The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 1 2022/23 Semester 2

Question 1.

(30 marks) Let $f(x) = x^3 - 6x^2 - 15x,\ -\infty < x < \infty$

(a) Find all critical points and inflection points.

(b) Find all local maximizers and minimizers.

(c) Sketch the curve of $f(x) = x^3 - 6x^2 - 15x,\ -\infty < x < \infty$.

My work :

(a) $f'(x) = 3x^2 - 12x - 15 = 3\,(x+1)\,(x-4) = 0 \iff x = -1, 5$

The critical points of f are $x = -1, 5$

$f''(x) = 6x - 12 = 0 \iff x = 2$

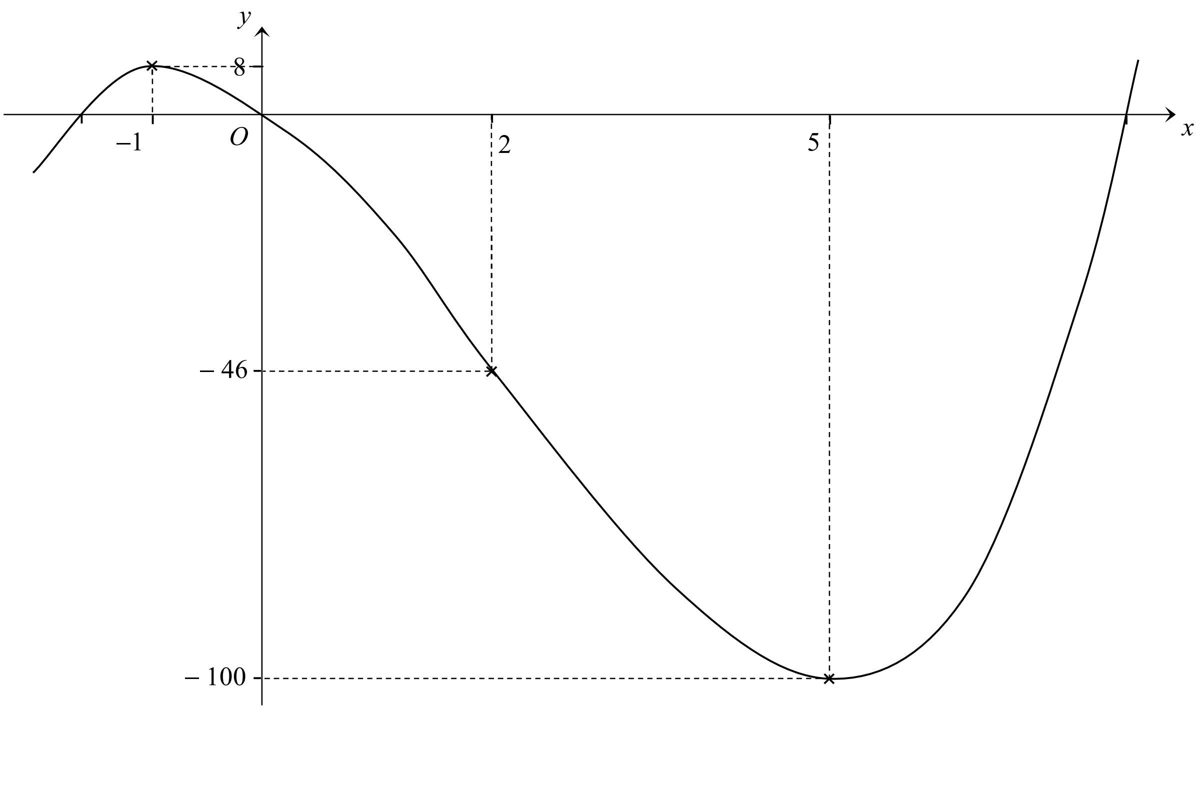
x	$(-\infty, -1)$	-1	$(-1, 2)$	2	$(2, 5)$	5	$(5, +\infty)$
f	↗	8	↘	-46	↘	-100	↗
f'	+	0	−	−	−	0	+
f''	−	−	−	0	+	+	+

The inflection point of f is $x = 2$ at which change of convexity occurs.

(b) f attains local minimum at $x = 5$ and $f(5) = -100$

f attains local maximum at $x = -1$ and $f(-1) = 8$

(c) y-intercept: $f(0) = 0$; x-intercepts: $f(x) = x^3 - 6x^2 - 15x = 0 \iff x \approx 0, 7.899, -1.899$



Question 2.

(30 marks) Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

My work :

Let $f(u) = 1 + \frac{1}{2}u - \sqrt{1+u}$, which is continuous on $[0, x]$ and differentiable in $(0, x)$, and

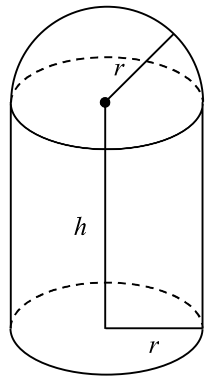
$f'(u) = \frac{1}{2} - \frac{\frac{1}{2}}{\sqrt{1+u}}$, $f''(u) = \frac{\frac{1}{4}}{(1+u)^{3/2}} > 0$ for $u \in (0, x)$ $\Rightarrow$ f' is strictly increasing on $(0, x)$.

Note that $f(0) = 0$. By Mean-Value Theorem, $\exists\, \xi \in (0, x)$ such that

$f(x) - f(0) = f'(\xi)(x-0) > f'(0)(x-0) = 0$, since $x > 0$; i.e., $\sqrt{1+x} < 1 + \frac{1}{2}x$, for all $x > 0$.

Question 3.

(40 marks) A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal? [Hint: Given a sphere of radius r, the sphere area is $S = 4\pi r^2$, and the sphere volume is $V = \frac{4}{3}\pi r^3$.]



My work :

Volume of the tank $= \frac{2}{3}\pi r^3 + \pi r^2 h = V \Rightarrow h = \frac{V}{\pi r^2} - \frac{2}{3}r \geq 0 \Rightarrow r \leq \sqrt[3]{\frac{V}{\frac{2}{3}\pi}} = \sqrt[3]{\frac{3V}{2\pi}}$

Surface area of the tank $= \pi r^2 + 2\pi r^2 + 2\pi r\, h = \frac{5}{3}\pi r^2 + \frac{2V}{r} =: S(r)$ on $r \in [0, \sqrt[3]{\frac{3V}{2\pi}}]$

$S'(r) = \frac{10}{3}\pi r - \frac{2V}{r^2} = 0 \Leftrightarrow r = \sqrt[3]{\frac{V}{\frac{5}{3}\pi}} = \sqrt[3]{\frac{3V}{5\pi}}$

R	$[0, \sqrt[3]{\frac{3V}{5\pi}})$	$\sqrt[3]{\frac{3V}{5\pi}}$	$(\sqrt[3]{\frac{3V}{5\pi}}, \sqrt[3]{\frac{3V}{2\pi}}]$
$S(r)$	↘	$\sqrt[3]{45V^2\pi}$	↗
$S'(r)$	$-$	0	+

When $r = \sqrt[3]{\frac{3V}{5\pi}}$, and $h = \frac{V}{\pi}\sqrt[3]{\frac{25\pi^2}{9V^2}} - \frac{2}{3}\sqrt[3]{\frac{3V}{5\pi}} = \frac{5}{3}\sqrt[3]{\frac{3V}{5\pi}} - \frac{2}{3}\sqrt[3]{\frac{3V}{5\pi}} = \sqrt[3]{\frac{3V}{5\pi}}$,

the tank requires the least amount of metal.