Uncertainty evaluation
Basics

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Terms and definitions

Terms and definitions

- Many terms are used ...
  - Error
  - Precision
  - Accuracy
  - Uncertainty
  - Resolution
  - Sensitivity
  - Systematic effect
  - …
- ... for historical reasons, but not only!

Terms and definitions

- **Error**
  - Used in the past, distinguishing between:
    - ✓ **systematic**: errors that affects a measurement always in the same way (same sign and same value)
    - ✓ **random**: errors due to phenomenon not under the operator's control
  - Today used as **measurement error** to indicate the difference between a measurement and a reference value
    - ✓ Example: difference between the nominal value of a standard resistor and its measured value

Terms and definitions

- (measurement) **Precision**
  - It expresses the capability of a measurement system to provide similar measurements in specified conditions
    - ✓ **Repeatability conditions**
      - – Same procedure, operator, isntruments and conditions
    - ✓ **Reproducibility conditions**
      - – Different procedures (inter-procedure reproducibility), or operators (inter-operator reproducibility), or instruments ...
  - Commonly used parameters: variance or standard deviation

Terms and definitions

- (measurement) **Accuracy**
  - It expresses the capability of a measurement system to provide measurements close to the «true» value of the parameter under measurement
    - ✓ Qualitative concept: the instrument **X** is more accurate than instrument **Y** if **X** provides lower errors
    - ✓ It is often confused with the term «uncertainty»
      - Manufacturers prefer to state
        «*accuracy specifications*»
        instead of
        «*uncertainty specifications*»

Terms and definitions

- (measurement) **Uncertainty**
  - It expresses the not perfect knowledge of the parameter under measurement (**measurand**)
    - ✓ It includes all the uncertainty contributions
    - ✓ It is stated by means of an interval of values
      - – The meaning of this interval depends on the model (deterministic or probabilistic)

Terms and definitions

- (measurement) **Resolution**
  - Smallest change in the input measurand of an instrument that causes a perceptible change in the output indication
    - ✓ The measurement unit is the same as the measurand
    - ✓ It is the capability of the instrument in detecting small changes in the measurand
      - – The resolution is an uncertainty contribution
      - – Small resolution can be required regardless of the uncertainty

Terms and definitions

- **Sensitivity**
  - Ratio between the output indication of a measuring device and the corresponding change in the input measurand
    - ✓ The measurement unit (m.u.) is the m.u. of the output divided by the m.u. of the input (e.g., mm/V or div/V for an oscilloscope)
    - ✓ For non-linear devices it depends on the measurand value
    - ✓ It is a constant value for linear devices
      - – The reciprocal value represents the calibration constant

Terms and definitions

- **Systematic effect**
  - Effect due to different sources that affects repeated measurements in the same way or changes in a predictable manner
    - ✓ If the systematic error can be evaluated, a correction is applied
    - ✓ The evaluation of a systematic effect is always affected by uncertainty

Terms and definitions

- **Systematic effect**
  - Main sources
    - Interaction between instruments and system under measurement (**load effect**)
    - Effects of the **influence quantities**
    - Instrument errors
      - Offset
      - Gain
      - ...

Terms and definitions

- **Systematic effect**
  - Its evaluation is not alway possible (not enough information) or convenient (time/cost constraints)
    - The load-effect evaluation is related to parameters of instrument and system under measurement, which could be unknown
    - Instrument-error evaluation requires known measurands
      - ✓ Null input for offset evaluation
      - ✓ Known inputs for gain-error evaluation

Terms and definitions

- **Systematic effect**
  - Two possible strategies
    i. Correction of the systematic effect
      - ✓ If it can be evaluated and if it is significative compared to the uncertainty
      - ✓ A residual uncertainty-contribution is related to this correction

    ii. Considering the systematic effect as an uncertainty contribution
      - ✓ Suitable assumption has to be made according to the available information

Uncertainty contributions

Uncertainty contributions

- Instrumentation
  - The **calibration function** of an instrument is not perfectly known
    - ✓ It suffers from several errors, such as offset, gain error and non linearity
  - The **calibration function** is affected by quantities different than the measurand (**influence quantities**)
    - ✓ Temperature, humidity, value and frequency of the supply voltage, vibration, electromagnetic fields, ...

Uncertainty evaluation - Basics

Uncertainty contributions

- Instrumentation
  - The instrument output (**indication**) is available with a finite **resolution** (quantization)
  - The interaction between an intrument and the system under meaurement always modify the system
    - ✓ The obtained measurement is different than the measurand in unloaded conditions
    - ✓ **Load effect**, which depends both on the instrument an the system under measurement

Uncertainty contributions

- Measurand
  - The definition of the measurand is provided by means of a finite amount of detail (**definitional uncertainty**)
    - ✓ It sets the minimum uncertainty achievable
    - ✓ The reduction of this contribution requires a more complex definition to be provided
    - ✓ A more complex definition makes the realization of the measurand more difficult

Uncertainty contributions

- Measurand
  - The way the measurand is defined identifies the quantities that describe the state of the system under measurement
    - ✓ Quantities that have to be specified when a measurement is provided
    - ✓ These quantities are not perfectly known (they are affected by uncertainty)

Uncertainty contributions

- Method
  - Different procedures can be used to measure the same parameter
  - Different instruments can be used to implement the same procedure
  - Different operator can implement the same procedure (with the same or with different instruments)
    - ✓ This contribution represents the **measurement reproducibility**
    - ✓ It is usually evaluated through specific experiments

Uncertainty evaluation - Basics

Uncertainty contributions

- Method
  - The repetition of the same procedure, with the same instruments, with the same operator and in the same nominal conditions does not provide the same results
    - ✓ This contribution represents the **measurement repeatability**
    - ✓ It is due to random effects (e.g., influence quantities variation or electronic noise)
    - ✓ It is usually evaluated through specific experiments

Classification of measurement methods

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods
  - **Single-reading** vs **multiple-readings** methods

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods

Direct method

↳ The measurement is obtained from the reading of one instrument (direct reading) or from the value of one or more standards (comparison methods)

Uncertainty evaluation - Basics

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods

Indirect method

- A **measurement model** is used that relates the measurand to other quantities, which are measured using direct methods
- Often used when a direct measurement is not convenient
  - Volumetric mass density of a substance (kg/m^3)
  - Electrical resistance (Ω)

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods

Indirect method

The **measurement model** is often expressed in explicit form

$$Y=f(X_1, X_2, \ldots , X_N)$$

- Y is the measurand
- $X_1, \ldots , X_N$ are the quantities measured through direct methods
- f is the **measurement function**

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods

Indirect method

- A further uncertainty contribution has to be taken into account
  - The measurement model could not perfectly describe the measurand
  - Often indicated as **model uncertainty**
    - Volume of a cube obtained as $V = L^3$
    - Sound speed in air obtained as a function of temperature and relative humidity ...

Classification of measurement methods

- According to our aim
  - **Direct** vs **indirect** methods
  - **Single-reading** vs **multiple-readings** methods

Single-reading methods

- The measurement is obtained starting from a single reading of one or more instruments

Multiple-readings methods

- A set of measurements is collected from one or more instruments in specified conditions
  - Repeatability conditions
  - Reproducibility conditions

Deterministic approach

Uncertainty evaluation - Basics

Deterministic approach

- When it is used
  - In the past (before the definition of the probabilistic approach)
  - Today
    - ✓ Uncertainty specifications of many commercial instruments
    - ✓ Industrial field
    - ✓ In worst-case design applications

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty interpretation
  - A measurement is stated as a closed interval of values, which is usually centered around the value assigned to the measurand
  - Example of measurement statement:

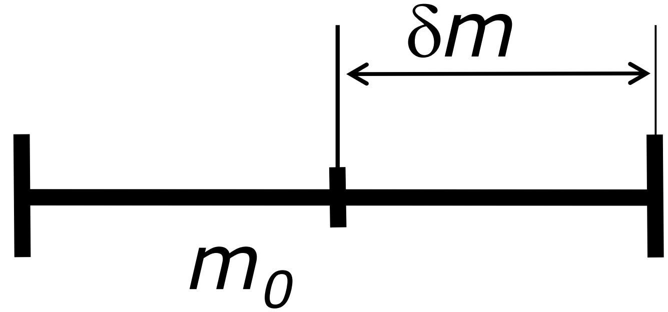
$$m = (m_0 \pm \delta m)\ \mathrm{U}$$

    - ✓ δm is the absolute uncertainty (half-width of the interval), which is expressed in the same measuring unit U of the measurand m_0

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty interpretation
  - According to the deterministic approach:
    - The measurand is included in the assigned interval of values
    - All the values of the assigned interval are equally representative of the measurand



Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty interpretation
  - Example of measurements:

$$r = (113 \pm 1)\ \Omega$$

$$m = (1.45 \pm 0.02)\ \text{g}$$

  - The absolute uncertainty does not higlight the «quality/cost» of a measurement

$$l_1 = (13.6 \pm 0.01)\ \text{m}$$

$$l_2 = (2320.1 \pm 0.01)\ \text{m}$$

    - ✓ Same absolute uncertainty, but very different cost!

Deterministic approach

- Uncertainty interpretation
  - Alternative uncertainty statements
    - Relative uncertainty or relative percentage uncertainty:

$$\varepsilon m = \delta m/m_0 \ ; \ \varepsilon m_{\%} = 100 \cdot \delta m/m_0$$

    - Measurement statement

$$m = m_0 \text{ U}, \pm \varepsilon m$$

    - For the previous example:

$$l_1 = 13.6 \text{ m}, \pm 0.07 \ \%$$

$$l_2 = 2320.1 \text{ m}, \pm 0.0004 \ \%$$

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty interpretation
  - Often manufacturers state relative uncertainty with respect to a conventional value m_C of the measurand

$$\varepsilon m = \delta m / m_C \ ; \ \varepsilon m_{\%} = 100 \cdot \delta m / m_C$$

    - **WARNING**: same symbol but different meaning
  - Voltmeter 1: range 50 V; relative uncertainty: 1 %
    Votmeter 2: range 100 V; relative uncetainty 0.5 % (with respect to the range)
    - ✓ Voltage to be measured = 30 V
      - $V_1 \rightarrow$ v = (30 ± 0.3) V
      - $V_2 \rightarrow$ v = (30 ± 0.5) V

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty propagation
  - The way the different uncertainty contributions are combined depends on the measurement method
    - Direct vs indirect methods
  - Direct measurement methods: sum of the contributions
    - Instrumental uncertainty
    - Measurement repeatability/reproducibility
    - Residual uncertainty of corrected systematic errors
    - Definitional uncertainty
    - ...

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty propagation
  - Indirect measurement methods

$$Y=f(X_1, X_2, \ldots, X_N)$$

- ✓ $x_1=(x_{10} \pm \delta x_1)\ U_1$
- ✓ ...
- ✓ $x_N=(x_{N0} \pm \delta x_N)\ U_N$

➢ Measurand evaluation

$$y_0=f(x_{10}, x_{20}, \ldots, x_{N0})$$

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty propagation
  - Indirect measurement methods
    - Absolute uncertainty evaluation

$$\delta y = \left|\frac{\partial f}{\partial x_1}\right|_{(x_{10},\ldots,x_{N0})} \cdot \delta x_1 + \ldots + \left|\frac{\partial f}{\partial x_N}\right|_{(x_{10},\ldots,x_{N0})} \cdot \delta x_N$$

    - Taylor series of the measurement function *f* estimated in the point (x_{10},..., x_{N0})
    - Approximation: only the first-order partial derivatives are used in the series

Uncertainty evaluation - Basics

Deterministic approach

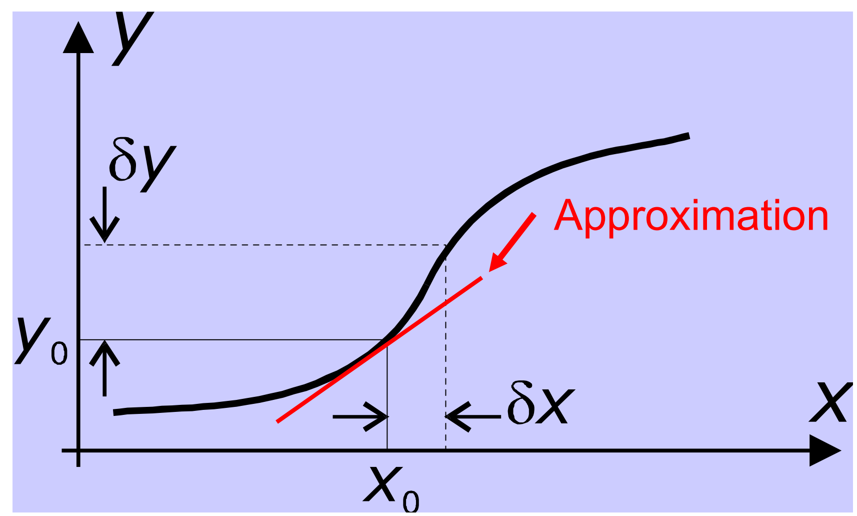
- Uncertainty propagation
  - Indirect measurement methods
    - Absolute uncertainty evaluation

$$\delta y = \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{(x_{10},\ldots,x_{N0})} \cdot \delta x_i = \sum_{i=1}^{N} |c_i| \cdot \delta x_i$$

    - c_i: sensitivity coefficients of the measurement function f

Deterministic approach

- Uncertainty propagation
  - Indirect measurement methods
    - Absolute uncertainty evaluation
      - The absolute uncertainty δx is considered as a deviation with respect to thw nominal value x_0, which corresponds to a deviation δy with respect to the nominal value y_0



Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty propagation
  - Indirect measurement methods
    - Absolute uncertainty evaluation
    - **WORST-CASE MODEL**

$$\delta y = \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{(x_{10},\ldots,x_{N0})} \cdot \delta x_i$$

The "absolute value" operator prevents any possible uncertainty compensation

Uncertainty evaluation - Basics

Deterministic approach

- Uncertainty propagation
  - Indirect measurement methods
    - Absolute uncertainty evaluation
    - **WORST-CASE MODEL**

$$\delta y = \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{(x_{10},\ldots,x_{N0})} \cdot \delta x_i$$

The maximum deviation δx from the nominal value x_0 is considered for each involved quantity

Example

$$\boxed{Y = X_1 + X_2 - X_3}$$

$$x_1 = \left(x_{10} \pm \delta x_1\right)\,\mathrm{U}$$

$$x_2 = \left(x_{20} \pm \delta x_2\right)\,\mathrm{U}$$

$$x_3 = \left(x_{30} \pm \delta x_3\right)\,\mathrm{U}$$

Measurand evaluation

$$y_0 = x_{10} + x_{20} - x_{30}$$

Uncertainty evaluation - Basics

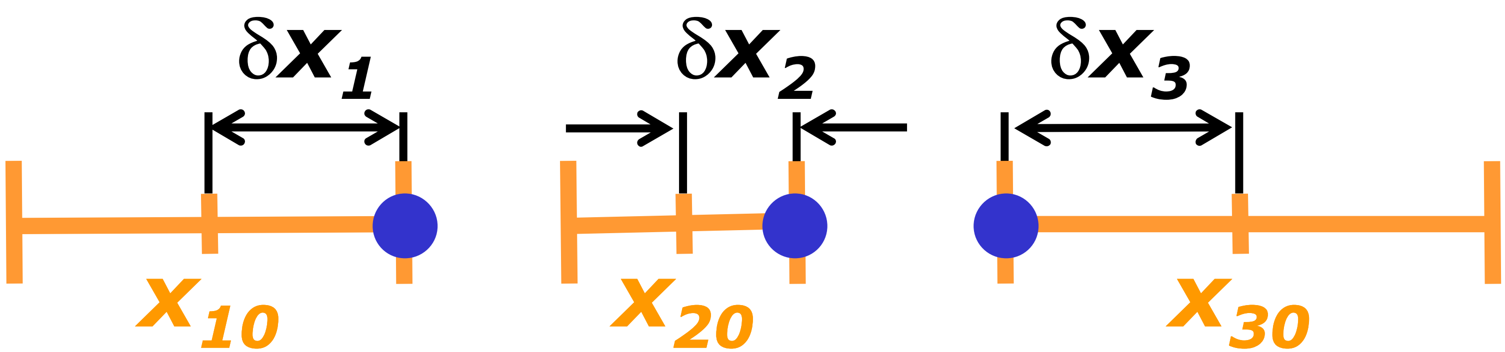
Example

$$Y = X_1 + X_2 - X_3$$

Uncertainty evaluation

$$\delta y = \delta x_1 + \delta x_2 + \delta x_3$$

WORST CASE



General rules

$$\boxed{Y = a \cdot X_1 - b \cdot X_2}$$

a and b constants > 0

$$\left|\frac{\partial f}{\partial x_1}\right| = a; \quad \left|\frac{\partial f}{\partial x_2}\right| = b$$

$$\delta y = a \cdot \delta x_1 + b \cdot \delta x_2$$

Uncertainty evaluation - Basics

General rules

$$\boxed{Y = \frac{a \cdot X_1 \cdot X_2}{b \cdot X_3}}$$

a and b constants > 0

$$\left|\frac{\partial f}{\partial x_1}\right| = \frac{a \cdot x_2}{b \cdot x_3}; \quad \left|\frac{\partial f}{\partial x_2}\right| = \frac{a \cdot x_1}{b \cdot x_3}; \quad \left|\frac{\partial f}{\partial x_3}\right| = \frac{a \cdot x_1 \cdot x_2}{b \cdot x_3^2}$$

General rules

$$\boxed{Y = \frac{a \cdot X_1 \cdot X_2}{b \cdot X_3}}$$

a and b constants > 0

y

$$\delta y = \frac{a \cdot x_1 \cdot x_2}{b \cdot x_3} \cdot \left(\frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \frac{\delta x_3}{x_3} \right)$$

$$\varepsilon y = \frac{\delta y}{y} = \varepsilon x_1 + \varepsilon x_2 + \varepsilon x_3$$

Uncertainty evaluation - Basics

General rules

$$\boxed{Y = a \cdot X^{\mathrm{n}}}$$

a constant > 0

$$Y = a \cdot \prod_{i=1}^{n} X_i$$

The rule of the product can be used

$$\varepsilon y = \mathrm{N} \cdot \varepsilon x$$

General rules

$$\boxed{Y = a \cdot \sqrt[n]{X}}$$

a constant > 0

$$Y = a \cdot X^{1/n}$$

The rule of the power can be used

$$\varepsilon y = \frac{1}{N} \cdot \varepsilon x$$