厦门大学《数学分析3》课程期中试卷

试卷类型：金融/统计　　　　考试日期　2017.11.19

一、求下列函数的极限（本题10分，每小题5分）：

（1）$\lim\limits_{\substack{x\to 0\\ y\to 0^+}}\dfrac{x^2y^{\frac{3}{2}}}{x^4+y^2}$；　　　　（2）$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^2+y^2+x^3+y^3)}{x^2+y^2}$.

解：（1）因为$\left|\dfrac{x^2y^{\frac{3}{2}}}{x^4+y^2}\right|\le\dfrac{1}{2}\dfrac{x^4+y^2}{x^4+y^2}y^{\frac{1}{2}}=\dfrac{1}{2}y^{\frac{1}{2}}$，因为$\lim\limits_{\substack{x\to 0\\ y\to 0^+}}\dfrac{1}{2}y^{\frac{1}{2}}=0$，故$\lim\limits_{\substack{x\to 0\\ y\to 0^+}}\dfrac{x^2y^{\frac{3}{2}}}{x^4+y^2}=0$.

（2）$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^2+y^2+x^3+y^3)}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2+y^2+x^3+y^3}{x^2+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}}(1+\dfrac{x^3+y^3}{x^2+y^2})$

因为$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^3+y^3}{x^2+y^2}=\lim\limits_{r\to 0}r(\cos^3\theta+\sin^3\theta)=0$，故$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^2+y^2+x^3+y^3)}{x^2+y^2}=1$.

二、已知函数$f(x,y)=\begin{cases}xy\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\ne 0\\ 0, & x^2+y^2=0\end{cases}$，证明：（1）偏导数$f_x(0,0)$，$f_y(0,0)$存在；（2）函数$f(x,y)$在$(0,0)$点可微；（3）偏导数$f_x(x,y)$，$f_y(x,y)$在$(0,0)$点不连续.（本题15分，每小题5分）

证明：（1）当$x^2+y^2\ne 0$时，$\left|\sin\dfrac{1}{\sqrt{x^2+y^2}}\right|\le 1$，而

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0.$$

故偏导数$f_x(0,0)$，$f_y(0,0)$存在.

（2）$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x\Delta y\sin\dfrac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$

因为$\left|\dfrac{\Delta x\Delta y\sin\dfrac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right|\le\dfrac{1}{2}\sqrt{(\Delta x)^2+(\Delta y)^2}$，

故$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0$，所以，$f(x,y)$在$(0,0)$点可微.

（3） $f_x(x,y)=\begin{cases} y\sin\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{x^2y}{\sqrt{(x^2+y^2)^3}}\cos\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\ne 0\\ 0, & x^2+y^2=0\end{cases}$，

$f_y(x,y)=\begin{cases} x\sin\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{xy^2}{\sqrt{(x^2+y^2)^3}}\cos\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\ne 0\\ 0, & x^2+y^2=0\end{cases}$

所以，$\lim\limits_{\substack{x\to 0\\ y=x}}f_x(x,y)=\lim\limits_{x\to 0}[x\sin\dfrac{1}{\sqrt{2x^2}}-\dfrac{x}{2\sqrt{2}|x|}\cos\dfrac{1}{\sqrt{2x^2}}]$不存在，

$\lim\limits_{\substack{x\to 0\\ y=x}}f_y(x,y)=\lim\limits_{x\to 0}[x\sin\dfrac{1}{\sqrt{2x^2}}-\dfrac{x}{2\sqrt{2}|x|}\cos\dfrac{1}{\sqrt{2x^2}}]$不存在，

即$\lim\limits_{\substack{x\to 0\\ y\to 0}}f_x(x,y)$和$\lim\limits_{\substack{x\to 0\\ y\to 0}}f_y(x,y)$都不存在，故$f_x(x,y)$，$f_y(x,y)$在$(0,0)$点不连续.

三、设$f(x,y)=\dfrac{1+xy}{1-xy}$，$(x,y)\in D=[0,1)\times[0,1)$，证明：函数$f(x,y)$在$D$上连续，但不一致连续.（本题**10**分）

证明：对任意$P_0(x_0,y_0)\in D=[0,1)\times[0,1)$，有$0<x_0y_0<1$，故

$$\lim_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{1+xy}{1-xy}=\frac{1+x_0y_0}{1-x_0y_0}=f(x_0,y_0)，$$

所以，$f(x,y)$在$P_0(x_0,y_0)$处连续，从而$f(x,y)$在D上连续.

取$\varepsilon_0=1>0$，则对任意的$0<\delta<\dfrac{1}{8}$，取$x_0=1-\delta,y_0=1-\delta$，及$x_1=1-\dfrac{\delta}{2},y_1=1-\dfrac{\delta}{2}$，则$P_0(x_0,y_0),P(x_1,y_1)\in D$，且

$$\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}=\sqrt{\frac{\delta^2}{4}+\frac{\delta^2}{4}}<\delta，$$

但
$$|f(x_1,y_1)-f(x_0,y_0)|=\frac{1}{1-x_1y_1}-\frac{1}{1-x_0y_0}=\frac{4}{\delta(4-\delta)}-\frac{1}{\delta(2-\delta)}$$

$$=\frac{4-3\delta}{\delta(4-\delta)(2-\delta)}>\frac{4-\frac{3}{8}}{8\delta}>1=\varepsilon_0,$$

故函数 $f(x,y)$ 在 D 上连续，但不一致连续.

四、设 $u(x,y),\ v(x,y)$ 是由方程组 $\begin{cases}u=f(ux,v+y)\\ v=g(u-x,v^2y)\end{cases}$ 所确定的隐函数，试求 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial v}{\partial x}$.（本题 **10** 分）

五、证明：由方程 $y=x\varphi(z)+\psi(z)$ 所确定的函数 $z=z(x,y)$ 满足方程

$$(\frac{\partial z}{\partial y})^2\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\partial y}+(\frac{\partial z}{\partial x})^2\frac{\partial^2 z}{\partial y^2}=0.$$（本题 **15** 分）

证明：对方程 $y=x\varphi(z)+\psi(z)$ 求一阶偏导数，得

$$\begin{cases}0=\varphi(z)+x\varphi'(z)\frac{\partial z}{\partial x}+\psi'(z)\frac{\partial z}{\partial x},\\ 1=x\varphi'(z)\frac{\partial z}{\partial y}+\psi'(z)\frac{\partial z}{\partial y},\quad (x\varphi'+\psi'\neq 0)\end{cases}$$

再对上式求偏导数，有

$$\begin{cases}0=2\varphi'\frac{\partial z}{\partial x}+(x\varphi''+\psi'')(\frac{\partial z}{\partial x})^2+(x\varphi'+\psi')\frac{\partial^2 z}{\partial x^2}, & (1)\\ 0=\varphi'\frac{\partial z}{\partial y}+(x\varphi''+\psi'')\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+(x\varphi'+\psi')\frac{\partial^2 z}{\partial x\partial y}, & (2)\\ 0=(x\varphi''+\psi'')(\frac{\partial z}{\partial y})^2+(x\varphi'+\psi')\frac{\partial^2 z}{\partial y^2}, & (3)\end{cases}$$

由 $(1)\times(\frac{\partial z}{\partial y})^2-(2)\times 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+(3)\times(\frac{\partial z}{\partial x})^2$，推出

$$(x\varphi'+\psi')[(\frac{\partial z}{\partial y})^2\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\partial y}+(\frac{\partial z}{\partial x})^2\frac{\partial^2 z}{\partial y^2}]=0,$$

因为 $x\varphi'+\psi'\neq 0$，故 $(\frac{\partial z}{\partial y})^2\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\partial y}+(\frac{\partial z}{\partial x})^2\frac{\partial^2 z}{\partial y^2}=0$.

六、求锥面 $z=xf(\frac{y}{x})$ 在任意一点 (x_0,y_0) $(x_0\neq 0)$ 处的切平面方程，并证明：所有切平面都经过原点.（本题 **10** 分）

证明：记 $F(x,y,z)=z-xf(\frac{y}{x})$，则所求法向量为

$$\vec{n}=(F_x(x_0,y_0,z_0),F_y(x_0,y_0,z_0),F_z(x_0,y_0,z_0))=(-f(\frac{y_0}{x_0})+\frac{y_0}{x_0}f(\frac{y_0}{x_0}),-f'(\frac{y_0}{x_0}),1)，$$

于是，所求切平面方程为

$$-f(\frac{y_0}{x_0})(x-x_0)+\frac{y_0}{x_0}f(\frac{y_0}{x_0})(x-x_0)-f'(\frac{y_0}{x_0})(y-y_0)+z-z_0=0，$$

即$-f(\frac{y_0}{x_0})x+\frac{y_0}{x_0}f(\frac{y_0}{x_0})x-f'(\frac{y_0}{x_0})y+z=0$，

因为$(0,0,0)$满足切平面方程，故所有切平面都经过原点.

七、求曲线$\begin{cases}x+y+z=0\\x^2+y^2+z^2=6\end{cases}$在点$(1,-2,1)$处的切线方程与法平面方程.（本题 **10** 分）

解：切向量为$T=\left(\begin{vmatrix}2y & 2z\\1 & 1\end{vmatrix},\begin{vmatrix}2z & 2x\\1 & 1\end{vmatrix},\begin{vmatrix}2x & 2y\\1 & 1\end{vmatrix}\right)\Bigg|_{(1,-2,1)}=(-6,0,6)$，所求切线方程为

$$\frac{x-1}{-6}=\frac{y+2}{0}=\frac{z-1}{6}\text{或}\begin{cases}x+z=2\\y+2=0\end{cases}.$$

法平面方程为$-6(x-1)+6(z-1)=0$，即$x-z=0$.

八、求由方程$z=xy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ \ (a>0,b>0)$的极值.（本题 **10** 分）

解：考虑函数$u=x^2y^2(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}),\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$，

解方程组$\begin{cases}\frac{\partial u}{\partial x}=2xy^2(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})-\frac{2}{a^2}x^3y^2=0\\\frac{\partial u}{\partial y}=2x^2y(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})-\frac{2}{b^2}x^2y^3=0\end{cases}$，得稳定点

$$P_0(0,0),P_1(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}}),P_2(-\frac{a}{\sqrt{3}},-\frac{b}{\sqrt{3}}),P_3(\frac{a}{\sqrt{3}},-\frac{b}{\sqrt{3}}),P_4(-\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}}).$$

由于z在$(0,0)$点附近变化，所以$(0,0)$不是极值点.

$$\frac{\partial^2 z}{\partial x^2}=2y^2(1-\frac{6x^2}{a^2}-\frac{y^2}{b^2})，\frac{\partial^2 z}{\partial y^2}=2x^2(1-\frac{x^2}{a^2}-\frac{6y^2}{b^2})，\frac{\partial^2 z}{\partial x\partial y}=4xy(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2})。$$

在P_1,P_2,P_3,P_4各点，得

$$A=-\frac{8}{9}b^2, B=\pm\frac{4}{9}ab, C=-\frac{8}{9}a^2$$

故 $AC-B^2=(\frac{64}{81}-\frac{16}{81})a^2b^2$

九、当 $x>0$，$y>0$，$z>0$ 时，求函数 $f(x,y,z)=\ln x+2\ln y+3\ln z$ 在球面 $x^2+y^2+z^2=6R^2$ 上的最大值。并由此证明：当 a, b, c 为正实数时，成立不等式 $ab^2c^3\le 108(\frac{a+b+c}{6})^6$.（本题 10 分）

解：令 $L(x,y,z,\lambda)=\ln x+2\ln y+3\ln z-\lambda(x^2+y^2+z^2-6R^2)$，求偏导数，得到

$$\begin{cases} L_x=\frac{1}{x}-2x\lambda=0 \\ L_y=\frac{2}{y}-2y\lambda=0 \\ L_z=\frac{3}{z}-2z\lambda=0 \\ L_\lambda=x^2+y^2+z^2-6R^2=0 \end{cases},$$

解得 $2\lambda=\frac{1}{x^2}=\frac{2}{y^2}=\frac{3}{z^2}$，代入约束条件 $x^2+y^2+z^2=6R^2$，可得

$$x^2=R^2, y^2=2R^2, z^2=3R^2.$$

由于目标函数无最小值，所以唯一驻点必是最大值点.

于是，$\ln x+2\ln y+3\ln z\le \ln[\sqrt{R^2}(2R^2)(3R^2)^{\frac{3}{2}}]=\ln(6\sqrt{3}R^6)$，即

$$xy^2z^3\le 6\sqrt{3}(\frac{x^2+y^2+z^2}{6})^3.$$

由前一式得到 $f_{\max}=f(R,\sqrt{2}R,\sqrt{3}R)=\ln(6\sqrt{3}R^6)$.

令 $a=x^2, b=y^2, z=c^2$，则

$$ab^2c^3=(xy^2z^3)^2\le 108(\frac{x^2+y^2+z^2}{6})^6=108(\frac{a+b+c}{6})^6.$$