



Question 2 (*Concept*)

Prove that $\lim_{x \to \infty} [\, f(x) - (mx + b)\,] = 0 \iff m = \lim_{x \to \infty} \dfrac{f(x)}{x}$ and $b = \lim_{x \to \infty} (\, f(x) - mx\,)$.

Question 3 (*Standard Level*)

(a) Find the asymptotes, if exist, of the graph $y = x^{2/3}(1 - x)^{1/3}$

(b) Find the asymptotes, if exist, of the graph $y = \dfrac{(x+1)^2}{x - 1}$.

(c) Find the asymptotes, if exist, of the graph $y = \dfrac{x^3}{x^2 - 3x + 2}$.

(d) Find the asymptotes, if exist, of the graph $y = e^{-x^2}$.

Procedure of Curve Sketching

Input: A function f

Output: A sketch of the graph $y = f(x)$

Procedure:

1. Find the domain of f and check whether f is odd or even or periodic.
2. Find the x-intercept (optional) and y-intercept, if exist, of the graph $y = f(x)$.
3. Find the stationary points (or critical points), if exist, of f.
4. Find the second-order stationary points (or points of inflection), if exist, of f.
5. Find the asymptotes, if exist, of the graph $y = f(x)$.
6. Tabulate the results and find intervals where $f' > 0$, $f' < 0$, $f'' > 0$, $f'' < 0$.
7. Sketch the graph $y = f(x)$ accordingly.

Question 4 (*Standard Level*)

(a) Sketch the curve of $f(x) = x^4 - 6x^2 + 8x + 10$.

(b) Sketch the curve of $f(x) = x^{2/3}(1 - x)^{1/3}$.

(c) Sketch the curve of $f(x) = \dfrac{(x+1)^2}{x-1}$.

(d) Sketch the curve of $f(x) = \dfrac{x^3}{x^2 - 3x + 2}$.

(e) Sketch the curve of $f(x) = e^{-x^2}$.

Question 5 (*Intermediate Level*)

Let $f(x) = \left|\dfrac{2x^2 + x - 1}{x - 1}\right|$, where $x \neq 1$.

(a) Find $f'(x)$ where $x \neq 1$.

(b) Show that $f'(-1)$ and $f'\left(\dfrac{1}{2}\right)$ do not exist.

(c) Find the relative extreme points of $f(x)$.

(d) Find the asymptotes, if exist, of $f(x)$.

(e) Sketch the graph of $f(x)$.

Question 6 (*Intermediate Level*)

Let $f(x) = x^2(2-x)e^{-x}$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the local extrema and the points of inflexion of $f(x)$.

(c) Find the asymptotes of the graph of $f(x)$.

(d) Sketch the graph of $f(x)$.

Question 7 (*Intermediate Level*)

Let $f(x) = \dfrac{36|x|}{(x-1)^2}$. Sketch the graph of $y = f(x)$.

Question 8 (*HKALE Level,* 1992 Paper II)

Let $f(x) = xe^{-x^2}$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find all relative extrema and the points of inflection of $f(x)$.

(c) Find the asymptote of the graph of $f(x)$.

(d) Sketch the graph of $y = f(x)$.

(e) Hence sketch the curve $x + y = (x-y)\,e^{-\frac{1}{2}(x-y)^2}$.

Question 9 (*Exam Level*)

Consider the function $g(x) = \dfrac{x^2-16}{x-5}$, where $x \neq 5$.

(a) Find all critical points of the function. Determine the intervals in which $g(x)$ is increasing and the intervals in which $g(x)$ is decreasing. Hence, or otherwise, find all the local (relative) maxima and local (relative) minima of the function.

[9 marks]

(b) Find the intervals in which $g(x)$ concaves up and the intervals in which $g(x)$ concaves down. Hence determine the points of inflection of the function.

[4 marks]

(c) Find all asymptotes of the function (including vertical, horizontal and inclined asymptotes). Sketch the graph of $g(x)$.

[7 marks]

Global Optimization

Consider a continuous function f on a closed and bounded interval $[a, b]$, it is not hard to see that the global maximum and the global minimum of the function f exist. Beyond the existence guarantee, one will be more interested in finding out those global (or absolute) maximum and minimum points. We state the procedure as follows:

Input: A continuous function f on a closed and bounded interval $[a, b]$

Output: The global maximum and minimum points and values of f

Procedure:

1. Find the stationary points of f and points where f is not differentiable, if exist.
2. Tabulate the function values of all the points in step 1, $f(a)$, and $f(b)$.
3. The largest value is the global maximum value, and the corresponding points are the global maximum points, while the smallest value is the global minimum value, and the corresponding points are the global minimum points.

Question 10 (*Intermediate Level*)

Let $f(x) = x^2 e^{-x}, -\infty < x < \infty$.

(a) Find all critical points and inflection points.

(b) Find the local and global maximum and minimum values, if exist.

(c) Find the intervals of increasing, decreasing, concave-up and concave-down.

(d) Sketch the graph $y = x^2 e^{-x}, -\infty < x < \infty$.

Question 11* (*Pure Global Optimization – Must-do!*)

(a) Find the absolute maximum and absolute minimum of $f(x) = x^{1/3} e^{-x/3}$ on $[-1, 27]$.

(b) Find the absolute maximum and absolute minimum of $f(x) = \dfrac{x}{x^3 + 2}$ on $[0, 2]$. What if we change the domain to $(0, \infty)$?

(c) Find the absolute maximum and minimum of $f(x) = 2\sin x + \sin 2x$ on $[0, 3\pi/2]$.

Question 12* (*Application Questions – Must-do!*)

(a) Find two positive numbers whose sum is 30 and the square of one number plus twice the square of the other is a minimum.

(b) You are asked to design a cylindrical container with volume 10π m^3. Suppose that the material for the base and side cost 8 dollars per square meter and the material for the top cost 2 dollars per square meter. Find the dimensions that minimize cost. Justify your answer.

(c) Find the point on the curve $y = \sqrt{x + \dfrac{20}{x}}$ that is closest to the origin (0, 0). Apply either the first or the second derivative test to verify that the answer you obtained is a local minimum.

(d) The sum of three positive numbers A, B and C is 20 (they do not have to be integers). It is also known that $2A = 3B$. Find A, B and C so that their product ABC is a maximum.

(e) A 30 m by 20 m park has a thirsty student Benny at one corner, and a drinking fountain at the opposite corner (diagonally across). The park consists of grass, on which Benny can walk at 0.8 m/s, surrounded by the sidewalk, on which Benny can walk at 1.0 m/s. If we know that Benny will first walk along the sidewalk of the long side and then cut across the grass in a straight line to reach the fountain, how many meters of sidewalk must Benny walk to reach the fountain in as little time as possible?

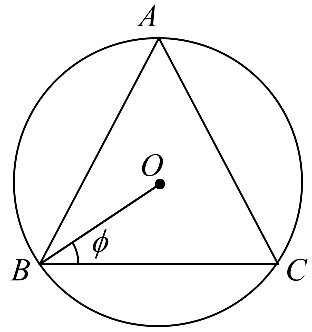
(f) Using a fare analysis model, a bus company finds that on one of its routes, for every dollar they raise (lower) the fare by, the ridership decreases (increases) by 23500 people. Currently the bus company charges \$11.3 on the route and the ridership is 250,000 per day.

(i) Write the daily revenue $R(x)$ (in \$) as a function of the fare adjustment x.

(ii) Find the fare (to the nearest cent) which gives the bus company maximum revenue on the route.

(g) Consider a cylinder whose surface area is 2π (including the areas of the top, the bottom, and the side). Find the largest possible value of its volume.

(h) The figure depicted shows a circle of radius r and center O circumscribing an isosceles triangle ABC with $AB = AC$. Show that the area of triangle ABC is the greatest when triangle ABC is equilateral.



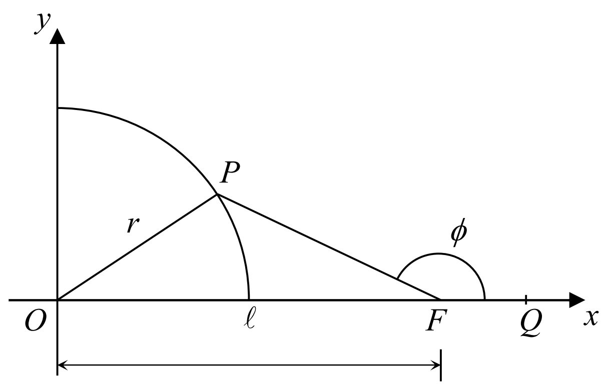
(i) P is a moving point on a circle with center at O in the first quadrant. The radius of the circle is r. F is a fixed point on the x-axis such that $OF = \ell$. Let

$$\angle POF = \theta, \quad \angle PFQ = \phi.$$

(a)* Show that $\tan\phi = \dfrac{r\sin\theta}{r\cos\theta - \ell}$.

(b) When $\ell = 2r$, find the least value of ϕ.

(c) When $r = 2\ell$, find the greatest value of ϕ.



Question 13*** (*Concept Level*)

(a) Suppose $f(x)$ is concave upwards everywhere, i.e. $f''(x) \geq 0$ for all x. It is known that by Taylor Theorem, if f is continuous on $[a, b]$ and twice continuously differentiable in (a, b), then $\exists\, \xi \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(\xi)(b-a)^2.$$

Show, by Taylor Theorem, that if $f(x)$ is concave upwards everywhere, then $f'(c) = 0$ if and only if $x = c$ is a global minimizer.

(b) Show that if $f(x)$ is concave downwards everywhere, then $f'(c) = 0$ if and only if $x = c$ is a global maximizer.