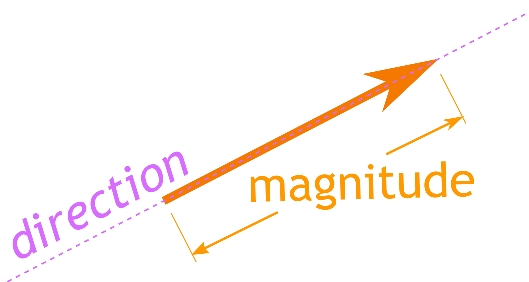
Vectors

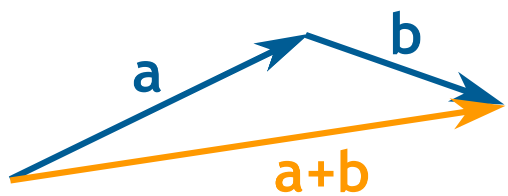
This is a vector:

A vector has **magnitude** (size) and **direction**:

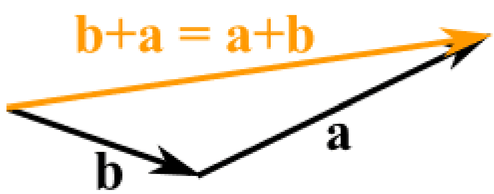


The length of the line shows its magnitude and the arrowhead points in the direction.

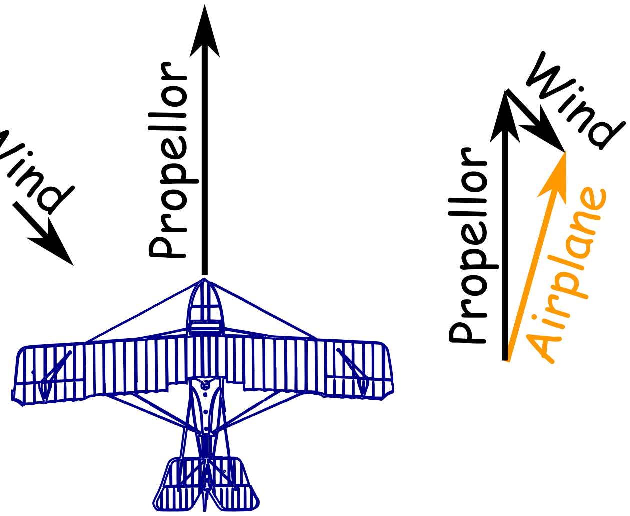
We can add two vectors by joining them head-to-tail:



And it doesn't matter which order we add them, we get the same result:

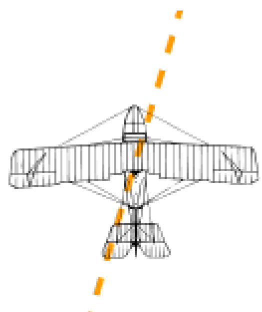


Example: A plane is flying along, pointing North, but there is a wind coming from the North-West.



The two vectors (the velocity caused by the propeller, and the velocity of the wind) result in a slightly slower ground speed heading a little East of North.

If you watched the plane from the ground it would seem to be slipping sideways a little.



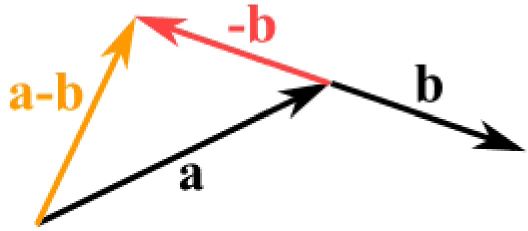
Have you ever seen that happen? Maybe you have seen birds struggling against a strong wind that seem to fly sideways. Vectors help explain that.

Velocity, acceleration, force and many other things are vectors.

Subtracting

We can also subtract one vector from another:

- first we reverse the direction of the vector we want to subtract,
- then add them as usual:

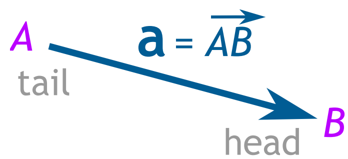


a − **b**

Notation

A vector is often written in **bold**, like **a** or **b**.

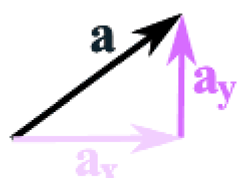
A vector can also be written as the letters of its head and tail with an arrow above it, like this:



Calculations

Now ... how do we do the calculations?

The most common way is to first break up vectors into x and y parts, like this:

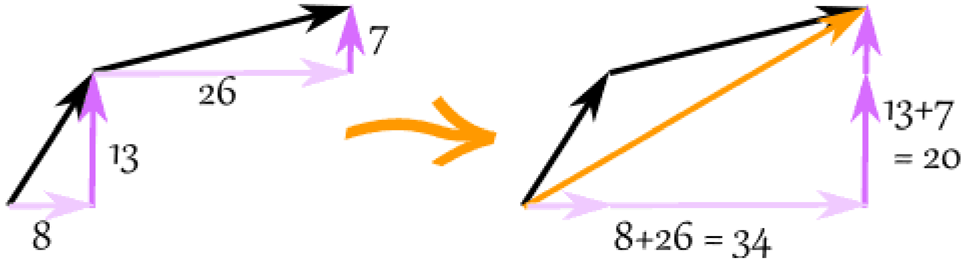


The vector **a** is broken up into the two vectors $\mathbf{a_x}$ and $\mathbf{a_y}$

(We see later how to do this.)

Adding Vectors

We can then add vectors by **adding the x parts** and **adding the y parts**:



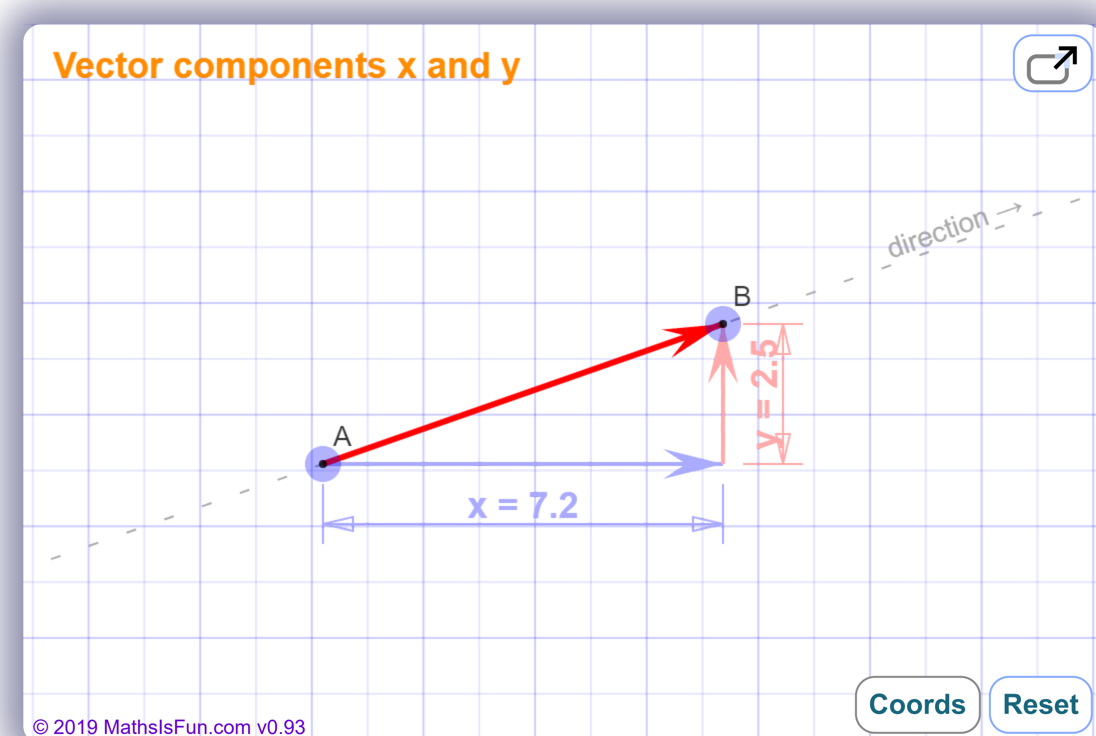
The vector (8, 13) and the vector (26, 7) add up to the vector (34, 20)

Example: add the vectors **a** = (8, 13) and **b** = (26, 7)

c = **a** + **b**

c = (8, 13) + (26, 7) = (8+26, 13+7) = (34, 20)

When we break up a vector like that, each part is called a **component**:



Subtracting Vectors

To subtract, first reverse the vector we want to subtract, then add.

Example: subtract **k** = (4, 5) from **v** = (12, 2)

a = **v** + −**k**

a = (12, 2) + −(4, 5) = (12, 2) + (−4, −5) = (12−4, 2−5) = (8, −3)

Magnitude of a Vector

The magnitude of a vector is shown by two vertical bars on either side of the vector:

$$|\mathbf{a}|$$

OR it can be written with double vertical bars (so as not to confuse it with absolute value):

$$||\mathbf{a}||$$

We use Pythagoras' theorem to calculate it:

$$|\mathbf{a}| = \sqrt{(x^2 + y^2)}$$

Example: what is the magnitude of the vector **b** = (6, 8) ?

$|\mathbf{b}| = \sqrt{(6^2 + 8^2)} = \sqrt{(36+64)} = \sqrt{100} = 10$

A vector with magnitude 1 is called a Unit Vector.

Vector vs Scalar

A **scalar** has **magnitude** (size) **only**.

Scalar: just a number (like 7 or −0.32) ... definitely not a vector.

A **vector** has **magnitude and direction**, and is often written in **bold**, so we know it is not a scalar:

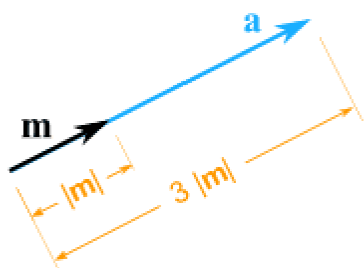
- so **c** is a vector, it has magnitude and direction
- but c is just a value, like 3 or 12.4

Example: k**b** is actually the scalar k times the vector **b**.

Multiplying a Vector by a Scalar

When we multiply a vector by a scalar it is called "scaling" a vector, because we change how big or small the vector is.

Example: multiply the vector **m** = (7, 3) by the scalar 3

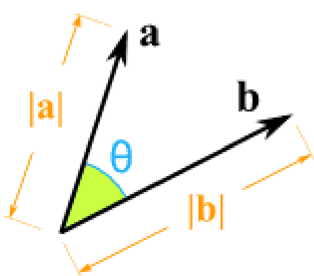


a = 3**m** = (3×7, 3×3) = (21, 9)

It still points in the same direction, but is 3 times longer

(And now you know why numbers are called "scalars", because they "scale" the vector up or down.)

Multiplying a Vector by a Vector (Dot Product and Cross Product)



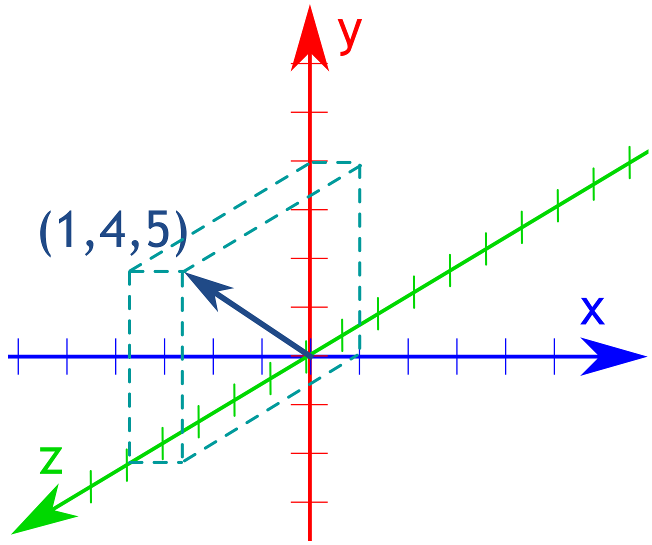
How do we **multiply two vectors** together? There is more than one way!

- The scalar or Dot Product (the result is a scalar).
- The vector or Cross Product (the result is a vector).

(Read those pages for more details.)

More Than 2 Dimensions

Vectors also work perfectly well in 3 or more dimensions:



The vector (1, 4, 5)

Example: add the vectors **a** = (3, 7, 4) and **b** = (2, 9, 11)

c = **a** + **b**

c = (3, 7, 4) + (2, 9, 11) = (3+2, 7+9, 4+11) = (5, 16, 15)

Example: what is the magnitude of the vector **w** = (1, −2, 3) ?

$$|\mathbf{w}| = \sqrt{(1^2 + (-2)^2 + 3^2)} = \sqrt{(1+4+9)} = \sqrt{14}$$

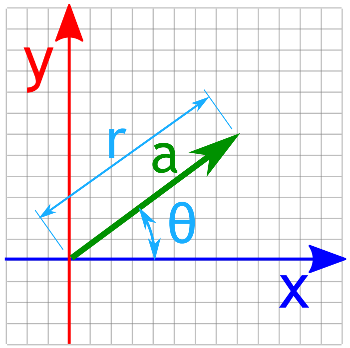
Here is an example with 4 dimensions (but it is hard to draw!):

Example: subtract (1, 2, 3, 4) from (3, 3, 3, 3)

$$\begin{aligned} &(3, 3, 3, 3) + -(1, 2, 3, 4) \\ &= (3, 3, 3, 3) + (-1,-2,-3,-4) \\ &= (3-1, 3-2, 3-3, 3-4) \\ &= (2, 1, 0, -1) \end{aligned}$$

Magnitude and Direction

We may know a vector's magnitude and direction, but want its x and y lengths (or vice versa):



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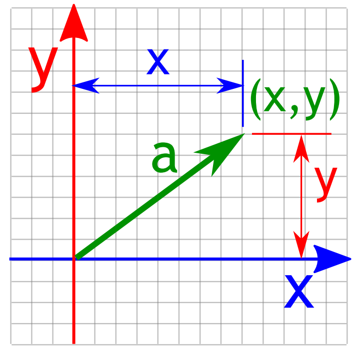


Vector **a** in Polar Coordinates

Vector **a** in Cartesian Coordinates

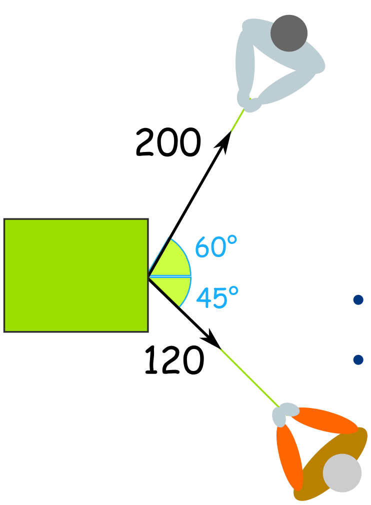
You can read how to convert them at Polar and Cartesian Coordinates, but here is a quick summary:

From Polar Coordinates (r,*θ*) to Cartesian Coordinates (x,y)

- **x = r × cos(*θ*)**
- **y = r × sin(*θ*)**

From Cartesian Coordinates (x,y) to Polar Coordinates (r,θ)

- **r = √ (x^2 + y^2)**
- ***θ* = $\tan^{-1}$ (y / x)**



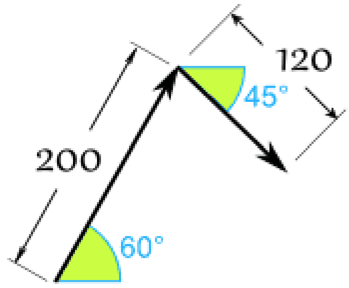
An Example

Sam and Alex are pulling a box.

- Sam pulls with 200 Newtons of force at 60°
- Alex pulls with 120 Newtons of force at 45° as shown

What is the combined force, and its direction?

Let us add the two vectors head to tail:



First convert from polar to Cartesian (to 2 decimals):

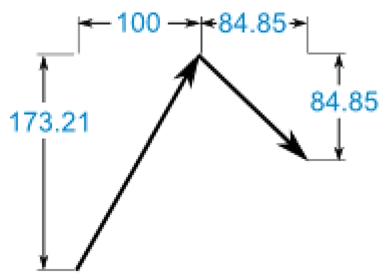
Sam's Vector:

- **$x = r \times \cos(\theta) = 200 \times \cos(60°) = 200 \times 0.5 = 100$**
- **$y = r \times \sin(\theta) = 200 \times \sin(60°) = 200 \times 0.8660 = 173.21$**

Alex's Vector:

- **$x = r \times \cos(\theta) = 120 \times \cos(-45°) = 120 \times 0.7071 = 84.85$**
- **$y = r \times \sin(\theta) = 120 \times \sin(-45°) = 120 \times -0.7071 = -84.85$**

Now we have:



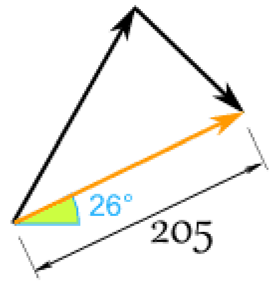
Add them:

$$(100, 173.21) + (84.85, -84.85) = (184.85, 88.36)$$

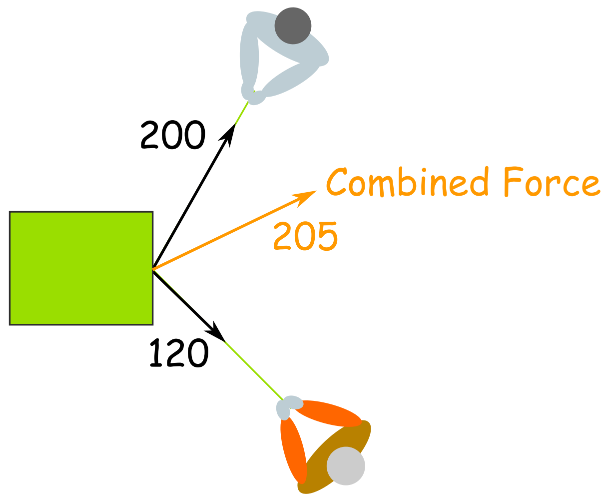
That answer is valid, but let's convert back to polar as the question was in polar:

- **$r = \sqrt{(x^2 + y^2)} = \sqrt{(184.85^2 + 88.36^2)} = 204.88$**
- **$\theta = \tan^{-1}(y / x) = \tan^{-1}(88.36 / 184.85) = 25.5°$**

And we have this (rounded) result:



And it looks like this for Sam and Alex:



They might get a better result if they were shoulder-to-shoulder!

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10