Introduction to Probability, Second Edition

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Preface

This book is an unofficial solution manual for the exercises in *Introduction to Probability, Second Edition* by Joseph Blitzstein and Jessica Hwang.

Chapter 1

Probability and Counting

1.1 Counting

1.1.1

Intuition

Imagine eleven empty slots to place the letters into.

How many ways are there to place the four I-s into the slots? For each placement of Is, can we figure out the number of ways to place the remaining letters into the 7 empty slots?

Solution

We have one M, four I-s, four S-s, and two P-s. There are $\binom{11}{4}$ ways to place the I-s, $\binom{7}{4}$ ways to place S-s, $\binom{3}{2}$ ways to place the P-s, and $\binom{1}{1}$ ways to place the M.

$$\binom{11}{4} \times \binom{7}{4} \times \binom{3}{2} \times \binom{1}{1}$$

1.1.2

a. **Intuition**

If the first digit can't be 0 or 1, how many choices are we left with for the first digit? For each choice of first digit, how many choices do we have for the remaining six digits?

Solution

If the first digit can't be 0 or 1, we have eight choices for the first digit - 2 to 9. The remaining six digits can be anything from 0 to 9. Hence, the solution is

$$8 \times 10^6$$

b. **Intuition**

How many phone numbers start with 911?

Can we use the answer from the previous part to find the desired quantity?

Solution

We can subtract the number of phone numbers that start with 911 from the total number of phone numbers we found in the previous part.

If a phone number starts with 911, it has ten choices for each of the remaining four digits.

$$8 \times 10^6 - 10^4$$

1.1.3

a. **Intuition**

How many choices of restaurants does Fred have on Monday?

Once Fred attends a restaurant on Monday, how many choices of restaurants does he have for the remainder of the week?

Solution

Fred has 10 choices for Monday, 9 choices for Tuesday, 8 choices for Wednesday, 7 choices for Thursday and 6 choices for Friday.

$$10 \times 9 \times 8 \times 7 \times 6$$

b. **Intution**

We are told that Fred will not attend a restaurant he went to the previous day, but can he go to a restaurant he went to two or more days ago?

Solution

For the first restaurant, Fred has 10 choices. For all subsequent days, Fred has 9 choices, since the only restriction is that he doesn't want to eat at the restaurant he ate at the previous day.

$$10 \times 9^4$$

1.1.4

a. **Intuition**

How many matches are there in a *round-robin* tournament?

How many outcomes are possible for each match?

Solution

There are $\binom{n}{2}$ matches.

For a given match, there are two outcomes. Each match has two possible outcomes. We can use the multiplication rule to count the total possible outcomes.

$$2^{\binom{n}{2}}$$

b. **Intuition**

How many opponents will every player play against?

How many times will a given pair of players face each other?

Solution

Since every player plays every other player exactly once, the number of games is the number of ways to pair up n people.

$$\binom{n}{2}$$

1.1.5

a. **Intuition**

How many players are left by the end of a round compared to the number of players at the start of the round?

How many rounds need to pass for a single player to be left standing?

Solution

By the end of each round, half of the players participating in the round are eliminated. So, the problem reduces to finding out how many times the number of players can be halved before a single player is left.

The number of times N can be divided by two is

$$\log_2 N$$

b. **Intuition**

Suppose there are N_r players at the start of round r. If every player plays exactly one game, how many games will be played in round r?

Solution

The number of games in a given round is $\frac{N_r}{2}$. We can sum up these values for all the rounds.

$$\begin{aligned} f(N) &= \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \cdots + \frac{N}{2^{\log_2 N}} \\ &= N \sum_{i=0}^{\log_2 N} \frac{1}{2^i} \\ &= N \times \frac{N-1}{N} \\ &= N - 1 \end{aligned} \tag{1.1}$$

c. **Intuition**

How many players need to be eliminated before the tournament is over?

How many players are eliminated as a result of a single match?

Solution

Tournament is over when a single player is left. Hece, $N-1$ players need to be eliminated. As a result of a match, exactly one player is eliminated. Hence, the number of matches needed to eliminate $N-1$ people is

$$N - 1$$

1.1.6

Intuition

How many ways can we match up twenty chess players if we don't care about who plays with white and who plays with black pieces?

Can we use the answer from the previous part to find the desired quantity?

Solution

There are $\binom{20}{2}$ ways to pair up twenty chess players. For each pairing, we can first let player A play with whites, then let player B play with whites. Thus, for each of the $\binom{20}{2}$ pairs, we have 2 matches for a total of

$$\binom{20}{2} \times 2$$

matches.

1.1.7

a. **Intuition**

How many ways are there to assign three wins to player A?

Out of the remaining four games, how many ways are there to assign two draws and two losses to A?

Solution

There are $\binom{7}{3}$ ways to assign three wins to player A. For a specific combination of three games won by A, there are $\binom{4}{2}$ ways to assign two draws to A. There is only one way to assign two losses to A from the remaining two games, namely, A losses both games.

$$\binom{7}{3} \times \binom{4}{2} \times \binom{2}{2}$$

b. **Intuition**

Can A get 4 points if A never wins? What if A wins more than 4 games?

List the possible outcomes of games that award 4 points to A.

Solution

If A were to draw every game, there would need to be at least 8 games for A to obtain 4 points, so A has to win at least 1 game. Similarly, if A wins more than 4 games, they will have more than 4 points.

Case 1: A wins 1 game and draws 6.

This case amounts to selecting 1 out of 7 for A to win and assigning a draw for the other 6 games. Hence, there are 7 possibilities.

Case 2: A wins 2 games and draws 4.

There are $\binom{7}{2}$ ways to assign 2 wins to A. For each of them, there are $\binom{5}{4}$ ways to assign four draws to A out of the remaining 5 games. Player B wins the remaining game. The total number of possibilites for this case is $\binom{7}{2} \times \binom{5}{4}$.

Case 3: A wins 3 games and draws 2.

There are $\binom{7}{3}$ ways to assign 3 wins to A. For each of them, there are $\binom{4}{2}$ ways to assign two draws to A out of the remaining 4 games. B wins the remaining 2 games. The total number of possibilites for this case is $\binom{7}{3} \times \binom{4}{2}$.

Case 4: A wins 4 games and loses 3.

There are $\binom{7}{4}$ ways to assign 4 wins to A. B wins the remaining 3 games. The total number of possibilites for this case is $\binom{7}{4}$.

Summing up the number of possibilities in each of the cases we get

$$\binom{7}{1} + \binom{7}{2} \times \binom{5}{4} + \binom{7}{3} \times \binom{4}{2} + \binom{7}{4}$$

c. **Intuition**

Given the final score of 4 to 3 and the fact that the match will end if either of the players reaches 4 points, could B have been the player to win the last game?

Suppose A wins the last game. Could A have won only 1 game out of the first 6?

Count the number of possibilites for the case when A wins the last game and the number of possibilities for the case when A draws the last game.

Solution

If B were to win the last game, that would mean that A had already obtained 4 points prior to the last game, so the last game would not be played at all. Hence, B could not have won the last game.

Case 1: A wins 3 out of the first 6 games and wins the last game.

There are $\binom{6}{3}$ ways to assign 3 wins to A out of the first 6 games. The other 3 games end in a draw. The number of possibilities then is $\binom{6}{3}$.

Case 2: A wins 2 and draws 2 out of the first 6 games and wins the last game.

There are $\binom{6}{2}$ ways to assign 2 wins to A out of the first 6 games. From the 4 remaining games, there are $\binom{4}{2}$ ways to assign 2 draws. The remaining 2 games are won by B. The number of possibilities is $\binom{6}{2} \times \binom{4}{2}$.

Case 3: The last game ends in a draw.

This case implies that A had 3.5 and B had 2.5 points by the end of game 6.

Case 3.1: A wins 3 and draws 1 out of the first 6 games.

There are $\binom{6}{3}$ ways to assign 3 wins to A out of the first 6 games. There are $\binom{3}{1}$ ways to assign a draw out of the remaining 3 games. B wins the other 2 games. The number of possibilities is $\binom{6}{3} \times \binom{3}{1}$.

Case 3.2: A wins 2 and draws 3 out of the first 6 games.

There are $\binom{6}{2}$ ways to assign 2 wins to A out of the first 6 games. There are $\binom{4}{3}$ ways to assign 3 draws out of the remaining 4 games. B wins the remaining game. The number of possibilities is $\binom{6}{2} \times \binom{4}{3}$.

Case 3.3: A wins 1 and draws 5 of the first 6 games.

There are $\binom{6}{1}$ ways to assign a win to A out of the first 6 games.

The total number of possibilities then is

$$\binom{6}{3} + \binom{6}{2} \times \binom{4}{2} + \binom{6}{3} \times \binom{3}{1} + \binom{6}{2} \times \binom{4}{3} + \binom{6}{1}$$