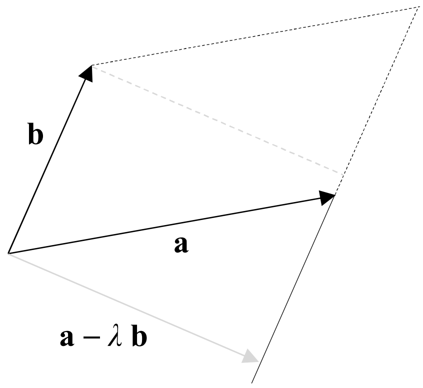
The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #08

Question 1. (*Concept Level*)

Consider two 2-vectors $\mathbf{a} := (a_1, a_2)^{\mathsf{T}}$ and $\mathbf{b} := (b_1, b_2)^{\mathsf{T}}$, the value of determinant is defined as $\det(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.



(a) Find $\lambda \in \mathbb{R}$ so that $\mathbf{a} - \lambda\,\mathbf{b} \perp \mathbf{b}$.

(b) Find $\|\,\mathbf{b}\,\|\,\|\,\mathbf{a} - \lambda\,\mathbf{b}\,\|$, and argue $|\det(\mathbf{a}, \mathbf{b})|$ is equal to the area of the parallelogram spanned by **a** and **b**.

Question 2. (*Concept Level*)

Given 3-vectors $\mathbf{a} = (a_1, a_2, a_3)^{\mathsf{T}}$ and $\mathbf{b} = (b_1, b_2, b_3)^{\mathsf{T}}$, the cross product $\mathbf{a} \times \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

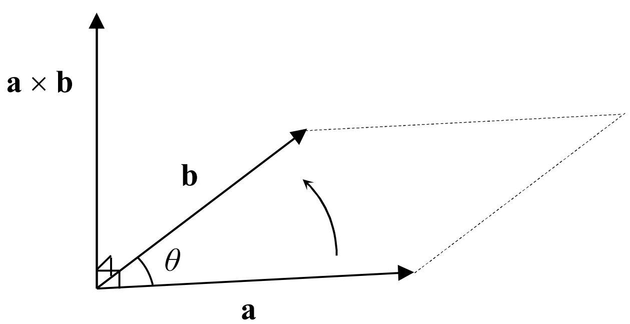
(a) Show that

(i) $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$

(ii) $(\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b}) = \alpha\,(\mathbf{a} \times \mathbf{b})$

(iii) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

(b) Explain that $\mathbf{a} \times \mathbf{b}$ has magnitude $\|\,\mathbf{a} \times \mathbf{b}\,\| = \|\,\mathbf{a}\,\|\,\|\,\mathbf{b}\,\| \sin\theta$, where $\theta \in [0, \pi]$ is the included angle between **a** and **b** ; and the direction is defined based on right hand grip rule:



Question 3. (*Basic Level*)

Compute the following cross products:

(a) $(2, 3, 6) \times (1, -4, 0)$ (b) $(1, -4, 0) \times (-3, 1, -2)$

(c) $(2, 3, 6) \times (-3, 1, -2)$

Question 4. (*Basic Level*)

Compute the following inner products (aka dot products):

(a) $\langle (2, 3, 6), (1, -4, 0) \rangle$ (b) $\langle (1, -4, 0), (-3, 1, -2) \rangle$

Question 5. (*Concept Level*)

Show, by the language of vectors and the knowledge of determinants, that

$$\frac{1}{2}\left| \begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix} \right|$$

is equal to the area of ΔABC where A, B and C are the points (a_1, b_1), (a_2, b_2) and (a_3, b_3).

Question 6. (*Exam Level*)

Given parallel planes $\Pi_1 :\ 2x - 2y + z = 5$ and $\Pi_2 : 2x - 2y + z = 20$.

(a) Show that $\mathbf{n} = (2, -2, 1)$ is normal to Π_1 and Π_2.

(b) Find the distance between the given planes.

Question 7. (*Standard Level*)

Consider the vectors $\mathbf{x} = (2, 3, 6)$, $\mathbf{y} = (1, -4, 0)$ and $\mathbf{z} = (-3, 1, -2)$.

(a) Denote the unit vector in the direction of $\mathbf{x}$ by $\hat{\mathbf{x}}$. Find $\hat{\mathbf{x}}$.

(b) Calculate the $\mathrm{proj}_{\hat{\mathbf{x}}}\, \mathbf{y}$ and $\mathrm{proj}_{\hat{\mathbf{x}}}\, \mathbf{z}$.

(c) Find the area of the triangle with vertices $P\,(2, 3, 6)$, $Q\,(1, -4, 0)$ and $R\,(-3, 1, -2)$.

Question 8. (*Standard Level*)

(a) Show that the vector $\mathbf{n} = (a, b)$ is perpendicular to the line defined by the equation $ax + by + c = 0$ in the plane.

(b) Show that the shortest distance between the point $P_0\ (x_0, y_0)$ to the line $ax + by + c = 0$ is

$$\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\,.$$

Question 9. (*Smart Level*)

Use determinants to find the equation of

(a) a straight line in $\mathbb{R}^2$ passing through (2, 1) and (3, 5);

(b) a plane in $\mathbb{R}^3$ passing through (1, 0, 0), (2, 1, 0) and (3, 5, 1);

(c) a circle in $\mathbb{R}^2$ passing through (1, 0), (2, 1) and (3, 5).

For part (a), find the equation of line in vector form.

Question 10. (*Challenging Level with Fun*)

Show that the area of polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ can be given by

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right| = \frac{1}{2}\left|(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_n y_1 - x_1 y_n)\right|$$

Question 11. (*Challenging Level with Fun, Not Linear Algebra*)

(a) Show that the area of regular n-sided polygon with radius of circumcircle R is given by

$$nR^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2} nR^2 \sin\frac{2\pi}{n}$$

(b) Show also that $\lim_{n\to\infty} \frac{1}{2} nR^2 \sin\frac{2\pi}{n} = \pi R^2$.

(c) Using excel or otherwise, try a numerical experiment to find the smallest integer n such that $\frac{1}{2} n \sin\frac{2\pi}{n}$ and π have no difference up to 5 decimal places.