The Hong Kong Polytechnic University
Department of Applied Mathematics
AMA1120 Tutorial Set #07

**Question 1** ( *Intermediate Level* )

Let $A$ be an $n \times n$ matrix, $n > 1$.

(a)* Show that $\operatorname{adj} A$ is nonsingular if and only if $A$ is nonsingular.

(b) Find $\det(\operatorname{adj} A)$.

(c) Show that if $A$ is nonsingular, then $(\operatorname{adj} A)^{-1} = \operatorname{adj}(A^{-1})$.

**Question 2** ( *Intermediate Level* )

Let $A = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & a \end{pmatrix}$.

(a) Find the determinant of $A$ in terms of $a$.

(b) Find all values of $a$ such that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Now suppose $a = 0$.

(c) Find the inverse of $A$.

(d) If $B$ is a $4 \times 4$ matrix such that $\operatorname{adj} B = A$. Find $\det B$ and $B$.

**Question 3** ( *Advanced Level* )

Let $A = \begin{pmatrix} -1-k & -1 & 3 \\ 3 & 2-k & -6 \\ -1 & 0 & 1-k \end{pmatrix}$, where $k$ is a real number and $k \neq 1$.

(a) Find $A^{-1}$ by using the formula $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$, where $\operatorname{adj} A$ is the adjoint matrix;

(b) Use part (a) of this question, or otherwise, to determine the matrix $B$ such that

$$\begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} + 2B = \begin{pmatrix} -1 & -1 & 3 \\ 3 & 2 & -6 \\ -1 & 0 & 1 \end{pmatrix} B$$

**Question 4** ( *Exam Level* )

(a) Find the value of $b$ such that the system of linear equations

$$\begin{pmatrix} 1 & 2 & -4 & 3 \\ 3 & 3 & 1 & 1 \\ -4 & -5 & -4 & 3 \\ -1 & -5 & 10 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ b \end{pmatrix}$$

is consistent. Find all solutions of the system when it is consistent. (8 marks)

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & p \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix}$, where $p, q$ are given scalars.

(i) Compute the $\det(\mathbf{A})$. Hence, or otherwise, find the value of $p$ such that $\mathbf{A}$ is singular.

(ii) Find $\mathbf{A}^{-1}$ when $p = 1$.

(iii) Determine the value(s) of $p$ and $q$ such that

(α) $\mathbf{Ax} = \mathbf{b}$ has one and only one solution;

(β) $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions;

(γ) $\mathbf{Ax} = \mathbf{b}$ has no solution.

Justify your answer. (12 marks)

**Question 5** ( *Exam Level* )

Let $\mathbf{A} = \begin{pmatrix} 0 & 2 & 6 & 4 \\ 1 & -2 & -13 & -4 \\ -2 & 4 & 17 & 1 \\ -2 & 2 & 11 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ p \\ q \\ -2 \end{pmatrix}$, where $p$, $q$ are given scalars.

Using elementary row operations, find the values of $p$ and $q$ such that

(i) $\mathbf{Ax} = \mathbf{b}$ has no solution;

(ii) $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions;

(iii) $\mathbf{Ax} = \mathbf{b}$ has one and only one solution. (10 marks)

**Question 6** ( *Standard Level* )

Consider the following system of linear equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & 4a-1 \\ 2 & a & a+1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Determine al possible value(s) of $a$ such that the system is

(a) consistent with infinitely many solutions;

(b) consistent with one and only one solution;

(c) inconsistent.

Solve the system when it has unique solution.

**Question 7** ( *Standard Level* )

Consider $\begin{bmatrix} 1 & 1 & -1 \\ -a & -1 & a \\ a^2 & 1 & -a \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(a) Find all values of $a$ such that the system is consistent for any values of $b_1$, $b_2$, and $b_3$.

(b) When $a = 0$, find all values of $b_1$, $b_2$, and $b_3$ so that the linear system is consistent.

(c) When $a = 1$ and $b_1 = b_2 = b_3 = 0$, find all solutions of the linear system.

**Question 8** ( *Smart Level* )

Find the determinant and inverse of the following matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $a_1, a_2, \ldots, a_n$ are nonzero real numbers.

**Question 9** ( *Puzzle Level* )

Consider $\begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Find conditions on $\alpha$, $\beta$ and $\gamma$ such that the linear system is consistent. When will the system have a unique solution? When will the system have infinitely many solutions?

**Question 10** ( *Puzzle Level* )

Consider

$$\begin{pmatrix} 1 & 1 & 1 \\ (b+1) & 1 & (ab+a) \\ (b+1)^2 & 1 & a^2(b+1)^2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ cb+c \\ c^2(b+1)^2 \end{pmatrix}$$

Here $a, b, c \in \mathbb{R}$. Determine the values of $a, b, c$ such that the linear system is

(a) consistent with infinitely many solutions;

(b) consistent with one and only one solution;

(c) inconsistent.

**Question 11** ( *Intermediate Level* )

Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $I$ the $3 \times 3$ identity matrix.

(a) Compute $A^n$ for positive integer $n$.

(b) Compute $(A+I)^{1120}$.

**Question 12** ( *Concept* )

Let $A$ and $B$ be $n \times n$ matrices. Is the following true? Justify your answer.

(a) $(A+B)(A-B) = A^2 - B^2$

(b) $(A+B)^2 = A^2 + 2AB + B^2$