2019—2020 学年第一学期《数学分析三》期中试卷解答

一、计算下列各题：

1. 设 $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x+y)^2}$，分别求出该函数在 $(0,0)$ 点处的累次极限和重极限.

解：$\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{x^2y^2}{x^2y^2+(x+y)^2}=\lim\limits_{x\to0}0=0$,

同理，$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{x^2y^2}{x^2y^2+(x+y)^2}=\lim\limits_{y\to0}0=0$.

因为 $\lim\limits_{\substack{y=x\\x\to0}}f(x,y)=\lim\limits_{x\to0}\dfrac{x^4}{x^4+4x^2}=\lim\limits_{x\to0}\dfrac{x^2}{x^2+4}=0$，而 $\lim\limits_{\substack{y=-x\\x\to0}}f(x,y)=\lim\limits_{x\to0}\dfrac{x^4}{x^4}=\lim\limits_{x\to0}1=1$，则重极限 $\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)$ 不存在.

2. 设 $z=z(x,y)$ 为方程 $x^2+y^2+z^2=yf(\dfrac{z}{y})$ 所确定的隐函数，求 $(x^2-y^2-z^2)\dfrac{\partial z}{\partial x}+2xy\dfrac{\partial z}{\partial y}$.

解：对式子 $x^2+y^2+z^2=yf(\dfrac{z}{y})$ 两边对 x 求导，得 $2x+2z\dfrac{\partial z}{\partial x}=yf'(\dfrac{z}{y})\dfrac{1}{y}\dfrac{\partial z}{\partial x}$，即

即
$$\frac{\partial z}{\partial x}=\frac{2x}{f'-2z}.$$

同理，对式子 $x^2+y^2+z^2=yf(\dfrac{z}{y})$ 两边对 y 求导，得

$$2y+2z\frac{\partial z}{\partial y}=f+yf'\cdot\frac{1}{y^2}(y\frac{\partial z}{\partial y}-z),$$

即
$$\frac{\partial z}{\partial y}=\frac{2y^2-yf+zf'}{y(f'-2z)}.$$

则 $(x^2-y^2-z^2)\dfrac{\partial z}{\partial x}+2xy\dfrac{\partial z}{\partial y}=\dfrac{2x(x^2-y^2-z^2)+2x(2y^2-yf+zf')}{f'-2z}$

$$=\frac{2x(x^2+y^2-z^2)+2x(-yf+zf')}{f'-2z}$$

$$=\frac{2x(-2)z^2+2xzf'}{f'-2z}=2xz.$$

3. 求曲线 $x=t, y=t, z=t^3$ 上的点，使在该点的切线平行于平面 $x+2y-z=4$.

解：设所求点的坐标为 (t_0,t_0,t_0^3)，该点处的切向量为 $\vec{T}=(1,1,3t_0^2)$.

因所求切线与平面 $x+2y-z=4$ 平行，则 $\vec{T}\perp\vec{n}=(1,2,-1)$，故有 $1+2-3t_0^2=0$，则 $t_0=\pm1$.

故所求点的坐标为 $(1,1,1)$ 和 $(-1,-1,-1)$.

4. 解：记 $F(x,y,z)=\mathrm{e}^z-z+xy-3$，则法向量

$$\vec{n}=(y,x,\mathrm{e}^z-1)\Big|_{(2,1,0)}=(1,2,0).$$

故所求切平面方程为 $(x-2)+2(y-1)=0$，即 $x+2y=4$.

法线方程为 $\dfrac{x-2}{1}=\dfrac{y-1}{2}=\dfrac{z-0}{0}$ 或 $\begin{cases}2x-y=3\\ z=0\end{cases}$.

5. 求函数

解：令 $\begin{cases}f_x=\mathrm{e}^{2x}(2x+4y^2+4y+1)=0\\ f_y=\mathrm{e}^{2x}(4y+2)=0\end{cases}$，则 $\begin{cases}x=0\\ y=-\dfrac{1}{2}\end{cases}$.

由 $f_{xx}=\mathrm{e}^{2x}(4x+8y^2+8y+4),\ f_{xy}=\mathrm{e}^{2x}(8y+4),\cdots f_{yy}=4\mathrm{e}^{2x}$ 可得

$$f_{xx}(0,-\frac{1}{2})=2,\ f_{xy}(0,-\frac{1}{2})=0,\ f_{yy}(0,-\frac{1}{2})=4.$$

因此，$f_{xx}(0,-\frac{1}{2})f_{yy}(0,-\frac{1}{2})-f_{xy}^2(0,-\frac{1}{2})=8>0$，$f_{xx}(0,-\frac{1}{2})=2>0$，故 $f(x,y)$ 在 $(0,-\frac{1}{2})$ 处取得极小值，极小值为 $f(0,-\frac{1}{2})=-\frac{1}{2}$.

二、证明：函数 $f(x,y)=\begin{cases}xy\cos\dfrac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$ 在 $(0,0)$ 点连续，可导且可微，但在 $(0,0)$ 的偏导数不连续.

证明：因为 $\left|xy\cos\dfrac{1}{x^2+y^2}\right|\le|xy|$，且 $\lim\limits_{\substack{x\to0\\y\to0}}xy=0$，则 $\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}xy\cos\dfrac{1}{x^2+y^2}=0=f(0,0)$.

因此，$f(x,y)$ 在 $(0,0)$ 处连续.

$$f_x(0,0)=\lim_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to0}\frac{0-0}{\Delta x}=0,$$

$$f_y(0,0)=\lim_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0,$$

所以，$f(x,y)$ 在 $(0,0)$ 处可导.

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{\Delta x\to 0}\frac{\Delta x\Delta y\cos\frac{1}{(\Delta x)^2+(\Delta y)^2}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}.$$

因为 $\left|\frac{\Delta x\Delta y\cos\frac{1}{(\Delta x)^2+(\Delta y)^2}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right|\le\left|\frac{\Delta x\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right|\le\frac{1}{2}\sqrt{(\Delta x)^2+(\Delta y)^2}$，且 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{1}{2}\sqrt{(\Delta x)^2+(\Delta y)^2}=0$，

故 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0$，即

$$f(\Delta x,\Delta y)=f(0,0)+f_x(0,0)\Delta x+f_y(0,0)\Delta y+o(\sqrt{(\Delta x)^2+(\Delta y)^2}).$$

所以，$f(x,y)$ 在 $(0,0)$ 处可微.

当 $(x,y)\neq(0,0)$ 时，

$$f_x(x,y)=y\cos\frac{1}{x^2+y^2}+\frac{2x^2y}{(x^2+y^2)^2}\sin\frac{1}{x^2+y^2},$$

$$f_y(x,y)=x\cos\frac{1}{x^2+y^2}+\frac{2xy^2}{(x^2+y^2)^2}\sin\frac{1}{x^2+y^2}.$$

因为 $\lim\limits_{\substack{y=x\\ x\to 0}}f_x(x,y)=\lim\limits_{x\to 0}[x\cos\frac{1}{2x^2}+\frac{1}{2x}\sin\frac{1}{2x^2}]$ 不存在，故 $\lim\limits_{\substack{x\to 0\\ y\to 0}}f_x(x,y)$ 不存在，所以，$f_x(x,y)$ 在 $(0,0)$ 处不连续.

同理，$f_y(x,y)$ 在 $(0,0)$ 处不连续.

三、设函数 $u=u(x,y)$ 满足方程 $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$，且 $u(x,2x)=x^2$，$u_x(x,2x)=x^2$，求 $u_{xx}(x,2x)$，$u_{xy}(x,2x)$ 和 $u_{yy}(x,2x)$.

解：由 $u(x,2x)=x$，两边对 x 求导，得

$$u_x(x,2x)+2u_y(x,2x)=1. \qquad (1)$$

再对 x 求导，得

$$u_{xx}(x,2x)+4u_{xy}(x,2x)+4u_{yy}(x,2x)=0.$$

由已知条件，$u_{xx}(x,2x)=u_{yy}(x,2x)$，即$u_{xx}(x,2x)=-\frac{4}{5}u_{xy}(x,2x)$

由$u_x(x,2x)=x^2$两端对x求导，得$u_{xx}(x,2x)+2u_{xy}(x,2x)=2x$，代入上式，得

$$(2-\frac{4}{5})u_{xy}(x,2x)=2x,$$

则$u_{xy}(x,2x)=\frac{5}{3}x$，从而$u_{xx}(x,2x)=u_{yy}(x,2x)=-\frac{4}{5}\cdot\frac{5}{3}x=-\frac{4}{3}x.$

四、设$\varphi(x)$和$\psi(x)$具有连续的二阶导数，$u=\varphi(\frac{y}{x})+x\psi(\frac{y}{x})$. 分别计算$\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial^2 u}{\partial x^2}$，$\frac{\partial^2 u}{\partial x\partial y}$和$\frac{\partial^2 u}{\partial y^2}$，

并由此计算$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$.

证明：$\frac{\partial u}{\partial x}=-\frac{y}{x^2}\varphi'(\frac{y}{x})+\psi(\frac{y}{x})-\frac{y}{x}\psi'(\frac{y}{x})$，$\frac{\partial u}{\partial y}=\frac{1}{x}\varphi'(\frac{y}{x})+\psi'(\frac{y}{x})$，

$$\frac{\partial^2 u}{\partial x^2}=\frac{2y}{x^3}\varphi'(\frac{y}{x})+\frac{y^2}{x^4}\varphi''(\frac{y}{x})-\frac{y}{x^2}\psi'(\frac{y}{x})+\frac{y}{x^2}\psi'(\frac{y}{x})+\frac{y^2}{x^3}\psi''(\frac{y}{x})$$

$$=\frac{2y}{x^3}\varphi'(\frac{y}{x})+\frac{y^2}{x^4}\varphi''(\frac{y}{x})+\frac{y^2}{x^3}\psi''(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{x^2}\varphi'(\frac{y}{x})-\frac{y}{x^3}\varphi''(\frac{y}{x})+\frac{1}{x}\psi'(\frac{y}{x})-\frac{1}{x}\psi'(\frac{y}{x})-\frac{y}{x^2}\psi''(\frac{y}{x})$$

$$=-\frac{1}{x^2}\varphi'(\frac{y}{x})-\frac{y}{x^3}\varphi''(\frac{y}{x})-\frac{y}{x^2}\psi''(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{x^2}\varphi''(\frac{y}{x})+\frac{1}{x}\psi''(\frac{y}{x}).$$

于是，$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

$$=x^2[\frac{2y}{x^3}\varphi'(\frac{y}{x})+\frac{y^2}{x^4}\varphi''(\frac{y}{x})+\frac{y^2}{x^3}\psi''(\frac{y}{x})]$$

$$+2xy[-\frac{1}{x^2}\varphi'(\frac{y}{x})-\frac{y}{x^3}\varphi''(\frac{y}{x})-\frac{y}{x^2}\psi''(\frac{y}{x})]$$

$$+y^2[\frac{1}{x^2}\varphi''(\frac{y}{x})+\frac{1}{x}\psi''(\frac{y}{x})]$$

$$=0.$$

五、设$y=f(x,t)$，而$t=t(x,y)$是由方程$F(x,y,t)=0$所确定的函数，其中f和F都具有一阶连续偏

导数，试证明：$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial t}-\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial x}}{\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial y}+\dfrac{\partial F}{\partial t}}$.

证明：由$\begin{cases} y=f(x,t) \\ F(x,y,t)=0 \end{cases}$两边对$x$求导，得

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial t}\dfrac{\mathrm{d}t}{\mathrm{d}x} \\ \dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial F}{\partial t}\dfrac{\mathrm{d}t}{\mathrm{d}x}=0 \end{cases},$$

解得$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial t}-\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial x}}{\dfrac{\partial f}{\partial t}\dfrac{\partial F}{\partial y}+\dfrac{\partial F}{\partial t}}$.

六、在空间曲线$\begin{cases} z=x^2+2y^2 \\ z=6-2x^2-y^2 \end{cases}$上求竖坐标分别为最大、最小的点.

解：目标函数为z，约束条件为$\begin{cases} z=x^2+2y^2 \\ z=6-2x^2-y^2 \end{cases}$.

作拉格朗日函数$L(x,y,z)=z+\lambda(x^2+2y^2-z)+\mu(6-2x^2-y^2)$，令

$$\begin{cases} L_x=2x\lambda+4\mu x=0 \\ L_x=4y\lambda+2\mu x=0 \\ L_z=1-\lambda+\mu=0 \\ L_\lambda=x^2+2y^2-z=0 \\ L_\mu=2x^2+y^2-z=0 \end{cases},$$

解得$(0,\sqrt{2},4)$，$(0,-\sqrt{2},4)$，$(\sqrt{2},0,2)$，$(-\sqrt{2},0,2)$.

B 比较这些点上的函数值，可得竖坐标最大的点为$(0,\sqrt{2},4)$和$(0,-\sqrt{2},4)$，竖坐标最小的点为$(\sqrt{2},0,2)$，$(-\sqrt{2},0,2)$.

七、设函数$F(x,y)$满足：（1）在$D=\{(x,y)\big||x-x_0|\le a,|y-y_0|\le b\}$上连续；（2）$F(x_0,y_0)=0$；（3）当$x$固定时，函数$F(x,y)$是$y$的严格单调减少函数. 证明：存在$\delta>0$，使得在$I_\delta=\{x\big||x-x_0|<\delta\}$上由方程$F(x,y)=0$确定了一个满足$y_0=f(x_0)$的隐函数，且$y=f(x)$在$I_\delta$上连续.

证明：由（3）知$F(x_0,y)$在$[y_0-b,y_0+b]$是y的严格单调减少函数.

因为$F(x_0,y_0)=0$，则由函数$F(x_0,y)$的连续性，有$F(x_0,y_0-b)>0$，$F(x_0,y_0-b)<0$.

对于一元连续函数$F(x,y_0-b)$，由于$F(x_0,y_0-b)>0$，则必存在$\delta_1>0$，使得当$|x-x_0|<\delta_1$时，

$F(x,y_0-b)>0$.

同理，存在$\delta_2>0$，使得当$|x-x_0|<\delta_2$时，$F(x,y_0+b)<0$.

取$\delta=\min(\delta_1,\delta_2)$，则当$|x-x_0|<\delta$时，$F(x,y_0-b)>0$，$F(x,y_0+b)<0$.

于是，对于固定的$\overline{x}$，由$F(\overline{x},y)$的连续性，存在$\overline{y}\in[y_0-b,y_0+b]$，使得$F(\overline{x},\overline{y})=0$.

由于$F(\overline{x},y)$关于y严格单调减少，则从而使$F(\overline{x},\overline{y})=0$的$\overline{y}$是唯一的.

再由$\overline{x}$的任意性，证明了对于$I_\delta=\{x||x-x_0|<\delta\}$内任意一点，总能从$F(x,y)=0$找到唯一的$y$与$x$相对应, 即存在函数关系$y=f(x)$. 从而证明了隐函数的存在性.

下面证明连续性.

设$\overline{x}$为$I_\delta=\{x||x-x_0|<\delta\}$上任意一点，记$\overline{y}=f(\overline{x})$，则$F(\overline{x},\overline{y})=0$.

由$F(\overline{x},y)$的严格单调性，$F(\overline{x},\overline{y}-\varepsilon)>0$，$F(\overline{x},\overline{y}+\varepsilon)<0$.

由$F(x,\overline{y}-\varepsilon)$和$F(x,\overline{y}+\varepsilon)$的连续性，存在$0<\eta<\delta$，使得当$|x-\overline{x}|<\eta$时，

$$F(x,\overline{y}-\varepsilon)>0，F(x,\overline{y}+\varepsilon)<0.$$

于是，在$(\overline{y}-\varepsilon,\overline{y}+\varepsilon)$内必存在唯一的$y$，使得$F(x,y)=0$.

即，对于满足$|x-\overline{x}|<\eta$的x，$|y-\overline{y}|<\varepsilon$.

所以函数$y=f(x)$在I_δ内连续.