The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #03

Question 1 (*Basic Level*)

(a) (i) $\int (1-x)(1-2x)(1-3x)\,dx = \int (1-6x+11x^2-6x^3)\,dx = x - 3x^2 + \frac{11}{3}x^3 - \frac{3}{2}x^4 + C$

(ii) $\int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + C$

(b) (i) $\int \sin x\,dx = -\cos x + C$

(ii) $\int \cos x\,dx = \sin x + C$

(iii) $\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = \int \frac{d(-\cos x)}{\cos x} = \ln|\sec x| + C$

(iv) $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$

(v) $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \frac{d(\tan x + \sec x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$

(vi) $\int \csc x\,dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\,dx = -\ln|\csc x + \cot x| + C$

(c) (i) $\int e^x\,dx = e^x + C$

(ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

(iii) $\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$

Question 2 (*Beginner's Level*)

(a) $\int (3 + 3\sqrt{x})\,dx = 3x + 2x^{3/2} + C$

(b) $\int (3 - \cos x + 3x^2)\,dx = 3x - \sin x + x^3 + C$

(c) $\int \left(2e^x + \frac{3}{x^2}\right) dx = 2e^x - \frac{3}{x} + C$

(d) $\int \frac{2}{9 + x^2}\, dx = \frac{2}{3} \int \frac{d\left(\frac{x}{3}\right)}{1 + \left(\frac{x}{3}\right)^2} = \frac{2}{3} \tan^{-1} \frac{x}{3} + C$

(e) $\int \frac{1}{5 - 3x}\, dx = -\frac{1}{3} \ln |\, 5 - 3x \,| + C$

(f) $\int (2x - 3)^{10}\, dx = \frac{1}{22} (2x - 3)^{11} + C$

(g) $\int (3e^x + \frac{2}{x} - \sin 2)\, dx = 3e^x + 2 \ln |\, x \,| - x \sin 2 + C$

(h) $\int (x^3 - 2)\,(\frac{1}{x} - 5)\, dx = \int (-\, 5x^3 + x^2 + 10 - \frac{2}{x})\, dx = -\frac{5}{4} x^4 + \frac{x^3}{3} + 10x - 2 \ln |\, x \,| + C$

(i) $\int \frac{x^2 + 3x - 2}{\sqrt{x}}\, dx = \int (\, x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\,)\, dx = \frac{2}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C$

(j) $\int (2 + \tan^2 x)\, dx = \int (1 + \sec^2 x)\, dx = x + \tan x + C$

(k) $\int (2 \sin t - 2 \cos t + t^{\frac{5}{4}})\, dt = -\, 2 \cos t - 2 \sin t + \frac{4}{9} t^{\frac{9}{4}} + C$

(l) $\int \cos x\, (\tan x + \sec x)\, dx = \int (\sin x + 1)\, dx = -\cos x + x + C$

(m) $\int \frac{\sin x}{\sin 2x}\, dx = \frac{1}{2} \int \sec x\, dx = \frac{1}{2} \ln |\, \sec x + \tan x \,| + C$

(n)* $\int \sinh x\, dx = \cosh x + C$

(o)* $\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx = \int \frac{d\,(\cosh x)}{\cosh x} = \ln |\, \cosh x \,| + C$

(p)* $\int \operatorname{sech} x\, dx = \int \frac{2}{e^x + e^{-x}}\, dx = 2 \int \frac{d\,(e^x)}{e^{2x} + 1} = 2 \tan^{-1} e^x + C$

Question 3 (*Intermediate Level*)

(a) $\int \sec (2x - 3) \tan (2x - 3)\, dx = \frac{1}{2} \sec (2x - 3) + C$

(b) $\int 2 \cos^3 x \sin x\, dx = -2 \int \cos^3 x\, d\,(\cos x) = -\frac{\cos^4 x}{2} + C$

(c) $\int \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}\int \frac{d\,(x^2+2)}{(x^2+2)^2} = -\frac{1}{2\,(x^2+2)} + C$

(d) $\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int \tan^{-1} x\; d\,(\tan^{-1} x) = \frac{(\tan^{-1} x)^2}{2} + C$

(e) Let $I = \int x\sqrt{2x+3}\,dx$

Let $u = 2x+3 \Rightarrow x = \frac{u-3}{2} \Rightarrow dx = \frac{1}{2}\,du$

$\therefore\; I = \frac{1}{2}\int (u-3)\,u^{\frac{1}{2}}\,\frac{1}{2}\,du = \frac{1}{4}\int (u^{\frac{3}{2}} - 3u^{\frac{1}{2}})\,du = \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{2}u^{\frac{3}{2}} + C$

$= \frac{1}{10}(2x+3)^{\frac{5}{2}} - \frac{1}{2}(2x+3)^{\frac{3}{2}} + C$

(f) *Method 1*

Let $I = \int \frac{1}{\sqrt{x^2+1}}\,dx$

Let $x = \tan\theta,\quad dx = \sec^2\theta\,d\theta$

$\therefore\; I = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|x + \sqrt{x^2+1}| + C$

Method 2

Let $I = \int \frac{1}{\sqrt{x^2+1}}\,dx$

Let $x = \sinh u,\quad dx = \cosh u\,du$

$\therefore\; I = \int \frac{\cosh u}{\cosh u}\,du = \int du = u + C = \sinh^{-1} x + C$

(g) $\int \frac{1}{x\,(\ln x)^3}\,dx = \int \frac{d\,(\ln x)}{(\ln x)^3} = -\frac{1}{2\,(\ln x)^2} + C$

(h) $\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos(x^4+2)\,d\,(x^4+2) = \frac{1}{4}\sin(x^4+2) + C$

(i) $\int \sin 4x\cos 5x\,dx = \frac{1}{2}\int (\sin 9x - \sin x)\,dx = -\frac{1}{18}\cos 9x + \frac{1}{2}\cos x + C$

(j) Let $I = \int x^5 \sqrt{x^2+1}\, dx = \dfrac{1}{2}\int x^4 \sqrt{x^2+1}\, d\,(x^2)$

Let $u = x^2 + 1,\ du = d\,(x^2)\,,\ x^2 = u - 1$

$\therefore\ I = \dfrac{1}{2}\int (u-1)^2\, u^{\frac{1}{2}}\, du = \dfrac{1}{2}\int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\, du = \dfrac{1}{7}u^{\frac{7}{2}} - \dfrac{2}{5}u^{\frac{5}{2}} + \dfrac{1}{3}u^{\frac{3}{2}} + C$

$= \dfrac{1}{7}(x^2+1)^{\frac{7}{2}} - \dfrac{2}{5}(x^2+1)^{\frac{5}{2}} + \dfrac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

(k) Let $I = \int \dfrac{1}{\sqrt{4x - x^2}}\, dx = \int \dfrac{dx}{\sqrt{4 - (x-2)^2}}$

Let $x - 2 = 2\sin\theta,\quad dx = 2\cos\theta\, d\theta$

$\therefore\ I = \int d\theta = \theta + C = \sin^{-1}\dfrac{x-2}{2} + C$

(l) Let $I = \int x \sin(\ln x^2)\, dx$

Let $x = e^u,\ dx = e^u\, du$

$\therefore\ I = \int e^{2u}\sin 2u\, du = -\dfrac{1}{4}e^{2u}\cos 2u + \dfrac{1}{4}e^{2u}\sin 2u + C$

$= -\dfrac{1}{4}x^2\cos(\ln x^2) + \dfrac{1}{4}x^2\sin(\ln x^2) + C$

(m) $\int \dfrac{dx}{1+\cos x} = \int \dfrac{dx}{2\cos^2\frac{x}{2}} = \dfrac{1}{2}\int \sec^2\dfrac{x}{2}\, dx = \tan\dfrac{x}{2} + C$

(n) Let $I = \int \sqrt{x^2 + 8x + 6}\, dx = \int \sqrt{(x+4)^2 - 10}\, dx$

Let $x + 4 = \sqrt{10}\sec\theta,\ \sqrt{(x+4)^2 - 10} = \sqrt{10}\tan\theta$
$dx = \sqrt{10}\sec\theta\tan\theta\, d\theta$

$\therefore\ I = \int (\sqrt{10}\tan\theta)(\sqrt{10}\sec\theta\tan\theta\, d\theta) = 10\int \sec\theta\tan^2\theta\, d\theta$

$= \dfrac{10}{2}\sec\theta\tan\theta - \dfrac{10}{2}\ln|\sec\theta + \tan\theta| + C$

$= \dfrac{1}{2}(x+4)\sqrt{(x+4)^2 - 10} - 5\ln|x + 4 + \sqrt{(x+4)^2 - 10}| + C$

(o) *Method 1*

Let $I = \int \frac{dx}{\sqrt{2x^2 + 3x + 5}} = \int \frac{dx}{\sqrt{2\left(x + \frac{3}{4}\right)^2 + \frac{31}{8}}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{31}{16}}}$

Let $x + \frac{3}{4} = \frac{\sqrt{31}}{4} \tan\theta$, $dx = \frac{\sqrt{31}}{4} \sec^2\theta \, d\theta$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{31}{16}} = \frac{\sqrt{31}}{4} \sec\theta$$

$$\therefore\ I = \frac{1}{\sqrt{2}} \int \sec\theta \, d\theta = \frac{1}{\sqrt{2}} \ln|\tan\theta + \sec\theta| + C$$

$$= \frac{1}{\sqrt{2}} \ln\left| \frac{4}{\sqrt{31}}\left(x + \frac{3}{4}\right) + \frac{4}{\sqrt{31}} \sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{31}{16}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \ln|4x + 3 + \sqrt{(4x + 3)^2 + 31}| + C$$

Method 2

Let $I = \int \frac{dx}{\sqrt{2x^2 + 3x + 5}} = \int \frac{dx}{\sqrt{2\left(x + \frac{3}{4}\right)^2 + \frac{31}{8}}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{31}{16}}}$

Let $x + \frac{3}{4} = \frac{\sqrt{31}}{4} \sinh u$, $dx = \frac{\sqrt{31}}{4} \cosh u \, du$, $\sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{31}{16}} = \frac{\sqrt{31}}{4} \cosh u$

$$\therefore\ I = \frac{1}{\sqrt{2}} \int du = \frac{1}{\sqrt{2}} u + C = \frac{1}{\sqrt{2}} \sinh^{-1} \frac{4}{\sqrt{31}}\left(x + \frac{3}{4}\right) + C$$

(p) $\int \sin^2 x \cos 2x \, dx$

$$= \frac{1}{2} \int (1 - \cos 2x) \cos 2x \, dx = \frac{1}{2} \int \cos 2x \, dx - \frac{1}{4} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{4} \sin 2x - \frac{1}{4} x - \frac{1}{16} \sin 4x + C$$

(q) *Method 1*

Let $I = \int \frac{dx}{2x\sqrt{1 + (\ln x)^2}} = \frac{1}{2} \int \frac{d(\ln x)}{\sqrt{1 + (\ln x)^2}}$

Let $\ln x = \tan\theta$, $d(\ln x) = \sec^2\theta \, d\theta$, $\sqrt{1 + (\ln x)^2} = \sec\theta$

$$\therefore\ I = \frac{1}{2} \int \sec\theta \, d\theta = \frac{1}{2} \ln|\sec\theta + \tan\theta| + C = \frac{1}{2} \ln|\sqrt{1 + (\ln x)^2} + \ln x| + C$$

Method 2

Let $I = \int \dfrac{dx}{2x\sqrt{1+(\ln x)^2}} = \dfrac{1}{2}\int \dfrac{d(\ln x)}{\sqrt{1+(\ln x)^2}}$

Let $\ln x = \sinh u$, $d(\ln x) = \cosh u \, du$, $\sqrt{1+(\ln x)^2} = \cosh u$

$\therefore I = \dfrac{1}{2}\int du = \dfrac{1}{2}u + C = \dfrac{1}{2}\sinh^{-1}(\ln x) + C$

(r) $\int (3x+1)\cos(3x^2+2x-1)\,dx$

$= \dfrac{1}{2}\int \cos(3x^2+2x-1)\,d(3x^2+2x-1) = \dfrac{1}{2}\sin(3x^2+2x-1) + C$

(s) $\int \sin 2x \cos^2 3x \, dx$

$= \dfrac{1}{2}\int \sin 2x(1+\cos 6x)\,dx = \dfrac{1}{2}\int \sin 2x\,dx + \dfrac{1}{4}\int (\sin 8x - \sin 4x)\,dx$

$= -\dfrac{1}{4}\cos 2x - \dfrac{1}{32}\cos 8x + \dfrac{1}{16}\cos 4x + C$

(t) $\int 3x[\sec(x^2+2)]^3 \tan(x^2+2)\,dx$

$= \dfrac{3}{2}\int [\sec(x^2+2)]^3 \tan(x^2+2)\; d(x^2+2) = \dfrac{3}{2}\int [\sec(x^2+2)]^2\, d[\sec(x^2+2)]$

$= \dfrac{1}{2}[\sec(x^2+2)]^3 + C$

(u) Let $I = \int \dfrac{2x^2}{\sqrt{9-x^2}}\,dx$

Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore I = 2\int (3\sin\theta)^2\,d\theta = 9\int (1-\cos 2\theta)\,d\theta = 9\theta - \dfrac{9}{2}\sin 2\theta + C$

$= 9\sin^{-1}\dfrac{x}{3} - x\sqrt{9-x^2} + C$

(v) Let $I = \int \frac{x^2}{\sqrt{9x - x^2}}\,dx = \int \frac{x^2\,dx}{\sqrt{\left(\frac{9}{2}\right)^2 - \left(x - \frac{9}{2}\right)^2}}$

Let $x - \frac{9}{2} = \frac{9}{2}\sin\theta$, $dx = \frac{9}{2}\cos\theta\,d\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore\ I = \frac{81}{4}\int (\sin\theta + 1)^2\,d\theta = \frac{81}{4}\int (\sin^2\theta + 2\sin\theta + 1)\ d\theta$

$= \frac{81}{8}\int (1 - \cos 2\theta)\,d\theta - \frac{81}{2}\cos\theta + \frac{81}{4}\theta$

$= \frac{243}{8}\theta - \frac{81}{16}\sin 2\theta - \frac{81}{2}\cos\theta + C$

$= \frac{243}{8}\sin^{-1}\left(\frac{2}{9}x - 1\right) - \frac{1}{2}\left(x - \frac{9}{2}\right)\sqrt{9x - x^2} - 9\sqrt{9x - x^2} + C$

(w) $\int x^5\sqrt{x^3 - 1}\,dx = \frac{1}{3}\int x^3\sqrt{x^3 - 1}\ d(x^3) = \frac{1}{3}\int (u + 1)\,u^{\frac{1}{2}}\,du$ (put $u = x^3 - 1$)

$= \frac{1}{3}\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{15}(x^3 - 1)^{\frac{5}{2}} + \frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C$

(x) $\int \frac{\sqrt{1 + \sin x}}{\sec x}\,dx = \int \sqrt{1 + \sin x}\ d(1 + \sin x) = \frac{2}{3}(1 + \sin x)^{\frac{3}{2}} + C$

(y) Let $I = \int \sqrt{x^2 - 9}\,dx$

Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\ d\theta$, where $\theta \in [0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$

$\therefore\ I = 9\int \sec\theta\tan^2\theta\,d\theta = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C$

$= \frac{1}{2}x\sqrt{x^2 - 9} - \frac{9}{2}\ln|\sqrt{x^2 - 9} + x| + C$

Note that

$\int \sec^3\theta\,d\theta - \int \sec\theta\tan^2\theta\,d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C$

$\int \sec^3\theta\,d\theta + \int \sec\theta\tan^2\theta\,d\theta = \sec\theta\tan\theta + C$

(z) Let $I = \int \frac{x^2}{\sqrt{x^2-25}}\,dx$. Let $x = 5\sec\theta \Rightarrow dx = 5\sec\theta\tan\theta\,d\theta,\ \theta \in [0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$

$$\therefore I = \int 25\sec^3\theta\,d\theta = \frac{25}{2}\sec\theta\tan\theta + \frac{25}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}x\sqrt{x^2-25} + \frac{25}{2}\ln|x + \sqrt{x^2-25}| + C$$

Question 4 (*Intermediate Level*)

(a) $\int xe^x\,dx = xe^x - e^x + C$

(b) *Method 1*

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - x + C$$

Method 2

Let $I = \int \ln x\,dx$. Let $x = e^u$, $dx = e^u\,du$

$$\therefore I = \int ue^u\,du = ue^u - e^u + C = x\ln x - x + C$$

Remark Method 2 gives an idea on how to integrate inverse function.

(c) $\int x^2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

(d) $\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$

(e) $\int 2x\sec^2 3x\,dx = \frac{2}{3}\int x\,d(\tan 3x) = \frac{2}{3}x\tan 3x - \frac{2}{3}\int \tan 3x\,dx$

$$= \frac{2}{3}x\tan 3x - \frac{2}{9}\ln|\sec 3x| + C$$

(f) Let $I = \int \sec^3 x \, dx$

$\therefore \; I = \int \sec x \, d(\tan x) = \sec x \tan x - \int \tan x \cdot \sec x \tan x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \; = \sec x \tan x - I + \int \sec x \, dx$

$\therefore \; I = \dfrac{1}{2} \sec x \tan x + \dfrac{1}{2} \ln | \sec x + \tan x | + C$

(g) Let $I = \int \sin^{-1} x \, dx$

Let $x = \sin \theta, \; \theta \in [-\pi/2, \pi/2]$

$\therefore \; I = \int \theta \, d(\sin \theta) = \theta \sin \theta - \int \sin \theta \, d\theta \; = \; \theta \sin \theta + \cos \theta + C$

$= x \sin^{-1} x + \sqrt{1 - x^2} + C$

(h) Let $I = \int \cos^{-1} x \, dx$

Let $x = \cos \theta, \; \theta \in [0, \pi]$

$\therefore \; I = \int \theta \, d(\cos \theta) = \theta \cos \theta - \int \cos \theta \, d\theta \; = \; \theta \cos \theta - \sin \theta + C$

$= x \cos^{-1} x - \sqrt{1 - x^2} + C$

(i) Let $I = \int \tan^{-1} x \, dx$

Let $x = \tan \theta, \; \theta \in (-\pi/2, \pi/2)$

$\therefore \; I = \int \theta \, d(\tan \theta) = \theta \tan \theta - \int \tan \theta \, d\theta \; = \; \theta \tan \theta - \ln | \sec \theta | + C$

$= x \tan^{-1} x - \ln \sqrt{x^2 + 1} + C$

(j) Let $I = \int \cot^{-1} x \, dx$

Let $x = \cot\theta,\ \theta \in (-\pi/2, \pi/2)$

$\therefore\ I = \int \theta \, d(\cot\theta) = \theta\cot\theta - \int \cot\theta \, d\theta \ = \ \theta\cot\theta - \ln|\sin\theta| + C$

$= x\cot^{-1} x + \ln\sqrt{x^2+1} + C$

(k) Let $I = \int \sec^{-1} x \, dx$

Let $x = \sec\theta,\ \theta \in [0, \pi/2) \cup (\pi/2, \pi]$

$\therefore\ I = \int \theta \, d(\sec\theta) = \theta\sec\theta - \int \sec\theta \, d\theta \ = \ \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$

$$= \begin{cases} x\sec^{-1} x - \ln|x + \sqrt{x^2-1}| + C \\ \qquad \text{if the interval of integration} \subseteq [1, +\infty) \\ x\sec^{-1} x - \ln|x - \sqrt{x^2-1}| + C \\ \qquad \text{if the interval of integration} \subseteq (-\infty, -1] \end{cases}$$

(l) Let $I = \int \csc^{-1} x \, dx$

Let $x = \csc\theta,\ \theta \in [-\pi/2, 0) \cup (0, \pi/2]$

$\therefore\ I = \int \theta \, d(\csc\theta) = \theta\csc\theta - \int \csc\theta \, d\theta \ = \ \theta\csc\theta + \ln|\csc\theta + \cot\theta| + C$

$$= \begin{cases} x\csc^{-1} x + \ln|x + \sqrt{x^2-1}| + C \\ \qquad \text{if the interval of integration} \subseteq [1, +\infty) \\ x\csc^{-1} x + \ln|x - \sqrt{x^2-1}| + C \\ \qquad \text{if the interval of integration} \subseteq (-\infty, -1] \end{cases}$$

Remark For (k) and (l), students should be alert to the domain of θ.

(m) $\int (x^2 + 3x + 1)\, e^x \, dx \ = \ x^2 e^x + x e^x + C$

(n) $\int e^{2\sin x} \sin^2 x \cos x \, dx \ = \ \int (\sin x)^2 \, e^{2\sin x} \ d(\sin x)$

$$= \frac{1}{2} e^{2\sin x} \sin^2 x - \frac{1}{2} e^{2\sin x} \sin x + \frac{1}{4} e^{2\sin x} + C$$

(o) $\int x^2 (\ln x)^3\, dx = \int (\ln x)^3 e^{3 \ln x}\, d(\ln x)$

$$= \frac{1}{3} x^3 (\ln x)^3 - \frac{1}{3} x^3 (\ln x)^2 + \frac{2}{9} x^3 \ln x - \frac{2}{27} x^3 + C$$

(p) $\int (x \sin x - x^2 \cos x)\, dx = -x^2 \sin x - 3x \cos x + 3 \sin x + C$

(q) $\int e^{2x} \sin x \cos x\, dx = \frac{1}{2} \int e^{2x} \sin 2x\, dx = -\frac{1}{8} e^{2x} \cos 2x + \frac{1}{8} e^{2x} \sin 2x + C$

(r) Let $I = \int (3x^2 - 5x + 1) \ln x\, dx$. Let $x = e^u$, $dx = e^u\, du$

$\therefore I = \int (3u\, e^{3u} - 5u\, e^{2u} + u\, e^u)\, du$

$= u\, e^{3u} - \frac{1}{3} e^{3u} - \frac{5}{2} u\, e^{2u} + \frac{5}{4} e^{2u} + u\, e^u - e^u + C$

$= (x^3 - \frac{5}{2} x^2 + x) \ln x - \frac{1}{3} x^3 + \frac{5}{4} x^2 - x + C$

(s) Let $I = \int e^{\sqrt{t}}\, dt$. Let $t = u^2$, $dt = 2u\, du$

$\therefore I = \int 2u\, e^u\, du = 2u\, e^u - 2e^u + C = 2\sqrt{t}\, e^{\sqrt{t}} - 2\, e^{\sqrt{t}} + C$

(t) $\int x\, e^{3x} \sin 5x\, dx$

$= -\frac{5}{34} x\, e^{3x} \cos 5x + \frac{3}{34} x\, e^{3x} \sin 5x + \frac{15}{578} e^{3x} \cos 5x + \frac{4}{289} e^{3x} \sin 5x + C$

(u) Let $I = \int \sin \sqrt{x}\, dx$. Let $x = u^2$, $dx = 2u\, du$

$\therefore I = \int 2u \sin u\, du = -2u \cos u + 2 \sin u + C = -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C$

(v) Let $I = \int \sec^{-1} \sqrt{x}\, dx$. Let $x = u^2,\ dx = 2u\, du$

$\therefore\ I = \int 2u \sec^{-1} u\, du$. Let $u = \sec\theta,\ du = d(\sec\theta)$ where $\theta \in [0, \pi/2)$

$$\therefore\ I = 2\int \theta \sec\theta\, d(\sec\theta) = \int \theta\, d(\sec^2\theta) = \theta\sec^2\theta - \int \sec^2\theta\, d\theta$$

$$= \theta\sec^2\theta - \tan\theta + C = x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C$$

(w) Let $I = \int x^{\frac{3}{2}} \tan^{-1} x^{\frac{1}{2}}\, dx$. Let $x = u^2,\ dx = 2u\, du$

$\therefore\ I = \int 2u^4 \tan^{-1} u\, du$. Let $u = \tan\theta,\ du = d(\tan\theta)$ where $\theta \in [0, \pi/2)$

$$\therefore\ I = \int 2\theta(\tan\theta)^4\, d(\tan\theta) = \frac{2}{5}\int \theta\, d(\tan^5\theta) = \frac{2}{5}\theta\tan^5\theta - \frac{2}{5}\int \tan^5\theta\, d\theta$$

$$\int \tan^5\theta\, d\theta + \int \tan^3\theta\, d\theta = \int \tan^3\theta\sec^2\theta\, d\theta = \int \tan^3\theta\, d(\tan\theta) = \frac{\tan^4\theta}{4} + C$$

$$\int \tan^3\theta\, d\theta + \int \tan\theta\, d\theta = \int \tan\theta\sec^2\theta\, d\theta = \int \tan\theta\, d(\tan\theta) = \frac{\tan^2\theta}{2} + C$$

$$\therefore\ \int \tan^5\theta\, d\theta = \frac{\tan^4\theta}{4} - \frac{\tan^2\theta}{2} + \int \tan\theta\, d\theta = \frac{\tan^4\theta}{4} - \frac{\tan^2\theta}{2} + \ln|\sec\theta| + C$$

$$\therefore\ I = \frac{2}{5}\theta\tan^5\theta - \frac{1}{10}\tan^4\theta + \frac{1}{5}\tan^2\theta - \frac{2}{5}\ln|\sec\theta| + C$$

$$= \frac{2}{5}x^{\frac{5}{2}}\tan^{-1}x^{\frac{1}{2}} - \frac{1}{10}x^2 + \frac{1}{5}x - \frac{2}{5}\ln\sqrt{x+1} + C$$

(x) Let $I = \int x^2 \exp x^{\frac{3}{2}}\, dx$. Let $u = x^{\frac{3}{2}},\ du = \frac{3}{2}x^{\frac{1}{2}}\, dx$

$$\therefore\ I = \frac{2}{3}\int u\, e^u\, du = \frac{2}{3}u\, e^u - \frac{2}{3}e^u + C = \frac{2}{3}x^{\frac{3}{2}}\exp x^{\frac{3}{2}} - \frac{2}{3}\exp x^{\frac{3}{2}} + C$$

(y) Let $I = \int e^x \sin^{-1}(e^x)\, dx = \int \sin^{-1}(e^x)\, d(e^x)$. Let $e^x = \sin\theta$, where $\theta \in (0, \pi/2]$

$$\therefore\ I = \int \theta\, d(\sin\theta) = \theta\sin\theta - \int \sin\theta\, d\theta = \theta\sin\theta + \cos\theta + C$$

$$= e^x\sin^{-1}e^x + \sqrt{1-e^{2x}} + C$$

(z) $\int \ln(1+\sqrt{x})\ dx = x\ln(1+\sqrt{x}) - \int \frac{x}{1+\sqrt{x}}\ d(\sqrt{x})$

$$= x\ln(1+\sqrt{x}) - \int \left(\sqrt{x} - 1 + \frac{1}{1+\sqrt{x}}\right)\ d(\sqrt{x})$$

$$= (x-1)\ln(1+\sqrt{x}) - \frac{x}{2} + \sqrt{x} + C$$

Question 5 (*Exam Level*)

(a) $J_n = \int (\ln x)^n\ dx = x(\ln x)^n - \int x \cdot n(\ln x)^n \frac{1}{x}\, dx = x(\ln x)^n - n\, J_{n-1}$

$\therefore\quad J_3 = x(\ln x)^3 - 3\,J_2 = x(\ln x)^3 - 3x(\ln x)^2 + 6\,J_1$

$\quad = x(\ln x)^3 - 3x(\ln x)^2 + 6\,x\ln x - 6x + C$

(b) $I_n = \int x^2(\ln x)^n\ dx = \int (\ln x)^n\ d\left(\frac{x^3}{3}\right) = \frac{1}{3}x^3(\ln x)^n - \frac{1}{3}\int x^3 \cdot n\ (\ln x)^{n-1}\frac{1}{x}\,dx$

$$= \frac{1}{3}x^3(\ln x)^n - \frac{n}{3}I_{n-1}$$

$\therefore\quad I_3 = \frac{1}{3}x^3(\ln x)^3 - \frac{3}{3}I_2 = \frac{1}{3}x^3(\ln x)^3 - \frac{1}{3}x^3(\ln x)^2 + \frac{2}{3}I_1$

$$= \frac{1}{3}x^3(\ln x)^3 - \frac{1}{3}x^3(\ln x)^2 + \frac{2}{9}x^3\ln x - \frac{2}{27}x^3 + C$$

(c) $I_n = \int (1+x)^n \sin 2x\ dx = -\frac{1}{2}\int (1+x)^n\ d(\cos 2x)$

$$= -\frac{1}{2}(1+x)^n\cos 2x + \frac{n}{2}\int (1+x)^{n-1}\cos 2x\ dx$$

$$= -\frac{1}{2}(1+x)^n\cos 2x + \frac{n}{4}\int (1+x)^{n-1}\ d(\sin 2x)$$

$$= -\frac{1}{2}(1+x)^n\cos 2x + \frac{n}{4}(1+x)^{n-1}\sin 2x - \frac{n(n-1)}{4}\ I_{n-2}$$

$\therefore\quad I_4 = -\frac{1}{2}(1+x)^4\cos 2x + (1+x)^3\sin 2x - 3I_2$

$$= -\frac{1}{2}(1+x)^4\cos 2x + (1+x)^3\sin 2x + \frac{3}{2}(1+x)^2\cos 2x - \frac{3}{2}(1+x)\sin 2x$$

$$-\frac{3}{4}\cos 2x + C$$

Question 6 (*Standard Level*)

(a) $I_n = \int \sin^n x\, dx = -\int \sin^{n-1} x\, d(\cos x) = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$

$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$ for $n \geq 2$

$\therefore \quad I_n = -\dfrac{1}{n} \sin^{n-1} x \cos x + \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$

(b) $I_n = \int \cos^n x\, dx = \int \cos^{n-1} x\, d(\sin x) = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x\, dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$ for $n \geq 2$

$\therefore \quad I_n = \dfrac{1}{n} \cos^{n-1} x \sin x + \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$

(c) $I_n + I_{n-2} = \int (\tan^n x + \tan^{n-2} x)\, dx = \int \tan^{n-2} x \sec^2 x\, dx = \int \tan^{n-2} x\, d(\tan x)$ for $n \geq 2$

$\therefore \quad I_n = \int \tan^{n-2} x\, d(\tan x) - I_{n-2} = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ for $n \geq 2$.

(d) $I_n + I_{n-2} = \int (\cot^n x + \cot^{n-2} x)\, dx = \int \cot^{n-2} x \csc^2 x\, dx = -\int \cot^{n-2} x\, d(\cot x)\ \forall\ n \geq 2$

$\therefore \quad I_n = -\int \cot^{n-2} x\, d(\cot x) - I_{n-2} = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$ for $n \geq 2$.

(e) $I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \tan^2 x\, dx + I_{n-2} \Rightarrow I_n - \int \sec^{n-2} x \tan^2 x\, dx = I_{n-2}$ if $n \geq 2$

$\dfrac{d}{dx}(\sec^{n-2} x \tan x) = (n-2) \sec^{n-2} x \tan^2 x + \sec^n x$

$\Rightarrow I_n + (n-2) \int \sec^{n-2} x \tan^2 x\, dx = \sec^{n-2} x \tan x + C$ for $n \geq 2$

$\therefore \quad I_n = \dfrac{1}{n-1} \sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$ for $n \geq 2$

(f) $I_n = \int \csc^n x\, dx = \int \csc^{n-2} x \cot^2 x\, dx + I_{n-2} \Rightarrow I_n - \int \csc^{n-2} x \cot^2 x\, dx = I_{n-2}$ if $n \geq 2$

$\dfrac{d}{dx}(\csc^{n-2} x \cot x) = -(n-2) \csc^{n-2} x \cot^2 x - \csc^n x$

$\Rightarrow I_n + (n-2) \int \csc^{n-2} x \cot^2 x\, dx = -\csc^{n-2} x \cot x + C$ for $n \geq 2$

$$\therefore \quad I_n = -\frac{1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2} \quad \text{for } n \ge 2$$

Question 7 (*Exam Level*)

(a) $$\int x^m (\ln x)^n \, dx = \int (\ln x)^n \, d\left(\frac{x^{m+1}}{m+1}\right) = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1} \, dx$$

(b) $$\int \sin^m x \cos^n x \, dx = -\int \sin^{m-1} x \cos^n x \, d(\cos x) = -\int \sin^{m-1} x \, d\left(\frac{\cos^{n+1} x}{n+1}\right)$$

$$= -\frac{1}{n+1}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x \, dx$$

$$= -\frac{1}{n+1}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x \,(1 - \sin^2 x) \, dx$$

$$\frac{m+n}{n+1}\int \sin^m x \cos^n x \, dx = -\frac{1}{n+1}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x \, dx$$

$$\therefore \int \sin^m x \cos^n x \, dx = -\frac{1}{m+n}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x \, dx$$

Question 8 (*Standard Level*)

(a) $$\int \frac{3x+2}{x^2+1} \, dx = \frac{3}{2}\int \frac{d(x^2)}{x^2+1} + 2\int \frac{dx}{x^2+1} = \frac{3}{2}\ln(x^2+1) + 2\tan^{-1} x + C$$

(b) $$\int \frac{x^2-2x-1}{(x-1)(x^2+1)} \, dx = \int \left(\frac{2x}{x^2+1} - \frac{1}{x-1}\right) dx = \ln(x^2+1) - \ln|x-1| + C$$

(c) $$\int \frac{x-11}{x^2+3x-4} \, dx = \int \left(\frac{3}{x+4} - \frac{2}{x-1}\right) dx = 3\ln|x+4| - 2\ln|x-1| + C$$

(d) $$\int \frac{x^2}{(x-1)(x-2)^2} \, dx = \int \left(\frac{4}{(x-2)^2} + \frac{1}{x-1}\right) dx = -\frac{4}{x-2} + \ln|x-1| + C$$

(e) $$\int \frac{2x^2-x+4}{x^3+4x} \, dx = \int \left(\frac{x-1}{x^2+4} + \frac{1}{x}\right) dx = \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + \ln|x| + C$$

(f) $$\int \frac{2x^2+11x+33}{(2x-3)(4x^2+9)} \, dx = \int \left(\frac{\frac{3}{2}}{x-\frac{3}{2}} + \frac{-\frac{5}{4}x - \frac{1}{2}}{x^2+\frac{9}{4}}\right) dx$$

$$= \frac{3}{2}\ln\left|x - \frac{3}{2}\right| - \frac{5}{8}\ln\left(x^2 + \frac{9}{4}\right) - \frac{1}{3}\tan^{-1}\frac{2x}{3} + C$$

(g) Let $I = \int \dfrac{x + \sqrt{x}}{x + 1}\,dx$

Let $x = u^2,\ dx = 2u\,du$

$$\therefore\ I = \int \frac{2u^3 + 2u^2}{u^2 + 1}\,du = \int \left(2u + 2 - \frac{2u + 2}{u^2 + 1} \right) du = u^2 + 2u - \ln (u^2 + 1) - 2\tan^{-1} u + C$$

$$= x + 2\sqrt{x} - \ln (x + 1) - 2\tan^{-1} \sqrt{x} + C$$

(h) $$\int \frac{2x^3 - 3x^2 + 5x - 9}{2x^2 - 3x - 2}\,dx = \int \left(x + \frac{1}{x - 2} + \frac{5}{2x + 1} \right) dx$$

$$= \frac{x^2}{2} + \ln |\, x - 2 \,| + \frac{5}{2} \ln |\, 2x + 1 \,| + C$$

(i) *Method 1*

Let $$I = \int \frac{dx}{x^4 + 1} = \int \frac{dx}{(x^2 + 1)^2 - 2x^2} = \int \frac{dx}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)}$$

Let $$\frac{1}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} \equiv \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}$$

$$1 \equiv (Ax + B)\,(x^2 - \sqrt{2}x + 1) + (Cx + D)\,(x^2 + \sqrt{2}x + 1)$$

From which we obtain

$A + C = 0$

$B + D = 1$

$-\sqrt{2}A + B + \sqrt{2}C + D = 0 \ \Rightarrow\ A - C = \frac{1}{\sqrt{2}}$

$A - \sqrt{2}B + C + \sqrt{2}D = 0 \ \Rightarrow\ B - D = 0$

$\therefore\quad A = \dfrac{1}{2\sqrt{2}},\ B = \dfrac{1}{2},\ C = -\dfrac{1}{2\sqrt{2}},\ D = \dfrac{1}{2}$

$$\therefore\ I = \int \left(\frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{(x + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} - \frac{\frac{1}{2\sqrt{2}}x - \frac{1}{2}}{(x - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \right) dx$$

$$= \frac{1}{4\sqrt{2}} \ln (x^2 + \sqrt{2}x + 1) + \frac{1}{2\sqrt{2}} \tan^{-1} (\sqrt{2}x + 1)$$

$$- \frac{1}{4\sqrt{2}} \ln (x^2 - \sqrt{2}x + 1) + \frac{1}{2\sqrt{2}} \tan^{-1} (\sqrt{2}x - 1) + C$$

Method 2 – Using Complex Numbers

Let $I = \int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2 - 2x^2} = \int \frac{dx}{(\,(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}\,)(\,(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}\,)}$

Let $\frac{1}{(\,(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}\,)(\,(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}\,)} \equiv \frac{A\,(x+\frac{\sqrt{2}}{2})+B}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{C\,(x-\frac{\sqrt{2}}{2})+D}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}$

$$\frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} \equiv A\,(x+\frac{\sqrt{2}}{2})+B+\frac{C\,(x-\frac{\sqrt{2}}{2})+D}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\left(\,(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}\right)$$

Put $x = -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, we have $\frac{1}{4}+\frac{1}{4}i \;= i\frac{\sqrt{2}}{2}A+B \Rightarrow\; A=\frac{1}{2\sqrt{2}},\; B=\frac{1}{4}$

$$\frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} \equiv \frac{A\,(x+\frac{\sqrt{2}}{2})+B}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\left(\,(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}\right)+C\,(x-\frac{\sqrt{2}}{2})+D$$

Put $x = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, we have $\frac{1}{4}-\frac{1}{4}i \;= i\frac{\sqrt{2}}{2}C+D \Rightarrow\; C=-\frac{1}{2\sqrt{2}},\, D=\frac{1}{4}$

$$\therefore\; I = \int\left(\frac{\frac{1}{2\sqrt{2}}(x+\frac{\sqrt{2}}{2})+\frac{1}{4}}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{-\frac{1}{2\sqrt{2}}(x-\frac{\sqrt{2}}{2})+\frac{1}{4}}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\ln(x^2+\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x+1)$$

$$-\frac{1}{4\sqrt{2}}\ln(x^2-\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x-1)+C$$

(j) $\int \frac{x^3}{x^4+1}dx = \frac{1}{4}\int\frac{d\,(x^4+1)}{x^4+1} = \frac{1}{4}\ln(x^4+1)+C$

Remark You may try using partial fractions to solve (j). I am sure you can still get the same result.

(k) $\int \frac{x^2-2}{x^2+1}dx = \int\left(1-\frac{3}{x^2+1}\right)dx \;= x-3\tan^{-1}x+C$

(l) $\int \frac{1}{x^5+2x^3+x}dx = \int\left(\frac{1}{x}-\frac{x}{(x^2+1)^2}-\frac{x}{x^2+1}\right)dx$

$$= \ln|x|+\frac{1}{2(x^2+1)}-\frac{1}{2}\ln(x^2+1)+C$$

(m) $\displaystyle\int \frac{2x^3 + 3x^2 + 4}{(x+1)^4}\,dx$

$$= \int \frac{2\,(x+1)^3 - 3\,(x+1)^2 + 5}{(x+1)^4}\,dx \;=\; \int \left(\frac{2}{x+1} + \frac{-3}{(x+1)^2} + \frac{5}{(x+1)^4} \right) dx$$

$$= \;2 \ln|\,x+1\,| + \frac{3}{x+1} - \frac{5}{3(x+1)^3} + C$$

Rough work: (successive Horner method)

-1	2	3	0	4
		-2	-1	1
-1	2	1	-1	$\boxed{5}$
		-2	1	
-1	2	-1	$\boxed{0}$	
		-2		
	2	$\boxed{-3}$		

(n) $$\int \frac{x^3 + x^2 + 2x + 1}{x^4 + 2x^2 + 1}\,dx \;=\; \int \frac{x^3 + x^2 + 2x + 1}{(x^2+1)^2}\,dx \;=\; \int \frac{(x+1)(x^2+1) + x}{(x^2+1)^2}\,dx$$

$$= \int \left(\frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx \;=\frac{1}{2}\ln(x^2+1) + \tan^{-1} x - \frac{1}{2(x^2+1)} + C$$

(o) $$\int \frac{5e^{-x}}{e^{-2x} + 4e^{-x} + 3}\,dx = \int \frac{-5}{(e^{-x})^2 + 4e^{-x} + 3}\,d\,(e^{-x}) = \int \left(\frac{\frac{5}{2}}{e^{-x}+3} + \frac{-\frac{5}{2}}{e^{-x}+1} \right) d\,(e^{-x})$$

$$= \frac{5}{2}\ln(e^{-x}+3) - \frac{5}{2}\ln(e^{-x}+1) + C \;=\; \frac{5}{2}\ln\frac{e^{-x}+3}{e^{-x}+1} + C$$

(p) $$\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx \;=\; \int \frac{e^x\,d\,(e^x)}{(e^x)^2 + 3e^x + 2} \;=\; \int \left(\frac{-1}{e^x+1} + \frac{2}{e^x+2} \right) d\,(e^x)$$

$$= 2\ln(e^x+2) - \ln(e^x+1) + C$$

(q) $$\int \frac{2\sin\theta}{\cos^2\theta + \cos\theta - 2}\,d\theta \;=\; \int \frac{-\,2\,d\,(\cos\theta)}{(\cos\theta)^2 + \cos\theta - 2} \;=\; \int \left(\frac{-\frac{2}{3}}{\cos\theta - 1} + \frac{\frac{2}{3}}{\cos\theta + 2} \right) d\,(\cos\theta)$$

$$= \;\frac{2}{3}\ln\frac{\cos\theta + 2}{1 - \cos\theta} + C$$

(r) $\displaystyle\int \frac{9\sec\theta}{1+\sin\theta}\,d\theta = \int \frac{9\,d(\sin\theta)}{(1-\sin\theta)(1+\sin\theta)^2}$

$$= \int \left(\frac{-\frac{9}{4}}{\sin\theta - 1} + \frac{\frac{9}{4}}{\sin\theta + 1} + \frac{\frac{9}{2}}{(\sin\theta+1)^2} \right) d(\sin\theta)$$

$$= \frac{9}{4}\ln\frac{1+\sin\theta}{1-\sin\theta} - \frac{9}{2(\sin\theta+1)} + C$$

(s) $\displaystyle\int \frac{dx}{1+e^x+e^{-x}} = \int \frac{d(e^x)}{(e^x)^2+e^x+1} = \int \frac{d(e^x)}{(e^x+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2e^x+1}{\sqrt{3}} + C$

(t) $\displaystyle\int \frac{dx}{1+e^x} = \int \frac{d(e^x)}{(e^x)^2+e^x} = \int \left(\frac{1}{e^x} + \frac{-1}{e^x+1} \right) d(e^x) = x - \ln(e^x+1) + C$

Question 9 (*AMA1500 Midterm Past Paper*)

By long division,

$$\frac{-3x^6+9x^5-12x^4+18x^3-22x^2+14x-6}{x^4-3x^3+4x^2-3x+1}$$

$$= -3x^2 + \frac{9x^3-19x^2+14x-6}{(x-1)^2(x^2-x+1)} = -3x^2 + \frac{5}{x-1} - \frac{2}{(x-1)^2} + \frac{4(x-\frac{1}{2})+3}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= -3x^2 + \frac{5}{x-1} - \frac{2}{(x-1)^2} + \frac{4x+1}{x^2-x+1}$$

Question 10 (*Concept Discussion*)

Let $\displaystyle I = \int \frac{1+2x^2}{x^5(1+x^2)^3}\,dx = \int \frac{x+2x^3}{(x^4+x^2)^3}\,dx$. Let $u = x^4+x^2$, $du = (4x^3+2x)\,dx$

$\displaystyle\therefore \quad I = \frac{1}{2}\int \frac{du}{u^3} = -\frac{1}{4(x^4+x^2)^2} + C$

Question 11 (*Challenging Level – Have Fun!*)

(a) Let $I = \int \frac{3 + \cos\theta}{2 - \cos\theta}\,d\theta$. Let $t = \tan\frac{\theta}{2}$, $\cos\theta = \frac{1 - t^2}{1 + t^2}$, $d\theta = \frac{2\,dt}{1 + t^2}$

$$\therefore\ I = \int \frac{3 + \frac{1 - t^2}{1 + t^2}}{2 - \frac{1 - t^2}{1 + t^2}}\,\frac{2\,dt}{1 + t^2} = \int \frac{8 + 4t^2}{(1 + t^2)(1 + 3t^2)}\,dt = \int \left(\frac{-2}{t^2 + 1} + \frac{\frac{10}{3}}{t^2 + \frac{1}{3}} \right) dt$$

$$= -2\tan^{-1} t + \frac{10}{\sqrt{3}}\tan^{-1}\sqrt{3}t + C = -\theta + \frac{10}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{\theta}{2}\right) + C$$

Remark: The technique $t = \tan\frac{\theta}{2}$ is called **Weierstrass' substitution**. Through this technique, one can turn <u>some</u> nontrivial trigonometric integration problems into rational function integration problems, which can be solved in theory by resolving the integrand into partial fractions. To use Weierstrass' substitution $t = \tan\frac{\theta}{2}$, one should remember:

$$\sin\theta = \frac{2t}{1 + t^2},\quad \cos\theta = \frac{1 - t^2}{1 + t^2},\quad \tan\theta = \frac{2t}{1 - t^2},\quad d\theta = \frac{2\,dt}{1 + t^2}$$

(b) Let $I = \int x\left(\frac{1 - x^2}{1 + x^2}\right)^{1/2} dx = \int \frac{x\sqrt{1 - x^4}}{1 + x^2}\,dx$. Let $x^2 = \sin\theta$, $2x\,dx = \cos\theta\,d\theta$, $\theta \in [0, \pi/2]$

$$\therefore\ I = \frac{1}{2}\int \frac{\cos\theta}{1 + \sin\theta}\cdot\cos\theta\,d\theta = \frac{1}{2}\int (1 - \sin\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{2}\cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}(x^2) + \frac{1}{2}\sqrt{1 - x^4} + C$$

(c) Let $I = \int \sqrt{1 + \sin t}\,dt$. Let $t = 2\theta$, $dt = 2\,d\theta$.

$$\therefore\ I = \int \sqrt{\sin^2\theta + \cos^2\theta + \sin 2\theta}\cdot 2\,d\theta = 2\int |\cos\theta + \sin\theta|\,d\theta$$

$$= \begin{cases} 2\int (\cos\theta + \sin\theta)\,d\theta & \text{if the interval of integration} \subseteq \{\,\theta : \cos\theta + \sin\theta \ge 0\,\} \\ -2\int (\cos\theta + \sin\theta)\,d\theta & \text{if the interval of integration} \subseteq \{\,\theta : \cos\theta + \sin\theta \le 0\,\} \end{cases}$$

$$= \begin{cases} 2\left(\sin\frac{t}{2} - \cos\frac{t}{2}\right) + C & \text{if the interval of integration} \subseteq \{\,t : \cos\frac{t}{2} + \sin\frac{t}{2} \ge 0\,\} \\ 2\left(\cos\frac{t}{2} - \sin\frac{t}{2}\right) + C & \text{if the interval of integration} \subseteq \{\,t : \cos\frac{t}{2} + \sin\frac{t}{2} \le 0\,\} \end{cases}$$

(d) $\int \sqrt{1+\cos t}\, dt = \int \sqrt{2}\,|\cos \frac{t}{2}|\, dt$

$$= \begin{cases} 2\sqrt{2}\int \cos\frac{t}{2}\, d\left(\frac{t}{2}\right) & \text{if the interval of integration} \subseteq \{\, t : \cos\frac{t}{2} \geq 0 \,\} \\ -2\sqrt{2}\int \cos\frac{t}{2}\, d\left(\frac{t}{2}\right) & \text{if the interval of integration} \subseteq \{\, t : \cos\frac{t}{2} \leq 0 \,\} \end{cases}$$

$$= \begin{cases} 2\sqrt{2}\sin\frac{t}{2} + C & \text{if the interval of integration} \subseteq \{\, t : \cos\frac{t}{2} \geq 0 \,\} \\ -2\sqrt{2}\sin\frac{t}{2} + C & \text{if the interval of integration} \subseteq \{\, t : \cos\frac{t}{2} \leq 0 \,\} \end{cases}$$

(e) Let $I = \int \frac{d\theta}{2 + 2\cos\theta + \sin\theta}$. Let $t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $d\theta = \frac{2\,dt}{1+t^2}$

$$\therefore\ I = \int \frac{\frac{2\,dt}{1+t^2}}{2 + \frac{2-2t^2}{1+t^2} + \frac{2t}{1+t^2}} = \int \frac{2\,dt}{4+2t} = \int \frac{dt}{2+t} = \ln|\,t+2\,| + C = \ln\left|\,\tan\frac{\theta}{2} + 2\,\right| + C$$

(f) Let $I = \int \frac{d\theta}{2 + 2\sin\theta + \cos\theta}$. Let $t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $d\theta = \frac{2\,dt}{1+t^2}$

$$\therefore\ I = \int \frac{\frac{2\,dt}{1+t^2}}{2 + \frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int \frac{2\,dt}{2+4t} = \int \frac{dt}{1+2t} = \frac{1}{2}\ln|\,2t+1\,| + C$$

$$= \frac{1}{2}\ln\left|\,2\tan\frac{\theta}{2} + 1\,\right| + C$$

(g) $\int \ln(x^2+x+1)\, dx$

$$= x\ln(x^2+x+1) - \int \frac{x(2x+1)\,dx}{x^2+x+1} = x\ln(x^2+x+1) - \int\left(2 + \frac{-(x+\frac{1}{2}) - \frac{3}{2}}{(x+\frac{1}{2})^2 + \frac{3}{4}}\right)dx$$

$$= x\ln(x^2+x+1) - 2x + \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

(h) $\int \frac{\tan^{-1} x}{x^2}\,dx = \int \tan^{-1} x \, d\left(-\frac{1}{x}\right) = -\frac{\tan^{-1} x}{x} + \int \frac{dx}{x\,(x^2+1)}$

$= -\frac{\tan^{-1} x}{x} + \int \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx = -\frac{\tan^{-1} x}{x} + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$

(i) $\int \frac{\tan^{-1} x}{(x-1)^3}\,dx = \int \tan^{-1} x \, d\left(-\frac{1}{2\,(x-1)^2}\right) = -\frac{\tan^{-1} x}{2\,(x-1)^2} + \frac{1}{2}\int \frac{dx}{(x-1)^2(x^2+1)}$

$= -\frac{\tan^{-1} x}{(x-1)^2} + \int \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{(x-1)^2} + \frac{\frac{1}{4}x}{x^2+1}\right) dx$

$= -\frac{\tan^{-1} x}{(x-1)^2} - \frac{1}{4}\ln|x-1| - \frac{1}{4\,(x-1)} + \frac{1}{8}\ln(x^2+1) + C$

(j) Let $I = \int \frac{\sin^{-1} x}{x^2}\,dx$. Let $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ \theta \in [-\pi/2, \pi/2]$

$\therefore\ I = \int \theta \cot\theta \csc\theta\,d\theta = -\int \theta\,d(\csc\theta) = -\theta\csc\theta + \int \csc\theta\,d\theta$

$= -\theta\csc\theta - \ln|\csc\theta + \cot\theta| + C = -\frac{\sin x}{x} - \ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$

(k) Let $I = \int \frac{\sqrt{1+\sin^2 x}}{\sec x \csc x}\,dx = \int (\sin x)\sqrt{1+\sin^2 x}\,d(\sin x)$

Let $\sin x = \tan\theta,\ d(\sin x) = \sec^2\theta\,d\theta,\ \theta \in [-\pi/4, \pi/4]$

$\therefore\ I = \int \tan\theta\sec^3\theta\,d\theta = \int \sec^2\theta\,d(\sec\theta) = \frac{1}{3}\sec^3\theta + C = \frac{1}{3}(1+\sin^2 x)^{\frac{3}{2}} + C$

(l) $\int \frac{e^{\sqrt{\sin x}}}{(\sec x)\sqrt{\sin x}}\,dx = \int e^{\sqrt{\sin x}} \frac{d(\sin x)}{\sqrt{\sin x}} = 2\int e^{\sqrt{\sin x}}\ d(\sqrt{\sin x}) = 2e^{\sqrt{\sin x}} + C$

(m) Let $I = \int \sqrt{\tan\theta}\ d\theta$. Let $u^2 = \tan\theta,\ 2u\,du = \sec^2\theta\,d\theta = (1+u^4)\,d\theta$

$\therefore\ I = \int \frac{2u^2}{1+u^4}\,du = \int \frac{2u^2}{((u+\frac{\sqrt{2}}{2})^2 + \frac{1}{2})((u-\frac{\sqrt{2}}{2})^2+\frac{1}{2})}\,du$

Let $\frac{2u^2}{((u+\frac{\sqrt{2}}{2})^2 + \frac{1}{2})((u-\frac{\sqrt{2}}{2})^2+\frac{1}{2})} \equiv \frac{A\,(u+\frac{\sqrt{2}}{2}) + B}{(u+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{C\,(u-\frac{\sqrt{2}}{2}) + D}{(u-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}$

Put $u = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, we have $\frac{1}{2} - \frac{1}{2}i = i\frac{\sqrt{2}}{2}A + B \Rightarrow A = -\frac{1}{\sqrt{2}},\ B = \frac{1}{2}$

Put $u = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, we have $\frac{1}{2} + \frac{1}{2}i \ = i\frac{\sqrt{2}}{2}C + D \ \Rightarrow \ C = \frac{1}{\sqrt{2}}, D = \frac{1}{2}$

$$\therefore \ I = \int \left(\frac{-\frac{1}{\sqrt{2}}(u + \frac{\sqrt{2}}{2}) + \frac{1}{2}}{(u + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} + \frac{\frac{1}{\sqrt{2}}(u - \frac{\sqrt{2}}{2}) + \frac{1}{2}}{(u - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}} \right) du$$

$$= -\frac{1}{2\sqrt{2}} \ln (u^2 + \sqrt{2}u + 1) + \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2}u + 1) + \frac{1}{2\sqrt{2}} \ln (u^2 - \sqrt{2}u + 1)$$

$$+ \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2}u - 1) + C$$

$$= -\frac{1}{2\sqrt{2}} \ln (\tan\theta + \sqrt{2\tan\theta} + 1) + \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2\tan\theta} + 1)$$

$$+ \frac{1}{2\sqrt{2}} \ln (\tan\theta - \sqrt{2\tan\theta} + 1) + \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2\tan\theta} - 1) + C$$

(n) Let $I = \int \frac{x^{1/3}}{x^{1/2} + x^{1/4}}\,dx$. Let $x = u^{12}$, $dx = 12u^{11}\,du$

$$\therefore \ I = \int \frac{12u^{15}\,du}{u^6 + u^3} = 12\int \frac{u^{12}\,du}{u^3 + 1} = 12\int \left(u^9 - u^6 + u^3 - 1 + \frac{1}{u^3 + 1} \right) du$$

$$= 12\int \left(u^9 - u^6 + u^3 - 1 + \frac{\frac{1}{3}}{u + 1} + \frac{-\frac{1}{3}(u - \frac{1}{2}) + \frac{1}{2}}{(u - \frac{1}{2})^2 + \frac{3}{4}} \right) du$$

$$= \frac{6}{5}u^{10} - \frac{12}{7}u^7 + 3u^4 - 12u + 4\ln|\,u + 1\,| - 2\ln(u^2 - u + 1) + 4\sqrt{3}\tan^{-1}\frac{2u - 1}{\sqrt{3}} + C$$

$$= \frac{6}{5}x^{\frac{5}{6}} - \frac{12}{7}x^{\frac{7}{12}} + 3x^{\frac{1}{3}} - 12x^{\frac{1}{12}} + 4\ln|\,x^{\frac{1}{12}} + 1\,|$$

$$- 2\ln(x^{\frac{1}{3}} - x^{\frac{1}{12}} + 1) + 4\sqrt{3}\tan^{-1}\frac{2x^{\frac{1}{12}} - 1}{\sqrt{3}} + C$$

Remark 12 is the least common multiple (LCM) of 2, 3 and 4.