The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Tutorial Set #08

Question 1. (*Concept Level*)

(a) If $\mathbf{b} = \mathbf{0}$, then we can pick any $\lambda \in \mathbb{R}$. If $\mathbf{b} \neq \mathbf{0}$, then $\langle\, \mathbf{a} - \lambda\, \mathbf{b}, \mathbf{b} \,\rangle = \langle\, \mathbf{a}, \mathbf{b} \,\rangle - \lambda \langle\, \mathbf{b}, \mathbf{b} \,\rangle = 0$
$\Rightarrow \quad \lambda = \langle\, \mathbf{a}, \mathbf{b} \,\rangle / \langle\, \mathbf{b}, \mathbf{b} \,\rangle$.

(b) WLOG consider $\mathbf{b} \neq \mathbf{0}$.

$\mathbf{a} - \lambda\, \mathbf{b}$

$$= \frac{(a_1, a_2)^{\mathsf{T}} (b_1^2 + b_2^2) - (b_1, b_2)^{\mathsf{T}} (a_1 b_1 + a_2 b_2)}{b_1^2 + b_2^2} = \frac{(\,(a_1 b_2 - a_2 b_1)\, b_2, (a_2 b_1 - a_1 b_2)\, b_1)^{\mathsf{T}}}{b_1^2 + b_2^2}$$

$\Rightarrow \quad \|\, \mathbf{b} \,\|\, \|\, \mathbf{a} - \lambda\, \mathbf{b} \,\| = |\, a_1 b_2 - a_2 b_1 \,| = |\, \det(\mathbf{a}, \mathbf{b}) \,|$

Hence $|\, \det(\mathbf{a}, \mathbf{b}) \,| = \|\, \mathbf{b} \,\|\, \|\, \mathbf{a} - \lambda\, \mathbf{b} \,\|$ = the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Question 2. (*Concept Level*)

(a) (i) $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\, \mathbf{b} \times \mathbf{a}.$$

(ii) $$(\alpha \mathbf{a}) \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha a_1 & \alpha a_2 & \alpha a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \alpha \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \alpha\, (\mathbf{a} \times \mathbf{b})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \end{vmatrix} = \mathbf{a} \times (\alpha \mathbf{b})$$

(iii) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

(b) Note that $\det(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}) = \|\, \mathbf{a} \times \mathbf{b} \,\|^2 \geq 0$, hence the direction of $\mathbf{a} \times \mathbf{b}$ is defined based on the right hand grip and is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ as depicted.

By definition or vector triple product, we have $\mathbf{b} \times (\mathbf{a} \times \mathbf{b}) = \|\, \mathbf{b} \,\|^2\, \mathbf{a} - \langle\, \mathbf{a}, \mathbf{b} \,\rangle\, \mathbf{b}$, hence or by direct expansion, $\|\, \mathbf{a} \times \mathbf{b} \,\|^2 = \|\, \mathbf{b} \,\|^2\, \|\, \mathbf{a} \,\|^2 - \langle\, \mathbf{a}, \mathbf{b} \,\rangle^2 = \|\, \mathbf{a} \,\|^2\, \|\, \mathbf{b} \,\|^2 \sin^2 \theta$, where θ is the included angle between $\mathbf{a}$ and $\mathbf{b}$.

Question 3. (*Basic Level*)

(a) $(2, 3, 6) \times (1, -4, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 6 \\ 1 & -4 & 0 \end{vmatrix} = (24, 6, -11)$

(b) $(1, -4, 0) \times (-3, 1, -2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 0 \\ -3 & 1 & -2 \end{vmatrix} = (8, 2, -11)$

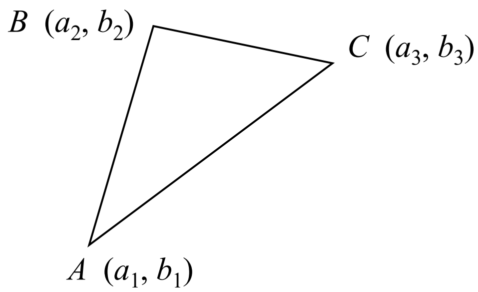
(c) $(2, 3, 6) \times (-3, 1, -2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 6 \\ -3 & 1 & -2 \end{vmatrix} = (-12, -14, 11)$

Question 4. (*Basic Level*)

(a) $(2, 3, 6) \cdot (1, -4, 0) = (2)(1) + (3)(-4) + (6)(0) = -10$

(b) $(1, -4, 0) \cdot (-3, 1, -2) = (1)(-3) + (-4)(1) + (0)(-2) = -7$

Question 5. (*Concept Level*)



Identify $A = (a_1, b_1, 0)$, $B = (a_2, b_2, 0)$, $C = (a_3, b_3, 0)$

Area of ΔABC

$$= \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 - a_1 & b_2 - b_1 & 0 \\ a_3 - a_1 & b_3 - b_1 & 0 \end{vmatrix} \right\| = \frac{1}{2} \left\| \begin{vmatrix} a_2 - a_1 & b_2 - b_1 \\ a_3 - a_1 & b_3 - b_1 \end{vmatrix} \mathbf{k} \right\|$$

$$= \frac{1}{2} \left| \begin{vmatrix} a_2 - a_1 & b_2 - b_1 \\ a_3 - a_1 & b_3 - b_1 \end{vmatrix} \right| = \frac{1}{2} \left| \begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix} \right|$$

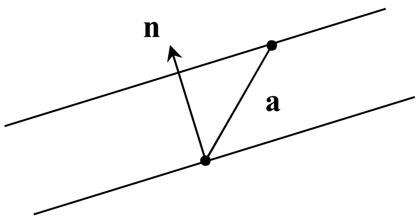
Question 6. (*Exam Level*)

(a) Let $\mathbf{p}_i, \mathbf{q}_i$ be points on $\Pi_i : 2x - 2y + z = \begin{cases} 5, & i = 1 \\ 20, & i = 2 \end{cases}$

$$\mathbf{n} \cdot \mathbf{p}_i = \mathbf{n} \cdot \mathbf{q}_i = \begin{cases} 5, & i = 1 \\ 20, & i = 2 \end{cases} \Leftrightarrow \mathbf{n} \cdot (\mathbf{p}_i - \mathbf{q}_i) \Leftrightarrow \mathbf{n} \perp \mathbf{p}_i - \mathbf{q}_i$$

Hence $\mathbf{n} = (2, -2, 1)$ is normal to Π_1 and Π_2 .

(b)



$(0, 0, 5) \in \Pi_1$ and $(0, 0, 20) \in \Pi_2$

Define $\mathbf{a} := (0, 0, 20) - (0, 0, 5) = (0, 0, 15)$

$$\text{Distance} = |\,\mathbf{a} \cdot \hat{\mathbf{n}}\,| = \left| \frac{(0, 0, 15) \cdot (2, -2, 1)}{\sqrt{2^2 + (-2)^2 + 1^2}} \right| = 5$$

Question 7. (*Standard Level*)

(a) $\hat{\mathbf{x}} = \dfrac{(2, 3, 6)}{\sqrt{2^2 + 3^2 + 6^2}} = \left(\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7} \right)$

(b) $\text{proj}_{\hat{\mathbf{x}}}\, \mathbf{y} = (\mathbf{y} \cdot \hat{\mathbf{x}})\, \hat{\mathbf{x}} = (1, -4, 0) \cdot \left(\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7} \right) \left(\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7} \right) = -\dfrac{10}{49}(2, 3, 6)$

$\text{proj}_{\hat{\mathbf{x}}}\, \mathbf{z} = (\mathbf{z} \cdot \hat{\mathbf{x}})\, \hat{\mathbf{x}} = (-3, 1, -2) \cdot \left(\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7} \right) \left(\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7} \right) = -\dfrac{15}{49}(2, 3, 6)$

(c) $\overrightarrow{PQ} = (-1, -7, -6), \quad \overrightarrow{PR} = (-5, -2, -8)$

$$\text{Area of } \Delta PQR = \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -7 & -6 \\ -5 & -2 & -8 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \sqrt{44^2 + 22^2 + (-33)^2} = \frac{\sqrt{3509}}{2}$$

Question 8. (*Standard Level*)

(a) Let $\mathbf{p} = (x_1, y_1)$, $\mathbf{q} = (x_2, y_2)$ be points on the line $ax + by + c = 0$.

$\mathbf{n} \cdot \mathbf{p} = ax_1 + by_1 = -c$ and $\mathbf{n} \cdot \mathbf{q} = ax_2 + by_2 = -c$

$\Leftrightarrow \quad \mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0 \quad \Leftrightarrow \quad \mathbf{n} \perp \mathbf{p} - \mathbf{q}$

Hence $\mathbf{n} = (a, b)$ is perpendicular to the line $ax + by + c = 0$.

(b) Let $P_1 = (x_1, y_1)$ be a point on the line $ax + by + c = 0$, $\overrightarrow{P_1P_0} = (x_0 - x_1, y_0 - y_1)$.

$$\text{The shortest distance} = |\overrightarrow{P_1P_0} \cdot \hat{\mathbf{n}}| = \left| \frac{(x_0 - x_1, y_0 - y_1) \cdot (a, b)}{\sqrt{a^2 + b^2}} \right|$$

$$= \left| \frac{ax_0 + by_0 - ax_1 - by_1}{\sqrt{a^2 + b^2}} \right| = \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$$

Question 9. (*Smart Level*)

(a) The equation of straight line is:

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & 5 & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \quad -4x + y + 7 = 0 \quad \Leftrightarrow \quad 4x - y - 7 = 0$$

The equation of the line passing through (2, 1) and (3, 5) vector form is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3-2 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$$

(b) The equation of plane is:

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 1 & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \quad x - y + 3z - 1 = 0$$

(c) The equation of circle is:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 1^2 + 0^2 & 1 & 0 & 1 \\ 2^2 + 1^2 & 2 & 1 & 1 \\ 3^2 + 5^2 & 3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 1 & 1 & 0 & 1 \\ 5 & 2 & 1 & 1 \\ 34 & 3 & 5 & 1 \end{vmatrix} = 0$$

$\Leftrightarrow \quad 3x^2 + 3y^2 + 13x - 25y - 16 = 0$