The Hong Kong Polytechnic University

Department of Applied Mathematics

AMA1120 Test 1 2020/21 Semester 2

Question 1.

(60 marks) Let $f(x) = 2e^{2x} - e^{x},\ -\infty < x < \infty$

(a) Find all critical points and inflection points.

(b) Find all local (i.e., relative) and global (i.e., absolute) maximum and minimum, if any.

(c) Find all intervals where the function is increasing, decreasing, concave-up (i.e., convex) or concave-down (i.e., concave).

(d) Find all asymptotes.

(e) Sketch the curve of the function $f(x)$.

My work :

(a) $f'(x) = 4e^{2x} - e^{x} = 0 \ \Leftrightarrow\ x = -\ln 4$

The critical point of f is: $x = -\ln 4$

$f''(x) = 8e^{2x} - e^{x} = 0 \ \Leftrightarrow\ x = -\ln 8$

x	$(-\infty, -\ln 8)$	$-\ln 8$	$(-\ln 8, -\ln 4)$	$-\ln 4$	$(-\ln 4, -\ln 2)$	$-\ln 2$	$(-\ln 2, +\infty)$
f	$\searrow$	$-\frac{3}{32}$	$\searrow$	$-\frac{1}{8}$	$\nearrow$	0	$\nearrow$
f'	$-$	$-$	$-$	0	$+$	$+$	$+$
f''	$-$	0	$+$	$+$	$+$	$+$	$+$

Change of convexity / concavity occurs at $x = -\ln 8$

The inflection point of f is: $x = -\ln 8$

x-intercept: $f(x) = 2e^{2x} - e^{x} = 0 \ \Leftrightarrow\ x = -\ln 2$

(b) f attains local minimum at $x = -\ln 4$ and $f(-1) = -\frac{1}{8}$

f has no local maximum

Since $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = +\infty$, f has no global maximum and

f attains global minimum at $x = -\ln 4$ with value $-\frac{1}{8}$.

(c) The interval where f is increasing is: $[-\ln 4, +\infty)$

The interval where f is decreasing is: $(-\infty, -\ln 4]$

The interval where f is convex is: $[-\ln 8, +\infty)$

The interval where f is concave is: $(-\infty, -\ln 8]$

(d) $\lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to -\infty} \dfrac{2e^{2x} - e^{x}}{x} = 0$, and $\lim\limits_{x \to -\infty} f(x) = 0$

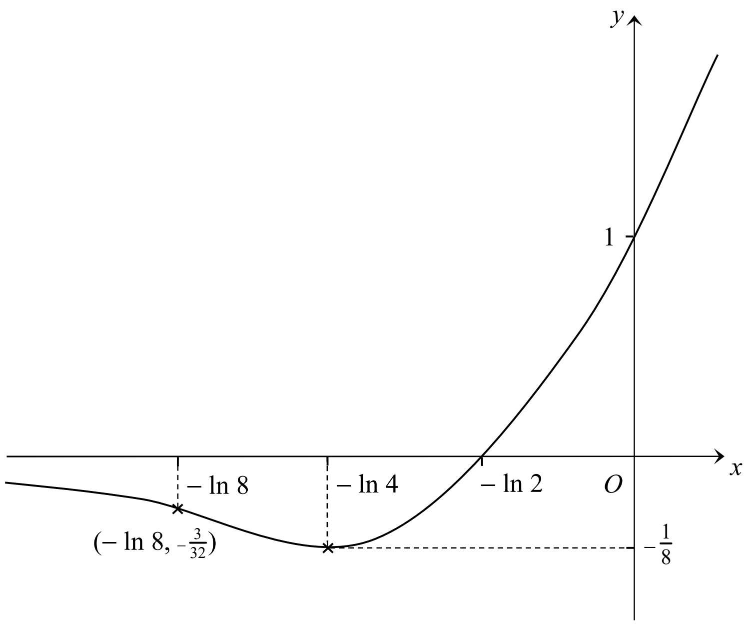
$\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to +\infty} \dfrac{2e^{2x} - e^{x}}{x} = \lim\limits_{x \to +\infty} (4e^{2x} - e^{x}) = +\infty$

$\Rightarrow$ $y = 0$ is a horizontal asymptote of $y = f(x)$, and f has no inclined asymptotes.

f is continuous on $(-\infty, +\infty)$ $\Rightarrow$ f has no vertical asymptotes.

(e) y-intercept: $f(0) = 2 - 1 = 1$

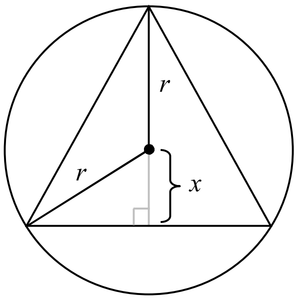
x-intercepts: $f(x) = 2e^{2x} - e^{x} = 0 \iff x = -\ln 2$ (see part (a))



Question 2.

(20 marks) Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r. Please specify the lengths of all the three sides.

My work :



Let x be the distance (taking downward positive) from the center of the inscribed circle to the base along the height of the isosceles triangle. Thus, $0 \le x < r$ (or $-r < x < r$).

Base length of the isosceles triangle $= 2\sqrt{r^2 - x^2}$

Area of the isosceles triangle $= \dfrac{1}{2}(r + x) \cdot 2\sqrt{r^2 - x^2} = (r + x)^{3/2}\,(r - x)^{1/2} =: A\,(x)$

$A'(x) = (r - 2x)\sqrt{\dfrac{r + x}{r - x}} = 0 \iff x = \dfrac{r}{2}$ (since $x \neq -r$)

x	$[0, \frac{r}{2})$	$\frac{r}{2}$	$(\frac{r}{2}, r)$
$A\,(x)$	↗	$\frac{\sqrt{3}r^2}{2}$	↘
$A'(x)$	$+$	0	$-$

or

x	$(-r, \frac{r}{2})$	$\frac{r}{2}$	$(\frac{r}{2}, r)$
$A\,(x)$	↗	$\frac{\sqrt{3}r^2}{2}$	↘
$A'(x)$	$+$	0	$-$

Thus, the area $A\,(x)$ attains global maximum at $x = \dfrac{r}{2}$, with the corresponding base length

$= 2\sqrt{r^2 - \left(\dfrac{r}{2}\right)^2} = \sqrt{3}\,r$, and the length of the equal leg $= \sqrt{(r + \frac{r}{2})^2 + \left(\dfrac{\sqrt{3}r}{2}\right)^2} = \sqrt{3}\,r$.

Question 3.

(20 marks) Let $f\,(x) = x^2 + |\,x - 1\,|$. Can you find a real constant c such that $f\,(2) - f\,(0) = 2f\,'(c)$, and why?

My work :

$f\,(2) = 5,\ f\,(0) = 1,\ \dfrac{f\,(2) - f\,(0)}{2 - 0} = 2.\ f\,'(x) = \begin{cases} 2x - 1, & \text{if } x < 1 \\ 2x + 1, & \text{if } x > 1 \end{cases}$

$f_-'(1) = \lim\limits_{x \to 1^-} (x + 1) + \dfrac{|\,x - 1\,|}{x - 1} = 1,\ f_+'(1) = \lim\limits_{x \to 1^+} (x + 1) + \dfrac{|\,x - 1\,|}{x - 1} = 3$

$\therefore\quad f_-'(1) \neq f_+'(1) \ \Rightarrow\ f\,'(1)$ does not exist.

Now assume $\exists\, c$ such that $f\,(2) - f\,(0) = 2f\,'(c) \iff f\,'(c) = 2$. First, $c \neq 1$ since $f\,'(1)$ does not exist. If $c < 1$, then $f\,'(c) = 2c - 1 = 2 \ \Rightarrow\ c = \frac{3}{2} > 1$, which is a contradiction. If $c > 1$, then $f\,'(c) = 2c + 1 = 2 \ \Rightarrow\ c = \frac{1}{2} < 1$, which is a contradiction again. Such c cannot exist.