The Hong Kong Polytechnic University
Department of Applied Mathematics
AMA1120 Final Exam 2019/20 Semester 2

Question 1.

(a) Consider the function $f(x) = \dfrac{|x|}{x^2+4}$, $x \in \mathbb{R}$. Find the open intervals where the function is increasing or decreasing. [5 marks]

(b) Consider a right circular cylinder whose volume is 1. Suppose that its bottom is a disk of radius r.

(i) Let S be the surface area of this cylinder, including the areas of the top, the bottom, and the side. Formulate S as a function of r. [5 marks]

(ii) Find the smallest possible surface area of this cylinder. [5 marks]

My work :

(a) $$f(x) = \begin{cases} \dfrac{x}{x^2+4}, & \text{if } x \ge 0 \\ -\dfrac{x}{x^2+4}, & \text{if } x < 0 \end{cases}$$

For $x > 0$, $f'(x) = \dfrac{(x^2+4) - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$. For $x < 0$, $f'(x) = \dfrac{x^2-4}{(x^2+4)^2}$.

$$f'_+(0) = \lim_{x\to 0^+} \frac{\dfrac{x}{x^2+4} - 0}{x-0} = \lim_{x\to 0^+} \frac{1}{x^2+4} = \frac{1}{4},\quad f'_-(0) = \lim_{x\to 0^+} \frac{\dfrac{-x}{x^2+4} - 0}{x-0} = \lim_{x\to 0^+} \frac{-1}{x^2+4} = -\frac{1}{4}$$

$\therefore$ f is not differentiable at $x = 0$.

$$f'(x) = 0 \iff |f'(x)| = \frac{|4-x^2|}{(x^2+4)^2} = 0 \iff x = \pm 2$$

x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	$+$	$-$	$+$	$-$

The open intervals where f is increasing are: $(-\infty, -2)$, $(0, 2)$
The open intervals where f is decreasing are: $(-2, 0)$, $(2, +\infty)$

(b) (i) Volume of the cylinder $= \pi r^2 h = 1 \;\Rightarrow\; h = \dfrac{1}{\pi r^2}$

$S(r) :=$ surface area of the cylinder $= 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{2}{r}$ on $(0, +\infty)$

(ii) $S'(r) = 4\pi r - \dfrac{2}{r^2} = 0 \;\Rightarrow\; r = \dfrac{1}{\sqrt[3]{2\pi}}$

r	$[0, \frac{1}{\sqrt[3]{2\pi}})$	$\frac{1}{\sqrt[3]{2\pi}}$	$(\frac{1}{\sqrt[3]{2\pi}}, +\infty)$
$S(r)$	↘	min	↗
$S'(r)$	$-$	0	$+$

The smallest possible surface area $= S(\sqrt[3]{2\pi}) = 3\sqrt[3]{2\pi}$

Question 2.

Evaluate the following limit and integrals.

(a) $\displaystyle\lim_{x\to 1} \frac{\int_x^1 e^{t^2}\,dt}{\ln x}$ [5 marks]

(b) $\displaystyle\int \frac{1}{\sqrt{x^2 + x + 1}}\,dx$ [5 marks]

(c) $\displaystyle\int \frac{\sin x + \cos x}{1 + \sqrt{\sin 2x}}\,dx$ [5 marks]

(d) $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)}$ [5 marks]

(e) $\displaystyle\int_0^{+\infty} (x+1)^2 e^{-x}\,dx$ [5 marks]

(f) $\displaystyle\int \frac{x^2 - 2x - 1}{(x^2 - 2x + 1)(x^2 + 1)}\,dx$ [5 marks]

My work :

(a) Since $\displaystyle\lim_{x\to 1}\int_x^1 e^{t^2}\,dt = 0$ and $\ln 1 = 0$, by l'Hôpital's rule,

$$\lim_{x\to 1} \frac{\int_x^1 e^{t^2}\,dt}{\ln x} = \lim_{x\to 1} \frac{-e^{x^2}}{1/x} = -e$$

(b) Let $I = \int \frac{1}{\sqrt{x^2+x+1}}\,dx = \int \frac{1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,dx$

Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta,\ dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta,\ \sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sec\theta$

$\therefore\ I = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{\sqrt{3}}{2}\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C$

(c) $\sin 2x = 2\sin x\cos x = 1-(\sin x-\cos x)^2$

Let $u = \sin x - \cos x,\ du = (\sin x+\cos x)\,dx,\ \sqrt{\sin 2x} = \sqrt{1-u^2}$

Let $I = \int \frac{\sin x+\cos x}{1+\sqrt{\sin 2x}}\,dx = \int \frac{du}{1+\sqrt{1-u^2}}$

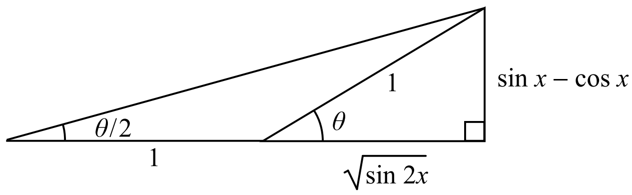
Let $u = \sin\theta,\ du = \cos\theta\,d\theta,\ \sqrt{1-u^2} = \cos\theta$

$\therefore\ I = \int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int\left(1-\frac{1}{2}\sec^2\frac{\theta}{2}\right)d\theta = \theta - \tan\frac{\theta}{2} + C$

$= \sin^{-1}(\sin x-\cos x) - \tan\frac{\sin^{-1}(\sin x-\cos x)}{2} + C$

Alternative Solution

$I = \theta - \tan\frac{\theta}{2} + C = \theta - \frac{\sin\theta}{1+\cos\theta} + C = \sin^{-1}(\sin x-\cos x) - \frac{\sin x-\cos x}{1+\sqrt{\sin 2x}} + C$



(d) Let $I = \int_1^{+\infty} \frac{dx}{\sqrt{x}\,(x+1)}$. Let $u = \sqrt{x},\ du = \frac{dx}{2\sqrt{x}}$.

When $x = 1$, $u = 1$; when $x \to +\infty$, $u \to +\infty$.

$\therefore\ I = 2\int_1^{+\infty} \frac{du}{u^2+1} = 2\lim_{b\to+\infty}\left[\tan^{-1}u\right]_1^b = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

(e) $\int_0^{+\infty}(x+1)^2 e^{-x}\,dx = \cdots = \lim_{b\to+\infty}\left[-(x+1)^2e^{-x} - 2(x+1)e^{-x} - 2e^{-x}\right]_0^b = 5$

(f) $\int \frac{x^2-2x-1}{(x^2-2x+1)(x^2+1)}\,dx = \cdots = \int\left(\frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{-x+1}{x^2+1}\right)dx$

$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$

Question 3.

Let $F(x) = \int_x^{x^2} e^{-\frac{t^2}{2}}\, dt$.

(a) Find $F'(0)$. [5 marks]

(b) Find $F''(0)$. [5 marks]

(c) For F, find the Taylor polynomial of degree 2 at 0. [5 marks]

My work :

(a) $F'(x) = 2x\, e^{-\frac{x^4}{2}} - e^{-\frac{x^2}{2}} \;\Rightarrow\; F'(0) = -1$

(b) $F''(x) = 2e^{-\frac{x^4}{2}} + 2x\, e^{-\frac{x^4}{2}}(-2x^3) - e^{-\frac{x^2}{2}}(-2x) \;\Rightarrow\; F''(0) = 2$

(c) $F(0) = 0$

The Taylor polynomial of degree 2 for F at 0 is given by

$$T_2(x) = F(0) + F'(0)\,x + \frac{F''(0)}{2}x^2 = -x + x^2$$

Question 4.

(a) Find the arc length of the curve $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2}$, where $1 \le x \le 2$. [5 marks]

(b) Find the volume of the solid obtained by rotating the region bounded by $x = 1$, $x = 2$, $y = 0$ and the curve in (a), about the x-axis. [5 marks]

My work :

(a) $y' = \dfrac{x^3}{4} - \dfrac{1}{x^3} \;\Rightarrow\; 1 + (y')^2 = 1 + \dfrac{x^6}{16} - \dfrac{1}{2} + \dfrac{1}{x^6} = \left(\dfrac{x^3}{4} + \dfrac{1}{x^3}\right)^2$

Arc length of the curve $= \displaystyle\int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_1^2 = \frac{21}{16}$

(b) Volume of the solid $= \displaystyle\int_1^2 \pi y^2\, dx = \pi \int_1^2 \left(\frac{x^8}{256} + \frac{x^2}{16} + \frac{1}{4x^4}\right) dx$

$$= \pi\left[\frac{x^9}{2304} + \frac{x^3}{48} - \frac{1}{12x^3}\right]_1^2 = \frac{1015}{2304}\pi$$

Question 5.

(a) Let $A=\begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 2 & 3 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

(i) Find $B^{\mathsf{T}}A$. [5 marks]

(ii) Find $(A+A^{-1})\,B$. Show the details of your calculation. [5 marks]

(b) Consider the system of linear equations $\begin{bmatrix} 3 & -1 & 1 \\ 3 & 1 & 4a-1 \\ 3 & a & 21 \end{bmatrix}\mathbf{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, where a is a scalar.

Determine the values of a such that the system is

(i) inconsistent;

(ii) consistent with infinitely many solutions;

(iii) consistent with a unique solution.

Solve the system when it is consistent. [20 marks]

My work :

(a) (i) $B^{\mathsf{T}}A=\begin{bmatrix} 5 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{bmatrix}=\begin{bmatrix} 4 & -12 & -9 \\ -1 & 9 & 5 \\ 3 & -11 & -8 \\ 3 & -13 & -9 \end{bmatrix}$

(ii) $\det A=\begin{vmatrix} 1 & -4 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix}=-1$, $\operatorname{adj} A=\begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$, $\therefore\ A^{-1}=\dfrac{\operatorname{adj} A}{\det A}=\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

$A+A^{-1}=\begin{bmatrix} 3 & -2 & 0 \\ 2 & -6 & -3 \\ -2 & 8 & 5 \end{bmatrix} \Rightarrow (A+A^{-1})\,B=\begin{bmatrix} 11 & 8 & 7 & 4 \\ -11 & 4 & -3 & -2 \\ 21 & -2 & 7 & 4 \end{bmatrix}$

(b) $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 3 & 1 & 4a-1 & 2 \\ 3 & a & 21 & 3 \end{array}\right]\to\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 4a-2 & 1 \\ 0 & a+1 & 20 & 2 \end{array}\right]\to\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 4a-2 & 1 \\ 0 & 0 & -4a^2-2a+42 & 3-a \end{array}\right]$

(iii) The system is consistent with unique solution

$\Leftrightarrow\ -4a^2-2a+42\neq 0\ \Leftrightarrow\ a\neq -\dfrac{7}{2}, 3$

In this case, the system becomes

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 4a-2 & 1 \\ 0 & 0 & 1 & \frac{1}{4a+14} \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & 0 & \frac{4a+13}{4a+14} \\ 0 & 2 & 0 & \frac{8}{2a+7} \\ 0 & 0 & 1 & \frac{1}{4a+14} \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 0 & \frac{4a+21}{4a+14} \\ 0 & 1 & 0 & \frac{4}{2a+7} \\ 0 & 0 & 1 & \frac{1}{4a+14} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4a+21}{12a+42} \\ 0 & 1 & 0 & \frac{4}{2a+7} \\ 0 & 0 & 1 & \frac{1}{4a+14} \end{array}\right], \quad \therefore \ \mathbf{x} = \begin{bmatrix} \frac{4a+21}{12a+42} \\ \frac{4}{2a+7} \\ \frac{1}{4a+14} \end{bmatrix}$$

When $a = -\frac{7}{2}$, the system becomes $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & -16 & 1 \\ 0 & 0 & 0 & \frac{13}{2} \end{array}\right]$

When $a = 3$, the system becomes

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 6 & \frac{3}{2} \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{2} \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow a = -\frac{7}{2}$.

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow a = 3$.

In this case, $\mathbf{x} = \begin{bmatrix} \frac{1}{2} - 2t \\ \frac{1}{2} - 5t \\ t \end{bmatrix}$ where $t \in \mathbb{R}$

Alternative Solution

(iii) The system is consistent with a unique solution

$$\Leftrightarrow \begin{vmatrix} 3 & -1 & 1 \\ 3 & 1 & 4a-1 \\ 3 & a & 21 \end{vmatrix} = -12a^2 - 6a + 126 = -6(2a+7)(a-3) \neq 0 \Leftrightarrow a \neq -\frac{7}{2}, 3$$

In this case, by Cramer's rule, $\mathbf{x} := (x_1, x_2, x_3)^\mathsf{T}$, where

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 4a-1 \\ 3 & a & 21 \end{vmatrix}}{-6(2a+7)(a-3)} = \frac{-4a^2 - 9a + 63}{-6(2a+7)(a-3)} = \frac{4a+21}{6(2a+7)}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 3 & 2 & 4a-1 \\ 3 & 3 & 21 \end{vmatrix}}{-6(2a+7)(a-3)} = \frac{-24a + 72}{-6(2a+7)(a-3)} = \frac{4}{2a+7}$$

$$x_3 = \frac{\begin{vmatrix} 3 & -1 & 1 \\ 3 & 1 & 2 \\ 3 & a & 3 \end{vmatrix}}{-6(2a+7)(a-3)} = \frac{-3a + 9}{-6(2a+7)(a-3)} = \frac{1}{2(2a+7)}$$

If $a = -\frac{7}{2}$, the system becomes

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 3 & 1 & -15 & 2 \\ 3 & -\frac{7}{2} & 21 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & -16 & 1 \\ 0 & -5 & 40 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & -16 & 1 \\ 0 & 0 & 0 & 13 \end{array}\right]$$

If $a = 3$, the system becomes

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 3 & 1 & 11 & 2 \\ 3 & 3 & 21 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 10 & 1 \\ 0 & 4 & 20 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 2 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 6 & \frac{3}{2} \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{2} \\ 0 & 1 & 5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(i) The system is inconsistent $\Leftrightarrow a = -\frac{7}{2}$

(ii) The system is consistent with infinitely many solutions $\Leftrightarrow a = 3$. In this case,

$\mathbf{x} := (\frac{1}{2} - 2t, \frac{1}{2} - 5t, t)^\mathsf{T}$, where $t \in \mathbb{R}$