Lecture 1

# Vectors and Vector Operations

Wednesday, Aug. 28

## Today's Goals

- You should be able to perform the operations of vector addition, scalar multiplication, and the dot product on vectors.
- You should be able to represent vectors graphically in 2- and 3-dimensional space.
- You should be able to calculate the magnitude of a vector and the angle between two vectors.

## Vectors

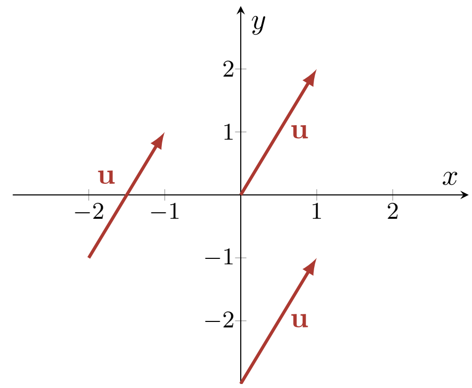
A **vector** is an ordered list of $n$ real numbers. The set of all $n$-dimensional vectors is denoted $\mathbb{R}^n$.

Some examples of vectors:

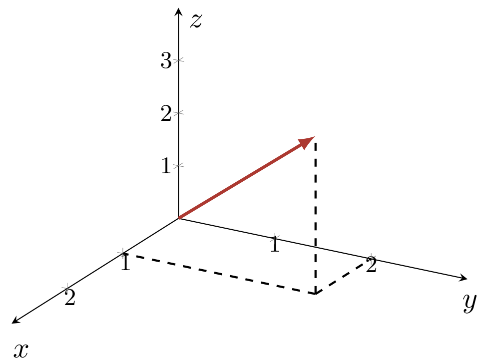
- $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (a vector in $\mathbb{R}^3$)
- $\mathbf{w} = [-7, \pi]$ (a vector in $\mathbb{R}^2$)
- $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$ (a general vector in $\mathbb{R}^5$)
- $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ (a general vector in $\mathbb{R}^n$)
- $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (the zero vector in $\mathbb{R}^2$)

Graphing vectors:

The vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$ is plotted several times below. Note that it's still the same vector $\mathbf{u}$ regardless of where its starting point is.




The vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ is plotted in 3D space below.




## Vector Operations

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let $k$ be a scalar (real number). We have the following operations:

- **Vector addition:** $\mathbf{v} + \mathbf{w}$ is the vector obtained by adding each pair of corresponding entries of $\mathbf{v}$ and $\mathbf{w}$.
- **Scalar multiplication:** $k\mathbf{v}$ is the vector obtained by multiplying every entry of $\mathbf{v}$ by $k$.

## Magnitude

The **magnitude** (or **length**) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

## Unit vector

A **unit vector** is a vector with magnitude 1.

For any nonzero vector $\mathbf{v}$, a unit vector in the same direction as $\mathbf{v}$ is $\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

**1** Let $\mathbf{v} = [4, 2, 2]$ and $\mathbf{w} = [-3, 1, 0]$. Find a unit vector in the direction of $\mathbf{v} + 2\mathbf{w}$.

**Solution**

First we determine the vector $\mathbf{u} = \mathbf{v} + 2\mathbf{w}$:

$$\mathbf{u} = [4, 2, 2] + 2[-3, 1, 0] = [-2, 4, 2]$$

Now we can divide $\mathbf{u}$ by its magnitude to get a unit vector in the same direction.

$$|\mathbf{u}| = \sqrt{(-2)^2 + 4^2 + 2^2} = 2\sqrt{6}$$
$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{2\sqrt{6}}[-2, 4, 2] = \boxed{\left[-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]}$$

## Standard Basis Vectors

The **standard basis vectors** in $\mathbb{R}^3$ are the vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Any vector in $\mathbb{R}^3$ can be written as a **linear combination** of the standard basis vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

**2** Four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ are acting on a box. The first three forces are $\mathbf{F}_1 = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{F}_2 = 7\mathbf{i} - 3\mathbf{k}$, and $\mathbf{F}_3 = -3\mathbf{i} - \mathbf{j} - \mathbf{k}$ (where the forces are all measured in Newtons). What vector must $\mathbf{F}_4$ be in order for the box to be in *static equilibrium*, where the sum of all forces is equal to 0?

**Solution**

We want to satisfy the condition

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0,$$

which we can solve for the force $\mathbf{F}_4$:

$$\begin{aligned} \mathbf{F}_4 &= -\mathbf{F}_1 - \mathbf{F}_2 - \mathbf{F}_3 \\ &= -(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) - (7\mathbf{i} - 3\mathbf{k}) - (-3\mathbf{i} - \mathbf{j} - \mathbf{k}) \\ &= -5\mathbf{i} - \mathbf{j} + 9\mathbf{k} \end{aligned}$$

Thus $\mathbf{F}_4 = \boxed{\begin{bmatrix} -5 \\ -1 \\ 9 \end{bmatrix}}$.

## Dot Product

The **dot product** of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

If $\theta$ is the angle between the two vectors, the dot product is

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta.$$

**3** Compute each of the following.

(a) $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

**Solution**

$4(1) + 2(-3) = \boxed{-2}$

(b) $[1, -2, 1] \cdot [4, 1, -2]$

**Solution**

$1(4) + (-2)(1) + 1(-2) = \boxed{0}$

What does the sign of the dot product tell us about the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$?

- If $\mathbf{v} \cdot \mathbf{w} > 0$, then the angle between $\mathbf{v}$ and $\mathbf{w}$ is smaller than $90°$.
- If $\mathbf{v} \cdot \mathbf{w} < 0$, then the angle between $\mathbf{v}$ and $\mathbf{w}$ is greater than $90°$.
- If $\mathbf{v} \cdot \mathbf{w} = 0$, then the angle between $\mathbf{v}$ and $\mathbf{w}$ is exactly $90°$. In this case, we say that the vectors are **orthogonal**.

Some useful properties of the dot product (for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$):

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$
- $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$
- $|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$