The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 1 2023/24 Semester 2

Question 1.

(60 marks) Let $f(x) = x\,(x+4)^3,\ -\infty < x < \infty$

(a) Find the intervals on which the function is increasing or decreasing, also find the relative (i.e. local) extrema.

(b) Find the intervals on which the function concaves up or concaves down and identify the inflection points.

(c) Find the intercepts and asymptotes (if any) of the graph $y = f(x)$ and sketch the graph.

My work :

(a) $f'(x) = x \cdot 3\,(x+4)^2 + (x+4)^3 = 4\,(x+1)\,(x+4)^2 = 0 \iff x = -1, -4$ (double)

The critical points of f are $x = -1, -4$

x	$(-\infty, -4)$	-4	$(-4, -1)$	-1	$(-1, +\infty)$
f'	$-$	0	$-$	0	$+$
f	↘	0	↘	-27	↗

The interval where f is increasing is: $[-1, +\infty)$

The interval where f is decreasing is: $(-\infty, -1]$

f attains a local minimum at $x = -1$ and $f(-1) = -27$

f has no local maximum.

(b) $f''(x) = 4\,(x+4)^2 + 4\,(x+1) \cdot 2\,(x+4) = (12x+24)\,(x+4) = 0 \iff x = -2, -4$

x	$(-\infty, -4)$	-4	$(-4, -2)$	-2	$(-2, +\infty)$
f''	$+$	0	$-$	0	$+$
f	convex	0	concave	-16	convex

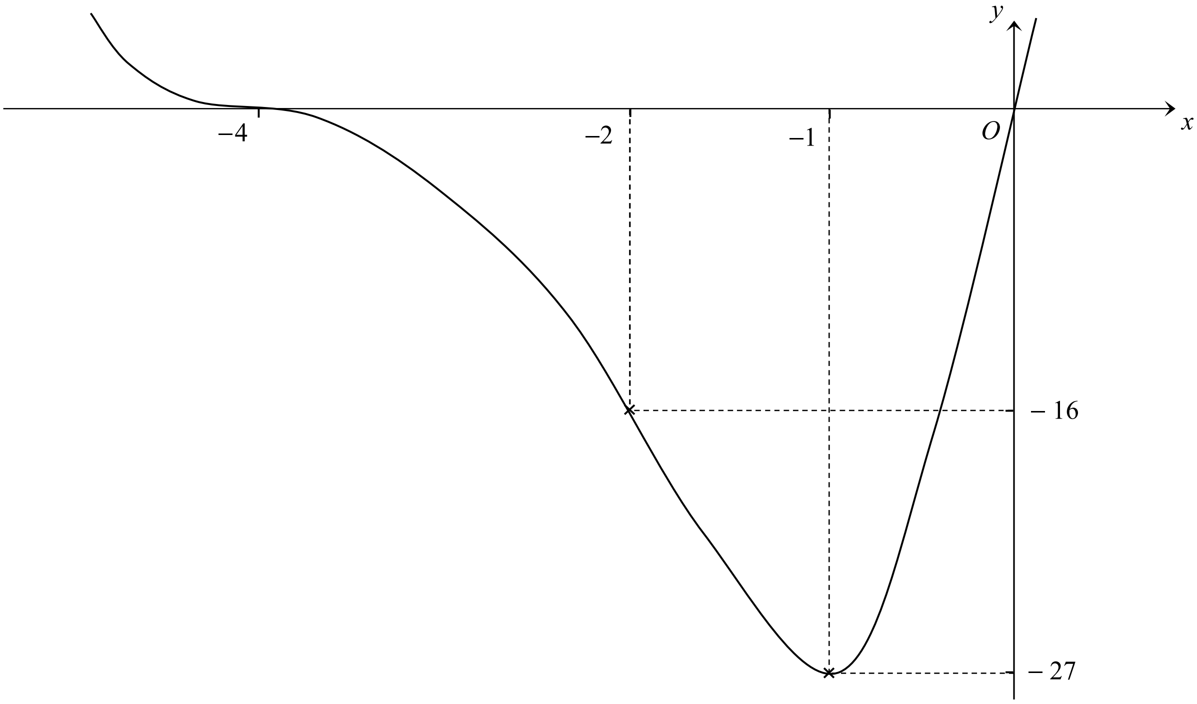
The intervals where f is concave up are: $(-\infty, -4], [-2, +\infty)$

The interval where f is concave down is: $[-4, -2]$

The inflection points of f are $x = -4, -2$ at which change of convexity occurs.

(c) y-intercept: $f(0) = 0$; x-intercepts: $f(x) = x\,(x+4)^3 = 0 \iff x = 0, -4$ (triple)

There are no vertical, horizontal and inclined asymptotes.



Remark:

	$x <$	-4	$< x <$	-2	$< x <$	-1	$< x$
f	↘	0	↘	-16	↘	-27	↗
f'	−	0	−	−	−	0	+
f''	+	−	−	0	+	+	+

Question 2.

(20 marks) You are asked to design a cylindrical container with total surface area of 10π m^2 including the top, the base and the side. Find the largest possible volume of the container. Justify your answer.

My work :

Surface area of the container $= 2\pi r^2 + 2\pi rh = 10\pi \;\Rightarrow\; r^2 + rh = 5 \;\Rightarrow\; h = \dfrac{5}{r} - r$

Volume of the container $= \pi r^2 h = \pi r^2\left(\dfrac{5}{r} - r\right) = \pi r\,(5 - r^2) =: V(r)$ on $r \in (0, \sqrt{5})$

$V'(r) = \pi\,(5 - r^2) + \pi r\,(-2r) = \pi\,(5 - 3r^2) = 0 \;\Leftrightarrow\; r = \sqrt{\dfrac{5}{3}}$

r	$(0, \sqrt{\frac{5}{3}})$	$\sqrt{\frac{5}{3}}$	$(\sqrt{\frac{5}{3}}, \sqrt{5})$
$V(r)$	↗	max	↘
$V'(r)$	$+$	0	$-$

The least possible value of the container $= V(\sqrt{\dfrac{5}{3}}) = \dfrac{10}{3}\sqrt{\dfrac{5}{3}}\,\pi \approx 13.5193$ m^3

Question 3.

(20 marks) Apply the Mean Value Theorem to prove that for all $1 \le a < b$,

$$\ln(b^2 + 1) - \ln(a^2 + 1) < b - a$$

My work :

Let $f(u) = \ln(u^2 + 1)$, which is continuous on $[a, b]$ and differentiable in (a, b), and

$f'(u) = \dfrac{2u}{u^2 + 1}$, $f''(u) = \dfrac{(u^2 + 1)\,2 - 2u\,(2u)}{(u^2 + 1)^2} = \dfrac{2\,(1 - u)\,(1 + u)}{(u^2 + 1)^2} < 0$ for all $u > 1$, which

implies f' is strictly decreasing in $(1, +\infty)$, i.e. $f'(u) < \lim\limits_{u \to 1^+} f'(u) = 1$ for all $u > 1$.

By Mean-Value Theorem, $\exists\, \xi \in (a, b)$ such that

$f(b) - f(a) = f'(\xi)\,(b - a) < b - a$ since $\xi > a > 1$.

Alternative Solution Skip blue part, and write:

Note that $\xi^2 + 1 - 2\xi = (\xi - 1)^2 > 0$ (since $\xi > a \ge 1 \Rightarrow \xi \ne 1$), i.e. $f'(\xi) = \dfrac{2\xi}{\xi^2 + 1} < 1$.

This implies that $\ln(b^2 + 1) - \ln(a^2 + 1) < b - a$ for all $1 \le a < b$.